The South African Mathematical Olympiad
Third Round 2013
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

Solutions

1. **Solution:** There are seven possible choices for the four digits (from $0, 1, 2, 3$ to $6, 7, 8, 9$), and for each choice we have $4! = 24$ different ways of arranging them. This gives us $7 \cdot 24 = 168$ possibilities. Now we still have to subtract those that start with 0 (there are six such possibilities: 0 followed by any of the $3!$ permutations of $1, 2, 3$), those that start with 1 (there are twelve such possibilities: 1 followed by any permutation of either $0, 2, 3$ or $2, 3, 4$), and 2013 itself, leaving us with $168 - 19 = 149$ such years that are still to come.

2. **Solution 1:** We write the condition as an equation relating A and B:

$$A \cdot \left(1 + \frac{B}{100}\right) = B \cdot \left(1 - \frac{A}{100}\right).$$

Now we multiply by 100:

$$A(100 + B) = B(100 - A).$$

Multiplying out, we find

$$2AB + 100A - 100B = 0.$$

Solving for A yields

$$A = \frac{100B}{2B + 100} = \frac{50B}{B + 50} = 50 - \frac{2500}{B + 50}.$$

Thus $B + 50$ has to be a divisor of 2500, and there are only 15 divisors: 1, 2, 4, 5, 10, 20, 25, 50, 100, 125, 250, 500, 625, 1250, 2500. Only three of these yield three-digit numbers B, namely 250, 500 and 625. The corresponding pairs (A, B) are:

$$(40, 200),\ (45, 450),\ (46, 575).$$

Solution 2: Note that $A\%$ of B equals $B\%$ of A (both are equal to $AB/100$). Since A increased by this amount equals B decreased by this amount, it must equal half their difference:

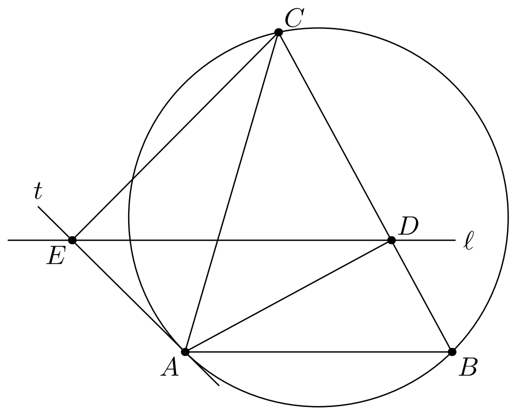
$$\frac{AB}{100} = \frac{B - A}{2}.$$

Once again, we obtain

$$AB + 50A - 50B = 0,$$

and we can continue in the same way as before.

3. **Solution:** By the tangent-chord theorem, $\angle CAE = \angle CBA$, and since AB and ℓ are parallel, we also have $\angle CAE = \angle CBA = \angle CDE$. It follows that $ADCE$ is a cyclic quadrilateral, which in turn means that $\angle CEA = 180° - \angle CDA = 90°$. This is exactly what we wanted to prove.



4. **Solution:** Write $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$. Then we obtain

$$x^2 g(x) = c_0 + c_1 g(x) + c_2 g(x)^2 + \cdots + c_n g(x)^n.$$

The left hand side is divisible by $g(x)$, and so are all terms on the right hand side except for c_0. Thus $c_0 = 0$ unless $g(x)$ is constant.

If $g(x) = K$ is constant, then we obtain

$$Kx^2 = f(K),$$

which is only possible if $K = 0$ (and $f(x)$ is any polynomial such that $f(0) = 0$).

If $g(x)$ is not constant, we have $c_0 = 0$, which allows us to divide by $g(x)$:

$$x^2 = c_1 + c_2 g(x) + c_3 g(x)^2 + \cdots + c_n g(x)^{n-1}.$$

Let m denote the degree of $g(x)$. Then the degree of the right hand side is $m(n-1)$, which must equal 2. This gives us two cases: $m = 1$ and $n = 3$, or $m = 2$ and $n = 2$.

- If $m = 1$ and $n = 3$, then $g(x) = ax + b$, and plugging in yields

  $$x^2 = c_1 + c_2(ax + b) + c_3(ax + b)^2.$$

  We substitute $y = ax + b$ (thus $x = (y - b)/a$) and get

  $$\frac{(y-b)^2}{a^2} = c_1 + c_2 y + c_3 y^2,$$

  hence $c_1 = b^2/a^2$, $c_2 = -2b/a^2$ and $c_3 = 1/a^2$.

- If $m = 2$ and $n = 2$, then we are left with

$$x^2 = c_1 + c_2 g(x),$$

which yields $g(x) = (x^2 - c_1)/c_2$.

Summarizing, there are three infinite families of solutions:

- $g(x) = 0$, $f(x)$ arbitrary such that $f(0) = 0$.
- $f(x) = \dfrac{x(x-b)^2}{a^2}$ and $g(x) = ax + b$, $a \neq 0$, b arbitrary.
- $g(x) = \dfrac{x^2 - c_1}{c_2}$ and $f(x) = c_1 x + c_2 x^2$, $c_2 \neq 0$, c_1 arbitrary.

It is easily verified that these are indeed solutions.

5. **Solution 1:** We prove the more general statement that on an $m \times n$ board (where $n, m > 1$), we can place $m + n$ coins and that this is the maximum possible. Such an arrangement is given by placing a coint on each square in the top row and on each square in the rightmost column as well as one on the bottom left square.

To prove that this is the maximum possible, we proceed by induction on $m + n$. Clearly, there can be no more than $4 = 2+2$ coins on a 2×2-board. So assume that the statement is true for all $m \times n$-boards with $m, n > 1$ and $m + n < k$. Consider an $m \times n$-board where $n + m = k$. Without loss of generality, we may assume that $m \leqslant n$. Consider an arrangement of $k + 1$ coins on the board. We have $k + 1 = m + n + 1 \geqslant 2m + 1$. By the Pigeonhole Principle, we may now deduce that there exists some row with at least 3 coins. Of these coins, one of the middle ones already has two neighbours, so it is alone in a column. If this column is deleted, we remain with an $m \times (n-1)$-board ($n - 1 > 1$, since we had three or more coins in the row we selected) with $k = m + n$ coins on it, which contradicts our induction hypothesis.

We conclude that $m + n$ is indeed the maximum number of coins, which in our case gives us $20 + 13 = 33$.

Solution 2: Again, we prove for a general $m \times n$-board that the maximum number is $n + m$, provided that $m, n > 1$. Let x denote the number of coins, k_i ($1 \leqslant i \leqslant m$) the number of coins in the i-th row and l_j ($1 \leqslant j \leqslant n$) the number of coins in the j-th column. The total number of neighbour pairs in the i-th row is obviously $\geqslant k_i - 1$ (where strict inequality only holds for $k_i = 0$), the number of neighbour pairs in the j-th column is $\geqslant l_j - 1$. The total number of neighbour pairs may not exceed $\frac{2x}{2} = x$ by the given condition, hence we have

$$\sum_{i=1}^{m}(k_i - 1) + \sum_{j=1}^{n}(l_j - 1) \leqslant x$$

or $x - m + x - n \leqslant x$ (since $\sum_i k_i = \sum_j l_j = x$). It follows that $m + n$ is indeed an upper bound. As in the first proof, explicit examples that yield this maximum are not hard to find.

6. **Solution 1:** Note first that $OX \cdot OY = OP \cdot OF$ (power of the point O with respect to the circumcircle of FXY). If we can prove that $OF^2 > OX \cdot OY$, then the statement follows immediately. Moreover, the power of F with respect to the circumcircle of ABC is

$$AF \cdot BF = R^2 - OF^2,$$

where R is the circumradius of ABC. Thus it remains to show that

$$R^2 - AF \cdot BF > OX \cdot OY. \tag{1}$$

Now note that $\angle CAX = 90^\circ - \angle ACX = \angle AOC/2 = \angle ABC$, which means that triangles AXC and BFC are similar. We obtain

$$\frac{AX}{AC} = \frac{BF}{BC}$$

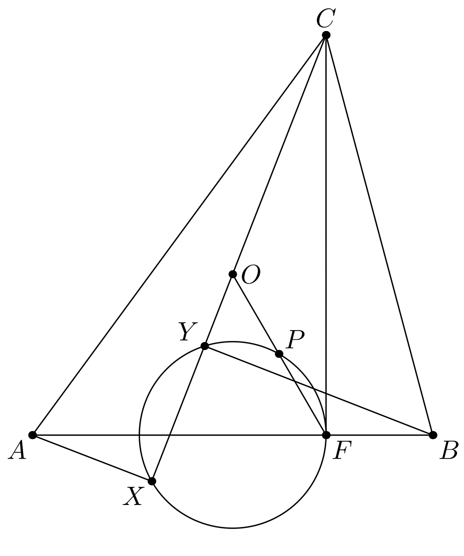
and analogously

$$\frac{BY}{BC} = \frac{AF}{AC}.$$

Combining the two, we get $AF \cdot BF = AX \cdot BY$, and we conclude

$$\begin{aligned} AF \cdot BF &= AX \cdot BY \\ &= \sqrt{R^2 - OX^2} \cdot \sqrt{R^2 - OY^2} \\ &= \sqrt{R^4 - (OX^2 + OY^2)R^2 + OX^2 \cdot OY^2} \\ &\leqslant \sqrt{R^4 - 2OX \cdot OY \cdot R^2 + OX^2 \cdot OY^2} \\ &= R^2 - OX \cdot OY, \end{aligned}$$

with equality if and only if $OX = OY$, which, however, would imply that $AX = BY$, meaning that OC is the perpendicular bisector of AB. But then $AC = BC$, which has been ruled out. This completes the proof of (1) and thus the entire statement.



Solution 2: By the power of point O with respect to the circumcircle of triangle FXY and Pythagoras' Theorem in triangles in AOX and BOY, we have

$$OP \cdot OF = OX \cdot OY = \sqrt{R^2 - AX^2} \cdot \sqrt{R^2 - BY^2},$$

where R is the circumradius of ABC. Now note that

$$AX = AC \cdot \cos B = 2R \cdot \sin B \cos B = R \sin 2B$$

(using the extended law of sines), and therefore

$$R^2 - AX^2 = R^2 - R^2 \sin^2 2B = R^2 \cos^2 2B.$$

By symmetry, we also have $R^2 - BY^2 = R^2 \cos^2 2A$. Hence

$$OP \cdot OF = R^2 \cos 2A \cos 2B.$$

We would like to compare this to OF^2. So let D be the midpoint of AB. Then by Pythagoras' Theorem in triangle OFD, we have

$$OF^2 = OD^2 + DF^2.$$

In triangle AOD, we have $\angle AOD = \angle ABC$, so $OD = R \cos C$. We also have $FD = AF - AD$ and $AD = AB/2 = (2R \sin C)/2 = R \sin C$, while $AF = AC \cos A = 2R \sin B \cos A$. Thus

$$OF^2 = R^2 \cos^2 C + (R \sin C - 2R \sin B \cos A)^2.$$

So to show that $OP < OF$, it suffices to show that

$$\frac{OP \cdot OF}{R^2} < \frac{OF^2}{R^2}$$

or equivalently that

$$\cos 2A \cos 2B < \cos^2 C + (\sin C - 2 \sin B \cos A)^2.$$

Noting that

$$\cos C = \cos(180^\circ - A - B) = -\cos(A + B) = \sin A \sin B - \cos A \cos B$$

and that

$$\sin C = \sin(180^\circ - A - B) = \sin(A + B) = \sin A \cos B + \cos A \sin B,$$

this is equivalent to

$$(\cos^2 A - \sin^2 A)(\cos^2 B - \sin^2 B) < (\sin A \sin B - \cos A \cos B)^2 + (\sin A \cos B - \cos A \sin B)^2,$$

which simplifies to

$$2(\sin A \cos B - \cos A \sin B)^2 = 2 \sin^2(A - B) > 0,$$

and this holds since the triangle is not isosceles (thus $\sin(A - B) \neq 0$).