Willmore Energy and Bifurcation

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Abstract

MSC-2020 classification: 49Q10; 53A05; 53A31

1 Introduction

FiXme Note: Would be a good idea to give a preliminary sketch of what we want to achieve here...

2 Notation

Andy: Andy is speaking.
Peter: Peter is speaking.
Henrik: Henrik is speaking.

symbol	meaning
$\mathbb{R}^3$	$\bar{\mathbb{R}}^3 := \mathbb{R}^3 \cup \{\infty\}$
C	$C := \{ f \in C^2(M;\mathbb{R}^3) \}$
$\bar{C}$	$\bar{C} := \{ f \in C^2(M;\bar{\mathbb{R}}^3) \}$
Möb(3)	group of conformal transformations on $\bar{\mathbb{R}}^3$
vol_f	Riemannian volume density induced by $f \in C$ on M
$\mathcal{A} \colon C \to \mathbb{R}$	area functional $\mathcal{A}(f) := \int_M \mathrm{vol}_f$.
ω	$\omega \in \Omega^2(\mathbb{R}^3)$, $\omega\vert_x := \frac{1}{3} \int_M \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk}\, x_i \,\mathrm{d}x_j \wedge \mathrm{d}x_k$, where ϵ_{ijk} denotes the Levi-Civita symbol. Observe that $\mathrm{d}\omega = \mathrm{vol}_{\mathbb{R}^3} := \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.
$\mathcal{V} \colon C \to \mathbb{R}$	enclosed volume functional $\mathcal{V}(f) := \frac{1}{3} \int_M f^\# \omega$

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3 Group action

Let M be a closed, 2-dimensional, oriented manifold and denote the space of embeddings of M into $\mathbb{R}^3$ by C. Note that by restricting the regularity of surfaces to some Banach space, say $C \subset C^2(M;\mathbb{R}^3)$ for the moment, C can be interpreted as an open set in a Banach space. The tangent space of C at a given point $f \in C$ is the space of $\mathbb{R}^3$-valued vector fields (not necessarily tangent to f):

$$T_f C = C = C^2(M;\mathbb{R}^3).$$

The Lie group $G = \text{Möb}(3)$ acts on C, i.e., each $g \in G$ induces a diffeomorphism

$$L_g \colon C \to C, \quad L_g(f) = g \circ f.$$

We can evaluate the fundamental vector fields of the group action of G on C at a given point $f \in C$: Each element $\xi of f$ $\text{Lie}(G) = T_1 G$ has a fundamental vector field K_ξ defined by

$$(K_\xi|_f)(x) := \frac{\mathrm{d}}{\mathrm{d}t} L_{\exp(t\,\xi)}(f)\Big|_{t=0} \quad \text{for each } x \in M.$$

By the chain rule, we have

$$(K_\xi|_f)(x) = K_\xi^{\mathbb{R}^3}|_{f(x)}. \tag{1}$$

For example, if $\xi \in \text{Lie}(\text{SO}(3)) \subset \text{Lie}(G)$ (it's nothing else but an antisymmetric 3×3 matrix), then

$$(K_\xi|_f)(x) = \xi \cdot f(x),$$

where $\cdot$ denotes the matrix-vector product. If $v \in \mathbb{R}^3 \cong \text{Lie}(\mathbb{R}^3) \subset \text{Lie}(G)$ is a generator of a translation, then

$$(K_\xi|_f)(x) = v.$$

For an infinitesimal dilation $\lambda \in \mathbb{R} = \text{Lie}(\mathbb{R}_{>0}) \subset \text{Lie}(G)$, the fundamental vector field looks like this:

$$(K_\xi|_f)(x) = \lambda\, f(x).$$

4 Questions

4.1 Andy: I do not have any intuition about preservation 'up to first order'.

Henrik: Through a somewhat intransparent process (in the sense that I did not present their derivation in the *Mathematica* notebook), I was able to determine the fundamental vector fields

$K^{\mathbb{R}^3}$ for the three "infinitesimal boosts" $u, v, w \in \mathrm{Lie}(G)$, i.e. for three further generators that complement $\mathrm{Lie}(\mathrm{SO}(3)) \oplus \mathrm{Lie}(\mathbb{R}^3) \oplus \mathrm{Lie}(\mathbb{R}_{>0}) \subset \mathrm{Lie}(G)$, so that we obtain a full basis of $\mathrm{Lie}(G)$. In the notebook, I denoted these three fundamental vector fields by U, V, W, i.e.,

$$U(x) := K_u^{\mathbb{R}^3}|_x = \begin{pmatrix} x_1^2 - x_2^2 - x_3^2 \\ 2\, x_1\, x_2 \\ 2\, x_1\, x_3 \end{pmatrix},$$

$$V(x) := K_v^{\mathbb{R}^3}|_x = \begin{pmatrix} 2\, x_2\, x_1 \\ x_2^2 - x_3^2 - x_1^2 \\ 2\, x_2\, x_3 \end{pmatrix},$$

$$W(x) := K_w^{\mathbb{R}^3}|_x = \begin{pmatrix} 2\, x_3\, x_1 \\ 2\, x_3\, x_2 \\ x_3^2 - x_1^2 - x_2^2 \end{pmatrix}.$$

Thus by (1), we have

$$K_u|_f = U \circ f, \quad K_v|_f = V \circ f, \quad \text{and} \quad K_w|_f = W \circ f.$$

We can now study area $\mathcal{A} \colon C \to \mathbb{R}$ and enclosed volume $\mathcal{V} \colon C \to \mathbb{R}$ along an orbit $O := (Gf)$ of G. By the chain rule we have for each differentiable curve $g \colon]-\epsilon, \epsilon[\to G$ with $g(0) = 1$ and $g'(0) \in \mathrm{Lie}(G)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{A}(L_{g(t)}(f))\Big|_{t=0} = D\,\mathcal{A}(f)\,(K_\xi|_f)$$

and analogously

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}(L_{g(t)}(f))\Big|_{t=0} = D\,\mathcal{V}(f)\,(K_\xi|_f).$$

And what I did in the notebook is just checking for each torus f of revolution with circular contour (TRCC for short) that

$$D\,\mathcal{A}(f)\,(K_u|_f) = 0, \tag{2}$$
$$D\,\mathcal{A}(f)\,(K_v|_f) = 0, \tag{3}$$
$$D\,\mathcal{A}(f)\,(K_w|_f) = 0, \tag{4}$$
$$D\,\mathcal{V}(f)\,(K_u|_f) = 0, \tag{5}$$
$$D\,\mathcal{V}(f)\,(K_v|_f) = 0, \tag{6}$$
$$D\,\mathcal{V}(f)\,(K_w|_f) = 0. \tag{7}$$

I did so by parameterizing the torus by

$$f(\varphi, \vartheta) \begin{pmatrix} \cos(\varphi)(r\cos(\vartheta) + R) \\ \sin(\varphi)(r\cos(\vartheta) + R) \\ r\,\sin(\vartheta) \end{pmatrix} \quad \text{with some } R > 0 \text{ and some } r > 0.$$

We know that

$$D\mathcal{A}(f)\,X = \int_M \langle H_f, X\rangle \,\mathrm{vol}_f \quad \text{and} \quad D\mathcal{A}(f)\,X = \int_M \langle \nu_f, X\rangle \,\mathrm{vol}_f,$$

where H_f and ν_f are mean curvature vector and outward pointing normal of f. Observe that

$$\mathrm{vol}_f = r\,\sqrt{(R + r\cos(\vartheta))^2}, \quad \nu_f = \tfrac{\partial}{\partial r} f(\varphi,\theta) = \begin{pmatrix} \cos(\vartheta)\,\cos(\varphi) \\ \cos(\vartheta)\,\sin(\varphi) \\ \sin(\vartheta) \end{pmatrix} \quad \text{and} \quad H_f(\varphi,\vartheta) = h(\theta)\,\nu_f,$$

where $h(\theta)$ denotes the magnitude of the mean curvature vector which – by rotation symmetry – does not depend on the angle φ. Thus, we can write down the (2)–(7) explicitly. Then we just have to exploit symmetry properties of the trigonometric functions sin and cos.

Of course, this does not necessarily imply that $\mathcal{A}$ and $\mathcal{V}$ are constant on the whole orbit $\mathcal{O} = (G f)$, not even on an arbitrarily small neighborhood of f. While the boost $\exp(t\,w)$ "in z-direction" (along the axis of revolution) sends a TRCC to a TRCC (the contour has to be circular again and boosting along the axis of revolution preserves rotation symmetry; the parameterization might change considerably, but we are not interested in the parameterization, right?), the boosts $\exp(t\,u)$ and $\exp(t\,v)$ along the x and y direction should deform TRCC into Dupin cyclides. And so I do not expect that $\exp(t\,u)$, $\exp(t\,v)$ preserve both area and volume. But I could be wrong.

4.2 Andy: With your symbolic setup in *Mathematica* Henrik, is it easy to look at the 2nd order change? Maybe we will even see what makes the Clifford torus isoperimetric ratio so special?

Henrik: Could be worth a try. Computing the Hessian at a critical point is typically not that difficult because it actually does not depend on the employed Riemannian metric and its connection:

Denote the orbit of f under G by $\mathcal{O} = L_G(f)$. The fundamental vector fields K span the tangent space $T_f\mathcal{O}$. If $\mathcal{F} : \mathcal{O} \to \mathbb{R}$ is twice differentiable, then for any Riemannian metric h on $\mathcal{O}$ we have

$$\mathrm{Hess}^h(\mathcal{F})(K_\xi, K_\eta) = (K_\xi K_\eta \mathcal{F})|_f - \mathrm{d}\mathcal{F}(f)\,\nabla^h_{K_\xi} K_\eta, \quad \text{for all } \xi, \eta \in \mathrm{Lie}(G),$$

where ∇^h is the Levi-Civita connection of h. So in general, the result depends on the choice of a metric! But if $\mathrm{d}\mathcal{F}\,|f = 0$ (like in the cases $\mathcal{F} = \mathcal{A}|_{\mathcal{O}}$ and $\mathcal{F} = \mathcal{V}|_{\mathcal{O}}$ for example), then the metric-dependent term just vanishes and we get

$$\mathrm{Hess}^h(\mathcal{F})(f)(K_\xi, K_\eta) = (K_\xi K_\eta \mathcal{F})|_f.$$

And we (or rather *Mathematica*) should be able to compute the latter, once we have figured out how the exponentials for the infinitesimal boost u, v, w actually look like …

Caution: If I am not mistaken, this does *not* imply

$$(K_\xi K_\eta \mathcal{A})|_f = D^2 \mathcal{A}(f)(K_\xi, K_\eta),$$

where D^2 denotes the second derivative on the surrounding Banach space! This has to do with the fact that the orbit is a curved submanifold of C and that $D\mathcal{A}(f) \neq 0$. (We only know that $D\mathcal{A}(f)$ vanishes when multiplied against fundamental vector fields $K_u|_f$, $K_v|_f$, $K_w|_f$ (and against any fundamental vector fields of the group of Euclidean motions, of course).

5 Questions

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D\mathcal{A}(f)\,(K_v|_f) &= 0,\\
D\mathcal{A}(f)\,(K_w|_f) &= 0,\\
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\end{aligned}$$

Of course, this does not necessarily imply that $\mathcal{A}$ and $\mathcal{V}$ are constant on the whole orbit $\mathcal{O} = (Gf)$, not even on an arbitrarily small neighborhood of f. While the boost $\exp(t\,w)$ "in z-direction" (along the axis of revolution) sends a TRCC to a TRCC (the contour has to be circular again and boosting along the axis of revolution preserves rotation symmetry; the parameterization might change considerably, but we are not interested in the parameterization, right?), the boosts $\exp(t\,u)$ and $\exp(t\,v)$ along the x and y direction should deform TRCC into Dupin cyclides. And so I do not expect that $\exp(t\,u)$, $\exp(t\,v)$ preserve both area and volume. But I could be wrong.

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where ∇^h is the Levi-Civita connection of h. So in general, the result depends on the choice of a metric! But if $\mathrm{d}\mathcal{F}|f = 0$ (like in the cases $\mathcal{F} = \mathcal{A}|_\mathcal{O}$ and $\mathcal{F} = \mathcal{V}|_\mathcal{O}$ for example), then the metric-dependent term just vanishes and we get

$$\operatorname{Hess}^h(\mathcal{F})(K_\xi, K_\eta) = (K_\xi K_\eta \mathcal{F})|_f. \bullet$$

And we (or rather *Mathematica*) should be able to compute the latter, once we have figured out how the exponentials for the infinitesimal boost u, v, w actually look like ...

6 Integrals

$$A(a) \coloneqq \int_{\partial\Omega} \lambda(x,a)^2 \, \mathrm{d}S(x)$$
$$V(a) \coloneqq \int_{\Omega} \lambda(x,a)^3 \, \mathrm{d}x$$

We would like to show that

$$\frac{\mathrm{d}}{\mathrm{d}a} \frac{V(a)^2}{A(a)^3} > 0.$$

This is equivalent to

$$2\,A(a)\,V'(a) - 3\,A'(a)\,V(a) > 0.$$

$$A'(a) = \int_{\partial\Omega} 2\,\lambda(x,a)\,\partial_a\lambda(x,a)\,\mathrm{d}S(x)$$
$$V'(a) = \int_{\Omega} 3\,\lambda(x,a)^2\,\partial_a\lambda(x,a)\,\mathrm{d}x$$

$$\begin{aligned}
3\,A'(a)\,V(a) &= 6 \int_{\partial\Omega}\int_{\Omega} \lambda(x,a)\,\partial_a\lambda(x,a)\,\lambda(y,a)^3\,\mathrm{d}y\,\mathrm{d}S(x) \\
2\,A(a)\,V'(a) &= 6 \int_{\Omega}\int_{\partial\Omega} \lambda(x,a)^2\,\partial_a\lambda(x,a)\,\lambda(y,a)^2\,\mathrm{d}S(y)\,\mathrm{d}x \\
&= 6 \int_{\partial\Omega}\int_{\Omega} \lambda(x,a)^2\,\partial_a\lambda(y,a)\,\lambda(y,a)^2\,\mathrm{d}y\,\mathrm{d}S(x)
\end{aligned}$$

Observe that

$$\partial(\Omega \times \Omega) = \partial\Omega \times \Omega \cup \Omega \times \partial\Omega.$$

So maybe we can find a nice 5-form on Ω such that

$$3\,A'(a)\,V(a) - 2\,A(a)\,V'(a) = \int_{\Omega\times\Omega} \mathrm{d}\omega.$$

and for which we can tell that the $\int_{\Omega\times\Omega} \mathrm{d}\omega$ must be positive.

$$\begin{aligned}
&3\,A'(a)\,V(a) - 2\,A(a)\,V'(a) \\
&\quad= 6 \int_M\int_M \big(\varphi(y,a) - \varphi(x,a)\big)\,\varphi(x,a)\,\varphi(y,a)^2\,\partial_a\varphi(x,a)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \\
&\quad= 3 \int_M\int_M \big(\varphi(y,a) - \varphi(x,a)\big)\,\varphi(x,a)\,\varphi(y,a)^2\,\partial_a\varphi(x,a)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) + \\
&\qquad 3 \int_M\int_M \big(\varphi(x,a) - \varphi(y,a)\big)\,\varphi(y,a)\,\varphi(x,a)^2\,\partial_a\varphi(y,a)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)
\end{aligned}$$

$$\lambda(x) := \frac{1}{1 + 2\,a\,x_1 + a^2\,|x|^2}$$

$$\operatorname{grad}\lambda^2(x) = \frac{-4}{(1 + 2\,a\,x_1 + a^2\,|x|^2)^3}\,(a\,e_1 + a^2\,x) = -4\,(a\,e_1 + a^2\,x)\,\lambda^3.$$

$$\begin{aligned}\int_{\partial\Omega} \lambda^2\,\mathrm{d}S\,(x) &= \int_{\partial\Omega} \langle N, \lambda^2\,X\rangle\,\mathrm{d}S\,(x)\\ &= \int_{\Omega} \operatorname{div}(\lambda^2\,X)\,\mathrm{d}x\\ &= \int_{\Omega} \langle \operatorname{grad}\lambda^2, X\rangle\,\mathrm{d}x + \int_{\Omega} \lambda^2\ \operatorname{div}(X)\,\mathrm{d}x,\end{aligned}$$

where $X = r\frac{\partial}{\partial r}$. Note that

$$\langle \operatorname{grad}\lambda^2(x), X\rangle = -\lambda^3\left(a\,r\ \cos(\varphi)\ \cos(\theta) + r\,(r + \sqrt{2}\ \cos(\theta))\right)$$

Orthonormal basis

$$e_1 := \frac{\partial}{\partial r}, \quad e_2 := \frac{1}{\sqrt{2} + r\cos(\theta)}\frac{\partial}{\partial\varphi} \quad e_3 := \frac{1}{r}\frac{\partial}{\partial\theta}.$$

$$\operatorname{div}(X) = \left\langle e_1, \frac{\partial}{\partial r}X\right\rangle + \left\langle e_2, \frac{1}{\sqrt{2} + r\cos(\theta)}\frac{\partial}{\partial\varphi}X\right\rangle + \left\langle e_3, \frac{1}{r}\frac{\partial}{\partial\theta}X\right\rangle = 2 + \frac{r\ \cos(\theta)}{\sqrt{2} + r\ \cos(\theta)}$$

$$\int_{\partial\Omega} \lambda(y,a)^2 \, \mathrm{d}S(y) = \int_\Omega \langle \operatorname{grad} \lambda^2(y,a), X(y)\rangle \, \mathrm{d}y + \int_\Omega \lambda^2(y,a) \; \operatorname{div}(X)(y) \, \mathrm{d}y,$$

$$\begin{aligned}
\frac{1}{2} A'(a) \, V(a) &= \int_\Omega \int_{\partial\Omega} \lambda(x,a)^3 \, \lambda(y,a) \, \partial_a \lambda(y,a) \, \mathrm{d}S(y) \, \mathrm{d}x \\
&= \int_\Omega \int_\Omega \lambda(x,a)^3 \, \lambda(y,a) \; \langle \partial_a \operatorname{grad} \lambda^2(y,a), X(y)\rangle \, \mathrm{d}y \, \mathrm{d}x \\
&\qquad + \int_\Omega \int_\Omega \lambda(x,a)^3 \, \lambda(y,a) \, \partial_a \lambda(y,a) \; \operatorname{div}(X)(y) \, \mathrm{d}y \, \mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
\frac{1}{3} A(a) \, V'(a) &= \int_\Omega \int_{\partial\Omega} \lambda(x,a)^2 \, \partial_a \lambda(x,a) \, \lambda(y,a)^2 \, \mathrm{d}S(y) \, \mathrm{d}x \\
&= \int_\Omega \int_\Omega \lambda(x,a)^2 \, \partial_a \lambda(x,a) \, \langle \operatorname{grad} \lambda^2(y,a), X(y)\rangle \, \mathrm{d}y \, \mathrm{d}x \\
&\qquad + \int_\Omega \int_\Omega \lambda(x,a)^2 \, \partial_a \lambda(x,a) \, \lambda(y,a)^2 \; \operatorname{div}(X)(y) \, \mathrm{d}y \, \mathrm{d}x
\end{aligned}$$

$$\partial_a \operatorname{grad} \lambda^2 = \operatorname{grad}(\partial_a \lambda^2) = 2 \; \operatorname{grad}(\lambda \, \partial_a \lambda) = 2 \, \lambda \operatorname{grad}(\partial_a \lambda) + 2 \, \partial_a \lambda \operatorname{grad} \lambda$$

$$\lambda(x,a)^3 \, \lambda(y,a) \; \langle \partial_a \operatorname{grad} \lambda^2(y,a), X(y)\rangle$$

$$\begin{aligned}
&\frac{1}{2} A'(a) \, V(a) - \frac{1}{3} A(a) \, V'(a) \\
&\quad = \int_\Omega \int_\Omega \Big(\lambda(x,a)^3 \, \lambda(y,a) \, \partial_a \lambda(y,a) - \lambda(x,a)^2 \, \partial_a \lambda(x,a)\Big) \langle \operatorname{grad} \lambda^2(y,a), X(y)\rangle \, \mathrm{d}y \, \mathrm{d}x \\
&\qquad + \int_\Omega \int_\Omega \Big(\lambda(x,a)^3 \, \lambda(y,a) \, \partial_a \lambda(y,a) - \lambda(x,a)^2 \, \partial_a \lambda(x,a) \, \lambda(y,a)^2\Big) \; \operatorname{div}(X)(y) \, \mathrm{d}y \, \mathrm{d}x \\
&\quad = \int_\Omega \int_\Omega \Big(\lambda(x,a) \, \lambda(y,a) \, \partial_a \lambda(y,a) - \partial_a \lambda(x,a)\Big) \lambda(x,a)^2 \, \langle \operatorname{grad} \lambda^2(y,a), X(y)\rangle \, \mathrm{d}y \, \mathrm{d}x \\
&\qquad + \int_\Omega \int_\Omega \Big(\lambda(x,a) \, \partial_a \lambda(y,a) - \partial_a \lambda(x,a) \, \lambda(y,a)\Big) \lambda(y,a) \, \lambda(x,a)^2 \; \operatorname{div}(X)(y) \, \mathrm{d}y \, \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
3\int_{\alpha}^{\beta} A'(a)\, V(a)\, \mathrm{d}a &= 6\int_{\alpha}^{\beta} \int_{\partial\Omega}\int_{\Omega} \lambda(x,a)\, \partial_a\lambda(x,a)\, \lambda(y,a)^3\, \mathrm{d}y\, \mathrm{d}S\,(x)\, \mathrm{d}a \\
&= 3\left[\int_{\partial\Omega}\int_{\Omega} \lambda(x,a)^2 \lambda(y,a)^3\, \mathrm{d}y\, \mathrm{d}S\,(x)\right]_{a=\alpha}^{a=\beta} \\
&\quad - 9\int_{\alpha}^{\beta} \int_{\partial\Omega}\int_{\Omega} \lambda(x,a)^2\, \partial_a\lambda(y,a)\, \lambda(y,a)^2\, \mathrm{d}y\, \mathrm{d}S\,(x)\, \mathrm{d}a \\
&= 3\left[\int_{\partial\Omega}\int_{\Omega} \lambda(x,a)^2 \lambda(y,a)^3\, \mathrm{d}y\, \mathrm{d}S\,(x)\right]_{a=\alpha}^{a=\beta} - 3\int_{\alpha}^{\beta} A(a)\, V'(a)\, \mathrm{d}a
\end{aligned}
$$

$$
\begin{aligned}
2\, A(a)\, V'(a) &= 6\int_{\Omega}\int_{\partial\Omega} \lambda(x,a)^2\, \partial_a\lambda(x,a)\, \lambda(y,a)^2\, \mathrm{d}S\,(y)\, \mathrm{d}x \\
&= 6\int_{\partial\Omega}\int_{\Omega} \lambda(x,a)^2\, \partial_a\lambda(y,a)\, \lambda(y,a)^2\, \mathrm{d}y\, \mathrm{d}S\,(x)
\end{aligned}
$$

7 Another take on the integrals

Denote the Willmoretorus by $\Sigma(0) \subset \mathbb{R}^3$.

$$\begin{aligned}\Sigma(x) &:= \iota(\iota(\Sigma(0) + (a, 0, 0))\\ U_a(x) &:= \frac{\mathrm{d}}{\mathrm{d}a}\iota(\iota(x) + (a, 0, 0)).\end{aligned}$$

$$\begin{aligned}A(a) &= \int_{\Sigma(a)} \mathrm{d}\mathcal{H}^2(x)\\ A'(a) &= -2\int_{\Sigma(a)} H(x)\,\langle N(x), U_a(x)\rangle\,\mathrm{d}\mathcal{H}^2(x)\\ V(a) &= \frac{1}{3}\int_{\Sigma(a)} \langle N(x), x\rangle\,\mathrm{d}\mathcal{H}^2(x)\\ V'(a) &= \int_{\Sigma(a)} \langle N(x), U_a(x)\rangle\,\mathrm{d}\mathcal{H}^2(x)\end{aligned}$$

$$\begin{aligned}2\,A(a)\,V'(a) &= 2\int_{\Sigma(a)}\int_{\Sigma(a)} \langle N(y), U_a(y)\rangle\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}\mathcal{H}^2(x)\\ 3\,A'(a)\,V(a) &= -2\int_{\Sigma(a)}\int_{\Sigma(a)} H(x)\,\langle N(x), U_a(x)\rangle\,\langle N(y), y\rangle\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}\mathcal{H}^2(x)\end{aligned}$$

$$\begin{aligned}&2\,A(a)\,V'(a) - 3\,A'(a)\,V(a)\\ &\qquad = 2\int_{\Sigma(a)} (1 + 3\,V(a)\,H(y))\,\langle N(y), U_a(y)\rangle\,\mathrm{d}\mathcal{H}^2(y)\end{aligned}$$