

Chapter 1

Functions and Graphs

Checkpoint Solutions

Checkpoint 1.1: Evaluating Functions

Instruction

For the function $f(x) = x^2 - 3x + 5$ evaluate

- (a) $f(1)$
- (b) $f(a + h)$

Solution

- (a) $f(1) = 1^2 - 3 \cdot 1 + 5 = 1 - 3 + 5 = 3.$
- (b) $f(a + h) = (a + h)^2 - 3(a + h) + 5 = a^2 + 2ah + h^2 - 3a - 3h + 5.$

Answer

- (a) $f(1) = 3.$
- (b) $f(a + h) = a^2 + 2ah + h^2 - 3a - 3h + 5.$

Checkpoint 1.2: Finding Domain and Range

Instruction

Find the domain and range for $f(x) = \sqrt{4 - 2x} + 5.$

Solution

- i To find the domain of f , we need the expression $4 - 2x \geq 0$, due to that real negative numbers do not have a square root. Solving this inequality, we conclude that the domain is $\{x \mid x \leq 2\}.$

ii To find the range of f , we note that since $\sqrt{4-2x} \geq 0$, it follows that $f(x) = \sqrt{4-2x} + 5 \geq 5$. Therefore, the range of f must be a subset of the set $\{y \mid y \geq 5\}$.

To show that every element in this set is in the range of f , we need to show that for all y in this set, there exists a real number x in the domain such that $f(x) = y$. Let $y \geq 5$. Then, $f(x) = y$ if and only if

$$\sqrt{4-2x} + 5 = y.$$

Solving this equation for x , we see that x must solve the equation

$$\sqrt{4-2x} = y - 5.$$

Since $y \geq 5$, such an x could exist. Squaring both sides of the above equation we have

$$4 - 2x = (y - 5)^2.$$

Therefore we need

$$-2x = (y - 5)^2 - 4,$$

which implies

$$x = 2 - \frac{(y - 5)^2}{2}.$$

We just need to verify that x is in the domain of f . Since the domain of f consists of all real numbers less or equal to 2, and

$$2 - \frac{(y - 5)^2}{2} \leq 2,$$

there does exist an x in the domain of f . We conclude that the range of f is $\{y \mid y \geq 5\}$.

Answer

Domain = $\{x \mid x \leq 2\}$, range = $\{y \mid y \geq 5\}$.

Checkpoint 1.3: Finding Zeroes

Instruction

Find the zeroes of $f(x) = x^3 - 5x^2 + 6x$.

Solution

The zeroes of a function are the values of x where $f(x) = 0$. To find the zeroes, we need to solve

$$f(x) = x^3 - 5x^2 + 6x = 0.$$

Factor out x

$$f(x) = x(x^2 - 5x + 6) = 0.$$

We can continue factoring by pure inspection, with the goal of finding a pair of numbers that add up to -5 and whose product is 6 . This pair of numbers turns out to be -2 and -3 , leading to the factoring

$$f(x) = x(x - 2)(x - 3) = 0.$$

From the above complete factoring of f , we conclude that there are three zeroes when x is 0 , 2 , and 3 .

Answer

$$x = 0, 2, 3.$$

Checkpoint 1.4: Combining Functions Using Mathematical Operations

Instruction

For $f(x) = x^2 + 3$ and $g(x) = 2x - 5$, find $(f/g)(x)$ and state its domain.

Solution

To find $(f/g)(x)$ we write the function with the quotient operator

$$\frac{f}{g}(x) = \frac{x^2 + 3}{2x - 5}.$$

The domain of this function is $\{x \mid x \neq \frac{5}{2}\}$.

Answer

$$\frac{f}{g}(x) = \frac{x^2 + 3}{2x - 5}. \text{ The domain is } \{x \mid x \neq \frac{5}{2}\}.$$

Checkpoint 1.5: Compositions of Functions

Instruction

Let $f(x) = 2 - 5x$. Let $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$.

Solution

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 2 - 5\sqrt{x}.$$

Answer

$$(f \circ g)(x) = 2 - 5\sqrt{x}.$$

Checkpoint 1.6: Application Involving a Composite Function

Instruction

If items are on sale for 10% off their original price, and a customer has a coupon for an additional 30% off, what will be the final price for an item that is originally x dollars, after applying the coupon to the sale price?

Solution

Since the sale price 10% off the original price, if an item is x dollars, its sale price is given by

$$f(x) = 0.90x.$$

Since the coupon entitles an individual to 30% off the price of any item, if an item is y dollars, the price after applying the coupon, is given by

$$g(y) = 0.70y.$$

Therefore, if the price is originally x dollars, its price after applying the coupon to the sale price will be

$$(g \circ f)(x) = g(f(x)) = (0.70)0.90x = 0.63x..$$

Answer

$$(g \circ f)(x) = 0.63x.$$

Exercise Solutions

Exercise 1.1.1

Instruction

Assuming the relation in table 1.1.

- (a) Determine the domain and the range of the relation.
- (b) State whether the relation is a function.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

Table 1.1: Relation between x and y in exercise 1.1.1

Solution

- (a) The domain of the relation is the set of unique x values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

The range of the relation is the set of unique y values,

$$\{0, 1, 4, 9\}.$$

- (b) This relation is a function, each input is assigned to exactly one output.

Answer

- (a) Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, range = $\{0, 1, 4, 9\}$.
(b) Yes, a function.

Exercise 1.1.2**Instruction**

Assuming the relation in table 1.2.

- (a) Determine the domain and the range of the relation.
(b) State whether the relation is a function.

x	-3	-2	-1	0	1	2	3
y	-2	-8	-1	0	1	8	-2

Table 1.2: Relation between x and y in exercise 1.1.2

Solution

- (a) The domain of the relation is the set of unique x values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

The range of the relation is the set of unique y values,

$$\{-8, -2, -1, 0, 1, 8\}.$$

- (b) This relation is a function, each input is assigned to exactly one output.

Answer

- (a) Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, range = $\{-8, -2, -1, 0, 1, 8\}$.
(b) Yes, a function.

Exercise 1.1.3**Instruction**

Assuming the relation in table 1.3.

- (a) Determine the domain and the range of the relation.
(b) State whether the relation is a function.

x	1	2	3	0	1	2	3
y	-3	-2	-1	0	1	2	3

Table 1.3: Relation between x and y in exercise 1.1.3

Solution

- (a) The domain of the relation is the set of unique x values,

$$\{0, 1, 2, 3\}.$$

The range of the relation is the set of unique y values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

- (b) This relation is not a function, each input is not assigned to exactly one output.
Take for example $x = 1$ that can cause both $y = -3$ and $y = 1$.

Answer

- (a) Domain = $\{0, 1, 2, 3\}$, range = $\{-3, -2, -1, 0, 1, 2, 3\}$.
(b) No, not a function.

Exercise 1.1.4**Instruction**

Assuming the relation in table 1.4.

- (a) Determine the domain and the range of the relation.
(b) State whether the relation is a function.

x	1	2	3	4	5	6	7
y	1	1	1	1	1	1	1

Table 1.4: Relation between x and y in exercise 1.1.4

Solution

- (a) The domain of the relation is the set of unique x values,

$$\{1, 2, 3, 4, 5, 6, 7\}.$$

The range of the relation is the set of unique y values,

$$\{1\}.$$

- (b) This relation is a function, each input is assigned to exactly one output.

Answer

- (a) Domain = $\{1, 2, 3, 4, 5, 6, 7\}$, range = $\{1\}$.

- (b) Yes, a function.

Exercise 1.1.5

Instruction

Assuming the relation in table 1.5.

- (a) Determine the domain and the range of the relation.

- (b) State whether the relation is a function.

x	3	5	8	10	15	21	33
y	3	2	1	0	1	2	3

Table 1.5: Relation between x and y in exercise 1.1.5

Solution

- (a) The domain of the relation is the set of unique x values,

$$\{3, 5, 8, 10, 15, 21, 33\}.$$

The range of the relation is the set of unique y values,

$$\{0, 1, 2, 3\}.$$

- (b) This relation is a function, each input is assigned to exactly one output.

Answer

- (a) Domain = $\{3, 5, 8, 10, 15, 21, 33\}$, range = $\{0, 1, 2, 3\}$.
(b) Yes, a function.

Exercise 1.1.6**Instruction**

Assuming the relation in table 1.6.

- (a) Determine the domain and the range of the relation.
(b) State whether the relation is a function.

x	-7	-2	-2	0	1	3	6
y	11	5	1	-1	-2	4	11

Table 1.6: Relation between x and y in exercise 1.1.6

Solution

- (a) The domain of the relation is the set of unique x values,

$$\{-7, -2, 0, 1, 3, 6\}.$$

The range of the relation is the set of unique y values,

$$\{-2, -1, 1, 4, 5, 11\}.$$

- (b) This relation is not a function, each input is not assigned to exactly one output.
See $x = -2$, that can cause both $y = 1$ and $y = 5$.

Answer

- (a) Domain = $\{-7, -2, 0, 1, 3, 6\}$, range = $\{-2, -1, 1, 4, 5, 11\}$.
(b) No, not a function.

Exercise 1.1.7**Instruction**

Find the below values for the function $f(x) = 5x - 2$, if they exist, then simplify.

- (a) $f(0)$

- (b) $f(1)$
- (c) $f(3)$
- (d) $f(-x)$
- (e) $f(a)$
- (f) $f(a + h)$

Solution

- (a) $f(0) = 5 \cdot 0 - 2 = 0 - 2 = -2.$
- (b) $f(1) = 5 \cdot 1 - 2 = 5 - 2 = 3.$
- (c) $f(2) = 5 \cdot 3 - 2 = 15 - 2 = 13.$
- (d) $f(-x) = 5(-x) - 2 = -5x - 2.$
- (e) $f(a) = 5a - 2.$
- (f) $f(a + h) = 5(a + h) - 2 = 5a + 5h - 2.$

Answer

- (a) $-2.$
- (b) $3.$
- (c) $13.$
- (d) $-5x - 2.$
- (e) $5a - 2.$
- (f) $5a + 5h - 2.$

Exercise 1.1.8

Instruction

Find the below values for the function $f(x) = 4x^2 - 3x + 1$, if they exist, then simplify.

- (a) $f(0)$
- (b) $f(1)$
- (c) $f(3)$
- (d) $f(-x)$
- (e) $f(a)$
- (f) $f(a + h)$

Solution

$$(a) f(0) = 4 \cdot 0^2 - 3 \cdot 0 + 1 = 4 \cdot 0 - 0 + 1 = 0 - 0 + 1 = 1.$$

$$(b) f(1) = 4 \cdot 1^2 - 3 \cdot 1 + 1 = 4 \cdot 1 - 3 + 1 = 4 - 3 + 1 = 2.$$

$$(c) f(3) = 4 \cdot 3^2 - 3 \cdot 3 + 1 = 4 \cdot 9 - 9 + 1 = 36 - 9 + 1 = 28.$$

$$(d) f(-x) = 4(-x)^2 - 3(-x) + 1 = 4x^2 + 3x + 1.$$

$$(e) f(a) = 4a^2 - 3a + 1.$$

$$\begin{aligned}(f) f(a+h) &= 4(a+h)^2 - 3(a+h) + 1 \\ &= 4(a^2 + 2ah + h^2) - 3a - 3h + 1 \\ &= 4a^2 + 4h^2 + 8ah - 3a - 3h + 1.\end{aligned}$$

Answer

$$(a) 1.$$

$$(b) 2.$$

$$(c) 28.$$

$$(d) 4x^2 + 3x + 1.$$

$$(e) 4a^2 - 3a + 1.$$

$$(f) 4a^2 + 4h^2 + 8ah - 3a - 3h + 1.$$

Exercise 1.1.15

Find the domain, range, and all zeros/intercepts, if any, of the function $g(x) = \sqrt{8x-1}$.

Solution

i The domain of the square root function is $[0, \infty)$, which implies $8x - 1 \geq 0$. Solving for x gives $x \geq \frac{1}{8}$.

ii To find the range of g , we note that $\sqrt{8x-1} \geq 0$. Therefore, the range of g must be a subset of the set $\{y \mid y \geq 0\}$. To show that every element in this set is in the range of g , we need to show that for a given y in this set, there exists a real number x in the domain such that $g(x) = y$.

Let $y \geq 0$. Then $g(x) = y$ if and only if

$$\sqrt{8x-1} = y.$$

We are interested in x , and will solve this equation for x . Since $y \geq 0$ such an x could exist. Squaring both sides of this equation, we have

$$8x - 1 = y^2.$$

Therefore, we need

$$8x = y^2 + 1,$$

which implies

$$x = \frac{y^2 + 1}{8}.$$

We just need to verify that x is in the domain of g . Since the domain of g consists of all real numbers greater than or equal to $1/8$, and

$$\frac{y^2 + 1}{8} \geq \frac{1}{8},$$

there does exist an x in the domain of g . We conclude that the range of g is $\{y \mid y \geq 0\}$.

- iii To find the zeroes, solve $g(x) = \sqrt{8x - 10}$. We discover that g have one zero at $x = -1/8$.
- iv The y -intercept is given by $(0, g(0))$. Since $x = 0$ isn't in the domain of g , it follows that that there aren't any intercepts.

Answer

Domain = $x \geq \frac{1}{8}$, range = $\{y \mid y \geq 0\}$, zeroes $x = -1/8$, no intercepts.

Exercise 1.1.23

Sketch the graph for the function $f(x) = 3x - 6$ with the aid of table 1.7.

x	-3	-2	-1	0	1	2	3
y	-15	-12	-9	-6	-3	0	3

Table 1.7: Relation between x and y in exercise 1.1.23

Solution

Begin by sketching the axes. We choose the same scale on both axes to not distort the graph. We choose the range for both axes to be -15 to 15, allowing us to plot all the points from table 1.7, see figure 1.1.



Figure 1.1: Empty graph with just the axes

After having sketched the axes we add markers based on the data in table 1.7, see figure 1.2.



Figure 1.2: Graph with added markers

We then connect the markers with line segments. In this particular case the result will be a single straight line so we can use a ruler when sketching, see figure 1.3.



Figure 1.3: Graph with connected markers

Answer



Figure 1.4: Answer to exercise 1.1.23

Exercise 1.1.29

Use the vertical line test to determine whether the graph in figure 1.5 represent a function. Assume that the graph continues at both ends beyond the given grid. If the graph represents a function, then determine the following for the graph:

- Domain and range
- x -intercept, if any (estimate where necessary)
- y -intercept, if any (estimate where necessary)
- The intervals for which the function is increasing

- (e) The intervals for which the function is decreasing
- (f) The intervals for which the function is constant
- (g) Symmetry about any axis and/or the origin
- (h) Whether the function is even, odd, or neither

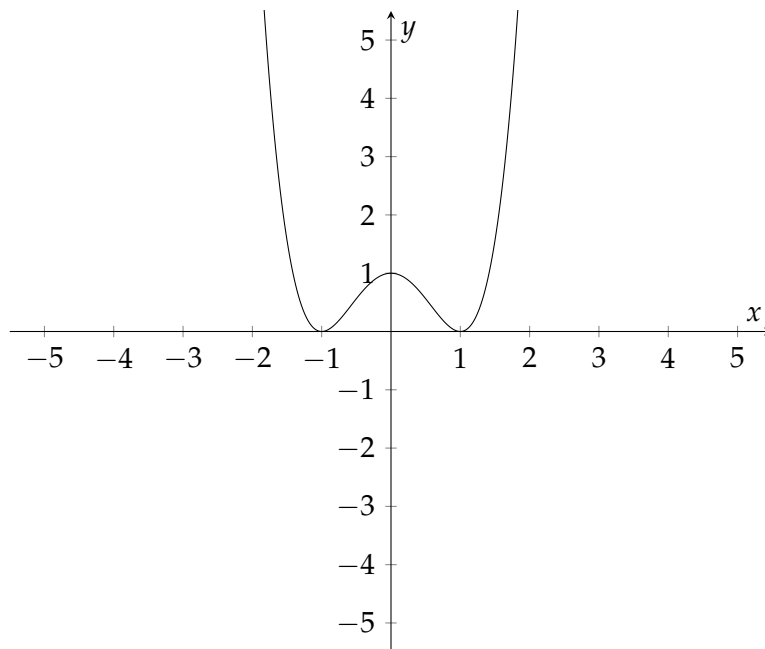


Figure 1.5: Graph for exercise 1.1.29

Solution

The graph in figure 1.5 do represent a function because every vertical line that may be drawn intersects the graph no more than once. See figure 1.6 for an example of a vertical line with one intersection of the graph. We could slide this line over the entire graph and there would always only be at most one intersection.

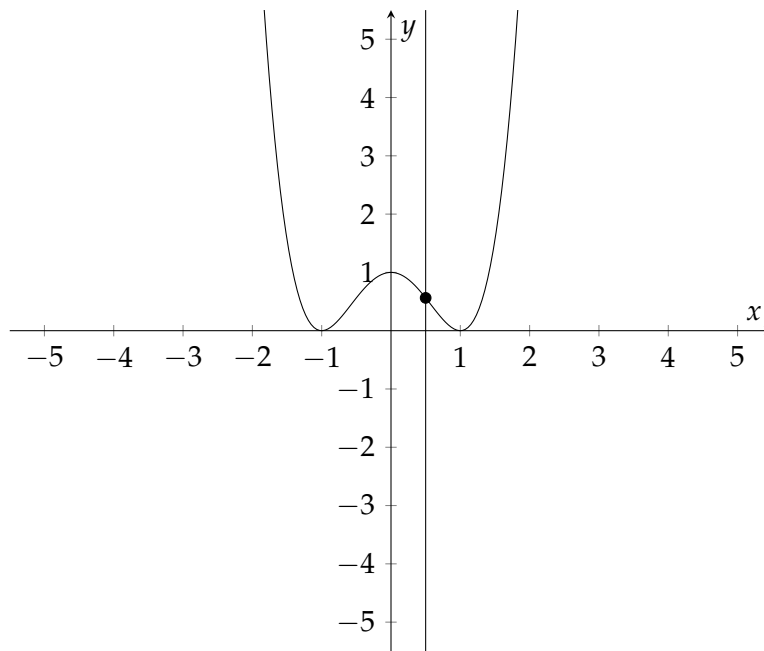


Figure 1.6: Vertical line test illustration

- (a)
 - i The function seems to grow rapidly as x goes towards $\pm\infty$, but there will still always be a y value. We conclude that the domain is all real numbers.
 - ii y is always greater or equal to 0, this is the range.
- (b) y is zero for $x = -1$, and $x = 1$, these are the x -intercepts.
- (c) The y -intercept is $y = 1$.
- (d) The function is increasing for the intervals $-1 < x < 0$ and $1 < x < \infty$.
- (e) The function is decreasing for the intervals $-\infty < x < -1$ and $0 < x < 1$.
- (f) The function changes from decreasing/increasing when x is -1 , 0 , and 1 , but there are no intervals for which the function is constant.
- (g) $(-x, y)$ is on the graph whenever (x, y) is on the graph, in other words the function is symmetric around the y -axis.
- (h) The function is not odd because $f(-x) \neq -f(x)$ for all x in the domain. Take for example $x = 0.5$ for which $f(-x) \approx 0.6$ and $-f(x) \approx -0.6$.
 The function is even because $f(-x) = f(x)$ for all x . Take for example $x = 0.5$ for which $f(-x) \approx 0.6$ and $f(x) \approx 0.6$.

Answer

Graph represents a function.

- (a) Domain: all real numbers, range: $y \geq 0$.
- (b) $x = -1$ and $x = 1$.
- (c) $y = 1$.
- (d) $-1 < x < 0$ and $1 < x < \infty$.
- (e) $-\infty < x < -1$ and $0 < x < 1$.
- (f) Not constant.
- (g) y -axis.
- (h) Even.