THE SOUTH AFRICAN
MATHEMATICS OLIMPIAD

in conjunction with THE SOUTH AFRICAN MATHEMATICAL SOCIETY and THE ASSOCIATION FOR MATHEMATICS EDUCATION OF SOUTH AFRICA

THE SOUTH AFRICAN MATHEMATICS OLIMPIAD

FIRST ROUND 1997

SENIOR SECTION: GRADES 10, 11 AND 12

(STANDARDS 8, 9 AND 10)

12 MARCH 1997

14:00

TIME: 60 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. A
2. D
3. C
4. C
5. C
6. D
7. C
8. A
9. B
10. A
11. C
12. B
13. E
14. D
15. E
16. A
17. A
18. C
19. A
20. E

SOLUTIONS

1. $2^n - 2^{n-1} = 2 \times 2^{n-1} - 2^{n-1} = 2^{n-1}$. Answer: A

2. $9971 - 1799 = 8172$. Answer: D

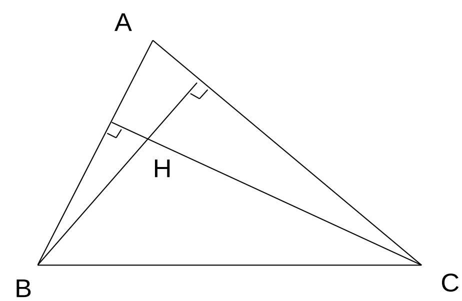
3. $0 = n^2 - 2n = n(n-2)$. Only $n = 0$ and $n = 2$ satisfy this equation. Answer: C

4. $\sqrt{\sqrt{36} - \sqrt{4}} = \sqrt{6-2} = \sqrt{4} = 2$. Answer: C.

5. 399×501 is approximately equal to $400 \times 500 = 200\ 000$. Answer: C

6. The graph cuts the x-axis where $y = 0$. That occurs only at $x = 2$ and $x = -3$, and these points are a distance 5 apart. Answer: D

7. $A\hat{C}H = 90^o - \hat{A}$ (angles in a right angled triangle). Therefore $B\hat{H}C = 90^o + (90^o - \hat{A}) = 180^o - \hat{A}$ (exterior angle of $\triangle HDC$).



Answer: C

8. Recall the factorization $x^2 - a^2 = (x-a)(x+a)$. This gives us $1997^2 - 3^2 = 1994 \times 2000$ and $1997^2 - 1^2 = 1996 \times 1998$. Therefore the given expression is

$$\frac{1996 \times 1994 \times 2000 \times 1998}{2000 \times 1996 \times 1998} = 1994.$$

Answer: A

9. If it takes Isaac 30 minutes to cycle both ways then it takes him 15 minutes to ride one way. He walks one way and rides back in 90 minutes. Therefore he must take 90 - 15 = 75 minutes to walk one way. It follows that the round trip by foot takes him 2×75 minutes, which is 2,5 hours. Answer: B

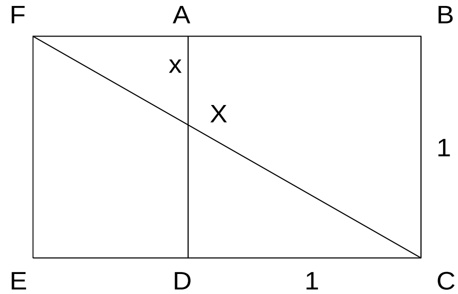
10. The primes less than 30 are 2,3,5,7,11,13,17,19,23 and 29. There are 10 of them, so $N(30) = 10$. There are 4 primes less than 10, so $N(N(30)) = N(10) = 4$. Answer: A

11. There are 10 terms in the numerator and 10 terms in the denominator. Each term in the numerator is a multiple of $1 \times 2 \times 4 = 8$ and each term in the denominator is a multiple of $1 \times 3 \times 9 = 27$. The given expression can therefore be written as

$$\frac{8(1 + 2^3 + 3^3 + \cdots + 10^3)}{27(1 + 2^3 + 3^3 + \cdots + 10^3)},$$

and the terms inside the two brackets cancel. Answer: C

12. Triangles FAX and FBC are similar. Hence, $\frac{AF}{FB} = \frac{AX}{BC} = \frac{x}{1} = x$.
But $FB = FA + AB = AF + 1$. It follows that $AF/(AF + 1) = x$ which can be solved for AF in terms of x: $AF = x(AF + 1)$, therefore $(1 - x)AF = x$, and finally $AF = x/(1 - x)$.



Answer: B

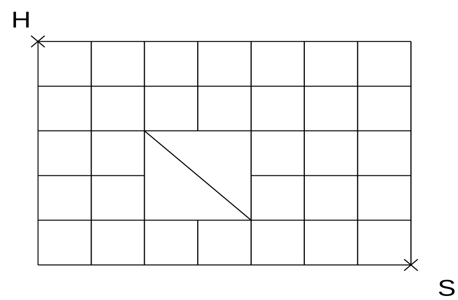
13. When we do the division then

$$111\ 222\ 333\ 444\ 555\ 666\ 777\ 888\ 999 = 111(1\ 002\ 003\ 004\ 005\ 006\ 007\ 008\ 009)$$

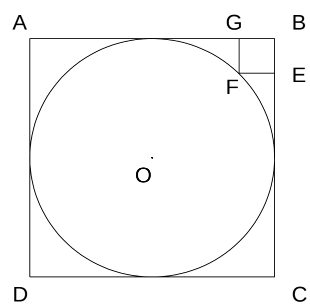
and the quotient, in the bracket, has 25 digits. Answer: E

14. Whether y is even or odd, $104y$ will be even. So, for the difference $187x - 104y$ to be odd, $187x$ must be odd. Hence if x is even it cannot satisfy the equation. Only (D) has an even x. Answer: D

15. The shortest route includes the road PQ through the park. To walk from H to P by taking the shortest route, Gloria has to walk 2 blocks East and 2 blocks South. That she can do by taking one of the six routes HABEP, HADEP, HADGP, HCFGP, HCDEP and HCDGP. Similarly, there are only 4 shortest routes from Q to S: QTUVS, QTUYS, QTXYS and QWXYS, for a total of $6 \times 4 = 24$ shortest routes, because for every one of the 6 shortest routes from H to P there are 4 shortest routes from Q to S. Answer: E



16. Let R be the radius of the circle. Then AB has length 2R, and OB is $\sqrt{2}R$. But OB is also $OF + FB = R + \sqrt{2}$, because FB is the diameter of a square of side 1. Therefore $\sqrt{2}R = R + \sqrt{2}$, and solving for R we obtain $R(\sqrt{2} - 1) = \sqrt{2}$, so that $R = \sqrt{2}/(\sqrt{2} - 1)$, and
$AB = 2R = \frac{2\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = 4 + 2\sqrt{2}$.



Answer: A

17. The differences between successive terms are 2, 3, 4, 5, 6, 7, 8, 9, 10, and so on. Writing out some more triangular numbers we will notice a pattern,

1	3	6	10	15
21	28	36	45	55
66	78	91	105	120
136	153			and so on.

The integers in the last two vertical columns are always divisible by 5. So in each horizontal row 2 out of the 5 integers are divisible by 5. Two fifths of 250 is 100.

Answer: A

[Why are the integers in the last two columns always divisible by 5? Notice that each triangular number can be written as the sum of numbers:

$$\begin{aligned} 1 &= 1 \\ 3 &= 1+2 \\ 6 &= 1+2+3 \\ 10 &= 1+2+3+4 \\ 15 &= 1+2+3+4+5, \end{aligned}$$

and so on.
From this pattern, or by any of several of other methods, you can observe that

$$\begin{aligned} 1 &= \tfrac{1}{2}(1 \times 2), \\ 3 &= \tfrac{1}{2}(2 \times 3), \\ 6 &= \tfrac{1}{2}(3 \times 4), \\ 10 &= \tfrac{1}{2}(4 \times 5), \\ 15 &= \tfrac{1}{2}(5 \times 6), \end{aligned}$$

and so on. The n-th triangular number is $\frac{1}{2}n(n+1)$. Clearly $\frac{1}{2}n(n+1)$ is divisible by 5 whenever n or its successor $n+1$, is divisible by 5.]

18. Let us suppose that the distance from home to Bizana is L km. In the time that George walks 7 km, Noxolo walks $L-7$ km.

Therefore in the time that George walks 1 km, Noxolo walks $(L-7)/7$ km.

In the time that George walks $L+4$ km, Noxolo walks $2L-4$ km. But, by what we worked out above, in the time that George walks $L+4$ km, Noxolo walks $\frac{L-7}{7} \times (L+4)$ km. This must equal $2L-4$ km. Solving the equation, $\frac{L-7}{7} \times (L+4) = 2L-4$ we get

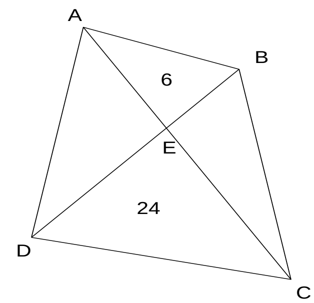
$(L-7)(L+4) = 7(2L-4)$. Therefore, $L^2 - 3L - 28 = 14L - 28$, and $L^2 - 17L = 0$. Hence $L = 17$ km. Answer: C

that of $\triangle AEB$ is $\frac{1}{2}EB.AX$. It follows that

19. Let the area of $\triangle AED$ and $\triangle BEC$ be x. The area of a triangle is $\frac{1}{2}$(base)$\times$(height). Hence, the area of $\triangle AED$ is $\frac{1}{2}DE.AX$, and

$$\frac{\text{Area}\triangle AED}{\text{Area}\triangle AEB} = \frac{\frac{1}{2}DE.AX}{\frac{1}{2}EB.AX} = \frac{DE}{EB},$$

that is the height AX cancels out, and we see that the ratio of the areas of these two triangles is the same as the ratio of their bases.



Similarly

$$\frac{\text{Area}\triangle DEC}{\text{Area}\triangle BEC} = \frac{DE}{EB}.$$

Putting in the areas we obtain $DE/EB = 24/x = x/6$. Solving for x: $x^2 = 6\times 24 = 12^2$. Therefore $x = 12$. Answer: A

20. Write the equation in the form,

$$3y^2 = x^2 - 1997.$$

3 is a factor of the lefthand side, therefore it must be a factor of the righthand side too. 3 is not a factor of 1997, therefore if x is a multiple of 3, then so is x^2, but $x^2 - 1997$ won't be. We conclude that x cannot be a multiple of 3.

There are only 2 other possibilities: (i) x leaves a remainder 1 when divided by 3, or (ii) x leaves a remainder 2 when divided by 3. We deal with each possibility in turn.

Possibility (i). x must be of the form $x = 3m + 1$ where n is an integer. Then $x^2 = 9n^2 + 6n + 1 = 3(n^2 + 2n) + 1$. We see that x^2 also leaves a remainder 1 when divided by 3. But 1997 leaves a remainder 2 when divided by 3. Hence the difference $x^2 - 1997$ cannot leave a remainder 0 when divided by 3. So $x^2 - 1997$ cannnot be divisible by 3.

Possibility (ii). x must be of the form $x = 3n + 2$ where m is an integer. Then $x^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1$. We see that in this case too x^2 leaves a remainder 1 when divided by 3, and, just as in (i), $x^2 - 1997$ cannot be divisible by 3. So there are no solutions. Answer: E