The South African Mathematical Olympiad
Third Round 2021
Senior Division (Grades 10 to 12)
Solutions

1. Find the smallest and largest integers with decimal representation of the form $ababa$ $(a \neq 0)$ that are divisible by 11.

   Solution: The number $N = \underline{ababa}$ is divisible by 11 if and only if $a - b + a - b + a = 3a - 2b$ is divisible by 11. To find the smallest possible N with this property, we observe that $a = 1$ (the smallest possible value for a) and $b = 7$ offer a solution: $3 \cdot 1 - 2 \cdot 7 = -11$ is divisible by 11 (and no smaller $b \in \{0, 1, \ldots, 9\}$ works). So $N = 17171$ is the smallest number of the required form that is divisible by 11. Similarly, the largest possible value for a is $a = 9$, and we do have a solution in this case, with $b = 8$: $3 \cdot 9 - 2 \cdot 8 = 11$ is divisible by 11 (and no larger $b \in \{0, 1, \ldots, 9\}$ works). So $N = 98989$ is the largest number of the required form that is divisible by 11.

2. Let PAB and PBC be two similar right-angled triangles (in the same plane) with $\angle PAB = \angle PBC = 90°$ such that A and C lie on opposite sides of the line PB. If $PC = AC$, calculate the ratio $\frac{PA}{AB}$.

   Solution: Consider Figure 1:

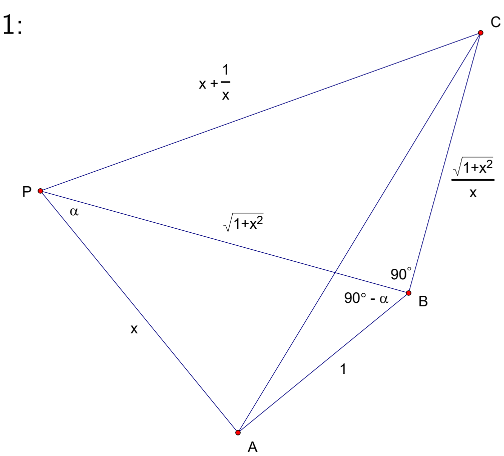

Figure 1

We may assume that $AB = 1$. Let $PA = x$, so that $PB = \sqrt{x^2+1}$, by Pythagoras. From the similarity of triangles PAB and PBC, we have $BC = \frac{\sqrt{x^2+1}}{x}$, so that $PC = x + \frac{1}{x}$, again by Pythagoras. Thus $AC = PC = x + \frac{1}{x}$.

Now let $\angle APB = \alpha$, so that $\angle PBA = 90° - \alpha$. Then $\cos\alpha = \frac{x}{\sqrt{x^2+1}}$ and from the cosine rule applied to triangle ABC, we get

$$\left(x + \tfrac{1}{x}\right)^2 = 1^2 + \tfrac{x^2+1}{x^2} - 2 \cdot \tfrac{\sqrt{x^2+1}}{x} \cdot \cos(180° - \alpha) = 1 + \tfrac{x^2+1}{x^2} + 2 \cdot \tfrac{\sqrt{x^2+1}}{x} \cdot \cos\alpha = 3 + \tfrac{x^2+1}{x^2}.$$

We now easily solve for x, and find that $\frac{PA}{AB} = x = \sqrt{2}$.

3. Determine the smallest integer $k > 1$ such that there exist k distinct primes whose squares sum to a power of 2.

   Solution: For $p_1^2 + p_2^2$ (where p_1 and p_2 are two distinct primes) to be equal to 2^n (where $n \geqslant 2$), we must have both p_1 and p_2 odd. This would give $p_1^2 + p_2^2 \equiv 2 \pmod 4$, while $2^n \equiv 0 \pmod 4$, a contradiction.

   For $p_1^2 + p_2^2 + p_3^2$ (where p_1, p_2 and p_3 are three distinct primes) to be equal to 2^n (where $n \geqslant 2$), we must have (say) $p_1 = 2$ and both p_2 and p_3 odd. This would give $p_1^2 + p_2^2 + p_3^2 \equiv 2 \pmod 4$, while $2^n \equiv 0 \pmod 4$, a contradiction.

   For $p_1^2 + p_2^2 + p_3^2 + p_4^2$ (where p_1, p_2, p_3 and p_4 are four distinct primes) to be equal to 2^n (where $n \geqslant 3$), we must have all four of these primes odd, say $p_i = 2m_i + 1,\ i = 1, 2, 3, 4$. This would give $\sum_{i=1}^4 p_i^2 = \sum_{i=1}^4 (2m_i + 1)^2 = \sum_{i=1}^4 (4m_i(m_i + 1) + 1) = 8L + 4 = 2^n$ for some integer L, since each $m_i(m_i + 1)$ is even. Then $2L + 1 = 2^{n-2}$ would be a contradiction, as 2^{n-2} is even if $n \geqslant 3$.

   We see that the smallest $k > 1$ that we are looking for must satisfy $k \geqslant 5$. Now, for the sum of the squares of five distinct primes to be a power of 2, one of them must be 2. The sum of the squares of the five smallest primes is not a power of two, but the very next choice of smallest primes does indeed satisfy the requirement: $2^2 + 3^2 + 5^2 + 7^2 + 13^2 = 256 = 2^8$. Hence $k = 5$ is the smallest k we are looking for.

4. Let ABC be a triangle with $\angle ABC \neq 90^\circ$ and AB its shortest side. Denote by H the intersection of the altitudes of triangle ABC. Let K be the circle through A with centre B. Let D be the other intersection of K and AC. Let K intersect the circumcircle of BCD again at E. If F is the intersection of DE and BH, show that BD is tangent to the circle through D, F and H.

   Solution: Consider Figure 2:

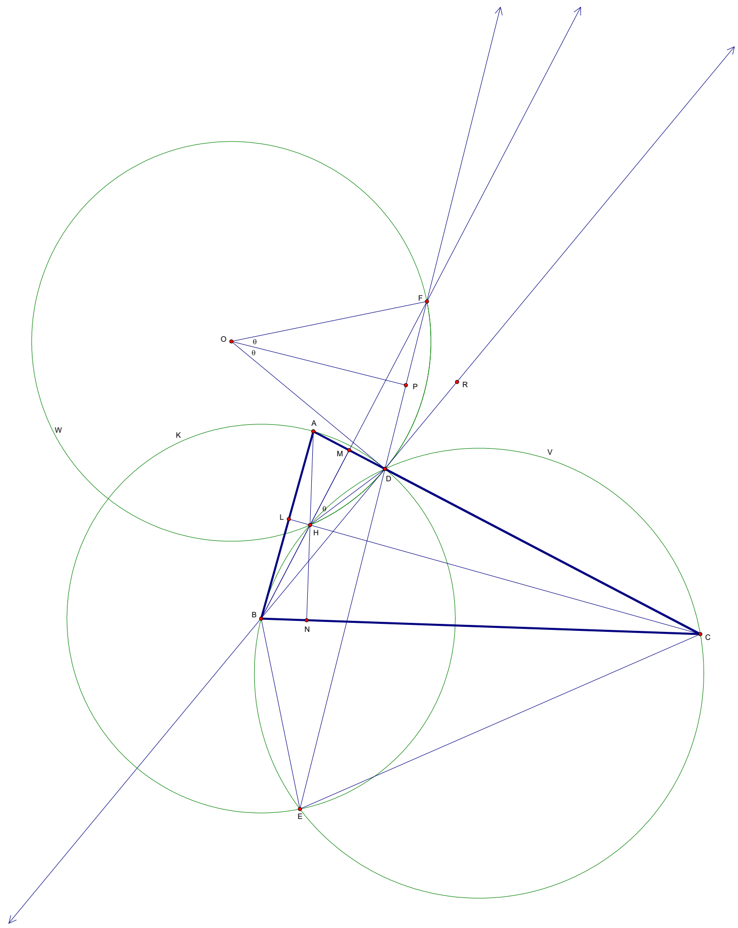

Figure 2

We note that H must also be on V, the circumcircle of triangle BDC. This is because triangles BLH and BMD are similar (note that triangle ABD is iscoceles, with $BA = BD$, and BM is a perpendicular bisector of AD, that also bisects $\angle ABD$), implying that angles BHC and BDC are equal, showing that $BHDC$ is a cyclic quadrilateral.

Let O be the centre of W, the circumcircle of triangle HDF and drop the perpendicular from O to DF, with foot P. Put $\theta = \angle DOP$, so that $\angle DOF = 2\theta$, giving $\angle DHF = \theta$. Since HM is a perpendicular bisector of AD, and $HA = HD$, it follows that $\angle HAC = 90^\circ - \theta$. Thus, $\angle ACN = \theta$. But the chords BD and BE of circle V have equal length (since they are both radii of circle K), hence subtend the same angle in circle V, giving $\angle BCE = \theta$. But then $\angle BDE = \theta$, so that $\angle FDR = \theta$, where R is an arbitrary point on the line through B and D, with R and B on opposites sides of D.

Finally, since $\angle ODP = 90^\circ - \theta$, we conclude that $\angle ODR = 90^\circ$, i.e. the line through B and D must be tangent to W, at the point D.

5. Find all polynomials $a(x), b(x), c(x), d(x)$ with real coefficients satisfying the simultaneous equations

$$b(x)c(x) + a(x)d(x) = 0$$
$$a(x)c(x) + (1 - x^2)b(x)d(x) = x + 1$$

Solution: We first show that it is not possible for all four polynomials to be non-zero. Suppose they are. Denote the leading coefficients of the polynomials $a(x), b(x), c(x), d(x)$ (which exist, because the polynomials are non-zero) by A, B, C, D, respectively. Then the first equation implies $BC = -AD$ and thus $ABCD = -(BC)^2 < 0$. In the second equation, the leading coefficient of $a(x)c(x)$ is AC and the leading coefficient of $(1-x^2)b(x)d(x)$ is $-BD$. Since the degree of $(1-x^2)b(x)d(x)$ is at least 2, these leading coefficients must cancel if we wish to end up with $x+1$ on the right. Thus $AC-BD = 0$, which implies that $ABCD = (AC)^2 > 0$. We have two contradictory inequalities, proving that this system has no solution when all four polynomials are non-zero.

Now suppose that $a(x) = 0$. Then, by the first equation, $b(x) = 0$ or $c(x) = 0$. If $b(x) = 0$, we see that the second equation is not satisfied. If $c(x) = 0$, we see that, again, the second equation is not satisfied, due to differences in degrees. Similarly, when $c(x) = 0$, we find, in a symmetrical way, that there are no solutions.

Henceforth, we assume that both $a(x)$ and $c(x)$ are non-zero. So either $b(x)$ or $d(x)$ (or both) must be zero. From the first equation, $b(x) = 0$ if and only if $d(x) = 0$. It follows that the complete set of solutions is given by all $(a(x), b(x), c(x), d(x))$ where $a(x)c(x) = x+1$ and $b(x) = d(x) = 0$, i.e.,

$\{(k, 0, \frac{x}{k} + \frac{1}{k}, 0) : k$ a non-zero real number$\} \cup \{(\frac{x}{k} + \frac{1}{k}, 0, k, 0) : k$ a non-zero real number$\}$.

6. Jacob and Laban take turns playing a game. Each of them starts with a list of square numbers $1, 4, 9, \ldots, 2021^2$, and there is a whiteboard in front of them with the number 0 on it. Jacob chooses a number x^2 from his list, removes it from his list, and replaces the number W on the whiteboard with $W + x^2$. Laban then does the same with a number from his list, and they repeat back and forth until both of them have no more numbers in their list. Now every time that the number on the whiteboard is divisible by 4 after a player has taken their turn, Jacob gets a sheep. Jacob wants to have as many sheep as possible by the end of the game, whereas Laban wants Jacob to have as few sheep as possible. What is the greatest number K such that Jacob can guarantee to get at least K sheep by the end of the game, no matter how Laban plays?

   Solution: Since $n^2 \equiv 0 \pmod 4$ if and only if n is even, and $n^2 \equiv 1 \pmod 4$ if and only if n is odd, we can simplify notation by replacing the even squares by 0 and the odd squares by 1, in each of the two lists. Thus Jacob and Laban each has a pool of 1010 zeros and 1011 ones to choose from when the game starts.

   Our first observation is that, no matter in which order they write down their ones and zeros on the board, after every fourth one, Jacob will earn another sheep. Since there are in total 2022 ones between the two players, and $2022 = 4 \times 505 + 2$, Jacob is guaranteed to have at least 505 sheep after the game. But since Jacob has the privilege of playing the first move, he can start with one of his zeros, which will secure a further sheep for himself, leaving him with a total of 506 (or more) sheep after the game.

   We now argue that if Laban plays intelligently, he can make sure that Jacob only get these 506 sheep, and nothing more. The secret for him is to make sure that no zeros are played (by anybody) immediately after every fourth one. Such zeros would add more sheep to Jacob's collection. So, immediately after every $4k$-th one (where $1 \leqslant k \leqslant 505$), another one should follow. Laban can easily achieve this by writing down a one after the zero with which Jacob has started the game, and after that simply copying each move of Jacob. This will ensure that Jacob must write down every $4k$-th one (for all $1 \leqslant k \leqslant 505$), and Laban then follows with another one immediately after each of these ones, achieving his goal.

By following this rule, Laban will not run out of zeros or ones before the 2020-th one has been used. After the first two moves (a zero by Jacob and a one by Laban), Laban has more zeros than Jacob, and one one less than Jacob in his pool, but since there are two spare ones left after 2020 ones have been used, this offers no problem for Laban. Any zeros or ones played after the 2021-th one will not contribute to a sum divisible by four, hence no further sheep for Jacob.

There is, of course, also the possibility that Jacob decides (for whatever reason) to start with a one, and not a zero. If this is the case, then Laban simply copies each move by Jacob right from the start, except the second one played by Jacob (which will be the third one of the game). After this one, Laban should proceed with a zero (if there are any zeros left in his list), and then continue to copy each move of Jacob after that. In doing so, no zero will be played directly after the $4k$-th one (for any $1 \leqslant k \leqslant 505$). If there are no zeros left in Laban's list after the third one of the game, then there will also be no zeros left in Jacob's list. So Laban simply continues with the ones from his list and Jacob cannot earn any more sheep than the gauranteed 505 produced by every fourth one. Hence, with this option where Jacob starts with a one, he is only gauranteed to earn 505 sheep.

We conclude that Jacob can guarantee (by starting with a zero) to collect 506 sheep for himself, but no more.

Die Suid Afrikaanse Wiskunde Olimpiade

Derde Rondte 2020

Senior Afdeling (Grade 10 tot 12)

Tyd : 4 ure

(Geen berekeningsapparate word toegelaat nie)

1.

2.

3.

4.

5.

6.

Elke probleem is 7 punte werd.