Prove $\vec{p} = A(A^TA)^{-1}A^T\vec{b}^T$ minimizes $\|\vec{b}-\vec{p}\| = \|\vec{e}\|$ 17

__Proof__ since $\vec{p} \in C(A)$, $\vec{p} = A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$

We need x so that $\|\vec{b}-\vec{p}\|^2 = \|\vec{b}-A\vec{x}\|^2$ is minimized.

$$
\begin{aligned}
f(\vec{x}) &= \|\vec{b}-A\vec{x}\|^2 \\
&= (\vec{b}-A\vec{x})^T(\vec{b}-A\vec{x}) \\
&= (\vec{b}^T - x^TA^T)(\vec{b}-A\vec{x}) \\
&= \vec{b}^T(\vec{b}-Ax) - x^TA^T(b-A\vec{x}) \\
&= \vec{b}^T\vec{b} - \vec{b}^TAx - x^TA^T\vec{b} + x^TA^TA\vec{x} \\
&= \vec{b}^T\vec{b} - (\vec{b}^TAx)^T - x^TA^T\vec{b} + x^TA^TA\vec{x} \\
&= \vec{b}^Tb - \vec{x}^TA^T\vec{b} - x^TA^Tb + x^TA^TAx \\
&= \vec{b}^Tb - 2x^TA^Tb + x^TA^TAx
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(b^Tb - 2\vec{x}^TA^Tb - x^TA^TAx) \\
&= 0 - 2A^Tb + \frac{\partial}{\partial x}\left[A\vec{x} \cdot Ax\right] \\
&= 0 - 2A^Tb + \frac{\partial}{\partial x}\left[x^TA^TAx\right] \\
&= -2A^Tb + 2A^TA\vec{x} \\
&= 2(A^TA\vec{x} - A^T\vec{b})
\end{aligned}
$$

set $f'(x) = 0$, solve for $\vec{x}$

$$
\begin{aligned}
f'(x) &= 2(A^TAx - A^Tb) = \vec{0} \\
&= A^TA\vec{x} - A^Tb = 0 \\
A^TAx &= A^Tb \\
(A^TA)^{-1}A^TAx &= (A^TA)^{-1}A^Tb \\
\vec{x} &= (A^TA)^{-1}A^T\vec{b}
\end{aligned}
$$

$$p = A\vec{x} = A(A^TA)^{-1}A^Tb$$