THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior Second Round 2012
Solutions

1. **Answer A**
   The fixed cost is R1000 and the variable cost for n CD's at R6 each is R6n, so the total cost in rands is $1000 + 6n$.

2. **Answer B**
   $(x+3)(x+a) = x^2 + (3+a)x + 3a = x^2 + bx - 12$, so $3 + a = b$ and $3a = -12$. Thus $a = -4$ and $b = -4 + 3 = -1$.

3. **Answer E**
   $\sqrt{5} \times \sqrt[3]{5} = 5^{1/2}5^{1/3} = 5^{3/6+2/6} = 5^{5/6}$.

4. **Answer C**
   If the radii of the two circles are r and $2r$, then their areas are πr^2 and $\pi(2r)^2 = 4\pi r^2$. The ratio is therefore $1 : 4$.

5. **Answer D**
   Among the numbers $1, \ldots, 6$ on the faces of the die, there are three odd numbers, three even numbers, three prime numbers (2, 3, and 5), five factors of 12 (all except 5), and four factors of 18 (all except 4 and 5). Therefore the number rolled is most likely to be a factor of 12 (with probability $\frac{5}{6}$).

6. **Answer E**
   The slope of the line $ax + 3y = 5$ is $-\frac{a}{3}$ and the slope of the line $2x + by = 3$ is $-\frac{2}{b}$. Since the lines are parallel, their slopes are equal, so $-\frac{a}{3} = -\frac{2}{b}$ and $ab = 6$.

7. **Answer D**
   $7^{99} + 7^{100} + 7^{101} = 7^{99}(1 + 7 + 7^2) = 7^{99} \times 57 = 7^{99} \times 3 \times 19$. The largest prime factor is 19.

8. **Answer B**
   Note that $B = 17^{14} > 16^{14} = (2^4)^{14} = 2^{56} = C$, so $B > C$. All the others are less than 2^{56}: firstly, $A = 31^{11} < 32^{11} = (2^5)^{11} = 2^{55}$; secondly, $D = 32^{10} = (2^5)^{10} = 2^{50}$, and finally $E = 127^8 < 128^8 = (2^7)^8 = 2^{56}$.

9. **Answer A**
   The area of the circle is π, so each side of the square is $\sqrt{\pi}$. If O is the centre of both figures and M is the midpoint of AB, then $\mathsf{OM} = \frac{1}{2}\sqrt{\pi}$, so by Pythagoras' theorem $\mathsf{AM} = \sqrt{1 - \frac{\pi}{4}}$ and $\mathsf{AB} = 2\mathsf{AM} = \sqrt{4 - \pi}$.

10. **Answer C**
    $(2012^2 - 2011^2) + (2010^2 - 2009^2) + \cdots + (4^2 - 3^2) + (2^2 - 1) = 4023 + 4019 + \cdots + 7 + 3$, using the formula for the difference of two squares. The series has 1006 terms, so its sum is $\frac{1}{2}(1006)(4023 + 3) = 1006 \times 2013 \approx 2 \times 10^6$.

11. **Answer A**
Suppose there are a girls and b boys, a muffin costs p rands and a sandwich costs q rands. We know that $ap + bq = bp + aq - 1$, so $(a - b)(p - q) = -1$. Since both factors are integers, one must be $+1$ and the other -1. We are also given that $a > b$, so $a - b = +1$, i.e., there is one more girl.

12. **Answer A**
The single digit primes are $2, 3, 5, 7$, of which we must choose two. The other two digits must be chosen from $0, 1, 4, 6, 8, 9$ so that the sum of the four digits is 19. Ignoring order, the possible choices for the four digits are:

Primes	Others	
2, 3	6, 8;	
2, 5	4, 8;	
2, 7	1, 9;	4, 6
3, 5	none	
3, 7	0, 9;	1, 8;
5, 7	1, 6;	

This gives seven choices for the four digits, and each choice can be arranged in $4 \times 3 \times 2 \times 1 = 24$ different ways, making a total of $7 \times 24 = 168$ passwords.

13. **Answer E**
$10^{99} - 1 = 999\ \dots\ 999$, with 99 digits. Subtracting another 98 gives $999\ \dots\ 901$, for which the sum of the digits is $97 \times 9 + 0 + 1 = 874$.

14. **Answer B**
Each square is half the area of the square just bigger than it (which is the other colour), so it is one-quarter of the area of the next larger square of the same colour. Taking the area of the whole figure to be 1, we see that the area of the four largest grey triangles is $\frac{1}{4}$, and each set of four grey triangles is $\frac{1}{4}$ of the area of the previous one. Thus the total grey area is $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1/4}{1-(1/4)} = \frac{1}{3}$. Alternatively, we could count grey area as positive and white area as negative. If we start with the largest grey square, which has area $\frac{1}{2}$, we get $\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \ldots = \frac{1/2}{1-(-1/2)} = \frac{1}{3}$. (In both series we used the formula $a/(1 - r)$ for the sum of an infinite geometric series with first term a and common ratio r, with $|r| < 1$.)

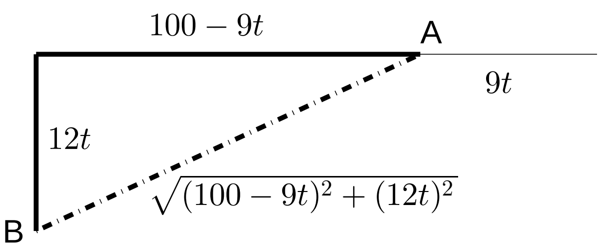
15. **Answer D**
There is a $\frac{1}{64}$ probability of choosing any one of the small cubes. Since none of the five visible faces is painted, the chosen cube either has no painted faces or has one painted face, which is out of sight (with probability $\frac{1}{6}$). There are $2 \times 2 \times 2 = 8$ small cubes with no painted faces (from the middle of the large cube), and $6 \times 4 = 24$ with one painted face (four from each of the six large faces). The probability of no painted face being visible is therefore $\frac{8}{64}.1 + \frac{24}{64}.\frac{1}{6} = \frac{3}{16}$.

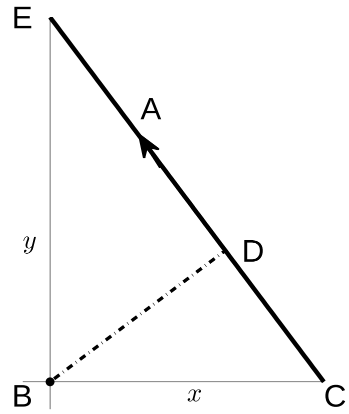
16. **Answer C**
Measured in km from the original position of boat B, after t hours boat A will be

$100 - 9t$ east and boat B will be $12t$ south (see the figure on the left, below). By Pythagoras' theorem, the square of the distance between the boats will be

$$(100-9t)^2+(12t)^2 = 225t^2-1800t+10000 = 225(t^2-8t)+10000 = 225(t-4)^2+6400.$$

This has a minimum value of 6400 when $t = 4$, so the minimum distance between the boats is $\sqrt{6400} = 80$ km.





Alternatively, for a geometrical approach one could fix the position of one of the boats, and consider the motion of the second boat relative to the first. Consider, for example, boat B at the origin of a fixed (x, y) coordinate system (see the figure on the right, above). Relative to an observer on boat B, boat A will travel along the line CE. Now $BE : BC = 12 : 9$ (the ratio of the two speeds) and therefore $BE = \frac{4}{3} \cdot 100$ and $CE = \sqrt{\frac{16}{9} \cdot 100^2 + 100^2} = \frac{5}{3} \cdot 100$. The distance is a minimum when BD is perpendicular to CE. From similar triangles $BD : BC = BE : CE$ and hence $BD = \frac{4}{3} \cdot 100^2 \div (\frac{5}{3} \cdot 100) = 80$ km.

17. **Answer D**
By the midpoint theorem $\mathsf{EG} = \frac{1}{2}$ and $\mathsf{GF} = 1$, so $\mathsf{EF} = 1\frac{1}{2}$ and $\mathsf{AC} = 3$. If the perpendicular from A to CD cuts CD extended at H, then by symmetry $\mathsf{DH} = \frac{1}{2}$, so $\mathsf{CH} = \frac{3}{2}$. By Pythagoras' theorem in triangle ACH we have $\mathsf{AH}^2 = 3^2 - (\frac{3}{2})^2 = \frac{27}{4}$. Then by Pythagoras' theorem in triangle ADH we have $\mathsf{AD}^2 = \mathsf{AH}^2 + (\frac{1}{2})^2 = \frac{27}{4} + \frac{1}{4} = 7$. Therefore $\mathsf{AD} = \sqrt{7}$. Finally, GFCD is a parallelogram (opposite sides are parallel and of equal length), so DG=CF, which equals DE, so $\mathsf{DG} = \mathsf{DE} = \frac{1}{2}\mathsf{AD} = \frac{1}{2}\sqrt{7}$. Alternatively, instead of Pythagoras' theorem, we can use Ptolemy's theorem for cyclic quadrilaterals. (Trapezium ABCD is cyclic because $\widehat{\mathsf{DAB}} + \widehat{\mathsf{DCB}} = \widehat{\mathsf{CBA}} + \widehat{\mathsf{CDA}}$, which equals $180°$, since $\mathsf{AB} \parallel \mathsf{CD}$.) Ptolemy's theorem says that the sum of the products of the two pairs of opposite sides of a cyclic quadrilateral is equal to the product of the diagonals, that is, $\mathsf{AB} \cdot \mathsf{CD} + \mathsf{AD} \cdot \mathsf{BC} = \mathsf{AC} \cdot \mathsf{BD}$. This simplifies to $2 \times 1 + \mathsf{AD}^2 = 3^2$, giving $\mathsf{AD}^2 = 7$, as before.

18. **Answer D**
Call the three-digit number A. It is helpful to write down a few factorials first: $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$ and digits larger than 6 can be excluded since $7!$ is already a four-digit number. We can also exclude 6, because if there is one 6 among the digits, then $A > 6! = 720$ and the hundreds digit is too large. The digit 5 has to be included, since $4! + 4! + 4! = 72$, which has

only two digits. Because $5! + 5! + 5! = 360$, the first digit is at most 3. Because $3! + 5! + 5! = 246$, the first digit is at most 2. Now $2! + 5! + 5! = 242 < 255$ and $2! + 4! + 5! = 146$ so the first digit is 1. Given that two of the digits are 1 and 5, we know $A = 121 + n!$, where n is the third digit. By checking $n = 0, 1, 2, 3, 4, 5$ we find only $n = 4$ works and $145 = 1! + 4! + 5!$. The sum of the three digits is therefore 10.

19. **Answer B**
Because the coefficients are symmetrical, it follows that if r is a root of the equation, then so is r^{-1}. (Divide the equation through by x^4 and you get the same equation, but with x replaced by x^{-1}.) The left hand side is therefore the product of two factors of the form $(x - r)(x - r^{-1}) = x^2 - ax + 1$, say, where $a = r + r^{-1}$. Thus we can write

$$x^4+2x^3-22x^2+2x+1 = (x^2-ax+1)(x^2-bx+1) = x^4-(a+b)x^3+(2+ab)x^2-(a+b)x+1,$$

giving $a+b = -2$ and $ab = -24$. Thus $a = -6$ and $b = 4$ (or the other way around), so the factors are $x^2 + 6x + 1$ and $x^2 - 4x + 1$. The roots are

$$x^2 + 6x + 1 = 0 \implies x = -3 \pm 2\sqrt{2}; \qquad x^2 - 4x + 1 = 0 \implies x = 2 \pm \sqrt{3}.$$

The largest root is therefore $2 + \sqrt{3}$. Alternatively, divide the equation by x^2 and group the first and last terms together, as well as the second and fourth terms

$$x^2 + \frac{1}{x^2} + 2\Big(x + \frac{1}{x}\Big) - 22 = 0 \quad\implies\quad \Big(x + \frac{1}{x}\Big)^2 + 2\Big(x + \frac{1}{x}\Big) - 24 = 0$$

where we have completed the square. Now put $t = x + x^{-1}$, then $t^2 + 2t - 24 = (t - 4)(t + 6) = 0$ which implies $t = 4$ or $t = -6$. Putting $4 = x + x^{-1}$ leads to $x^2 - 4x + 1 = 0$ as above, and putting $-6 = x + x^{-1}$ leads to $x^2 + 6x + 1 = 0$. (Both these approaches work for polynomial equations of degree 4 with a symmetric coefficient pattern.)

20. **Answer C**
First note that if $f(n) \le n$, then $f(f(n)) \le f(n)$ (because if $m < n$ then $f(m) < f(n)$). It follows that $f(f(n)) \le n$, that is, $3n \le n$, which is impossible, since n is positive. Thus $f(n) > n$. Further, $1 < f(1) < f(f(1)) = 3$, so $f(1) = 2$ and $f(2) = f(f(1)) = 3$. Then $f(3) = f(f(2)) = 6$, and in the same way we can start building up a table of values as follows.

n	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	2	3	6			9			18			

Next $6 = f(3) < f(4) < f(5) < f(6) = 9$, so $f(4) = 7$ and $f(5) = 8$. This, together with the fact that $f(f(n)) = 3n$, enables us to extend our table to

n	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	2	3	6	7	8	9	12	15	18			21

Finally, as before, $18 = f(9) < f(10) < f(11) < f(12) = f(f(7)) = 21$, so $f(10) = 19$ and $f(11) = 20$.