The South African Mathematical Olympiad

Junior Third Round 2016

Solutions

1. (a) 2112, since the next smaller palindrome is 2002, which is less than 2016.
   (b) 2002, since 2112 is the next palindrome.

2. 6123 is the smallest number starting with a '6' and 3621 is the largest number starting with a '3'. The difference between these is $6123 - 3621 = 2502$. Adjacent numbers between 1236 and 3621 will be less than 2500 apart, while adjacent numbers larger than 6123 will be less than 1000 apart. Thus 2502.

3. Let the side of the square be x, then the square has area x^2, but x is also the height of the triangle. Since the area of the square and the area of the triangle are equal, we have

$$x^2 = \frac{1}{2}(16)x = 8x,$$

which means $x = 8$. Thus the area of the square is 64.

4. Label the 4×4 grid as shown:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Either 12 or 16 must be an A, since A must be the first seen from the bottom direction and there is exactly one empty cell in each column. Similarly, either 4 or 3 must be an A. However, since the 12 or 16 is in the same column as 4 the A cannot be in 4 and A must be in 3. Similarly, since either 9 or 13 is a B we have that B cannot be in 1. Hence 1 is a C and 2 is a B. Since 4 is empty, 16 is an A. Similarly, since either 9 or 10 is a B, 12 cannot be a B and hence 12 is a C and therefore 8 is a B. The rest of the grid can be completed in a similar fashion. The completed grid is:

C	B	A	
A		C	B
B	A		C
	C	B	A

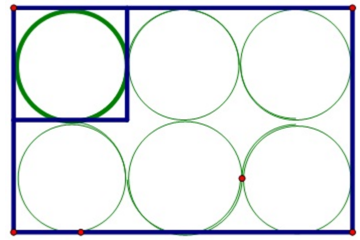
5. The bottom gives $4 \times 4 = 16$. From the top view, all the top surfaces are in a 4×4 square so also 16. From a side there are 10 cubes of 1×1 and hence the 4 sides contribute $4 \times 10 = 40$. Thus $16 + 16 + 40 = 72$.

6. Let there be x R1 coins and y R5 coins.
Hence $4x = 9y$ and $0 < x + 5y < 50$.
So $x = \frac{9y}{4}$ and we have

$$\begin{aligned}
& 0 < x + 5y < 50 \\
\Leftrightarrow \quad & 0 < \frac{9y}{4} + 5y < 50 \\
\Leftrightarrow \quad & 0 < 9y + 20y < 200 \\
\Leftrightarrow \quad & 0 < 29y < 200 \\
\Leftrightarrow \quad & 0 < y < \frac{200}{29} = 6\frac{26}{29}.
\end{aligned}$$

Furthermore, y must be divisible by 4, since 9 is odd. Thus $y = 4$. Hence $x = 9$.
A check gives $9 \times 1 + 4 \times 5 = 29$. Hence you have R29.

7. Consider the square in the diagram alongside. It is made up of four congruent shapes, each having area x say, and a circle having area 4π, so $4^2 = 4\pi + 4x$, hence $4x = 16 - 4\pi$. Our region is made up 2 circles and 8 shapes each having area x. Thus region has area $8\pi + 8x = 8\pi + 2(16 - 4\pi) = 32$.



8. Suppose Cee tells the truth, then Bee said that only one of the three tells the truth, so Bee can't tell the truth or otherwise there will be at least two truth tellers which would be a contradiction. Hence, if Cee tells the truth Bee lies and there must be more than one truth teller, so Vee must be telling the truth. However, Vee says Cee lies contradicting our assumption. Hence Cee can't tell the truth and lies.

   We don't know what Bee said, so we don't know if he tells the truth or lies. Thus Cee lies, Vee tells the truth and we don't know about Bee.

9.
$$\begin{aligned}\frac{n}{3} + \frac{n^2}{2} + \frac{n^3}{6} &= \frac{2n + 3n^2 + n^3}{6} \\ &= \frac{n(2 + 3n + n^2)}{6} \\ &= \frac{n(n+1)(n+2)}{6}\end{aligned}$$

   Since n, $n+1$, $n+2$ are three consecutive integers at least one is divisible by 2 and one is divisible by 3. Hence $n(n+1)(n+2)$ is divisible by 6 and therefore $\frac{n}{3} + \frac{n^2}{2} + \frac{n^3}{6}$ must be an integer.

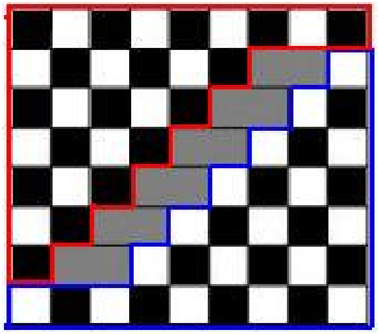
10. Since the game must end (finite number of blocks and can't play on a previously occupied block) and only one person can move last, there will be a winner and hence a winning strategy.

    We claim that the first player (Zola) has the winning strategy and it is to always move clockwise. There are 7 positions left in the outer ring so only player 1 (Zola) can complete the outer ring and then player 2 (Ron) will be forced to go inwards. Note at any stage player 2 (Ron) can opt to go inwards, but if he moves inwards, there will be 7 positions left in that ring and again only player 1 (Zola) can complete that ring. This is the same for each ring. Hence only player 1 (Zola) can complete the inner ring and move last and hence win.

11. $u_1 = 0$
    $v_1 = 1$
    $u_{n+1} = \frac{1}{2}(u_n + v_n)$
    $v_{n+1} = \frac{1}{4}(u_n + 3v_n)$

$$\begin{aligned}v_{2016} - u_{2016} &= \frac{1}{4}(u_{2015} + 3v_{2015}) - \frac{1}{2}(u_{2015} + v_{2015}) \\ &= \frac{1}{4}v_{2015} - \frac{1}{4}u_{2015} \\ &= \frac{1}{4}(v_{2015} - u_{2015}) \\ &= \frac{1}{4}(\frac{1}{4}(v_{2014} - u_{2014})) \\ &= \frac{1}{4}(\frac{1}{4}(\frac{1}{4})^{2013}(v_1 - u_1)) \\ &= \frac{1}{4^{2015}}(v_1 - u_1) \\ &= \frac{1}{4^{2015}}(1 - 0) \\ &= \frac{1}{4^{2015}}.\end{aligned}$$

12. Colour the board as shown below.



Consider the blue region. There are 17 white squares and 13 black squares, but each domino must cover exactly one white and one black square. Hence, at most 15 of the white squares can be covered by dominoes and therefore there must be at least 2 white squares remaining uncovered. Similarly in the red region at least 2 black squares must remain uncovered. Thus there must be at least 4 uncovered squares and hence a maximum of

$$\frac{8 \times 9 - 4}{2} = \frac{68}{2} = 34$$

dominoes can be placed on the board.

Therefore only a further 28 dominoes can still be placed. To see that the maximum of 28 can be achieved one needs to provide an example of how 28 dominoes can be placed.

13. (a) By Pythagoras we have $x^2 + r^2 = R^2$. Hence, $x = \sqrt{R^2 - r^2}$.

(b) Let h be the perpendicular distance between the line connecting the centres of the circles and the line marked y. Let the centre of the left hand circles be O_1 and the centre of the right hand circles be O_2. Let the point where the outer right hand circle intersect the inner left hand circle be A and the foot of the altitude from A onto the line O_1O_2 be B. Hence $AB = h$ and AO_1B and ABO_2 are right angled triangles. Let the distance from O_1 to B be a. We then have

$$\begin{aligned}
h^2 &= R^2 - (2r - a)^2 = R^2 - 4r^2 + 4ra - a^2 \\
a^2 &= r^2 - h^2 = r^2 - (R^2 - 4r^2 + 4ra - a^2) = 5r^2 - R^2 - 4ra + a^2 \\
4ra &= 5r^2 - R^2 \\
a &= \frac{5r^2 - R^2}{4r} \\
y &= 2r - 2a = 2r - \frac{5r^2 - R^2}{2r} = \frac{R^2 - r^2}{2r}.
\end{aligned}$$

14. Note that $221 = 13 \times 17$ and therefore $222 = 13 \times 17 + 1$ is in the arithmetic sequence.
Furthermore, 222 222 221 is divisible by 13 and hence 222 222 222 is in the sequence, since

$$222222 = 222 \times 1001 = 222 \times 7 \times 11 \times 13.$$

In fact, any number consisting of $6m + 3$ 2s will be one more than a multiple of 13, since the number consisting of $6m$ 2s will be a multiple of 1001 and hence a multiple of 13. Thus there are infinitely many terms in the sequence that only uses the digit 2.

15. The sum from 2 to 22 is $2 + 3 + \cdots + 22 = \frac{22(23)}{2} - 1 = 11 \times 23 - 1 = 252$.
Therefore the new number must be even or otherwise the sum can't be exactly divisible by 2.
The smallest possible even number we can use is 24. To see that this is indeed possible take

$$24 + 22 + 21 + 20 + 19 + 12 + 6 + 5 + 4 + 3 + 2 = 138 = 18 + 17 + 16 + 15 + 14 + 13 + 11 + 10 + 9 + 8 + 7.$$

To find the greatest number possible, we try and put the 10 smallest numbers with the greatest number. The sum of the 11 numbers from 12 to 22 is

$$12 + 13 + \cdots + 22 = \frac{22(23)}{2} - \frac{11(12)}{2} = 11 \times 23 - 11 \times 6 = 11 \times 17 = 187.$$

Hence the largest total can be 187. Subtracting the numbers 2 to 11 from this gives us

$$187 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 - 10 - 11 = 122,$$

which is an even number. Thus 122 is the largest possible jersey number to achieve this.