THE SOUTH AFRICAN MATHEMATICS OLYMPIAD
Senior Second Round 2013
Solutions

1. **Answer 006**
If the child's age is x, then $x+3 = (x-3)^2$, which simplifies to $x^2-7x+6 = 0$ or $(x-1)(x-6) = 0$, so $x = 1$ or $x = 6$. However, if $x = 1$, then $x-3 = -2$, which is *not* the square root of 4, since by definition a square root must be positive. Thus $x = 6$.
Alternatively, by trial and error, the age is 3 less than a perfect square, so the only possibilities for a child are 1 or 6 or 13, and 6 is the only one that satisfies the other condition.

2. **Answer 008**
$$\frac{2^1+2^0+2^{-1}}{2^{-2}+2^{-3}+2^{-4}} = \frac{2^3(2^{-2}+2^{-3}0+2^{-4}}{2^{-2}+2^{-3}+2^{-4})} = 2^3 = 8.$$

3. **Answer 909**
When written out, $10^{101}-1$ has 101 digits, all equal to 9. Thus the sum of the digits is 909.

4. **Answer 016**
If r is the radius of the circles, then the side of the small square is $2r$ and its area is $4r^2$, which is equal to 4, so $r = 1$. Each side of the large square is $4r$, which is 4, so the area of the large square is 16.

5. **Answer 477**
There are 75 two-digit numbers (from 25 to 99 inclusive) and 109 three-digit numbers (from 100 to 208 inclusive), so the number of digits is $75\times 2+109\times 3 = 150+327 = 477$.

6. **Answer 008**
The number $A1234567B$ is divisible by 45, so it must be divisible by 9 and also by 5. Since it is divisible by 9, the sum of the digits must also be divisible by 9, which means that $A+28+B$ is divisible by 9. Since the last digit B must be 0 or 5, there are two possibilities:

   - If $B = 0$, then $A+28 = 36$, so $A = 8$. ($0 \le A < 10$, so $28 \le A+28 < 38$.)
   - If $B = 5$, then $A+33 = 36$, so $A = 3$. ($0 \le A < 10$, so $33 \le A+33 < 43$.)

   Thus $A+B = 8$ and the two possibilities for (A, B) are $(8, 0)$ or $(3, 5)$.

7. **Answer 169**
   $(2013)^2 - (2)(2013)(2000) + (2000)^2 = (2013 - 2000)^2 = 13^2 = 169.$

8. **Answer 996**
   If we insert brackets, $(2013-2009)+(2005-2001)+\cdots+(29-25)$, then each bracket is equal to 4, and the number of brackets is $(2013 - 29)/8 + 1 = 249$, since the first term decreases by 8 from one bracket to the next. Thus the expression is equal to $4 \times 249 = 996$.
   Alternatively, the expression is equal to $(2013 + 2005 + \cdots + 29) - (2009 + 2001 + \cdots + 25) = \frac{249}{2}[(2013 + 29) - (2009 + 25)] = 996$.

   [Note that if an arithmetic series has n terms, with the first term equal to a and the last term equal to l, then the sum is $\frac{n}{2}(a+l)$. To calculate n, we could use the fact that, for an arithmetic sequence, $T_n = a + (n - 1)d$, which in the current case means $2013 = 29 + (n - 1)(8)$, and hence $n = 249$.]

9. **Answer 012**
   The outside triangle, of area 81, is divided into nine congruent triangles of area 9, one of which is divided into nine congruent triangles of area 1. Thus the area of the shaded region is $1 \times 9 + 3 \times 1 = 12$.

10. **Answer 033**
    To avoid having ten marbles of the same colour, Lindiwe can remove up to nine of each of the colours red, green, blue, plus the five other marbles, making a maximum total of 32 marbles. Thus in order to ensure ten of the same colour, she needs to remove 33 marbles.

11. **Answer 001**
    The left hand side can be factorized to give $(2^x - 3^y)(2^x + 3^y) = 55$, so either $2^x - 3^y = 1$ and $2^x + 3^y = 55$ or $2^x - 3^y = 5$ and $2^x + 3^y = 11$. The first possibility gives $2^x = 28$ and $3^y = 27$, which does not have an integer solution for x. The second possibility gives $2^x = 8$ and $3^y = 3$, which has one integer solution $(x, y) = (3, 1)$.

12. **Answer 036**
    First arrange B, C and D next to each other. There are $3 \times 2 \times 1 = 6$ ways in which this can be done. Now regard B, C and D standing together as a single unit P. Now we arrange A, P and E, which can be done in $3 \times 2 \times 1 = 6$ ways. Thus there are a total of $6 = 36$ for the students to line up.

    Alternatively one can reason as follows: If B, C, D are next to one another, they can line up in $3 \times 2 \times 1 = 6$ different arrangements. With each arrangement, there are two with A and E both in front of them, two more with A and E both behind them, and two more with one in front and one behind. Thus the total number of arrangements is $6 \times (2 + 2 + 2) = 36$.

13. **Answer 040**
The two tangents from an external point to a circle are of equal length, so $AB = AC$ and $RC = RQ$. Therefore the perimeter $AP + PR + RA = AP+(PQ+QR)+RA = (AP+PB)+(CR+RA) = AB+CA = 2\times 20 = 40$.

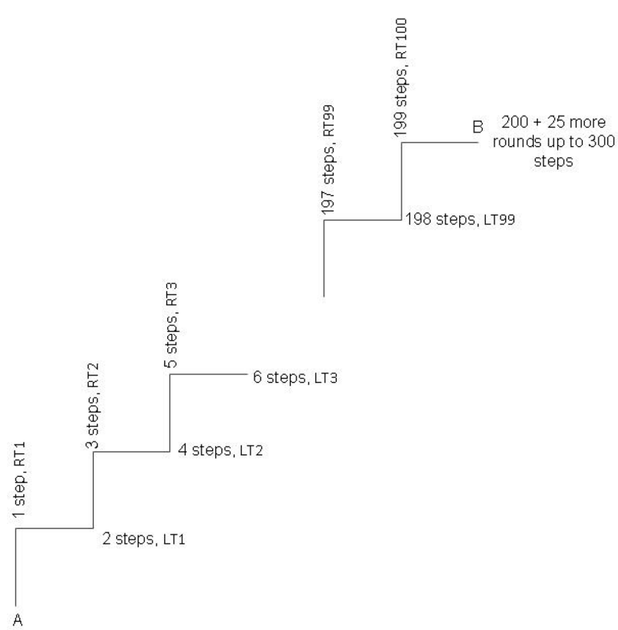
Here is an alternative solution: Using the fact that two tangents from an external point to a circle are of equal length, we have the following equalities: $PB = PQ = 3$, $RQ = RC = x$ and $AB = AC = 20$. Hence the perimeter of triangle APR is equal to $AP + PR + AR = 17 + (3 + x) + (20 - x) = 40$.

14. **Answer 002**
Putting $b = 1$, we see that $f(a + 1) = f(a)$ for all a, so $f(1) = f(2) = f(3) = \cdots = f(2013)$. Thus $f(2013) = 2$.

15. **Answer 030**
It follows from the fact that B says 'One of us present is lying', that he *must* be telling the truth. Indeed, suppose that B were lying. Then his utterance 'One of us present is lying' would be a false statement. Hence both present would be telling the truth, which would contradict the assumption that A is lying. Since we now know that what B is saying is true, we can deduce that A must be lying, hence that the triplets are not between 20 and 29 years old, and hence that they must be 30 years old.

16. **Answer 100**

The maximum distance from A to B is obtained by walking as illustrated in the accompanying diagram. From A to B a sequence of 100 isosceles right triangles is formed. If these triangles are drawn as graphs with A at the origin, the hypotenuses all have gradient 1. The triangles are connected as shown in the diagram. The length of one step is $\frac{1}{\sqrt{2}}$m and therefore the hypotenuse of each triangle is 1m. Since there are 100 triangles between A and B the distance from A to B is 100m.



LTn — nth left turn; RTn — nth right turn.

Now we give a more formal argument, proving that this is indeed the maximum distance that can be obtained:

Instead of working with moves of length $1/\sqrt{2}$ we scale everything up by a factor of $\sqrt{2}$ and prove that the maximum distance is $100\sqrt{2}$. In fact, we prove the stronger statement that both the horizontal and vertical displacement can be at most 100. In what follows we suppose without loss of generality that the first move is vertically upwards from the origin (i.e. from $(0,0)$ to $(0,1)$).

Firstly, let us show that it is possible to obtain a distance of $100\sqrt{2}$. The sequence

$$\underbrace{RLRL\cdots RL}_{\text{99 times}}\underbrace{RR\cdots R}_{\text{101 times}}$$

provides an example. Here follows the location after each of the first few moves: $(0,0),(0,1),(1,1),(1,2),(2,2),(2,3),(3,3),\ldots$. It is clear that after the 99 RL's the location will be $(99,100)$ and after the next R it will be $(100,100)$. However, any sequence of 4 consecutive R's has no effect, since you turn in a circle and get back where you started. Hence, the next 100 R's end up having no effect, so we end up at position $(100,100)$ which is $100\sqrt{2}$ units from the origin.

To prove that this is the best possible we need to keep track of a few things. Define $R(n)$ (resp. $L(n)$) to be the number of right turns (resp. left turns) in the first n moves. (Note that in the first n moves there will be $n-1$ turns.) Also define $x(n)$ and $y(n)$ to be the x- and y-coordinates after n moves. Note that $x(0)=x(1)=y(0)=0$ and $y(1)=1$. Also note that the x-coordinate changes only on even numbered moves, and the y-coordinate changes only on odd numbered moves (e.g. move 1 is vertically upwards, move 2 will be horizontal, move 3 vertical, etc.). In fact, we can be more precise:

$$x(2n+2)=\begin{cases} x(2n)+1 & \text{if } R(2n+1)-L(2n+1)\equiv 1 \pmod 4 \\ x(2n)-1 & \text{if } R(2n+1)-L(2n+1)\equiv 3 \pmod 4 \end{cases}$$

$$y(2n+1)=\begin{cases} y(2n-1)+1 & \text{if } R(2n)-L(2n)\equiv 0 \pmod 4 \\ y(2n-1)-1 & \text{if } R(2n)-L(2n)\equiv 2 \pmod 4 \end{cases}$$

If we let N_i $(i=0,1,2,3)$ denote the number of integers $n\in\{1,2,\ldots,299\}$ for which $R(n)-L(n)\equiv i \pmod 4$ we can write down an explicit formula for the final position: $x(300)=N_1-N_3$ and $y(300)=N_0-N_2$. Note, however, that $N_1+N_3=N_0+N_2=150$, so that $|x(300)|=150-2\min(N_1,N_3)$ and $|y(300)|=150-2\min(N_0,N_2)$. It thus suffices to show that $\min(N_0,N_1,N_2,N_3)\geq 25$.

This is not hard. Note that $R(1)-L(1)=0$, while $R(300)-L(300)=200-99=101$. Also note that this quantity can only increase or decrease by 1 at a time. Therefore $R(n)-L(n)$ takes on all the values $0,1,\ldots,101$. Since there are 25 elements of each congruence class (mod 4) in this set of values, the quantities N_0,N_1,N_2,N_3 are all greater or equal to 25.

17. **Answer 036** Drop a perpendicular from the vertex X to the square base. The slant height XY is the altitude of equilateral triangle ABX. Therefore, height

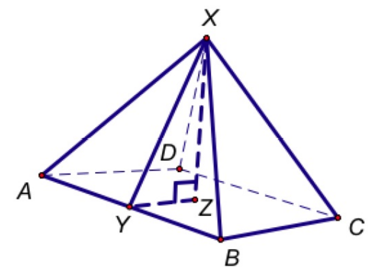
$$XY = 3\sqrt{2} \times \sin 60^\circ = 3\sqrt{2} \times \frac{\sqrt{3}}{2} = \frac{s\sqrt{3}}{\sqrt{2}}$$

(or by using the Theorem of Pythagoras). From the Theorem of Pythagoras,

$$XZ = \sqrt{XY^2 - YZ^2} = \sqrt{\frac{3^2 \cdot 3}{2} - \frac{9}{2}} = 3.$$

Therefore, the volume, which is twice the volume of $ABCDX$ is given by

$$2 \times \frac{1}{3} \text{ Area } ABCD \times XZ = 2 \times \frac{1}{3}(3\sqrt{2})^2 \times 3 = 36.$$



18. **Answer 018**
If two sides of a triangle are of given lengths, then the triangle has maximum area when the two sides are at right angles. If the sides have lengths 7 and 4, then the maximum area is 14 and the hypotenuse is of length $\sqrt{65}$. If the sides have lengths 8 and 1, then the maximum area is 4 and the hypotenuse is also of length $\sqrt{65}$. We can put the hypotenuses together to make a quadrilateral of maximum area $14 + 4 = 18$.

19. **Answer 300**
Let the train to Muizenberg have length x m and speed y m/min, and let the train to Cape Town have speed z m/min. Then $x = 8y$ and $150 = 12z$, giving $z = 12.5$. Also $x + 150 = 9(y + z)$, so $8y + 150 = 9y + 112.5$. This gives $y = 37.5$ and $x = 300$.

20. **Answer 013**
Let the other points of tangency be E (on BC) and F (on CA). Then $AF = 5$ and $BE = 3$. Let $CE = y = CF$. By the Cosine Rule we have $BC^2 = AB^2 + CA^2 - 2.AB.CA.\cos A = AB^2 + CA^2 - AB.CA$, or $(y + 3)^2 = 8^2 + (y + 5)^2 - 8(y + 5)$. This simplifies to $4y = 40$, so $y = 10$. Therefore, $BC = BE + CE = 3 + 10 = 13$.