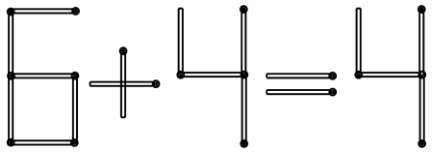
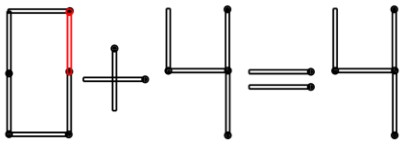
MEMO

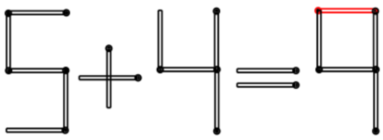
1) Move one matchstick and replace it elsewhere to make the statement true. The committee could only find three possible solutions, so bonus marks if you find more! (3 + 1 bonus)



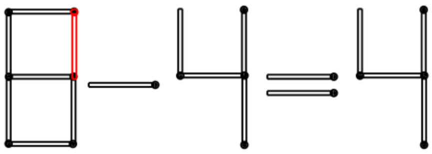
Solution 1:



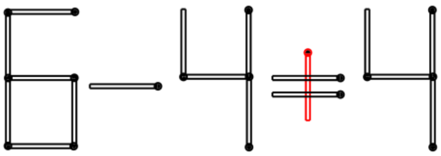
Solution 2:



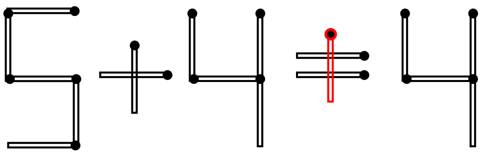
Solution 3:



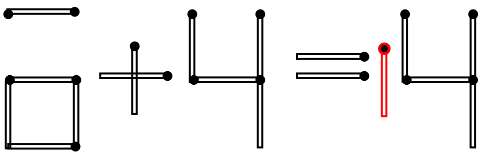
Possible Solution 4:



Possible Solution 5:

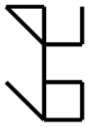


Possible Solution 6:



2) The Cistercian monks invented a numbering system in the 13th century which meant that any number between 1 and 9999 could be written using a single symbol.

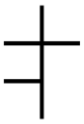
a) What is the numerical value of the Cistercian symbol? (2)



Solution: 3958

b) Write the number 2022 using a Cistercian numbering symbol. (3)

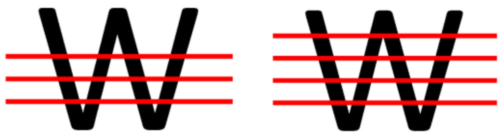
Solution:



3) Two horizontal lines cut the letter W into 9 pieces.

a) Using four horizontal lines, how many pieces do you get? (2)

Solution: 1 line = 5 pieces
2 lines = 9 pieces



3 lines = 13 pieces
4 lines = 17 pieces

b) Determine the smallest number of lines needed to cut the W into at least 2022 pieces. (3)

Solution: General "term" is $P_n = 4n + 1$
$4n + 1 \geq 2022$
$4n \geq 2021$
$n \geq 505{,}25$
$n = 506$ lines

4) Four friends were racing side by side down a dusty staircase. Ambete went down two steps at a time, Bruce three steps at a time, Claire four steps at a time and Divakaran five steps at a time. If the only steps with all four's footprints were at the top and the bottom, how many steps had just one person's footprints? (5)

Solution:

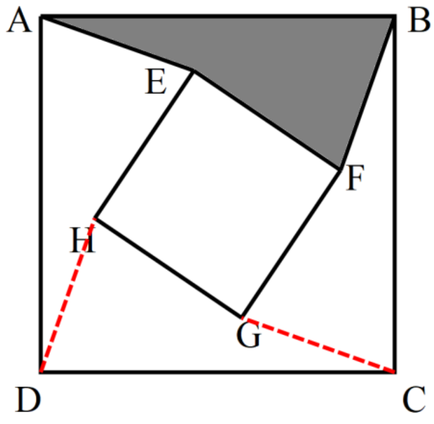
	Ambete	Bruce	Claire	Divakaran
1				
2	A			
3		B		
4	A		C	
5				D
6	A	B		
7				
8	A		C	
9		B		
10	A			D
11				
12	A	B	C	
13				
14	A			
15		B		D
16	A		C	
17				
18	A	B		
19				
20	A		C	D
21		B		
22	A			
23				
24	A	B	C	
25				D
26	A			
27		B		
28	A		C	
29				
30	A	B		D
31				
32	A		C	
33		B		
34	A			
35				D
36	A	B	C	
37				
38	A			
39		B		
40	A		C	D
41				
42	A	B		
43				
44	A		C	
45		B		D
46	A			
47				
48	A	B	C	
49				
50	A			D
51		B		
52	A		C	
53				
54	A	B		
55				D
56	A		C	
57		B		
58	A			
59				
60	A	B	C	D

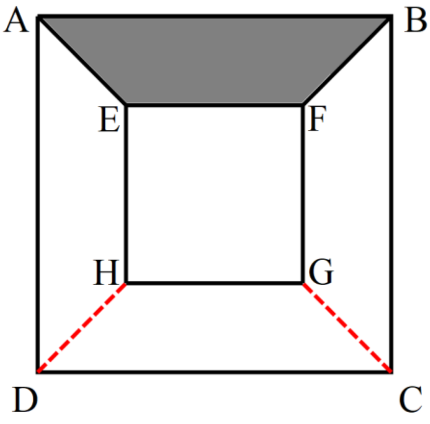
20 steps have only 1 person's footprint

5) In the figure below square ABCD has side length 12 and square EFGH has side length 6. Both of **which have the same centre**.
Interesting fact: the area of the shaded quadrilateral ABFE is always constant and does not depend on the rotation of square EFGH.

What is the area of the shaded region? (4)

Solution: By rotating the small square, you are always creating 4 congruent shapes!





Area of large square = 12 x 12 = 144 sq. units
Area of small square = 6 x 6 = 36 sq. units
Difference of areas = 144 – 36 = 108 sq. units
Area of grey area = 108 / 4 = 27 sq. units

6) Consider the numbers: 24, 55, 27, 64 and x

- The average (mean) of these five numbers is prime
- The median is a multiple of 3

Calculate the sum of all the possible positive whole number values of x. (6)

Solution: Sum of given values = $170 + x$

For x , 24 , 27 , 55 , 64
Only possible solution for x is 15 (average / mean = 37)

For 24 , x , 27 , 55 , 64
No possible solutions for x (no average / mean = prime)

For 24 , 27 , x , 55 , 64
Only possible solution for x is 45 (average / mean = 43)
x = 35 (average / mean = 41) not a solution because 35 a multiple of 3

For 24 , 27 , 55 , x , 64
No possible solutions for x (median is not a multiple of 3)

For 24 , 27 , 55 , 64 , x
No possible solutions for x (median is not a multiple of 3)

Sum of all possible solutions = 15 + 45 = 60

7) All positive whole numbers n for which $n(n+1)(n+2)$ is a multiple of 5 are listed in increasing order.

a) Give the first three possible values of n. (2)

Solution: First 3 possible values are n = 3, n = 4 and n = 5

b) What is the 2022^{nd} number in this list? (4)

Solution: Second 3 possible values are n = 8, n = 9 and n = 10
So, solutions exist in sets of 3 every multiple of 5

So, for the 2022^{nd} number, we need the 2022 / 3 = 674^{th} set
So, 2022^{nd} number is 674 x 5 = 3370

8) Find the smallest whole number which when multiplied by 123 yields a product that ends in 2022. (6)

Solution: For the product to end with digit 2 (202**2**), we need our whole number to end with digit 4
(123 x 4 =492)

Then, for the second-last digit of the product to be 2 (20**2**2), we need our whole number to have a second-last digit of 1
(123 x 14 = 1722)

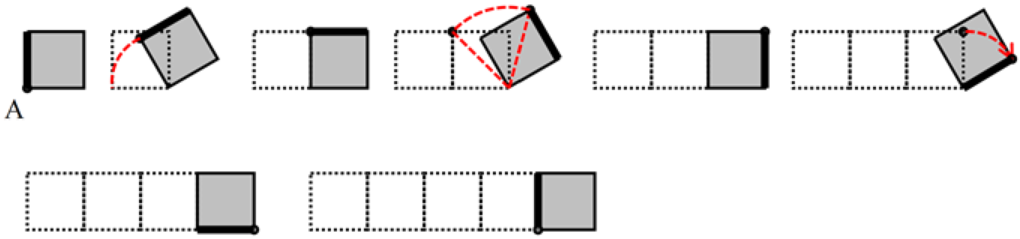
Then, for the third-last digit of the product to be 0 (2**0**22), we need our whole number to have a third-last digit of 1
(123 x 114 = 14022)

Then, for the fourth-last digit of the product to be 2 (**2**022), we need our whole number to have a fourth-last digit of 6
(123 x 6114 = 752022)

Thus, smallest whole number = 6114

9. A square “wheel” rolls without slipping along a straight level road until it has completed one revolution. If the side length is 1 cm, how far does point A travel? (6)

Solution:



For the first quarter-turn, r = 1,
so A moves 2π(1) / 4 = π/2

For the second quarter-turn, r = sqrt(2),
so A moves 2π(sqrt(2)) / 4 = sqrt(2)π/2

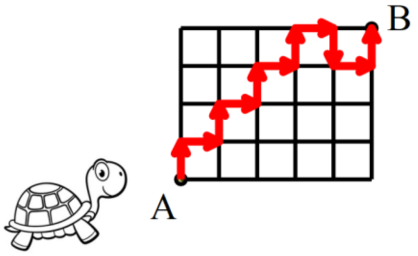
For the third quarter-turn, r = 1,
so A moves 2π(1) / 4 = π/2

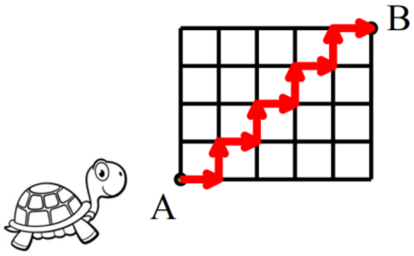
For the fourth quarter-turn, r = 0 (A doesn’t move on the road!)

So in total, A moves = π/2 + π/2 + sqrt(2)π/2 = π + sqrt(2)π/2

10) A tortoise moves around on a chequered grid with unit squares by changing direction each time it reaches a new square.

a) What is the shortest distance it travels from A to B on a 4×5 grid? (2)





Solution: Shortest route = 4 (up) + 5 (right) = 9

b) If the tortoise travels from lower left to upper right corner of a 2022×4044 grid, what is the shortest distance it travels? (5)

Solution: From above, we can get to the "top" of the 2022×4044 grid by moving 2022 (up) + 2023 (right) = 4045 units

But, we still have 2021 units horizontally right to move

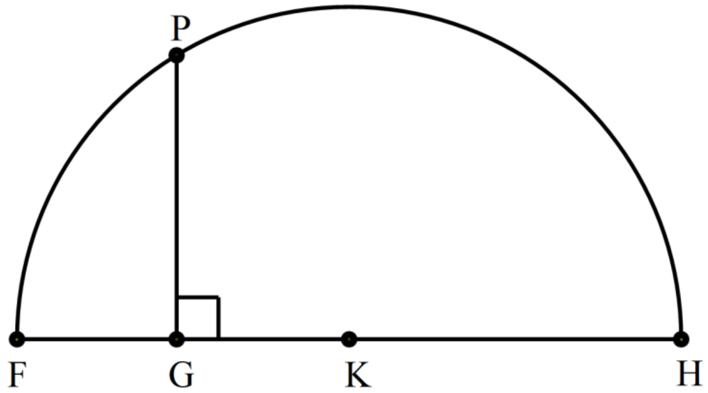
We can only do this by moving down – right – up – right
which moves us 2 units to the right (4 moves = 2 units right)
So, number of moves = 2020 x (4 / 2) + 3 (only for the last one)
Thus, number of moves = 4043

Thus, shortest distance travelled = 4045 + 4043 = 8088 units

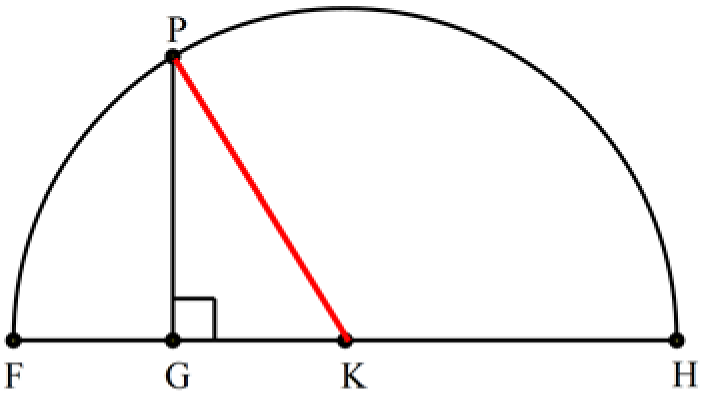
11) Here's a cool way to find the square root of a number!

Suppose you want to find $\sqrt{12}$.

- You draw a line segment of length 12 units. (GH in the diagram).
- You extend GH by a length of 1 unit. (FG in the diagram).
- With a diameter FH and K as a centre you draw a semicircle.
- You construct PG perpendicular to FH to touch the circle at P.
- Then $\text{PG} = \sqrt{12}$.



Prove that this construction method works for finding the square root of any number. (7)



Given $\text{GH} = x$

Prove: $\text{PG} = \sqrt{x}$

Construction: Draw PK

Radius: $\text{KH} = \dfrac{x+1}{2}$ and therefore $\text{GK} = \left(\dfrac{x+1}{2} - 1\right)$

In ΔPGK: $\text{PG}^2 = \text{PK}^2 - \text{GK}^2$ Pythagoras

$$\therefore \text{PG} = \sqrt{\left(\frac{x+1}{2}\right)^2 - \left(\frac{x+1}{2} - 1\right)^2}$$

$$\therefore \text{PG} = \sqrt{\left(\frac{x^2+2x+1}{4}\right) - \left(\frac{x^2-2x+1}{4}\right)}$$

$$\therefore \text{PG} = \sqrt{\frac{4x}{4}} = \sqrt{x}$$

12) Let a sequence $a_0, a_1, \ldots$ be defined by

$$a_0 = 2022 \qquad a_{n+1} = \frac{1+a_n}{1-a_n} \qquad \text{for } n = 0,\ 1,\ \ldots$$

What is the value of a_{2022} ? (6)

Solution: $n = 0,\ a_1 = \dfrac{1+2022}{1-2022} = -\dfrac{2023}{2021}$

$$n = 1,\ a_2 = \frac{1+\left(-\dfrac{2023}{2021}\right)}{1-\left(-\dfrac{2023}{2021}\right)} = -\frac{2}{4044} = -\frac{1}{2022}$$

$$n = 2,\ a_3 = \frac{1+\left(-\dfrac{1}{2022}\right)}{1-\left(-\dfrac{1}{2022}\right)} = \frac{2021}{2023}$$

$$n = 3,\ a_4 = \frac{1+\left(\dfrac{2021}{2023}\right)}{1-\left(\dfrac{2021}{2023}\right)} = \frac{4044}{2} = 2022$$

$$n = 4,\ a_5 = \frac{1+2022}{1-2022} = -\frac{2023}{2021}$$

Thus, pattern occurs in ‘4’s’.
So, 2022 / 4 = 505 full sets of 4 rem 2

Thus, $a_{2022} = -\dfrac{1}{2022}$

13. A two-digit number is divided by the sum of its digits (they are not the same).

What is the largest possible remainder? (7)

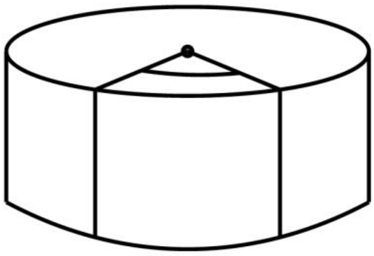
Solution: The biggest remainder will always be the divisor – 1
or in this case the sum of the digits – 1

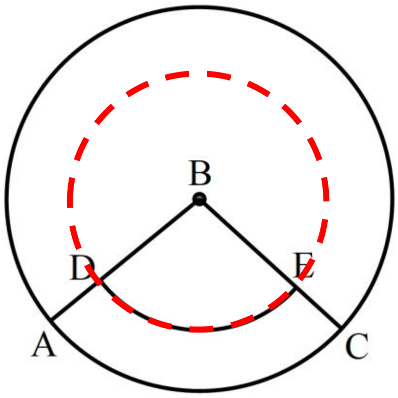
If the digits are x and y, then sum is $x + y$
For $x = y = 9$, 99 mod 18 = 9

For $x = 9$ and $y = 8$ and vice versa, 98 mod 17 = 13 and
89 mod 17 = 4
For $x = 8$ and $y = 8$, 88 mod 16 = 8

For $x = 9$ and $y = 7$ and vice versa, 97 mod 16 = 1 and
79 mod 16 = 15
(1 less than the divisor and thus also the largest possible remainder)!

14) You want to divide a birthday cake in a non-traditional way.
If area of BDE = area of DACE, what is the ratio of AD:BD? (7)





Solution 1: Let small radius (BD and BE) = r and large radius (BA and BC) = R
Area of BDE (full circle) = πr^2
Area of BAC (full circle) = πR^2
Area of DACE (full ring) = $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

For equal areas, $\pi(R^2 - r^2) = \pi r^2$
$R^2 = 2r^2$
R = sqrt(2)r

Thus, AD:BD = (R – r) / r = (sqrt(2)r – r) / r = r(sqrt(2) – 1) / r
Thus, AD:BD = (sqrt(2) – 1) : 1

OR

Q14 Solution 2:

$$\frac{\text{AreaBAC}}{\text{AreaBDE}} = \left(\frac{\text{BA}}{\text{BD}}\right)^2$$

$$\therefore \frac{\text{BDE + ACED}}{\text{BDE}} = \left(\frac{\text{BD + DA}}{\text{BD}}\right)^2$$

$$\therefore 1 + \frac{\text{ACED}}{\text{BDE}} = \left(1 + \frac{\text{DA}}{\text{BD}}\right)^2$$

$$\therefore 1 + \frac{1}{1} = \left(1 + \frac{\text{DA}}{\text{BD}}\right)^2$$

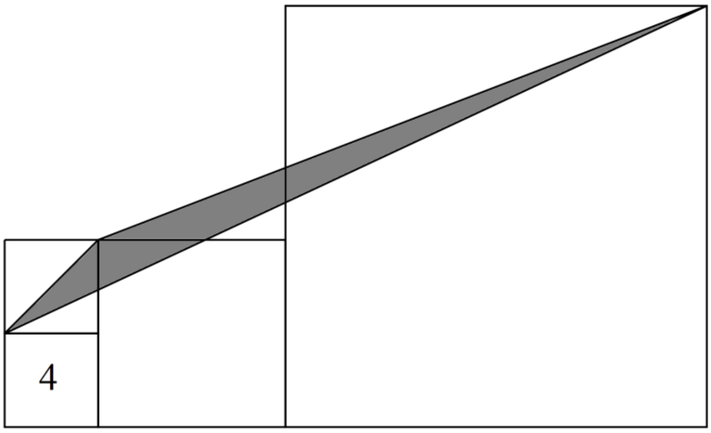
$$\therefore \sqrt{2} = 1 + \frac{\text{DA}}{\text{BD}}$$

$$\therefore \frac{\text{DA}}{\text{BD}} = \frac{\sqrt{2} - 1}{1}$$

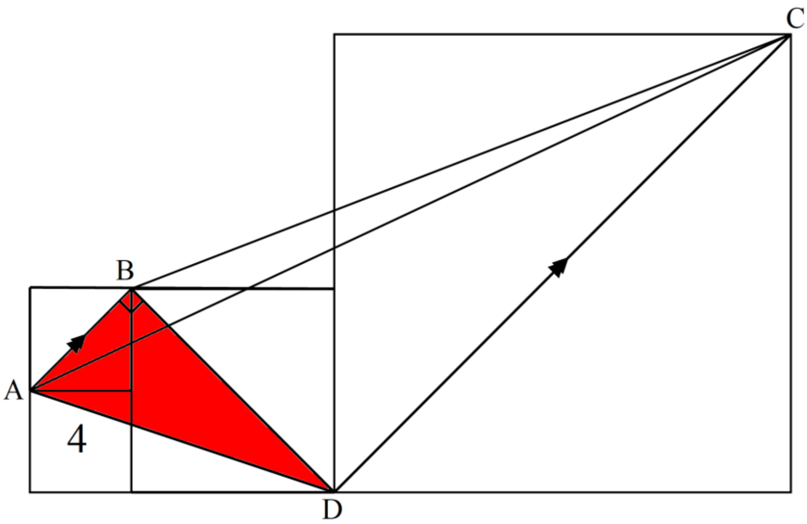
$$\therefore \text{AD : BD} = \left(\sqrt{2} - 1\right) : 1$$

15. Four squares are drawn below, with the area of the bottom left square being 4 units^2.

Determine the area of the shaded triangle. (10)



Solution:



Construct diameter CD such that $AB \parallel DC$
Move point C to D over the diagonal DC

Area of ΔABC = Area of ΔABD (parallel diagonals → equal height, same base AB).

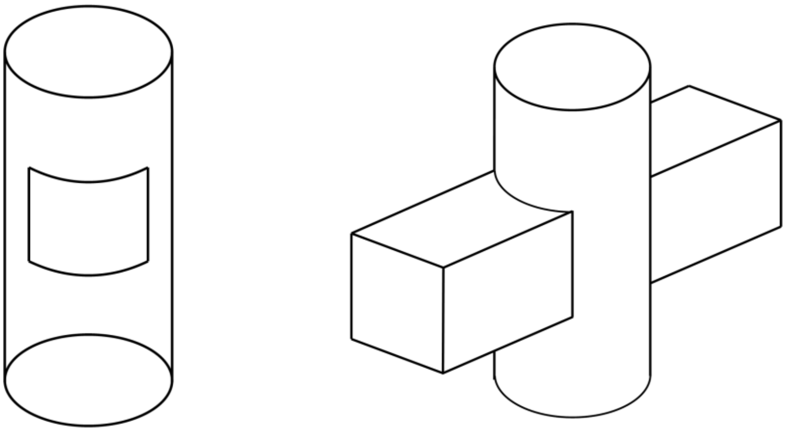
$AB \perp BD$ (diagonals of two squares meeting together)

$AB = \sqrt{8} = 2\sqrt{2}$ $\left(\sqrt{2^2 + 2^2}\right)$ small squares

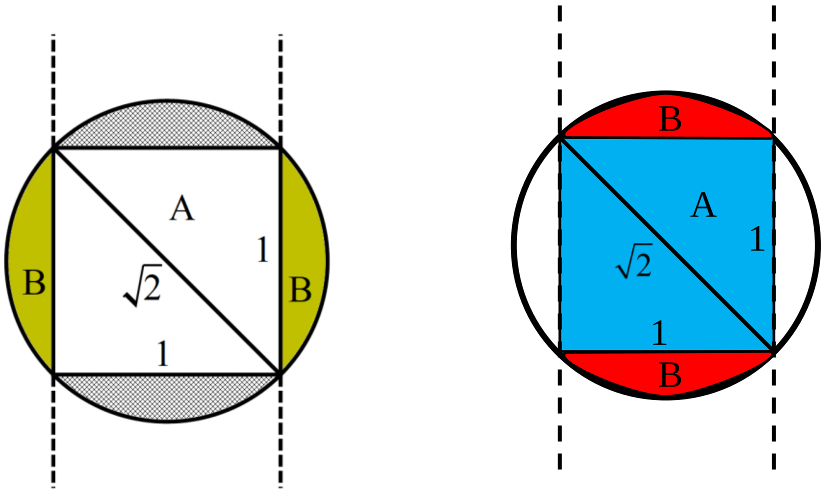
$BD = \sqrt{32} = 4\sqrt{2}$ $\left(\sqrt{4^2 + 4^2}\right)$ next larger square

$$\text{Area } \Delta ABD = \frac{1}{2}(AB)(BD) = \frac{1}{2}\left(2\sqrt{2}\right)\left(4\sqrt{2}\right) = 8 \text{ units}^2$$

16) You have a cylinder, with diameter $\sqrt{2}$.
A square tubing with side length 1 is cut through the centre of a cylinder as shown.
Determine the volume removed from the cylinder. (10)



Solution:



Required Area = A + 2B and area of A = 1 x 1 =1

$$B = \frac{Area\ of\ circle - 1}{4}$$

$$= \frac{\pi\left(\frac{\sqrt{2}}{2}\right)^2 - 1}{4}$$

$$= \left(\frac{\pi - 2}{2}\right) \times \frac{1}{4}$$

$$= \frac{\pi - 2}{8}$$

$$Area = 1 + 2\left(\frac{\pi - 2}{8}\right)$$

$$= 1 + \frac{\pi - 2}{4}$$

$$Volume = 1 \times Area$$

$$= 1 + \frac{\pi - 2}{4}$$

or

$$\frac{2 + \pi}{4}$$

or

$$\frac{1}{2} + \frac{\pi}{4}$$