1. If n is even, then x is obviously divisible by 101. If n is odd, then

$$x = 1111\ldots1 \times 909090\ldots91,$$

where the first factor has n ones and the second $(n-1)/2$ nines. The only prime case is $x = 101$, i.e. $n = 2$.

2. By inspection, $x = -1$ is a solution. Divide by $x+1$ to get $36x^3 - 7x - 1 = 0$. This begs to be written as $x(36x^2-1) - (6x+1) = 0$. Divide by $6x+1$ (which gives $x = -\frac{1}{6}$) to get $6x^2 - x - 1 = 0$. Factorize: $(3x+1)(2x-1) = 0$, giving $x = -\frac{1}{3}$ and $x = \frac{1}{2}$.

3. Express c in the form $c = \frac{1}{2}(x + \frac{1}{x})$ and a_n in the form $a_n = y_n + \frac{1}{y_n}$. This is possible for $c \geq 1$ and for a_1, with $y_1 = 1$, and since the recursion implies $a_{n+1} \geq a_n$, it is possible for all n. Then $c^2 - 1 = \left(\frac{1}{2}(x - \frac{1}{x})\right)^2$ and $a_n^2 - 4 = (y_n - \frac{1}{y_n})^2$. Hence

$$\begin{aligned}\left(y_{n+1} + \frac{1}{y_{n+1}}\right) &= \tfrac{1}{2}\left(x + \frac{1}{x}\right)\left(y_n + \frac{1}{y_n}\right) + \tfrac{1}{2}\left(x - \frac{1}{x}\right)\left(y_n - \frac{1}{y_n}\right) \\ &= xy_n + \frac{1}{xy_n}.\end{aligned}$$

This last equation is clearly satisfied by $y_n = x^n$, which gives $a_n = x^n + x^{-n}$. But it is well known that if $x + x^{-1}$ is an integer, the same is true for all $x^n + x^{-n}$.

4. Assign coordinates $A(0,0)$, $B(1,0)$, $C(1,1)$, $D(0,1)$, $P(x,0)$ and $Q(1,y)$. Then

$$\begin{aligned}\tan \widehat{PDA} &= x; \\ \tan \widehat{QDC} &= 1 - y.\end{aligned}$$

Since $\widehat{PDA} + \widehat{QDC} = 45°$, we have

$$\begin{aligned}\frac{\tan \widehat{PDA} + \tan \widehat{QDC}}{1 - \tan \widehat{PDA} \tan \widehat{QDC}} &= \frac{x + 1 - y}{1 - x + xy} \\ &= \tan(\widehat{PDA} + \widehat{QDC}) \\ &= 1\end{aligned}$$

and hence $y = \frac{2x}{1+x}$. We can now calculate the required perimeter as

$$1 - x + y + \sqrt{(1-x)^2 + y^2}$$

which simplifies to 2 since

$$(1-x)^2 + y^2 = (1+x-y)^2 - 4x + 2y(1+x) = (1+x-y)^2.$$

5. Consider the orbit of any $x \in \mathbb{Z}$, that is to say, the sequence

$$a_1 = x, \quad a_2 = f(a_1), \quad a_3 = f(a_2), \ldots .$$

By substituting successively a_2, a_3 etc. in the functional equation, one finds that

$$2000a_{n+2} - 3999a_{n+1} + 1999a_n, \ \texttt{for} \ n = 1, 2, 3, \ldots .$$

The corresponding quadratic equation is

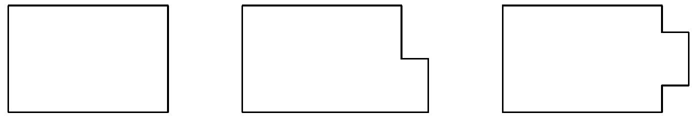
$$\begin{aligned} 0 &= 2000\lambda^2 - 3999\lambda + 1999 \\ &= (\lambda - 1)(2000\lambda - 1999) \end{aligned}$$

and its roots are therefore $\lambda_1 = 1$ and $\lambda_2 = \frac{1999}{2000}$. It is well-known that the general solution to the recurrence is given by

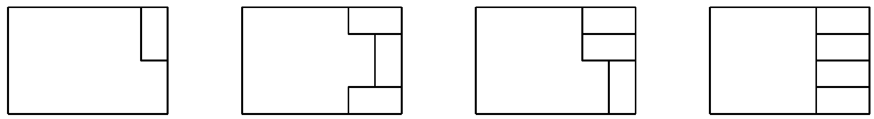
$$\begin{aligned} a_n &= K\lambda_1^n + L\lambda_2^n \\ &= K + L\left(\frac{1999}{2000}\right)^n \\ &\to K \ \text{as} \ n \to \infty. \end{aligned}$$

Since all a_n are integers, it follows that K must be an integer and $L = 0$. In other words, the sequence a_n is constant, and in particular $a_2 = a_1$, which means $f(x) = x$.

6. Let the number of ways to tile the following shapes,



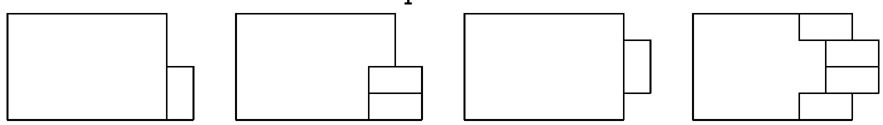
where the height of each shape is 4 and the width without the 2×1 protuberance is n, respectively be A_n, B_n and C_n. Starting to tile at the top right-hand corner of the first shape, we must reach one of the following mutually exclusive possibilities:



Hence

$$A_n = B_{n-1} + C_{n-2} + B_{n-2} + A_{n-2}.$$

Similarly, for the second and third shapes we must reach one of:



and therefore

$$B_n = A_n + B_{n-1}$$
$$C_n = A_n + C_{n-2}.$$

We can now solve the recursion, starting from the obvious facts that there is no way to tile when $n = -1$ and only one way (use no tiles) when $n = 0$. (If you are squeamish about this, you can with more effort start later.) Since the question involves divisibility by 2 and 3, it suffices to work modulo 6. (But it might build confidence to do the first few cases without reducing.) We obtain:

n	A_n	B_n	C_n
-1	0	0	0
0	1	1	1
1	1	2	1
2	5	1	0
3	5	0	0
4	0	0	0
5	5	5	5

Now note that $5 \equiv_6 -1$, so that the values for $n = 4$ and $n = 5$ are the negatives of those for $n = -1$ and $n = 0$. We will therefore obtain $A_n \equiv_6 -A_{n-5}$, etc. Therefore A_n can only be 0, 1 or 5 modulo 6, which proves the assertion.