The South African Mathematical Olympiad
Third Round 2020
Senior Division (Grades 10 to 12)
Solutions

1. Find the smallest positive multiple of 20 with exactly 20 positive divisors.

   Solution 1: Consider, for a positive integer k, the multiple $M_k = k\times 20$. If $k = 2^{a_1}\cdot 3^{a_2}\cdot 5^{a_3}\cdots$ is the prime factorization of k, where $a_i \geqslant 0$ for all $i \geqslant 1$, then $M_k = 2^{a_1+2}\cdot 3^{a_2}\cdot 5^{a_3+1}\cdot 7^{a_4}\cdots$ is the prime factorization of M_k.

   For M_k to have 20 positive divisors, we need to have $(a_1+3)(a_2+1)(a_3+2)\times\prod_{i\geqslant 4}(a_i+1) = 20 = 2^2\cdot 5$. This forces $a_1+3 \in \{4,5,10\}$ and $a_3+2 \in \{2,4,5\}$. Also, since we want the smallest such k, we may assume that k has no prime divisors larger than 5. (Any $a_i+1 = 2$, with $i \geqslant 4$, can be replaced by $a_2+1 = 2$, resulting in a smaller k, but without changing the number of divisors of M_k.) Henceforth, we assume that $a_i = 0$ for all $i \geqslant 4$. All the possibilities are summarised in the following table:

a_1+3	a_2+1	a_3+2	a_1	a_2	a_3	$k = 2^{a_1}\cdot 3^{a_2}\cdot 5^{a_3}$
4	1	5	1	0	3	250
5	1	4	2	0	2	100
5	2	2	2	1	0	12
10	1	2	7	0	0	128

   We conclude that the smallest positive multiple of 20 with exactly 20 positive divisors, is $12\times 20 = 240$.

   Solution 2: Of course, one can try to be lucky, and simply start to determine the numbers of divisors of $1\times 20, 2\times 20, 3\times 20, \ldots$, and soon end up with 12×20 having 20 positive divisors (for the first time in this sequence). Hence, 12×20 is the smallest such multiple.

2. Let S be a square with sides of length 2 and R be a rhombus with sides of length 2 and angles measuring 60° and 120°. These quadrilaterals are arranged to have the same centre and the diagonals of the rhombus are parallel to the sides of the square. Calculate the area of the region on which the figures overlap.

   Solution: Let S be the square $ABCD$, with centre O. Let R be the rhombus $EFGH$, with EG the short diagonal, and O the midpoint of EG. Since the diagonals of R bisect the angles of R, we have that $\angle OFE = 30^\circ$, so that $\sin 30^\circ = \frac{1}{2}$ forces EG to have length 2. We may therefore assume that E is the midpoint of AD and G is the midpoint of BC. See Figure 1.

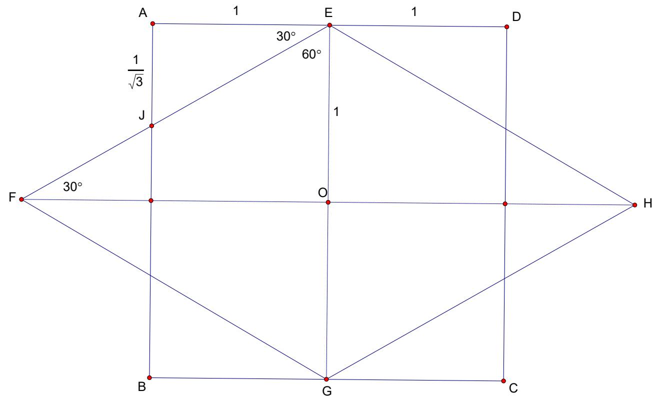

Figure 1

Consequently, $\angle AEJ = 30°$ so that $\tan 30° = \frac{1}{\sqrt{3}}$ implies that $AJ = \frac{1}{\sqrt{3}}$. The area of the region where R and S overlap is therefore, by symmetry, equal to the area of S minus four times the area of triangle AJE, i.e.,

$$4 - 4 \cdot \tfrac{1}{2} \cdot 1 \cdot \tfrac{1}{\sqrt{3}} = 4 \cdot \left(1 - \tfrac{1}{2\sqrt{3}}\right).$$

3. If x, y, z are real numbers satisfying

$$\begin{aligned}(x+1)(y+1)(z+1) &= 3\\ (x+2)(y+2)(z+2) &= -2\\ (x+3)(y+3)(z+3) &= -1,\end{aligned}$$

find the value of

$$(x+20)(y+20)(z+20).$$

Solution: By putting $X = x+2, Y = y+2$ and $Z = z+2$, the system of equations translates to

$$\begin{aligned}(X-1)(Y-1)(Z-1) &= 3 & \ldots(1)\\ XYZ &= -2 & \ldots(2)\\ (X+1)(Y+1)(Z+1) &= -1. & \ldots(3)\end{aligned}$$

By expanding the left hand sides of (1) and (3), and substituting $XYZ = -2$, we obtain the system

$$\begin{aligned}T_1 - T_2 &= 6\\ T_1 + T_2 &= 0,\end{aligned}$$

where $T_1 = X + Y + Z$ and $T_2 = XY + YZ + ZX$. Form this, we solve $T_1 = 3$ and $T_2 = -3$, so that

$$\begin{aligned}(x+20)(y+20)(z+20) &= (X+18)(Y+18)(Z+18)\\ &= -2 + 18 \cdot T_2 + 18^2 \cdot T_1 + 18^3\\ &= -2 + 18 \cdot (-3) + 18^2 \cdot (3) + 18^3\\ &= 6748.\end{aligned}$$

4. A positive integer k is said to be *visionary* if there are integers $a > 0$ and $b \geqslant 0$ such that $a \cdot k + b \cdot (k+1) = 2020$. How many visionary integers are there?.

   Solution: All lower case variables in this solution denote integers. Let X denote the set of visionary integers. We show that $X = \{\lfloor \frac{2020}{n} \rfloor : 1 \leqslant n \leqslant 2020\}$.

   If k is a visionary integer, then there exist $a > 0$ and $b \geqslant 0$ such that $ak + b(k+1) = 2020$, i.e., $2020 = k(a+b) + b$, where $0 \leqslant b < a+b$. This implies that $k = \lfloor \frac{2020}{a+b} \rfloor$ and we also have $1 \leqslant a+b \leqslant 2020$, since $a \geqslant 1$ and $k \geqslant 1$. Hence $k \in X$. Conversely, let $k \in X$, i.e., $k = \lfloor \frac{2020}{n} \rfloor$ for some $1 \leqslant n \leqslant 2020$. Then $2020 = kn + r$, where $0 \leqslant r < n$. Put $b = r$ and $a = n - r$. Then $a > 0$ and $b \geqslant 0$, and we see that $ak + b(k+1) = 2020$, i.e., k is visionary.

   In order to solve the problem, we need to find the cardinality of the set X. To this end, write X as the disjoint union of $X_1 = \{\lfloor \frac{2020}{n} \rfloor : 1 \leqslant n < \sqrt{2020}\}$ and $X_2 = \{\lfloor \frac{2020}{n} \rfloor : \sqrt{2020} < n \leqslant 2020\}$. (Note that 2020 is not a square.) Also, since the smallest element of X_1 is $\left\lfloor \frac{2020}{\lfloor \sqrt{2020} \rfloor} \right\rfloor = \lfloor \frac{2020}{44} \rfloor = 45$, and the largest element of X_2 is $\left\lfloor \frac{2020}{\lfloor \sqrt{2020} \rfloor + 1} \right\rfloor = \lfloor \frac{2020}{45} \rfloor = 44$, the sets X_1 and X_2 are indeed disjoint.

   Let $1 \leqslant n_1 < n_2 \leqslant \lfloor \sqrt{2020} \rfloor$. If $\lfloor \frac{2020}{n_1} \rfloor = \lfloor \frac{2020}{n_2} \rfloor$, then $0 < \frac{2020}{n_1} - \frac{2020}{n_2} < 1$, implying that $0 < n_2 - n_1 < 1$, an impossibility. This shows that X_1 has exactly $\lfloor \sqrt{2020} \rfloor = 44$ elements.

   Next, consider any n such that $1 \leqslant n \leqslant \lfloor \sqrt{2020} \rfloor$. We show that there exists a q, where $\sqrt{2020} < q \leqslant 2020$, such that $\lfloor \frac{2020}{q} \rfloor = n$. By the Division Algorithm, there exist (unique) q and r such that $2020 = qn + r$, where $0 \leqslant r < n$. Now if $q \leqslant \lfloor \sqrt{2020} \rfloor$, then $2020 = qn + r < qn + n = n(q+1) \leqslant \lfloor \sqrt{2020} \rfloor \cdot (\lfloor \sqrt{2020} \rfloor + 1) = 44 \cdot 45 = 1980$, a contradiction. So we have $\sqrt{2020} < q \leqslant 2020$. Moreover, $n = \lfloor \frac{2020}{q} \rfloor$, as $2020 = nq + r$, where $0 \leqslant r < n < q$. Finally, if $\sqrt{2020} < n \leqslant 2020$, then $1 \leqslant \frac{2020}{n} < \sqrt{2020}$, so that also $1 \leqslant \lfloor \frac{2020}{n} \rfloor < \sqrt{2020}$, i.e., all elements of X_2 lie in the interval $[1, \lfloor \sqrt{2020} \rfloor]$. This shows that $|X_2| = |X_1|$, and we conclude that $|X| = |X_1| + |X_2| = 44 + 44 = 88$.

5. Let ABC be a triangle, and let T be a point on the extension of AB beyond B, and U a point on the extension of AC beyond C, such that $BT = CU$. Moreover, let R and S be points on the extensions of AB and AC beyond A such that $AS = AT$ and $AR = AU$. Prove that R, S, T, U lie on a circle whose centre lies on the circumcircle of ABC.

   Solution 1: Consider Figure 2:

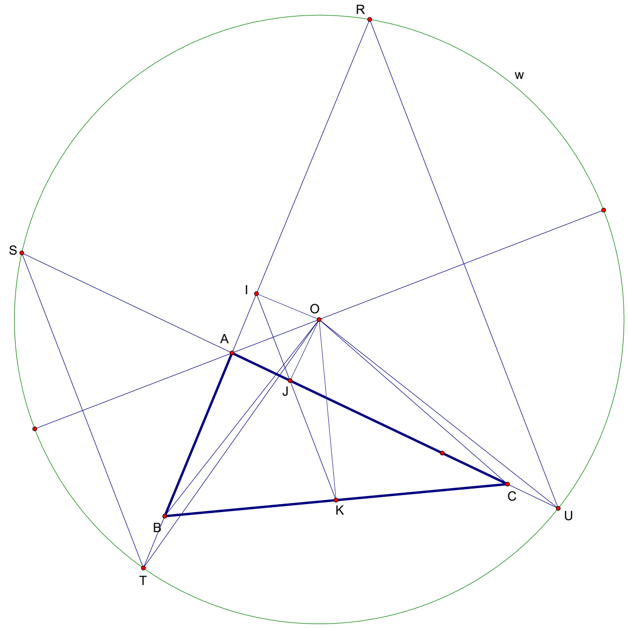

Figure 2

Since $AS = AT$, we have $\angle AST = \angle ATS = \frac{1}{2}\angle BAC$. Similarly, $\angle ARU = \angle AUR = \frac{1}{2}\angle BAC$, so that $\angle S = \angle R$. This implies that R, S, T and U are concylic.

We now show that the centre O of the circle ω through R, S, T and U, lies on the circumcircle of ABC. We know that O lies on the perpendicular bisector of ST (which is also the perpendicular bisector of RU, since $ST \parallel RU$). This perpendicular bisector forms a diameter of ω, and contains A.

If $O = A$, then we are finished, since A is certainly on the circumcircle of ABC. So suppose that the points O and A are different, as shown in Figure 2. Drop perpendiculars from O to BR (which is BA extended), to AC, and to CB. Let the feet of these perpendiculars be I, J and K, respectively. It is known that O is on the circumcircle of ABC if and only if the points I, J and K are collinear. (In case this happens, the line through I, J, K is called the *Simson line* of ABC determined by O.)

In order to show that I, J, K are collinear, it is sufficient to show that $\angle JKO = \angle IKO$. Since $\angle OJC = \angle OKC = 90°$, $OJKC$ is a cyclic quadrilateral, so that $\angle JKO = \angle JCO$. Our next observation is that $IT = JU$. This follows from the fact that $OI = OJ$ (from symmetry – recall that triangle RAU is isosceles, and AO is on the perpendicular bisector of RU), and $OT = OU$, giving $IT^2 = OT^2 - OI^2 = OU^2 - OJ^2 = JU^2$. Hence, $IB = IT - BT = JU - CU = JC$, from which we get that triangles OIB and OJC are congruent. We therefore have $\angle IBO = \angle JCO$. Finally, since $IOKB$ is a cyclic quadrilateral ($\angle BIO = \angle BKO = 90°$), we also have $\angle IBO = \angle IKO$.

Putting everything together, we conclude that $\angle JKO = \angle JCO = \angle IBO = \angle IKO$, and we are done.

Solution 2: The same strategy as in Solution 1 shows that RSTU is cyclic.

The centre O must lie on the four perpendicular bisectors of RS, ST, SU and RU. But since triangles RAU and SAT are isosceles, the perpendicular bisectors of ST and RU pass through A and bisect the angle RAU.

Now $\angle TOU = 2\angle TSU$, since O is the centre of circle RSTU, which equals $\angle TSU + \angle ATS$, since AS=AT, which in turn equals $\angle TAU$, the exterior angle in triangle AST. This proves that AOUT is cyclic and hence $\angle BTO = \angle ATO = \angle AUO = \angle CUO$.

Along with OT=OU (radii) and the given BT=CU, we conclude that triangles BTO and CUO are congruent, so $\angle BOT = \angle COU$. Finally $\angle BAC = \angle TAU = \angle TOU = \angle BOC - \angle BOT + \angle COU = \angle BOC$ and thus ABCO is cyclic.

(The problem could also be finished by noting that O lies on the perpendicular bisector of BC and the exterior angle bisector of A and thus lies on the circumcircle, but this assumes some extra knowledge that the above presentation doesn't.)

6. Marjorie is the drum major of the world's largest marching band, with more than one million members. She would like the band members to stand in a square formation. To this end, she determines the smallest integer n such that the band would fit in an $n \times n$ square and lets the members form rows of n people. However, she is dissatisfied with the result, since some empty positions remain. Therefore, she tells the entire first row to go home and repeats the process with the remaining members. Her aim is to continue it until the band forms a perfect square, but as it happens, she does not succeed until the last members are sent home. Determine the smallest possible number of members in this marching band.

   Solution: The answer is 1000977. Let M be the number of members of the marching band. We prove by induction that Marjorie's approach always yields a perfect square at some point, unless M is of the form $M = (2^a + b)^2 + 2b + 1$ (a, b nonnegative integers, $0 \le b < 2^a$) or $(2^a + b)^2 + 2^a + 3b + 2$ (a, b nonnegative integers, $0 \le b < 2^a - 1$), in which case all members are eventually sent home.

   This is true for $M = 1$ (which is not of either form), since the single member forms a 1×1 square, and for $M = 2$ (which is of the form $M = (2^a + b)^2 + 2b + 1$ with $a = b = 0$), in which case the two members form an incomplete 2×2 square and are sent home.

   For the induction step, suppose that $M > 2$ and take n to be the unique positive integer for which $(n-1)^2 < M \le n^2$. Then the M members will stand in an $n \times n$ square, and if $M \ne n^2$, then n members are sent home. We write $n - 1 = 2^a + b$, where 2^a is the greatest power of 2 less than or equal to $n - 1$, and $0 \le b < 2^a$. We claim that the process reaches a perfect square if and only if neither $M = (2^a + b)^2 + 2b + 1$ nor $M = (2^a + b) + 2^a + 3b + 2$ (the latter only for $b < 2^a - 1$).

   Suppose first that $M \le n^2 - n + 1$, so that $M - n \le (n-1)^2 = (2^a + b)^2$. By the induction hypothesis, the process never reaches a perfect square if and only if $M - n = (2^a + b - 1)^2 + 2b - 1$ or $M - n = (2^a + b - 1)^2 + 2^a + 3b - 1$. The former equation is equivalent to $M = (2^a + b - 1)^2 + 2^a + 3b$. However, this is impossible since it gives

$$M = (2^a + b - 1)^2 + 2^a + 3b = (2^a + b)^2 - (2^a - b - 1) \le (n-1)^2.$$

   The latter equation yields

$$M = (2^a + b - 1)^2 + 2^{a+1} + 4b = (2^a + b)^2 + 2b + 1,$$

which is what we wanted to prove.

Likewise, if $M > n^2 - n + 1$, then $M - n > (n-1)^2$. So by the induction hypothesis, the process never reaches a perfect square if and only if either $M - n = (2^a + b)^2 + 2b + 1$ or $M - n = (2^a + b)^2 + 2^a + 3b + 2$. In the former case, we get

$$M = (2^a + b)^2 + 2^a + 3b + 2,$$

which is exactly the desired statement. Note, however, that $M \neq n^2$ (otherwise, the band forms a perfect square immediately) requires $b < 2^a - 1$. In the latter case, we obtain

$$M = (2^a + b)^2 + 2^{a+1} + 4b + 3 = (2^a + b + 1)^2 + 2(b+1) > n^2,$$

which is impossible.

This completes the induction. Now note that $1000000 = 1000^2$ and $1000 = 2^9 + 488$, so the smallest number greater than 1000000 that is of the form $(2^a + b)^2 + 2b + 1$ or $(2^a + b)^2 + 2^a + 3b + 2$ is $(2^9 + 488)^2 + 2 \cdot 488 + 1 = 1000977$.