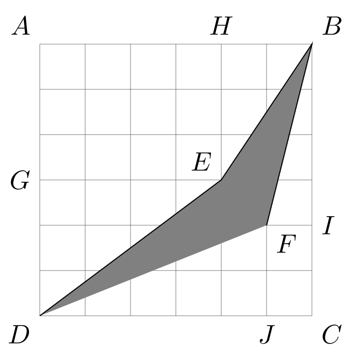
2014 Junior Round 3 Solutions

Question 1

For a number to be a multiple of 9, the sum of the digits must be a multiple of 9. If we try to make the first three digits as large as possible, we can try to start with 987..., which gives us $9+8+7=24$, the possible multiples of 9 that we can form are 27 and 36, since 45 won't work $(45-24=21>9+9)$. If we try 6 as the fourth digit, we need 6 as the fifth, which is invalid. If we try 5 as fourth digit, then we need 7 as fifth, which is invalid. If we try 4 as fourth digit, we need 8 as fifth, which is invalid. If we try 3 as fourth digit, we need 0 as fifth, which is indeed valid and gives us the largest satisfying the needed conditions.

Answer: 98730

Question 2



$$\begin{aligned}
\text{Area}\, DEBF &= \text{total area} - \text{unshaded region} \\
&= 36 - (DGE + AHEG + EHB + BFI + FICJ + DFJ) \\
&= 36 - (\frac{1}{2}\cdot 12 + 12 + \frac{1}{2}\cdot 6 + \frac{1}{2}\cdot 4 + 2 + \frac{1}{2}\cdot 10) \\
&= 36 - 30 \\
&= 6.
\end{aligned}$$

Answer: 6 cm^2

Question 3

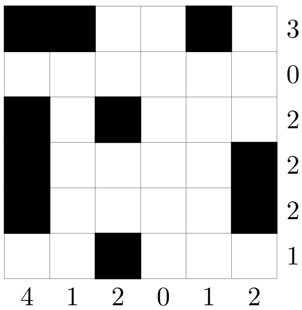
$10! = (5\times 2)\times(3\times 3)\times(2\times 2\times 2)\times 7!$

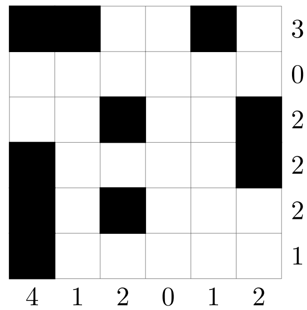
Since 7 is prime, one of the numbers a, b, c must contain 7! or higher. The additional factor of 5 must come from 10!, 6! or 5!. In the first case, this gives $10!\times 1!\times 1!$, which is invalid since we need a, b and c to be distinct, and in the second case we get $7!\times 6!\times 1!$, and in the the third case, we get $7!\times 5!\times 3!$.

Answer: $(7!\times 5!\times 3!)$ and $(7!\times 6!\times 1!)$

Question 4

There are two solutions:





Question 5

Let $a+b+c+d=396$. We have that

$$a+5=b-5=5c=d/5,$$

so $b=a+10$, $c=(a+5)/5$ and $d=5a+25$. Substituting,

$$\begin{aligned} a+b+c+d &= a+(a+10)+(a+5)/5+5a+25 \\ &= \frac{36a}{5}+36. \end{aligned}$$

So $a=(396-36)\times 5/36=50$. Solving for the other numbers gives $a=50$, $b=60$, $c=11$, $d=275$.

Answer: $(50, 60, 11, 275)$

Question 6

Since B must be twice C, we can immediately see that B is 6 and C is 3. Hence on the left we either have $4\times 2+9=17$, $5\times 2+9=19$ or $7\times 2+9=23$. We can't have E as 7, as that will already give us 21. If we have D as 7, we get $14+3\times 4=26$ or $14+3\times 5=29$, which don't work. So A should be 7. Then D should be 4 and E should be 5. Checking we have $7\times 2+9=2\times 4+3\times 5$, which is indeed true.

Answer: A is 7, B is 6, C is 3, D is 4, E is 5

Question 7

We calculate everything in Johannesburg time first. Starting from 06h00 and flying 15 hrs to New York, means arriving at 21h00 (still in Johannesburg time). After waiting 24 hours, it is again 21h00. The flight to Bangkok takes 17 hours, making it 14h00 Johannesburg time. Since Bangkok is 7 hours ahead, it is 21h00 Bangkok time.

Answer: 21h00

Question 8

The upper triangle is covered by A, two triangles with area half of A, and one triangle with area a quarter of A. The lower triangle is covered by B and two triangles with area half of B. The upper and lower triangles have the same area, so

$$(1+2(\frac{1}{2})+1(\frac{1}{4})) \times \text{Area } A = (1+2(\frac{1}{2})) \times \text{Area } B$$
$$\Leftrightarrow \text{Area } A = \frac{8}{9} \times \text{Area } B$$

Answer: $\frac{8}{9}$

Question 9

Let $N = p \times q$ where p and q are prime. The positive divisors of N that are less than N are 1, p and q, so $p + q + 1 = 2014$. Hence $p + q = 2013$. One of p and q must be even, and so can only be 2. Then the other one is 2011, which is prime. So $N = 2 \times 2011 = 4022$.

Answer: 4022

Question 10

a) Yes. Jill can colour in the middle 3×3 square. There are then 6 white squares remaining, none of which form a 2×2 square, so they will each have to be coloured in individually, finishing with Jill.

b) Yes. Jill can colour in the middle 14×14 square, leaving a 14×3 rectangle on either side. It is now Jack's turn, who will have to choose one rectangle in which to colour in a square. Whatever Jack does, Jill can do the same in the other rectangle, leaving the two rectangles with the same colouring. If Jill continues copying Jack's move in the opposite rectangle, then after her turn she will either have coloured the last white square and won, or she will leave at least one white square in each rectangle that Jack can't colour simultaneously. This means Jack can never win, so Jill will win.

c) Let the grid have dimensions $m \times n$ where $m \leq n$ and m and n are odd. If $m = n$ then Jill can colour in the whole grid and win. Otherwise, Jill can colour in the middle $m \times m$ square, leaving a $m \times \frac{n-m}{2}$ rectangle on either side. Similarly to part b), Jill can copy Jack's moves in the opposite rectangle and force a win.

Question 11

a)

$$\frac{4(4+1)}{2} + \frac{k(k+1)}{2} = \frac{10(10+1)}{2}$$
$$\Leftrightarrow 10 + \frac{k(k+1)}{2} = 55$$
$$\Leftrightarrow k(k+1) = 90$$
$$\Leftrightarrow k = 9$$

Answer: 9

b)
$$\frac{(b+1)(b+2)}{2} - \frac{b(b+1)}{2} = \frac{x(x+1)}{2}$$
$$\Leftrightarrow b+1 = \frac{x(x+1)}{2},$$

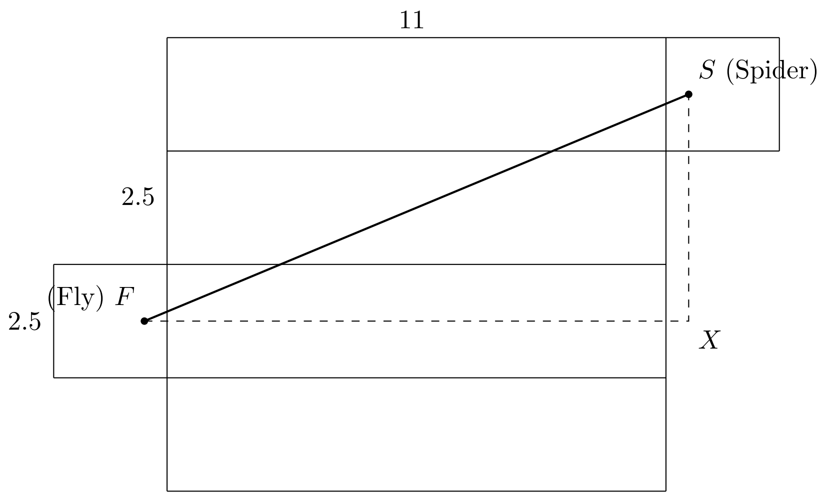
so $b+1$ is the smallest triangular number greater than 2015.

$$\frac{(62)(63)}{2} = 1953 < 2015 < 2016 = \frac{(63)(64)}{2},$$

so $b+1$ is the triangular number 2016.
Answer: 2015

Question 12

Unwrap the box as shown.



Then the length of the distance travelled by the spider is SF, where

$$\begin{aligned} SF^2 &= SX^2 + FX^2 \\ &= (1.25 + 2.5 + 1.25)^2 + (0.5 + 11 + 0.5)^2 \\ &= 5^2 + 12^2 = 169, \end{aligned}$$

so $SF = \sqrt{169} = 13$.
We also just need to check that any other way we fold it open leads to a greater distance.

Answer: 13 m

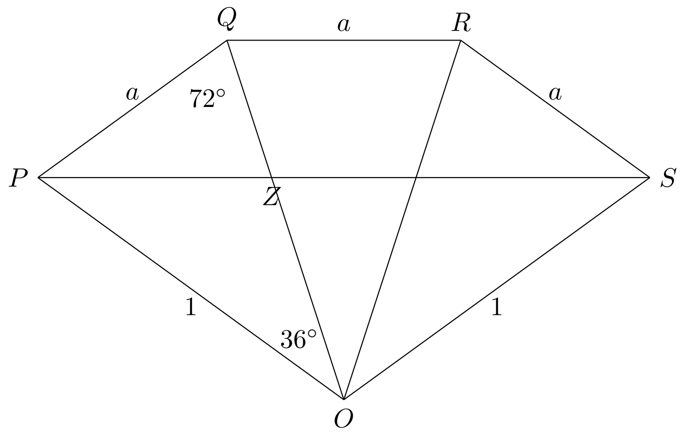
Question 13

Let O be the centre of the circle, and let the lines PS and OQ meet at Z.
Then $P\hat{O}Q = Q\hat{O}R = R\hat{O}S = 36°$ and $P\hat{Q}O = Q\hat{P}O = 72°$. Since triangle POS is isosceles, $O\hat{P}S = \frac{1}{2}(180° - 108°) = 36°$, and $Q\hat{P}Z = Q\hat{P}O - O\hat{P}S = 72° - 36° = 36°$. Therefore triangle PQZ is isosceles and $PZ = PQ = a$.

Since $S\hat{Z}O = 72° = S\hat{O}Z$, triangle SZO is isosceles, so $SZ = SO = 1$. Then

$$PS - PQ = PZ + ZS - PQ = a + 1 - a = 1.$$

Answer: 1



Question 14

The biggest total less than 1 will have smallest difference

$$1-\left(\frac{a}{b}+\frac{c}{d}\right)=\frac{bd-ad-bc}{bd}.$$

Both numerator and denominator are integers, so the smallest possible numerator is 1, and the largest possible denominator using digits 1 to 9 is $8\times 9=72$. In this case, $b=8$, $d=9$ (or the reverse). For the numerator to be 1, we must have $bd-ad+bc=72-9a-8c=1$, so $9a+8c=71$. After trial and error with the possible digits 1 to 7, we get $a=7$, $c=1$. The sum is therefore

$$\frac{a}{b}+\frac{c}{d}=\frac{7}{8}+\frac{1}{9}=\frac{71}{72}.$$

Answer: $\frac{71}{72}$

Question 15

In each row, every second circle is yellow. If n is odd, then in each row there will be $\frac{n+1}{2}$ yellow circles, so $\frac{(n+1)^2}{2}$ yellow circles in total. If n is even, then each row will have either $\frac{n}{2}+1$ or $\frac{n}{2}$ yellow circles, and the two sums will alternate. If the first row has $\frac{n}{2}$ yellow circles, then there will be $\frac{n}{2}+1$ rows with $\frac{n}{2}$ yellow circles and $\frac{n}{2}$ rows with $\frac{n}{2}+1$ yellow circles, giving

$$\frac{n^2+2n}{4}+\frac{n^2+2n}{4}=\frac{n^2+2n+1}{2}-\frac{1}{2}=\frac{(n+1)^2}{2}-\frac{1}{2}$$

yellow circles in total. If the first row has $\frac{n}{2}+1$ yellow circles, then there will be $\frac{n}{2}+1$ rows with $\frac{n}{2}+1$ yellow circles and $\frac{n}{2}$ rows with $\frac{n}{2}$ yellow circles, giving

$$\frac{n^2+4n+4}{4}+\frac{n^2}{4}=\frac{(n+1)^2}{2}+\frac{1}{2}$$

yellow circles in total.

Answer: n odd: $\frac{(n+1)^2}{2}$ and n even: $\frac{(n+1)^2}{2}\pm\frac{1}{2}$