

# Chapter 1

## Functions and Graphs

### Checkpoint Solutions

#### Checkpoint 1.1: Evaluating Functions

##### Instruction

For the function  $f(x) = x^2 - 3x + 5$  evaluate

- (a)  $f(1)$
- (b)  $f(a + h)$

##### Solution

- (a)  $f(1) = 1^2 - 3 \cdot 1 + 5 = 1 - 3 + 5 = 3.$
- (b)  $f(a + h) = (a + h)^2 - 3(a + h) + 5 = a^2 + 2ah + h^2 - 3a - 3h + 5.$

##### Answer

- (a)  $f(1) = 3.$
- (b)  $f(a + h) = a^2 + 2ah + h^2 - 3a - 3h + 5.$

#### Checkpoint 1.2: Finding Domain and Range

##### Instruction

Find the domain and range for  $f(x) = \sqrt{4 - 2x} + 5.$

##### Solution

- i To find the domain of  $f$ , we need the expression  $4 - 2x \geq 0$ , due to that real negative numbers do not have a square root. Solving this inequality, we conclude that the domain is  $\{x \mid x \leq 2\}.$

ii To find the range of  $f$ , we note that since  $\sqrt{4-2x} \geq 0$ , it follows that  $f(x) = \sqrt{4-2x} + 5 \geq 5$ . Therefore, the range of  $f$  must be a subset of the set  $\{y \mid y \geq 5\}$ .

To show that every element in this set is in the range of  $f$ , we need to show that for all  $y$  in this set, there exists a real number  $x$  in the domain such that  $f(x) = y$ . Let  $y \geq 5$ . Then,  $f(x) = y$  if and only if

$$\sqrt{4-2x} + 5 = y.$$

Solving this equation for  $x$ , we see that  $x$  must solve the equation

$$\sqrt{4-2x} = y - 5.$$

Since  $y \geq 5$ , such an  $x$  could exist. Squaring both sides of the above equation we have

$$4 - 2x = (y - 5)^2.$$

Therefore we need

$$-2x = (y - 5)^2 - 4,$$

which implies

$$x = 2 - \frac{(y - 5)^2}{2}.$$

We just need to verify that  $x$  is in the domain of  $f$ . Since the domain of  $f$  consists of all real numbers less or equal to 2, and

$$2 - \frac{(y - 5)^2}{2} \leq 2,$$

there does exist an  $x$  in the domain of  $f$ . We conclude that the range of  $f$  is  $\{y \mid y \geq 5\}$ .

### Answer

Domain =  $\{x \mid x \leq 2\}$ , range =  $\{y \mid y \geq 5\}$ .

### Checkpoint 1.3: Finding Zeroes

#### Instruction

Find the zeroes of  $f(x) = x^3 - 5x^2 + 6x$ .

#### Solution

The zeroes of a function are the values of  $x$  where  $f(x) = 0$ . To find the zeroes, we need to solve

$$f(x) = x^3 - 5x^2 + 6x = 0.$$

Factor out  $x$

$$f(x) = x(x^2 - 5x + 6) = 0.$$

We can continue factoring by pure inspection, with the goal of finding a pair of numbers that add up to  $-5$  and whose product is  $6$ . This pair of numbers turns out to be  $-2$  and  $-3$ , leading to the factoring

$$f(x) = x(x - 2)(x - 3) = 0.$$

From the above complete factoring of  $f$ , we conclude that there are three zeroes when  $x$  is  $0$ ,  $2$ , and  $3$ .

**Answer**

$$x = 0, 2, 3.$$

### Checkpoint 1.4: Combining Functions Using Mathematical Operations

**Instruction**

For  $f(x) = x^2 + 3$  and  $g(x) = 2x - 5$ , find  $(f/g)(x)$  and state its domain.

**Solution**

To find  $(f/g)(x)$  we write the function with the quotient operator

$$\frac{f}{g}(x) = \frac{x^2 + 3}{2x - 5}.$$

The domain of this function is  $\{x \mid x \neq \frac{5}{2}\}$ .

**Answer**

$$\frac{f}{g}(x) = \frac{x^2 + 3}{2x - 5}. \text{ The domain is } \{x \mid x \neq \frac{5}{2}\}.$$

### Checkpoint 1.5: Compositions of Functions

**Instruction**

Let  $f(x) = 2 - 5x$ . Let  $g(x) = \sqrt{x}$ . Find  $(f \circ g)(x)$ .

**Solution**

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 2 - 5\sqrt{x}.$$

**Answer**

$$(f \circ g)(x) = 2 - 5\sqrt{x}.$$

## Checkpoint 1.6: Application Involving a Composite Function

### Instruction

If items are on sale for 10% off their original price, and a customer has a coupon for an additional 30% off, what will be the final price for an item that is originally  $x$  dollars, after applying the coupon to the sale price?

### Solution

Since the sale price 10% off the original price, if an item is  $x$  dollars, its sale price is given by

$$f(x) = 0.90x.$$

Since the coupon entitles an individual to 30% off the price of any item, if an item is  $y$  dollars, the price after applying the coupon, is given by

$$g(y) = 0.70y.$$

Therefore, if the price is originally  $x$  dollars, its price after applying the coupon to the sale price will be

$$(g \circ f)(x) = g(f(x)) = (0.70)0.90x = 0.63x..$$

### Answer

$$(g \circ f)(x) = 0.63x.$$

## Exercise Solutions

### Exercise 1.1.1

#### Instruction

Assuming the relation in table 1.1.

- (a) Determine the domain and the range of the relation.
- (b) State whether the relation is a function.

$x$	-3	-2	-1	0	1	2	3
$y$	9	4	1	0	1	4	9

Table 1.1: Relation between  $x$  and  $y$  in exercise 1.1.1

**Solution**

- (a) The domain of the relation is the set of unique  $x$  values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

The range of the relation is the set of unique  $y$  values,

$$\{0, 1, 4, 9\}.$$

- (b) This relation is a function, each input is assigned to exactly one output.

**Answer**

- (a) Domain =  $\{-3, -2, -1, 0, 1, 2, 3\}$ , range =  $\{0, 1, 4, 9\}$ .  
(b) Yes, a function.

**Exercise 1.1.2****Instruction**

Assuming the relation in table 1.2.

- (a) Determine the domain and the range of the relation.  
(b) State whether the relation is a function.

$x$	-3	-2	-1	0	1	2	3
$y$	-2	-8	-1	0	1	8	-2

Table 1.2: Relation between  $x$  and  $y$  in exercise 1.1.2

**Solution**

- (a) The domain of the relation is the set of unique  $x$  values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

The range of the relation is the set of unique  $y$  values,

$$\{-8, -2, -1, 0, 1, 8\}.$$

- (b) This relation is a function, each input is assigned to exactly one output.

**Answer**

- (a) Domain =  $\{-3, -2, -1, 0, 1, 2, 3\}$ , range =  $\{-8, -2, -1, 0, 1, 8\}$ .  
(b) Yes, a function.

**Exercise 1.1.3****Instruction**

Assuming the relation in table 1.3.

- (a) Determine the domain and the range of the relation.  
(b) State whether the relation is a function.

$x$	1	2	3	0	1	2	3
$y$	-3	-2	-1	0	1	2	3

Table 1.3: Relation between  $x$  and  $y$  in exercise 1.1.3

**Solution**

- (a) The domain of the relation is the set of unique  $x$  values,

$$\{0, 1, 2, 3\}.$$

The range of the relation is the set of unique  $y$  values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

- (b) This relation is not a function, each input is not assigned to exactly one output.  
Take for example  $x = 1$  that can cause both  $y = -3$  and  $y = 1$ .

**Answer**

- (a) Domain =  $\{0, 1, 2, 3\}$ , range =  $\{-3, -2, -1, 0, 1, 2, 3\}$ .  
(b) No, not a function.

**Exercise 1.1.4****Instruction**

Assuming the relation in table 1.4.

- (a) Determine the domain and the range of the relation.  
(b) State whether the relation is a function.

$x$	1	2	3	4	5	6	7
$y$	1	1	1	1	1	1	1

Table 1.4: Relation between  $x$  and  $y$  in exercise 1.1.4

### Solution

- (a) The domain of the relation is the set of unique  $x$  values,

$$\{1, 2, 3, 4, 5, 6, 7\}.$$

The range of the relation is the set of unique  $y$  values,

$$\{1\}.$$

- (b) This relation is a function, each input is assigned to exactly one output.

### Answer

- (a) Domain =  $\{1, 2, 3, 4, 5, 6, 7\}$ , range =  $\{1\}$ .

- (b) Yes, a function.

### Exercise 1.1.5

#### Instruction

Assuming the relation in table 1.5.

- (a) Determine the domain and the range of the relation.

- (b) State whether the relation is a function.

$x$	3	5	8	10	15	21	33
$y$	3	2	1	0	1	2	3

Table 1.5: Relation between  $x$  and  $y$  in exercise 1.1.5

### Solution

- (a) The domain of the relation is the set of unique  $x$  values,

$$\{3, 5, 8, 10, 15, 21, 33\}.$$

The range of the relation is the set of unique  $y$  values,

$$\{0, 1, 2, 3\}.$$

- (b) This relation is a function, each input is assigned to exactly one output.

**Answer**

- (a) Domain =  $\{3, 5, 8, 10, 15, 21, 33\}$ , range =  $\{0, 1, 2, 3\}$ .  
(b) Yes, a function.

**Exercise 1.1.6****Instruction**

Assuming the relation in table 1.6.

- (a) Determine the domain and the range of the relation.  
(b) State whether the relation is a function.

$x$	-7	-2	-2	0	1	3	6
$y$	11	5	1	-1	-2	4	11

Table 1.6: Relation between  $x$  and  $y$  in exercise 1.1.6

**Solution**

- (a) The domain of the relation is the set of unique  $x$  values,

$$\{-7, -2, 0, 1, 3, 6\}.$$

The range of the relation is the set of unique  $y$  values,

$$\{-2, -1, 1, 4, 5, 11\}.$$

- (b) This relation is not a function, each input is not assigned to exactly one output.  
See  $x = -2$ , that can cause both  $y = 1$  and  $y = 5$ .

**Answer**

- (a) Domain =  $\{-7, -2, 0, 1, 3, 6\}$ , range =  $\{-2, -1, 1, 4, 5, 11\}$ .  
(b) No, not a function.

**Exercise 1.1.7****Instruction**

Find the below values for the function  $f(x) = 5x - 2$ , if they exist, then simplify.

- (a)  $f(0)$



- (b)  $f(1)$
- (c)  $f(3)$
- (d)  $f(-x)$
- (e)  $f(a)$
- (f)  $f(a + h)$

**Solution**

- (a)  $f(0) = 5 \cdot 0 - 2 = 0 - 2 = -2.$
- (b)  $f(1) = 5 \cdot 1 - 2 = 5 - 2 = 3.$
- (c)  $f(2) = 5 \cdot 3 - 2 = 15 - 2 = 13.$
- (d)  $f(-x) = 5(-x) - 2 = -5x - 2.$
- (e)  $f(a) = 5a - 2.$
- (f)  $f(a + h) = 5(a + h) - 2 = 5a + 5h - 2.$

**Answer**

- (a)  $-2.$
- (b)  $3.$
- (c)  $13.$
- (d)  $-5x - 2.$
- (e)  $5a - 2.$
- (f)  $5a + 5h - 2.$

**Exercise 1.1.8**

**Instruction**

Find the below values for the function  $f(x) = 4x^2 - 3x + 1$ , if they exist, then simplify.

- (a)  $f(0)$
- (b)  $f(1)$
- (c)  $f(3)$
- (d)  $f(-x)$
- (e)  $f(a)$
- (f)  $f(a + h)$

**Solution**

$$(a) f(0) = 4 \cdot 0^2 - 3 \cdot 0 + 1 = 4 \cdot 0 - 0 + 1 = 0 - 0 + 1 = 1.$$

$$(b) f(1) = 4 \cdot 1^2 - 3 \cdot 1 + 1 = 4 \cdot 1 - 3 + 1 = 4 - 3 + 1 = 2.$$

$$(c) f(3) = 4 \cdot 3^2 - 3 \cdot 3 + 1 = 4 \cdot 9 - 9 + 1 = 36 - 9 + 1 = 28.$$

$$(d) f(-x) = 4(-x)^2 - 3(-x) + 1 = 4x^2 + 3x + 1.$$

$$(e) f(a) = 4a^2 - 3a + 1.$$

$$\begin{aligned}(f) f(a+h) &= 4(a+h)^2 - 3(a+h) + 1 \\ &= 4(a^2 + 2ah + h^2) - 3a - 3h + 1 \\ &= 4a^2 + 4h^2 + 8ah - 3a - 3h + 1.\end{aligned}$$

**Answer**

$$(a) 1.$$

$$(b) 2.$$

$$(c) 28.$$

$$(d) 4x^2 + 3x + 1.$$

$$(e) 4a^2 - 3a + 1.$$

$$(f) 4a^2 + 4h^2 + 8ah - 3a - 3h + 1.$$

**Exercise 1.1.15**

Find the domain, range, and all zeros/intercepts, if any, of the function  $g(x) = \sqrt{8x-1}$ .

**Solution**

i The domain of the square root function is  $[0, \infty)$ , which implies  $8x - 1 \geq 0$ . Solving for  $x$  gives  $x \geq \frac{1}{8}$ .

ii To find the range of  $g$ , we note that  $\sqrt{8x-1} \geq 0$ . Therefore, the range of  $g$  must be a subset of the set  $\{y \mid y \geq 0\}$ . To show that every element in this set is in the range of  $g$ , we need to show that for a given  $y$  in this set, there exists a real number  $x$  in the domain such that  $g(x) = y$ .

Let  $y \geq 0$ . Then  $g(x) = y$  if and only if

$$\sqrt{8x-1} = y.$$

We are interested in  $x$ , and will solve this equation for  $x$ . Since  $y \geq 0$  such an  $x$  could exist. Squaring both sides of this equation, we have

$$8x - 1 = y^2.$$

Therefore, we need

$$8x = y^2 + 1,$$

which implies

$$x = \frac{y^2 + 1}{8}.$$

We just need to verify that  $x$  is in the domain of  $g$ . Since the domain of  $g$  consists of all real numbers greater than or equal to  $1/8$ , and

$$\frac{y^2 + 1}{8} \geq \frac{1}{8},$$

there does exist an  $x$  in the domain of  $g$ . We conclude that the range of  $g$  is  $\{y \mid y \geq 0\}$ .

- iii To find the zeroes, solve  $g(x) = \sqrt{8x - 10}$ . We discover that  $g$  have one zero at  $x = -1/8$ .
- iv The  $y$ -intercept is given by  $(0, g(0))$ . Since  $x = 0$  isn't in the domain of  $g$ , it follows that that there aren't any intercepts.

### Answer

Domain =  $x \geq \frac{1}{8}$ , range =  $\{y \mid y \geq 0\}$ , zeroes  $x = -1/8$ , no intercepts.

### Exercise 1.1.23

Sketch the graph for the function  $f(x) = 3x - 6$  with the aid of table 1.7.

$x$	-3	-2	-1	0	1	2	3
$y$	-15	-12	-9	-6	-3	0	3

Table 1.7: Relation between  $x$  and  $y$  in exercise 1.1.23

### Solution

Begin by sketching the axes. We choose the same scale on both axes to not distort the graph. We choose the range for both axes to be -15 to 15, allowing us to plot all the points from table 1.7, see figure 1.1.



Figure 1.1: Empty graph with just the axes

After having sketched the axes we add markers based on the data in table 1.7, see figure 1.2.



Figure 1.2: Graph with added markers

We then connect the markers with line segments. In this particular case the result will be a single straight line so we can use a ruler when sketching, see figure 1.3.



Figure 1.3: Graph with connected markers

**Answer**



Figure 1.4: Answer to exercise 1.1.23

### Exercise 1.1.29

Use the vertical line test to determine whether the graph in figure 1.5 represent a function. Assume that the graph continues at both ends beyond the given grid. If the graph represents a function, then determine the following for the graph:

- Domain and range
- $x$ -intercept, if any (estimate where necessary)
- $y$ -intercept, if any (estimate where necessary)
- The intervals for which the function is increasing

- (e) The intervals for which the function is decreasing
- (f) The intervals for which the function is constant
- (g) Symmetry about any axis and/or the origin
- (h) Whether the function is even, odd, or neither

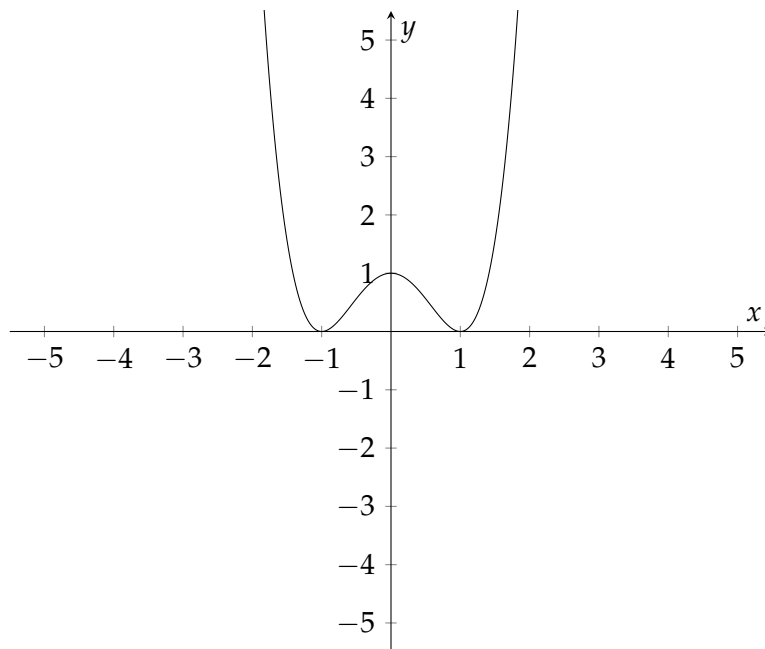


Figure 1.5: Graph for exercise 1.1.29

### Solution

The graph in figure 1.5 do represent a function because every vertical line that may be drawn intersects the graph no more than once. See figure 1.6 for an example of a vertical line with one intersection of the graph. We could slide this line over the entire graph and there would always only be at most one intersection.

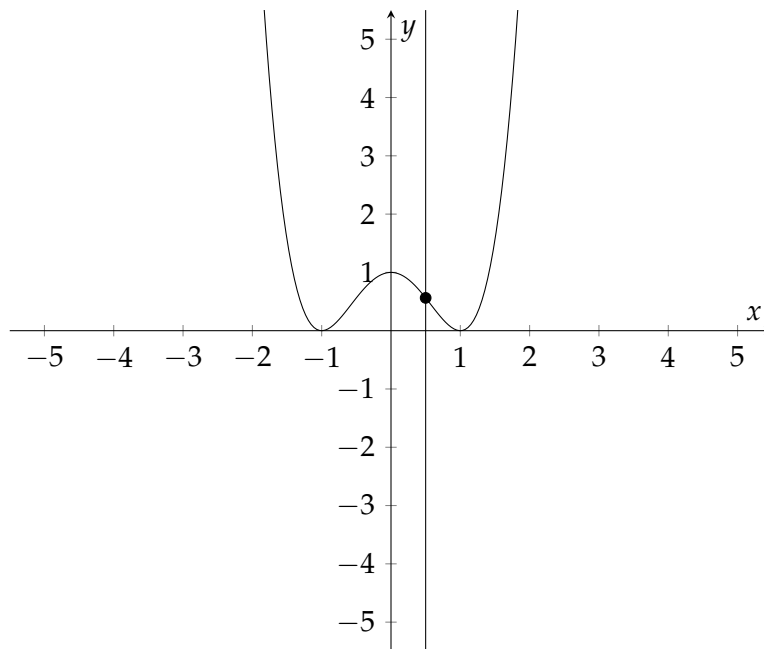


Figure 1.6: Vertical line test illustration

- (a)
  - i The function seems to grow rapidly as  $x$  goes towards  $\pm\infty$ , but there will still always be a  $y$  value. We conclude that the domain is all real numbers.
  - ii  $y$  is always greater or equal to 0, this is the range.
- (b)  $y$  is zero for  $x = -1$ , and  $x = 1$ , these are the  $x$ -intercepts.
- (c) The  $y$ -intercept is  $y = 1$ .
- (d) The function is increasing for the intervals  $-1 < x < 0$  and  $1 < x < \infty$ .
- (e) The function is decreasing for the intervals  $-\infty < x < -1$  and  $0 < x < 1$ .
- (f) The function changes from decreasing/increasing when  $x$  is  $-1$ ,  $0$ , and  $1$ , but there are no intervals for which the function is constant.
- (g)  $(-x, y)$  is on the graph whenever  $(x, y)$  is on the graph, in other words the function is symmetric around the  $y$ -axis.
- (h) The function is not odd because  $f(-x) \neq -f(x)$  for all  $x$  in the domain. Take for example  $x = 0.5$  for which  $f(-x) \approx 0.6$  and  $-f(x) \approx -0.6$ .  
 The function is even because  $f(-x) = f(x)$  for all  $x$ . Take for example  $x = 0.5$  for which  $f(-x) \approx 0.6$  and  $f(x) \approx 0.6$ .

**Answer**

Graph represents a function.

- (a) Domain: all real numbers, range:  $y \geq 0$ .
- (b)  $x = -1$  and  $x = 1$ .
- (c)  $y = 1$ .
- (d)  $-1 < x < 0$  and  $1 < x < \infty$ .
- (e)  $-\infty < x < -1$  and  $0 < x < 1$ .
- (f) Not constant.
- (g)  $y$ -axis.
- (h) Even.

**Exercise 1.1.37****Instruction**

For the pair of functions  $f(x) = x - 8$  and  $g(x) = 5x^2$ , find each of the below new functions. Also determine the domain for each of these new functions.

- (a)  $f + g$
- (b)  $f - g$
- (c)  $f \cdot g$
- (d)  $f/g$

**Solution**

- (a) Add the two given functions to form the requested function,

$$f + g = x - 8 + 5x^2 = 5x^2 + x - 8.$$

The domain of the above new function is all real numbers.

- (b) Subtract the two given functions to form the requested function,

$$f - g = x - 8 - 5x^2 = -5x^2 + x - 8.$$

The domain of the above new function is all real numbers.



(c) Multiply the two given functions to form the requested function,

$$f \cdot g = (x - 8)5x^2 = 5x^3 - 8x^2.$$

The domain of the above new function is all real numbers.

(d) Divide the two given functions to form the requested function,

$$\frac{f}{g} = \frac{x - 8}{5x^2}.$$

The division is defined except for for  $x = 0$ , the domain is hence  $x \neq 0$ .

### Answer

(a)  $5x^2 + x - 8$ , domain: all real numbers.

(b)  $-5x^2 + x - 8$ , domain: all real numbers.

(c)  $5x^3 - 8x^2$ , domain: all real numbers.

(d)  $\frac{x-8}{5x^2}$ , domain:  $x \neq 0$ .

### Exercise 1.1.43

#### Instruction

For the pair of functions  $f(x) = x + 4$  and  $g(x) = 4x - 1$ , find the below listed compositions. Simplify the results. Find the domain of each of the results.

(a)  $(f \circ g)(x)$

(b)  $(g \circ f)(x)$

#### Solution

(a) The composition is given by

$$(f \circ g)(x) = f(g(x)) = (4x - 1) + 4 = 4x - 1 + 4 = 4x + 3.$$

The domain of the above composition is all real numbers.

(b) The composition is given by

$$(g \circ f)(x) = g(f(x)) = 4(x + 4) - 1 = 4x + 16 - 1 = 4x + 15.$$

The domain of the above composition is all real numbers.

### Answer

(a)  $4x + 3$ , domain: all real numbers.

(b)  $4x + 15$ , domain: all real numbers.