THE SOUTH AFRICAN MATHEMATICS OLYMPIAD
SENIOR SECOND ROUND 2015
Solutions

1. **Answer 010**
   Since Total cost = Number of sweets $\times$ Cost of each sweet, it follows that the number of sweets is (R4.00) $\div$ (40c) $=$ (R4.00) $\div$ (R0.40) $=$ $4 \div 0.4 = 10$.

2. **Answer 001**
   $2015-(2015-(2015-(2015-1))) = 2015-(2015-(2015-2014)) = 2015-(2015-1) = 2015-2014 = 1$.
   Alternatively, $2015-(2015-(2015-(2015-1))) = 2015-2015+2015-2015+1 = 1$.

3. **Answer 027**
   Since 11:45 is 45 minutes after 11:00 and 12:12 is $60+12=72$ minutes after 11:00, the length of break is $72-45=27$ minutes.
   Alternatively, there are 15 minutes from 11:45 to 12:00 and 12 minutes from 12:00 to 12:12, so the length of break is $15+12=27$ minutes.

4. **Answer 020**
   Suppose Steve breaks n eggs, so $1000-n$ eggs are unbroken. He receives R0.20 $\times (1000-n)$, but has to repay R1 $\times n$. Thus $0.2(1000-n)-n = 176$, giving $200-1.2n = 176$, so $1.2n = 200-176 = 24$ and $n = 20$.

5. **Answer 012**
   We must decide whether the books are to be packed flat, on their spines, or upright (or possibly a combination). In this case it is easy, because if they are packed upright, then their height is 20 cm, the same as the depth of the box. The width of each book is 15 cm, which goes twice into the width of the box; the thickness of each book is 6 cm, which goes six times into the length of the box. Thus $2\times 6 = 12$ upright books will fill the box completely, so no higher number is possible.

6. **Answer 017**
   A node is a point where two or more paths meet. Starting at A, we label each node with the number of paths from A to that node. This number is the sum of the numbers below and/or to the left of it, omitting the

node C. This builds up the following table from the bottom left to the top right.

$$\begin{array}{ccccc} 1 & 4 & 4 & 8 & B \\ 1 & 3 & C & 4 & 9 \\ 1 & 2 & 3 & 4 & 5 \\ A & 1 & 1 & 1 & 1 \end{array}$$

It follows that the number of paths from A to B is $8 + 9 = 17$.

7. **Answer 828**
In order to obtain the least sum, we would like to have the smallest possible digits (1, 2, 3) in the hundreds column and the largest possible digits in the units (or ones) column. However, 9 must be in the middle (or tens) column, since no larger digit is available. We can put 8 in the units column next to 9, but then 7 must be in the middle column again, with 6 next to it. We then have 5 in the middle column and 4 in the units column. The sum is then
$100 \times (1 + 2 + 3) + 10 \times (5 + 7 + 9) + (4 + 6 + 8) = 828$.

8. **Answer 019**
Since 144 leaves remainder 11 when divided by n, it follows that n divides exactly into $144 - 11$. Thus n is a factor of $133 = 7 \times 19$. Similarly, n is a factor of $220 - 11 = 209 = 11 \times 19$. Since also $n > 11$ (because the remainder must be less than the divisor), it follows that $n = 19$.
Alternatively, since 144 and 220 leave the same remainder after division by n, it follows that n is a factor of $220 - 144 = 76 = 2^2 \times 19$. We can then use the original information to eliminate other factors of 76.

9. **Answer 012**
Two copies of the shape can be fitted together to make a solid 5×2 rectangle, or a hollow 4×3 rectangle (with two squares empty in the middle). Four 5×2 rectangles (eight copies of the shape) will completely cover five rows of the board, and two 4×3 rectangles (four copies) will cover the remaining three rows (or columns), leaving four squares uncovered. Thus the total is $8 + 4 = 12$ copies.
As a check, note that each copy covers five squares of the board, so 12 copies will cover $12 \times 5 = 60$ squares. This is the largest multiple

of 5 that is less than 64 (the total number of squares on the board), so 12 copies must be the maximum possible number.

10. **Answer 385**
The n-th pyramid is made by putting the $(n-1)$-st pyramid on an $n \times n$ square of balls. Thus the number of balls in the n-th pyramid is $1^2 + 2^2 + \cdots + n^2$. The number of balls in the 10th pyramid is therefore $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = 385$.
Remark. It can be shown that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all natural numbers n.

11. **Answer 720**
The sum of the exterior angles of any polygon is 360 degrees, going in one direction. In the figure, $a + g + e + c = 360$ degrees (clockwise), and $b + d + f + h = 360$ degrees (anti-clockwise), so the total sum is 720 degrees.

12. **Answer 021**
If I buy x chocolate bars and y cool-drinks, then $25x + 9y = 839$ with $x > y$. If we put $x = y$ as a first guess, then $x = y = 839/34 \approx 24$, so we can write $x = 24 + a$ and $y = 24 - b$. The equation becomes $25a - 9b = 23$, which by easy trial and error (guess and check) has a solution $a = 2$ and $b = 3$. Thus $x = 24 + 2 = 26$ and $y = 24 - 3 = 21$, which is the required answer.

13. **Answer 096**
If the number in the bottom right square is x, then from the bottom row $k = 31 + 28 + x = 59 + x$. Similarly, from one diagonal the number in the middle is $59 + x - 31 - 33 = x - 5$. It is now possible to fill in all the squares to obtain

$x-10$	36	33
38	$x-5$	26
31	28	x

Finally, from the main diagonal we have $(x - 10) + (x - 5) + x = k$, so $3x - 15 = x + 59$. Thus $2x = 74$, so $x = 37$ and $k = x + 59 = 96$.
Alternatively, if you are aware that for a 3×3 magic square the value

of k is always three times the number in the middle, it is possible to obtain the equation $3x - 15 = x + 59$ more quickly.

14. **Answer 007**
If the number of balls in the boxes are denoted $x_1, x_2, \ldots$, then
$$x_n + x_{n+1} + x_{n+2} + x_{n+3} = 30 = x_{n+1} + x_{n+2} + x_{n+3} + x_{n+4}.$$
Thus $x_n = x_{n+4}$ for all n, which means that the numbers recur in cycles of length four. Since $2015 = 4 \times 503 + 3$, it follows that $x_{2015} = x_3 = 7$.

15. **Answer 024**
Denote the side lengths of squares A, B, C, D, E by a, b, c, d, e respectively. The two right-angled triangles enclosed by squares C, D, E and the line are congruent to each other and have sides c, d, e, so by Pythagoras' theorem $d^2 = c^2 + e^2$. We are given that $c^2 = 4^2 = 16$ and $d^2 = 2 \times 20 = 40$, so $e^2 = 40 - 16 = 24$.

16. **Answer 031**
Since each of the given sums involves exactly three of the numbers, it follows that the sum of the four sums is three times the sum of the four numbers. Thus the required sum is equal to $(20 + 22 + 24 + 27)/3 = 93/3 = 31$.

17. **Answer 015**
We are given $3 = f(500) = f(100 \times 5) = \frac{f(100)}{5}$, so $f(100) = 3 \times 5 = 15$.

18. **Answer 050**
If the length and breadth of a rectangle are divided into the same number of pieces, then it can be seen that its perimeter is equal to the sum of the perimeters of the diagonal subrectangles. Applying this to the four subrectangles at the top left of the figure, we see that the first subrectangle has perimeter $18 + 18 - 22 = 14$. Similarly, at the bottom right, the last subrectangle has perimeter $20 + 16 - 22 = 14$. Finally, by the same principle, the perimeter of the whole rectangle is equal to $14 + 22 + 14 = 50$.

19. **Answer 100**
Rotate triangle DBC about point D to a position $DB'C'$, where C' coincides with E (since $DC' = DC = DE$). Then AEB' is a straight

line, since $A\widehat{E}B' = A\widehat{E}D + D\widehat{C'}B' = A\widehat{E}D + D\widehat{C}B = 180°$. Also $AB' = AE + CB = AB$ (given) and $DB' = DB$. Therefore triangles DAB' and DAB are congruent (SSS), so $B'\widehat{D}B = 2.A\widehat{D}B = 2(180 - 70 - 60)° = 100°$. Finally, since $B'\widehat{D}E = B\widehat{D}C$, it follows that $E\widehat{D}C = B'\widehat{D}B = 100°$.

Remark. Note that since triangles DAB' and DAB are congruent as shown above, we can reflect DAB' around DA to map onto DAB. This implies that if triangles EDA and CDB are reflected in the lines DA and DB respectively, then the reflections of points E and C will coincide at the foot of the perpendicular from D to AB. From this, the result also follows easily.

An alternative proof, using trigonometry, is as follows. Denote the sides of triangle ABD by a, b, d in the usual way, and let $x = ED$ and $y = EA$. Then $CD = x$ and $CB = d - y$. Let F be the foot of the perpendicular from D to AB. By Pythagoras' theorem we have $x^2 + y^2 = b^2$ and $x^2 + (d-y)^2 = a^2$, which gives $a^2 = x^2 + y^2 - 2dy + d^2 = b^2 + d^2 - 2dy$. From the Cosine Rule (or Apollonius' theorem) we also have $a^2 = b^2 + d^2 - 2bd \cos A$, from which it follows that $y = b \cos A = AF$. Thus the right-angled triangles DAE and DAF are congruent, as are triangles DBC and DBF. It follows that $E\widehat{D}C = 2A\widehat{D}B = 100°$.

20. **Answer 138**

The required numbers are of the form $3^a 4^b 5^c 6^d 7^e$, where

$$0 \le a \le 1, \quad 0 \le b \le 2, \quad 0 \le c \le 2, \quad 0 \le d \le 1, \quad 0 \le e \le 3,$$

and $a + b + c + d + e \ge 2$. This gives two possible values of a, three of b, three of c, two of d, and four of e, making a total of $2 \times 3 \times 3 \times 2 \times 4 = 144$ possible numbers if we ignore the last restriction. To satisfy it, we must exclude one possibility with $a + b + c + d + e = 0$ and five possibilities with $a + b + c + d + e = 1$, so the final total is $144 - 1 - 5 = 138$.