# 1 Eigenfaces

## 1.1 PCA

### 1.1.1 Power method

It iteratively finds the approximation of eigenvalues in a matrix. To use this method then we have to assume that a matrix has a dominant eigenvalue with corresponding dominant eigenvectors. Afterwards an initial approximation value $x_0$ is chosen for one of the dominant eigenvectors of the matrix. The power method can thus iteratively approximate the result for finding the dominant eigenvalues by calculating the new approximation using the old one such as this:

$$x_{new} = A \cdot x_{old}$$

Where A is a matrix and $x_{old}$ is either the initial approximation or one derived thereof.[1] [2]

One can further improve upon the convergence rate by normalizing the approximated eigenvector. This will also result in lower numbers being generated during each iteration without loss of progress, which means less space is being used.[3]

Therefore, a method which iteratively scales is preferable.
The eigenvalue of a matrix can be found by normalizing the approximated eigenvector, however this approach introduces a minor problem about the sign which can be circumvented. When the power method is almost done then to make sure the right eigenpair is achieved a value $w_{old}$ is to be stored to check if the eigenvalue is negative, this check can be done by using the fact that the signs fluctuate on the eigenvector when a negative eigenvalue is the dominant one. Thereby one can divide $w_n$ with $w_{n-1}$ to achieve a number denoting if the eigenvalue is negative.[4]
**Algorithm 1** displays the iterative process of the Power method. Where $x_0$ is a random initialization vector and $w_0$ is a scaled initial approximation used to approximate the eigenvector, $w$ related to the eigenvalue $\lambda$.
The power method is capable of finding an eigenpair but what is needed to complete the PCA is to find all the eigenpairs for a given covariance matrix.

### 1.1.2 Eigenshift method

The eigen value shift technique is used on a covariance matrix to "shift" its dominant eigenpair so that another matrix can be formed which has different dominant eigenpairs. This makes it possible to use the power method combined with the eigenshift method for finding all eigenpairs for a given covariance matrix. One can derive a shifted matrix by using the following formula:

$$C_{shift} = C - C \cdot w \cdot w^T$$

Where $C$ is the covariance matrix and $w$ is the dominant eigenvector.
Since a homomorphic setting is intended then one can alter the formula slightly

**Algorithm 1:** Power method

**Input:** C : Covariance Matrix.
**Output:** $\lambda, w$ : Dominant eigenpair of C.
$w_0 = \frac{x_0}{||x_0||}$
**for** $i \leftarrow 0$ **to** $n$ **do**
  $x_i = C \cdot w_{i-1}$
  $\lambda_i = ||x_i||$
  $w_i = \frac{x_i}{\lambda_i}$
  **if** $i+2 == n$ **then**
    $w_{old} = w_i[0]$
  **end**
**end**
**if** $\frac{w_n[0]}{w_{old}} < 0$ **then**
  $\lambda = -1 \cdot \lambda_n$
  $w = \frac{x_n}{\lambda}$
**end**
**return** $\lambda, w$

to achieve a faster computation, this is possible due to the following relationship: $A \cdot w = \lambda \cdot w$ Thereby the formula becomes:

$$C_{shift} = C - \lambda \cdot w \cdot w^T$$

Using $\lambda$ instead of $C$ makes the algorithm have to calculate less matrix multiplications making it faster.[5]

**Algorithm 2:** Eigenshift method

**Input:** $C, \lambda, w$ : Covariance Matrix, Dominant eigenpair.
**Output:** $C_{shift}$ : Shifted covariance matrix.
$C_{shift} = C - \lambda \cdot w \cdot w^T$
**return** $C_{shift}$

**Algorithm 2** shows the method used to shift the dominant eigenpair on a covariance matrix. By combining algorithm 1 and 2 then all eigenpairs can be found for a given covariance matrix.

# References


[1] *Power method for approximating eigenvalues.* [Online]. Available: `https://ergodic.ugr.es/cphys/lecciones/fortran/power_method.pdf`.

[2] R. V. Mises and H. Pollaczek-Geiringer, "Praktische verfahren der gleichungsauflösung .," *ZAMM - Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 9, no. 1, 152–164, 1929. DOI: `10.1002/zamm.19290090105`.

[3] G. Sanchez, *A matrix algebra companion for statistical learning (matrix4sl)*, 2018. [Online]. Available: `https://www.gastonsanchez.com/matrix4sl/power-method.html`.

[4] W. L. Perry, *Elementary Linear Algebra*. McGraw-Hill, 1988.

[5] H. V. L. Pereira and D. F. Aranha, "Principal component analysis over encrypted data using homomorphic encryption," Ph.D. dissertation, ResearchGate, 2017.