NTNU | Norwegian University of Science and Technology

SMALL-SIGNAL STABILITY:

Linearization and eigenvalue analysis

Gilbert Bergna-Diaz

September 17, 2024

Objectives of this lecture

Lecture goals:

- Learn how to linearize your nonlinear system,
  - ...and validate your linear model numerically in Matlab/Simulink.
- Understand how linear systems are expected to behave.
- Learn how to perform an eigenvalue-based stability analysis,
  - ...using your linearized system,
  - ...and use this tool to help you tune your control parameters.

Recommended reading:

- Sections 2.2-2.3, 3.1-3.3, 6.1, 6.4 (p.410).

Outline

Recap from previous lectures

Linearization of nonlinear systems

Bloc 3 Challenge $\times 1/2$

Response of a linear system

Modal transformation & eigenvalues

Bloc 3 (full) Challenge

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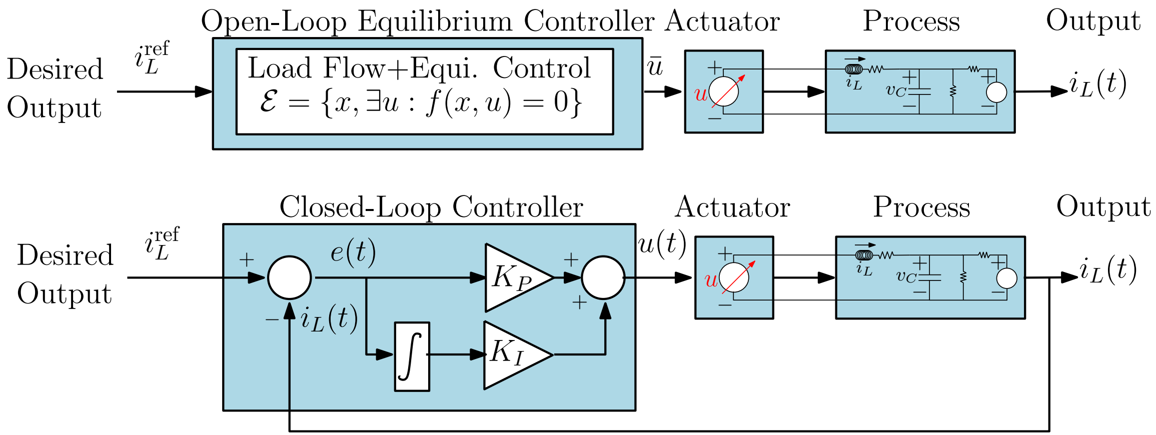
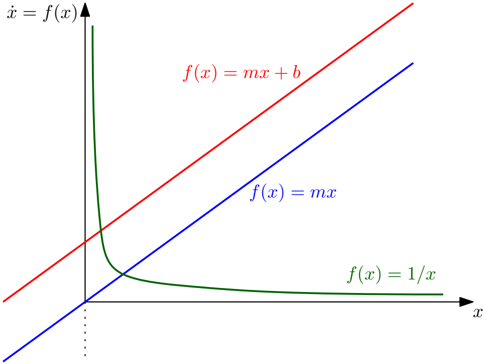


Is my model linear or nonlinear? (Chapt. 2.3)



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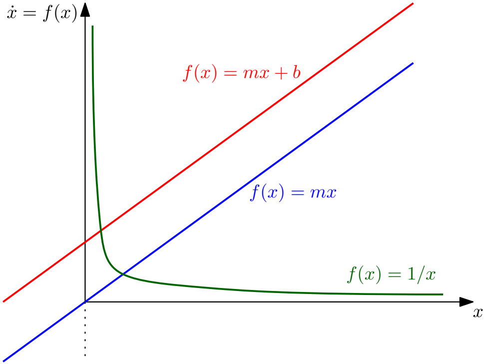

- A system (vetor field or function) $f(x)$ is linear when it satisfies the properties of:
  - **Superposition** $\rightarrow f(x_1 + x_2) = f(x_1) + f(x_2)$.
  - **Homogeneity** $\rightarrow f(\alpha x) = \alpha f(x)$.

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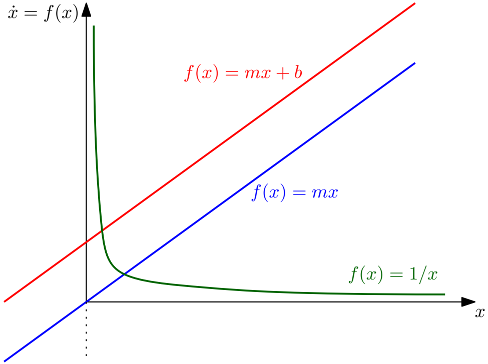

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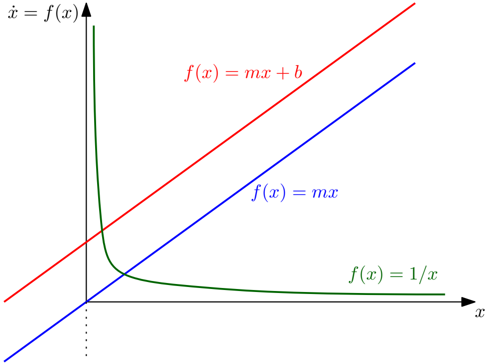

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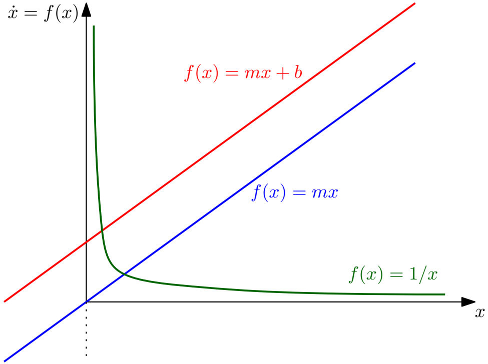

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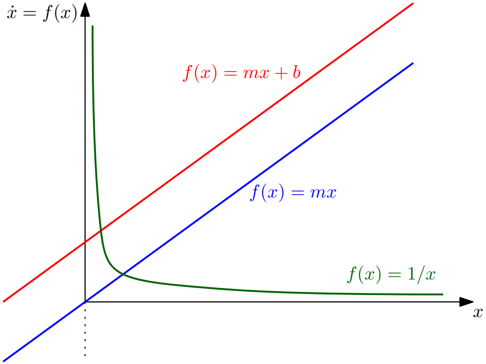

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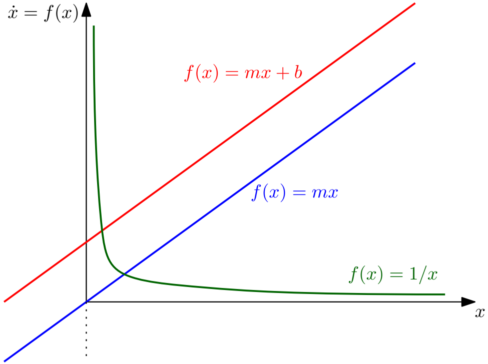

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- $\dfrac{1}{\underbrace{x_1 + x_2}_{f(x_1+x_2)}} \neq \dfrac{1}{\underbrace{x_1}_{f(x_1)}} + \dfrac{1}{\underbrace{x_2}_{f(x_2)}}$; $\dfrac{1}{\underbrace{\alpha x}_{f(\alpha x)}} \neq \underbrace{\alpha \dfrac{1}{x}}_{\alpha f(x)}$

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However, the system $\dot{x} = mx + b$ may be considered linear about an operating point $\bar{x}$.
Define $\tilde{x} := x - \bar{x}$, then we have that:

$$\begin{aligned} \dot{x} &= mx + b \\ - \quad & \\ \dot{\bar{x}} &= m\bar{x} + b \\ \hline \dot{\tilde{x}} &= m\tilde{x} \end{aligned}$$

which is linear!

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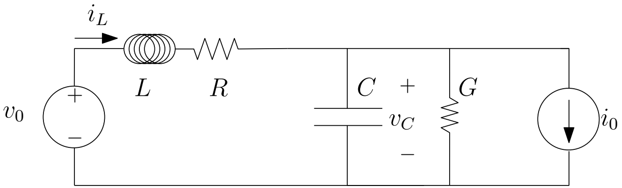

State-Space Model:

$$
\begin{aligned}
\frac{d}{dt} i_L &= -\frac{R}{L} i_L - \frac{1}{L} v_C + \frac{1}{L} v_0 \\
\frac{d}{dt} v_C &= -\frac{G}{C} v_C + \frac{1}{C} i_L - \frac{1}{C} i_0
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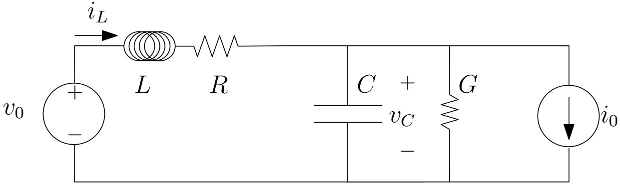

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$$\frac{d}{dt} \underbrace{\begin{bmatrix} i_L \\ v_C \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{G}{C} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} i_L \\ v_C \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \frac{v_0}{L} \\ -\frac{i_0}{C} \end{bmatrix}}_{E}$$

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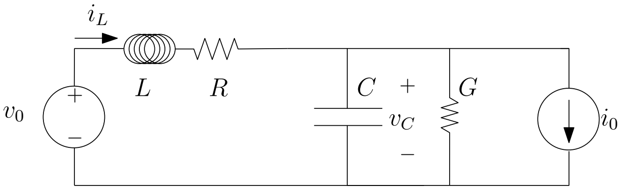

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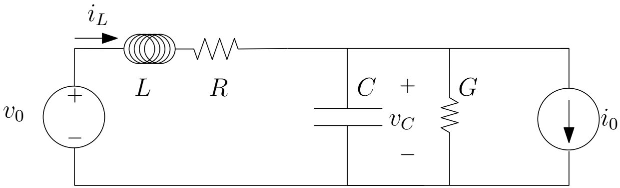

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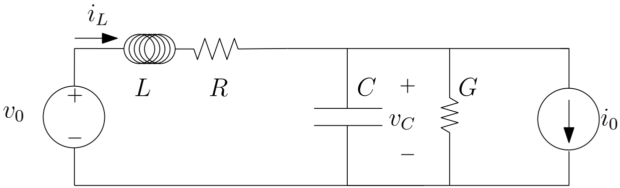

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- **Incremental** model: $\frac{d}{dt} \tilde{x} = A\tilde{x} \leftarrow$ linear!
  - (with $\tilde{x} := x - \bar{x}$).

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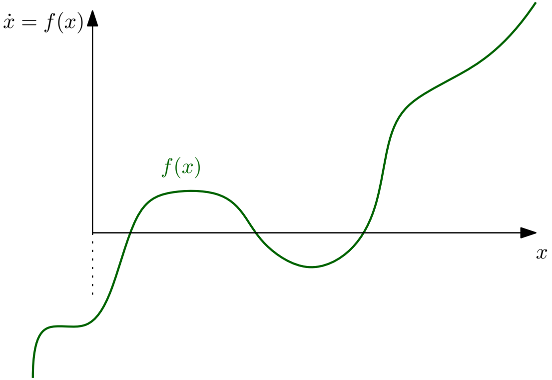

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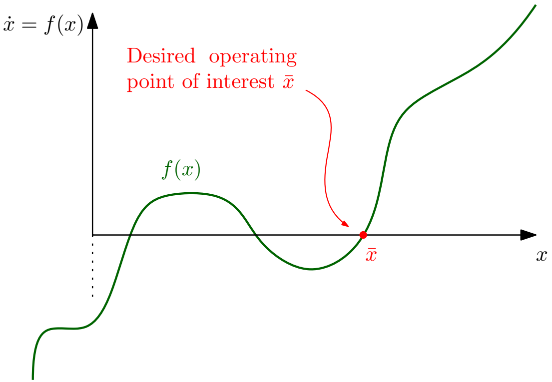

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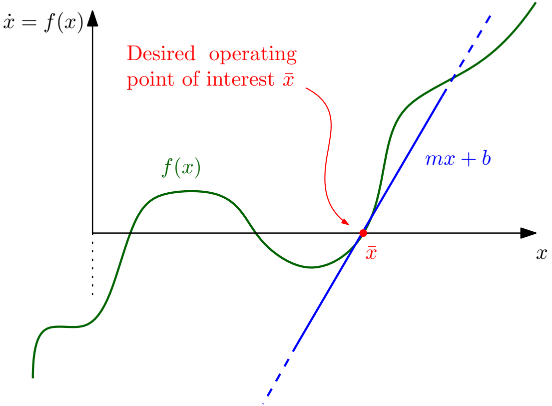

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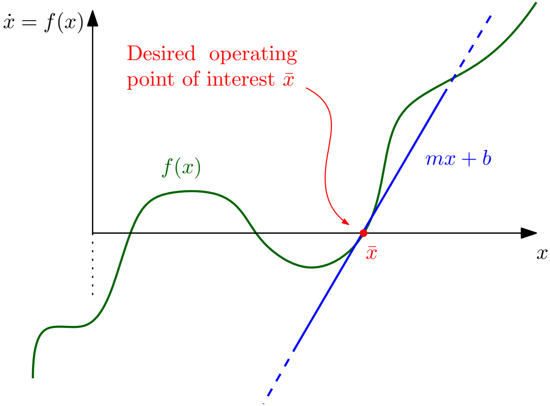

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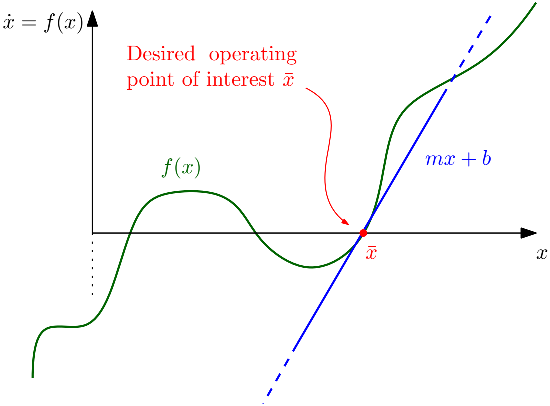

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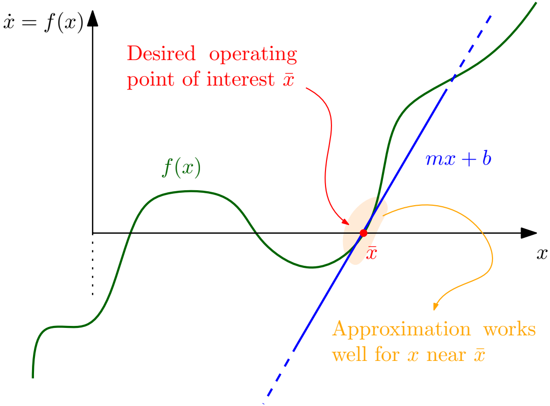

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since $$\underbrace{y}_{\dot{x}} - \underbrace{\bar{y}}_{f(\bar{x})=0} = m(\underbrace{x - \bar{x}}_{\tilde{x}}),$$

then $$\dot{x} \approx m(x - \bar{x}) \Leftrightarrow \dot{\tilde{x}} \approx m\tilde{x}$$

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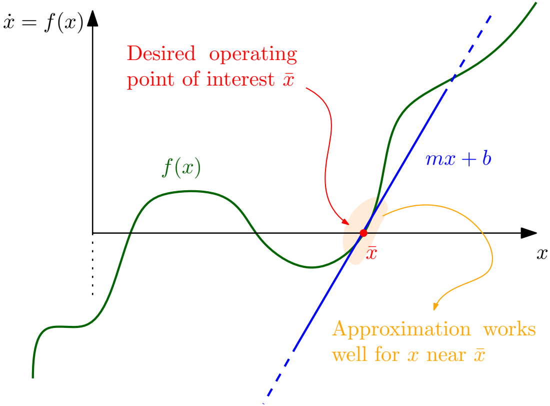

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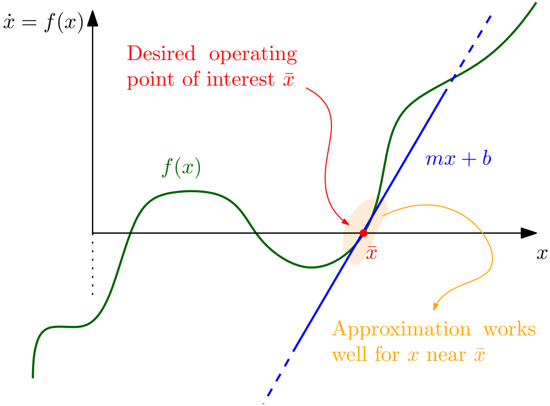
$$\text{then} \quad \dot{x} \approx m(x - \bar{x}) \Leftrightarrow \dot{\tilde{x}} \approx m\tilde{x}$$

In summary

- We approximate our system $\dot{x} = f(x)$ with $\dot{x} \approx m\tilde{x}$, which holds sufficiently close to the operating points $\leftrightarrow$ under small deviations.
- This gives rise to *small-signal* (stability) studies.

Linearization of closed systems - multidimensional case



Now we have as state variable $x \in \mathbb{R}^n$ and vector field $f : \mathbb{R}^n \to \mathbb{R}^n$; i.e.,

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix}$$

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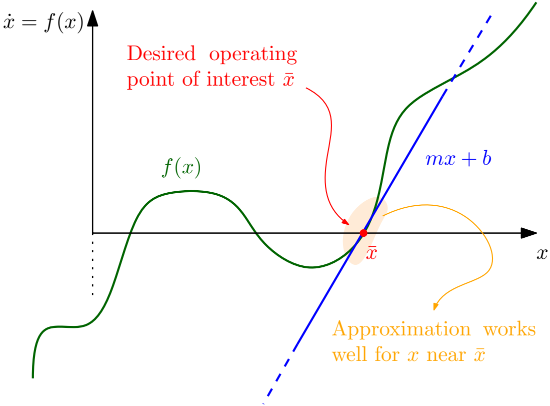

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Like the scalar case, we can linearize $f(x)$ around $\bar{x}$ as $\dot{x} \approx A\underbrace{(x - \bar{x})}_{\tilde{x}}$, with

$A \in \mathbb{R}^{n \times n}$ the *Jacobian* of $f(x)$, **evaluated at** $\bar{x}$.

Linearization of closed systems - multidimensional case

More precisely, the A-matrix can be computed by first calculating the Jacobian:

$$\frac{\partial}{\partial x} f(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \frac{\partial f_n(x)}{\partial x_2} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix},$$

and then **evaluating** the Jacobian at $x = \bar{x}$; i.e., $A := \frac{\partial}{\partial x} f(x)\big|_{x=\bar{x}}$. We then have the following approximated linear system,

$$\dot{\tilde{x}} = A\tilde{x} \longleftrightarrow \dot{x} = A(x - \bar{x})$$

for small-signal studies.

Step-by-step example

Consider the following nonlinear system: $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 5x_1^2 + 2x_2^3 \\ x_1 x_2 + 5 \end{bmatrix}$$

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2. derivate: $\frac{\partial f_1(x_1,x_2)}{\partial x_1} = 10x_1$, $\frac{\partial f_1(x_1,x_2)}{\partial x_2} = 6x_2$, $\frac{\partial f_2(x_1,x_2)}{\partial x_1} = x_2$ and $\frac{\partial f_2(x_1,x_2)}{\partial x_2} = x_1$

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Consider the following nonlinear system: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 5x_1^2 + 2x_2^3 \\ x_1 x_2 + 5 \end{bmatrix}$

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Linearization of open systems - multidimensional case

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with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ defined as:

$$A := \frac{\partial}{\partial x} f(x, u)\bigg|_{x=\bar{x}, u=\bar{u}}, \quad \text{and} \quad B := \frac{\partial}{\partial u} f(x, u)\bigg|_{x=\bar{x}, u=\bar{u}}$$

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Outline

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Bloc 3 Challenge$\times 1/2$

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Bloc 3 (full) Challenge

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4. Some more stuff to do...see last slide for full challenge.

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Response of a linear system - scalar (easy) case

Consider a general scalar, closed linear system

$$\dot{z} = \lambda z + p \longleftrightarrow \dot{\tilde{z}} = \lambda \tilde{z}, \quad \text{with} \quad \bar{z} := -p/\lambda,$$

with state variable $z \in \mathbb{R}$ with initial value $z(0)$, and a constant parameters $\lambda, p \in \mathbb{R}$.

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$$\lim_{t \to \infty} z(t) = \lim_{t \to \infty} \left([z(0) - \bar{z}]e^{\lambda t} + \bar{z} \right) = \bar{z}, \quad \forall \quad \lambda < 0$$

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Response of a linear system - multidimensional case

However, your closed linearized systems $\dot{x} = A(x - \bar{x})$ will look a bit different$\rightarrow$ A matrix is not diagonal!$\rightarrow$ hard to tell if it is stable by just looking at it.

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What is the system response $x(t)$, for a given initial value $x(0)$?

Response of a linear system - multidimensional case

However, your closed linearized systems $\dot{x} = A(x - \bar{x})$ will look a bit different$\rightarrow$ A matrix is not diagonal!$\rightarrow$ hard to tell if it is stable by just looking at it.

$$\underbrace{\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \vdots \\ \dot{\tilde{x}}_n \end{bmatrix}}_{\dot{\tilde{x}}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}}_{\tilde{x}} \longleftrightarrow \dot{\tilde{x}} = A\tilde{x}$$

What is the system response $x(t)$, for a given initial value $x(0)$?

- we propose $\tilde{x}(t) = e^{St} \cdot c$ with $c \in \mathbb{R}^n$ and $S \in \mathbb{R}^{n \times n}$ and replace it in $\dot{\tilde{x}} = A\tilde{x}$, where we get $Se^{St} \cdot c = A \cdot e^{St} \cdot c \implies S = A$.
- evaluating our (updated) solution $x(t) = e^{At} \cdot c + \bar{x}$ at $t = 0$ gives $x(0) = c + \bar{x}$. Thus, $x(t) = e^{At}[x(0) - \bar{x}] + \bar{x}$

Outline

Recap from previous lectures

Linearization of nonlinear systems

Bloc 3 Challenge$\times 1/2$

Response of a linear system

Modal transformation & eigenvalues

Bloc 3 (full) Challenge

Modal transformation: diagonal form

Question: How can we then know if our linearized system $\dot{\tilde{x}} = A\tilde{x}$ is stable or not when our A-matrix is nondiagonal?

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- **Strategy:** Let's choose Q such that $\Lambda := Q^{-1}AQ$ becomes diagonal

$$\rightarrow \Lambda = \mathrm{diag}(\lambda_i) \in \mathbb{R}^{n\times n}$$

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- Then $\dot{\tilde{z}} = \Lambda \tilde{z}$ would look like n-independent linear systems; i.e.:

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- If trajectories of $\tilde{z}(t)$ are stable, and since $\tilde{z} := Q^{-1}\tilde{x} \leftrightarrow \tilde{x} = Q\tilde{z}$, then the trajectories of $\tilde{x}(t)$ are also stable.

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- **ok, but how? :** we achieve this by selecting Q as a collection of all the (right) eigenvectors of the A-matrix
- ...leading to $\Lambda = Q^{-1}AQ = \mathrm{diag}(\lambda_i)$ being a diagonal matrix containing all eigenvalues of the A-matrix.

Eigenvalues and Eigenvectors

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- To find the eigenvalue of A, we write $Av = \lambda v = \lambda \mathbb{I} v$ (with $\mathbb{I} \in \mathbb{R}^{n\times n}$ the identity matrix) as

$$(A - \lambda \mathbb{I})v = 0_n \tag{1}$$

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- if the matrix $(A - \lambda \mathbb{I})$ is *nonsingular* then the only solution of (1) is $v = 0_n$.
- to have a (more interesting) non zero solution, $(A - \lambda \mathbb{I})$ must be *singular*:

$$\Delta(\lambda) := \det(A - \lambda \mathbb{I}) = 0$$

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since $\tilde{x}(t) = Q\tilde{z}(t)$, we have that:

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- the slowest (smallest) rate of convergence will then dominate the system dynamics.

Outline

Recap from previous lectures

Linearization of nonlinear systems

Bloc 3 Challenge $\times 1/2$

Response of a linear system

Modal transformation & eigenvalues

Bloc 3 (full) Challenge

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5. Include example of how α and β appear in the simulation plots.