Take-home Assignment for Advanced Computational Methods (GRA6561)

General Instructions

1. The exam counts 30% into the final grade.
2. In each section, the points are evenly distributed between the questions unless otherwise stated; in each question, the points are evenly distributed between the sub-questions unless otherwise stated.
3. Unless otherwise specified, all rates are annually and continuously compounded.
4. In each exercise you should explain which numerical method you use and why you think the method is appropriate for the problem. Write down the key formulas of the algorithm. If the method involves formulas that need to be adapted to the problem, write down the derivation.
5. Your final submission should include a pdf file and a zip file. The pdf file consists of the explanation of your code, the figures and tables, and the results. The zip file consists of the programming files and should be uploaded as appendix material.
6. Show your final numerical results with at least 4 decimal points unless otherwise specified.
7. You are not allowed to use your software's built-in toolboxes or downloaded online toolboxes unless explicitly stated.
8. If you believe that not enough information is provided to answer a question or the information is wrong then you should ask immediately by email. Answers or follow-up clarifications will be publicly announced to all students through itslearning.

1 Basics in Numerical Methods (35 points)

1. (5 points) Assume that a simple derivative is of the form $\Phi(S_T) = \frac{1}{S_T}$, where the risk-neutral dynamics of S is a geometric Brownian motion with drift rS_t and volatility σS_t. Calculate its price $F(t, s)$ using Gauss-Hermite quadrature for 100 uniform nodes of $s \in [1, 3]$. Use the following parameters:

$$r = .02, \sigma = .2, T - t = .5.$$

   Plot F against s.

2. (5 points) Compute the discounted utility

$$U = \int_0^\infty e^{-\rho t} \log(1 - e^{-\lambda t}) dt,$$

   for 100 uniform nodes of $\lambda \in [0.02, 0.04]$, where $\rho = 0.05$. Plot U against λ.

3. (5 points) Solve the following differential equation with projection method

$$y'(x) = x + y, \quad y(0) = 0.$$

   Use the 5th order monomial basis, (1, ..., x^5) as the basis for approximation. Plot the solution for $x \in [0, 4]$.

4. (20 points) Consider the life-cycle consumption problem. A consumer lives for T years and dies. She maximizes her life-time utility

$$\int_0^T e^{-\rho t} \log c_t dt.$$

   She has an initial asset $a_0 = 1$, and receives interest payment ra_t at each period. The evolution of her net worth is

$$\dot{a}_t = ra_t - c_t.$$

   We let $a_T = 0$ so she does not leave any wealth behind. This optimization problem gives us a two point boundary value problem (TPBVP)

$$\begin{cases} \dot{c}_t = c_t(r - \rho) \\ \dot{a}_t = ra_t - c_t \end{cases}, \tag{1}$$

   with $a_0 = 1$, $a_T = 0$. Take $\rho = 0.05$, $T = 100$, and $r = .06$ (annual). The problem is to solve for the optimal consumption and asset path.

   (a) Solve the TPBVP by forward shooting. Plot the time paths of c and a.

   (b) Solve the TPBVP by projection method. Use monomial basis to at least the 5th order (1, ..., t^5) as the basis for approximation. Plot the time paths of c and a.

2 Derivatives Pricing (40 points)

1. For this question you are allowed to use uniform and normal random number generator, low-discrepancy sequence generator, and statistics toolbox for the purpose of calculating variance, covariance, or confidence interval.

   Consider an arithmetic average Asian option that expires in 5 months. There are 5 monitoring dates at the end of each month $j = 1, ..., 5$. The payoff function is

   $$\max\left\{\frac{1}{5}\sum_{j=1}^{5} S_j - K, 0\right\}.$$

   The risk-neutral dynamics of the stock price is

   $$dS = rSdt + \sigma S dz.$$

   The current price of the underlying is 50. The annual risk free rate is 0.02. The strike price is 50. $\sigma = 0.2$.

   (a) Find the price of the Asian option using the crude Monte Carlo method. Report the variance and confidence interval.

   (b) Redo exercise (a) using antithetic sampling as the variance reduction technique. Comment on your findings.

   (c) Redo exercise (a) using control variates as the variance reduction technique. Comment on your findings.

   (d) Redo exercise (a) using Quasi-Monte Carlo method. Use Halton sequence. Comment on your findings.

2. Consider a call on call compound option (C^a on C^c). The first call, C^a, is an arithmetic Asian option expiring in 5 months with monitoring dates at the end of each month and a strike price of $K^a = 3$. Therefore, its payoff is

   $$\max\left\{\frac{1}{5}\sum_{j=1}^{5} C_j^c - K^a, 0\right\}.$$

   The second call, C^c, is an American call option on a dividend-paying underlying stock with risk-neutral dynamics

   $$dS_t = (r - \rho)S_t dt + \sigma S_t dz_t.$$

   C^c expires in 5 months with the strike price $K^c = 50$. The current stock price is 50. The parameters are $r = 0.03$ (annual), $\rho = 0.01$ (annual), and $\sigma = 0.4$.

   (a) (15 points) Find the price of this option C^a.

   (b) (5 points) Find the Delta and Gamma of this option.

3 Dynamic Programming (25 points)

Consider the Ramsey-Cass-Koopmans optimal growth problem. The economy has a representative household which lives forever and a technology to produce output from capital inputs. The household consumes an amount c_t each period. The household's lifetime utility is

$$\sum_{t=0}^{\infty} \beta^t \log c_t.$$

The production technology takes the form $y_t = k_t^{\alpha}$. The resource constraint is

$$y_t = c_t + i_t,$$

where i_t is the investment. The capital stock accumulates according to:

$$k_{t+1} = k_t(1 - \delta) + i_t.$$

The parameters are $\alpha = 1/3$, $\beta = 0.95$, $\delta = 0.1$.

1. (5 points) What are the state variable and control variable in this model? Set up the Bellman equation and derive the Euler equation.

2. (20 points) Solve this problem by value function iteration. Plot the value function. Plot the policy function (the optimal k_{t+1} or c_t as a function of k_t). The discretized state space has to contain at least 20 nodes.