Regression Method

In [58]:
```
# Necessary packages
import pandas as pd
import numpy as np
from numpy.linalg import inv
import matplotlib.pyplot as plt

# Importing the CSV file
df_reg = pd.read_csv("treasury_data_all.csv")
df_reg.head()
```

Out[58]:

	CUSIP	ISSUE_DATE	MATURITY_DATE	COUPON_RATE	BID_PRICE	ASK_PRICE	TIME_TO_MATURITY
0	912810EB	22/11/1988	15/11/2018	9.000	103.093750	103.218750	0.5
1	912828M6	16/11/2015	15/11/2018	1.250	99.609375	99.648438	0.5
2	912828JR	17/11/2008	15/11/2018	3.750	100.734375	100.757812	0.5
3	912796QJ	17/05/2018	15/11/2018	0.000	99.069000	99.073667	0.5
4	912828R4	16/05/2016	15/05/2019	0.875	98.679688	98.718750	1.0

This is an overview over the different variables which is located in this dataframe

- CUSIP: A unique security-level identifier
- ISSUE_DATE: The date the Treasury bond was issued
- MATURITY_DATEThe date the Treasury bond matures
- COUPON_RATEThe bond’s coupon rate in percent
- BID_PRICEThe price at which dealers are willing to buy
- ASK_PRICEThe price at which dealers are willing to sell
- TIME_TO_MATURITYThe time-to-maturity measured in years

In [59]:
```
df_reg["Mid_Price"] = (df_reg.ASK_PRICE + df_reg.BID_PRICE)/2
df_reg.head()
```

Out[59]:

	CUSIP	ISSUE_DATE	MATURITY_DATE	COUPON_RATE	BID_PRICE	ASK_PRICE	TIME_TO_MATURITY	Mid_Price
0	912810EB	22/11/1988	15/11/2018	9.000	103.093750	103.218750	0.5	103.156250
1	912828M6	16/11/2015	15/11/2018	1.250	99.609375	99.648438	0.5	99.628906
2	912828JR	17/11/2008	15/11/2018	3.750	100.734375	100.757812	0.5	100.746094
3	912796QJ	17/05/2018	15/11/2018	0.000	99.069000	99.073667	0.5	99.071333
4	912828R4	16/05/2016	15/05/2019	0.875	98.679688	98.718750	1.0	98.699219

In [60]:
```
## Creating the Cash Flow Matrix
unique_maturities = sorted(df_reg["TIME_TO_MATURITY"].unique())

# Initialize the cash flow matrix with zeroes
cash_flow_matrix = np.zeros((len(df_reg), len(unique_maturities)))

# Iterative process to create a vector row of each bonds casf flows at different maturities
for i, bond in df_reg.iterrows():
    for j, maturity in enumerate(unique_maturities):
        # Coupon payment for maturities before, and at the bonds maturity
        cash_flow = 100*bond["COUPON_RATE"] / 100 / 2  # Assuming semiannual
        if maturity == bond["TIME_TO_MATURITY"]:
            cash_flow += 100    # Adding principal
        cash_flow_matrix[i, j] = cash_flow

# We convert the cash flow matrix to Dataframe
C = pd.DataFrame(cash_flow_matrix, columns = unique_maturities)
C.head()
```

Out[60]:

	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
0	104.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000
1	100.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250	0.6250
2	101.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750
3	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.4375	100.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375

In [79]:
```
## Step 2. Prepare the Price Vector
price_vector = df_reg["Mid_Price"].values
```

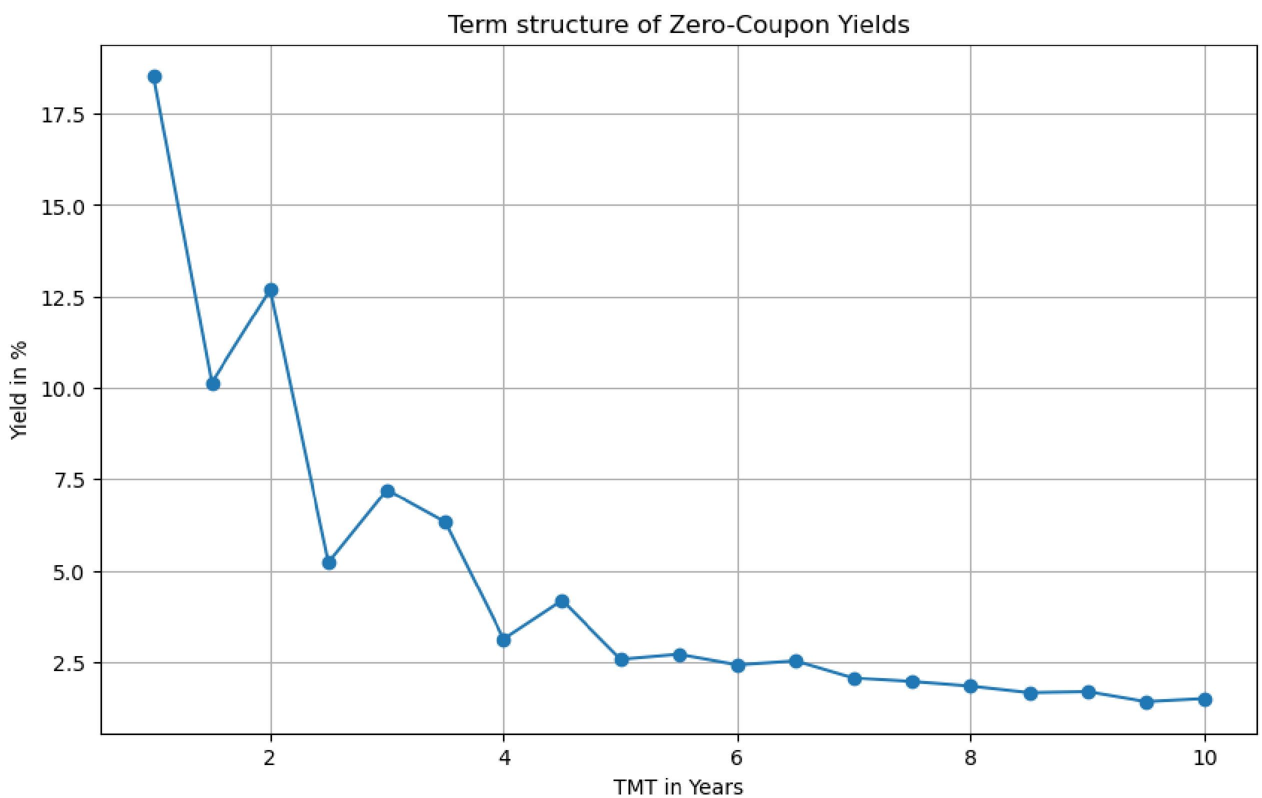
In [81]:

```
# Transpose and multiply the cash flow matrix
CT_C = np.dot(C.T, C)

# Invert the resulting matrix
inv_CT_C = inv(CT_C)

# Multiply by tranposed cash flow matrix and price vector
Z_0 = np.dot(np.dot(inv_CT_C, C.T), price_vector)

# Convert the discount factors to DataFrame
df_reg_discount = pd.DataFrame(Z_0)
df_reg_discount["Discount_Factors"] = df_reg_discount[0]
df_reg_discount = df_reg_discount.drop(df_reg_discount.columns[0], axis = 1)
df_reg_discount["TMT"] = unique_maturities
df_reg_discount
```

Out[81]:

	Discount_Factors	TMT
0	0.733487	0.5
1	0.830802	1.0
2	0.816418	1.5
3	0.683638	2.0
4	0.811555	2.5
5	0.697612	3.0
6	0.684373	3.5
7	0.803780	4.0
8	0.715933	4.5
9	0.793584	5.0
10	0.762665	5.5
11	0.766401	6.0
12	0.738448	6.5
13	0.765793	7.0
14	0.760918	7.5
15	0.760296	8.0
16	0.768178	8.5
17	0.751902	9.0
18	0.776729	9.5
19	0.753493	10.0

In [82]:
```
## Task 2 is to calculate the continously compounded zero-coupon bond yields for each maturity

# We will be using the same discount factors as calculated above
df_reg_discount["Continously_Compounded_Yield"] = -np.log(df_reg_discount["Discount_Factors"]) / df_reg_discount.index

# Convert the yield to percent
df_reg_discount["Continously_Compounded_Yield_in%"] = df_reg_discount.Continously_Compounded_Yield * 100


# Plot the term structure
plt.figure(figsize = (10, 6))
plt.plot(df_reg_discount.TMT, df_reg_discount["Continously_Compounded_Yield_in%"], marker = "o")
plt.title("Term structure of Zero-Coupon Yields")
plt.xlabel("TMT in Years")
plt.ylabel("Yield in %")
plt.grid(True)
plt.show()
```



Nelson Siegel

In [64]:
```
# Necessary package to optimize
from scipy.optimize import minimize
```

In [65]:
```
# We define the df_NS as same as the df_reg
df_NS = df_reg

# Define the NS model functions
def nelson_siegel_yield(theta_0, theta_1, theta_2, lambda_, T):
    # Calculates the yield for a given maturity T based on NS
    return theta_0 + (theta_1 + theta_2)*(1-np.exp(-T / lambda_)) / (T / lambda_) - theta_2 * np.exp(-T / lambda_)

def discount_factor(theta_0, theta_1, theta_2, lambda_, T):
    # Calculates the discount factor for a given maturity
    r = nelson_siegel_yield(theta_0, theta_1, theta_2, lambda_, T)
    return np.exp(-r * T)
```

In [66]:
```
# Defining the objective function for Optimization
def objective_function(params, df_NS):
    theta_0, theta_1, theta_2, lambda_ = params
    squared_errors = []

    for index, row in df_NS.iterrows():
        T = row["TIME_TO_MATURITY"]
        coupon = row["COUPON_RATE"] / 100
        mid_price = row["Mid_Price"]
        face_value = 100     # We assume that the standard face value is 100

        # Calculate the price of the bond under the NS model
        # We will assume semi-annual coupons
        cash_flows = [coupon * face_value / 2 for _ in range(int(2*T))]
        cash_flows[-1] += face_value     # Add the Face value as the last payment
        model_price = sum(discount_factor(theta_0, theta_1, theta_2, lambda_, t/2) * cf for t, cf in enumerate(cash_flows, start=1))

        # Calculate the squared error between the model price and the observed mid price
        squared_errors.append((model_price - mid_price) ** 2)

    return np.sum(squared_errors)
```

In [67]:
```
# We will initialize the optimization model by a guess for our thetas, lambda and T
initial_guess = [0.03, 0, 0, 3]

# Start the optimization
result = minimize(objective_function, initial_guess, args = (df_NS, ), method = 'L-BFGS-B')

# Print out the result
print("Optimized Parameters: θ0 = {:.4f}, θ1 = {:.4f}, θ2 = {:.4f}, λ = {:.4f}".format(*result.x))
```

```
Optimized Parameters: θ0 = 0.0300, θ1 = -0.0092, θ2 = 0.0054, λ = 3.0000
```

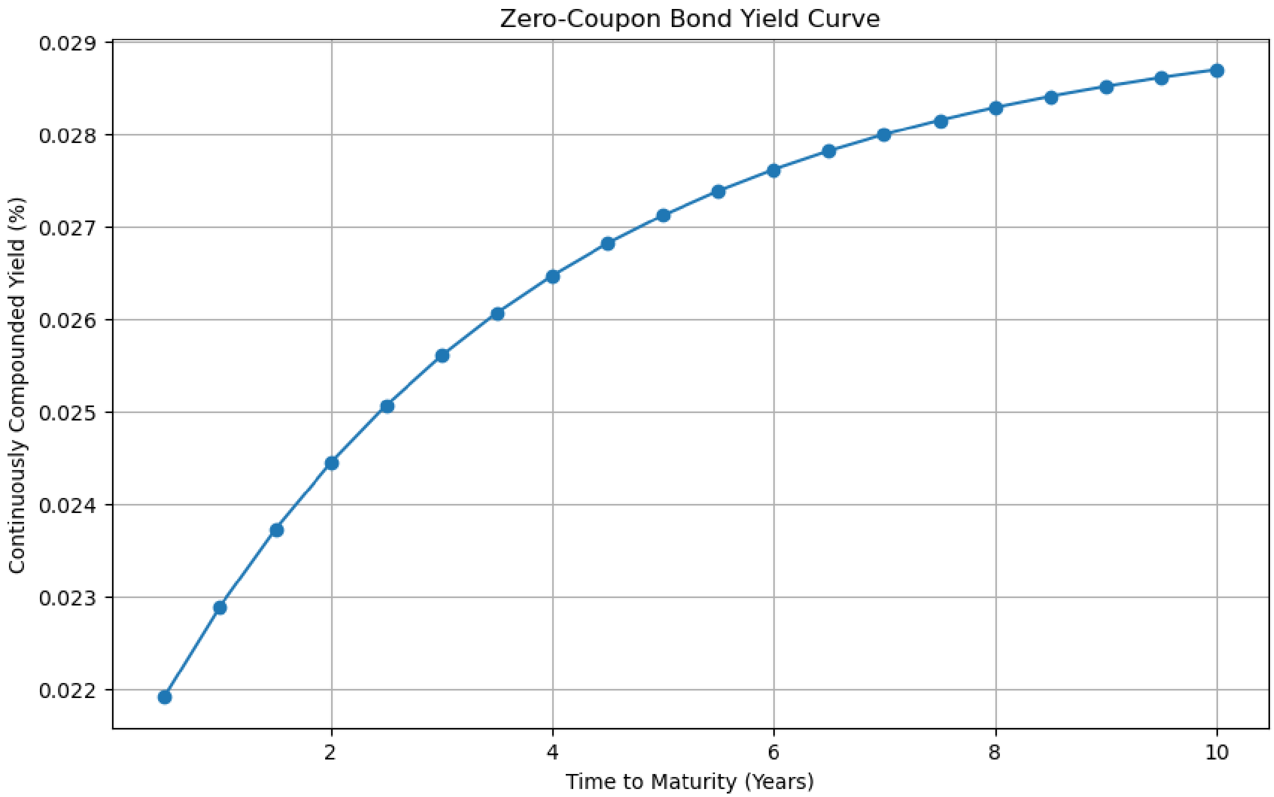
In [68]:
```
### Next task is to se the estimated model parameters to calculate the zero-coupon bond yield for each maturity.
## We extract our optimized parameters so it could be later used
optimized_params = result.x
theta_0, theta_1, theta_2, lambda_ = optimized_params
```

In [69]:
```
# Extract the unique maturity values
unique_maturities = df_NS["TIME_TO_MATURITY"].unique()

# We then calculate yeuekds for each maturity
zero_cp_yields = {T: nelson_siegel_yield(theta_0, theta_1, theta_2, lambda_, T) for T in unique_maturities}

# We convert the dictionary to a DF
df_NS_model = pd.DataFrame(list(zero_cp_yields.items()), columns = ["Maturity", "Continously_Compounded_Yield_in%"])
df_NS_model.head()
```

Out[69]:

	Maturity	Continously_Compounded_Yield_in%
0	0.5	0.021928
1	1.0	0.022898
2	1.5	0.023732
3	2.0	0.024451
4	2.5	0.025071

In [70]:
```
# We can then plot the term structure of interest rates obtained in our calculation

plt.figure(figsize=(10, 6))
plt.plot(df_NS_model['Maturity'], df_NS_model['Continously_Compounded_Yield_in%'], marker='o')
plt.title('Zero-Coupon Bond Yield Curve')
plt.xlabel('Time to Maturity (Years)')
plt.ylabel('Continuously Compounded Yield (%)')
plt.grid(True)
plt.show()
```



Bootstrap Method

In [71]:
```
df_bot = pd.read_csv("treasury_data_bootstrap.csv")
df_bot.head()
```

Out[71]:

	CUSIP	ISSUE_DATE	MATURITY_DATE	COUPON_RATE	BID_PRICE	ASK_PRICE	TIME_TO_MATURITY
0	912810EB	22/11/1988	15/11/2018	9.000	103.093750	103.218750	0.5
1	912828R4	16/05/2016	15/05/2019	0.875	98.679688	98.718750	1.0
2	912828LY	16/11/2009	15/11/2019	3.375	101.382812	101.429688	1.5
3	912828X9	15/05/2017	15/05/2020	1.500	98.218750	98.257812	2.0
4	9128283G	15/11/2017	15/11/2020	1.750	98.195312	98.234375	2.5

In [72]:
```
df_bot["Mid_Price"] = (df_bot.ASK_PRICE + df_bot.BID_PRICE)/2
df_bot.head()
```

Out[72]:

	CUSIP	ISSUE_DATE	MATURITY_DATE	COUPON_RATE	BID_PRICE	ASK_PRICE	TIME_TO_MATURITY	Mid_Price
0	912810EB	22/11/1988	15/11/2018	9.000	103.093750	103.218750	0.5	103.156250
1	912828R4	16/05/2016	15/05/2019	0.875	98.679688	98.718750	1.0	98.699219
2	912828LY	16/11/2009	15/11/2019	3.375	101.382812	101.429688	1.5	101.406250
3	912828X9	15/05/2017	15/05/2020	1.500	98.218750	98.257812	2.0	98.238281
4	9128283G	15/11/2017	15/11/2020	1.750	98.195312	98.234375	2.5	98.214844

In [73]:
```
## Step 1. Creating the Cash Flow Matrix
unique_maturities = sorted(df_bot["TIME_TO_MATURITY"].unique())

# Initialize the cash flow matrix with zeroes
cash_flow_matrix = np.zeros((len(df_bot), len(unique_maturities)))

# Iterative process to create a vector row of each bonds casf flows at different maturities
for i, bond in df_bot.iterrows():
    for j, maturity in enumerate(unique_maturities):
        # Coupon payment for maturities before, and at the bonds maturity
        cash_flow = 100*bond["COUPON_RATE"] / 100 / 2  # Assuming semiannual
        if maturity == bond["TIME_TO_MATURITY"]:
            cash_flow += 100    # Adding principal
        cash_flow_matrix[i, j] = cash_flow

# We convert the cash flow matrix to Dataframe
df_cash_flow_matrix = pd.DataFrame(cash_flow_matrix, columns = unique_maturities)
df_cash_flow_matrix.head()
```

Out[73]:

	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
0	104.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000	4.5000
1	0.4375	100.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375	0.4375
2	1.6875	1.6875	101.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875	1.6875
3	0.7500	0.7500	0.7500	100.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500	0.7500
4	0.8750	0.8750	0.8750	0.8750	100.8750	0.8750	0.8750	0.8750	0.8750	0.8750	0.8750	0.8750	0.8750	0.8750	0.8750	0.8750	0.8750	0.8750	0.8750	0.8750

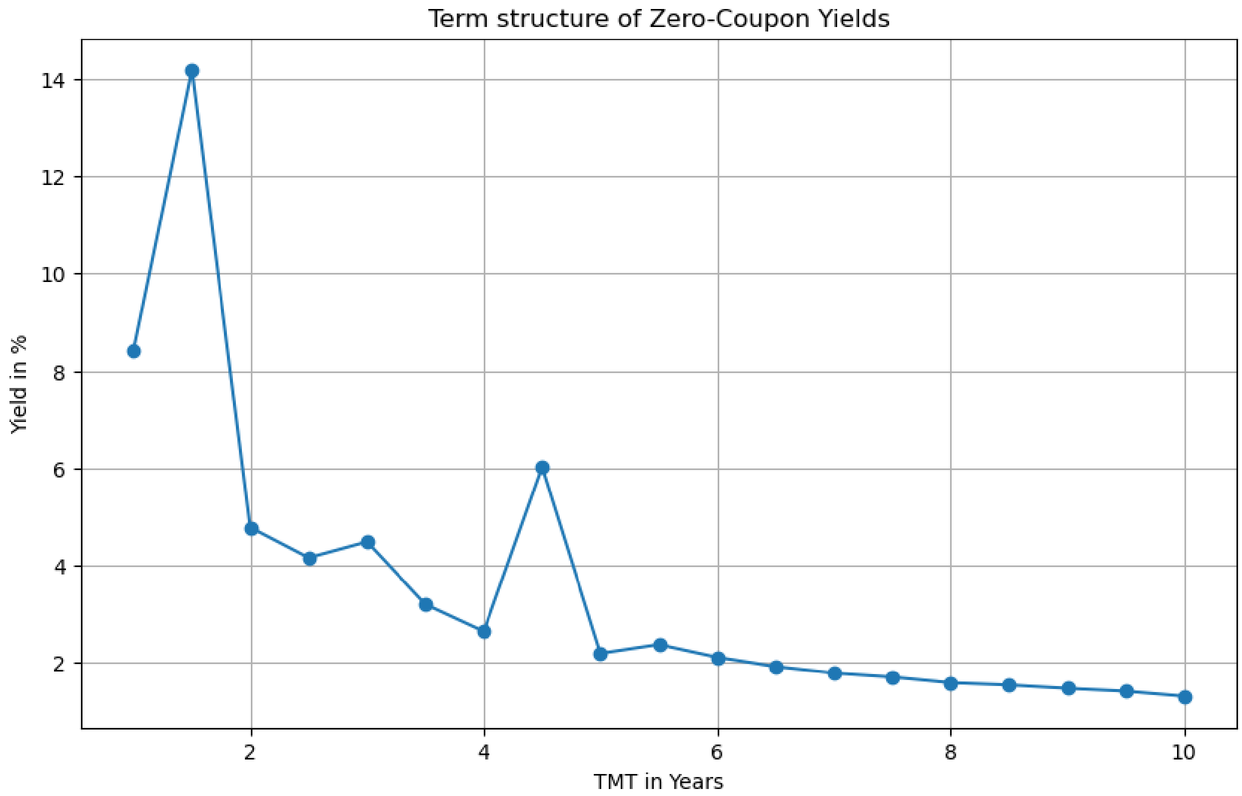
In [74]:
```
## Step 2. Prepare the Price Vector
price_vector = df_bot["Mid_Price"].values
```

In [75]:
```
## Step 3.Calculate Discount Factors

# Invert the cash flow matrix
cash_flow_matrix_inv = inv(cash_flow_matrix)

# Calculate the discount factors by multiplying the inverted matrix by the price vector
discount_factors = np.dot(cash_flow_matrix_inv, price_vector)

# Convert disocunt factors to Dataframe
df_discount_factors = pd.DataFrame(discount_factors)
df_discount_factors["Discount_Factors"] = df_discount_factors[0]
df_bootstrap = df_discount_factors.drop(df_discount_factors.columns[0], axis = 1)
df_bootstrap["TMT"] = df_bot["TIME_TO_MATURITY"]
df_bootstrap.head()
```

Out[75]:

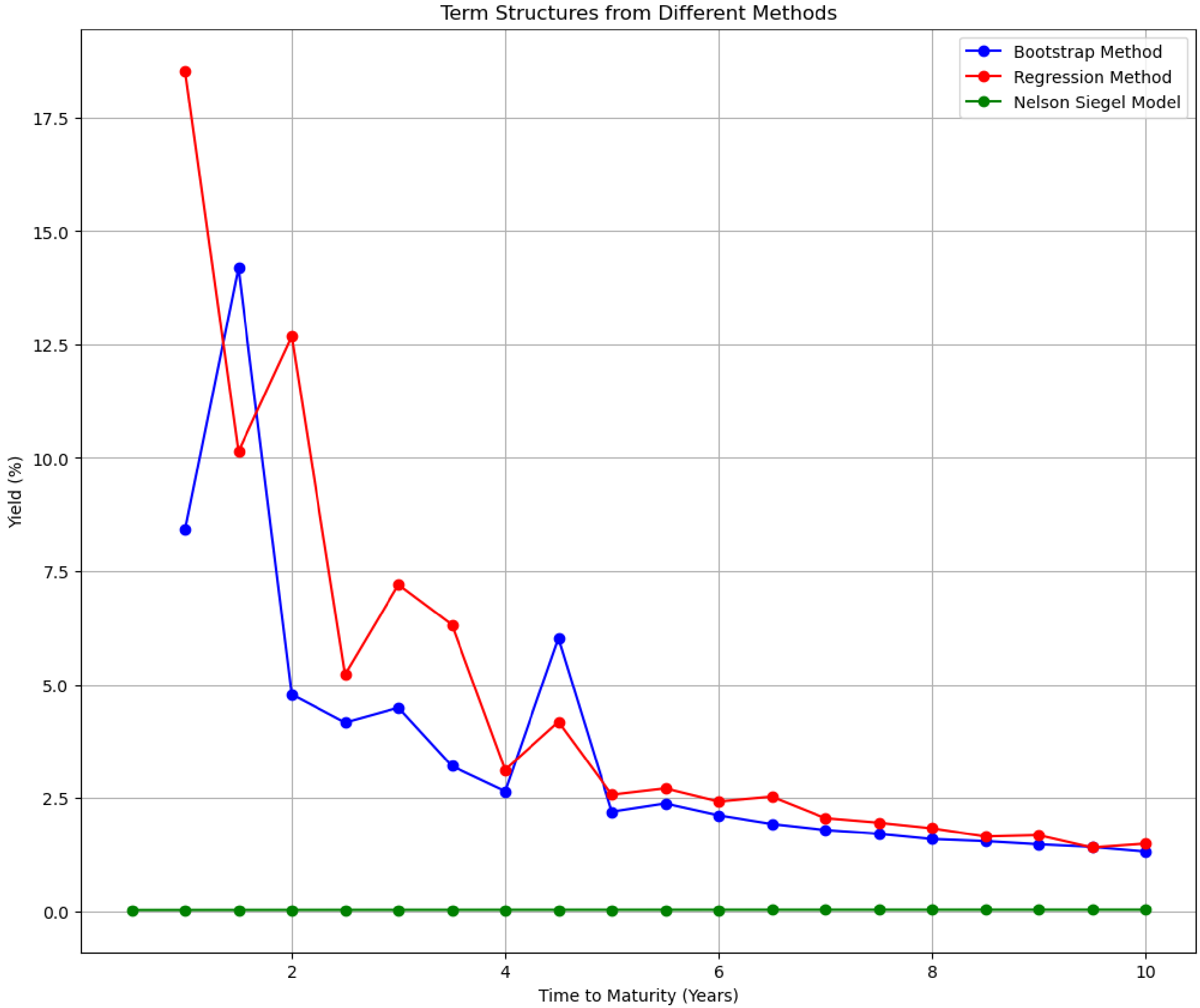
	Discount_Factors	TMT
0	0.335418	0.5
1	0.919311	1.0
2	0.753008	1.5
3	0.866359	2.0
4	0.846787	2.5

In [76]:
```
## Task 2 is to calculate the continously compounded zero-coupon bond yields for each maturity

# We will be using the same Discount_Factors as calculated above
df_bootstrap["Continously_Compounded_Yield"] = -np.log(df_bootstrap["Discount_Factors"]) / df_bootstrap.index

# Convert the yield to percent
df_bootstrap["Continously_Compounded_Yield_in%"] = df_bootstrap.Continously_Compounded_Yield * 100

# Plot the term structure
plt.figure(figsize = (10, 6))
plt.plot(df_bootstrap["TMT"], df_bootstrap["Continously_Compounded_Yield_in%"], marker = "o")
plt.title("Term structure of Zero-Coupon Yields")
plt.xlabel("TMT in Years")
plt.ylabel("Yield in %")
plt.grid(True)
plt.show()
```



Plotting BootStrap and Regression

In [84]:
```
nitialize the figure
.figure(figsize=(12,10))

lotting the Bootstrap
.plot(df_bootstrap.TMT, df_bootstrap["Continously_Compounded_Yield_in%"], label = "Bootstrap Method", marker="o", linestyle = "-", color="blue")

lotting Regression
.plot(df_reg_discount.TMT, df_reg_discount["Continously_Compounded_Yield_in%"], label = "Regression Method", marker="o", linestyle = "-", color="red")

lotting Nelson Siegel Model
.plot(df_NS_model['Maturity'], df_NS_model['Continously_Compounded_Yield_in%'], label = "Nelson Siegel Model", marker='o', linestyle = "-", color ="green")


dding titles and labels
.title('Term Structures from Different Methods')
.xlabel('Time to Maturity (Years)')
.ylabel('Yield (%)')
.legend()
.grid(True)

how plot
.show()
```


Term Structures from Different Methods
Bootstrap Method
Regression Method
Nelson Siegel Model
Yield (%)
Time to Maturity (Years)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
2
4
6
8
10