Chapter 1

Functions and Graphs

Checkpoint Solutions

Checkpoint 1.1: Evaluating Functions

Instruction

For the function $f(x) = x^2 - 3x + 5$ evaluate

- (a) f(1)
- (b) f(a+h)

Solution

(a)
$$f(1) = 1^2 - 3 \cdot 1 + 5 = 1 - 3 + 5 = 3$$
.

(b)
$$f(a+h) = (a+h)^2 - 3(a+h) + 5 = a^2 + 2ah + h^2 - 3a - 3h + 5$$
.

Answer

- (a) f(1) = 3.
- (b) $f(a+h) = a^2 + 2ah + h^2 3a 3h + 5$.

Checkpoint 1.2: Finding Domain and Range

Instruction

Find the domain and range for $f(x) = \sqrt{4 - 2x} + 5$.

Solution

i To find the domain of f, we need the expression $4 - 2x \ge 0$, due to that real negative numbers do not have a square root. Solving this inequality, we conclude that the domain is $\{x \mid x \le 2\}$.

ii To find the range of f, we note that since $\sqrt{4-2x} \ge 0$, it follows that $f(x) = \sqrt{4-2x} + 5 \ge 5$. Therefore, the range of f must be a subset of the set $\{y \mid y \ge 5\}$.

To show that every element in this set is in the range of f, we need to show that for all y in this set, there exists a real number x in the domain such that f(x) = y. Let $y \ge 5$. Then, f(x) = y if and only if

$$\sqrt{4-2x}+5=y.$$

Solving this equation for x, we see that x must solve the equation

$$\sqrt{4-2x} = y - 5.$$

Since $y \ge 5$, such an x could exist. Squaring both sides of the above equation we have

$$4 - 2x = (y - 5)^2.$$

Therefore we need

$$-2x = (y-5)^2 - 4,$$

which implies

$$x = 2 - \frac{(y-5)^2}{2}.$$

We just need to verify that x is in the domain of f. Since the domain of f consists of all real numbers less or equal to 2, and

$$2 - \frac{(y-5)^2}{2} \le 2,$$

there does exist an x in the domain of f. We conclude that the range of f is $\{y \mid y \ge 5\}$.

Answer

Domain = $\{x \mid x \le 2\}$, range = $\{y \mid y \ge 5\}$.

Checkpoint 1.3: Finding Zeroes

Instruction

Find the zeroes of $f(x) = x^3 - 5x^2 + 6x$.

Solution

The zeroes of a function are the values of x where f(x) = 0. To find the zeroes, we need to solve

$$f(x) = x^3 - 5x^2 + 6x = 0.$$

Factor out *x*

$$f(x) = x(x^2 - 5x + 6) = 0.$$

We can continue factoring by pure inspection, with the goal of finding a pair of numbers that add up to -5 and whose product is 6. This pair of numbers turns out to be -2 and -3, leading to the factoring

$$f(x) = x(x-2)(x-3) = 0.$$

From the above complete factoring of f, we conclude that there are three zeroes when x is 0, 2, and 3.

Answer

x = 0, 2, 3.

Checkpoint 1.4: Combining Functions Using Mathematical Operations

Instruction

For $f(x) = x^2 + 3$ and g(x) = 2x - 5, find (f/g)(x) and state its domain.

Solution

To find (f/g)(x) we write the function with the quotient operator

$$\frac{f}{g}(x) = \frac{x^2+3}{2x-5}.$$

The domain of this function is $\{x \mid x \neq \frac{5}{2}\}$.

Answer

 $\frac{f}{g}(x) = \frac{x^2+3}{2x-5}$. The domain is $\{x \mid x \neq \frac{5}{2}\}$.

Checkpoint 1.5: Compositions of Functions

Instruction

Let
$$f(x) = 2 - 5x$$
. Let $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$.

Solution

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 2 - 5\sqrt{x}.$$

Answer

$$(f \circ g)(x) = 2 - 5\sqrt{x}.$$

Checkpoint 1.6: Application Involving a Composite Function

Instruction

If items are on sale for 10% off their original price, and a customer has a coupon for an additional 30% off, what will be the final price for an item that is originally x dollars, after applying the coupon to the sale price?

Solution

Since the sale price 10% off the original price, if an item is *x* dollars, its sale price is given by

$$f(x) = 0.90x$$
.

Since the coupon entitles an individual to 30% off the price of any item, if an item is *y* dollars, the price after applying the coupon, is given by

$$g(y) = 0.70y$$
.

Therefore, if the price is originally *x* dollars, its price after applying the coupon to the sale price will be

$$(g \circ f)(x) = g(f(x)) = (0.70)0.90x = 0.63x..$$

Answer

$$(g \circ f)(x) = 0.63x.$$

Exercise Solutions

Exercise 1.1.1

Instruction

Assuming the relation in table 1.1.

- (a) Determine the domain and the range of the relation.
- (b) State whether the relation is a function.

\bar{x}	-3	-2	-1	0	1	2	3
\overline{y}	9	4	1	0	1	4	9

Table 1.1: Relation between *x* and *y* in exercise 1.1.1

(a) The domain of the relation is the set of unique *x* values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

The range of the relation is the set of unique *y* values,

$$\{0,1,4,9\}.$$

(b) This relation is a function, each input is a assigned to exactly one output.

Answer

- (a) Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, range = $\{0, 1, 4, 9\}$.
- (b) Yes, a function.

Exercise 1.1.2

Instruction

Assuming the relation in table 1.2.

- (a) Determine the domain and the range of the relation.
- (b) State whether the relation is a function.

\bar{x}	-3	-2	-1	0	1	2	3
\overline{y}	-2	-8	-1	0	1	8	-2

Table 1.2: Relation between *x* and *y* in exercise 1.1.2

Solution

(a) The domain of the relation is the set of unique *x* values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

The range of the relation is the set of unique *y* values,

$$\{-8, -2, -1, 0, 1, 8\}.$$

(b) This relation is a function, each input is a assigned to exactly one output.

Answer

- (a) Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, range = $\{-8, -2, -1, 0, 1, 8\}$.
- (b) Yes, a function.

Exercise 1.1.3

Instruction

Assuming the relation in table 1.3.

- (a) Determine the domain and the range of the relation.
- (b) State whether the relation is a function.

Table 1.3: Relation between x and y in exercise 1.1.3

Solution

(a) The domain of the relation is the set of unique *x* values,

$$\{0,1,2,3\}.$$

The range of the relation is the set of unique *y* values,

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

(b) This relation is not a function, each input is not assigned to exactly one output. Take for example x = 1 that can cause both y = -3 and y = 1.

Answer

- (a) Domain = $\{0,1,2,3\}$, range = $\{-3,-2,-1,0,1,2,3\}$.
- (b) No, not a function.

Exercise 1.1.4

Instruction

Assuming the relation in table 1.4.

- (a) Determine the domain and the range of the relation.
- (b) State whether the relation is a function.

х	1	2	3	4	5	6	7
y	1	1	1	1	1	1	1

Table 1.4: Relation between x and y in exercise 1.1.4

(a) The domain of the relation is the set of unique *x* values,

$$\{1,2,3,4,5,6,7\}.$$

The range of the relation is the set of unique *y* values,

{1}.

(b) This relation is a function, each input is a assigned to exactly one output.

Answer

- (a) Domain = $\{1, 2, 3, 4, 5, 6, 7\}$, range = $\{1\}$.
- (b) Yes, a function.

Exercise 1.1.5

Instruction

Assuming the relation in table 1.5.

- (a) Determine the domain and the range of the relation.
- (b) State whether the relation is a function.

\bar{x}	3	5	8	10	15	21	33
\overline{y}	3	2	1	0	1	2	3

Table 1.5: Relation between x and y in exercise 1.1.5

Solution

(a) The domain of the relation is the set of unique *x* values,

$${3,5,8,10,15,21,33}.$$

The range of the relation is the set of unique *y* values,

$$\{0,1,2,3\}.$$

(b) This relation is a function, each input is a assigned to exactly one output.

Answer

- (a) Domain = $\{3, 5, 8, 10, 15, 21, 33\}$, range = $\{0, 1, 2, 3\}$.
- (b) Yes, a function.

Exercise 1.1.6

Instruction

Assuming the relation in table 1.6.

- (a) Determine the domain and the range of the relation.
- (b) State whether the relation is a function.

Table 1.6: Relation between x and y in exercise 1.1.6

Solution

(a) The domain of the relation is the set of unique *x* values,

$$\{-7, -2, 0, 1, 3, 6\}.$$

The range of the relation is the set of unique y values,

$$\{-2, -1, 1, 4, 5, 11\}.$$

(b) This relation is not a function, each input is not assigned to exactly one output. See x = -2, that can cause both y = 1 and y = 5.

Answer

- (a) Domain = $\{-7, -2, 0, 1, 3, 6\}$, range = $\{-2, -1, 1, 4, 5, 11\}$.
- (b) No, not a function.

Exercise 1.1.7

Instruction

Find the below values for the function f(x) = 5x - 2, if they exist, then simplify.

(a) f(0)

- (b) f(1)
- (c) f(3)
- (d) f(-x)
- (e) *f*(*a*)
- (f) f(a+h)

- (a) $f(0) = 5 \cdot 0 2 = 0 2 = -2$.
- (b) $f(1) = 5 \cdot 1 2 = 5 2 = 3$.
- (c) $f(2) = 5 \cdot 3 2 = 15 2 = 13$.
- (d) f(-x) = 5(-x) 2 = -5x 2.
- (e) f(a) = 5a 2.
- (f) f(a+h) = 5(a+h) 2 = 5a + 5h 2.

Answer

- (a) -2.
- (b) 3.
- (c) 13.
- (d) -5x 2.
- (e) 5a 2.
- (f) 5a + 5h 2.

Exercise 1.1.8

Instruction

Find the below values for the function $f(x) = 4x^2 - 3x + 1$, if they exist, then simplify.

- (a) f(0)
- (b) f(1)
- (c) f(3)
- (d) f(-x)
- (e) *f*(*a*)
- (f) f(a+h)

(a)
$$f(0) = 4 \cdot 0^2 - 3 \cdot 0 + 1 = 4 \cdot 0 - 0 + 1 = 0 - 0 + 1 = 1$$
.

(b)
$$f(1) = 4 \cdot 1^2 - 3 \cdot 1 + 1 = 4 \cdot 1 - 3 + 1 = 4 - 3 + 1 = 2$$
.

(c)
$$f(3) = 4 \cdot 3^2 - 3 \cdot 3 + 1 = 4 \cdot 9 - 9 + 1 = 36 - 9 + 1 = 28$$
.

(d)
$$f(-x) = 4(-x)^2 - 3(-x) + 1 = 4x^2 + 3x + 1$$
.

(e)
$$f(a) = 4a^2 - 3a + 1$$
.

(f)
$$f(a+h) = 4(a+h)^2 - 3(a+h) + 1$$

= $4(a^2 + 2ah + h^2) - 3a - 3h + 1$
= $4a^2 + 4h^2 + 8ah - 3a - 3h + 1$.

Answer

- (a) 1.
- (b) 2.
- (c) 28.

(d)
$$4x^2 + 3x + 1$$
.

(e)
$$4a^2 - 3a + 1$$
.

(f)
$$4a^2 + 4h^2 + 8ah - 3a - 3h + 1$$
.

Exercise 1.1.15

Find the domain, range, and all zeros/intercepts, if any, of the function $g(x) = \sqrt{8x - 1}$.

Solution

- i The domain of the square root function is $[0, \infty)$, which implies $8x 1 \ge 0$. Solving for x gives $x \ge \frac{1}{8}$.
- ii To find the range of g, we note that $\sqrt{8x-1} \ge 0$. Therefore, the range of g must be a subset of the set $\{y \mid y \ge 0\}$. To show that every element in this set is in the range of g, we need to show that for a given g in this set, there exists a real number g in the domain such that g(g) = g.

Let
$$y \ge 0$$
. Then $g(x) = y$ if and only if

$$\sqrt{8x-1}=y.$$

We are interested in x, and will solve this equation for x. Since $y \ge 0$ such an x could exist. Squaring both sides of this equation, we have

$$8x - 1 = y^2$$
.

Therefore, we need

$$8x = y^2 + 1,$$

which implies

$$x = \frac{y^2 + 1}{8}.$$

We just need to verify that x is in the domain of g. Since the domain of g consists of all real numbers greater than or equal to 1/8, and

$$\frac{y^2+1}{8}\geq \frac{1}{8},$$

there does exist an x in the domain of g. We conclude that the range of g is $\{y \mid y \ge 0\}$.

- iii To find the zeroes, solve $g(x) = \sqrt{8x 10}$. We discover that g have one zero at x = -1/8.
- iv The y-intercept is given by (0, g(0)). Since x = 0 isn't in the domain of g, it follows that that there aren't any intercepts.

Answer

Domain = $x \ge \frac{1}{8}$, range = $\{y \mid y \ge 0\}$, zeroes x = -1/8, no intercepts.

Exercise 1.1.23

Sketch the graph for the function f(x) = 3x - 6 with the aid of table 1.7.

\bar{x}	-3	-2	-1	0	1	2	3
\overline{y}	-15	-12	-9	-6	-3	0	3

Table 1.7: Relation between *x* and *y* in exercise 1.1.23

Solution

Begin by sketching the axes. We choose the same scale on both axes to not distort the graph. We choose the range for both axes to be -15 to 15, allowing us to plot all the points from table 1.7, see figure 1.1.

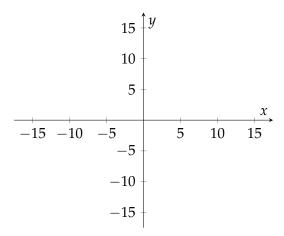


Figure 1.1: Empty graph with just the axes

After having sketched the axes we add markers based on the data in table 1.7, see figure 1.2.

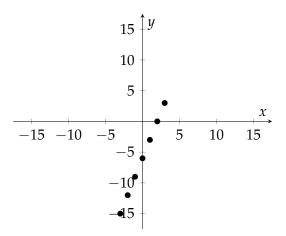


Figure 1.2: Graph with added markers

We then connect the markers with line segments. In this particular case the result will be a single straight line so we can use a ruler when sketching, , see figure 1.3.



Figure 1.3: Graph with connected markers

Answer

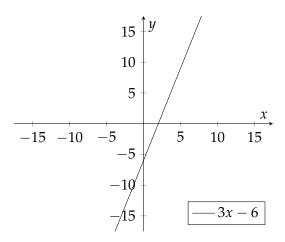


Figure 1.4: Answer to exercise 1.1.23

Exercise 1.1.29

Use the vertical line test to determine whether the given graph represent a function. Assume that the graph continues at both ends beyond the given grid. If the graph represents a function, then determine the following for the graph:

- (a) Domain and range
- (b) *x*-intercept, if any (estimate where necessary)
- (c) *y*-intercept, if any (estimate where necessary)
- (d) The intervals for which the function is increasing

- (e) The intervals for which the function is decreasing
- (f) The intervals for which the function is constant
- (g) Symmetry about any axis and/or the origin
- (h) Whether the function is even, odd, or neither