THE OLD MUTUAL SOUTH AFRICAN MATHEMATICS OLYMPIAD

SAMF
SOUTH AFRICAN MATHEMATICS FOUNDATION

Organised by the
SOUTH AFRICAN MATHEMATICS FOUNDATION

2021 THIRD ROUND
JUNIOR SECTION: GRADES 8 AND 9

28 July 2021 | **Time: 4 Hours** | **Number of questions: 15**

TOTAL: 100

Instructions

- Answer all the questions.
- All working details and explanations must be shown. Answers alone will not be awarded full marks.
- The neatness in your presentation of the solutions may be taken into account.
- Diagrams are not necessarily drawn to scale.
- No calculator of any form, or any geometric instruments may be used.
- Use your time wisely and do not spend all your time on only a few questions.
- Questions are not necessarily arranged in order of difficulty.
- Answers and solutions will be made available at: www.samf.ac.za

Do not turn the page until you are told to do so.
Draai die boekie om vir die Afrikaanse vraestel.

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Organisations involved: AMESA, SA Mathematical Society,
SA Akademie vir Wetenskap en Kuns

OLDMUTUAL

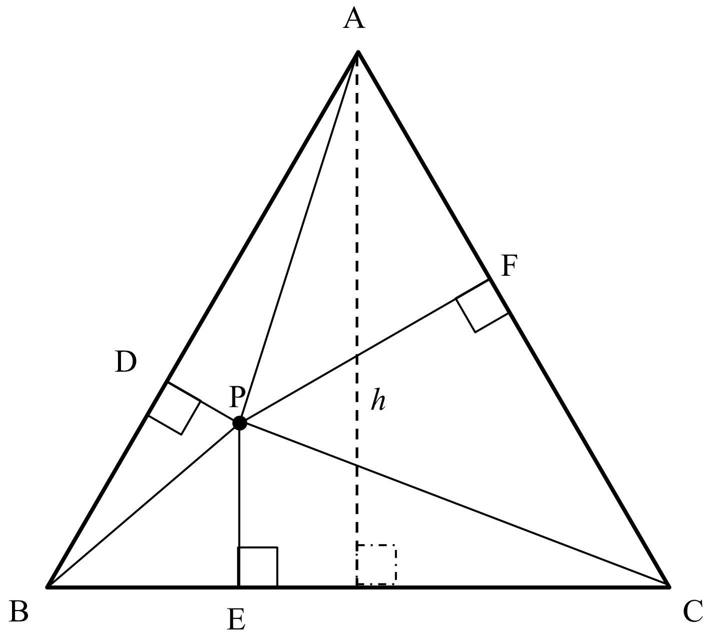
1) Which number is larger: $2\sqrt{2021}$ or $20\sqrt{21}$? Motivate your answer.

2) To convert between the Fahrenheit and Celsius temperature scales, we use the following formula: F $= \frac{9}{5}C + 32$, where F is the temperature reading in degrees Fahrenheit and C is the temperature reading in degrees Celsius.

   At which temperature (in either scale) do the two temperature scales give the same reading?

3) Let ABC be an equilateral triangle with side length $6\sqrt{3}$ and height h, and let P be a point inside the triangle. Let D, E and F be the feet of perpendiculars dropped from P to the sides AB, BC and AC, respectively. It is given that PD = 1 and PE = 3.

   a) What are the areas of triangles APB and BPC?
   b) An interesting result for any equilateral triangle is that $PD + PE + PF = h$. Prove it.
   c) What is the length of PF?



4) Steve adds consecutive odd numbers 1 + 3 + 5 + 7 + … up to a certain point and claims that the total sum is 2020.

   a) Show that Steve made a mistake.
   b) Could it be that Steve accidentally skipped a number? If so, which one?
   c) Show that Steve didn't accidentally add the same number twice.

5) Some binomials, such as the difference of two squares, can be factorised, for example, $x^2 - 4y^2 = (x - 2y)(x + 2y)$.
   We have been taught that others such as $x^2 + 4y^2$ cannot be factorised.
   It turns out, somewhat surprisingly, that $x^4 + 4y^4$ *can indeed* be factorised!

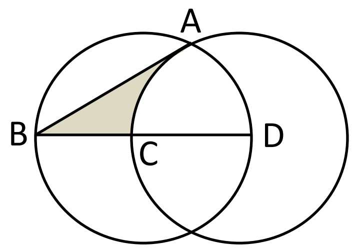
   a) Find integers m and n such that $x^4 + 4y^4 = (x^2 + my^2)^2 - (nxy)^2$
   b) Hence, or otherwise, factorise $x^4 + 4y^4$
   c) Show that $5^4 + 2^{14}$ is divisible by 73.

6) Two players play a game on a $1 \times n$ strip of squares. Initially, the first player has a counter on the left-most square, and the second player has a counter on the right-most square. They take turns to move their counters. A player can move their counter either one square to the left, or one square to the right. If a player can't make a move, that player loses.

   a) If both players play optimally on a 1×4 strip of squares, which player always wins?
   b) If both players play optimally on a 1×5 strip of squares, which player always wins?
   c) If both players play optimally on a $1 \times n$ strip of squares, which player always wins? Explain why.

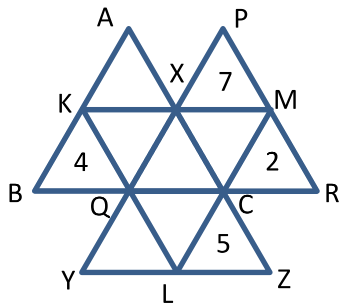
7) Two circles with radius 1 and centres C and D intersect at point A. BD is a diameter of the circle with centre C.

a) What is the size of angle ABC?
b) Find the area of the shaded region.



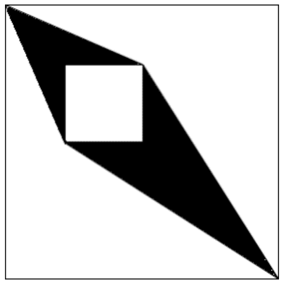
8) The diagram shows four triangles, $ABC, KLM, XYZ,$ and $PQR,$ each of which is divided up into four smaller triangles. The diagram is to be completed so that the positive integers from 1 to 10 (inclusive) are placed in the ten small triangles, one number per small triangle. The sums of the numbers in the four triangles ABC, XYZ, KLM and PQR should be the same. Numbers 2, 4, 5 and 7 have already been placed.

There are two different ways in which to complete the diagram – what are the possible values for the number in triangle QCL?



9) Find three pairs of integers m and n with $m \geq n$ such that $\frac{1}{m} + \frac{1}{n} = \frac{1}{5}$.
(You do not need to prove that these are the only solutions, but you must still show all your working).

10) A square with side length 2 is inside a square with side length 7. The sides of the smaller square are parallel to the sides of the larger one.



Show that the area of the shaded region does not depend on the position of the small square.

11) a) What is the remainder when 7^{2021} is divided by 6?

b) Let $n = 7 + 7^2 + 7^3 + \ldots + 7^{2021}$.
What are the last two digits of $6n$? (Hint: $6n = 7n - n$)

12) A 20×20 grid is divided into unit squares, some of which are painted black.

a) If none of the black squares share an edge (but some may touch diagonally), what is the largest number of squares that can be black? Prove your answer.

b) If none of the black squares touch (not even diagonally), what is the largest number of squares that can be black? Prove your answer.

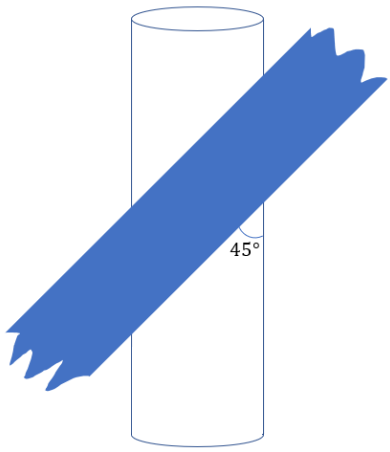
13) Solve the ALPHAMETIC puzzle below. Remember to write down your workings and observations – even if you don't solve the alphametic completely, marks will still be given for correct reasoning and observations. (Each letter represents a unique digit and leading zeros are not allowed).

$$\begin{array}{rcccccc} & & V & I & R & U & S \\ + & & V & I & R & U & S \\ \hline & C & O & R & O & N & A \\ \hline \end{array}$$

14) Alice, Brad, Christine, David, Erin and Farah went to a breakfast together and sat around a circular table eating pancakes. Alice ate more pancakes than the person to her right, but fewer than Christine. Brad ate fewer pancakes than the person to his right, but more than Christine. Christine ate fewer pancakes than the person to her left, but more than David. David ate more pancakes than the person to his left, but fewer than Erin. Farah was not hungry and did not eat any pancakes at all.

Determine the seating arrangement of the friends around the table and prove that it is the only solution.

15) We wrap a blue ribbon around a cylindrical white pole, initially placed at an angle of 45 degrees. In this way, we get a blue spiral around the pole, with a white spiral where the ribbon doesn't cover the pole.
The radius of the pole is 2 cm.
It turns out that the white and blue spirals have the same widths.

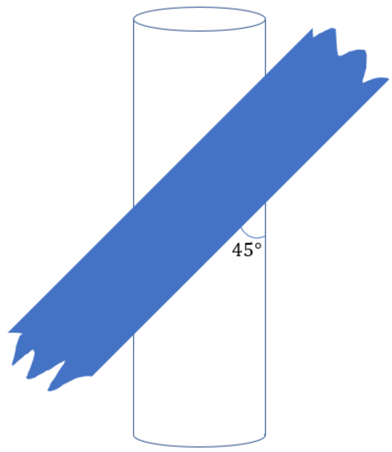


What is the width of the ribbon?

14) Alice, Brad, Christine, David, Erin en Farah gaan ontbyt eet en sit aan by ‘n ronde tafel waar hulle pannekoeke eet. Alice eet meer pannekoeke as die persoon aan haar regterkant, maar minder as Christine. Brad eet minder pannekoeke as die persoon aan sy regterkant, maar meer as Christine. Christine eet minder pannekoeke as die persoon aan haar linkerkant, maar meer as David. David eet meer pannekoeke as die persoon aan sy linkerkant, maar minder as Erin. Farah was nie honger nie en het niks pannekoeke geëet nie.

Bepaal die tafelplasing van die vriende rondom die tafel en bewys dat dit die enigste oplossing is.

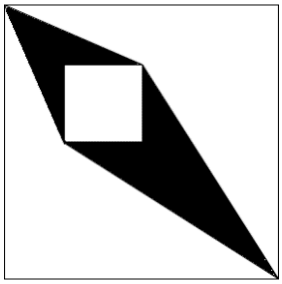
15) Ons het ‘n blou lint om ‘n silindervormige wit paal gedraai. Die lint is aanvanklik teen ‘n hoek van 45 grade geplaas. Op hierdie manier kry ons ‘n blou spiraal om die paal, met ‘n wit spiraal waar die lint nie die paal bedek nie.
Die radius van die paal is 2 cm.
Vervolgens word waargeneem dat die blou en wit spirale dieselfde breedte het.



Wat is die breedte van die lint?

10) ‘n Vierkant met sylengte 2 is binne-in ‘n vierkant met sylengte 7. Die sye van die kleiner vierkant is ewewydig aan die sye van die groter vierkant.

Toon aan dat die oppervlakte van die ingekleurde gebied nie afhanklik van die posisie van die klein vierkant is nie.



11) a) Wat is die res as 7^{2021} deur 6 gedeel word?

b) Laat $n = 7 + 7^2 + 7^3 + \ldots + 7^{2021}$.
Wat die laaste twee syfers van $6n$? (Wenk: $6n = 7n - n$)

12) ‘n 20×20 rooster word in eenheidsvierkante verdeel, waarvan sommige swart geverf is.

a) Indien geen swart vierkant ‘n sy met ‘n ander swart vierkant deel nie (sommige mag wel diagonaal raak), wat is die grootste aantal vierkante wat swart kan wees?
Bewys jou antwoord.

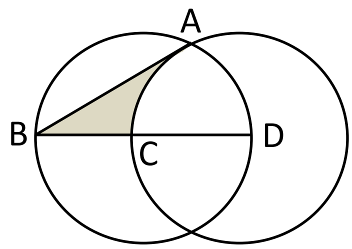
b) Indien geen swart vierkant ‘n ander swart vierkant raak nie (nie eens diagonaal nie), wat is die grootste aantal vierkant wat swart kan wees?
Bewys jou antwoord.

13) Los die ALPHAMETIC hieronder op. Onthou om jou bewerkings en waarnemings neer te skryf – selfs al sou jy nie die alphametic volledig kan oplos nie, sal jy nogtans punte vir korrekte redenasies en waarnemings verdien. (Elke letter stel ‘n unieke syfer voor en leidende nulle word nie toegelaat nie).

$$\begin{array}{cccccc} & V & I & R & U & S \\ + & V & I & R & U & S \\ \hline C & O & R & O & N & A \\ \hline \end{array}$$

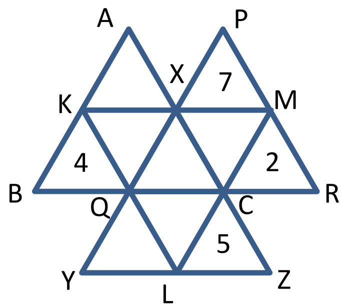
7) Twee sirkels met radiusse 1 en middelpunte C en D sny mekaar in punt A. BD is ‘n middellyn van die sirkel met middelpunt C.

a) Wat is die grootte van hoek ABC?
b) Bepaal die area van die ingekleurde gebied.



8) Die diagram toon vier driehoeke, ABC, KLM, XYZ, en PQR aan. Elke driehoek is in vier kleiner driehoeke verdeel. Die diagram moet voltooi word sodat die positiewe heelgetalle van 1 tot 10 (beide ingesluit) in die klein driehoeke geplaas word, met een getal in elke klein driehoek. Die som van die getalle in elk van die vier driehoeke ABC, KLM, XYZ, en PQR, moet dieselfde wees. Die getalle 2, 4, 5 en 7 is reeds geplaas.

Daar is twee verskillende maniere om die diagram te voltooi.
Wat is die moontlike waardes van die getal in driehoek QCL?



9) Bepaal drie pare heelgetalle m en n met $m \geq n$ sodat $\frac{1}{m} + \frac{1}{n} = \frac{1}{5}$.
(Jy hoef nie te bewys dat dit die enigste oplossings is nie, maar jy moet nogtans al jou berekeninge aantoon).

4) Steve tel die opeenvolgende onewe getalle $1 + 3 + 5 + 7 + \ldots$ tot by ‘n sekere punt op, en beweer dan dat die som van die getalle 2020 is.

a) Toon aan dat Steve ‘n fout begaan het.
b) Is dit moontlik dat Steve per ongeluk ‘n getal oorgeslaan het? Indien wel, watter getal?
c) Toon dat Steve nie per ongeluk dieselfde getal tweemaal getel het nie.

5) Sommige tweeterme, soos die verskil van twee vierkante, kan gefaktoriseer word, byvoorbeeld, $x^2 - 4y^2 = (x - 2y)(x + 2y)$.
Ons het ook geleer dat ander tweeterme soos $x^2 + 4y^2$, nie gefaktoriseer kan word nie.
Tog, verbasend genoeg, kan $x^4 + 4y^4$ *wel* gefaktoriseer word!

a) Bepaal die heelgetalle m en n sodat $x^4 + 4y^4 = (x^2 + my^2)^2 - (nxy)^2$.
b) Faktoriseer vervolgens -of andersins, $x^4 + 4y^4$.
c) Toon aan dat $5^4 + 2^{14}$ deelbaar is deur 73.

6) Twee spelers speel ‘n bordspel op ‘n $1 \times n$ strook van vierkante. Aanvanklik begin die eerste speler met sy merker op die blokkie heel links. Die tweede speler begin met sy merker op die blokkie heel regs. Hulle neem beurte om hulle merkers te beweeg. ‘n Speler kan sy merker slegs een blokkie na links of een blokkie na regs beweeg. Indien ‘n speler nie meer kan beweeg nie, verloor hy.

a) Indien beide spelers optimaal op ‘n 1×4 strook van vierkante speel, watter speler sal altyd wen?
b) Indien beide spelers optimaal op ‘n 1×5 strook van vierkante speel, watter speler sal altyd wen?
c) Indien beide spelers optimaal op ‘n $1 \times n$ strook van vierkante speel, watter speler sal altyd wen? Verduidelik hoekom.

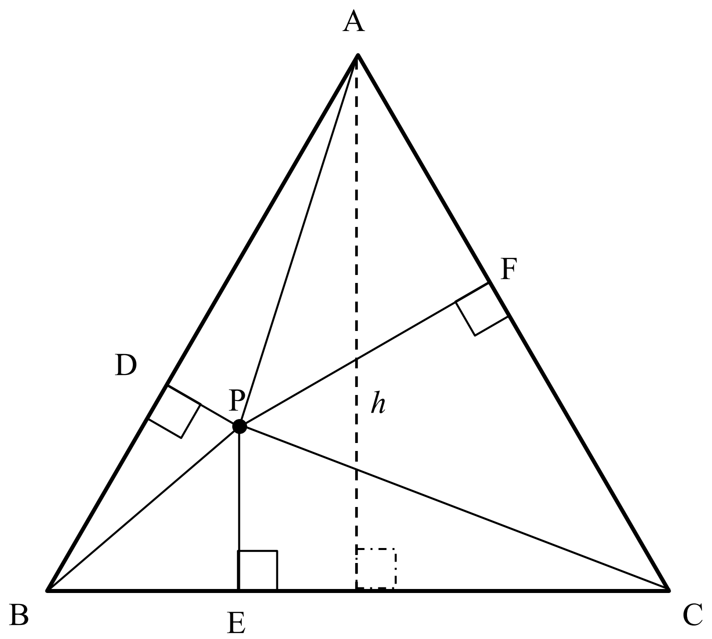
1) Watter een van die getalle $2\sqrt{2021}$ of $20\sqrt{21}$ is die grootste?
Motiveer jou antwoord.

2) Om omskakelings tussen die Fahrenheit- en Celsius-temperatuurskale te doen, gebruik ons die volgende formule: $F = \frac{9}{5}C + 32$, waar F die temperatuurlesing in grade Fahrenheit en C die temperatuurlesing in grade Celsius aandui.

By watter temperatuur (op enige skaal) sal die twee temperatuurskale dieselfde lesing gee?

3) Gestel ABC is ‘n gelyksydige driehoek met sylengte $6\sqrt{3}$ en hoogte h, met P ‘n punt binne die driehoek. Laat D, E en F die eindpunte van die loodlyne, getrek vanuit punt P, tot by die sye van die driehoek wees. Verder word gegee dat PD = 1 en PE = 3.

a) Bepaal die oppervlakte van driehoek APB en driehoek BPC.
b) ‘n Interessante gevolgtrekking wat vir enige gelyksydige driehoek gemaak kan word is dat
$PD + PE + PF = h$. Bewys dit.
c) Bepaal die lengte van lynstuk PF.



DIE OLD MUTUAL SUID-AFRIKAANSE WISKUNDE-OLIMPIADE

Georganiseer deur die
SOUTH AFRICAN MATHEMATICS FOUNDATION

2021 DERDE RONDE
JUNIOR AFDELING: GRAAD 8 EN 9

28 Julie 2021 | **Tyd: 4 Ure** | **Aantal vrae: 15**

TOTAAL: 100

Instruksies

- Beantwoord al die vrae.
- Alle berekeninge en motiverings moet getoon word. Antwoorde sonder motivering sal nie volpunte verdien nie.
- Die netheid van jou oplossings mag in ag geneem word.
- Diagramme is nie noodwendig volgens skaal geteken nie.
- Geen sakrekenaar, in welke vorm ook al, of enige meetkunde instrumente, mag gebruik word nie.
- Gebruik jou tyd oordeelkundig en moenie al jou tyd op slegs 'n paar vrae spandeer nie.
- Vrae is nie noodwendig in volgorde van maklik na moeilik gerangskik nie.
- Die antwoorde en oplossings sal beskikbaar wees by: www.samf.ac.za

Moenie omblaai voordat daar vir jou gesê word om dit te doen nie.

Turn the booklet over for the English paper.

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