2019 Junior Third Round - Solutions

25 July 2019

1.

$$\begin{aligned} & 1 < (x-2)^2 < 25 \\ \implies & 1 < x-2 < 5 \quad \text{or} \quad -5 < x-2 < -1 \\ \implies & 3 < x < 7 \quad \text{or} \quad -3 < x < 1 \\ \implies & x = -2, -1, 0, 4, 5, 6. \end{aligned}$$

2. From the way the multiplication table is defined, every number in the top row must divide all the numbers in the column below it, and every number in the first column must divide all the numbers in the row to the right of it. In particular, a_1 must be a common divisor of 15 and 18, and so must be a divisor of 3, the greatest common divisor (gcd) of 15 and 18. Similarly, b_2 must be a common divisor of 15 and 40, and so must be a factor of $\gcd(15, 40) = 5$. Since $a_1 \times b_2 = 15$, the only possibility is $a_1 = 3$ and $b_2 = 5$. From $b_2 \times a_3 = 40$ we find that $a_3 = 8$, and so $a_3 \times b_5 = 56$ it follows that $b_5 = 7$. We continue in the same fashion to determine the rest of the numbers in the first row and column.

X	a_1	a_2	a_3	a_4	a_5
b_1	**A**	10		20	
b_2	15	**B**	40		
b_3	18		**C**	60	
b_4		20		**D**	24
b_5			56		**E**

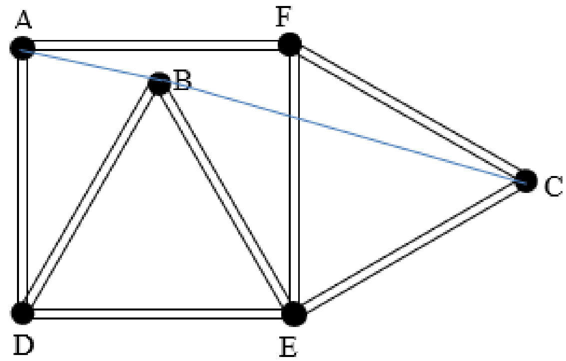
$\implies$

X	3	5	8	10	6
2	**A**	10		20	
5	15	**B**	40		
6	18		**C**	60	
4		20		**D**	24
7			56		**E**

It then follows that

$$A + B + C + D + E = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 = 6 + 25 + 48 + 40 + 42 = 161.$$

3. Since all the matchsticks are the same length, we know that the angles of triangles BDE and EFC are all $60°$. Hence $\angle ADB = 30°$ and since $AD = BD$, it follows that $\angle ABD = \angle BAD = \frac{180° - 30°}{2} = 75°$.



Similarly, $\angle BEF = 30°$, and so $\angle BEC = 30° + 60° = 90$. Since $BE = CE$, it follows that $\angle EBC = \angle ECB = 45°$. Hence $\angle ABC + \angle DBE + \angle EBC = 75° + 60° + 45° = 180°$, which shows that A, B and C lie on a straight line.

4. The maximum possible answer for $THIS + IS + HIS = 9876 + 76 + 876 = 10828$, which forces $C = 1$ and $L = 0$. Next, if $T \neq 9$, then the largest sum is $8976 + 76 + 976 = 10028$, which will force $A = 0$, contradicting $L = 0$. Hence $T = 9$.

   Next, we consider the possible values of I. Since either 0, 1 or 2 can be carried from the sum $S + S + S$, we have that the last digit of $3I$, $3I + 1$ or $3I + 2$ must be I. In the first case, this gives $I = 0$ (not possible since $L = 0$) or $I = 5$. The second case is not possible since if I is odd then $3I + 1$ is even, and vice versa. In the third case $I = 4$ or $I = 9$ (not possible, since $T = 9$). This leaves $I = 4$ or $I = 5$.

   If $I = 4$, then it means that the sum $S + S + S$ carries a 2 across, that is, $3S >= 20$, so $S = 7$ or $S = 8$ ($S = 9$ is not possible since $T = 9$). However, if $S = 7$ then $M = 1 = C$, a contradiction, and if $S = 8$, then $M = 4 = I$, a contradiction. So this case is not possible and hence $I = 5$.

   This means that $S + S + S < 10$ (otherwise it would carry a 1), so $S = 2$ (if $S = 3$, then $M = 9 = T$, which is not possible, and $S \neq 1$ since $C = 1$) and $M = 6$.

   The sum $I + I + I$ carries a 1, and so $2H + 1 = 10 + A$, which is an odd number. Since $C = 1$, $I = 5$ and $T = 9$, this leaves only the possibilities $A = 3$ (in which case $H = 6 = M$, a contradiction) and $A = 7$ (in which case $H = 8$).

   The sum is thus $9852 + 52 + 852 = 10756$.

5. From right triangle APD we know that $PD^2 + AP^2 = AD^2 = 17$, and from the hint we know that $AP^2 + 8^2 = PD^2 + 7^2$, which simplifies to $PD^2 = AP^2 + 8^2 - 7^2 = AP^2 + 15$. Substituting this into the first equation, we get

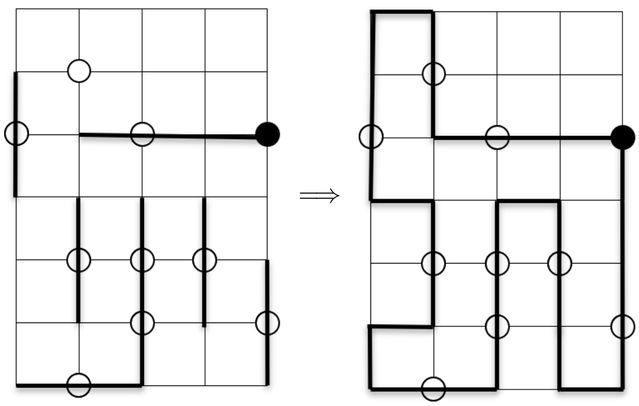
   $$AP^2 + 15 + AP^2 = 17 = \Longrightarrow \; 2AP^2 = 2 \implies AP = 1,$$

   which gives $PD^2 = AP^2 + 15 = 16$ and so $PD = 4$. Since $\angle APD$ is a right angle, it follows that the area of triangle APD equals $\frac{1}{2} \cdot AP \cdot PD = \frac{1 \cdot 4}{2} = 2$.

6. Consider the three adjacent white circles (third row from the bottom): the loop must pass straight through all three circles, but if the segments were horizontal, it will have to pass right across the grid, which would violate condition (c) for the middle circle. Hence three vertical segments pass through those three white circles.

   Next, consider the four circles on the edge of the grid - there is only one way the loop can pass through the white circles, and at the black circle, the loop has to turn towards the inside of the grid (that is, a horizontal segment).

   From this starting position, it's easy to complete the rest of the loop.



7. Note that $84 = 3 \times 4 \times 7$.

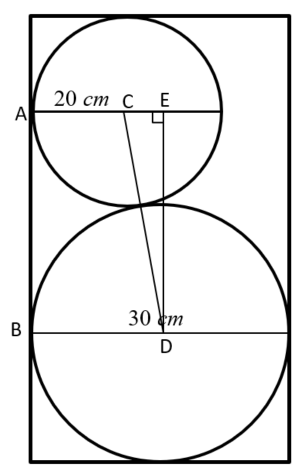
   (a) Since 84 is divisible by 4, the last two digits of N must be divisible by 4. Thus N must end in a 6 (to be even) and of 66 and 76, only 76 is divisible by 4.

   (b) Since 84 is divisible by 3, N must also be divisible by 3, and so the sum of the digits of N must be divisible by 3. Suppose that N has a digits 6 and b digits 7. Then the sum of the digits of N is $6a + 7b$, and because $6a$ is already divisible by 4, $7a$ must also be divisible by 3, and so a is divisible by 3.

   (c) By part (a), N contains at least one 7, and so by part (b) must contain at least three digits 7. As long as N satisfies conditions (a) and (b) we guarantee that it is divisible by 3 and 4, so to be divisible by 84, we only need to check whether it is divisible by 7.

   The smallest number satisfying parts (a) and (b) is 7776, which is not divisible by 7. The next smallest number is 67776, which isn't divisible by 7. The next smallest number is 76776, which is divisble by 7, and so $N = 76776$.

8. (a) Connect the centres of the pizzas and drop a perpendicular from the centre of the one pizza, as shown.



Then $AC = 10$cm, $AE = BD = 15$cm and $CD = 10\text{cm}+15\text{cm} = 25$cm. By Pythagoras,

$$DE^2 = CD^2 - CE^2 = 25^2 - (15 - 10)^2 = 625 - 25 = 600,$$

and so $DE = \sqrt{600} = 10\sqrt{6}$. Hence the length of the tray is equal to $10 + 10\sqrt{6} + 15 = 25 + 10\sqrt{6}$.

(b) Suppose the round pizza has radius r; then the square pizza has side length $2r$, and their areas are πr^2 and $(2r)^2 = 4r^2$, respectively. The ratio of the area of the square pizza to that of the round pizza equals $\frac{4r^2}{\pi r^2} = \frac{4}{\pi}$. Now, $\pi \approx 3.1415 > 3.1$ and so

$$1.3\pi > 1.3 \times 3.1 = 4.03 > 4 \implies \frac{4}{\pi} < 1.3.$$

This means that the 30% figure is an exaggeration.

9. (a) Note that if you are given 7 different digits, there is only one way to arrange them in increasing order, so we only need to count the number of ways one can choose 7 different digits from the 9 available (0 is not allowed, since it would necessarily be a leading zero). This is the same as choosing two digits to leave out: for the first digit to leave out, there are 9 possibilities, and for the second digit, there are 8 remaining possibilities, so $9 \times 8 = 72$ in total. However, since the order in which we choose the two digits doesn't matter, we have to divide by 2 to give a total of 36 numbers.

   (b) Since the numbers are 7-digit numbers whose digits increase from left to right, the smallest such number is 1234567, which has a 6 in the hundreds position. Thus no such number can have a 5 (or smaller) in the hundreds position, so the probability is zero.

10. Let $2n-1$ and $2n+1$ be two consecutive numbers (for $n \geq 1$). Then

$$\frac{1}{2n-1} + \frac{1}{2n+1} = \frac{2n+1+2n-1}{(2n-1)(2n+1)} = \frac{4n}{4n^2-1},$$

and

$$(4n)^2 + (4n^2-1)^2 = 16n^2 + 16n^4 - 8n^2 + 1 = 16n^4 + 8n^2 + 1 = (4n^2+1)^2,$$

which is a perfect square.

11. (a) Since E and F are the midpoints of sides AB and BC in triangle ABC, it follows by the midpoint rule that EF is parallel to AC. A similar argument in triangle ADC shows that GH is parallel to AC, and so GH and EF are parallel. In the same way, EH and FG are parallel, which shows that $EFGH$ is a parallelogram.

   (b) The midpoint rule in triangle ABD implies that $EH = \frac{1}{2}BD$, and so the sides of triangle AEH are all half the sides of triangle ABD. This means that the perpendicular height of triangle AEH is also half that of triangle ABD, which means that the area of triangle AEH is a quarter of triangle ABD. Applying the same argument in triangle CBD shows that the area of triangle GFC is a quarter that of triangle BCD. Hence, the sum of the areas of triangles AEH and CFG is a quarter of the sum of the areas of triangles ABD and CBD, which is the area of $ABCD$. Exactly the same argument shows that the sum of the areas of triangles BEF and DHG is a quarter of area $ABCD$, and so that sum of the four smaller triangles equals two quarters, or a half, of $ABCD$. Hence $EFGH$ accounts for the other half.

   (c) By part (b), the area of the trapezium equals $2 \times 64 = 128$. Using the formula for the area of a trapezium, we find that

$$128 = \frac{a + 4\sqrt{2}}{2} \cdot 8\sqrt{2} = 4\sqrt{2}a + 32 \implies 4\sqrt{2}a = 96 \implies a = 12\sqrt{2}.$$

   Let K be the foot of the perpendicular from D to AB. Then

$$AK = AB - CD = 12\sqrt{2} - 4\sqrt{2} = 8\sqrt{2},$$

   and using Pythagoras in triangle AKD we find that

$$b^2 = AK^2 + KD^2 = AK^2 + BC^2 = (8\sqrt{2})^2 + (8\sqrt{2})^2 = 256,$$

   which implies that $b = 16$.

12. (a) If $\left\lfloor \frac{x}{20} \right\rfloor = 5$ it means that $5 \leq \frac{x}{20} < 6$ and so $100 \leq x < 120$. Since x is an integer, it follows that $100 \leq x \leq 119$.

    (b) Similar to part (a), $\left\lfloor \frac{x}{20} \right\rfloor = m$ leads to $20m \leq x \leq 20m + 19$, and $\left\lfloor \frac{x}{19} \right\rfloor = m$ leads to $19m \leq x \leq 19m+18$. Note that since m is non-negative, it follows that $19m \leq 20m$ and $19m+18 < 20m+19$, so if x satisfies both equations, it follows that $20m \leq x \leq 19m+18$. This only has a solution if $20m \leq 19m + 18 \implies m \leq 18$, in which case there are exactly $(19m+18) - (20m) + 1 = 19 - m$ integers between $20m$ and $19m+18$ inclusive.

    (c) To count all the solutions to the equation $\left\lfloor \frac{x}{20} \right\rfloor = \left\lfloor \frac{x}{19} \right\rfloor$ is the same as counting the solutions to $\left\lfloor \frac{x}{20} \right\rfloor = \left\lfloor \frac{x}{19} \right\rfloor = m$ for all m. By part (b), we only need to consider $0 \leq m \leq 18$, and for each such m there are $19 - m$ solutions. Adding them all up from $m = 0$ to $m = 18$ gives $19 + 18 + \cdots + 1 = \frac{19 \cdot 20}{2} = 190$ solutions.

13. (a) Arithmetic Mean $= \frac{8+18}{2} = 13$. Geometric Mean $= \sqrt{8 \cdot 18} = 18$. Harmonic Mean $= \frac{2}{\frac{1}{8}+\frac{1}{18}} = \frac{2\cdot 8\cdot 18}{8+18} = \frac{144}{13}$.

    (b) If a and b are positive, $\sqrt{a}$ and $\sqrt{b}$ are defined, and

$$(\sqrt{a} - \sqrt{b})^2 \geq 0 \implies (\sqrt{a})^2 - 2\sqrt{ab} + (\sqrt{b})^2 \geq 0 \implies a + b \geq 2\sqrt{ab} \implies \frac{a+b}{2} \geq \sqrt{ab}.$$

    (c)

$$\frac{a+b}{2} \geq \sqrt{ab} \implies \frac{a+b}{ab} \geq \frac{2\sqrt{ab}}{ab} \implies \frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}} \implies \sqrt{ab} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}}.$$

14. The first player can always win by placing matches in the following way on their first move.

    «Diagram»

    This places all possible vertical matches, and hence from this point onwards, only horizontal matches can be placed. This means that during a player's turn, matches can either be placed above the centre line of the grid, or below it, but not both. Note that after the first move, the grid is symmetric when rotated through 180° around the centre, so whatever move the second player makes, the first player can make the same move rotated through 180°. (This move is always possible, since the two moves must necessarily be on opposite sides of the grid's centre line, which guarantees that the two moves cannot possibly overlap.) If the first player continues in this way, the grid will always again be rotationally symmetric after their turn, and so guarantee that they will be able to play at again. This means that player 2 can never place the last match, which means that player 1 will win.

15. (a)

$$\phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1 + 2\sqrt{5} + 5}{4} = \frac{6 + 2\sqrt{5}}{4} = \frac{3 + \sqrt{5}}{2} = \frac{1 + \sqrt{5} + 2}{2} = \frac{1+\sqrt{5}}{2} + 1 = \phi + 1.$$

    (b) The interior angles of a pentagon are equal to $180° - \frac{360°}{5} = 180° - 72° = 108°$. Now, since $AE = AB$, it follows that $\angle AEB = ABE = \frac{180° - 108°}{2} = 36°$. Similarly, $\angle FAB = 36° = \angle EAG$, which means that $\angle GAF = 108° - 2 \cdot 36° = 36°$. Hence $\angle GAB = 36° + 36° = 72°$ and $\angle AGB = \angle GAE + \angle GEA = 72°$, which implies that $GB = AB = a$. Also, since the figure is symmetric, $EG = BF = b$ and so $EB = EG + GB = b + a$.

    Next, in triangles AFB and EAB, $\angle FAB = 36° = \angle AEB$ and $\angle ABF = \angle EBA$ is common, it follows that triangles AFB and EAB are similar. Hence $\frac{AB}{BF} = \frac{EB}{BA}$ and so $\frac{a}{b} = \frac{a+b}{a} = 1 + \frac{b}{a}$. Let $x = \frac{a}{b}$. Then $x = 1 + \frac{1}{x} \implies x^2 = x + 1$. We know from part (a) that $\phi^2 = \phi + 1$, so $x = \phi$ is a solution. We need to make sure it's the only solution.

Suppose that $y \neq \phi$ also satisfies $y^2 = y + 1$. Then $\phi^2 - \phi = 1 = y^2 - y$ and so

$$\phi^2 - \phi = y^2 - y \implies \phi^2 - y^2 = \phi - y \implies (\phi - y)(\phi + y) = \phi - y \implies \phi + y = 1$$

since we assumed that $y \neq \phi$. Hence $y = 1 - \phi = \frac{1-\sqrt{5}}{2}$, but since $\sqrt{5} > 2$, it follows that $y < 0$. However, since the ratio $x = \frac{a}{b}$ is clearly positive, it follows that $x = \phi$ is the only solution.