Reference Frames, Relativity, and Mathematics for the Global Positioning System

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November 19, 2017

Introduction

Linear algebra has numerous applications that are essential to keep our technological world running. As of today, one of the biggest systems we take advantage of are satellite navigational systems. More specifically, the Global Positioning System (GPS) which is the American satellite navigational system and the first satellite system to be launched back in 1974. Millions of people each day use this system for means to find a quick get around, track transit schedules, and just an all around profound system that makes getting from point A to point B simplistic. However, we never stop to realize just how complex the mathematics and physics used in this system really are. To ensure that GPS is accurate general and special relativity has to be taken into account to fix the affects caused by special relativistic time dilation. Without the time corrections the satellites would become considerably off. Each day the satellites will gain 45 microseconds while the clocks on earth will fall behind by 7 microseconds, the difference of the two tell us that the satellites will be off by 38 microseconds. This is a considerably low amount of time, however, over the span of one day the calculations become so inaccurate that a calculated position on earth would be 10 kilometers off if general relativity was not taken into account [2]. Rendering this system completely useless, but the correct time is only one part of the problem. Knowing the location of the 31 satellites orbiting the globe are obvisouly essential in the calculation process. To determine a location on earth, the satellites need to know the location of other satellites around it. Thus, determining a location on earth using GPS is deeply rooted in navigational algorithms that involve coordinate frames and transformations of coordinates between them [1].

Coordinate Transformations

For a person on earth to receive navigation information with respect to earth, the position of the satellite and it's velocity are transformed to an earth fixed frame [2]. To understand coordinate transformations we first have to understand the various coordinate frames. The first is the Earth-Centered Inertial

Frame (ECI), this is the frame responsible for calculating the satellite's position and velocity. This frame will be denoted as the i-frame. The next is the Earth-Centered Earth-Fixed Frame (ECEF), this frame is similar to the previous frame but serves a different purpose. The purpose of this frame is to rotate along with the earth, hence, it is fixed to the earth. This frame is denoted as e-frame. Finally, the Local-Level Frame (LLF), used for altitude and velocity this frame will be denoted as the l-frame.There is also the Wander Frame, and the Computational Frame, and the Body Frame. Of which the frames are purposeful for orientation of the satellite, any reference frame used for calculating motion, and accelerometer sensors respectively. However, there is a lot of material that just go into the transformations for these frames alone, therefore the basic transformation will be between ECI and ECEF, then LLF and ECEF.

ECI and ECEF

We begin to define the angular velocity and acceleration as,

$$\vec{\omega}_{ie}^{i} = \begin{bmatrix} 0 \\ 0 \\ \omega_e \end{bmatrix}. \tag{1}$$

Where ω_e is the magnitude of the Earth's rotation rate. Now to find the transformation from the i-frame to the e-frame we have to derive the elementary rotational matrices. To be able to transform a vector from a frame δ to η, the frames a and b are alined using rotation angle γ about the z-axis, β about the x-axis, and α about the y-axis. These angles are referred to as Euler angles. So, the new coordinates are represented as

$$x^{\eta} = r_1 cos(\theta - \gamma) \tag{2}$$

$$x^{\delta} = r_1 cos(\theta) \tag{3}$$

$$y^{\eta} = r_1 sin(\theta - \gamma) \tag{4}$$

$$y^{\delta} = r_1 sin(\theta) \tag{5}$$

$$z^{\eta} = y^{\delta}. \tag{6}$$

The way we can look at this is that the earth is a sphere and thus we should think in spherical coordinates. With that being said we have a coordinate that is fixed to the earth and a coordinate that floats around the earth. Therefore moving in the x direction changes along the equator while moving in the y direction changes along the prime meridian and z stays constant since there are no angles needed to change in the z direction. Using the sum and difference trigonometric identity and substituting equation (3) into (2) and (5) into (4) we get,

$$x^{\eta} = x^{\delta} cos\gamma + y^{\delta} sin\gamma \tag{7}$$

$$y^{\eta} = -x^{\delta} cos\gamma + y^{\delta} sin\gamma. \tag{8}$$

In matrix form we get the elementary rotational matrices to be,

$$R_{\delta}^{\eta} = \begin{bmatrix} cos\omega_e t & sin\omega_e t & o \\ -sin\omega_e t & cos\omega_e t & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

The transformation from the i-frame to the e-frame is the same as the elementary rotational matrix and becomes

$$R_i^e = \begin{bmatrix} x^{\eta} \\ y^{\eta} \\ z^{\eta} \end{bmatrix}_{ECI} = \begin{bmatrix} cos\omega_e t & sin\omega_e t & o \\ -sin\omega_e t & cos\omega_e t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^{\delta} \\ y^{\delta} \\ z^{\delta} \end{bmatrix}_{EFC} \tag{10}$$

Where t is the time since we are now dealing with velocity. This transformation can now be calculated in MATLAB for real time coordinates. Using the command dcmeci2ecef, we are able to find the position of the satellite. Lets say we enter in the commands

```
dcm = dcmeci2ecef('IAU-2000/2006',[2017 11 15 1 00 0])
dcm = dcmeci2ecef('IAU-2000/2006',[2017 11 15 1 20 0])
```

where we enter a specific time. Let's say November 15^{th}, 2017 at 1:00:00 UTC and then November 15^{th}, 2017 at 1:20:00 UTC.

The matrix returned is with the coordinates

$$\begin{bmatrix} 0.3570 & 0.9341 & 0.0000 \\ -0.9341 & 0.3570 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

and 20 minutes later,

$$\begin{bmatrix} 0.2740 & 0.9617 & 0.0000 \\ -0.9617 & 0.2740 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

the transformations coordinates change slightly. We see the actual transformation of these reference frames calculated.

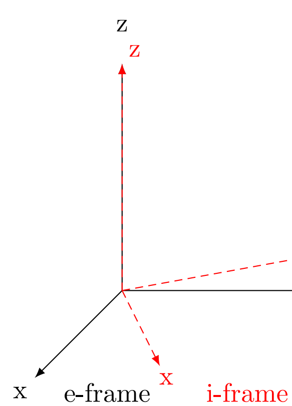

Figure 1: An example of the ECI and ECEF transformation graph.

Conclution

The Global Positioning System is a complex system that involves an immense amount of extreme engineering, physics, and mathematics. In this paper, only a brief subsection of math was covered to talk about this complex system. However small, these transformation play an important roll in a vastly larger system. There is a great deal more of mathematics that is involved in this system from launch to orbit to calculating your location. But none the less each piece as a whole allows for out modern society to function each day.

References

[1] Noureldin, Aboelmagd, et al. Fundamentals of Inertial Navigation, Satellite-Based Positioning and Their Integration. Springer-Verlag, 2013.

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[3] Friedberg, Stephen H., et al. Linear Algebra. Pearson, 2016.

[4] MATLAB Release 2016b, The MathWorks, Inc., Natick, Massachusetts, USA.