Advanced Topics in Algebraic Graph Theory

1 Distance-regular graphs

Throughout these notes, we let $X = (V, E)$ be a simple undirected loopless graph with n vertices and m edges, and adjacency matrix A. We let $X_i = \{(u, v) \in V \times V | D(u, v) = i\}$ denote the set of all pairs of vertices at distance i, and if $u \in V$, then $X_i(u) = \{v \in V | (u, v) \in X_i\}$. We let A_i be the adjacency matrix of X_i, that is, the distance-i matrix of X.

Definition 1. We say that a connected regular graph X with diameter d and degree k is *distance-regular* if there are constants p^l_{ij}, for all $i, j, l \in \{0, 1, ..., d\}$, such that for any pair of vertices u, v at distance l from one another, the number of vertices w at distance i from u and j from l equals p^l_{ij}. In other words, for any $u, v \in V$ such that $D(u, v) = l$, then

$$|X_i(u) \cap X_j(v)| = p^l_{ij}.$$

More concretely, for any $i, j \in \{0, 1, ..., d\}$, we have

$$A_i A_j = \sum_{k=0}^{d} p^l_{ij} A_l.$$

We note that since the graph is assumed to be undirected, it follows that $p^l_{ij} = p^l_{ji}$. We also note that $p^0_{ij} = 0$ if $i \neq j$, $p^l_{0j} = 0$ if $j \neq l$, and $p^j_{0j} = 1$ for all j. We denote by $k_l = p^0_{ll}$ as the degree of l-distance graph of X. We also note that $p^i_{j1} = 0$ if $i \neq 0$ and $j \notin \{i-1, i, i+1\}$, and we denote by $c_i = p^i_{(i-1)1}, b_i = p^i_{(i+1)1}$. It will be useful to let $c_0 = b_d = 0$,$A_{-1} = A_{d+1} = 0$, and to let c_{d+1}, b_{-1} be undefined, and we note that $b_0 = k, c_1 = 1$. If we fix a vertex $v \in V$, there are exactly k_i vertices at distance i from v, and each of these vertices has b_i neighbors at distance $i + 1$ from v, hence there are $k_i b_i$ edges from $X_i(v)$ to $X_{i+1}(v)$. On the other hand, there are k_{i+1} vertices in $X_{i+1}(v)$, each with c_{i+1} neighbors in $X_i(v)$, thus we obtain that

$$k_{i+1} = \frac{k_i b_i}{c_{i+1}},$$

and it also follows that $|V| = 1 + k_1 + ... + k_d$. We can also count triples of vertices in X to obtain certain equalities. Let $u, v, w \in V$ such that $D(u, v) = l, D(u, w) = i, D(w, v) = j$, then

$$k_l p^l_{ij} = k_i p^i_{jl} = k_j p^j_{il}.$$

From these definitions, we can prove our first result.

Theorem 2. *If X is a connected k-regular graph with diameter d, and if there are integers k_i, b_i, c_i such that for any vertex $v \in V(X)$ we have*

$$|X_i(v)| = k_i,$$

and for any $u \in V(X)$ at distance i from v,

$$|X_1(u) \cap X_{i-1}(v)| = c_i$$
$$|X_1(u) \cap X_{i+1}(v)| = b_i,$$

then X is distance-regular.

Proof. As the set $\{I, A, A_2, ..., A_d\}$ is a symmetric-closed set that contains I, it suffices to show that $A_i A_j \in \text{span}(\{I, A, A_2, ..., A_d\})$ for any pair i, j. We first note that, if $i \in \{0, ..., d\}$, and letting $a_i = k - b_i - c_i$, then from the assumptions it follows that

$$(AA_i)_{uv} = |X_1(u) \cap X_i(v)| = \begin{cases} b_{i-1}, & \text{if } u \in X_{i-1}(v) \\ a_i, & \text{if } u \in X_i(v) \\ c_{i+1}, & \text{if } u \in X_{i+1}(v), \end{cases}$$

thus

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}$$

We can apply induction on i to show that A_i is always a polynomial in A of degree i, and also that $A^i \in \text{span}(\{I, A, A_2, ..., A_i\})$. Indeed, the cases for $i = 0$ or $i = 1$ are immediate, and if $i = 2$ we have

$$A^2 = b_0 I + a_1 A + c_2 A_2,$$

hence A^2 is a linear combination of I, A, A_2, and since $c_2 \neq 0$, we have that A_2 is a polynomial $\nu_2(A)$ of degree 2 in A given by

$$c_2 A_2 = c_2 \nu_2(A) = A^2 - a_1 A - b_0 I.$$

If the result holds for i, then it clearly follows that

$$c_{i+1}A_{i+1} = c_{i+1}\nu_{i+1}(A) = A\nu_i(A) - a_i\nu_i(A) - b_{i-1}\nu_{i-1}(A),$$

hence A_{i+1} is a polynomial of degree $i+1$ in A. Now let $A^i = \sum_{l=0}^{i} \alpha_l A_l$, hence

$$A^{i+1} = \sum_{l=0}^{i} \alpha_l A_l A,$$

and using that $A_l A$ is a linear combination of A_{l-1}, A_l, A_{l+1}, it follows that A^{i+1} is in $\text{span}(\{I, A, A_2, ..., A_{i+1}\})$, which concludes the induction. This shows that A_iA_j is a polynomial in A, and since

$$AA_d = b_{d-1}A_{d-1} + a_d A_d,$$

it follows that this polynomial has degree at most d, hence $A_iA_j \in \text{span}(\{I, A, A_2, ..., A_d\})$, as desired. ■

The polynomials defined in the previous result can be recursively defined by

$$\nu_{-1}(x) = 0, \quad \nu_0(x) = 1, \quad \nu_1(x) = x,$$
$$c_{i+1}\nu_{i+1}(x) = (x - a_i)\nu_i(x) - b_{i-1}\nu_{i-1}(x).$$

The previous result shows that a distance-regular graph X is completely determined by the parameters k, b_i, c_i, and we define the *intersection array* of X as

$$\iota(X) := \{b_0, b_1, ..., b_{d-1}; c_1, ..., c_d\}.$$

The Petersen graph has intersection array given by $\{3, 2; 1, 1\}$, and in fact it is determined by this array as a DRG, that is, if X is a DRG with $\iota(X) = \{3, 2; 1, 1\}$, then X must be the Petersen graph. The smallest intersection array that corresponds to more than one graph is $\{6, 3; 1, 2\}$, as both the Hamming graph $H(2, 4)$ and the Shrikhande graph – constructed as the Cayley graph on $\mathbb{Z}_4 \times \mathbb{Z}_4$ with connection set $\{\pm(1, 0), \pm(0, 1), \pm(1, 1)\}$ – both have this same intersection array. The Shrikhande graph is also the smallest distance-regular graph that is not distance-transitive.

The coefficients of the intersection array also satisfy some important properties:

Proposition 3. *If X is a distance-regular graph with intersection array $\iota(X)$, then:*

(i) $1 = c_1 \leq c_2 \leq ... \leq c_d$;

(ii) $b_{d-1} \geq ... \geq b_1 \geq b_0 = k$;

(iii) if $i + j \leq D$, *then* $c_i \leq b_j$;

(iv) The sequence $k_0 = 1, k_1 = k, ..., k_d$ *is unimodal.*

Proof. For (i) and (ii), we consider a pair of vertices u, v at distance i, and a vertex w at distance 1 from u and $i-1$ from v. If z is a neighbor of v at distance $i-2$ from w (note that there are precisely c_{i-1} of these), then the triples wuz and zuv guarantee that $D(z,u) = i-1$, hence z is one of the c_i neighbors of v that are at distance $i-1$ from u, implying that $c_{i-1} \leq c_i$. Similarly, if we consider a neighbor z of v at distance $i+1$ from u (again noting that there are b_i of these), then triples uwz, vwz guarantee that $D(z,w) = i$, hence z is one of the b_{i-1} neighbors of v at distance i from w, implying that $b_i \leq b_{i+1}$.

For (iii), we consider a pair u, v at distance $i+j$, and a vertex w at distance j from u and i from v. If z is one of the c_i neighbors of w at distance $i-1$ from v, then triples uwz, uvz show that $D(z,u) = j+1$, hence z is one of the b_j neighbors of w at distance $j+1$ from u, implying that $c_i \leq b_j$.

For (iv), it suffices to note that from $(i), (ii)$ it follows that

$$\frac{k_i k_i}{k_{i+1}k_{i-1}} = \frac{c_{i+1}b_{i-1}}{c_i b_i} \geq 1,$$

hence $k_i^2 \geq k_{i+1}k_{i-1}$, and thus $k_0, ..., k_d$ is unimodal. ∎

If X is an arbitrary graph with diameter d, the adjacency algebra $\mathbb{C}[A]$ of A – that is, the algebra of all polynomials in A – is a commutative $*$-subalgebra of the full matrix algebra $M_n(\mathbb{C})$. The dimension of this algebra is equal to the number of distinct eigenvalues of A, and since $I, A, A^2, ..., A^d$ forms a linearly independent set in $\mathbb{C}[A]$, it follows that A has at least $d+1$ distinct eigenvalues. The previous observations regarding distance-regular graphs allows us to prove the following result:

Proposition 4. *Let X be a connected k-regular graph with diameter d. Then $\mathbb{C}[A] = \mathbb{C}[I, A, A_2, ..., A_d]$ if, and only if, X is distance-regular, and in this case X has exactly $d+1$ distinct eigenvalues.*

Proof. If X is distance-regular, then the dimension of $\mathbb{C}[I, A, A_2, ..., A_d]$ is exactly $d+1$, and since this algebra contains $\mathbb{C}[A]$ which has dimension at least $d+1$, it follows that they are equal. As the dimension of $\mathbb{C}[A]$ is the number of distinct eigenvalues of A, it also follows that this number is $d+1$. The other direction is immediate. ∎

We let

$$L = \begin{pmatrix} 0 & b_0 & 0 & 0 & \dots & 0 \\ c_1 & a_1 & b_1 & 0 & \dots & 0 \\ 0 & c_2 & a_2 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & 0 & \dots & 0 & c_d & a_d \end{pmatrix}$$

be the tridiagonal $(d+1) \times (d+1)$ *intersection matrix.* It is interesting to observe that this is precisely the partition matrix w.r.t. the distance partition induced by any fixed vertex $v \in V(X)$ – this partition is equitable, and in fact, a graph is distance-regular iff the distance partition w.r.t. any fixed vertex is equitable. If we let $\Delta = \mathrm{Diag}(k_0, ..., k_d)$, noting that

$$\frac{b_i\sqrt{k_i}}{\sqrt{k_{i+1}}} = \frac{c_{i+1}\sqrt{k_{i+1}}}{\sqrt{k_i}},$$

we conclude that $\Delta^{1/2}L\Delta^{-1/2}$ is symmetric, hence L is diagonalizable. We note that if $x = (x_0, ..., x_D)$ is an θ-eigenvector of L, then

$$\theta x_i = c_i x_{i-1} + a_i x_i + b_i x_{i+1},$$

where $c_0 = b_d = 0$, hence we don't need to define x_{-1} and x_{d+1}. Moreover, we note that $x_0 \neq 0$, since otherwise $x = 0$, thus we may always assume that $x_0 = 1$, which in turn implies that $x_1 = \theta/k$ – this way of writing x is usually called the *standard sequence* w.r.t. θ.

We can now prove that L has precisely $d+1$ distinct eigenvalues, and that $\mathrm{Dspec}(A) = \mathrm{Dspec}(L)$.

Proposition 5. *If X is a distance-regular graph with diameter d, then*

$$Dspec(L) = Dspec(A).$$

Proof. Let $x = (1, \theta/k, x_2, ..., x_d)$ be the standard sequence w.r.t. θ, and fix a node v from X. Define a n-dimensional vector z by $z_u = x_{D(u,v)}$, for all $u \in X$, thus if $\mathbf{a}_i$ denotes the v-th column of the i-distance matrix A_i, we have

$$z = \sum_{i=0}^{d} x_i \mathbf{a}_i.$$

We now note that

$$\begin{aligned} Az &= \sum_{i=0}^{d} x_i A \mathbf{a}_i \\ &= \sum_{i=0}^{d} x_i (b_{i-1}\mathbf{a}_{i-1} + a_i \mathbf{a}_i + c_{i+1}\mathbf{a}_{i+1}) \\ &= \sum_{i=0}^{d} (c_i x_{i-1} + a_i x_i + b_i x_{i+1}) \mathbf{a}_i \\ &= \theta z, \end{aligned}$$

where the second equality follows from the identity

$$AA_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1},$$

and the third equality follows from noting that each entry in $\mathbf{a}_j$ appears exactly three times in the sum when $i \in \{j-1, j, j+1\}$. This shows that $\mathrm{Dspec}(L) \subseteq \mathrm{Dspec}(A)$. Now note that

$$L - \theta I = \begin{pmatrix} -\theta & b_0 & 0 & 0 & \dots & 0 \\ c_1 & a_1 - \theta & b_1 & 0 & \dots & 0 \\ 0 & c_2 & a_2 - \theta & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & c_{d-1} & a_{d-1} - \theta & b_{d-1} \\ 0 & 0 & \dots & 0 & c_d & a_d - \theta \end{pmatrix},$$

and since the $c_i's$ are all nonzero, it follows that $\mathrm{rk}(L - \theta I) \geq d$, thus $\mathrm{null}(L - \theta I) \leq 1$, implying that the eigenspace of each eigenvalue has dimension 1, thus $|\mathrm{Dspec}(L)| = d + 1$, which concludes the proof. ∎

We can also compute the multiplicities of each eigenvalue of A from the standard sequences of L.

Theorem 6 (Bigg's Formula). *Let X be a distance-regular graph with diameter d, and let $x = (1, \theta/k, ..., x_d)$ be the standard sequence w.r.t. some eigenvalue θ of L. Then the multiplicity $m(\theta)$ of θ as an eigenvalue of A is given by:*

$$m(\theta) = \frac{n}{\sum_{i=0}^{d} x_i^2 k_i}.$$

Proof. Let E be the projection onto the θ-eigenspace of A, and note that since X is distance-regular, $E \in \mathbb{C}[I, A, ..., A_d]$, hence we may write

$$\sum_{i=0}^{d} \alpha_i A_i.$$

We note that the diagonal entries of E are thus constant and equal to α_0, hence $m(\theta) = n\alpha_0$. We also note that

$$\begin{aligned}\theta E = AE &= \sum_{i=0}^{d} \alpha_i A A_i \\ &= \sum_{i=0}^{d} \alpha_i (b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1}) \\ &= \sum_{i=0}^{d} (c_i\alpha_{i-1} + a_i\alpha_i + b_i\alpha_{i+1})A_i.\end{aligned}$$

On the other hand, $\theta E = \sum_{i=0}^{d} \theta\alpha_i A_i$, hence

$$\theta\alpha_i = c_i\alpha_{i-1} + a_i\alpha_i + b_i\alpha_{i+1},$$

from which it follows by induction on i that $\alpha_i = \alpha_0 x_i$. From expression of E as a linear combination of the $A_i's$ we get

$$E^2 = \sum_{l=0}^{d}\sum_{i=0}^{d}\sum_{j=0}^{d} (\alpha_i\alpha_j p_{ij}^l)A_l,$$

and noting that $p_{ij}^0 = 0$ if $i \neq j$ and that $p_{ii}^0 = k_i$, we conclude that the diagonal entries of E^2 are constant and equal to $\sum_{i=0}^{d} \alpha_i^2 k_i$. As $E^2 = E$, and since the diagonal entries of E are α_0, it follows that $\alpha_0 = \sum_{i=0}^{d} \alpha_i^2 k_i$. Using that $\alpha_i = \alpha_0 x_i$, we get

$$\alpha_0 = \frac{1}{\sum_{i=0}^{d} x_i^2 k_i}.$$

We can thus conclude that

$$m(\theta) = \mathrm{tr}(E) = n\alpha_0 = \frac{n}{\sum_{i=0}^{d} x_i^2 k_i},$$

as desired. ∎

We can also define representations for the vertices of V associated with the eigenvalues θ of A. If $E = \sum_{i=1}^{l} y_l y_l^T$ is the orthogonal projection onto the θ-eigenspace written in terms of an orthonormal basis $\{y_i\}$, where $l = m(\theta)$, then we can map each vertex v to Ee_v. Thus if u, v are vertices of X, then

$$\langle Ee_u, Ee_v\rangle = E_{uv} = \alpha_{D(u,v)} = \alpha_0 x_{D(u,v)},$$

where x is the standard sequence w.r.t. θ. Moreover, the vectors Ee_v satisfy:

$$||Ee_v||^2 = \langle Ee_v, Ee_v\rangle = E_{vv} = \alpha_0,$$

hence $||Ee_v|| = \sqrt{\alpha_0}$.

We say that a graph X is *imprimitive* if X_i is disconnected for some i. Bipartite graphs with diameter at least 2 are always imprimitive, since A_2 is disconnected, and so are antipodal graphs – graphs such that X_d is a disjoint union of cliques. We then have the following result:

Theorem 7 (Smith's Theorem)**.** *If X is an imprimitive distance-regular graph with degree $k > 2$, then it is bipartite or antipodal (or both).*

Proof. Let X be as stated with diameter i. We say that a triple of vertices $u, v, w \in V$ is of type (l, i, j) if $D(u, v) = l, D(u, w) = i, D(v, w) = j$. Note then that if $j > 0$ and $p_{ii}^j > 0$, then any pair of vertices adjacent in X_j have at least one common neighbor in X_i, thus any path in X_j gives rise to a path in X_i, implying that if X_j is connected then so is X_i. If we choose i to be the minimal index such that X_i is disconnected, then it follows that if $j < i$, then $p_{ii}^j = 0$, i.e., there are no triples of type (j, i, i). Since X is connected by assumption, then $i > 1$.

We first consider the case where $i = d$. Note that $p^j_{dd} = 0$ for all $j < d$, that is, all triples of type (j, d, d) are forbidden. This means that if $D(u, w) = D(u, v) = d$, then $D(w, v) = d$, i.e., X_d is a disjoint union of cliques, thus X is antipodal.

We now further subdivide the possibilities into two cases:

(i) If $2 = i < d$, then we will show that X is bipartite. Indeed, we first consider a triangle wv_0v_1, and let $v_0v_1v_2v_3$ be a path of length 3 between vertices v_0, v_3 at distance 3. If we look at the triple w, v_0, v_2, then since $D(w, v_0) = 1, D(v_0, v_2) = 2$, it follows that $D(w, v_2) \in \{1, 2, 3\}$, however it cannot be 3 since wv_1v_2 is a path of length 2 from w to v_2, and it cannot be 2 since otherwise the triple would be of type $(1, 2, 2)$ which is forbidden, thus it must be 1. Now if we look at the triple w, v_1, v_3, similarly we have that $D(w, v_3) \in \{1, 2, 3\}$, however it cannot be 3 as wv_2v_3 is a path of length 2 from w to v_3, and it cannot be 1 since otherwise v_0wv_3 would be a path of length 2 between v_0 and v_3, hence $D(w, v_3) = 2$ and wv_1v_3 is a forbidden triple of type $(1, 2, 2)$. This shows that there are no triangles in X. Now let C be an odd cycle of length > 3, and note then that all of its vertices lie in the same connected component Δ w.r.t. X_2. If u, v are adjacent vertices in C and w is adjacent to u, then it cannot be adjacent to v as there are no triangles, meaning that w, v are adjacent in X_2 and thus w belongs to Δ. We can repeat this for any path connecting a vertex in X to C to conclude that all vertices in the path must be in Δ, and since X is connected, this implies that X_2 is connected, which is a contradiction. Hence there are no odd cycles and X is bipartite;

(ii) If $2 < i < d$, we consider a path of length d between vertices v_0, v_d at distance d, and note that since $k \geq 3$, we can find a vertex w adjacent to v_i that is distinct from v_{i-1}, v_i. By looking at the triple w, v_0, v_i, we get that $D(w, v_0) \in \{i-1, i, i+1\})$. If $D(w, v_0) = i$, then the triple w, v_i, v_0 is of forbidden type $(1, i, i)$. If $D(w, v_0) = i + 1$, then the triples w, v_i, v_1 and w, v_1, v_0 show that $D(w, v_1) = i$, then the triple w, v_{i+1}, v_1 is of forbidden type (j, i, i) with with $j \leq 2$. This implies that any neighbor from v_i other than v_{i+1} must be at distance $i - 1$ from v_0, hence $c_i = k - 1$ and $b_i = 1$. Now as the graph is distance-regular, it follows that v_{i+1} must also have c_i neighbors at distance $i - 1$ from v_1, and as $k - 1 \geq 2$, we can find a vertex z adjacent to v_{i+1} distinct from v_i which is at distance $i - 1$ from v_1. The triples z, v_0, v_1 and z, v_0, v_{i+1} show that $D(z, v_0) = i$, hence $z \neq w$, and the triple z, v_i, v_0 is of forbidden type (j, i, i), with $j \leq 2$.

The previous cases show that i is either d or 2, which in turn imply that X is either bipartite or antipodal, as desired. ■

We note that a there are imprimitive distance-regular graphs with degree $k = 2$ that are not bipartite nor antipodal, e.g., C_9. Also, the complete bipartite graphs $K_{d,d}$ are examples of imprimitive distance-regular graphs that are both bipartite and antipodal.

As a final remark, the previous theorem allows us to construct primitive graphs from imprimitive graphs. If X is an imprimitive bipartite graph with degree at least 3 and partitions V_1, V_2, then V_i is a connected component of X_2. The graphs induced by the components V_i in X_2 are called the *halved graphs*, and are denoted by X^+, X^-. If X is antipodal, then we can obtain a graph X' with vertex set given by the equivalence classes of $X_0 \cup X_d$ such that two classes are adjacent if they contain adjacent vertices in X. This is called the *folded graph* of X. It can be shown that both halved and folded graphs of a DRG are also DRGs, and that after at most two steps of halving and/or folding, we obtain a primitive DRG.

2 Association Schemes

Definition 8. We say that a set $S = \{A_0, ..., A_d\}$ of nonzero $n \times n$ matrices with entries in $\{0, 1\}$ is an *association scheme* if the following hold:

(i) $A_0 = I$;

(ii) each A_i is a symmetric matrix;

(iii) $\sum_i A_i = J$;

(iv) there are constants p_{ij}^l such that

$$A_i A_j = \sum_{l=0}^{d} p_{ij}^l A_l.$$

The span of the matrices in S is denoted by $\mathcal{A}$ and is called the *Bose-Mesner algebra*, and it forms a commutative $*$-subalgebra of $M_n(\mathbb{C})$ that is also closed under the Schur product. From the previous lecture, we see that DRGs are precisely the graphs whose distance partition forms an association scheme. Similarly to the case of DRGs, we can show that for general association schemes, we can also find a basis $E_0, ..., E_d$ of orthogonal projection matrices.

Theorem 9. *Let $\{I, ..., A_d\}$ be an association scheme with Bose-mesner algebra $\mathcal{A}$, then there exists an orthogonal matrix U such that $U^T \mathcal{A} U$ is a set of diagonal matrices. In other words, we can find a common basis of orthogonal eigenvectors for all matrices in $\mathcal{A}$.*

Proof. We prove the result via induction on the dimension of $\mathcal{A}$. If $d = 1$, then we have a basis $\{I, A_1\}$, and since A_1 is symmetric, it follows that any basis of eigenvectors for A_1 diagonalizes $\mathcal{A}$. The case where $d = 2$ follows similarly, by noting that two symmetric matrices commute iff they share an orthogonal basis of eigenvectors. For the general case, we note that a set of commuting matrices in $M_n(\mathbb{C})$ always share a common eigenvector v_1, hence we may decompose

$$\mathbb{C}^n = \mathbb{C}v_1 \oplus W,$$

where $W = (\mathbb{C}v_1)^{\perp}$. As W is invariant w.r.t. all A_i, we may write

$$A_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & A_i|_W \end{pmatrix}.$$

The blocks $A_i|_W$ will also commute, and thus we may apply induction to conclude the desired result. ■

The previous result shows that we can find orthogonal projectors $E_0, ..., E_d$ such that

$$\mathbb{C}^n = E_0\mathbb{C}^n \oplus \dots E_d\mathbb{C}^n,$$

where each subspace $E_i\mathbb{C}^n$ is a subeigenspace of each matrix in $\mathcal{A}$, with dimension $m_i = \operatorname{tr}(E_i)$. Moreover, if $A \in \mathcal{A}$, we may also write

$$A = \sum_{i=0}^{d} \lambda_i E_i,$$

where λ_i are the eigenvalues of A – not necessarily distinct, although this shows that each matrix in $\mathcal{A}$ has at most $d+1$ distinct eigenvalues –, hence $\mathcal{A}E_i = \mathbb{C}E_i$, and we thus obtain the following decomposition for $\mathcal{A}$:

$$\mathcal{A} = \mathcal{A}E_0 \oplus \dots \oplus \mathcal{A}E_d = \mathbb{C}E_0 \oplus \dots \oplus \mathbb{C}E_d.$$

In other words, there are two canonical basis for a given association scheme: (i) its *Schur idempotents* given by the symmetric matrices $I, A_1, ..., A_d$, which are orthogonal idempotents w.r.t. the Schur product; and (ii) its *matrix idempotents* given by the symmetric matrices $E_0, ..., E_d$, which are orthogonal idempotents w.r.t. the usual matrix product. Since each A_i can be seen as the adjacency matrix of a k_i-regular graph, it follows that $\mathbb{1}$ is the unique k_i-eigenvector for each A_i, hence we may assume WLOG that $E_0 = (1/n)J$. From these observations, we can define the *eigenmatrix* P and *dual eigenmatrix* Q of the scheme as follows:

$$A_i = \sum_{j=0}^{d} P_{ji} E_j,$$

$$E_i = \frac{1}{n} \sum_{j=0}^{d} Q_{ji} A_j.$$

From this, we can see that

$$A_iE_j = P_{ji}E_j \quad \text{and} \quad E_i \circ A_j = \frac{Q_{ji}}{n}A_j.$$

The first row of P contains the degrees k_i of the scheme, the first row of Q contains the multiplicities m_j of the scheme, and note that the multiplicity of P_{ji} is m_j for any i. Moreover, we have

$$E_iE_j = (\sum_{l=0}^{d} \frac{P_{il}Q_{lj}}{n})E_i = (PQ)_{ij}E_i,$$

$$A_i \circ A_j = (\sum_{l=0}^{d} \frac{Q_{il}P_{lj}}{n})A_i = (QP)_{ij}A_i$$

hence $PQ = nI = QP$. Noting that $\mathrm{tr}(AB) = \mathrm{sum}(A \circ B)$, we can also obtain the relations $m_jP_{ji} = k_iQ_{ij}$ for any $i, j \in \{0, ..., d\}$ from the trace of A_iE_j.

As $\mathcal{A}$ is Schur-closed, we can also express $E_i \circ E_j$ as a linear combination of the E_i's, that is,

$$E_i \circ E_j = (1/n)\sum_{l=0}^{d} q_{ij}^l E_l,$$

where the real numbers q_{ij}^l are called the *Krein parameters* of the scheme. Noting that both E_i and E_j are PSD matrices, it follows that $E_i \circ E_j$ is also PSD, hence the Krein parameters are all nonnegative. From this, we can obtain the following bound for the multiplicities of the scheme:

Proposition 10. *If $m_0, ...m_d$ are the multiplicities of the association scheme $\{I, A_1, ..., A_d\}$, then*

$$\sum_{q_{ij}^l \neq 0} m_l \leq \begin{cases} m_im_j, & \text{if } i \neq j \\ \frac{m_i(m_i+1)}{2}, & \text{if } i = j \end{cases}$$

Proof. Let A, B be $n \times m$ matrices with rank m_A, m_B, respectively, and let

$$A = \sum_{i=1}^{m_A} v_iu_i^* \quad \text{and} \quad B = \sum_{i=1}^{m_B} x_iy_i^*$$

be their SVD decomposition. Hence

$$A \circ B = \sum_{i,j}(v_i \circ x_j)(u_i \circ y_j)^*,$$

thus $A \circ B$ is a sum of at most m_Am_B linearly independent rank 1 matrices, implying that $\mathrm{rk}(A \circ B) \leq \mathrm{rk}(A)\mathrm{rk}(B)$. If $A = B$, then we note that there are at most $\binom{m_A+1}{2}$ linearly independent rank 1 matrices, hence $\mathrm{rk}(A \circ A) \leq \binom{\mathrm{rk}(A)+1}{2}$. Combining this with the fact that

$$\mathrm{rk}(E_i \circ E_j) = \sum_{q_{ij}^l \neq 0} m_l,$$

we conclude the proof. ∎

An association scheme is called *P-polynomial* if there is an ordering of the Schur basis such that each A_i is a polynomial in A_1 of degree i. Similarly, we call it *Q-polynomial* if there's an ordering of the matrix idempotents such that each E_i is a polynomial (w.r.t. the Schur product) in E_1 with degree i.

3 Coherent Configurations

Definition 11. We say that a set $C = \{A_0, ..., A_d\}$ of nonzero $n \times n$ matrices with entries in $\{0, 1\}$ is a *coherent configuration* if the following hold:

(i) $\sum_i A_i = J$;

(ii) for each i, $A_i^T \in C$;

(iii) for each i, if A_i has a nonzero diagonal entry, then it is a diagonal matrix;

(iv) there are constants p_{ij}^l such that
$$A_i A_j = \sum_{l=0}^{d} p_{ij}^l A_l.$$

Coherent configurations generalize the notion of association schemes, and as we shall see, they are especially useful in the context of finite groups. From condition (iii), it follows that there are diagonal matrices $I_1, ..., I_m$ in the configuration that partition the identity matrix I, and these are called the *fibers* of the configuration. If C contains the identity, that is, has only one fiber, we say that the configuration is *homogeneous*. If each element of C is symmetric, we say that the configuration is *symmetric*.

It also follows from the definition that A_i is either symmetric or anti-symmetric – that is, $A_i \circ A_i^T$ is either A_i or 0. If $i \in \{0, ..., d\}$, we denote by i' the unique matrix in C such that $A_{i'}^T = A_i$, and similarly to what we did with distance-regular graphs, we let X_i denote the set of pairs related by A_i, and $X_i(\alpha)$ to be the set of elements in X such that $(\alpha, \beta) \in X_i$. From this it follows that if $(\alpha, \beta) \in X_l$, then
$$(A_i A_j)_{\alpha\beta} = |X_i(\alpha) \cap X_{j'}(\beta)| = p_{ij}^l.$$

We now turn our attention to the algebras associated with coherent configurations. We say that a subset $\mathcal{A} \subseteq M_n(\mathbb{C})$ is a *coherent algebra* if:

(i) $\mathcal{A}$ closed w.r.t. the conjugate-transpose map;

(ii) $\mathcal{A}$ is an algebra w.r.t. the usual matrix product with unit given by I;

(iii) $\mathcal{A}$ is an algebra w.r.t. the Schur product with unit given by J;

It is clear then that the linear span of the matrices in a coherent configuration gives rise to a coherent algebra, however we can actually prove a one-to-one correspondence.

Proposition 12. *If $\mathcal{A}$ is a coherent algebra of dimension d, then there is a unique coherent configuration $C = \{A_0, ..., A_d\}$ such that $\mathcal{A} = span_{\mathbb{C}}(A_0, ..., A_d)$.*

Proof. If $A \in \mathcal{A}$, we can write it as a linear combination of orthogonal 01 matrices w.r.t. its Schur product, and by using Lagrange's polynomials w.r.t. the Schur product we can see that each of these components belongs to $\mathcal{A}$. Hence the 01 components of each matrix in $\mathcal{A}$ belongs to it, and thus we can consider the minimal components – that is, that cannot be further decomposed as the sum of two distinct components in $\mathcal{A}$ – and note that these must form a unique basis for $\mathcal{A}$. ■

We note that if I_i, I_j are fibers of C, then $I_i J I_j \in \mathcal{A}$, hence
$$I_i J I_j = \sum_l \alpha_l A_l,$$
but since $I_i J I_j$ is a 01 matrix, it follows that $\alpha_l \in \{0, 1\}$, hence if $A_l \circ I_i J I_j \neq 0$, it follows that $A_l \circ I_i J I_j = A_l$. This shows us that each relation X_i is always contained in the Cartesian product of two fibers of C.

We now turn our attention to permutation groups.

Example 13. Let G be a permutation group acting on a set $X = \{1, ..., n\}$. This action induces an action on $X \times X$ as follows:

$$\sigma(\alpha, \beta) = (\sigma(\alpha), \sigma(\beta)),$$

for any $\sigma \in G, \alpha, \beta \in X$. The orbits $\text{Orb}(G, X \times X) = \{X_0, ..., X_d\}$ on $X \times X$ are called the *orbitals* of G, and they partition $X \times X$. If A_i is the adjacency matrix of the relation X_i, we claim that $C = \{A_0, ..., A_d\}$ is a coherent configuration. Indeed, C clearly satisfies (i), and we note that if $(\alpha, \alpha), (\gamma, \theta) \in X_i$, then there exists some $\sigma \in G$ such that $\sigma(\alpha) = \sigma(\gamma) = \sigma(\theta)$, hence $\gamma = \theta$, and thus C satisfies (iii). We note that if $(\alpha, \beta) \in X_i$, we can then write $X_i = G(\alpha, \beta)$, hence

$$(\gamma, \theta) \in G(\alpha, \beta) \iff (\theta, \gamma) \in G(\beta, \alpha),$$

implying that the orbit $X_{i'} = G(\beta, \alpha)$ is the transpose of X_i – that is, $A_{i'} = A_i^T$ –, and so C also satisfies (ii). We now note that if $\gamma \in X_i(\alpha)$ then $\sigma(\gamma) \in X_i(\sigma(\alpha))$, and conversely if $\gamma \in X_i(\sigma(\alpha))$, then $\gamma \in \sigma(X_i(\alpha))$, hence $X_i(\sigma(\alpha)) = \sigma(X_i(\alpha))$. Thus, if we fix some orbital $X_l = G(\alpha, \beta)$, then

$$\begin{aligned}
(A_i A_j)_{\alpha\beta} &= |X_i(\alpha) \cap X_{j'}(\beta)| \\
&= |\sigma(X_i(\alpha) \cap X_{j'}(\beta))| \\
&= |\sigma(X_i(\alpha)) \cap \sigma(X_{j'}(\beta))| \\
&= |X_i(\sigma(\alpha)) \cap X_{j'}(\sigma(\beta))|,
\end{aligned}$$

hence $(A_i A_j)_{\alpha\beta}$ is constant for all $(\alpha, \beta) \in X_l$, which shows that C is indeed a coherent configuration, and we denote this coherent configuration by $\text{Inv}(G)$. It is clear that the fibers of this configuration are given by the orbits $\text{Orb}(G, X)$ of the action of G on X, hence $\text{Inv}(G)$ is homogeneous iff G is transitive on X. We also claim that

$$\text{Orb}(G_\alpha, X) = \{X_i(\alpha) | i \in \{0, ..., d\}\} \setminus \{\emptyset\}.$$

We first note that each set $X_i(\alpha)$ is G_α-invariant, hence it is a disjoint union of G_α-orbits. If $\gamma, \gamma' \in X_i(\alpha)$, then both (α, γ) and (α, γ') belong to X_i, and since this is an orbital, it follows that there exists some $\sigma \in G$ such that

$$\sigma(\alpha, \gamma) = (\alpha, \gamma'),$$

thus $\sigma \in G_\alpha$, implying that $X_i(\alpha)$ is contained in some G_α-orbit, thus $X_i(\alpha)$ must itself be an G_α-orbit. Now if Δ is a G_α-orbit, then any pairs $\gamma, \gamma' \in \Delta$ are such that $(\alpha, \gamma), (\alpha, \gamma')$ belong to the same orbital X_i, hence $\Delta = X_i(\alpha)$, which proves the claim.

If $S \subseteq M_n(\mathbb{C})$ is a set of matrices, we define its *centralizer* (or *commutant*) as

$$C(S) = \{B \in M_n(\mathbb{C}) | BA = AB, \forall A \in S\}.$$

If G is a group of permutation matrices, then $C(G)$ is a coherent algebra, and in fact, we can show that $C(G)$ is the coherent algebra associated with the configuration $\text{Inv}(G)$.

Theorem 14. *If G is a group of permutation matrices on $GL(n, \mathbb{C})$, then the coherent algebra $\mathcal{A}$ generated by $Inv(G)$ is precisely $C(G)$.*

Proof. We first note that if $A_0, ..., A_d$ are the adjacency matrices of the orbitals of G, and if $(\alpha, \beta) \in X_i$, then

$$(P^T A_i P)_{\alpha\beta} = (A_i)_{P(\alpha,\beta)} = (A_i)_{\alpha\beta},$$

hence $A_i \in C(G)$, and thus $\mathcal{A} \subseteq C(G)$. Conversely, if $P^T A_i P = A$ for any $P \in G$, then the 01 components of A are precisely the adjacency matrices of the orbitals of G on X, hence $A \in \mathcal{A}$. ∎

If X is a DRG with automorphism group G, then this means that

$$\mathbb{C}[A] = \mathbb{C}[I, A, ..., A_d] \subseteq C(G).$$

The group G clearly acts on the sets of ordered tuples X_i of X, and it this action is transitive on each X_i, that is, if for any pair $(\alpha, \beta), (\gamma, \theta) \in X_i$, there exists some $\sigma \in G$ such that $(\sigma\alpha, \sigma\beta) = (\gamma, \theta)$, then we say that X is *distance-transitive*. Every distance-transitive graph is distance regular, however we cannot in general determine if a DRG is distance-transitive from its intersection array. We can however note that if X is distance-transitive with diameter d, then the coherent algebra generated by $\mathrm{Inv}(G)$ has precisely $d+1$ orbitals, hence it has dimension $d+1$, implying that $C(G) = \mathbb{C}[A]$.

References

[Bailey, 2004] Bailey, R. (2004). *Association Schemes: Designed Experiments, Algebra, and Combinatorics*. Cambridge studies in advanced mathematics. Cambridge University Press.

[Brouwer et al., 2011] Brouwer, A., Cohen, A., and Neumaier, A. (2011). *Distance-Regular Graphs*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg.

[Chen and Ponomarenko, 2023] Chen, G. and Ponomarenko, I. (2023). Lectures on coherent configurations.

[Faradzev et al., 2013] Faradzev, I., Ivanov, A., Klin, M., and Woldar, A. (2013). *Investigations in Algebraic Theory of Combinatorial Objects*. Mathematics and its Applications. Springer Netherlands.

[Van Dam et al., 2016] Van Dam, E. R., Koolen, J. H., and Tanaka, H. (2016). Distance-regular graphs. *The Electronic Journal of Combinatorics*, 1000.