Inner Product (,)

$(\vec{v}, \vec{w})$ = inner product of $\vec{v}, \vec{w}$

1. $(\vec{x}+\vec{y}, \vec{z}) = (\vec{x}, \vec{z}) + (\vec{y}, \vec{z})$
2. $(\vec{x}, c\vec{y}) = c(\vec{x}, \vec{y})$

~~3~~2 $(c\vec{x}, \vec{y}) = \bar{c}(\vec{x}, \vec{y})$

3. $(\vec{y}, \vec{x}) = \overline{(\vec{x}, \vec{y})}$
4. $(\vec{x}, \vec{x}) \geq 0$

Examples

1) Dot product on $\mathbb{R}^n$

2) $(\vec{z}, \vec{w}) = \vec{z}^* \vec{w}$ where z^* = conjugate transpose of $\vec{z}$

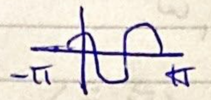
$$\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$$

$$\vec{z}^* = \begin{bmatrix} \bar{z}_1 & \cdots & \bar{z}_n \end{bmatrix}$$

3) If S is a $n \times n$ symmetric positive def matrix, the inner product $(,)_S$ on $\mathbb{R}^n$ is $(\vec{x}, \vec{y})_S = \vec{x}^T S \vec{y}$

Example $\vec{x}, \vec{y} \in \mathbb{R}^2$. $S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

4. $V = \{$ continuous real-valued functions defined on $[-\pi, \pi]\}$



Define an inner product of 2 functions f, g in V by

$$(f,g) = \int_{-\pi}^{\pi} f(t)g(t)\,dt$$

(IP1) $(f+g, h) \stackrel{?}{=} (f,h) + (g,h)$

proof

$$(f+g,h) = \int_{-\pi}^{\pi} (f(t)+g(t))h(t)\,dt$$

$$= \int_{-\pi}^{\pi} f(t)h(t) + g(t)h(t)\,dt$$

$$= \int_{-\pi}^{\pi} f(t)h(t)\,dt + \int_{-\pi}^{\pi} g(t)h(t)\,dt$$

$$= (f,h) + (g,h) \checkmark$$

Cosines and Sines

Let m, n be positive integers

$$f(t) = \cos(mt) \quad g(t) = \cos(nt)$$

$$(f,g) = \int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (\cos(m+n)t\,dt + \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)t\,dt$$

$$= 0 + \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)t\,dt$$

$$= \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$(\cos(mt), \cos(nt)) = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$(\sin(mt), \sin(nt)) = \begin{cases} 0 \\ \pi \end{cases}$$

$$(\cos(mt), \sin(nt)) = 0$$

Reusing the inner product,

$$(f, g) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(t) \, dt$$

$$(\cos(mt), \cos(nt)) = \begin{cases} 0 \\ 1 \end{cases}$$

orthonormal set of functions

$$\{ \cos(t), \cos(2t), \cos(3t) \ldots$$
$$\sin(t), \sin(2t), \sin(3t) \ldots \}$$

- recall if $\{\vec{q}_1 \ldots \vec{q}_n\}$ is an "on basis" for $\mathbb{R}^n$, any vector $\vec{v}$ can be written as $\vec{v} = c_1\vec{q}_1 + c_2\vec{q}_2 + \cdots + c_n\vec{q}_n$

where $c_k = \vec{v} \cdot \vec{q}_k$

Forier Series: if $f \in C([-\pi,\pi]) = V$, then f can be written as

$$f(t) = a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + \dots$$

to find a_k, b_k use the orthogonality of sins and cosines.

$$(f, \cos(t)) = ((a_0 + a_1\cos t + b_1 \sin t + \dots), \cos t)$$
$$= a_1(\cos(t), \cos(t)) = a_1$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} fg\,dt$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos t\,dt$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt$$

average over $[-\pi,\pi]$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

Simpler Method: use complex numbers

Let $W = \{$ continuous functions $f: [-\pi,\pi] \to \underline{\mathbb{C}} \}$

Complex output

e.g $f(t) = e^{it} = \cos t + i\sin t$

Inner product on W

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\overline{g(t)}\,dt$$

Note: this satisfies (IP1–IP4) (in particular)

$$(f,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{f(t)}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt \geq 0$$

With respect to this inner product, the functions $e^{it}, e^{-it}, e^{2it}, e^{-2it}, \dots$ are an orthonormal set

$$(e^{imt}, e^{int}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imt} \cdot \overline{e^{int}}\,dt$$

$$I = \frac{1}{2\pi}\int e^{imt} e^{-int}\,dt$$

$$\boxed{\begin{aligned} e^{int} &= \cos(nt) + i\sin(nt) \\ e^{-int} &= \cos(-nt) + i\sin(-nt) \\ &= \cos(nt) - i\sin(nt) \end{aligned}}$$

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)t}\,dt$$

if $m \neq n$

$$I = \frac{1}{2\pi}\cdot\frac{1}{i(m-n)}\left(e^{i(m-n)\pi} - e^{i(m-n)(-\pi)}\right)$$

$\downarrow$

$I = 0.$

if $m = n$

$I = 1$

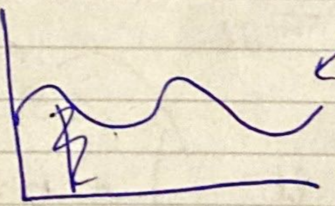
The forier series in powers of e^{it}

$$f(t) = c_0 + c_1 e^{it} + c_{-1} e^{-it} + \dots$$

$$c_k = (f, e^{ikt}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt$$

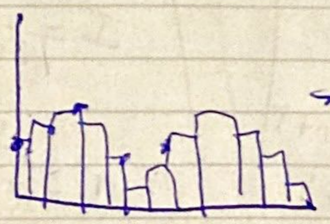
$$\boxed{f(t) = \sum_{k=0}^{\infty} c_k e^{ikt}}$$

Discrete Forier Transform (DFT)



continuous case

integrate with e^{ikt}



discrete case

$$\left\{ \begin{aligned} f(0) &= 2 \\ f(\pi/2) &= 4 \\ f(\pi) &= 6 \\ f(3\pi/2) &= 8 \end{aligned} \right\}$$

$$f(0) = c_0 + c_1 + c_2 + c_3 = 2$$

$$f(\tfrac{\pi}{2}) = c_0 + ic_1 - c_2 - ic_3 = 4$$

$$f(\pi) = c_0 + c_1 + c_2 - c_3 = 6$$

$$f(\tfrac{3\pi}{2}) = c_0 - ic_1 - c_2 + ic_3 = 8$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & +i \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}$$

$$\bar{F} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$\bar{F} F = 4I \Rightarrow F^{-1} = \frac{1}{4} \bar{F}$$