Chapter 1

Functions and Graphs

Checkpoint Solutions

Checkpoint 1.1: Evaluating Functions

Instruction

For the function $f(x) = x^2 - 3x + 5$ evaluate

(a) $f(1)$

(b) $f(a+h)$

Solution

(a) $f(1) = 1^2 - 3 \cdot 1 + 5 = 1 - 3 + 5 = 3.$

(b) $f(a+h) = (a+h)^2 - 3(a+h) + 5 = a^2 + 2ah + h^2 - 3a - 3h + 5.$

Answer

(a) $f(1) = 3.$

(b) $f(a+h) = a^2 + 2ah + h^2 - 3a - 3h + 5.$

Checkpoint 1.2: Finding Domain and Range

Instruction

Find the domain and range for $f(x) = \sqrt{4-2x} + 5.$

Solution

i To find the domain of f, we need the expression $4 - 2x \geq 0$, due to that real negative numbers do not have a square root. Solving this inequality, we conclude that the domain is $\{x \mid x \leq 2\}$.

ii To find the range of f, we note that since $\sqrt{4-2x} \geq 0$, it follows that $f(x) = \sqrt{4-2x} + 5 \geq 5$. Therefore, the range of f must be a subset of the set $\{y \mid y \geq 5\}$.

To show that every element in this set is in the range of f, we need to show that for all y in this set, there exists a real number x in the domain such that $f(x) = y$. Let $y \geq 5$. Then, $f(x) = y$ if and only if

$$\sqrt{4-2x} + 5 = y.$$

Solving this equation for x, we see that x must solve the equation

$$\sqrt{4-2x} = y - 5.$$

Since $y \geq 5$, such an x could exist. Squaring both sides of the above equation we have

$$4 - 2x = (y-5)^2.$$

Therefore we need

$$-2x = (y-5)^2 - 4,$$

which implies

$$x = 2 - \frac{(y-5)^2}{2}.$$

We just need to verify that x is in the domain of f. Since the domain of f consists of all real numbers less or equal to 2, and

$$2 - \frac{(y-5)^2}{2} \leq 2,$$

there does exist an x in the domain of f. We conclude that the range of f is $\{y \mid y \geq 5\}$.

Answer

Domain $= \{x \mid x \leq 2\}$, range $= \{y \mid y \geq 5\}$.

Checkpoint 1.3: Finding Zeroes

Instruction

Find the zeroes of $f(x) = x^3 - 5x^2 + 6x$.

Solution

The zeroes of a function are the values of x where $f(x) = 0$. To find the zeroes, we need to solve

$$f(x) = x^3 - 5x^2 + 6x = 0.$$

Factor out x

$$f(x) = x(x^2 - 5x + 6) = 0.$$

We can continue factoring by pure inspection, with the goal of finding a pair of numbers that add up to -5 and whose product is 6. This pair of numbers turns out to be -2 and -3, leading to the factoring

$$f(x) = x(x-2)(x-3) = 0.$$

From the above complete factoring of f, we conclude that there are three zeroes when x is 0, 2, and 3.

Answer

$x = 0, 2, 3.$

Checkpoint 1.4: Combining Functions Using Mathematical Operations

Instruction

For $f(x) = x^2 + 3$ and $g(x) = 2x - 5$, find $(f/g)(x)$ and state its domain.

Solution

To find $(f/g)(x)$ we write the function with the quotient operator

$$\frac{f}{g}(x) = \frac{x^2+3}{2x-5}.$$

The domain of this function is $\{x \mid x \neq \frac{5}{2}\}$.

Answer

$\frac{f}{g}(x) = \frac{x^2+3}{2x-5}$. The domain is $\{x \mid x \neq \frac{5}{2}\}$.

Checkpoint 1.5: Compositions of Functions

Instruction

Let $f(x) = 2 - 5x$. Let $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$.

Solution

$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 2 - 5\sqrt{x}.$

Answer

$(f \circ g)(x) = 2 - 5\sqrt{x}.$

Checkpoint 1.6: Application Involving a Composite Function

Instruction

If items are on sale for 10% off their original price, and a customer has a coupon for an additional 30% off, what will be the final price for an item that is originally x dollars, after applying the coupon to the sale price?

Solution

Since the sale price 10% off the original price, if an item is x dollars, its sale price is given by

$$f(x) = 0.90x.$$

Since the coupon entitles an individual to 30% off the price of any item, if an item is y dollars, the price after applying the coupon, is given by

$$g(y) = 0.70y.$$

Therefore, if the price is originally x dollars, its price after applying the coupon to the sale price will be

$$(g \circ f)(x) = g(f(x)) = (0.70)0.90x = 0.63x..$$

Answer

$(g \circ f)(x) = 0.63x.$

Exercise Solutions

Exercise 1.1.1

Instruction

Assuming the relation in table 1.1.

(a) Determine the domain and the range of the relation.

(b) State whether the relation is a function.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

Table 1.1: Relation between x and y in exercise 1.1.1

Solution

(a) The domain of the relation is the set of unique x values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

The range of the relation is the set of unique y values,

$$\{0, 1, 4, 9\}.$$

(b) This relation is a function, each input is a assigned to exactly one output.

Answer

(a) Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, range = $\{0, 1, 4, 9\}$.

(b) Yes, a function.

Exercise 1.1.2

Instruction

Assuming the relation in table 1.2.

(a) Determine the domain and the range of the relation.

(b) State whether the relation is a function.

x	-3	-2	-1	0	1	2	3
y	-2	-8	-1	0	1	8	-2

Table 1.2: Relation between x and y in exercise 1.1.2

Solution

(a) The domain of the relation is the set of unique x values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

The range of the relation is the set of unique y values,

$$\{-8, -2, -1, 0, 1, 8\}.$$

(b) This relation is a function, each input is a assigned to exactly one output.

Answer

(a) Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, range = $\{-8, -2, -1, 0, 1, 8\}$.

(b) Yes, a function.

Exercise 1.1.3

Instruction

Assuming the relation in table 1.3.

(a) Determine the domain and the range of the relation.

(b) State whether the relation is a function.

x	1	2	3	0	1	2	3
y	-3	-2	-1	0	1	2	3

Table 1.3: Relation between x and y in exercise 1.1.3

Solution

(a) The domain of the relation is the set of unique x values,

$$\{0, 1, 2, 3\}.$$

The range of the relation is the set of unique y values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

(b) This relation is not a function, each input is not assigned to exactly one output. Take for example $x = 1$ that can cause both $y = -3$ and $y = 1$.

Answer

(a) Domain = $\{0, 1, 2, 3\}$, range = $\{-3, -2, -1, 0, 1, 2, 3\}$.

(b) No, not a function.

Exercise 1.1.4

Instruction

Assuming the relation in table 1.4.

(a) Determine the domain and the range of the relation.

(b) State whether the relation is a function.

x	1	2	3	4	5	6	7
y	1	1	1	1	1	1	1

Table 1.4: Relation between x and y in exercise 1.1.4

Solution

(a) The domain of the relation is the set of unique x values,

$$\{1, 2, 3, 4, 5, 6, 7\}.$$

The range of the relation is the set of unique y values,

$$\{1\}.$$

(b) This relation is a function, each input is a assigned to exactly one output.

Answer

(a) Domain = $\{1, 2, 3, 4, 5, 6, 7\}$, range = $\{1\}$.

(b) Yes, a function.

Exercise 1.1.5

Instruction

Assuming the relation in table 1.5.

(a) Determine the domain and the range of the relation.

(b) State whether the relation is a function.

x	3	5	8	10	15	21	33
y	3	2	1	0	1	2	3

Table 1.5: Relation between x and y in exercise 1.1.5

Solution

(a) The domain of the relation is the set of unique x values,

$$\{3, 5, 8, 10, 15, 21, 33\}.$$

The range of the relation is the set of unique y values,

$$\{0, 1, 2, 3\}.$$

(b) This relation is a function, each input is a assigned to exactly one output.

Answer

(a) Domain = $\{3, 5, 8, 10, 15, 21, 33\}$, range = $\{0, 1, 2, 3\}$.

(b) Yes, a function.

Exercise 1.1.6

Instruction

Assuming the relation in table 1.6.

(a) Determine the domain and the range of the relation.

(b) State whether the relation is a function.

x	-7	-2	-2	0	1	3	6
y	11	5	1	-1	-2	4	11

Table 1.6: Relation between x and y in exercise 1.1.6

Solution

(a) The domain of the relation is the set of unique x values,

$$\{-7, -2, 0, 1, 3, 6\}.$$

The range of the relation is the set of unique y values,

$$\{-2, -1, 1, 4, 5, 11\}.$$

(b) This relation is not a function, each input is not assigned to exactly one output. See $x = -2$, that can cause both $y = 1$ and $y = 5$.

Answer

(a) Domain = $\{-7, -2, 0, 1, 3, 6\}$, range = $\{-2, -1, 1, 4, 5, 11\}$.

(b) No, not a function.

Exercise 1.1.7

Instruction

Find the below values for the function $f(x) = 5x - 2$, if they exist, then simplify.

(a) $f(0)$

(b) $f(1)$

(c) $f(3)$

(d) $f(-x)$

(e) $f(a)$

(f) $f(a+h)$

Solution

(a) $f(0) = 5 \cdot 0 - 2 = 0 - 2 = -2.$

(b) $f(1) = 5 \cdot 1 - 2 = 5 - 2 = 3.$

(c) $f(2) = 5 \cdot 3 - 2 = 15 - 2 = 13.$

(d) $f(-x) = 5(-x) - 2 = -5x - 2.$

(e) $f(a) = 5a - 2.$

(f) $f(a+h) = 5(a+h) - 2 = 5a + 5h - 2.$

Answer

(a) $-2.$

(b) $3.$

(c) $13.$

(d) $-5x - 2.$

(e) $5a - 2.$

(f) $5a + 5h - 2.$

Exercise 1.1.8

Instruction

Find the below values for the function $f(x) = 4x^2 - 3x + 1$, if they exist, then simplify.

(a) $f(0)$

(b) $f(1)$

(c) $f(3)$

(d) $f(-x)$

(e) $f(a)$

(f) $f(a+h)$

Solution

(a) $f(0) = 4 \cdot 0^2 - 3 \cdot 0 + 1 = 4 \cdot 0 - 0 + 1 = 0 - 0 + 1 = 1.$

(b) $f(1) = 4 \cdot 1^2 - 3 \cdot 1 + 1 = 4 \cdot 1 - 3 + 1 = 4 - 3 + 1 = 2.$

(c) $f(3) = 4 \cdot 3^2 - 3 \cdot 3 + 1 = 4 \cdot 9 - 9 + 1 = 36 - 9 + 1 = 28.$

(d) $f(-x) = 4(-x)^2 - 3(-x) + 1 = 4x^2 + 3x + 1.$

(e) $f(a) = 4a^2 - 3a + 1.$

(f) $$\begin{aligned} f(a+h) &= 4(a+h)^2 - 3(a+h) + 1 \\ &= 4(a^2 + 2ah + h^2) - 3a - 3h + 1 \\ &= 4a^2 + 4h^2 + 8ah - 3a - 3h + 1. \end{aligned}$$

Answer

(a) 1.

(b) 2.

(c) 28.

(d) $4x^2 + 3x + 1.$

(e) $4a^2 - 3a + 1.$

(f) $4a^2 + 4h^2 + 8ah - 3a - 3h + 1.$

Exercise 1.1.15

Find the domain, range, and all zeros/intercepts, if any, of the function $g(x) = \sqrt{8x - 1}$.

Solution

i The domain of the square root function is $[0, \infty)$, which implies $8x - 1 \geq 0$. Solving for x gives $x \geq \frac{1}{8}$.

ii To find the range of g, we note that $\sqrt{8x-1} \geq 0$. Therefore, the range of g must be a subset of the set $\{y \mid y \geq 0\}$. To show that every element in this set is in the range of g, we need to show that for a given y in this set, there exists a real number x in the domain such that $g(x) = y$.

Let $y \geq 0$. Then $g(x) = y$ if and only if

$$\sqrt{8x - 1} = y.$$

We are interested in x, and will solve this equation for x. Since $y \geq 0$ such an x could exist. Squaring both sides of this equation, we have

$$8x - 1 = y^2.$$

Therefore, we need

$$8x = y^2 + 1,$$

which implies

$$x = \frac{y^2 + 1}{8}.$$

We just need to verify that x is in the domain of g. Since the domain of g consists of all real numbers greater than or equal to $1/8$, and

$$\frac{y^2 + 1}{8} \geq \frac{1}{8},$$

there does exist an x in the domain of g. We conclude that the range of g is $\{y \mid y \geq 0\}$.

iii To find the zeroes, solve $g(x) = \sqrt{8x - 1}0$. We discover that g have one zero at $x = -1/8$.

iv The y-intercept is given by $(0, g(0))$. Since $x = 0$ isn't in the domain of g, it follows that that there aren't any intercepts.

Answer

Domain = $x \geq \frac{1}{8}$, range = $\{y \mid y \geq 0\}$, zeroes $x = -1/8$, no intercepts.

Exercise 1.1.23

Sketch the graph for the function $f(x) = 3x - 6$ with the aid of table 1.7.

x	-3	-2	-1	0	1	2	3
y	-15	-12	-9	-6	-3	0	3

Table 1.7: Relation between x and y in exercise 1.1.23

Solution

Begin by sketching the axes. We choose the same scale on both axes to not distort the graph. We choose the range for both axes to be -15 to 15, allowing us to plot all the points from table 1.7, see figure 1.1.

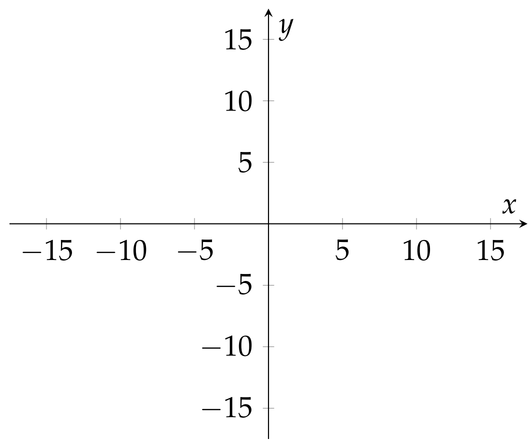

Figure 1.1: Empty graph with just the axes

After having sketched the axes we add markers based on the data in table 1.7, see figure 1.2.

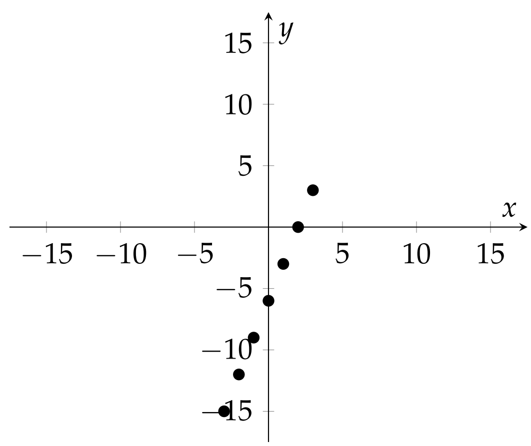

Figure 1.2: Graph with added markers

We then connect the markers with line segments. In this particular case the result will be a single straight line so we can use a ruler when sketching, , see figure 1.3.

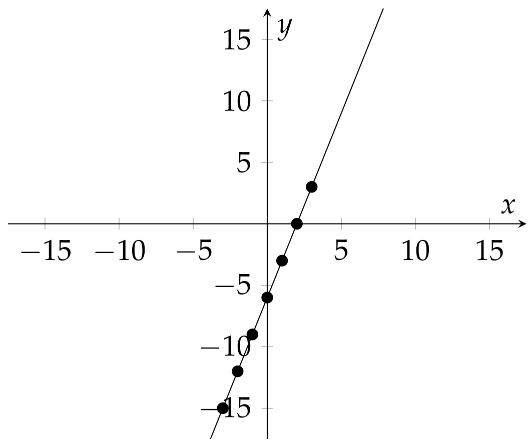

Figure 1.3: Graph with connected markers

Answer

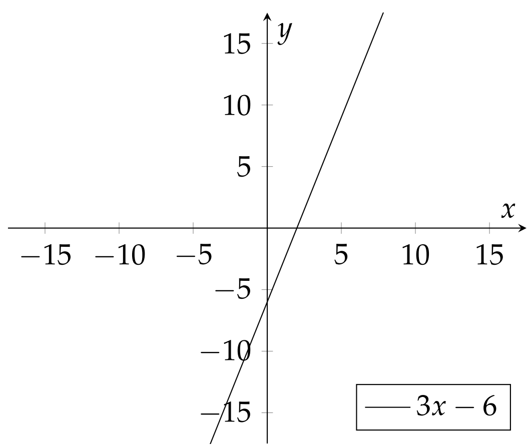

Figure 1.4: Answer to exercise 1.1.23

Exercise 1.1.29

Use the vertical line test to determine whether the graph in figure 1.5 represent a function. Assume that the graph continues at both ends beyond the given grid. If the graph represents a function, then determine the following for the graph:

(a) Domain and range

(b) x-intercept, if any (estimate where necessary)

(c) y-intercept, if any (estimate where necessary)

(d) The intervals for which the function is increasing

(e) The intervals for which the function is decreasing

(f) The intervals for which the function is constant

(g) Symmetry about any axis and/or the origin

(h) Whether the function is even, odd, or neither

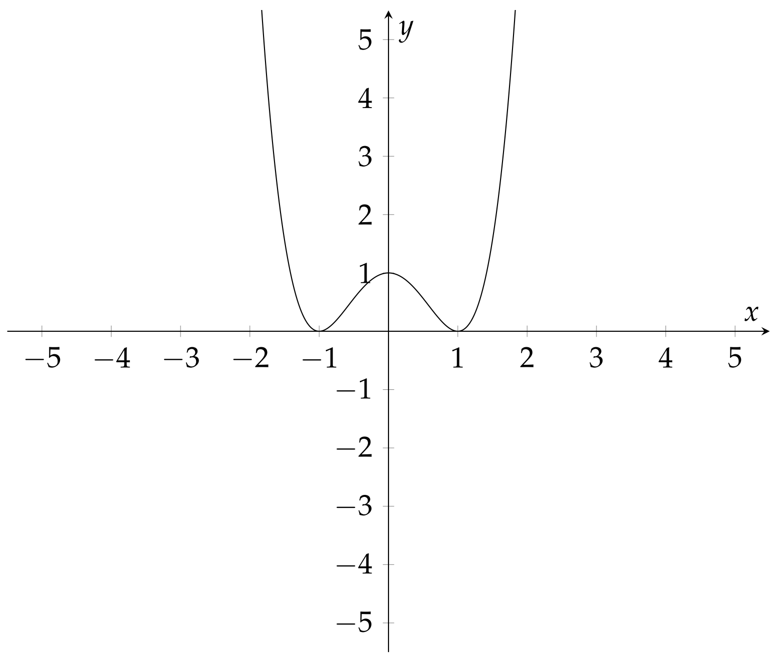

Figure 1.5: Graph for exercise 1.1.29

Solution

The graph in figure 1.5 do represent a function because every vertical line that may be drawn intersects the graph no more than once. See figure 1.6 for an example of a vertical line with one intersection of the graph. We could slide this line over the entire graph and there would always only be at most one intersection.

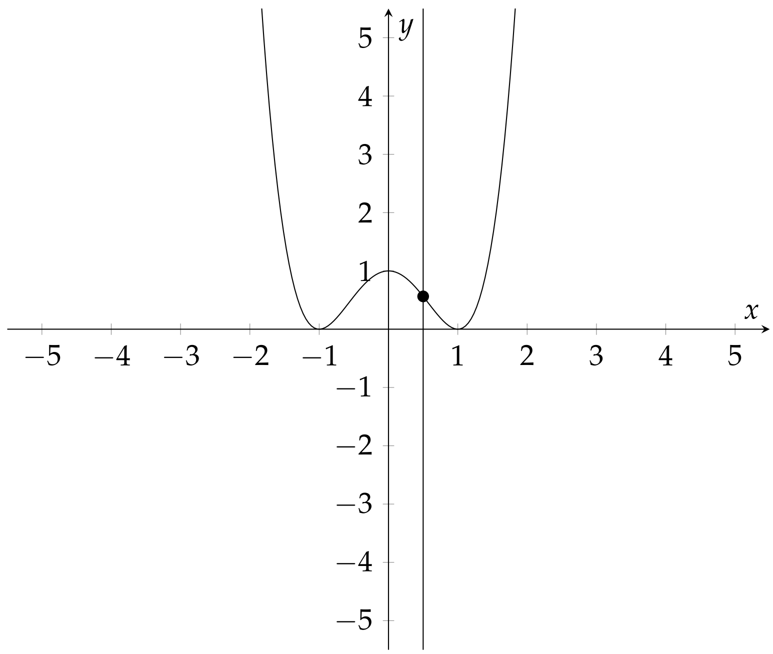

Figure 1.6: Vertical line test illustration

(a)
   i The function seems to grow rapidly as x goes towards $\pm\infty$, but there will still always be a y value. We conclude that the domain is all real numbers.

   ii y is always greater or equal to 0, this is the range.

(b) y is zero for $x = -1$, and $x = 1$, these are the x-intercepts.

(c) The y-intercept is $y = 1$.

(d) The function is increasing for the intervals $-1 < x < 0$ and $1 < x < \infty$.

(e) The function is decreasing for the intervals $-\infty < x < -1$ and $0 < x < 1$.

(f) The function changes from decreasing/increasing when x is -1, 0, and 1, but there are no intervals for which the function is constant.

(g) $(-x, y)$ is on the graph whenever (x, y) is on the graph, in other words the function is symmetric around the y-axis.

(h) The function is not odd because $f(-x) \neq -f(x)$ for all x in the domain. Take for example $x = 0.5$ for which $f(-x) \approx 0.6$ and $-f(x) \approx -0.6$.

The function is even because $f(-x) = f(x)$ for all x. Take for example $x = 0.5$ for which $f(-x) \approx 0.6$ and $f(x) \approx 0.6$.

Answer

Graph represents a function.

(a) Domain: all real numbers, range: $y \geq 0$.

(b) $x = -1$ and $x = 1$.

(c) $y = 1$.

(d) $-1 < x < 0$ and $1 < x < \infty$.

(e) $-\infty < x < -1$ and $0 < x < 1$.

(f) Not constant.

(g) y-axis.

(h) Even.

Exercise 1.1.37

Instruction

For the pair of functions $f(x) = x - 8$ and $g(x) = 5x^2$, find the below new listed functions. Also determine the domain for each of these new functions.

(a) $f + g$

(b) $f - g$

(c) $f \cdot g$

(d) f/g

Solution

(a) Add the two given functions to form the requested function,

$$f + g = x - 8 + 5x^2 = 5x^2 + x - 8.$$

The domain of the above new function is all real numbers.

(b) Subtract the two given functions to form the requested function,

$$f - g = x - 8 - 5x^2 = -5x^2 + x - 8.$$

The domain of the above new function is all real numbers.

(c) Multiply the two given functions to form the requested function,

$$f \cdot g = (x-8)5x^2 = 5x^3 - 8x^2.$$

The domain of the above new function is all real numbers.

(d) Divide the two given functions to form the requested function,

$$\frac{f}{g} = \frac{x-8}{5x^2}.$$

The division is defined except for for $x = 0$, the domain is hence $x \neq 0$.

Answer

(a) $5x^2 + x - 8$, domain: all real numbers.

(b) $-5x^2 + x - 8$, domain: all real numbers.

(c) $5x^3 - 8x^2$, domain: all real numbers.

(d) $\frac{x-8}{5x^2}$, domain: $x \neq 0$.