THE SOUTH AFRICAN
MATHEMATICS OLYMPIAD

organised by the SOUTH AFRICAN ACADEMY OF SCIENCE AND ARTS
in collaboration with OLD MUTUAL, AMESA and SAMS

SPONSORED BY OLD MUTUAL

FIRST ROUND 1998

SENIOR SECTION: GRADES 10, 11 AND 12
(STANDARDS 8, 9 AND 10)

10 MARCH 1998

TIME: 60 MINUTES

NUMBER OF QUESTIONS: 20

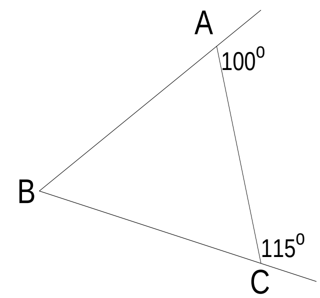
ANSWERS

1. E
2. C
3. E
4. B
5. C
6. C
7. B
8. A
9. C
10. B
11. B
12. D
13. C
14. D
15. D
16. E
17. D
18. B
19. D
20. B

SOLUTIONS

1. $\frac{1\times9\times9\times8}{1+9+9+8} = \frac{9\times9\times8}{27} = 24.$

2. $\widehat{BAC} = 180^o - 100^o = 80^o$. Therefore, $\widehat{ABC} = 115^o - 80^o = 35^o$.



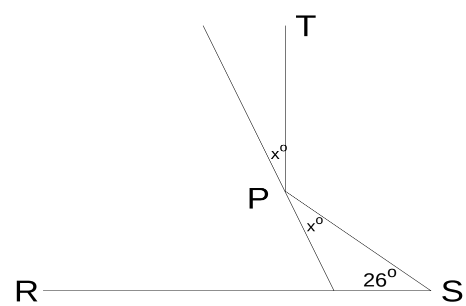
3. Squaring both sides of the equation $\sqrt{x+1} = 3$, we obtain $x + 1 = 9$. Then $x + 2 = 10$ and $(x + 2)^2 = 100$.

4. $18 - 10 = 8$.

5. We certainly obtain the largest number of Mondays if the first day is a Monday, and also the 7th, the 14th, etc, day after that. So we need to have Monday fall on the 1st, 8th 15th,..., 43rd day. A total of 7 days.

6. The ball travels 144 000 metres in 60x60 seconds. Therefore the speed in metres per second is $\frac{144\ 000}{60\times60} = 40$. The ball takes $\frac{20}{40} = \frac{1}{2}$ seconds to reach the bat.

7. $\widehat{QPU} = x^o$. In $\triangle QPS$, $2x^o + 26^o = 90^o$. Therefore $2x = 64$, or $x = 32^o$.

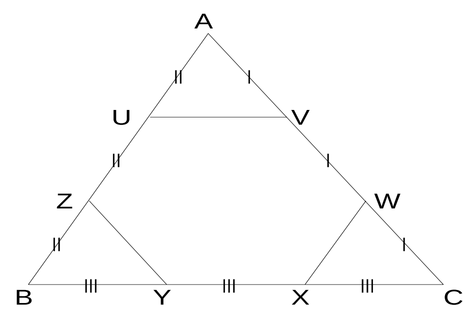


8. Add the three given equations: $(a + b) + (b + c) + (c + a) = 3 + 4 + 5$. Therefore $2(a + b + c) = 12$ and $a + b + c = 6$.

9. Let the lengths of the sides, in centimetres, be $2x, 3x$ and $5x$. Then $(2x)(3x)(5x) = 6\ 480$. It follows that $30x^3 = 6\ 480$ and $x^3 = 6\ 480/30 = 216 = 6^3$. Therefore $x = 6$cm, and the length of the shortest side is $2x = 12$.

10. Note that there is only route to get to each of A, B, C, D and H. There are 2 routes to E, 3 routes to I and F, 6 routes to J, 4 routes to G and finally, 10 routes to CC.

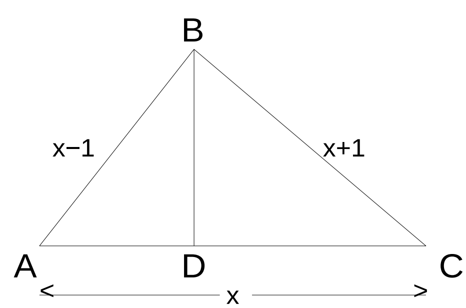
11. The area of $\triangle AUV$ is $\frac{1}{2}(AU) \times$ (the height of $\triangle AUV$). But AU is one third of AB, and the height of $\triangle AUV$ is one third the height of $\triangle ABC$. Therefore the area of $\triangle AUV$ is $(\frac{1}{3})^2 = \frac{1}{9}$ of the total area of $\triangle ABC$. Similarly the areas of $\triangle BZY$ and $\triangle CXW$ are $\frac{1}{9}$ of the area of $\triangle ABC$. By removing the three triangles we obtain the area of the hexagon $UVWXYZ$ to be $(1 - \frac{3}{9}) = \frac{2}{3}$ of $\triangle ABC$.



12. $1 = (\frac{1}{p} + \frac{1}{q}) + \frac{1}{r} = \frac{p+q}{pq} + \frac{1}{r} = \frac{p+q}{s} + \frac{1}{r}$. Hence, $\frac{p+q}{s} = 1 - \frac{1}{r} = \frac{1-r}{r}$, and $p + q = s(\frac{r-1}{r})$.

13. $2^x = 41 - 3^y$ where 2^x has to be positive. Therefore $41 - 3^y$ has to be positive. This only happens for $y = 1, 2$ and 3.
If $y = 1$, $2^x = 41 - 3 = 38$ which is not a power of 2.
If $y = 2$, $2^x = 41 - 9 = 32 = 2^5$, i.e. $x = 5$.
If $y = 3$, $2^x = 41 - 27 = 14$ which is also not a power of two.
Therefore $y = 2$, $x = 5$, and $x + y = 7$.

14. $BD^2 = (x-1)^2 - AD^2$ and $BD^2 = (x+1)^2 - DC^2$. Therefore $DC^2 - AD^2 = (x+1)^2 - (x-1)^2 = 4x$. But $DC^2 - AD^2 = (DC - AD)(DC + AD) = (DC - AD)x$, since $AC = x = AD + DC$. So we have, $(DC - AD)x = 4x$ and $DC - AD = 4$.

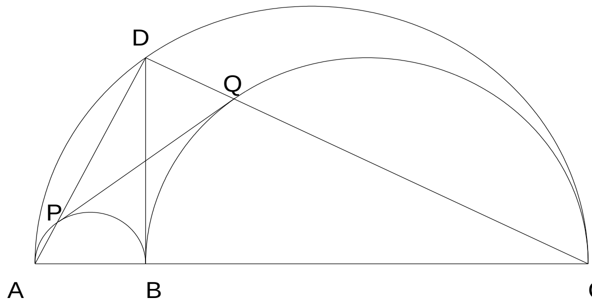


15. Let the dimensions of the rectangle be m and n, where m and n are both positive integers. We are given that $mn = 4(m+n)$. This can be rearranged into $mn - 4m = 4n$, hence $m(n-4) = 4n$. This tells us that n is larger than 4 (otherwise the left hand side is negative and the right hand side is positive). Solving for m,

$$m = \frac{4n}{n-4} = 4 + \frac{16}{n-4},$$

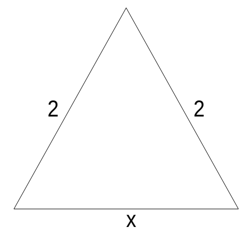
when divided out. Next we use the fact that both m and n are integers. It follows that $n - 4$ must be an integer that divides exactly into 16. So $n - 4$ can only be 1, 2, 4, 8 or 16, which give $n = 5, 6, 8, 12$ or 20 and $m = 20, 12, 8, 6$ or 5. The only 3 rectangles are 20×5, 12×6 and 8×8.
Note: A square is a special case of a rectangle.

16. **Method 1**: $\widehat{ADC}$ is an angle in a semicircle. Therefore it is 90^o. Join PB and QB. Then $\widehat{APB}$ and $\widehat{BQC}$ are also right angles. Therefore $PDQB$ is a rectangle and the diagonal PQ equals the diagonal BD. So PQ is also 10.

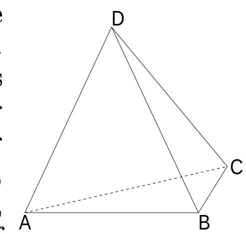


Method 2: Notice that you are asked to choose between 4 numbers and "None of these". So the implication is that the answer is a number. Next notice that we have not been given the radii of the 3 semicircles. This implies that the length of PQ does not depend on these radii. Let us therefore choose the diameters AB and BC to be equal. Then $PQ = 10$.

17. The area of $\triangle ABC$ is $\frac{1}{2}$(base CB) $\times$ (vertical height). The base length is fixed (at the value 2) but the vertical height will be largest when $\widehat{ABC} = 90^o$, then $AC^2 = AB^2 + BC^2 = 4 + 4 = 8$. Therefore $AC = 2\sqrt{2}$.



18. Remember that in a triangle the sum of the lengths of 2 sides of the triangle must exceed the length of the third side. We call that the *triangle inequality.* If AB has length 41 then one side of $\triangle ABC$ and one side of $\triangle ABD$ has length 41. So in each of these 2 triangles the sum of the lengths of the other 2 sides must exceed 41. There are 2 possible sets of 2 pairs of lengths for the other 2 sides of $\triangle ABC$ and $\triangle ABD$: 18, 27 and 13, 36, or 18, 27 and 7, 36. Each of these pairs contains 18 and 27. So let $\triangle ABC$ have lengths 41, 18 ($= AC$) and 27 ($= BC$). If $\triangle ABD$ has lengths 13 and 36 then $DC = 7$. If we let $AB = 13$ and $BD = 36$ then $\triangle BCD$ has sides with lengths 36, 27, and 7 which is impossible because $27 + 7 < 36$. If we let $AD = 36$ and $BD = 13$ then $\triangle BCD$ has sides 27, 13 and 7 which is also impossible since $13 + 7 < 27$. So $\triangle ABD$ must have lengths 41, 36 and 7, so that $CD = 13$. You can check that $AD = 7, BD = 36, AC = 18, BC = 27$ and $DC = 13$ works.



19. **Method 1**:

$$\begin{aligned} x^4 + 6x^2 + 25 &= (x^4 + 10x^2 + 25) - 4x^2 \\ &= (x^2 + 5)^2 - (2x)^2 \\ &= (x^2 + 5 - 2x)(x^2 + 5 + 2x). \end{aligned}$$

Therefore $p(x)$ is either $x^2 - 2x + 5$ or $x^2 + 2x + 5$. By using long division we find that only $x^2 - 2x + 5$ is a factor of $3x^4 + 4x^2 + 28x + 5$. (The factorization is $(x^2 - 2x + 5)(3x^2 + 6x + 1)$). Therefore $p(x) = x^2 - 2x + 5$ and $p(1) = 4$.

Method 2: Notice that neither of the 2 given quantities has a term in x^3. So if $p(x)$ divides into both of them then $p(x)$ also divides into

$$\begin{aligned} 3x^4 + 4x^2 + 28x + 5 - 3(x^4 + 6x^2 + 25) &= -14x^2 + 28x - 70 \\ &= -14(x^2 - 2x + 5), \end{aligned}$$

which is a quadratic. So again $p(x) = x^2 - 2x + 5$ and $p(1) = 4$.

20.

	Hours	Minutes
	x	y
	z	x
Add	y	z

In the minutes column either $x + y = z$ or, if $x + y \geq 60$, then $x + y = z + 60$. Then in the hours column, if $x + y = z$ then $x + z = y$. Adding these two equations gives us that $x = 0$. Or, if $x + y = z + 60$ then in the hours column $x + z + 1 = y$. Again adding the last 2 equations we get $2x + 1 = 60$ which does not give us integer solutions. So the only solution is $x = 0$. There is only one possible value of x.