The South African Mathematical Olympiad
Third Round 2012
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Given that
$$\frac{1+3+5+\cdots+(2n-1)}{2+4+6+\cdots+(2n)}=\frac{2011}{2012},$$
determine n.

2. Let $ABCD$ be a square and X a point such that A and X are on opposite sides of CD. The lines AX and BX intersect CD in Y and Z respectively. If the area of $ABCD$ is 1 and the area of XYZ is $\frac{2}{3}$, determine the length of YZ.

3. Sixty points, of which thirty are coloured red, twenty are coloured blue, and ten are coloured green, are marked on a circle. These points divide the circle into sixty arcs. Each of these arcs is assigned a number according to the colours of its endpoints: an arc between a red and a green point is assigned a number 1, an arc between a red and a blue point is assigned a number 2, and an arc between a blue and a green point is assigned a number 3. The arcs between two points of the same colour are assigned a number 0. What is the greatest possible sum of all the numbers assigned to the arcs?

4. Let p and k be positive integers such that p is prime and $k > 1$. Prove that there is at most one pair (x, y) of positive integers such that
$$x^k + px = y^k.$$

5. Let ABC be a triangle such that $AB \neq AC$. We denote its orthocentre by H, its circumcentre by O and the midpoint of BC by D. The extensions of HD and AO meet in P. Prove that triangles AHP and ABC have the same centroid.

   (Note: The *orthocentre* of a triangle is the point where its altitudes intersect; the *circumcentre* of a triangle is the midpoint of its circumcribed circle; the *centroid* of a triangle is the point where its medians intersect.)

6. Find all functions $f : \mathbb{N} \to \mathbb{R}$ such that
$$f(km) + f(kn) - f(k)f(mn) \geqslant 1$$
for all $k, m, n \in \mathbb{N}$.

   ($\mathbb{N}$ is the set of all positive integers and $\mathbb{R}$ the set of all real numbers.)