# 2017 Junior Third Round - Solutions

1. The 2-digit squares are 16, 25, 36, 49, 64, 81. Starting with 16 we get 1649. Starting with 25 we get 25. Starting with 36 we get 3649. Starting with 49 we get 49 and starting with 64 we get 649. However, starting with 81 gives us 81649, which is the longest string of digits such that every two consecutive digits form a 2-digit square number. Thus 81649.

2. Subtracting 1 from each term of the sum (except the very first term), the sum becomes
$$1 + 10 + 100 + \cdots + 100\ldots0 = 111\ldots1,$$
composed of 52 ones. Adding back 51 (the ones subtracted from 51 terms) we obtain $11\ldots162$, which has digit sum $50 + 6 + 2 = 58$.

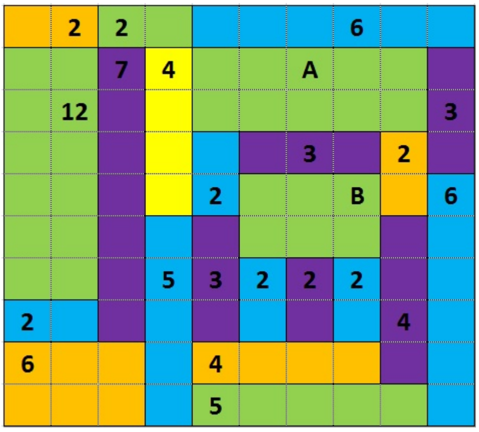
3. We factorise $1728 = 3^3 \cdot 2^6$. In order to form the smallest number with digit product equal to 1728, we would like to form as few digits as possible from the factors of 1728. Note that since $9 \times 9 \times 9 = 729 < 1728$, we cannot form a 3-digit number. The possible groupings of the factors of 1728 into four digits are (3,8,8,9) and (4,6,8,9), from which the smallest number that can be formed is 3889.

4. Of the visible faces, there are 7 pairs of opposite faces (one set each in the bottom 5 dice, and two in the top die), which must each add up to 7. Of the remaining faces, two pairs are adjacent faces on the same die (the bottom left and centre left dice), which can sum to a maximum of 5+6 = 11 each. The remaining three faces can all be chosen to be 6, and so the greatest sum is equal to $7 \times 7 + 2 \times 11 + 3 \times 6 = 89$.

5. In order for the number to be divisible by 99, the number must be divisible by both 9 and 11. Divisibility by 9 requires that the digit sum is a multiple of 9 - since all the digits are equal to 7, this means that the digit sum is equal to 7 times the number of digits, which is only divisible by 9 if the number of digits is divisible by 9. Divisibility by 11 requires that the difference between the sum of the digits in odd positions and the sum of the digits in the even positions is divisible by 11. Since all the digits are 7, this difference is either equal to 7 (if there is an odd number of digits) or 0 (if there is an even number of digits), with the latter being divisible by 11. Hence the number of digits must be even and divisible by 9, which means that the smallest such number consists of eighteen 7s.

6. The cut cube consists of 8 corner cubes of side length 1, and 12 edge cubes of side length 1. Each corner cube contributes 3 squares of surface area 1 to the total surface area, and each edge cube contributes 4 such squares, for a total of $8 \times 3 + 12 \times 4 = 72$.

7. Diagram for Part (a).




As can be seen from the above $A = 10$ and $B = 6$.

8. (a) $30 \times 31 = 930$ and $5^2 = 25$, so $305^2 = 93025$.

   (b) Let the original number be $10x + 5$. Then

$$(10x + 5)^2 = 100x^2 + 100x + 5^2 = 100x(x + 1) + 5^2,$$

   so the correct answer is given by appending $5^2 = 25$ to $x(x + 1)$.

9. Extend $GB$ to meet $AE$ in $H$, and let $\angle BGC = \alpha$. Then, since $BG = BC$, $\angle BCG = \alpha$, and so $\angle HBC = 2\alpha$. Now, since $\angle ABC = 90° = \angle HBE$, it follows that $\angle ABE = 2\alpha$. Since $AB = BE$, it follows that $\angle AEB = \frac{1}{2}(180° - \angle ABE) = \frac{1}{2}(180° - 2\alpha) = 90° - \alpha$. Now, since $\angle HBE = 90°$, $\angle BHE = \alpha = \angle BGC$. Since $HBG$ is a straight line and the alternate angles are equal, it follows that $AE \parallel GC$.

10. Let the distance between cities $A$ and $B$ be $d$. When the busses meet the first time, Bus 1 has travelled 7km and Bus 2 has travelled $d - 7$km. The second time they meet, Bus 1 has travelled $d + 4$km and Bus 2 has travelled $2d - 4$km. If Bus 1 travels at a speed of $x$ and Bus 2 at a speed of $y$, it follows that $\frac{7}{x} = \frac{d-7}{y}$ and $\frac{d+4}{x} = \frac{2d-4}{y}$. Multiplying the two equations we get

$$\begin{aligned} \frac{7}{x} \cdot \frac{2d - 4}{y} &= \frac{d - 7}{y} \cdot \frac{d + 4}{x} \\ \implies 7(2d - 4) &= (d - 7)(d + 4) \\ \implies d^2 - 17d &= 0 \\ \implies d &= 17 \text{ (since } d \neq 0). \end{aligned}$$

11. (a) It suffices to find the largest prime $p$ such that $3p \leq 275 \implies p < 92$. The largest prime less than 92 is 89.

(b) Similarly to the above, the largest prime $p$ such that $p^3$ divides $15!$ is $p = 5$. Hence the prime factors of $n$ are all less than or equal to 5. We determine the largest power of each of the primes 2, 3 and 5 that divides into $15!$. As noted, $5^3$ divides into $15!$. The multiples of 3 less than or equal to 15 are 3, 6, 9, 12, 15, which contain 6 factors of 3, so $3^6$ divides $15!$. Similarly, $2^{11}$ divides into $15!$. Hence the largest $n$ such that $n^3$ divides into $15!$ is equal to $2^3 \cdot 3^2 \cdot 5 = 360$.

12. (a) Looking horizontally, the square's side length equals 4 times the radius of the AA battery, so the radius is equal to $\frac{30}{4} = 7.5$mm.

(b) Let $A$ and $B$ be the centres of the two AA batteries, and $C$ the centre of the lower $AAA$ battery. Let $D$ be the midpoint of the square, and $E$ the point where the AAA battery touches the square. Suppose that the AA battery has radius $R$ and the AAA battery has radius $r$. Then $AD = R$, $AC = R+r$, $CE = r$ and $DE = 2R$. We work out $CD$ in two ways to get

$$\begin{aligned}
CD = DE - CE &= \sqrt{AC^2 - AD^2} \\
\implies 2R - r &= \sqrt{(R+r)^2 - R^2} \\
\implies (2R-r)^2 &= (R+r)^2 - R^2 \\
\implies 4R^2 - 4Rr + r^2 &= R^2 + 2Rr + r^2 - R^2 \\
\implies 4R^2 &= 6Rr \\
\implies r &= \frac{2R}{3}.
\end{aligned}$$

Since $R = 7.5$mm, it follows that $r = 5$mm.

(c) Let the midpoints of the two AA batteries be $A$ and $B$, let $O$ be the midpoint of the square and let $CD$ be the diagonal of the square through $A$ and $B$. Let $E$ be the point where the AA battery with mipoint $A$ touches the square. Let $R$ be the radius of the AA battery. We compute the length of the diagonal $CD$:

$$\begin{aligned}
CD &= 2(AC + CO) \\
&= 2(\sqrt{AE^2 + EC^2} + R) \\
&= 2(\sqrt{2R^2} + R) \\
&= 2R(\sqrt{2} + 1).
\end{aligned}$$

Now, if the side length of the square is $x$, then the length of the diagonal of the square is equal to $\sqrt{2}x$, and so $x = \frac{2R(\sqrt{2}+1)}{\sqrt{2}} = \sqrt{2}R(\sqrt{2}+1) = \sqrt{2}(\sqrt{2}+1) \cdot 7.5$.

13. The second player can force a win. Since the starting number is odd, the first player is forced to subtract an odd factor, which results in an even number on the board. The second player can then simply subtract 1 to write an odd number on the board again. The first player is thus again forced to subtract an odd factor, and so on. Since the second player always subtract 1 from the (necessarily even) number on the board, and write an odd number, the second player can never write the number zero, which is an even number, and hence cannot lose.

14. (a) Let $DE = a$ and $AD = BC = b$. Then $AB = a + 9$ and so

$$\begin{aligned}(a+9)^2 &= AB^2 = AE^2 + BE^2 = AD^2 + DE^2 + EC^2 + BC^2 \\ &= b^2 + a^2 + 9^2 + b^2 \\ \implies a^2 + 18a + 81 &= a^2 + 2b^2 + 81 \\ \implies 9a &= b^2.\end{aligned}$$

Since 9 is a square, it follows that $a$ must also be a square. Then, in triangle $ADE$, $AE^2 = a^2 + b^2 = a^2 + 9a = a(a+9)$. Since $a$ is a square, it follows that $a + 9$ is also a square. The only perfect squares which differ by 9 are 16 and 25, and so $a = 16$, $b = \sqrt{9a} = 12$ which leads to $AB = 9 + 16 = 25$, $AD = BC = 12$, $AE = \sqrt{a^2 + b^2} = \sqrt{16^2 + 12^2} = 20$ and $BE = \sqrt{9^2 + b^2} = \sqrt{9^2 + 12^2} = 15$.

(b) Similarly to above, we find that $7a = b^2$, and hence $a$ must be a square multiple of 7, say $a = 7x^2$. Then in triangle $CBE$, $BE^2 = 7^2+b^2 = 7^2+7a = 7^2+7^2x^2 = 7^2(x^2+1)$. This implies that $x^2+1$ is a perfect square, which is impossible for any integer $x > 0$.

15. (a) Each term of the sequence is at least twice the last, so if there are $n$ terms in the sequence, the last term would be at least $2^{n-1}$. Now, $2^{10} = 1024 < 2017 < 2048 = 2^{11}$, so $n \leq 11$. A possible sequence of length 11 is given by 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024.

(b) The first term in the sequence must be 1 for the last term to to be less than 2017. Since $2^9 \times 3 = 1526 < 2017 < 2^8 \times 3^2$, each other term is either 2 or 3 times greater than the previous term, with at most one being 3 times. There are 10 choices for the term that is 3 times the previous one, plus the sequence in part (a), giving 11 sequences in total.