THE HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

Organised by the SOUTH AFRICAN MATHEMATICS FOUNDATION
Sponsored by HARMONY GOLD MINING

FIRST ROUND 2005

SENIOR SECTION: GRADES 10, 11 AND 12

15 MARCH 2005

TIME: 60 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. D
2. E
3. D
4. C
5. A
6. B
7. E
8. B
9. A
10. C
11. B
12. B
13. B
14. E
15. B
16. C
17. A
18. C
19. D
20. E

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SOLUTIONS

1. **Answer D.** $0.001 \div 0.05 = \dfrac{0.001}{0.05} = \dfrac{1}{50}$.
   Alternatively, $0.001 \div 0.05 = \dfrac{1}{1000} \div \dfrac{5}{100} = \dfrac{1}{1000} \times \dfrac{100}{5} = \dfrac{1}{50}$

2. **Answer E.** $\dfrac{1}{\frac{1}{3}+\frac{1}{4}} = \dfrac{1}{(\frac{4+3}{12})} = \dfrac{1}{(\frac{7}{12})} = \dfrac{12}{7}$.

3. **Answer D.** Angle $A\widehat{F}E = 180^\circ - A\widehat{F}G = 180^\circ - 120^\circ = 60^\circ$. Next, angle $F\widehat{A}E = 45^\circ$, since AC is a diagonal of the square. Thus angle $A\widehat{E}F = 180^\circ - (60^\circ + 45^\circ) = 75^\circ$.

4. **Answer C.** The shaded area is half of a 10×12 rectangle, with a 4×4 square removed. Thus its area is $\frac{1}{2} \times 10 \times 12 - 4 \times 4 = 60 - 16 = 44$.

5. **Answer A.** First, $\dfrac{x}{h} + \dfrac{y}{h} = \dfrac{x+y}{h} = \dfrac{5}{h}$. Next, $\dfrac{x}{h} + \dfrac{y}{h} = \dfrac{1}{12} + \dfrac{1}{8} = \dfrac{5}{24}$. Comparing the denominators of the right-hand sides gives $h = 24$.

6. **Answer B.** The least number is the one that is furthest to the left on the number line, which is $-8 \times 4 = -32$. (The number -40 is not possible, since $-8 \times -5 = +40$.)

7. **Answer E.** The circumference of each wheel $= 2\pi r = 140\pi$ cm $= 1.4\pi$ m $\approx 1.4 \times \frac{22}{7}$ m $=$ 4.4 m. Thus when the car travels 100 m, the number of revolutions is approximately $100 \div 4.4 = \frac{250}{11} = 22.727\ldots \approx 23$.

8. **Answer B.** If $2^a 2^b 2^c = 256$, then $2^{a+b+c} = 2^8$, so $a+b+c = 8$. The average is $\dfrac{a+b+c}{3} = \dfrac{8}{3}$.

9. **Answer A.** If x is close enough to 0, then $(x-y)^2$ is close to y^2, which is greater than x^2. (If you need an example, try $x = 0.1$ and $y = 0.9$.) There is no need to consider the other choices.

10. **Answer C.** If the hexagon is cut up into six equilateral triangles, then one-and-a-half triangles are shaded. Thus the shaded area makes up $\frac{1.5}{6} = \frac{1}{4}$ of the hexagon.

11. **Answer B.** The numbers from 20 to 29 lie above the numbers from 21 to 30, so all of their units digits are different from the digits below them, but nine of their tens digits are the same. (The only one that differs is the 2 of 29, which lies above the 3 of 30.) The same applies with the remaining three groups of ten numbers, so the total number of digits that are the same as those below them is $9 \times 4 = 36$.

12. **Answer B.** The n strings have a total of $2n$ ends. One boy picks up an end; this leaves $2n-1$ ends for the second boy to choose, of which only one is correct. Thus the probability is $\dfrac{1}{2n-1}$.

13. **Answer B.** If the expressions are all simplified in exponent form, then they are:

    $$\text{(A) } 5^{1+5} = 5^6 \quad \text{(B) } 5^{3125} \quad \text{(C) } 5^{5\times 5} = 5^{25} \quad \text{(D) } 5^{55} \quad \text{(E) } (5^2)^5 = 5^{2\times 5} = 5^{10},$$

    and 3125 is the largest exponent.

14. **Answer E.** There are six possible pairs of non-zero digits: $(1;6), (2;5), \ldots, (6;1)$. Each pair $(a;b)$ can form three numbers: $ab00$, $a0b0$, $a00b$. Thus the total is $6 \times 3 = 18$.

15. **Answer B.** If we multiply out $(x+x^{-1})^3$, we find that

    $$(x+x^{-1})^3 = x^3 + 3x + 3x^{-1} + x^{-3} = (x^3 + x^{-3}) + 3(x + x^{-1}).$$

    Thus $x^3 + x^{-3} = (x+x^{-1})^3 - 3(x+x^{-1}) = 4^3 - 3 \times 4 = 64 - 12 = 52$.

16. **Answer C.** Let $x =$ area of $\triangle APQ$ and $y =$ area of $\triangle ARS$. Then $x+1+y =$ area of $\triangle APS = 4$, so $x + y = 3$. The six triangles and their areas are:

$\triangle APQ$	$\triangle APR$	$\triangle APS$	$\triangle AQR$	$\triangle AQS$	$\triangle ARS$
x	$x+1$	$x+1+y$	1	$1+y$	y

The sum of the areas is $3(x + y) + 4 = 9 + 4 = 13$, so the average area is $\frac{13}{6}$.

17. **Answer A.** The differences between the numbers at the ends of successive rows form an arithmetic sequence: $5-1 = 4$, $11-5 = 6$, $19-11 = 8$, etc. Thus the numbers at the ends of the rows are one less than sums of arithmetic sequences with first term $a = 2$ and common difference $d = 2$:

$$-1 + 2 = 1, \quad -1 + (2 + 4) = 5, \quad -1 + (2 + 4 + 6) = 11, \quad -1 + (2 + 4 + 6 + 8) = 19, \ldots$$

In the 80th row, the sequence has $n = 80$ terms, so the number at the end is

$$-1 + \frac{n}{2}(2a + (n-1)d) = -1 + \frac{80}{2}(4 + 79 \times 2) = -1 + 80 \times 81 = 6479.$$

(Alternatively, the general formula $n^2 + n - 1$ can be found quite easily by inspection.)

18. **Answer C.** Let their ages now be A, B, and C. Then we are given the following facts: (i) $A = 3(C + 1)$, (ii) $B + 12 = 3A$, (iii) $B = 8C$. Substituting from (i) and (iii) into (ii) gives $8C + 12 = 9(C + 1)$, so $C = 3$. We then obtain $A = 12$ and $B = 24$.

19. **Answer D.** Consider the right-angled triangle with two vertices at the centres of the circles and two sides parallel to PQ and QR. Since the centres lie on the bisectors of angle $\widehat{R}$, the triangle has an angle of $30°$. If the larger circle has radius r, then the hypotenuse of the triangle is $r+1$ and the shortest side (parallel to PQ) is $r-1$. In any $(30°, 60°, 90°)$ triangle, the hypotenuse is twice the shortest side ($\sin 30° = \frac{1}{2}$), so $r + 1 = 2(r - 1)$, giving $r = 3$.

20. **Answer E.** Suppose ℓ% of the population are lung cancer sufferers, s% of the population are smokers, and x% are both. We are given that $x = 10\%$ of s and $x = 90\%$ of ℓ, so $s = 9\ell$. Since $s = 20$, it follows that $\ell = \dfrac{20}{9}$.