3-24-21

II.2 # 1-11

1.

2. the row space of A^+ is the column space of A^T and viceversa column rank and row rank are always equal. Hence $\text{rank}(A^+) == \text{rank}(A^T)$

3. $A^+A = \sum \frac{v_i u_i^T}{\sigma_i} \cdot \sum \sigma_i u_i v_i^T$

$= \sum \frac{v_i u_i^T}{\cancel{\sigma_i}} \cancel{\sigma_i} u_i v_i^T = \sum v_i u_i^T u_i v_i^T = \sum v_i (1) v_i^T$

$= \sum v_i v_i^T$ ✓

the dot product of a vector with itself is the square of its magnitude. since u and v are unit vectors, their magnitude = 1.

3 cont.

if $i \neq j$
$v_i v_j^T = 0$

$$(A^+A)^2 = \left(\sum v_i v_i^T\right)^2 = (v_i v_i^T)(v_j v_j^T) + \dots$$

$$= v_1 v_1^T v_1 v_1^T + v_2 v_2^T v_2 v_2^T + \dots$$

$$= v_1 (1) v_1^T + v_2 (1) v_2^T + \dots$$

$$= \sum v_i v_i^T = A^+ A \checkmark$$

why does this prove / show a projection?

4. if A is $m \times n$, then A^+ is $n \times m$

if $A^+ = A$ then $m = n$ ~~then~~ they are square.

if $A^+ = A$, then

$$A = U \Sigma V^T = V \Sigma^+ U^T = A^+ \text{ so}$$

$$A^+A = V \Sigma^+ U^T U \Sigma V^T = V \Sigma^+ (I) \Sigma V^T$$

$$= V \cdot \begin{bmatrix} I & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} V^T$$

$A^+A =$

$A^+ = A$ when A is independent.

if Σ^+ has $\emptyset$s then we can't complete this

5) $A = U\Sigma V^T$

a)

$$= \underset{m\times m}{\begin{bmatrix} & \end{bmatrix}} \underset{m\times n}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline 0 & \cdots & 0 \end{bmatrix}} \underset{n\times n}{\begin{bmatrix} & \end{bmatrix}}$$

$\boxed{\text{there are } n \text{ non-zero } \sigma\text{'s}}$

b)

$$(\Sigma^T\Sigma)^{-1} = \Bigg(\underset{n\times m}{\left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_n & \end{array}\right]} \underset{m\times n}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline & 0 & \end{bmatrix}} \Bigg)^{-1}$$

$$= \Bigg(\underset{n\times n}{\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}} \Bigg)^{-1}$$

$$= \underset{n\times n}{\begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{bmatrix}} \checkmark$$

c) $(\Sigma^T\Sigma)^{-1}\Sigma^T =$

$$\underset{n \times n}{\begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{bmatrix}} \underset{n \times m}{\left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_n & \end{array}\right]}$$

$$= \underset{n \times m}{\left[\begin{array}{ccc|c} \frac{1}{\sigma_1} & & & \\ & \ddots & & 0 \\ & & \frac{1}{\sigma_n} & \end{array}\right]} = \Sigma^+$$

d) $A^+ = (A^TA)^{-1}A^T$

$A^+ = ((U\Sigma V^T)^T U\Sigma V^T)^{-1} V\Sigma^T U^T$

$= (V\Sigma^T U^T U\Sigma V^T)^{-1} V\Sigma^T U^T$

$= (V\Sigma^T\Sigma V^T)^{-1} V\Sigma^T U^T$

$= (V^T)^{-1}(\Sigma^T\Sigma)^{-1} V^{-1} V\Sigma^T U^T$

$= (V^T)^{-1}(\Sigma^T\Sigma)^{-1}\Sigma^T U^T$

$= (V^T)^{-1}\Sigma^+ U^T$

$A = V\Sigma^+ U^T$

why does $(V^T)^{-1} = V$?

6)

$$r = a - 2\,\frac{(a-r)(a-r)^T}{(a-r)^T(a-r)}\,a$$

$$= a - 2\,\cancel{\frac{(a-r)(a-r)^T}{(a_1-r_1)^2+\dots+(a_n-r_n)^2}}\,a$$

7)

8)

$$(b - ca) \cdot a = 0$$

$$\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix} - c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$(4 - c)(1) + (-c)(1) = 0$$

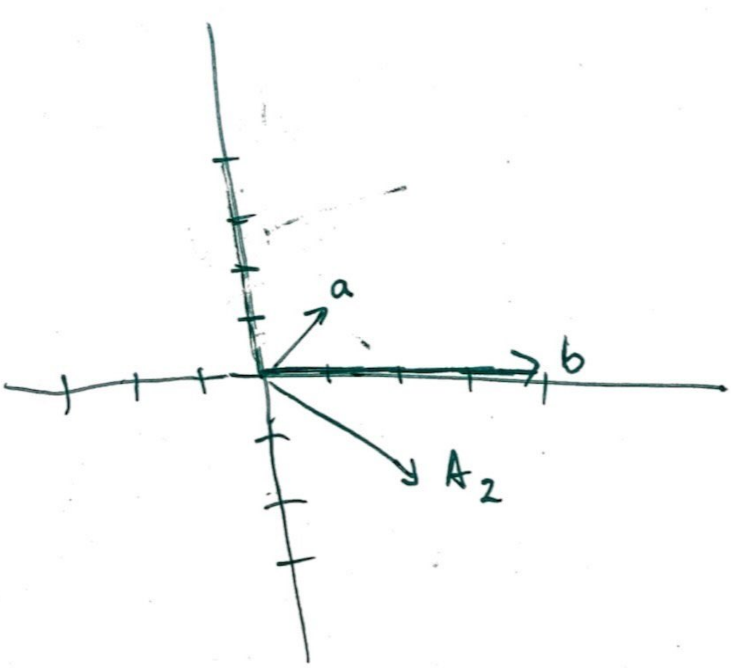
$$4 - c - c = 0$$

$$4 - 2c = 0$$

$$-2c = -4$$

$$\boxed{c = +2}$$

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$



9) $q_1 = \dfrac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$$A_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - 2\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$q_2 = \frac{\begin{bmatrix} 2 \\ -2 \end{bmatrix}}{\sqrt{8}} = \frac{\begin{bmatrix} 2 \\ -2 \end{bmatrix}}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}}$$

$$r_{12} = q_1^T a_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= 2\sqrt{2}$$

10) Since R is a upper triangle matrix, $R^T R$ is lower times upper

11)

$Q^T Q = I$

$Q^T Q Q^{-1} = Q^{-1}$

$Q^T = Q^{-1}$

hence $Q^T = Q^+$

$A = QR$

~~$AR^{-1} = Q$~~

~~$AA^T = QRR^T Q^T$~~

~~$AA^T = QR^{-1} R R^T Q^T$~~

$AA^T = QR^T Q^T$