The South African Mathematical Olympiad
Third Round 2014
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Determine the last two digits of the product of the squares of all positive odd integers less than 2014.

   Solution 1: Since the product of the odd integers less than 2014 contains 25 as a factor, it is clearly divisible by 25. Also, since it is odd, its last two digits have to be 25 or 75. So the product is either of the form $100n + 25$ or of the form $100n + 75$. In either case, the last two digits of the squared product (which is the same as the product of the squares) are 25:

   $$(100n + 25)^2 = 10000n^2 + 5000n + 625 = 100(100n^2 + 50n + 6) + 25$$

   or

   $$(100n + 75)^2 = 10000n^2 + 15000n + 5625 = 100(100n^2 + 150n + 56) + 25.$$

   In fact, we see that the last three digits have to be 625.

   Solution 2: Our number has $25 = 5^2$ as a factor, so its last two digits are 00, 25, 50 or 75. Moreover, it is the square of an odd number, which we can write as $(2n+1)^2 = 4n^2 + 4n + 1$. This means that the remainder upon division by 4 has to be 1. Now note that $100n + 0$, $100n + 25$, $100n + 50$ and $100n + 75$ leave a remainder of $0, 1, 2, 3$ respectively when divided by 4. This means that the last two digits are in fact 25.

   Remark: $(2n + 1)^2 = 4n^2 + 4n + 1 = 4n(n + 1) + 1$, and since n and $n + 1$ are consecutive, their product is even. Hence the remainder of our number upon division by 8 is also necessarily 1, and since it is also divisible by 125, we see again that its last three digits are 625.

2. Given that

   $$\frac{a-b}{c-d} = 2 \qquad \text{and} \qquad \frac{a-c}{b-d} = 3$$

   for certain real numbers a, b, c, d, determine the value of

   $$\frac{a-d}{b-c}.$$

   Solution 1: Set $x = c - d$ and $y = b - d$. We have

   $$a - d = (c - d) + (a - c) = x + 3y$$

   and

   $$a - d = (b - d) + (a - b) = y + 2x,$$

hence $x + 3y = y + 2x$, which implies $x = 2y$. Now we get $a - d = 5y$ and

$$b - c = (b - d) - (c - d) = y - x = -y,$$

so

$$\frac{a - d}{b - c} = -5.$$

Solution 2: We are given that

$$a - b = 2c - 2d, \quad \text{thus} \quad a + 2d = b + 2c,$$

and

$$a - c = 3b - 3d, \quad \text{thus} \quad a + 3d = 3b + c.$$

Multiply the first equation by 4 and the second one by 3, and subtract:

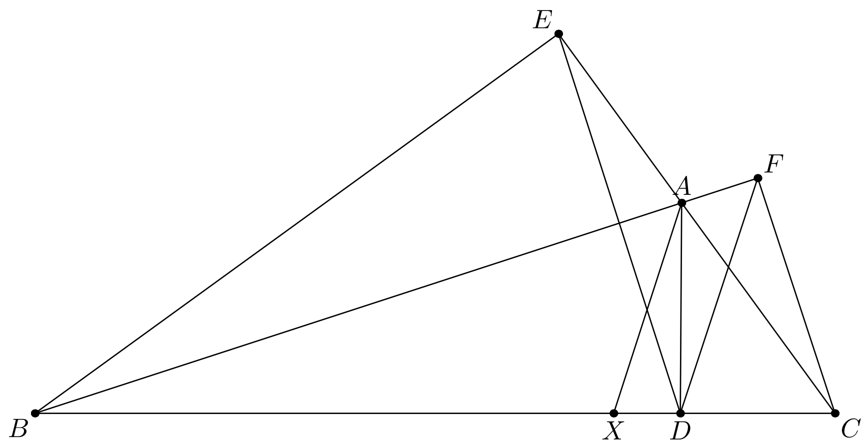
$$a - d = 4(a + 2d) - 3(a + 3d) = 4(b + 2c) - 3(3b + c) = -5b + 5c,$$

thus

$$\frac{a - d}{b - c} = -5.$$

3. In obtuse triangle ABC, with the obtuse angle at A, let D, E, F be the feet of the altitudes through A, B, C respectively. DE is parallel to CF, and DF is parallel to the angle bisector of $\angle BAC$. Find the angles of the triangle.

Solution:



We denote the angles of the triangle by $\alpha = \angle BAC$, $\beta = \angle ABC$ and $\gamma = \angle ACB$. X is the intersection of BC with the angle bisector of $\angle BAC$. Since $\angle BDA = \angle BEA = 90^\circ$, both D and E lie on the circle with diameter AB. Thus $AEBD$ is cyclic, which implies that

$$\angle EDB = \angle EAB = 180^\circ - \alpha.$$

It is given that DE and CF are parallel, hence

$$\angle EDB = \angle FCB = 90^\circ - \angle FBC = 90^\circ - \beta,$$

so $\beta = \alpha - 90^\circ$. Likewise,

$$\angle FDC = \angle FAC = 180^\circ - \alpha,$$

and

$$\angle FDC = \angle AXC = 180^\circ - \angle ACX - \angle XAC = 180^\circ - \gamma - \alpha/2,$$

so $\gamma = \alpha/2$. Since $\alpha + \beta + \gamma = 180^\circ$, this gives us

$$\alpha + (\alpha - 90^\circ) + \alpha/2 = 180^\circ,$$

and thus $\alpha = 108^\circ$, $\beta = 18^\circ$ and $\gamma = 54^\circ$.

4. (a) Let a, x, y be positive integers. Prove: if $x \neq y$, then also

$$ax + \gcd(a, x) + \operatorname{lcm}(a, x) \neq ay + \gcd(a, y) + \operatorname{lcm}(a, y).$$

(b) Show that there are no two positive integers a and b such that

$$ab + \gcd(a, b) + \operatorname{lcm}(a, b) = 2014.$$

Solution:

(a) Suppose that

$$ax + \gcd(a, x) + \operatorname{lcm}(a, x) = ay + \gcd(a, y) + \operatorname{lcm}(a, y)$$

for certain positive integers a, x, y. It follows that

$$\gcd\left(a, ax + \gcd(a, x) + \operatorname{lcm}(a, x)\right) = \gcd\left(a, ay + \gcd(a, y) + \operatorname{lcm}(a, y)\right).$$

Since a divides both ax and $\operatorname{lcm}(a, x)$, we have

$$\gcd\left(a, ax + \gcd(a, x) + \operatorname{lcm}(a, x)\right) = \gcd(a, \gcd(a, x)) = \gcd(a, x)$$

and likewise

$$\gcd\left(a, ay + \gcd(a, y) + \operatorname{lcm}(a, y)\right) = \gcd(a, \gcd(a, y)) = \gcd(a, y).$$

Therefore, we must have $\gcd(a, x) = \gcd(a, y) = d$ for some positive integer d. Since $\operatorname{lcm}(a, x) = ax/\gcd(a, x)$ and $\operatorname{lcm}(a, y) = ay/\gcd(a, y)$, this gives us

$$ax + d + \frac{ax}{d} = ay + d + \frac{ay}{d},$$

so

$$ax\left(1 + \frac{1}{d}\right) = ay\left(1 + \frac{1}{d}\right),$$

which implies $x = y$. This proves the first statement.

(b) Suppose that $ax+\gcd(a,x)+\mathrm{lcm}(a,x) = 2014$. Note that the left hand side is divisible by $\gcd(a,x)$, so $\gcd(a,x)$ has to be a divisor of 2014, i.e., one of $1, 2, 19, 38, 53, 106, 1007, 2014$. On the other hand,

$$(\gcd(a,x)+1)(\mathrm{lcm}(a,x)+1) = ax + \gcd(a,x) + \mathrm{lcm}(a,x) + 1 = 2015,$$

so $\gcd(a,x)+1$ has to divide 2015. Since $2, 3, 20, 39, 54, 107, 1008$ are all not divisors of 2015, this leaves us with $\gcd(a,x) = 2014$. But then $ax + \mathrm{lcm}(a,x) = 0$, which is clearly impossible since the left hand side is positive.

5. Let $n > 1$ be an integer. An $n \times n$-square is divided into n^2 unit squares. Of these smaller squares, n are coloured green and n are coloured blue. All remaining squares are coloured white. Are there more such colourings for which there are no two green squares in a row, and no two blue squares in a column, or colourings for which there are neither two green squares in a row nor two blue squares in a row?

Solution: Suppose that n squares have been coloured green, no two of them in the same row. This leaves $n-1$ empty squares in each row, which means that there are $(n-1)^n$ ways to colour n of the remaining n^2-n squares blue such that there are no two blue squares in the same row.

On the other hand, let x_i be the number of squares in column i that have not been coloured green. Clearly, $x_1+x_2+\cdots+x_n = n^2-n$. The number of ways to colour n of the remaining n^2-n squares blue in such a way that there are no two blue squares in the same column is

$$x_1x_2\cdots x_n \le \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n = \left(\frac{n^2-n}{n}\right)^n = (n-1)^n$$

by the inequality between the arithmetic and geometric mean. For some configurations (e.g., all green squares in one column), this holds with strict inequality.

Hence we can conclude that there are more possible colourings for which there are no two green squares and no two blue squares in the same row than there are colourings for which there are no two green squares in the same row and no two blue squares in the same column.

6. Let O be the centre of a two-dimensional coordinate system, and let $A_1, A_2, \ldots, A_n$ be points in the first quadrant and $B_1, B_2, \ldots, B_m$ points in the second quadrant. We associate numbers $a_1, a_2, \ldots, a_n$ to the points $A_1, A_2, \ldots, A_n$ and numbers $b_1, b_2, \ldots, b_m$ to the points $B_1, B_2, \ldots, B_m$, respectively. It turns out that the area of triangle OA_jB_k is always equal to the product a_jb_k, for any j and k. Show that either all the A_j or all the B_k lie on a single line through O.

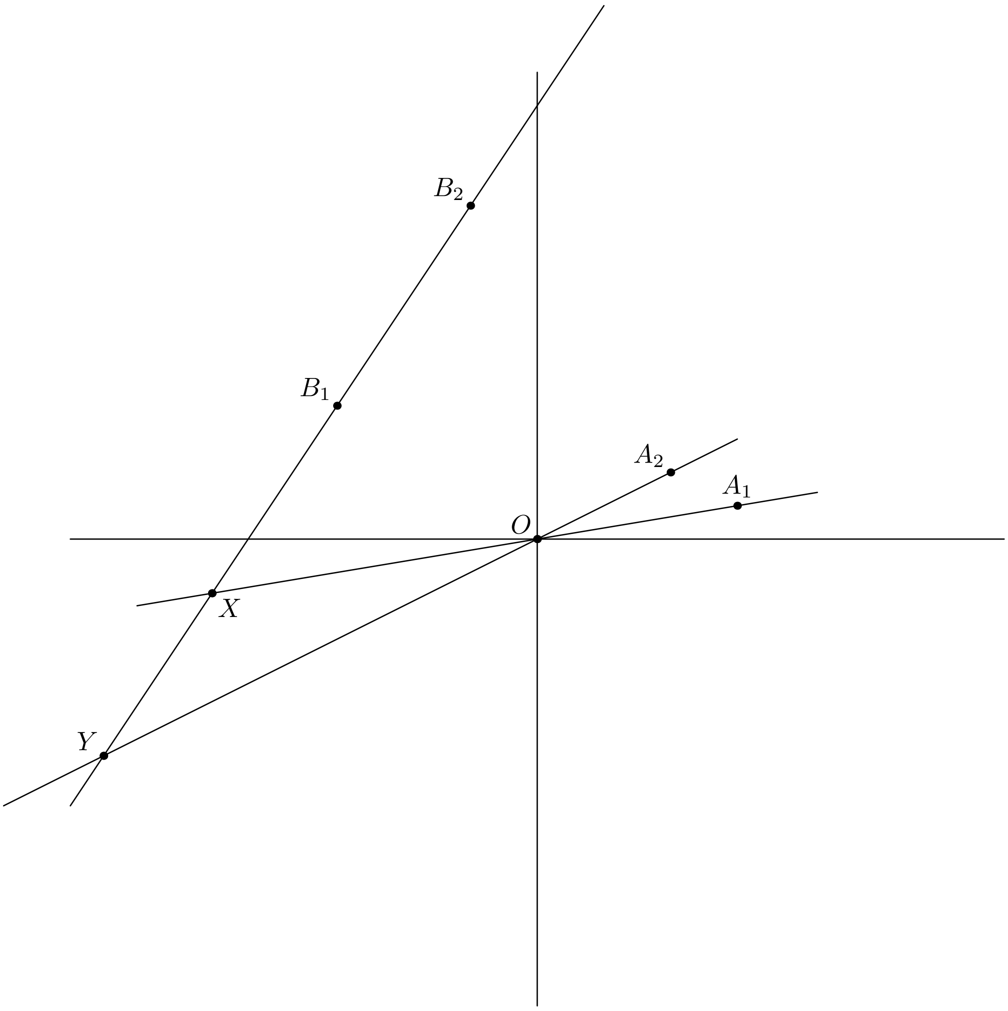
Solution 1: Consider first the case that one of the areas is zero, e.g. area$(OA_1B_1) = 0$. Then either $a_1 = 0$ or $b_1 = 0$. If $a_1 = 0$, then area$(OA_1B_k) = a_1b_k = 0$ for all k, which means that all B_k lie on a straight line through O and A_1. Likewise, if $b_1 = 0$, then all A_j lie on a straight line through O and B_1. So we can assume from now on that none of the areas is zero.

Suppose there are two points among $A_1, A_2, \ldots, A_n$ that are not on the same line through O (since the numbering does not matter, we can assume that A_1 and A_2 have this property), and at the same time there are two points (again, we can assume them to be B_1 and B_2) that are not on the same line through O. We have

$$\frac{\text{area}(OA_1B_2)}{\text{area}(OA_1B_1)} = \frac{a_1b_2}{a_1b_1} = \frac{b_2}{b_1} = \frac{a_2b_2}{a_2b_1} = \frac{\text{area}(OA_2B_2)}{\text{area}(OA_2B_1)}.$$

Since triangles OA_1B_1 and OA_1B_2 have the same base (OA_1), their heights must be in a b_1/b_2-ratio. The same is true for the heights of triangles OA_2B_1 and OA_2B_2. If B_1B_2 is parallel to OA_1, then $b_1 = b_2$, so B_1B_2 is parallel to OA_2. In this case, O, A_1 and A_2 lie on one line, contradicting the assumption. The same argument applies if B_1B_2 is parallel to OA_2.

If neither OA_1 nor OA_2 is parallel to B_1B_2, let X be the intersection of OA_1 and B_1B_2, and let Y be the intersection of OA_2 and B_1B_2.



Since A_1 and A_2 are in the first quadrant, X and Y are either in the first or third quadrant. In either case, they are not between B_1 and B_2. Using similar triangles, we see that the ratio of the heights of OA_1B_1 and OA_1B_2 is $|XB_1|/|XB_2|$, which must be b_1/b_2. The same is true (analogously) for the ratio $|YB_1|/|YB_2|$. Hence

$$\frac{|XB_1|}{|XB_2|} = \frac{b_1}{b_2} = \frac{|YB_1|}{|YB_2|}.$$

If X and Y are on different sides of B_1B_2, one of these ratios is greater than 1, the other less than 1, a contradiction. Thus we assume that B_1 is closer to both X and Y (otherwise, we just interchange the roles of B_1 and B_2), as in the figure. We get

$$1 - \frac{|B_1B_2|}{|XB_2|} = \frac{|XB_2| - |B_1B_2|}{|XB_2|} = \frac{|YB_2| - |B_1B_2|}{|YB_2|} = 1 - \frac{|B_1B_2|}{|YB_2|}$$

and thus $|XB_2| = |YB_2|$. This means that X and Y coincide, so X, Y, O, A_1, A_2 lie on one line, and we get a contradiction to our assumption again. This completes the proof.

Solution 2: We argue as in the first proof and assume again that A_1 and A_2 do not lie on a common line through O, and that B_1 and B_2 do not lie on a common line through O. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the angles enclosed by OA_1, OA_2, OB_1 and OB_2 with the x-axis. Then

$$\text{area}(OA_1B_1)\text{area}(OA_2B_2) = a_1b_1a_2b_2 = \text{area}(OA_1B_2)\text{area}(OA_2B_1)$$

and thus

$$\frac{|OA_1||OB_1|\sin(\beta_1-\alpha_1)}{2} \cdot \frac{|OA_2||OB_2|\sin(\beta_2-\alpha_2)}{2} = \frac{|OA_1||OB_2|\sin(\beta_2-\alpha_1)}{2} \cdot \frac{|OA_2||OB_1|\sin(\beta_1-\alpha_2)}{2}.$$

It follows that

$$\sin(\beta_1-\alpha_1)\sin(\beta_2-\alpha_2) = \sin(\beta_2-\alpha_1)\sin(\beta_1-\alpha_2).$$

Now we use the trigonometric identity $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$ to get

$$\frac{1}{2}\left(\cos(\beta_1-\alpha_1+\alpha_2-\beta_2) - \cos(\beta_1-\alpha_1+\beta_2-\alpha_2)\right) = \frac{1}{2}\left(\cos(\beta_2-\alpha_1+\alpha_2-\beta_1) - \cos(\beta_2-\alpha_1+\beta_1-\alpha_2)\right).$$

This implies

$$\cos(\beta_1-\alpha_1+\alpha_2-\beta_2) = \cos(\beta_2-\alpha_1+\alpha_2-\beta_1),$$

and by the addition theorem for the cosine

$$\cos(\beta_1-\beta_2)\cos(\alpha_1-\alpha_2) + \sin(\beta_1-\beta_2)\sin(\alpha_1-\alpha_2) = \cos(\beta_1-\beta_2)\cos(\alpha_1-\alpha_2) - \sin(\beta_1-\beta_2)\sin(\alpha_1-\alpha_2),$$

so finally

$$\sin(\beta_1-\beta_2)\sin(\alpha_1-\alpha_2) = 0,$$

which means that either $\beta_1 = \beta_2$ or $\alpha_1 = \alpha_2$, contradicting our assumption and thus completing the proof.