THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

SENIOR SECOND ROUND 2019

Solutions

1. **Answer 012**
   The rise in temperature is $8 - (-4) = 8 + 4 = 12$ degrees Celsius.

2. **Answer 120**
   30% of $400 = \frac{30}{100} \times 400 = 120$.

3. **Answer 006**
   If 6 cats catch 6 rats in 6 minutes, then 6 cats will catch 12 rats in 12 minutes.

4. **Answer 006**
   $2^n = \frac{256}{4} = 64 = 2^6$, so $n = 6$.

5. **Answer 060**
   From the sum of angles in a triangle, $\widehat{RQP} = 180° - 20° - 100° = 60°$, and from the sum of angles on a line, $\widehat{RQP} + \widehat{PQT} + \widehat{TQS} = 180°$. Therefore $60° + 2\widehat{PQT} = 180°$ (since $\widehat{PQT} = \widehat{TQS}$), so $\widehat{PQT} = \frac{1}{2}(180° - 60°) = 60°$.

6. **Answer 024**
   Let n be the number. Then $n + \frac{1}{4}n = 30$, so $\frac{5}{4}n = 30$, giving $n = \frac{4}{5} \times 30 = 24$.

7. **Answer 008**
   The base BC is a horizontal line of length $5 - 1 = 4$ (difference of x-values), and the vertical height of point A above that line is $5 - 1 = 4$ (difference of y-values), so the area is $\frac{1}{2}bh = \frac{1}{2} \times 4 \times 4 = 8$.

8. **Answer 035**
   Since the numbers are consecutive odd integers, the two middle ones must be 32 ± 1 and the two outside ones must be 32 ± 3. The largest number is therefore $32 + 3 = 35$.

9. **Answer 025**
   Since triangle HAR is isosceles (given), it follows that $\widehat{HAR} = \widehat{HRA} = x$, say, in degrees. Then $\widehat{YAR} = \widehat{YRA} = x - 40$ (given), so triangle YAR is isosceles, and $2(x - 40) + 130 = 180$, giving $x = 65$. Finally, triangles HAY and HRY are congruent (SAS), so $\widehat{AHY} = \frac{1}{2}AHR = \frac{1}{2}(180 - 2 \times 65) = 25$.

10. **Answer 012**
    There are $4 \times 3 \times 2 \times 1 = 4! = 24$ ways of arranging the four letters in SAMO. In exactly half of these, the vowels A and O will be in alphabetical order, giving $\frac{1}{2} \times 24 = 12$ arrangements.

11. **Answer 027**
    When a point is reflected in the x-axis, its y-co-ordinate changes sign. Therefore the reflection of the line $x + 3y + 80 = 0$ is $x - 3y + 80 = 0$. Thus $y = \frac{1}{3}(x + 80)$, so $m = \frac{1}{3}$ and $b = \frac{80}{3}$, giving $m + b = \frac{81}{3} = 27$.

12. **Answer 005**
If the rectangles have length ℓ and breadth b, then $\ell \times b = 12$ and $2\ell + 2b = 14$, so (by inspection or by solving the equations) $\ell = 4$ and $b = 3$. Since the angle in the circle where the two rectangles meet is $90°$, it follows from the angle in a semicircle that the other two vertices on the circle are at opposite ends of a diameter, which is of length $d = \sqrt{3^2 + 4^2} = 5$. The circumference of the circle is then $\pi d = 5\pi = n\pi$, so $n = 5$.

13. **Answer 006**
If $kx - 12 = 3k$, then $x = 3 + \frac{12}{k}$. If x and k are integers, then k must be a divisor or factor of 12. Since k is given to be positive, the possible values of k are $1, 2, 3, 4, 6, 12$, giving six possible values for x.

14. **Answer 009**

$$a + \cfrac{1}{b + \frac{1}{c}} = \frac{30}{13} = 2 + \frac{4}{13}, \quad \text{so} \quad b + \frac{1}{c} = \frac{13}{4} = 3 + \frac{1}{4}.$$

Thus $a = 2$, $b = 3$, $c = 4$, and therefore $a + b + c = 9$.

15. **Answer 012**
There are three ways to choose the two colours on any tile. For each choice, there are two tiles with three squares of one colour and one square of the other colour, plus two tiles with two squares of each colour (arranged either diagonally or side by side). This gives a total of $3 \times (2 + 2) = 12$ different colourings.

16. **Answer 200**
First note that $\frac{x^{52}+201x}{x} = x^{51} + 201$. Next, suppose

$$\frac{x^{51} + 201}{x + 1} = q(x) + \frac{r}{x + 1}, \quad \text{that is,} \quad x^{51} + 201 = (x + 1)q(x) + r.$$

(Note that the remainder r is constant, because it must have lower degree than the divisor $x + 1$.) If we substitute $x = -1$, then we obtain $(-1)^{51} + 201 = 0 \times q(-1) + r = r$, so $r = -1 + 201 = 200$.

17. **Answer 024**
Let S, C, R denote the sets of members playing soccer, cricket, rugby, respectively. The number of members who play at least one sport is $n(S \cup C \cup R) = 100 - 5 = 95$, so if $x = n(S \cap C \cap R)$ (that is, the number who play all three sports), then

$$95 = n(S) + n(C) + n(R) - n(S \cap C) - n(C \cap R) - n(S \cap R) + n(S \cap C \cap R)$$

$$= 44 + 45 + 50 - (8 + x) - (9 + x) - (7 + x) + x = 115 - 2x.$$

Thus $x = 10$, and the number of members playing rugby only is $50 - 10 - 9 - 7 = 24$. (Note: The way $n(S \cup C \cup R)$ is calculated is first to find $n(S) + n(C) + n(R)$; however, the numbers of people playing at least two sports have then been counted twice, so we must subtract $(n(S \cap C) + n(C \cap R) + n(S \cap R))$; now the number of people playing all three sports has been added three times and subtracted three times, so we must add it once to get the correct total. This is called the Principle of Inclusion and Exclusion.)

18. **Answer 002**
The numbers satisfying the criteria are: 1111 (with four digits), 121 and 211 (with three digits), and 13 and 31 (with two digits). Now $1111 = 11 \times 101$ and $121 = 11^2$. Trial and error shows that 211 is not divisible by any of the primes $2, 3, 5, 7, 1, 13$, so it is prime, as are 13 and 31. Thus exactly two such numbers are not prime.

19. **Answer 072**
Triangles MAC, MAD, MCD, MDB are all isosceles, because each has two sides that are radii. Triangle CAE is also isosceles (given). Working in degrees, let $\widehat{BAD} = x$ and $\widehat{DAC} = y$. In triangle CAM we have $\widehat{ACE} = x + y$ and in triangle CAE we have $\widehat{CEA} = y$ (AC=EC given). The angle sum of triangle CAE then gives $x + 3y = 180$ (eq 1).
Next, $\widehat{ADC} = x$ (alternate angles) and $\widehat{DCE} = y - x$ (external angle AEC of triangle CAE), but $\widehat{AMC} = 2 \cdot \widehat{ADC}$ (angle at the centre), so $y - x = 2x$, giving $y = 3x$ (eq 2).
By substituting of the second equation in the first we see that $x + 9x = 180$, so $x = 18$ and $y = 3x = 54$. Finally, $\widehat{CAM} = x + y = 18 + 54 = 72$.

20. **Answer 100**
Let b be the length of the slant side of the parallelogram with height 5. Then $5 = b \sin 30^\circ = \frac{1}{2}b$, so $b = 10$. Regarding b as the base of the shaded parallelogram, which is of height 10, we see that its area is $10b = 10 \times 10 = 100$.

21. **Answer 072**
If the distance between Durban and Pietermaritzburg is d km, then the time in hours for the uphill journey is $\frac{1}{60}d$ and for the downhill journey is $\frac{1}{90}d$, giving a total time of $(\frac{1}{60} + \frac{1}{90})d = \frac{5}{180}d$. The average speed is then $(2d) \div (\frac{5}{180}d) = \frac{360}{5} = 72$ km/h.

22. **Answer 001**
Suppose the roots are p, q, r, where $q - p = r - q$, so $p + r = 2q$. Then

$$64x^3 - 144x^2 + 92x - 15 = 64(x-p)(x-q)(x-r)$$

$$= 64(x^3 - (p+q+r)x^2 + \cdots - pqr).$$

Therefore $144 = 64(p+q+r)$ from the coefficient of x^2, and $15 = 64pqr$ from the constant term.
But $p + q + r = 2q + q = 3q$, so $3q = \frac{144}{64} = \frac{9}{4}$, giving $q = \frac{3}{4}$ and $p + r = 2q = \frac{3}{2}$.
Next, $15 = 64pqr = \frac{3}{4}(64pr) = 48pr$, so $pr = \frac{15}{48} = \frac{5}{16}$.
Finally, $(p-r)^2 = (p+r)^2 - 4pr = (\frac{3}{2})^2 - 4\frac{5}{16} = \frac{9}{4} - \frac{5}{4} = 1$, so $|p - r| = 1$.

23. **Answer 022**

$$f(6) = \frac{3-1}{3+1} = \frac{1}{2}, \quad f(9) = \frac{\frac{1}{2} - 1}{\frac{1}{2} + 1} = -\frac{1}{3},$$

$$f(12) = \frac{-\frac{1}{3} - 1}{-\frac{1}{3} + 1} = -2, \quad f(15) = \frac{-2-1}{-2+1} = 3 = f(3).$$

Therefore $f(n + 12) = f(n)$ for all positive integers n divisible by 3. Now $2019 = 168 \times 12 + 3$, so $f(2019) = f(3) = 3$. $f(2019) + 19 = f(3) + 19 = 22$.

24. **Answer 340**
When the dice are rolled, the sum can be any integer from n to $6n$. The sum $n + k$ can be obtained in the same number of ways as the sum $6n - k$, and this number of ways increases as k increases. Minimize $S = n + k$ by choosing n and k as small as possible with $6n - k = 2019$. Since the least multiple of 6 that is greater than or equal to 2019 is $2022 = 6(337)$, S is smallest when $n = 337$ and $k = 3$. Consequently, $S = n + k = 340$.

25. **Answer 648**
If the side length of T is x, then the diagonal of S is equal to $x + \frac{1}{2}x$, so $27\sqrt{2} = \frac{3}{2}\,x$, giving $x = 18\sqrt{2}$. The area of T is therefore $18^2 \times 2 = 324 \times 2 = 648$.

Dus is $f(n+12) = f(n)$ vir alle positiewe heelgetalle n deelbaar deur 3. Nou is $2019 = 168 \times 12 + 3$, sodat $f(2019) = f(3) = 3$ en $f(2019) + 19 = f(3) + 19 = 3 + 19 = 22$

24. **Antwoord 340**
Wanneer die dobbelstene gegooi word kan die som enige heelgetal tussen n en $6n$ wees. 'n Som van $n + k$ kan op dieselfde aantal maniere verkry word as 'n som van $6n - k$, en hierdie aantal maniere neem toe soos k toeneem. Minimeer $S = n + k$ deur n en k so klein as moontlik te kies sodanig dat $6n - k = 2019$. Die kleinste veelvoud van 6 groter of gelyk aan 2019 is $2022 = 6(337)$, en dus is S die kleinste wanneer $n = 337$ en $k = 3$. Gevolglik is $S = n + k = 340$.

25. **Antwoord 648**
As die sylengte van T gelyk is aan x, dan is die hoeklyn van S gelyk aan $x + \frac{1}{2}x$, sodat $27\sqrt{2} = \frac{3}{2}x$, wat $x = 18\sqrt{2}$ gee. Die oppervlakte van T is dus $18^2 \times 2 = 324 \times 2 = 648$.

18. **Antwoord 002**
Die getalle wat aan die vereistes voldoen, is: 1111 (met vier syfers), 121 en 211 (met drie syfers), en 13 en 31 (met twee syfers). Nou is $1111 = 11 \times 101$ en $121 = 11^2$. Probeer en Verbeter toon dat 211 nie deelbaar is deur enige van die priemgetalle $2, 3, 5, 7, 11, 13$ nie, sodat hulle dus priemgetalle is, net soos 13 en 31. Dus is daar presies twee sulke getalle wat nie priem is nie.

19. **Antwoord 072**
Driehoeke MAC, MAD, MCD, MDB is almal gelykbenig, want elkeen het twee sye wat radii is. Driehoek CAE is ook gelykbenig (gegee). Ons werk in grade en laat $\widehat{BAD} = x$ en $\widehat{DAC} = y$. In driehoek CAM is $\widehat{ACE} = x + y$ en in driehoek CAE is $\widehat{CEA} = y$ (AC=EC gegee). Die som van die hoeke van driehoek CAE gee dan $x + 3y = 180$ (vgl. 1).
Volgende, $\widehat{ADC} = x$ (verwisselende hoeke) en $\widehat{DCE} = y - x$ (buitehoek AEC van driehoek CAE), maar $\widehat{AMC} = 2 \cdot \widehat{ADC}$ (hoek by middelpunt), sodat $y - x = 2x$, wat $y = 3x$ (vgl. 2) gee.
Deur substitusie van die tweede vergelyking in die eerste sien ons dat $x + 9x = 180$, sodat $x = 18$ en $y = 3x = 54$. Ten slotte is $\widehat{CAM} = x + y = 18 + 54 = 72$.

20. **Antwoord 100**
Laat b die lengte van die skuinssy van die parallelogram met hoogte 5 wees. Dan is $5 = b \sin 30^\circ = \frac{1}{2}b$, sodat $b = 10$. Deur b as die basis van die ingekleurde parallelogram te neem, wat 'n hoogte van 10 het, sien ons dat die oppervlakte gelyk is aan $10b = 10 \times 10 = 100$.

21. **Antwoord 072**
As die afstand tussen Durban en Pietermaritzburg gelyk is aan d km, dan is die tyd, in uur, vir die opdraande rit met vrag gelyk aan $\frac{1}{60}d$ en vir die afdraande rit sonder vrag is dit $\frac{1}{90}d$, wat 'n totale tyd gee van $(\frac{1}{60} + \frac{1}{90})d = \frac{5}{180}d$. Die gemiddelde spoed is dan $(2d) \div (\frac{5}{180}d) = \frac{360}{5} = 72$ km/h.

22. **Antwoord 001**
Laat die wortels p, q, r wees waar $q - p = r - q$, sodat $p + r = 2q$. Dan is

$$64x^3 - 144x^2 + 92x - 15 = 64(x-p)(x-q)(x-r)$$

$$= 64(x^3 - (p+q+r)x^2 + \cdots - pqr).$$

Dus is $144 = 64(p + q + r)$ deur die koëffisient van x^2 te gebruik, en $15 = 64pqr$ deur die konstante term te gebruik.
Maar, $p + q + r = 2q + q = 3q$, sodat $3q = \frac{144}{64} = \frac{9}{4}$, wat $q = \frac{3}{4}$ en $p + r = 2q = \frac{3}{2}$ gee.
Volgende, $15 = 64pqr = \frac{3}{4}(64pr) = 48pr$, sodat $pr = \frac{15}{48} = \frac{5}{16}$.
Ten slotte is $(p - r)^2 = (p + r)^2 - 4pr = (\frac{3}{2})^2 - 4\frac{5}{16} = \frac{9}{4} - \frac{5}{4} = 1$, sodat $|p - r| = 1$.

23. **Antwoord 022**

$$f(6) = \frac{3-1}{3+1} = \frac{1}{2}, \quad f(9) = \frac{\frac{1}{2}-1}{\frac{1}{2}+1} = -\frac{1}{3},$$

$$f(12) = \frac{-\frac{1}{3}-1}{-\frac{1}{3}+1} = -2, \quad f(15) = \frac{-2-1}{-2+1} = 3 = f(3).$$

12. **Antwoord 005**
As die reghoeke lengte ℓ en breedte b het, dan is $\ell \times b = 12$ en $2\ell + 2b = 14$, sodat (deur inspeksie of deur die vergelykings op te los) $\ell = 4$ en $b = 3$. Omdat die hoek in die sirkel waar die twee reghoeke ontmoet, gelyk is aan $90°$, volg dit uit die hoek in 'n semisirkel dat die ander twee hoekpunte op die sirkel op die teenoorsgestelde punte van 'n middellyn moet wees, wat 'n lengte van $d = \sqrt{3^2 + 4^2} = 5$ het. Die omtrek van die sirkel is dan $\pi d = 5\pi = n\pi$, sodat $n = 5$.

13. **Antwoord 006**
As $kx - 12 = 3k$, dan is $x = 3 + \frac{12}{k}$. As x en k heelgetalle is, dan moet k 'n deler of faktor van 12 wees. Omdat k positief moet wees, is die moontlike waardes vir k dan $1, 2, 3, 4, 6, 12$, wat ses moontlike waardes vir x gee.

14. **Antwoord 009**
$$a + \frac{1}{b + \frac{1}{c}} = \frac{30}{13} = 2 + \frac{4}{13}, \quad \text{so} \quad b + \frac{1}{c} = \frac{13}{4} = 3 + \frac{1}{4}.$$
Dus is $a = 2$, $b = 3$, $c = 4$, en $a + b + c = 9$.

15. **Antwoord 012**
Daar is drie maniere om twee kleure vir enige teël te kies. Vir elke keuse is daar twee teëls met drie vierkante van een kleur en een vierkant van 'n ander kleur, plus twee teëls met twee vierkante van elke kleur (diagonaal of langs mekaar). Dit gee 'n totaal van $3 \times (2 + 2) = 12$ verskillende maniere om die teëls in te kleur.

16. **Antwoord 200**
Let op dat $\frac{x^{52}+201x}{x} = x^{51} + 201$. Volgende, laat
$$\frac{x^{51} + 201}{x + 1} = q(x) + \frac{r}{x + 1}, \quad \text{en dit is,} \quad x^{51} + 201 = (x + 1)q(x) + r.$$
(Let op dat die res r 'n konstante is omdat dit van 'n laer graad moet wees as die deler $x + 1$.) As ons $x = -1$ instel, kry ons $(-1)^{51} + 201 = 0 \times q(-1) + r = r$, sodat $r = -1 + 201 = 200$.

17. **Antwoord 024**
Laat S, C, R die versamelings aandui van die lede wat onderskeidelik sokker, krieket en rugby speel. Die aantal lede wat aan ten minste een sportsoort deelneem, is $n(S \cup C \cup R) = 100 - 5 = 95$, en dus as $x = n(S \cap C \cap R)$ (d.i. die aantal wat aan al drie sportsoorte deelneem), dan is
$$\begin{aligned} 95 &= n(S) + n(C) + n(R) - n(S \cap C) - n(C \cap R) - n(S \cap R) + n(S \cap C \cap R) \\ &= 44 + 45 + 50 - (8 + x) - (9 + x) - (7 + x) + x = 115 - 2x. \end{aligned}$$
Dus is $x = 10$, en die aantal lede wat slegs rugby speel, is $50 - 10 - 9 - 7 = 24$.
(Let op: Die manier waarop $n(S \cup C \cup R)$ bereken word, is om eerstens $n(S) + n(C) + n(R)$ te vind; maar die getal mense wat aan ten minste twee sportsoorte deelneem, is twee keer getel en $(n(S \cap C) + n(C \cap R) + n(S \cap R))$ moet dus afgetrek word; en nou is die aantal mense wat aan al drie sportsoorte deelneem drie keer bygetel en drie keer afgetrek en dit moet dus weer een keer bygetel word. Dit word die Beginsel van Insluiting en Uitsluiting genoem.)

DIE SUID-AFRIKAANS WISKUNDE OLIMPIADE
SENIOR TWEEDE RONDTE 2019
Oplossings

1. **Antwoord 012**
   Die styging in temperatuur is $8 - (-4) = 8 + 4 = 12$ grade Celsius.

2. **Antwoord 120**
   30% van $400 = \frac{30}{100} \times 400 = 120$.

3. **Antwoord 006**
   As 6 katte 6 rotte in 6 minute vang, sal 6 katte 12 rotte in 12 minute vang.

4. **Antwoord 006**
   $2^n = \frac{256}{4} = 64 = 2^6$, dus $n = 6$.

5. **Antwoord 060**
   Uit die som van die hoeke van 'n driehoek het ons $\widehat{RQP} = 180° - 20° - 100° = 60°$, en uit die som van die hoeke op 'n reguit lyn het ons $\widehat{RQP} + \widehat{PQT} + \widehat{TQS} = 180°$. Dus, $60° + 2\widehat{PQT} = 180°$ (omdat $\widehat{PQT} = \widehat{TQS}$), sodat $\widehat{PQT} = \frac{1}{2}(180° - 60°) = 60°$.

6. **Antwoord 024**
   Laat n die getal wees. Dan is $n + \frac{1}{4}n = 30$, sodat $\frac{5}{4}n = 30$, wat $n = \frac{4}{5} \times 30 = 24$ gee.

7. **Antwoord 008**
   Die basis BC is 'n horisontale lyn met lengte $5 - 1 = 4$ (verskil tussen x-waardes), en die vertikale hoogte van punt A bokant BC is $5 - 1 = 4$ (verskil tussen y-waardes), en dus is die oppervlakte $\frac{1}{2}bh = \frac{1}{2} \times 4 \times 4 = 8$.

8. **Antwoord 035**
   Omdat die getalle opeenvolgende onewe getalle is, is die twee in die middel 32 ± 1, en die twee aan die buitekant is dan 32 ± 3. Die grootste getal is dus $32 + 3 = 35$.

9. **Antwoord 025**
   Omdat driehoek HAR gelykbenig is (gegee), volg dit dat $\widehat{HAR} = \widehat{HRA} = x$, in grade. Dan is $\widehat{YAR} = \widehat{YRA} = x - 40$ (gegee), en dus is driehoek YAR gelykbenig, en $2(x - 40) + 130 = 180$, wat $x = 65$ gee. Ten slotte is driehoeke HAY en HRY kongruent (SHS), sodat $\widehat{AHY} = \frac{1}{2}AHR = \frac{1}{2}(180 - 2 \times 65) = 25$.

10. **Antwoord 012**
    Daar is $4 \times 3 \times 2 \times 1 = 4! = 24$ maniere om die vier letters in SAMO te rangskik. In presies die helfte van hierdie gevalle is die klinkers A en O in alfabetiese volgorde, sodat ons $\frac{1}{2} \times 24 = 12$ rangskikkings het.

11. **Antwoord 027**
    Wanneer 'n punt in die x-as gereflekteer word, verander die teken van die y-koördinaat. Die refleksie van die lyn $x + 3y + 80 = 0$ is dan $x - 3y + 80 = 0$. Dus is $y = \frac{1}{3}(x + 80)$, sodat $m = \frac{1}{3}$ en $b = \frac{80}{3}$, wat $m + b = \frac{81}{3} = 27$ gee.