SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior First Round 2016

Solutions

1. **Answer C.**

$$\frac{3 \times 2016 + 13 \times 2016}{1008} = \frac{(3+13) \times 2016}{1008} = (3+13) \times \frac{2016}{1008} = 16 \times 2 = 32.$$

2. **Answer D.**
If all terms are brought to 60, the least common multiple (LCM), then all the numerators except 6 have a factor 5. Since the required numerator is 60, which also has a factor 5, the term that must be dropped is $\frac{1}{10}$. Alternatively,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{10} + \frac{1}{12} = \frac{30+15+10+6+5}{60} = \frac{66}{60} = 1 + \frac{1}{10}.$$

3. **Answer B.**
Factorise 2016. It is divisible by 2, because it is an even number and it is also divisibel by 9 since the sum of the digits is 9. The full factorisation is $2016 = 2^5 \times 3^2 \times 7$. Each prime factor of a complete square must have an even number as the exponent (i.e. every prime factor must appear an even number of times) so that 2016 multiplied by a number n will be a complete square. Therefore, n must be equal to 2×7 which is equal to 14.

4. **Answer C.**

$$\sqrt{5050^2 - 4950^2} = \sqrt{(5050+4950)(5050-4950)} = \sqrt{10000 \times 100} = \sqrt{10^6} = 10^3.$$

5. **Answer B.**
Since triangle YTR is isosceles, it follows that $Y\widehat{R}T = 36^\circ$, so $S\widehat{R}T = 24+36 = 60^\circ$. Then, since triangle SRT is isosceles, we have $S\widehat{T}R = 24+36 = 60^\circ$ also. Therefore $T\widehat{S}R = 180 - 60 - 60 = 60^\circ$.

6. **Answer C.**
Any three consecutive integers are of the form $n, n+1, n+2$. The difference between the largest and the smallest is $(n+2) - n = 2$, and the square of the difference is 4.

7. **Answer E.**
One rand is $\frac{1}{12}$ of a dollar, so one pound is $\frac{21}{12} = \frac{7}{4} = 1.75$ dollars.

8. **Answer D.**
We need $34 - 5x - 7y$ to be a multiple of 3, so $5x + 7y$, like 34, must have remainder 1 after division by 3. For the the largest possible multiple of 3 we need the smallest such value of $5x + 7y$, which is $0 \times 5 + 1 \times 7 = 7$. Then $(34 - 7) \div 3 = 9$.

9. **Answer E.**
Since $a = \frac{2}{3}b$ and $a + b = 80$, we have $80 = \frac{2}{3}b + b = \frac{5}{3}b$, so $b = \frac{3}{5} \times 80 = 48$. Then $a = 80 - 48 = 32$ and $ab = 32 \times 48 = 1536$.

10. **Answer D.**
There are $64 - 24 = 40$ detective stories which are not about Sherlock Holmes. The probability of choosing one of these at random out of 1024 books is $\frac{40}{1024} = \frac{5}{128}$.

11. **Answer B .**
Since $d = 8\sqrt{\frac{h}{5}}$, it follows that $d^2 = \frac{64}{5}h$, so $h = \frac{5}{64}d^2$. With $d = 80$, this gives $d^2 = 6400$, so $h = \frac{5}{64} \times 6400 = 500$.

12. **Answer E.**
Mrs Habana will be $45 + x$ years old and her son will be $17 + x$ old where x is the number of years from 2016. Therefore $45 + x = 2(17 + x)$, giving $45 + x = 34 + 2x$, so $x = 45 - 34 = 11$. The year will then be $2016 + 11 = 2027$.

13. **Answer E.**
Since 0.4 mm $= 0.4 \times 10^{-3}$ m and 1 km $= 10^3$ m, it follows that the area of the line is $(0.4 \times 10^{-3}) \times 10^3 = 0.4\ \mathrm{m}^2$, and the area of the square will be the same.

14. **Answer A.**
Let $h = RY$ and $x = UY$, so $GY = 21 - x$. By Pythagoras' theorem, $h^2 = 10^2 - x^2$ and also $h^2 = 17^2 - (21 - x)^2$, so $100 - x^2 = 289 - 441 + 42x - x^2$. This gives $42x = 100 - 289 + 441 = 252$ and therefore $x = 252 \div 42 = 6$. Finally, $h^2 = 100 - 6^2 = 64$, so $h = 8$.

15. **Answer C.**
The first six terms of the sequence are $3, 3, 6, 9, 15, 24, ...$, which are Odd, Odd, Even, Odd, Odd, Even, $\ldots$, since Odd + Odd = Even and Odd + Even = Odd. The pattern continues in the same way in cycles of length 3, with two odd numbers and one even number in each cycle. After division by 2, the remainders in each cycle are $1, 1, 0$, so the sum of the three remainders is 2. The sum of the first 2016 remainders is therefore $\frac{2}{3} \times 2016 = 1344$.

16. **Answer D.**
There are $5+6+4 = 15$ pens in the box, so the probability of first picking a red pen is $\frac{5}{15} = \frac{1}{3}$. This leaves 14 pens (since the red pen is not replaced), and the probability of next picking a blue pen is $\frac{6}{14} = \frac{3}{7}$. Finally, the probability of picking a green pen is $\frac{4}{13}$. The events are not independent, and the probability is $\frac{1}{3} \times \frac{3}{7} \times \frac{4}{13} = \frac{4}{7 \times 13} = \frac{4}{91}$.

17. **Answer C.**
Since 13 is a prime number, the only factors in 169! supplying powers of 13 to the product are $13, 26, 39, \ldots, 156, 169$. The first 12 of these provide one power of 13 each, but since $169 = 13^2$, it follows that the total power of 13 is $12 + 2 = 14$.

18. **Answer D.**
By joining opposite vertices, the hexagon can be divided into six congruent equilateral triangles. If we form a rectangle around the hexagon by drawing lines through A and B parallel to CD, then the area of the rectangle is $AB \times CD$. Next, the portion of the rectangle outside the hexagon is composed of four right-angled triangles, which can be combined into two equilateral triangles congruent to the first six. Thus the area of the rectangle is $\frac{8}{6} \times 126 = 168$, which is also equal to $AB \times CD$.

19. **Answer E.**
The initial 5 g of mixture contains $10\% \times 5 = 0.5$ g of gold. If we add x g of pure gold, then the mass of gold is $(0.5 + x)$ g and the total mass is $(5 + x)$ g. Therefore $0.5 + x = 20\% \times (5 + x) = 1 + 0.2x$, so $0.8x = 0.5$ and $x = \frac{5}{8} = 0.625$.

20. **Answer B.**
If we let $F\widehat{B}C = \theta$, then $E\widehat{B}F = \theta$ also, and $A\widehat{E}B = 2\theta$ (alternate angles). If x denotes the side of the square, then $\tan\theta = \frac{3}{x}$ from triangle BCF and $\tan 2\theta = \frac{x}{2}$ from triangle BEA. Now from the double-angle formulae $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$ it is easy to show that if $t = \tan\theta$, then $\tan 2\theta = 2t/(1 - t^2)$. Thus we have
$$\frac{x}{2} = \frac{6/x}{1 - (3/x)^2} = \frac{6x}{x^2 - 9}.$$
This simplifies to $x^2 - 9 = 12$ (since $x \neq 0$), so $x^2 = 21$. Finally, by Pythagoras' theorem $BE^2 = AB^2 + AE^2 = x^2 + 4 = 21 + 4 = 25$, so $BE = 5$.

Alternatively, it is possible to solve the problem using only similar triangles. Let G be the point on EB such that $EG = EA$ and let H be the foot of the perpendicular from G to AB. Then triangles BHG and BAE are similar. Also $E\widehat{A}G = 90 - \theta$, since triangle EAG is isosceles, so $G\widehat{A}H = \theta$ and triangle AHG is similar to triangle BCF. Using these facts, we have
$$\frac{GB}{EG} = \frac{HB}{AH} = \frac{HB}{HG}.\frac{HG}{AH} = \frac{AB}{AE}.\frac{CF}{BC} = \frac{CF}{AE}, \text{ since } AB = BC.$$
It follows that $GB = CF$, since $EG = AE$ by construction, and finally $EB = EG + GB = AE + CF = 5$, as required.

An even quicker and more elegant solution is as follows. Rotate triangle ABE through 90° clockwise around B, so that A coincides with C and E is at position E' on DC produced. (In other words, let E' be the point on DC produced such that $CE' = AE$, and join B and E' so that triangles BAE and BCE' are congruent.) Then $E'\widehat{F}B = 90 - \theta$ and $E'\widehat{B}F = E'\widehat{B}C + C\widehat{B}F = (90 - 2\theta) + \theta = 90 - \theta$. Thus triangle $E'BF$ is isosceles, and therefore $E'B = E'F$. Since $E'B = EB$ and $E'F = E'C + CF = AE + CF$ it follows that $EB = AE + CF = 5$.
