HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

Organised by the SOUTH AFRICAN MATHEMATICS FOUNDATION

THIRD ROUND 2013
JUNIOR SECTION: GRADES 8 AND 9

9 SEPTEMBER 2013
TIME: 4 HOURS
NUMBER OF QUESTIONS: 15
TOTAL: 100

Instructions

- Answer all the questions.
- All working details and explanations must be shown. Answers alone will not be awarded full marks.
- This paper consists of 15 questions for a total of 100 marks as indicated.
- The neatness in your presentation of the solutions may be taken into account.
- Diagrams are not necessarily drawn to scale.
- No calculator of any form may be used.
- Use your time wisely and do not spend all your time on one question.
- Answers and solutions will be available at: www.samf.ac.za

DO NOT TURN THE PAGE
UNTIL YOU ARE TOLD TO DO SO.

PRIVATE BAG X173, PRETORIA, 0001
TEL: (012) 392-9323 FAX: (012) 392-9312
E-mail:ellie@samf.ac.za

Organizations involved: AMESA, SA Mathematical Society, SA Akademie vir Wetenskap en Kuns

Question 1

How many digits does the number 20^{13} have?

[4]

Question 2

A palindromic number is a number that reads the same when its digits are reversed,

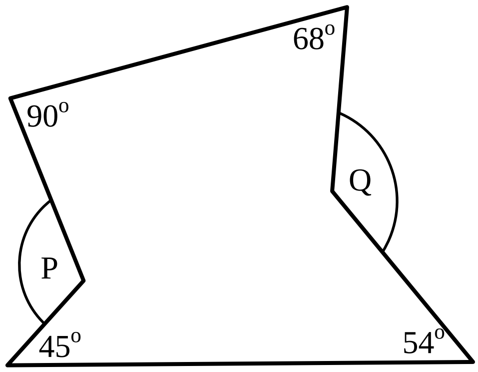
e.g. 1623261.

What is the largest palindromic 8-digit number which is exactly divisible by 45?

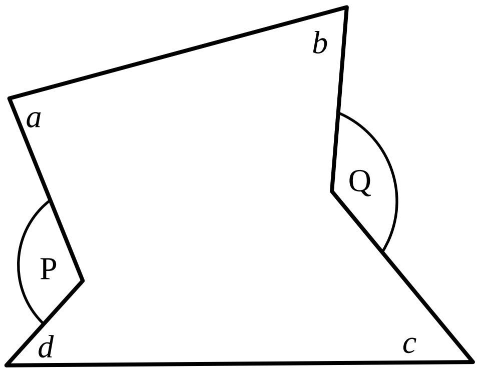
[6]

Question 3

a) Calculate the value of P + Q.



b) Prove that P + Q = $a + b + c + d$.



[6]

Question 4

This is a real image of a Zimbabwean bank note:

The thickness of one of these Zimbabwean bank notes is 0,1 mm. If we stack twenty trillion (short scale – see below) such notes on top of each other, how high would the pile be?

a) To the roof of your classroom?
b) As high as the Telkom tower in Pretoria?
c) As high as Table Mountain?
d) As high as Mount Everest?
e) As high as a 747 Jet flies from King Shaka International to Oliver Thambo International airport?
f) Higher than to the moon?

Explain your answer. [4]

Nice to know:
There are different number naming systems which sometimes create confusion.

Value in Scientific notation	Value in numerals	Short Scale		Long Scale	
		Name	Logic	Name	Logic
10^6	1 000 000	million	$1\,000 \times 1\,000^1$	million	$1\,000\,000^1$
10^9	1 000 000 000	**bi**llion	$1\,000 \times 1\,000^2$	thousand million or milliard	
10^{12}	1 000 000 000 000	**tri**llion	$1\,000 \times 1\,000^3$	**bi**llion	$1\,000\,000^2$
10^{15}	1 000 000 000 000 000	**quadr**illion	$1\,000 \times 1\,000^4$	thousand billion or billiard	
10^{18}	1 000 000 000 000 000 000	**quint**illion	$1\,000 \times 1\,000^5$	**tri**llion	$1\,000\,000^3$
etc.	etc.	To get from one named order of magnitude to the next: multiply by 1 000		To get from one named order of magnitude to the next: multiply by 1 000 000	

The word *milliard*, or its translation, is found in many European languages and is used in those languages for 10^9. However, it is unknown in American English, which uses *billion*, and not used in British English, which preferred to use *thousand million.*

The root *mil* in "million" refers to the Latin word for "thousand" (milia).
The word *million* derives from the Old French *milion* from the earlier Old Italian *milione*, an intensification of *mille*, a thousand.

Question 5

The Fibonacci sequence is given by 1; 1; 2; 3; 5; 8; 13; 21; …, where the next number is generated by summing the previous two. Fibonacci numbers were made famous by the rabbit problem, because it explained rabbit breeding. It is less well-known that one can also use Fibonacci numbers to convert miles to kilometers.

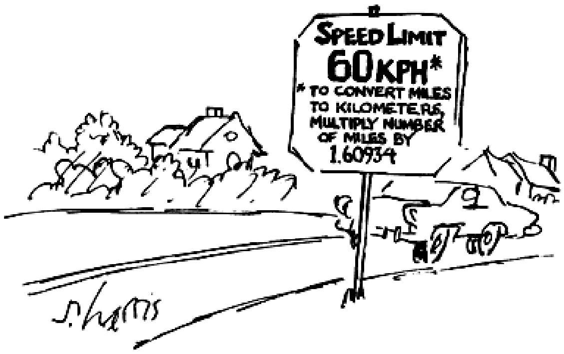
To do so, one must realise that every positive integer can be uniquely expressed as the sum of different, non-consecutive Fibonacci numbers. To convert integer miles into kilometres, miles are expressed as the unique sum of non-consecutive Fibonacci numbers, then each Fibonacci number is changed to the next Fibonacci number. The new sum approximately gives the kilometres.

For example, 50 miles = 34 + 13 + 3 miles, where each number on the right hand side is a Fibonacci number. Using the conversion above, the right-hand side becomes
55 + 21 + 5 km = 81 km.

Now use this method to convert 120 miles into kilometres. Show your working.

[6]

[You might want to check your answer by using the conversion on the cartoon.]



Question 6

All boxes in a 3 × 3 table are occupied by zeroes. Suppose that we can choose any

2 × 2 sub-table and increase all the numbers in it by 1.

Example: Possible next two moves:

0	0	0
0	0	0
0	0	0

1	1	0
1	1	0
0	0	0

1	1	0
1	2	1
0	1	1

Prove that we cannot obtain the table below using these operations.

4	9	5
10	18	12
6	13	7

[6]

Question 7

a) Find all integers a and b such that

$$(a+b\sqrt{15})^2 = 31-8\sqrt{15}$$

b) Find the value of

$$\sqrt{31-8\sqrt{15}}+\sqrt{31+8\sqrt{15}}$$

[6]

Question 8

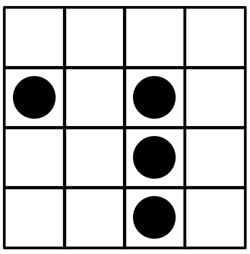
The game of TacTic is a board game for two people.

i) During a turn, a player selects a row or column and removes at least one stone, or any number of adjacent stones from that row or column.

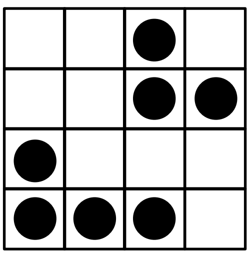
ii) The player, who removes the last stone(s) from the board, wins.

It is your turn.

a) Describe all possible first moves to guarantee a win for this configuration.



b) Find a winning move for this configuration and explain why it works.



[8]

Question 9

In a certain city all the skyscrapers are arranged in a square grid and each has either one, two, three or four storeys. Jonathan is surveying 4×4 sections of the city and noting how many skyscrapers he can see from a certain position in a certain direction. For example, Jonathan came across the following section of the city (viewed from the top):

Each number represents the number of storeys in the building located there.

1	3	4	2
4	2	1	3
2	1	3	4
3	4	2	1

He jots down the following data:

E.g. Viewing from the circled position Jonathan can see three buildings, one with 2 storeys, one with 3 storeys and one with 4 storeys. The building with 1 storey is hidden behind the building with 2 storeys.

	2	2	1	3	
3	1	3	4	2	2
1	4	2	1	3	2
3	2	1	3	4	1
2	3	4	2	1	3
	2	1	3	2	

Suppose Jonathan jots down the following data for a different city:

	4	1	2	2	3	
2						3
2						2
2						3
1						4
5						1
	2	3	2	2	1	

Jonathan knows that each row and each column in the grid contains exactly one of each of the numbers 1; 2; 3; 4 and 5 (storeys of the buildings). Complete the grid by filling in the missing numbers (storeys).

[6]

Question 10

Molly and Fred had an argument about the next term in the sequence:

$$2\ ;\ 4\ ;\ 6\ ;\ \ldots$$

Molly says it is 8.
Fred says it is – 4, which he got by using the formula:

$$T_n = 2n - k(n-1)(n-2)(n-3)\text{, where } k = 2.$$

a) What is the fifth term in Fred's sequence?

b) Find a formula that defines the sequence:

$$2\ ;\ 4\ ;\ 6\ ;\ 38\ ;\ \ldots?$$

[6]

Question 11

a) The number 15 can be written as 7 + 8 or 4 + 5 + 6 or 1 + 2 + 3 + 4 + 5 (the sum of 2 consecutive integers, 3 consecutive integers and 5 consecutive integers).
Find a positive integer that can be expressed as the sum of 3 consecutive integers, 5 consecutive integers and 7 consecutive integers.

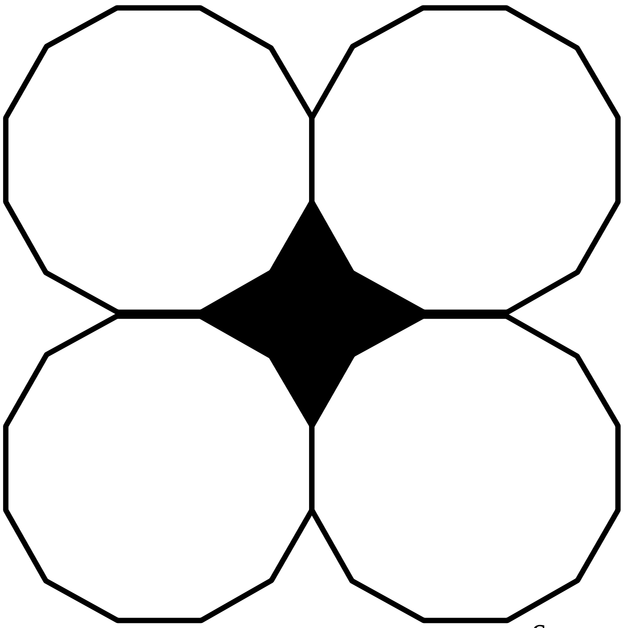
b) Prove that if an integer n can be written as the sum of p consecutive integers where p is odd, then n is divisible by p.

[8]

Question 12

Four regular dodecagons (12-sided shapes) are placed as shown.

If each side has length 1, what is the area of the black region?



You will be given four Canadian 1c coins (dodecagons) which you may use.

You may keep the coins as a souvenir.

[6]

Question 13

Let A be any set of 19 numbers chosen from the arithmetic progression

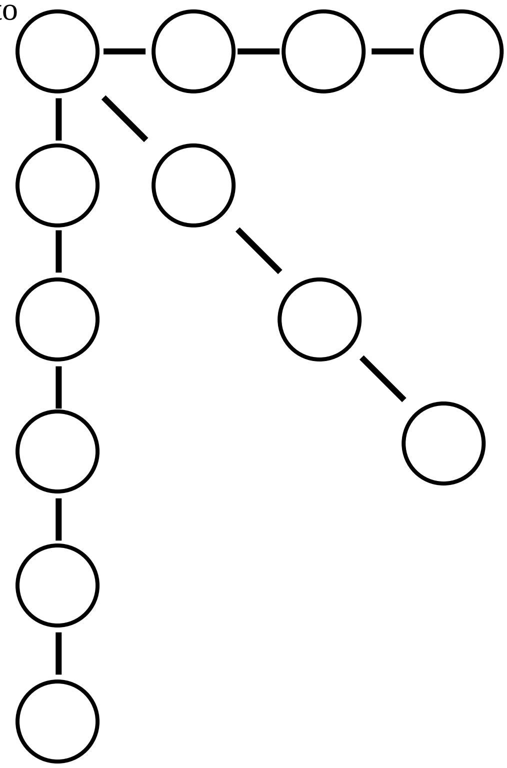
1; 4; 7; 10; 13; 16: … ; 100.

Prove that there will always be two numbers in A whose sum is 104.

[8]

Question 14

a) The numbers 1 through 12 must be filled into the circles such that the sum along each line is 32. Do it!



b) Prove that it is impossible to arrange the numbers 1 through 12 such that each sum is 31.

c) The numbers 1 through 12 are arranged in the circles such that each sum is S. What are the possible values of S?

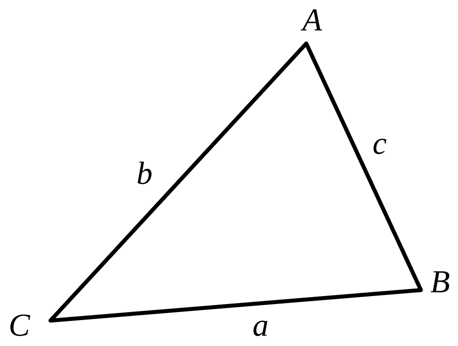
[10]

Question 15

If ΔABC has sides of length *a*, *b* and *c*,

then we define the semi-perimeter, *s*, by:

$$s = \frac{a+b+c}{2}$$



The area of ΔABC is given by Heron's Formula: $K = \sqrt{s(s-a)(s-b)(s-c)}$

a) If ΔABC has sides of lengths with the consecutive integers 13, 14, and 15, show that the area is twice the perimeter.

b) Find all sets of 3 consecutive *even* integers such that the triangle with sides of these lengths has an area exactly equal to its perimeter.

[10]

Total: 100

THE END