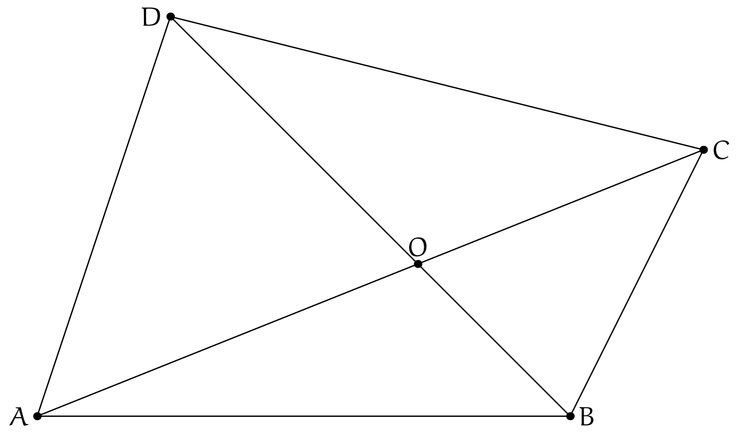
1.



Since $A\widehat{O}D + C\widehat{O}D = 180°$, we have without loss of generality $B\widehat{O}C = A\widehat{O}D = \theta \leqslant 90°$ and $A\widehat{O}B = C\widehat{O}D = 180° - \theta \geqslant 90°$. By the Pythagoras inequality, we therefore have:

$$AO^2 + DO^2 \geqslant AD^2, \quad BO^2 + CO^2 \geqslant BC^2, \quad AO^2 + BO^2 \leqslant AB^2, \quad CO^2 + DO^2 \leqslant CD^2,$$

in all cases with equality if and only if $\theta = 90°$.

Therefore

$$AD^2 + BC^2 \leqslant (AO^2 + DO^2) + (BO^2 + CO^2) = (AO^2 + BO^2) + (CO^2 + DO^2) \leqslant AB^2 + CD^2,$$

with equality if and only if $\theta = 90°$. Since $AD^2 + BC^2 = AB^2 + CD^2$ is given, it follows that $AC \perp BD$.

Solution by Ingrid von Glehn.

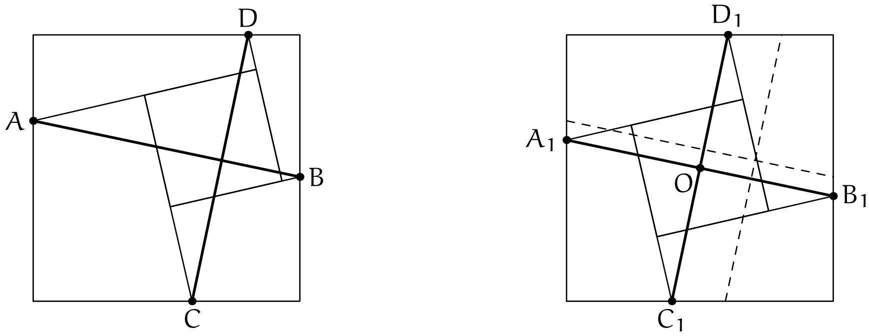
2. Let the numbers be a, ar, ar^2 with $r = x/y$ and x and y coprime. Since $a(x/y)^2$ is an integer, $a = ky^2$ for some integer k. Therefore $k(y^2+xy+x^2) = 111 = 3\times 37$, and (y^2+xy+x^2) is one of 3, 37 or 111.

Here is a table of values of $(y^2 + xy + x^2)$ for $y \leqslant x$, with entries over 111 omitted.

	1	2	3	4	5	6	7	8	9	10
1	**3**	7	13	21	31	43	57	73	91	**111**
2		12	19	28	39	52	67	84	103	
3			27	**37**	49	63	79	97		
4				48	61	76	93			
5					75	91				
6						108				

We get the solutions $x = y = 1,\ k = 37;\quad x = 10,\ y = 1,\ k = 1;\quad x = 4,\ y = 3\ k = 3$. Two more solutions are obtained by swapping x and y. So the solution triples are $(37, 37, 37)$, $(1, 10, 100)$, $(100, 10, 1)$, $(27, 36, 48)$, $(48, 36, 27)$.

3.



Move the small square so that its sides remain parallel to their original position until its centre coincides with that of the big one. Then AB and CD move to A_1B_1 and C_1D_1, where $AB \parallel A_1B_1$ and $CD \parallel C_1D_1$. The angle between A_1B_1 and C_1D_1 is the same as the angle between AB and CD; and $A_1B_1 = AB$ and $C_1D_1 = CD$ since ABB_1A_1 and CDD_1C_1 are parallelograms. It therefore suffices to show that $A_1B_1 = C_1D_1$ and $A_1B_1 \perp C_1D_1$.

Imagine that you now rotate everything by $90°$ around that common centre O. Both squares are superimposed on their old positions, and therefore A_1 and B_1 are superimposed on C_1 and D_1 respectively. In other words, C_1D_1 is just A_1B_1 rotated by $90°$ around O. But that implies $A_1B_1 = C_1D_1$ and $A_1B_1 \perp C_1D_1$.

4. First count the number of ways to factorize $1\,000\,000 = abc$ if (a, b, c) is regarded as different from (a, c, b) etc. and the factors are allowed to equal 1 — call this the "rough" count. Note that $1\,000\,000 = 2^6 5^6$. It is well known that there are $\binom{n+k-1}{k-1}$ ways to distribute n identical objects into k distinct piles,[1] so there are $\binom{8}{2} = 28$ ways to distribute the 2's among the three factors and similarly for the 5's. The rough count therefore gives $28^2 = 784$ ways.

Next eliminate duplicates.

Case $a = b = c$: This possibility is unique, and was counted once.

Case $a = b \neq c$: c must be of the form $2^m 3^n$, with $0 \leqslant m, n \leqslant 3$. There are 16 such cases, but one of these has $a = b = c$. So there remain 15 distinct cases, which were each counted three times in the rough count.

Case $a \neq b \neq c \neq a$: There are $784 - 1 - 15 \times 3 = 738$ items left in the rough count, representing items of this type that were counted six times. So there are only $738/6 = 123$ distinct cases here.

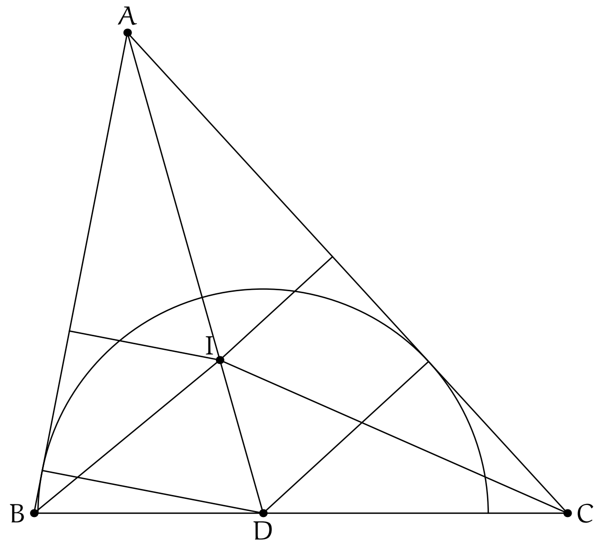
We are left with $1 + 15 + 123 = 139$ distinct cases. Finally, eliminate the cases with $a = 1$, i.e. $bc = 2^6 5^6$. There are $\binom{7}{1} = 7$ ways to distribute the $2's$ and similarly for the 5's, giving 49 cases. One of these has $b = c$, but the other 48 cases consist of 24 cases that have been counted twice. So the total number of cases containing a 1 is 24+1=25.

The solution is $139 - 25 = 114$ cases.

Solution by Jon Smit.

[1] To see this for yourself, imagine the objects arranged in a line with $(k-1)$ neutral objects among them, acting as separators, and count the number of ways to place the separators.

5.



Let I be the incentre and D the centre of the semicircle on BC. Then the length of the altitudes from I and D to to AB are respectively r and r_a, and the same applies to the altitudes to AC. Hence

$$2[ABD] = cr_a, \quad 2[ACD] = br_a, \quad 2[ABC] = (b+c)r_a$$

and

$$2[IAB] = cr, \quad 2[IAC] = br.$$

Similarly,

$$2[ABC] = (c+a)r_b, \quad 2[ABC] = (a+b)r_c, \quad 2[IBC] = ar,$$

which gives $2[ABC] = (a+b+c)r$. Therefore

$$(b+c)r_a = (c+a)r_b = (a+b)r_c = (a+b+c)r.$$

This gives

$$(a+b+c)\left(\frac{r}{r_a} + \frac{r}{r_b} + \frac{r}{r_c}\right) = (a+b) + (b+c) + (c+a) = 2(a+b+c),$$

and the desired conclusion follows upon dividing by $(a+b+c)r$.

Problem by Bruce Merry

6. Die vergelykings kan geskryf word as:

$$(2a-3b+7)(4a+b-5) + (5c+2d-7)(c+8d+1) = 0$$
$$(3a-b+1)(5a+3b-11) + (7c-d-11)(3c+5d-3) = 0.$$

[Dirk Basson en Garrick Orchard het hierdie formules gevind.]

Stel nou $u = 2a-3b+7,\ v = 3a-b+1,\ x = c+8d+1, y = 3c+5d-3$. Dit is duidelik dat u, v, x, y rasionaal moet wees. Die vergelykings word:

$$u(2v-u) + x(2y-x) = 0$$
$$v(3v-2u) + y(3y-2x) = 0.$$

Tel bymekaar:

$$3v^2 - u^2 + 3y^2 - x^2 = 0 \quad \implies 3(y^2 + v^2) = u^2 + x^2.$$

As u en x nie albei 0 is nie, is y en v ook nie albei nul nie, en ons kan met die KGV van die noemers vermeningvuldig (of, as dit 1 is, met die GGD van die tellers deel) om 'n "minimale" heeltallige oplossing te verkry waarin $\mathrm{GGD}(u, v, x, y) = 1$. Maar $u^2 + x^2$ is deelbaar deur 3, wat slegs moontlik is wanneer u en x albei deelbaar is deur 3, sê $u = 3p, v = 3q$. Dit gee $y^2 + v^2 = 3(p^2 + q^2)$, en volgens dieselfde argument is y en v albei deelbaar deur 3, wat strydig met die bestaan van 'n minimale heeltallige oplossing is. Daar bestaan dus nie 'n nie-nul rasionale oplossing nie.

Die enigste oplossing kom voor wanneer $u = x = v = y = 0$, naamlik

$$a = \frac{4}{7},\ b = \frac{19}{7},\ c = \frac{29}{19},\ d = -\frac{6}{19}.$$