# Regression Method

In [1]:
```
# Necessary packages
import pandas as pd
import numpy as np
from numpy.linalg import inv
import matplotlib.pyplot as plt

# Importing the CSV file
df_reg = pd.read_csv("treasury_data_all.csv")
df_reg.head()
```

Out[1]:

| | CUSIP | ISSUE_DATE | MATURITY_DATE | COUPON_RATE | BID_PRICE | ASK_PRICE | TIME_TO_MATURITY |
|---|---|---|---|---|---|---|---|
| **0** | 912810EB | 22/11/1988 | 15/11/2018 | 9.000 | 103.093750 | 103.218750 | 0.5 |
| **1** | 912828M6 | 16/11/2015 | 15/11/2018 | 1.250 | 99.609375 | 99.648438 | 0.5 |
| **2** | 912828JR | 17/11/2008 | 15/11/2018 | 3.750 | 100.734375 | 100.757812 | 0.5 |
| **3** | 912796QJ | 17/05/2018 | 15/11/2018 | 0.000 | 99.069000 | 99.073667 | 0.5 |
| **4** | 912828R4 | 16/05/2016 | 15/05/2019 | 0.875 | 98.679688 | 98.718750 | 1.0 |

This is an overview over the different variables which is located in this dataframe

- CUSIP: A unique security-level identifier
- ISSUE_DATE: The date the Treasury bond was issued
- MATURITY_DATEThe date the Treasury bond matures
- COUPON_RATEThe bond’s coupon rate in percent
- BID_PRICEThe price at which dealers are willing to buy
- ASK_PRICEThe price at which dealers are willing to sell
- TIME_TO_MATURITYThe time-to-maturity measured in years

In [2]:
```
df_reg["Mid_Price"] = (df_reg.ASK_PRICE + df_reg.BID_PRICE)/2
df_reg.head()
```

Out[2]:

| | CUSIP | ISSUE_DATE | MATURITY_DATE | COUPON_RATE | BID_PRICE | ASK_PRICE | TIME_TO_MATURITY | Mid_Price |
|---|---|---|---|---|---|---|---|---|
| **0** | 912810EB | 22/11/1988 | 15/11/2018 | 9.000 | 103.093750 | 103.218750 | 0.5 | 103.156250 |
| **1** | 912828M6 | 16/11/2015 | 15/11/2018 | 1.250 | 99.609375 | 99.648438 | 0.5 | 99.628906 |
| **2** | 912828JR | 17/11/2008 | 15/11/2018 | 3.750 | 100.734375 | 100.757812 | 0.5 | 100.746094 |
| **3** | 912796QJ | 17/05/2018 | 15/11/2018 | 0.000 | 99.069000 | 99.073667 | 0.5 | 99.071333 |
| **4** | 912828R4 | 16/05/2016 | 15/05/2019 | 0.875 | 98.679688 | 98.718750 | 1.0 | 98.699219 |

In [3]:
```
## Creating the Cash Flow Matrix
unique_maturities = sorted(df_reg["TIME_TO_MATURITY"].unique())

# Initialize the cash flow matrix with zeroes
cash_flow_matrix = np.zeros((len(df_reg), len(unique_maturities)))

# Iterative process to create a vector row of each bonds casf flows at different maturities
for i, bond in df_reg.iterrows():
    for j, maturity in enumerate(unique_maturities):
        # Coupon payment for maturities before, and at the bonds maturity
        cash_flow = 100*bond["COUPON_RATE"] / 100 / 2  # Assuming semiannual
        if maturity == bond["TIME_TO_MATURITY"]:
            cash_flow += 100    # Adding principal
        cash_flow_matrix[i, j] = cash_flow

# We convert the cash flow matrix to Dataframe
C = pd.DataFrame(cash_flow_matrix, columns = unique_maturities)
C.head()
```

Out[3]:

| | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 | 5.5 | 6.0 | 6.5 | 7.0 | 7.5 | 8.0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | 104.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4. |
| **1** | 100.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0.6250 | 0. |
| **2** | 101.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1.8750 | 1. |
| **3** | 100.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0. |
| **4** | 0.4375 | 100.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0. |

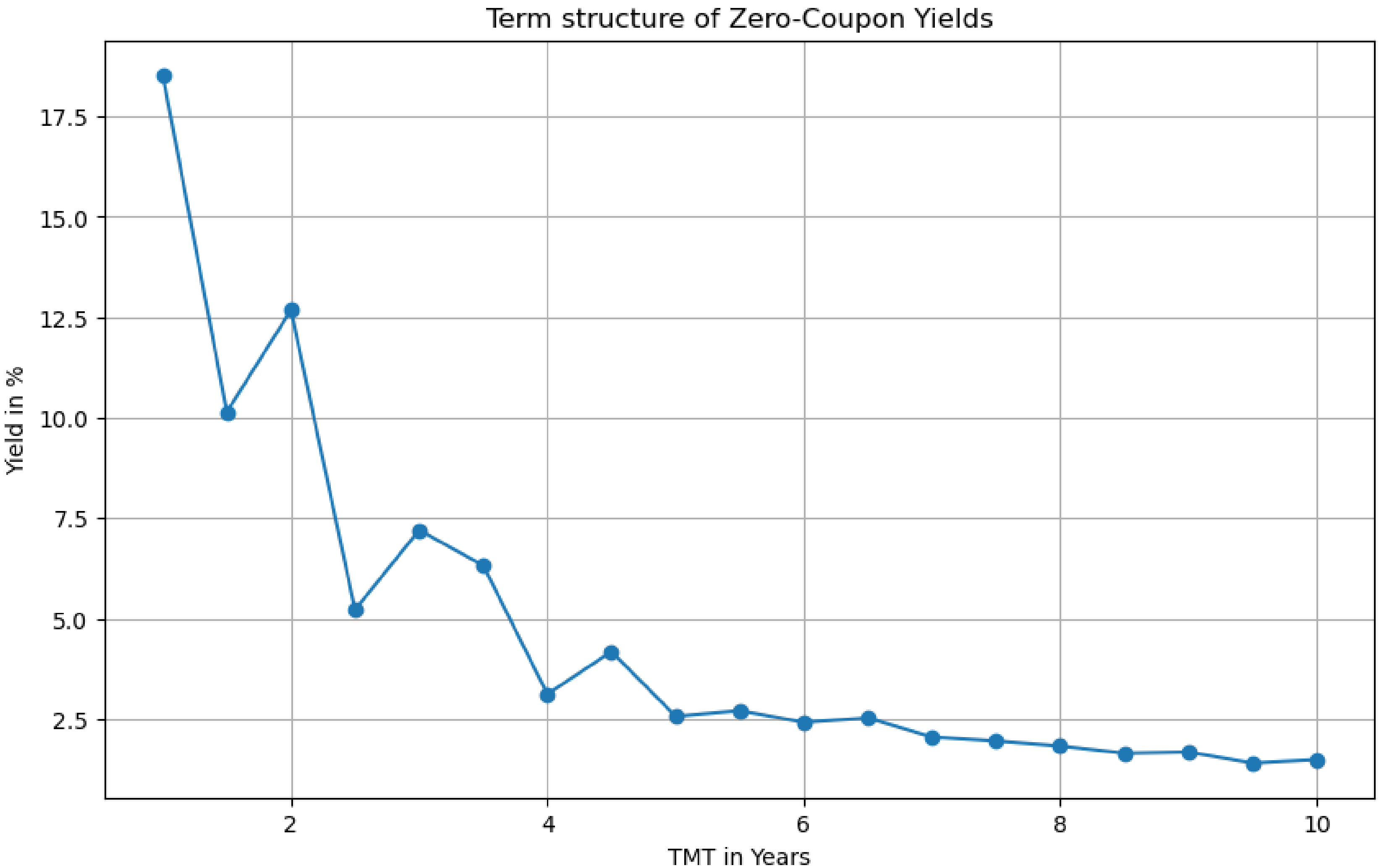
In [4]:
```
## Step 2. Prepare the Price Vector
price_vector = df_reg["Mid_Price"].values
```

In [5]:
```
# Transpose and multiply the cash flow matrix
CT_C = np.dot(C.T, C)

# Invert the resulting matrix
inv_CT_C = inv(CT_C)

# Multiply by tranposed cash flow matrix and price vector
Z_0 = np.dot(np.dot(inv_CT_C, C.T), price_vector)

# Convert the discount factors to DataFrame
df_reg_discount = pd.DataFrame(Z_0)
df_reg_discount["Discount_Factors"] = df_reg_discount[0]
df_reg_discount = df_reg_discount.drop(df_reg_discount.columns[0], axis = 1)
df_reg_discount["TMT"] = unique_maturities
df_reg_discount
```

Out[5]:

| | Discount_Factors | TMT |
|---|---|---|
| 0 | 0.733487 | 0.5 |
| 1 | 0.830802 | 1.0 |
| 2 | 0.816418 | 1.5 |
| 3 | 0.683638 | 2.0 |
| 4 | 0.811555 | 2.5 |
| 5 | 0.697612 | 3.0 |
| 6 | 0.684373 | 3.5 |
| 7 | 0.803780 | 4.0 |
| 8 | 0.715933 | 4.5 |
| 9 | 0.793584 | 5.0 |
| 10 | 0.762665 | 5.5 |
| 11 | 0.766401 | 6.0 |
| 12 | 0.738448 | 6.5 |
| 13 | 0.765793 | 7.0 |
| 14 | 0.760918 | 7.5 |
| 15 | 0.760296 | 8.0 |
| 16 | 0.768178 | 8.5 |
| 17 | 0.751902 | 9.0 |
| 18 | 0.776729 | 9.5 |
| 19 | 0.753493 | 10.0 |

In [6]:
```
## Task 2 is to calculate the continously compounded zero-coupon bond yields for each maturity

# We will be using the same discount factors as calculated above
df_reg_discount["Continously_Compounded_Yield"] = -np.log(df_reg_discount["Discount_Factors"]) / df_reg_disco

# Convert the yield to percent
df_reg_discount["Continously_Compounded_Yield_in%"] = df_reg_discount.Continously_Compounded_Yield * 100


# Plot the term structure
plt.figure(figsize = (10, 6))
plt.plot(df_reg_discount.TMT, df_reg_discount["Continously_Compounded_Yield_in%"], marker = "o")
plt.title("Term structure of Zero-Coupon Yields")
plt.xlabel("TMT in Years")
plt.ylabel("Yield in %")
plt.grid(True)
plt.show()
```




## Nelson Siegel

In [7]:
```
# Necessary package to optimize
from scipy.optimize import minimize
```

In [8]:
```
# We define the df_NS as same as the df_reg
df_NS = df_reg

# Define the NS model functions
def nelson_siegel_yield(theta_0, theta_1, theta_2, lambda_, T):
    # Calculates the yield for a given maturity T based on NS
    return theta_0 + (theta_1 + theta_2)*(1-np.exp(-T / lambda_)) / (T / lambda_) - theta_2 * np.exp(-T / lam

def discount_factor(theta_0, theta_1, theta_2, lambda_, T):
    # Calculates the discount factor for a given maturity
    r = nelson_siegel_yield(theta_0, theta_1, theta_2, lambda_, T)
    return np.exp(-r * T)
```

In [9]:
```
efining the objective function for Optimization
 objective_function(params, df_NS):
 theta_0, theta_1, theta_2, lambda_ = params
 squared_errors = []

 for index, row in df_NS.iterrows():
     T = row["TIME_TO_MATURITY"]
     coupon = row["COUPON_RATE"]
     mid_price = row["Mid_Price"]
     face_value = 100     # We assume that the standard face value is 100

     # Calculate the price of the bond under the NS model
     # We will assume semi-annual coupons
     cash_flows = [coupon * face_value / 2 for _ in range(int(2*T))]
     cash_flows[-1] += face_value     # Add the Face value as the last payment
     model_price = sum(discount_factor(theta_0, theta_1, theta_2, lambda_, t/2) * cf for t, cf in enumerate(ca

     # Calculate the squared error between the model price and the observed mid price
     squared_errors.append((model_price - mid_price) ** 2)

 return np.sum(squared_errors)
```

In [10]:
```
# We will initialize the optimization model by a guess for our thetas, lambda and T
initial_guess = [0.03, 0, 0, 3]

# Start the optimization
result = minimize(objective_function, initial_guess, args = (df_NS, ), method = 'L-BFGS-B')

# Print out the result
print("Optimized Parameters: θ0 = {:.4f}, θ1 = {:.4f}, θ2 = {:.4f}, λ = {:.4f}".format(*result.x))
```

```
Optimized Parameters: θ0 = 290.3481, θ1 = -287.3081, θ2 = -335.8996, λ = 41.6076
```

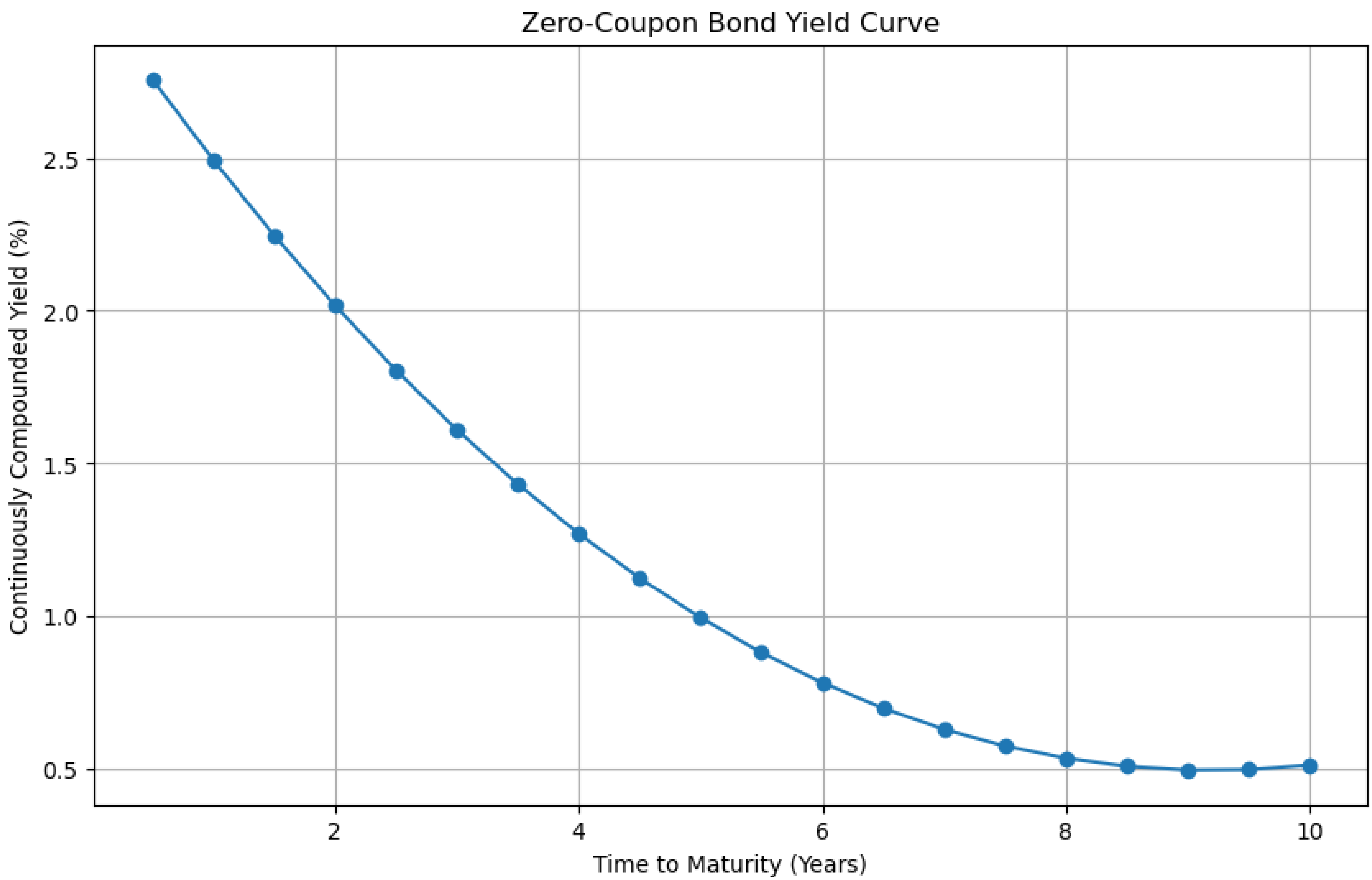
In [11]:
```
### Next task is to se the estimated model parameters to calculate the zero-coupon bond yield for each maturi
## We extract our optimized parameters so it could be later used
optimized_params = result.x
theta_0, theta_1, theta_2, lambda_ = optimized_params
```

In [12]:
```
# Extract the unique maturity values
unique_maturities = df_NS["TIME_TO_MATURITY"].unique()

# We then calculate yeuekds for each maturity
zero_cp_yields = {T: nelson_siegel_yield(theta_0, theta_1, theta_2, lambda_, T) for T in unique_maturities}

# We convert the dictionary to a DF
df_NS_model = pd.DataFrame(list(zero_cp_yields.items()), columns = ["Maturity", "Continously_Compounded_Yield
df_NS_model.head()
```

Out[12]:

| | Maturity | Continously_Compounded_Yield_in% |
|---|---|---|
| **0** | 0.5 | 2.757297 |
| **1** | 1.0 | 2.492734 |
| **2** | 1.5 | 2.246063 |
| **3** | 2.0 | 2.016982 |
| **4** | 2.5 | 1.805194 |

In [13]:
```
# We can then plot the term structure of interest rates obtained in our calculation

plt.figure(figsize=(10, 6))
plt.plot(df_NS_model['Maturity'], df_NS_model['Continously_Compounded_Yield_in%'], marker='o')
plt.title('Zero-Coupon Bond Yield Curve')
plt.xlabel('Time to Maturity (Years)')
plt.ylabel('Continuously Compounded Yield (%)')
plt.grid(True)
plt.show()
```



# Bootstrap Method

In [14]:
```
df_bot = pd.read_csv("treasury_data_bootstrap.csv")
df_bot.head()
```

Out[14]:

| | CUSIP | ISSUE_DATE | MATURITY_DATE | COUPON_RATE | BID_PRICE | ASK_PRICE | TIME_TO_MATURITY |
|---|---|---|---|---|---|---|---|
| **0** | 912810EB | 22/11/1988 | 15/11/2018 | 9.000 | 103.093750 | 103.218750 | 0.5 |
| **1** | 912828R4 | 16/05/2016 | 15/05/2019 | 0.875 | 98.679688 | 98.718750 | 1.0 |
| **2** | 912828LY | 16/11/2009 | 15/11/2019 | 3.375 | 101.382812 | 101.429688 | 1.5 |
| **3** | 912828X9 | 15/05/2017 | 15/05/2020 | 1.500 | 98.218750 | 98.257812 | 2.0 |
| **4** | 9128283G | 15/11/2017 | 15/11/2020 | 1.750 | 98.195312 | 98.234375 | 2.5 |

In [15]:
```
df_bot["Mid_Price"] = (df_bot.ASK_PRICE + df_bot.BID_PRICE)/2
df_bot.head()
```

Out[15]:

| | CUSIP | ISSUE_DATE | MATURITY_DATE | COUPON_RATE | BID_PRICE | ASK_PRICE | TIME_TO_MATURITY | Mid_Price |
|---|---|---|---|---|---|---|---|---|
| **0** | 912810EB | 22/11/1988 | 15/11/2018 | 9.000 | 103.093750 | 103.218750 | 0.5 | 103.156250 |
| **1** | 912828R4 | 16/05/2016 | 15/05/2019 | 0.875 | 98.679688 | 98.718750 | 1.0 | 98.699219 |
| **2** | 912828LY | 16/11/2009 | 15/11/2019 | 3.375 | 101.382812 | 101.429688 | 1.5 | 101.406250 |
| **3** | 912828X9 | 15/05/2017 | 15/05/2020 | 1.500 | 98.218750 | 98.257812 | 2.0 | 98.238281 |
| **4** | 9128283G | 15/11/2017 | 15/11/2020 | 1.750 | 98.195312 | 98.234375 | 2.5 | 98.214844 |

In [16]:
```
## Step 1. Creating the Cash Flow Matrix
unique_maturities = sorted(df_bot["TIME_TO_MATURITY"].unique())

# Initialize the cash flow matrix with zeroes
cash_flow_matrix = np.zeros((len(df_bot), len(unique_maturities)))

# Iterative process to create a vector row of each bonds casf flows at different maturities
for i, bond in df_bot.iterrows():
    for j, maturity in enumerate(unique_maturities):
        # Coupon payment for maturities before, and at the bonds maturity
        cash_flow = 100*bond["COUPON_RATE"] / 100 / 2  # Assuming semiannual
        if maturity == bond["TIME_TO_MATURITY"]:
            cash_flow += 100    # Adding principal
        cash_flow_matrix[i, j] = cash_flow

# We convert the cash flow matrix to Dataframe
df_cash_flow_matrix = pd.DataFrame(cash_flow_matrix, columns = unique_maturities)
df_cash_flow_matrix.head()
```

Out[16]:

| | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 | 5.5 | 6.0 | 6.5 | 7.0 | 7.5 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | 104.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5( |
| **1** | 0.4375 | 100.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4375 | 0.4: |
| **2** | 1.6875 | 1.6875 | 101.6875 | 1.6875 | 1.6875 | 1.6875 | 1.6875 | 1.6875 | 1.6875 | 1.6875 | 1.6875 | 1.6875 | 1.6875 | 1.6875 | 1.6875 | 1.6 |
| **3** | 0.7500 | 0.7500 | 0.7500 | 100.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7 |
| **4** | 0.8750 | 0.8750 | 0.8750 | 0.8750 | 100.8750 | 0.8750 | 0.8750 | 0.8750 | 0.8750 | 0.8750 | 0.8750 | 0.8750 | 0.8750 | 0.8750 | 0.8750 | 0.8 |

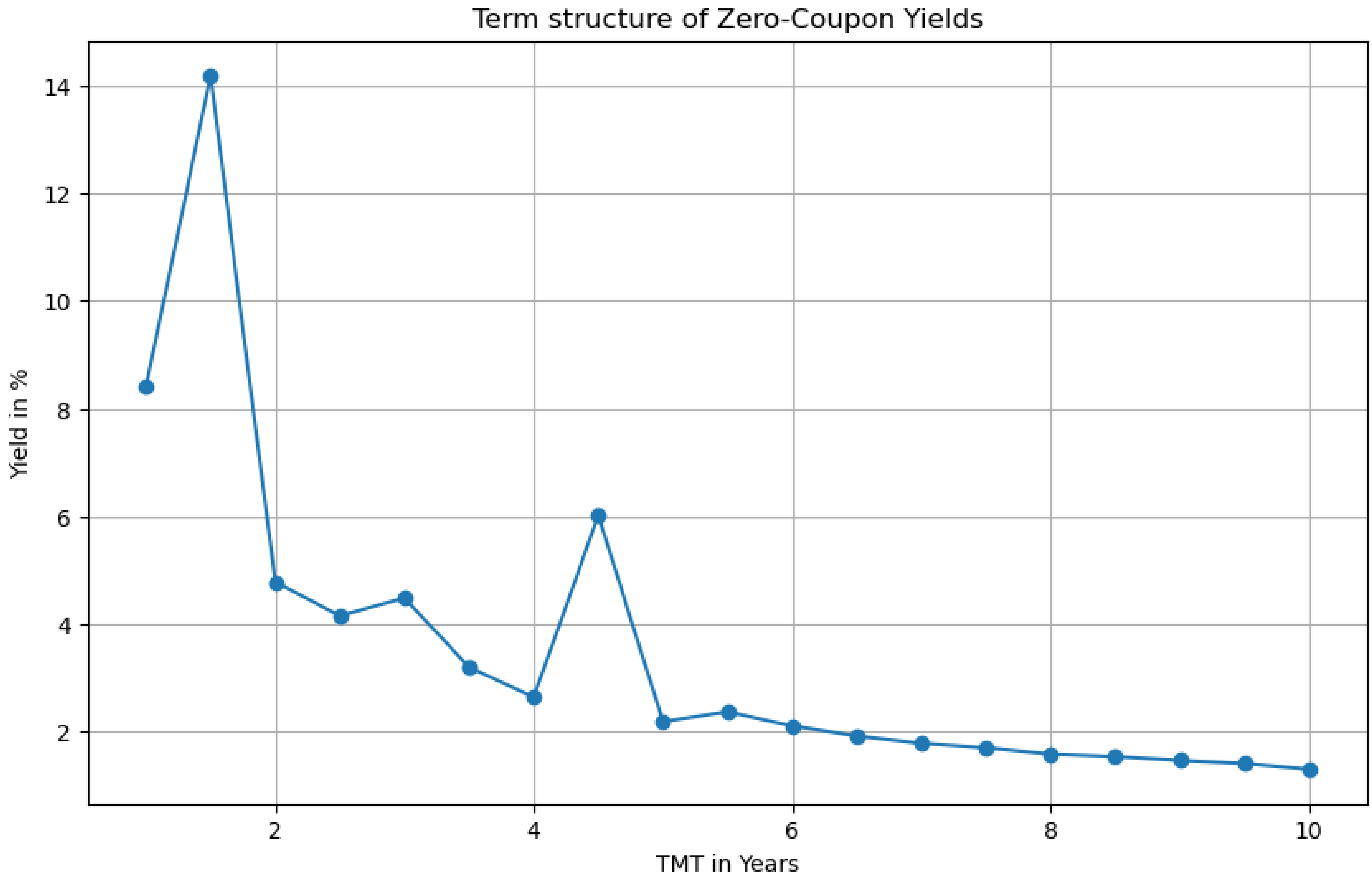
In [17]:
```
## Step 2. Prepare the Price Vector
price_vector = df_bot["Mid_Price"].values
```

In [18]:
```
## Step 3.Calculate Discount Factors

# Invert the cash flow matrix
cash_flow_matrix_inv = inv(cash_flow_matrix)

# Calculate the discount factors by multiplying the inverted matrix by the price vector
discount_factors = np.dot(cash_flow_matrix_inv, price_vector)

# Convert disocunt factors to Dataframe
df_discount_factors = pd.DataFrame(discount_factors)
df_discount_factors["Discount_Factors"] = df_discount_factors[0]
df_bootstrap = df_discount_factors.drop(df_discount_factors.columns[0], axis = 1)
df_bootstrap["TMT"] = df_bot["TIME_TO_MATURITY"]
df_bootstrap.head()
```

Out[18]:

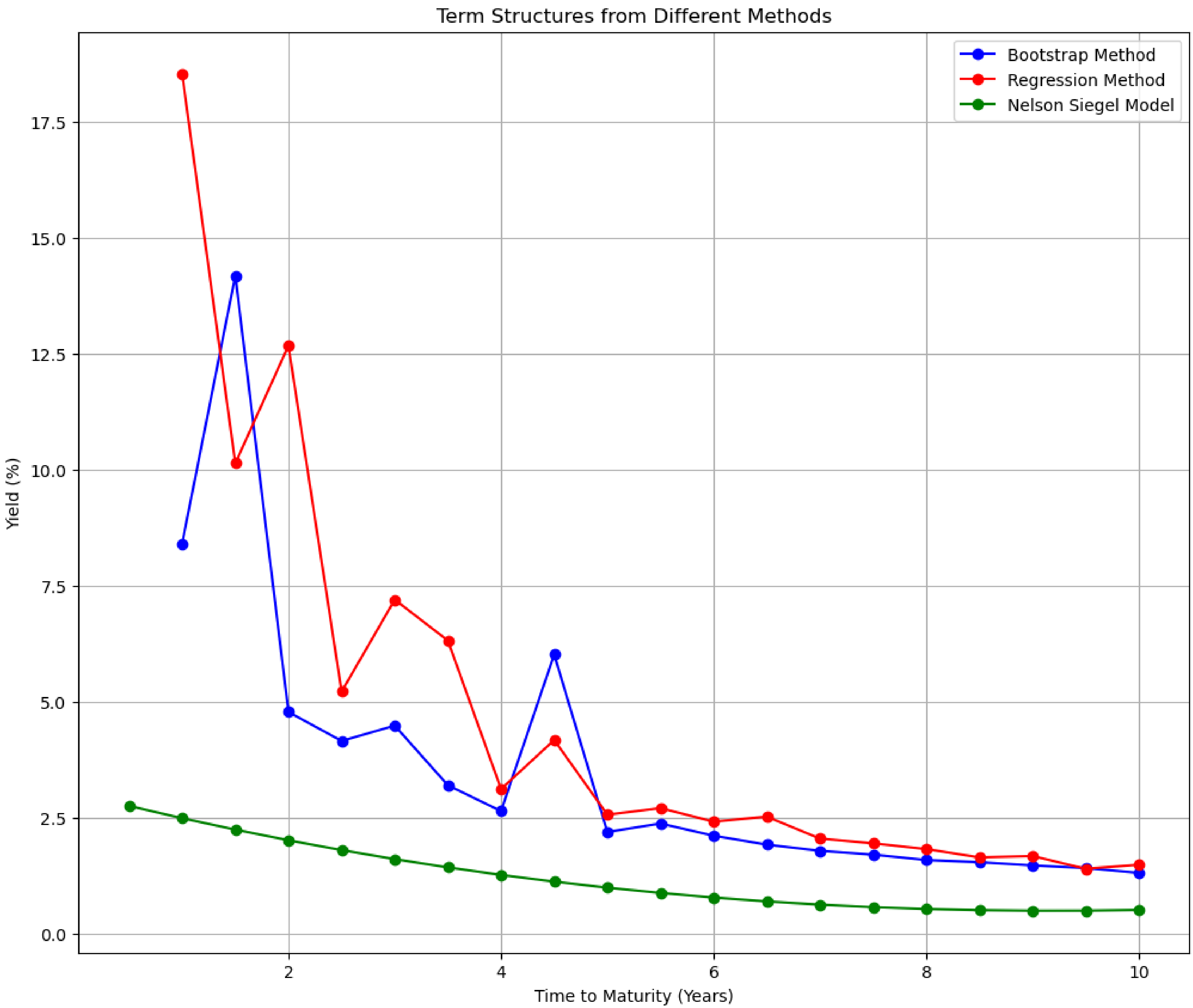
| | Discount_Factors | TMT |
|---|---|---|
| **0** | 0.335418 | 0.5 |
| **1** | 0.919311 | 1.0 |
| **2** | 0.753008 | 1.5 |
| **3** | 0.866359 | 2.0 |
| **4** | 0.846787 | 2.5 |

In [19]:
```
## Task 2 is to calculate the continously compounded zero-coupon bond yields for each maturity

# We will be using the same Discount_Factors as calculated above
df_bootstrap["Continously_Compounded_Yield"] = -np.log(df_bootstrap["Discount_Factors"]) / df_bootstrap.index

# Convert the yield to percent
df_bootstrap["Continously_Compounded_Yield_in%"] = df_bootstrap.Continously_Compounded_Yield * 100

# Plot the term structure
plt.figure(figsize = (10, 6))
plt.plot(df_bootstrap["TMT"], df_bootstrap["Continously_Compounded_Yield_in%"], marker = "o")
plt.title("Term structure of Zero-Coupon Yields")
plt.xlabel("TMT in Years")
plt.ylabel("Yield in %")
plt.grid(True)
plt.show()
```


Term structure of Zero-Coupon Yields
Yield in %
TMT in Years
14
12
10
8
6
4
2
2
4
6
8
10

# Plotting BootStrap and Regression

In [20]:
```
# Initialize the figure
plt.figure(figsize=(12,10))

# Plotting the Bootstrap
plt.plot(df_bootstrap.TMT, df_bootstrap["Continously_Compounded_Yield_in%"], label = "Bootstrap Method", mark

# Plotting Regression
plt.plot(df_reg_discount.TMT, df_reg_discount["Continously_Compounded_Yield_in%"], label = "Regression Method

# Plotting Nelson Siegel Model
plt.plot(df_NS_model['Maturity'], df_NS_model['Continously_Compounded_Yield_in%'], label = "Nelson Siegel Mod


# Adding titles and labels
plt.title('Term Structures from Different Methods')
plt.xlabel('Time to Maturity (Years)')
plt.ylabel('Yield (%)')
plt.legend()
plt.grid(True)

# Show plot
plt.show()
```


Term Structures from Different Methods
Bootstrap Method
Regression Method
Nelson Siegel Model
Yield (%)
Time to Maturity (Years)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
2
4
6
8
10