Lecture 3: Layer potential techniques

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Layer potential techniques

- Layer potential techniques:
  - Conductivity equation;
  - Helmholtz equation.

Layer potential techniques

- Fundamental solution to the Laplacian:

$$\Gamma_0(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x| \, , & d = 2, \\ \dfrac{1}{(2-d)\omega_d} |x|^{2-d} \, , & d \geq 3, \end{cases}$$

- ω_d: area of the unit sphere in $\mathbb{R}^d$.
- Ω: bounded domain in $\mathbb{R}^d$, $d \geq 2$, of class $\mathcal{C}^{1,\eta}$ for some $\eta > 0$.
- $\nu(y)$: outward unit normal to $\partial\Omega$ at y.

Layer potential techniques

- Single- and double-layer potentials of $\varphi \in L^2(\partial\Omega)$:

$$\mathcal{S}_\Omega^0[\varphi](x) := \int_{\partial\Omega} \Gamma_0(x-y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d,$$

$$\mathcal{D}_\Omega^0[\varphi](x) := \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)}\Gamma_0(x-y)\varphi(y)\, d\sigma(y)\ , \quad x \in \mathbb{R}^d \setminus \partial\Omega.$$

- Neumann-Poincaré operator: $\mathcal{K}_\Omega^0 : L^2(\partial\Omega) \to L^2(\partial\Omega)$:

$$\mathcal{K}_\Omega^0[\varphi](x) := \frac{1}{\omega_d}\int_{\partial\Omega} \frac{\langle y-x, \nu(y)\rangle}{|x-y|^d}\varphi(y)\, d\sigma(y).$$

- $(\mathcal{K}_\Omega^0)^*$: L^2-adjoint of $\mathcal{K}_\Omega^0$.
- $\mathcal{K}_\Omega^0$ and $(\mathcal{K}_\Omega^0)^*$: compact in $L^2(\partial\Omega)$.

Layer potential techniques

- Jump relations: For $\varphi \in L^2(\partial\Omega)$,

$$(\mathcal{D}_\Omega^0[\varphi])\big|_\pm(x) = \left(\mp\frac{1}{2}I + \mathcal{K}_\Omega^0\right)[\varphi](x) \quad \text{a.e. } x \in \partial\Omega;$$

$$\mathcal{S}_\Omega^0[\varphi]\Big|_+(x) = \mathcal{S}_\Omega^0[\varphi]\Big|_-(x) \quad \text{a.e. } x \in \partial\Omega;$$

$$\frac{\partial}{\partial\nu}\mathcal{S}_\Omega^0[\varphi]\Big|_\pm(x) = \left(\pm\frac{1}{2}I + (\mathcal{K}_\Omega^0)^*\right)[\varphi](x) \quad \text{a.e. } x \in \partial\Omega.$$

- For $\varphi \in L^2(\partial\Omega)$, $\partial\mathcal{D}_\Omega^0[\varphi]/\partial\nu$ exists (in $H^{-1}(\partial\Omega)$) and has no jump across $\partial\Omega$:

$$\frac{\partial}{\partial\nu}\mathcal{D}_\Omega^0[\varphi]\Big|_+ = \frac{\partial}{\partial\nu}\mathcal{D}_\Omega^0[\varphi]\Big|_- .$$

- For $\varphi \in L^2(\partial\Omega)$,

$$\frac{\partial}{\partial\nu}\mathcal{S}_\Omega^0[\varphi]\Big|_+(x) - \frac{\partial}{\partial\nu}\mathcal{S}_\Omega^0[\varphi]\Big|_-(x) = \varphi(x) \quad \text{a.e. } x \in \partial\Omega.$$

Layer potential techniques

- Dirichlet-to-Neumann operator $\mathcal{N} : L^2(\partial\Omega) \rightarrow H^{-1}(\partial\Omega)$:

$$\mathcal{N}[\varphi] = \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega};$$

- u: solution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

- Identity:

$$\frac{\partial}{\partial \nu}\mathcal{D}^0_\Omega[\varphi]\bigg|_\pm = (\frac{1}{2} + (\mathcal{K}^0_\Omega)^*)\mathcal{N}[\varphi].$$

Layer potential techniques

- Capacity:
  - $d = 2$; $(\varphi_e, a) \in L^2(\partial\Omega) \times \mathbb{R}$:

$$\begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{\partial\Omega} \ln|x-y|\varphi_e(y)d\sigma(y) + a = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} \varphi_e(y)d\sigma(y) = 1. \end{cases}$$

  - Logarithmic capacity of $\partial\Omega$: $\mathrm{cap}(\partial\Omega) := e^{2\pi a}$.
  - $d = 3$; $\varphi_e \in L^2(\partial\Omega)$:

$$\begin{cases} \displaystyle\int_{\partial\Omega} \frac{\varphi_e(y)}{|x-y|} d\sigma(y) = \text{constant} \quad \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} \varphi_e(y)d\sigma(y) = 1. \end{cases}$$

  - Capacity of $\partial\Omega$: $\dfrac{1}{\mathrm{cap}(\partial\Omega)} := -\mathcal{S}_\Omega^0[\varphi_e]$.

Layer potential techniques

- Spectrum of the Neumann–Poincaré Operator:
  - $(\mathcal{K}_\Omega^0)^* : L^2(\partial\Omega) \to L^2(\partial\Omega)$.
  - Spectrum of $(\mathcal{K}_\Omega^0)^*$:

$$\sigma(\mathcal{K}_\Omega^0)^*) \subset (-1/2, 1/2].$$

  - $(1/2)\, I + \mathcal{K}_\Omega^0$: invertible on $L^2(\partial\Omega)$.
  - $-(1/2)\, I + \mathcal{K}_\Omega^0$: invertible on $L_0^2(\partial\Omega)$.
  - $L_0^2(\partial\Omega) := \left\{ \varphi \in L^2(\partial\Omega) : \int_{\partial\Omega} \varphi \, d\sigma = 0 \right\}$.

Layer potential techniques

- Proof by contradiction:
  - $\lambda \in (-\infty, -1/2] \cup (1/2, +\infty)$; $\varphi \in L^2(\partial\Omega)$ satisfies $(\lambda I - (\mathcal{K}_\Omega^0)^*)[\varphi] = 0$ and $\varphi \neq 0$.
  - $\mathcal{K}_\Omega^0[1] = 1/2 \Rightarrow$

$$0 = \int_{\partial\Omega} (\lambda I - (\mathcal{K}_\Omega^0)^*)[\varphi] \, d\sigma = \int_{\partial\Omega} \varphi(\lambda - \frac{1}{2}) \, d\sigma.$$

  - $\Rightarrow \int_{\partial\Omega} \varphi d\sigma = 0$.
  - $\Rightarrow \mathcal{S}_\Omega^0[\varphi](x) = O(|x|^{1-d})$ and $\nabla \mathcal{S}_\Omega^0[\varphi](x) = O(|x|^{-d})$, $|x| \to +\infty$ for $d \geq 2$.
  - $\varphi \neq 0 \Rightarrow (A, B)$ cannot be zero:

$$A = \int_\Omega |\nabla \mathcal{S}_\Omega^0[\varphi]|^2 \, dx \text{ and } B = \int_{\mathbb{R}^d \setminus \overline{\Omega}} |\nabla \mathcal{S}_\Omega^0[\varphi]|^2 \, dx.$$

  - By contradiction: if A and B are zero, then $\mathcal{S}_\Omega^0[\varphi] = \text{constant}$ in Ω and in $\mathbb{R}^d \setminus \overline{\Omega} \Rightarrow \varphi = 0$.

Layer potential techniques

- Divergence theorem $\Rightarrow$

$$A=\int_{\partial\Omega}(-\frac{1}{2}I+(\mathcal{K}_\Omega^0)^*)[\varphi]\,\mathcal{S}_\Omega^0[\varphi]\,d\sigma \text{ and } B=-\int_{\partial\Omega}(\frac{1}{2}I+(\mathcal{K}_\Omega^0)^*)[\varphi]\,\mathcal{S}_\Omega^0[\varphi]\,d\sigma.$$

- $(\lambda I-(\mathcal{K}_\Omega^0)^*)[\varphi]=0 \Rightarrow$

$$\lambda=\frac{1}{2}\frac{B-A}{B+A}\Rightarrow|\lambda|<1/2\Rightarrow \text{ contradiction.}$$

- For $\lambda\in(-\infty,-\frac{1}{2}]\cup(\frac{1}{2},+\infty)$, $\lambda I-(\mathcal{K}_\Omega^0)^*$: one to one on $L^2(\partial\Omega)$.
- If $\lambda=1/2$, then $A=0\Rightarrow\mathcal{S}_\Omega^0[\varphi]=$ constant in Ω.
- $\Rightarrow$
  - $\mathcal{S}_\Omega^0[\varphi]$: harmonic in $\mathbb{R}^d\setminus\partial\Omega$;
  - $\mathcal{S}_\Omega^0[\varphi](x)=O(|x|^{1-d})$, $|x|\to+\infty$ (since $\varphi\in L_0^2(\partial\Omega)$);
  - $\mathcal{S}_\Omega^0[\varphi]$: constant on $\partial\Omega$.
- $(\mathcal{K}_\Omega^0)^*[\varphi]=(1/2)\,\varphi\Rightarrow$

$$B=-\int_{\partial\Omega}\varphi\,\mathcal{S}_\Omega^0[\varphi]\,d\sigma=C\int_{\partial\Omega}\varphi\,d\sigma=0,$$

- $\Rightarrow\varphi=0\Rightarrow(1/2)\,I-(\mathcal{K}_\Omega^0)^*$: one to one on $L_0^2(\partial\Omega)$.

Layer potential techniques

- Symmetrization of $(\mathcal{K}_\Omega^0)^*$:
  - Non-self-adjoint operator $(\mathcal{K}_\Omega^0)^*$: can be realized as a self-adjoint operator on $H^{-1/2}(\partial\Omega)$ by introducing a new inner product.
  - $\mathcal{S}_\Omega^0$ in $H^{-1/2}(\partial\Omega)$: self-adjoint and $-\mathcal{S}_\Omega^0 \geq 0$ on $H^{-1/2}(\partial\Omega)$.
  - $(\mathcal{K}_\Omega^0)^* : H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$: compact.
  - Calderón identity $\Rightarrow$

$$\mathcal{S}_\Omega^0(\mathcal{K}_\Omega^0)^* = \mathcal{K}_\Omega^0\mathcal{S}_\Omega^0 \quad \text{on } H^{-1/2}(\partial\Omega)\,.$$

Layer potential techniques

- Kernel of $\mathcal{S}_\Omega^0$:
  - $d \geq 3$; $\mathcal{S}_\Omega^0 : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ has a bounded inverse.
  - $d = 2$; If $\phi_0 \in \mathrm{Ker}(\mathcal{S}_\Omega^0)$, then u:

$$u(x) := \mathcal{S}_\Omega^0[\phi_0](x), \quad x \in \mathbb{R}^2$$

    satisfies $u = 0$ on $\partial\Omega \Rightarrow u(x) = 0$ for all $x \in \Omega$.
  - Jump condition $\Rightarrow$

$$(\mathcal{K}_\Omega^0)^*[\phi_0] = \frac{1}{2}\phi_0 \quad \text{on } \partial\Omega\,.$$

  - If $\langle \chi(\partial\Omega), \phi_0 \rangle_{1/2,-1/2} = 0$, then $u(x) \rightarrow 0$ as $|x| \rightarrow \infty \Rightarrow$ $u(x) = 0$ for $x \in \mathbb{R}^2 \setminus \Omega \Rightarrow \phi_0 = 0$.
  - Eigenfunctions: one dimensional subspace of $H^{-1/2}(\partial\Omega)$.
  - $\Rightarrow \mathrm{Ker}(\mathcal{S}_\Omega^0)$: of at most one dimension.
  - $\mathcal{S}_\Omega^0 : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ has a bounded inverse iff $\ln \mathrm{cap}(\partial\Omega) \neq 0$.

Layer potential techniques

- $d = 3$; inner product:

$$\langle u, v \rangle_{\mathcal{H}^*} = -\langle \mathcal{S}_\Omega^0[v], u \rangle_{\frac{1}{2},-\frac{1}{2}} \, ,$$

- Equivalent: $H^{-1/2}(\partial\Omega)$.
- $(\mathcal{K}_\Omega^0)^*$: self-adjoint in $\mathcal{H}^*(\partial\Omega)$;
- (λ_j, φ_j), $j = 0, 1, 2, \ldots$: eigenvalue and normalized eigenfunction pair of $(\mathcal{K}_\Omega^0)^*$ in $\mathcal{H}^*(\partial\Omega)$ with $\lambda_0 = 1/2$.
- $\lambda_j \in (-\frac{1}{2}, \frac{1}{2})$ for $j \geq 1$ with $|\lambda_1| \geq |\lambda_2| \geq \ldots \to 0$ as $j \to \infty$;
- Spectral representation formula: for any $\psi \in H^{-1/2}(\partial\Omega)$,

$$(\mathcal{K}_\Omega^0)^*[\psi] = \sum_{j=0}^{\infty} \lambda_j \langle \varphi_j, \psi \rangle_{\mathcal{H}^*} \varphi_j \, .$$

- $\mathcal{H}(\partial\Omega)$: $H^{1/2}(\partial\Omega)$ equipped with the equivalent inner product

$$\langle u, v \rangle_{\mathcal{H}} = \langle v, (-\mathcal{S}_\Omega^0)^{-1}[u] \rangle_{\frac{1}{2},-\frac{1}{2}} \, .$$

- $\mathcal{S}_\Omega^0$: isometry between $\mathcal{H}^*(\partial\Omega)$ and $\mathcal{H}(\partial\Omega)$.

Layer potential techniques

- $d = 2$; $\mathcal{S}_\Omega^0 : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$: not injective (in general).
- Substitute:

$$\widetilde{\mathcal{S}}_\Omega[\psi] := \begin{cases} \mathcal{S}_\Omega^0[\psi] & \text{if } \langle \chi(\partial\Omega), \psi \rangle_{\frac{1}{2},-\frac{1}{2}} = 0\,, \\ -\chi(\partial\Omega) & \text{if } \psi = \varphi_0\,, \end{cases}$$

- φ_0: unique eigenfunction of $(\mathcal{K}_\Omega^0)^*$ associated with eigenvalue $1/2$ s.t. $\langle \chi(\partial\Omega), \varphi_0 \rangle_{\frac{1}{2},-\frac{1}{2}} = 1$.
- $\widetilde{\mathcal{S}}_\Omega : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$: invertible.
- Calderón identity:

$$\mathcal{K}_\Omega^0 \widetilde{\mathcal{S}}_\Omega = \widetilde{\mathcal{S}}_\Omega (\mathcal{K}_\Omega^0)^*.$$

Layer potential techniques

- $(\mathcal{K}_\Omega^0)^*$: compact self-adjoint in $\mathcal{H}^*(\partial\Omega)$ equipped with
$$\langle u, v\rangle_{\mathcal{H}^*} = -\langle \widetilde{\mathcal{S}}_\Omega[v], u\rangle_{\frac{1}{2},-\frac{1}{2}} .$$
- (λ_j, φ_j), $j = 0, 1, 2, \ldots,$: eigenvalue and normalized eigenfunction pair of $(\mathcal{K}_\Omega^0)^*$ with $\lambda_0 = \frac{1}{2}$. $\lambda_j \in (-\frac{1}{2}, \frac{1}{2})$ with $|\lambda_1| \geq |\lambda_2| \geq \ldots \to 0$ as $j \to \infty$;
- Twin property: For any $j \geq 1$, $\pm\lambda_j$: eigenvalues of $(\mathcal{K}_\Omega^0)^*$;
- $\mathcal{H}^*(\partial\Omega) = \mathcal{H}_0^*(\partial\Omega) \oplus \{\mu\varphi_0,\ \mu \in \mathbb{C}\}$, where $\mathcal{H}_0^*(\partial\Omega)$: zero mean subspace of $\mathcal{H}^*(\partial\Omega)$;
- For any $\psi \in H^{-1/2}(\partial\Omega)$,
$$(\mathcal{K}_\Omega^0)^*[\psi] = \sum_{j=0}^{\infty} \lambda_j \langle \varphi_j, \psi\rangle_{\mathcal{H}^*} \varphi_j .$$
- $\mathcal{H}(\partial\Omega)$: $H^{1/2}(\partial\Omega)$ equipped with the equivalent inner product:
$$\langle u, v\rangle_{\mathcal{H}} = \langle v, -\widetilde{\mathcal{S}}_\Omega^{-1}[u]\rangle_{\frac{1}{2},-\frac{1}{2}} .$$
- $\widetilde{\mathcal{S}}_\Omega$: isometry between $\mathcal{H}^*(\partial\Omega)$ and $\mathcal{H}(\partial\Omega)$.

Layer potential techniques

- Neumann function: For Ω: smooth bounded domain in $\mathbb{R}^d$,

$$\begin{cases} -\Delta_x N(x,z) = \delta_z \quad \text{in } \Omega \,, \\ \displaystyle \frac{\partial N}{\partial \nu_x}\Big|_{\partial\Omega} = -\frac{1}{|\partial\Omega|} \,, \int_{\partial\Omega} N(x,z)\, d\sigma(x) = 0 \quad \text{for } z \in \Omega \,. \end{cases}$$

- $U(x) := \int_{\partial\Omega} N(x,z) g(z)\, d\sigma(z)$: solution to

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega \,, \\ \displaystyle \frac{\partial U}{\partial \nu}\Big|_{\partial\Omega} = g \,, \\ \displaystyle \int_{\partial\Omega} U \, d\sigma = 0. \end{cases}$$

Layer potential techniques

- N: symmetric in its arguments, $N(x,z) = N(z,x)$ for $x \neq z \in \Omega$.
- $$N(x,z) = \begin{cases} -\dfrac{1}{2\pi} \ln |x-z| + R_2(x,z) & \text{if } d = 2\,, \\ \dfrac{1}{(d-2)\omega_d} \dfrac{1}{|x-z|^{d-2}} + R_d(x,z) & \text{if } d \geq 3\,, \end{cases}$$
- $R_d(\cdot, z) \in H^{\frac{3}{2}}(\Omega)$ for any $z \in \Omega, d \geq 2$.

Layer potential techniques

- For $D \Subset \Omega$,

$$N_D[f](x) := \int_{\partial D} N(x,y) f(y)\, d\sigma(y), \quad x \in \Omega \,.$$

- Relation between the fundamental solution Γ_0 and the Neumann function N:

  For $z \in \Omega$ and $x \in \partial\Omega$, let $\Gamma_z(x) := \Gamma_0(x - z)$ and $N_z(x) := N(x, z)$. Then

$$\left(-\frac{1}{2} I + \mathcal{K}_\Omega^0\right)[N_z](x) = \Gamma_z(x) \quad \text{modulo constants}, \quad x \in \partial\Omega \,.$$

Layer potential techniques

- Outgoing fundamental solution $\Gamma_\omega(x)$ to the Helmholtz operator $\Delta + \omega^2$ in $\mathbb{R}^d$, $d = 2, 3$: $(\Delta_x + \omega^2)\Gamma_\omega(x) = \delta_0(x)$

$$\Gamma_\omega(x) = \begin{cases} -\dfrac{i}{4} H_0^{(1)}(\omega|x|)\ , & d = 2, \\ -\dfrac{e^{i\omega|x|}}{4\pi|x|}\ , & d = 3, \end{cases}$$

- $H_0^{(1)}$: Hankel function of the first kind of order 0.
- Behavior of $H_0^{(1)}$ near 0:

$$-\frac{i}{4} H_0^{(1)}(\omega|x|) \sim \frac{1}{2\pi} \ln|x| \quad \left[\Delta_x\left(\frac{1}{2\pi}\ln|x|\right) = \delta_0(x)\right].$$

- Sommerfeld radiation condition:

$$\left|\frac{x}{|x|} \cdot \nabla \Gamma_\omega(x) - i\omega\Gamma_\omega(x)\right| = \begin{cases} O(|x|^{-3/2}), & d = 2, \\ O(|x|^{-2}), & d = 3. \end{cases}$$

Layer potential techniques

- Single- and double-layer potentials: For $\varphi \in L^2(\partial D)$,

$$\mathcal{S}_D^\omega[\varphi](x) = \int_{\partial D} \Gamma_\omega(x-y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d,$$
$$\mathcal{D}_D^\omega[\varphi](x) = \int_{\partial D} \frac{\partial \Gamma_\omega(x-y)}{\partial \nu(y)}\varphi(y)\, d\sigma(y)\,, \quad x \in \mathbb{R}^d \setminus \partial D.$$

- $\Gamma_\omega(x)$ outgoing fundamental solution to the Helmholtz operator $\Rightarrow$
  - $\mathcal{S}_D^\omega[\varphi]$ and $\mathcal{D}_D^\omega[\varphi]$ satisfy the Helmholtz equation

$$(\Delta + \omega^2)u = 0 \quad \text{in } D \text{ and in } \mathbb{R}^d \setminus \overline{D}.$$

  - $\mathcal{S}_D^\omega[\varphi]$ and $\mathcal{D}_D^\omega[\varphi]$ satisfy the Sommerfeld radiation condition.

Layer potential techniques

- Jump relations: For $\varphi \in L^2(\partial D)$,

$$\frac{\partial(\mathcal{S}_D^\omega[\varphi])}{\partial \nu}\bigg|_{\pm}(x) = \left(\pm \frac{1}{2} I + (\mathcal{K}_D^\omega)^* \right)[\varphi](x) \quad \text{a.e. } x \in \partial D,$$

$$(\mathcal{D}_D^\omega[\varphi])\bigg|_{\pm}(x) = \left(\mp \frac{1}{2} I + \mathcal{K}_D^\omega \right)[\varphi](x) \quad \text{a.e. } x \in \partial D,$$

- $\mathcal{K}_D^\omega$ and $(\mathcal{K}_D^\omega)^*$:

$$\mathcal{K}_D^\omega[\varphi](x) = \int_{\partial D} \frac{\partial \Gamma_\omega(x-y)}{\partial \nu(y)} \varphi(y)\, d\sigma(y);$$

$$(\mathcal{K}_D^\omega)^*[\varphi](x) = \int_{\partial D} \frac{\partial \Gamma_\omega(x-y)}{\partial \nu(x)} \varphi(y)\, d\sigma(y).$$

- $(\mathcal{K}_D^\omega)^*$: L^2-adjoint of $\mathcal{K}_D^{-\omega}$ (complex inner product).
- $\mathcal{K}_D^\omega$ and $(\mathcal{K}_D^\omega)^*$: compact on $L^2(\partial D)$.

Layer potential techniques

- Ω: smooth bounded domain in $\mathbb{R}^d$; ω^2: not a Dirichlet eigenvalue in Ω.
- Dirichlet function:

$$\begin{cases} \Delta_x G_\omega(x,z) + \omega^2 G_\omega(x,z) = \delta_z & \text{in } \Omega\,, \\ G_\omega(x,z)\Big|_{\partial\Omega} = 0 & \text{for } z \in \Omega\,. \end{cases}$$

- $V(x) := \int_{\partial\Omega} G_\omega(x,z) f(z)\, d\sigma(z)$: solution to

$$\begin{cases} \Delta V + \omega^2 V = 0 & \text{in } \Omega\,, \\ V\Big|_{\partial\Omega} = f\,. \end{cases}$$

- Relation between Γ_ω and G_ω:

$$\left(\frac{1}{2} I + (\mathcal{K}_\Omega^\omega)^*\right)[\frac{\partial G_\omega}{\partial \nu}] = \frac{\partial \Gamma_\omega}{\partial \nu}\,.$$