THE HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

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in collaboration with HARMONY GOLD MINING, AMESA and SAMS

FIRST ROUND 2003

SENIOR SECTION: GRADES 10, 11 AND 12

18 MARCH 2003

TIME: 60 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. B
2. B
3. E
4. A
5. A
6. E
7. E
8. B
9. D
10. E
11. D
12. D
13. E
14. E
15. E
16. B
17. A
18. B
19. D
20. A

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SOLUTIONS

1. **Answer B.** First, evaluate the bracket: $(1-2)^{2003} = (-1)^{2003}$. Next, note that -1 raised to any odd power is equal to -1. Since 2003 is an odd number, it follows that $(-1)^{2003} = -1$.

2. **Answer B.** This is best done using exponents: $0.1 = 10^{-1}$ and $0.01 = 10^{-2}$. The expression is therefore equal to

$$(10^{-1})^3 \div 10^{-2} = 10^{-3} \div 10^{-2} = 10^{-3-(-2)} = 10^{-3+2} = 10^{-1} = 0.1.$$

3. **Answer E.**

$$(2^{-1}+3^{-1})^{-1} = (\frac{1}{2}+\frac{1}{3})^{-1} = (\frac{5}{6})^{-1} = \frac{6}{5}.$$

4. **Answer A.** We must substitute $a=1$, $b=2$, and $c=3$. Thus $1\#2\#3 = \dfrac{1+2}{3-1} = \dfrac{3}{2}$.

5. **Answer A.** List the powers of 2: $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $\ldots$, $2^9 = 512$, $2^{10} = 1024$, $2^{11} = 2048$. This is the closest to 2003, so the closest integer to x is 11.

6. **Answer E.** If we put $y = \sqrt{x\sqrt{x\sqrt{x}}}$, then $y^2 = x\sqrt{x\sqrt{x}}$, $y^4 = x^2x\sqrt{x}$, and $y^8 = x^4x^2x = x^7$, so $y = x^{7/8}$.

7. **Answer E.** If the team has an average of 28 points per game after five games, then the total points scored is $28 \times 5 = 140$. If the average is to increase by two points per game after six games, then the total will have to be $(28+2) \times 6 = 180$. Thus the number of points they must score in the sixth game is $180 - 140 = 40$.

8. **Answer B.** If you know (or can work out) that $\sqrt{3} \approx 1.7$, then $q \approx 5.1$, $r \approx 4.4$, and $s \approx 4.7$. In fact, all you need to know is that $2 > \sqrt{3}$: then, by adding 1 and multiplying by $\sqrt{3}$ on both sides we see that $3\sqrt{3} > \sqrt{3}+3$, so $q > s$. Also by adding $1+\sqrt{3}$ to both sides we see that $3+\sqrt{3} > 1+2\sqrt{3}$, so $s > r$.

9. **Answer D.** Squaring both sides gives $\frac{1}{6}(4+\sqrt{x+3})^2 + 3 = 9$, so $(4+\sqrt{x+3})^2 = 36$, and $4+\sqrt{x+3} = \pm 6$. Thus $\sqrt{x+3} = 2$ or -10, but the second answer is impossible, since a square root cannot be negative. Therefore the only possible answer is $\sqrt{x+3} = 2$, giving $x+3 = 4$, so $x = 1$.

10. **Answer E.** Note that all the expressions are products of exactly 100 factors. To evaluate 100! we multiply together *all* the integers from 1 to 100. To evaluate any of the other expressions, we multiply together the integers from 1 up to some integer less than 100, then continue multiplying, starting again at 1, so the product is obviously less than 100!.

11. **Answer D.** If an integer is divisible by 9, then the sum of its digits is also divisible by 9. (This follows from the fact that every power of 10 has remainder 1 after being divided by 9.) Thus $a+6+a+4+1$ is divisible by 9, that is, $2a+11 = 9$ or 18 or 27 and so on. However, the digit a must be an integer between 0 and 9, so the only possible equation is $2a+11 = 27$, which gives $a = 8$.

12. **Answer D.** If the year of Sophie's first marathon is y, then the next six years are $y+2$, $y+4$, $y+6$, $y+8$, $y+10$, $y+12$, giving a total of $7y+42$, which equals 13951. Thus $y = (13951-42)/7 = 1987$.

13. **Answer E.** We need to know the values of E for which $60 + 0.2E < 0.3(E-50)$, that is, $60+0.2E < 0.3E - 15$. This is the same as $75 < 0.1E$, or $750 < E$.

14. **Answer E.** The first triangle has two sides of length 1, and each of the remaining 99 triangles has one side of length 1. This gives 101 sides of length 1 to start with. What are the lengths of the unmarked sides, which are the hypotenuses of the triangles? By Pythagoras' theorem we see that they are $\sqrt{2}$, $\sqrt{3}$, $\ldots$, $\sqrt{101}$. How many of these are integers? Obviously only $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{16} = 4$, $\ldots$, $\sqrt{100} = 10$, giving nine more. Thus the total number of sides of integer length is $101 + 9 = 110$.

15. **Answer E.** We can write $k^3 + 2k^2 = k^2(k+2)$. If this is to be the square of an odd integer, then k must be odd (as well as positive) and $k+2$ must be an odd perfect square. The smallest possibility is $k = 7$. (If it were not stated that k must be positive, then $k = -1$ would be the answer.)

16. **Answer B.** Let $A\widehat{B}F = A\widehat{F}B = x^\circ$, and $C\widehat{B}D = C\widehat{D}B = y^\circ$. Then $D\widehat{B}F = (180-x-y)^\circ$. Also $\widehat{A} = (180-2x)^\circ$ and $\widehat{C} = (180-2y)^\circ$, so $360 - 2x - 2y = 90$, since triangle ACE is right-angled. Thus $180 - x - y = 45$, so $D\widehat{B}F = 45^\circ$.

17. **Answer A.** By Pythagoras' theorem, the length of the diagonal BD is $\sqrt{5}$. Now observe that triangles BCD and BCE are similar. Thus $BE/BC = CD/BD = 2/\sqrt{5}$, and since $BC = 1$, it follows that $BE = 2/\sqrt{5}$. The area of rectangle $BDFE$ is therefore $\sqrt{5} \times 2/\sqrt{5} = 2$.

18. Answer B. The number of paths to each letter after H is equal to the total number of paths to the preceding letter or letters. Thus the numbers of paths to the letters in the pattern are:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1 \\
 & 4 & & 6 & & 4 & \\
 & & 10 & & 10 & & \\
 & & & 20 & & &
\end{array}$$

and the last number is the required answer. (You may recognize these numbers as being part of Pascal's Triangle.)

19. Answer D. The quickest way is to see if there is a pattern: clearly $3^2 + 2 = 11$, and $33^2 + 22 = 11(99 + 2) = 1111$, with twice as many digits. It is now a reasonable guess (especially if you are short of time) that the expression is equal to $11\ldots1$ (with ten digits), so the sum of its digits is 10. For a formal proof, note that $99\ldots9$ (with n digits) is equal to $10^n - 1$, so $11\ldots1$ (with n digits) is equal to $(10^n - 1)/9$. Thus the general expression is equal to

$$\left(\frac{10^n - 1}{3}\right)^2 + \frac{2(10^n - 1)}{9} = \frac{10^n - 1}{9}[(10^n - 1) + 2] = \frac{(10^n - 1)(10^n + 1)}{9} = \frac{10^{2n} - 1}{9},$$

which has $2n$ digits, all equal to 1. In our case, with $n = 5$, the sum of the digits is 10.

20. Answer A. Since k, m, n are digits, they must be integers from 0 to 9. If we write the equation as $64k + 8m = 403 - n$, then we see that the left hand side is divisible by 8. The right hand side must therefore also be divisible by 8, which means that $n = 3$. Now divide through by 8 to get $8k + m = 50$, from which the same argument shows that $m = 2$ and $k = 6$.
