# Willmore Energy and Bifurcation

Peter McGrath[*], Andy Sageman-Furnas[†], and Henrik Schumacher[‡]

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**Abstract**

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## 1 Introduction

FiXme Note: Would be a good idea to give a preliminary sketch of what we want to achieve here...

## 2 Notation

Andy: Andy is speaking.
Peter: Peter is speaking.
Henrik: Henrik is speaking.

| symbol | meaning |
|---|---|
| $\bar{\mathbb{R}}^3$ | $\bar{\mathbb{R}}^3 := \mathbb{R}^3 \cup \{\infty\}$ |
| $\mathcal{C}$ | $\mathcal{C} := \{ f \in C^2(M;\mathbb{R}^3) \}$ |
| $\bar{\mathcal{C}}$ | $\bar{\mathcal{C}} := \{ f \in C^2(M;\bar{\mathbb{R}}^3) \}$ |
| Möb(3) | group of conformal transformations on $\bar{\mathbb{R}}^3$ |
| $\mathrm{vol}_f$ | Riemannian volume density induced by $f \in \mathcal{C}$ on $M$ |
| $\mathcal{A} \colon \mathcal{C} \to \mathbb{R}$ | area functional $\mathcal{A}(f) := \int_M \mathrm{vol}_f$. |
| $\omega$ | $\omega \in \Omega^2(\mathbb{R}^3)$, $\omega\|_x := \frac{1}{3} \int_M \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk}\, x_i \,\mathrm{d}x_j \wedge \mathrm{d}x_k$, where $\epsilon_{ijk}$ denotes the Levi-Civita symbol. Observe that $\mathrm{d}\omega = \mathrm{vol}_{\mathbb{R}^3} := \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$. |
| $\mathcal{V} \colon \mathcal{C} \to \mathbb{R}$ | enclosed volume functional $\mathcal{V}(f) := \frac{1}{3} \int_M f^\# \omega$ |

[*]–please insert your affiliation here–
[†]–please insert your affiliation here–
[‡]Chemnitz University of Technology, Faculty of Mathematics, 09107 Chemnitz, Germany, henrik.schumacher@math.tu-chemnitz.de

## 3 Group action

Let $M$ be a closed, 2-dimensional, oriented manifold and denote the space of embeddings of $M$ into $\mathbb{R}^3$ by $C$. Note that by restricting the regularity of surfaces to some Banach space, say $C \subset C^2(M;\mathbb{R}^3)$ for the moment, $C$ can be interpreted as an open set in a Banach space. The tangent space of $C$ at a given point $f \in C$ is the space of $\mathbb{R}^3$-valued vector fields (not necessarily tangent to $f$):

$$T_f C = C = C^2(M;\mathbb{R}^3).$$

The Lie group $G = \text{Möb}(3)$ acts on $C$, i.e., each $g \in G$ induces a diffeomorphism

$$L_g \colon C \to C, \quad L_g(f) = g \circ f.$$

We can evaluate the fundamental vector fields of the group action of $G$ on $C$ at a given point $f \in C$: Each element $\xi o f$ Lie$(G) = T_1 G$ has a fundamental vector field $K_\xi$ defined by

$$(K_\xi|_f)(x) := \frac{\mathrm{d}}{\mathrm{d}t} L_{\exp(t\,\xi)}(f)\big|_{t=0} \quad \text{for each } x \in M.$$

By the chain rule, we have

$$(K_\xi|_f)(x) = K_\xi^{\mathbb{R}^3}|_{f(x)}. \tag{1}$$

For example, if $\xi \in \mathrm{Lie}(\mathrm{SO}(3)) \subset \mathrm{Lie}(G)$ (it's nothing else but an antisymmetric $3 \times 3$ matrix), then

$$(K_\xi|_f)(x) = \xi \cdot f(x),$$

where $\cdot$ denotes the matrix-vector product. If $v \in \mathbb{R}^3 \cong \mathrm{Lie}(\mathbb{R}^3) \subset \mathrm{Lie}(G)$ is a generator of a translation, then

$$(K_\xi|_f)(x) = v.$$

For an infinitesimal dilation $\lambda \in \mathbb{R} = \mathrm{Lie}(\mathbb{R}_{>0}) \subset \mathrm{Lie}(G)$, the fundamental vector field looks like this:

$$(K_\xi|_f)(x) = \lambda\, f(x).$$

## 4 Questions

### 4.1 Andy: I do not have any intuition about preservation 'up to first order'.

Henrik: Through a somewhat intransparent process (in the sense that I did not present their derivation in the *Mathematica* notebook), I was able to determine the fundamental vector fields

$K^{\mathbb{R}^3}$ for the three "infinitesimal boosts" $u, v, w \in \mathrm{Lie}(G)$, i.e. for three further generators that complement $\mathrm{Lie}(\mathrm{SO}(3)) \oplus \mathrm{Lie}(\mathbb{R}^3) \oplus \mathrm{Lie}(\mathbb{R}_{>0}) in \, \mathrm{Lie}(G)$, so that we obtain a full basis of $\mathrm{Lie}(G)$. In the notebook, I denoted these three fundamental vector fields by $U$, $V$, $W$, i.e.,

$$U := K_u^{\mathbb{R}^3}, \quad V := K_v^{\mathbb{R}^3}, \quad \text{and} \quad W := K_w^{\mathbb{R}^3}.$$

Thus by (1), we have

$$K_u|_f = U \circ f, \quad K_v|_f = V \circ f, \quad \text{and} \quad K_w|_f = W \circ f.$$

We can now study area $\mathcal{A} \colon C \to \mathbb{R}$ and enclosed volume $\mathcal{V} \colon C \to \mathbb{R}$ along an orbit $\mathcal{O} := (G f)$ of $G$. By the chain rule we have for each differentiable curve $g \colon ]-\epsilon, \epsilon[ \to G$ with $g(0) = 1$ and $g'(0) \in \mathrm{Lie}(G)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{A}(L_{g(t)}(f))\Big|_{t=0} = D\mathcal{A}(f)\,(K_\xi|_f)$$

and analogously

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}(L_{g(t)}(f))\Big|_{t=0} = D\mathcal{V}(f)\,(K_\xi|_f).$$

And what I did in the notebook is just checking for each torus $f$ of revolution with circular contour (TRCC for short) that

$$\begin{aligned}
D\mathcal{A}(f)\,(K_u|_f) &= 0.\\
D\mathcal{A}(f)\,(K_v|_f) &= 0,\\
D\mathcal{A}(f)\,(K_w|_f) &= 0,\\
D\mathcal{V}(f)\,(K_u|_f) &= 0,\\
D\mathcal{V}(f)\,(K_v|_f) &= 0,\\
D\mathcal{V}(f)\,(K_w|_f) &= 0.
\end{aligned}$$

Of course, this does not necessarily imply that $\mathcal{A}$ and $\mathcal{V}$ are constant on the whole orbit $\mathcal{O} = (G f)$, not even on an arbitrarily small neighborhood of $f$. While the boost $\exp(t\,w)$ "in $z$-direction" (along the axis of revolution) sends a TRCC to a TRCC (the contour has to be circular again and boosting along the axis of revolution preserves rotation symmetry; the parameterization might change considerably, but we are not interested in the parameterization, right?), the boosts $\exp(t\,u)$ and $\exp(t\,v)$ along the $x$ and $y$ direction should deform TRCC into Dupin cyclides. And so I do not expect that $\exp(t\,u)$, $\exp(t\,v)$ preserve both area and volume. But I could be wrong.

## 4.2 Andy: With your symbolic setup in *Mathematica* Henrik, is it easy to look at the 2nd order change? Maybe we will even see what makes the Clifford torus isoperimetric ratio so special?

Henrik: Could be worth a try. Computing the Hessian at a critical point is typically not that difficult because it actually does not depend on the employed Riemannian metric and its connection:

Denote the orbit of $f$ under $G$ by $O = L_G(f)$. The fundamental vector fields $K$ span the tangent space $T_f O$. If $\mathcal{F} : O \to \mathbb{R}$ is twice differentiable, then for any Riemannian metric $h$ on $O$ we have

$$\mathrm{Hess}^h(\mathcal{F})(K_\xi, K_\eta) = (K_\xi K_\eta \mathcal{F})|_f - \mathrm{d}\mathcal{F}(f)\, \nabla^h_{K_\xi} K_\eta, \quad \text{for all } \xi, \eta \in \mathrm{Lie}(G),$$

where $\nabla^h$ is the Levi-Civita connection of $h$. So in general, the result depends on the choice of a metric! But if $\mathrm{d}\mathcal{F}\,|f = 0$ (like in the cases $\mathcal{F} = \mathcal{A}|_O$ and $\mathcal{F} = \mathcal{V}|_O$ for example), then the metric-dependent term just vanishes and we get

$$\mathrm{Hess}^h(\mathcal{F})(f)(K_\xi, K_\eta) = (K_\xi K_\eta \mathcal{F})|_f.$$

And we (or rather *Mathematica*) should be able to compute the latter, once we have figured out how the exponentials for the infinitesimal boost $u$, $v$, $w$ actually look like ...

Caution: If I am not mistaken, this does *not* imply

$$(K_\xi K_\eta \mathcal{A})|_f = D^2 \mathcal{A}(f)(K_\xi, K_\eta),$$

where $D^2$ denotes the second derivative on the surrounding Banach space! This has to do with the fact that the orbit is a curved submanifold of $C$ and that $D\mathcal{A}(f) \neq 0$. ( We only know that $D\mathcal{A}(f)$ vanishes when multiplied against fundamental vector fields $K_u|_f$, $K_v|_f$, $K_w|_f$ (and against any fundamental vector fields of the group of Euclidean motions, of course).