The South African Mathematical Olympiad
Third Round 2005
Senior Division (Grades 10 to 12)
Solutions

1. The five numbers have the form

$$1+3a, 16+3b, 31+3c, 46+3d, 61+3e$$

where $\{a,b,c,d,e\} \subseteq \{0,1,2,3,4\}$. Bot no two of a,b,c,d,e are the same, so $\{a,b,c,d,e\} = \{0,1,2,3,4\}$. Hence $a+b+c+d+e = 10$, and the sum is therefore always equal to $155 + 3 \times 10 = 185$.

2. We are looking for $m/n \in F$ with $16/23 - m/n = (16n - 23m)/23n$ as small as possible (and positive). Since 16 and 23 are relatively prime, we opt to find positive integers m and n such that $16n - 23m = 1$, with n is as large as possible, provided that $m + n \leq 2005$.

   Solving the linear Diophantine equation $16n - 23m = 1$, we find that $m = -7 + 16k$ and $n = -10 + 23k$ for arbitrary $k \in \mathbb{Z}$. To ensure that both m and n are positive and $m + n \leq 2005$, we must have $1 \leq k \leq 51$. So the largest n is obtained when $k = 51$, and this gives $n = 1163$. The corresponding m when $k = 51$ is $m = 809$. So the desired $a = m/n \in F$ is $a = 809/1163$.

   (Note that the only way in which we could hope to make the fraction $(16n - 23m)/23n$ even smaller, is to have $16n - 23m \geq 2$ and n at least twice 1163. But this would violate $m + n \leq 2005$, so the solution $a = m/n = 809/1163$ is indeed the correct one.)

3. Let $n = x + y + z$, where x, y and z denote the number of usable no. 8, no. 9 and no. 10 pairs, respectively, and assume that $n \in \{50, 51\}$. Lets put the n usable pairs aside and refer to these boots as 'used', and to the remaining boots as 'unused'.

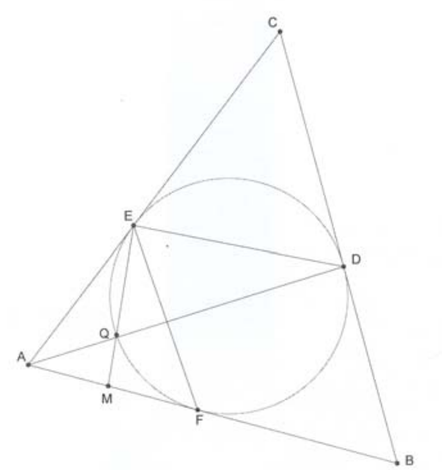
   Then $250-n$ left boots and $300-n$ right boots are unused. Since $300-n \geq 249$, there must be at least two sizes of unused right boots (200 is the maximum number of boots in any size). If the $250 - n$ unused left boots also contain at least two sizes amongst them, there has to be one size in which there are both left and right unused boots, a contradiction, since otherwise we would have more than n usable pairs. This means that the $250 - n$ unused left boots are all of the same size, and this size has to be 10, since $250 - n \geq 199$ (only 175 boots in each of sizes 8 and 9 are available).

So the total number of size 10 boots is given by

$$250 - n + 2z = 200, \tag{1}$$

immediately disqualifying $n = 51$, being an odd number. For $n = 50$, (1) implies that $z = 0$, and we simply have to solve for $x + y = 50$ in nonnegative integers to see that there are many ways in which $n = 50$ is possible. In fact, the complete set of solutions (x, y, z) in this case is given by $\{(x, 50 - x, 0) : x = 0, 1, \ldots, 50\}$.

4. Construct EF and ED and let M be the point where EQ extended meets AF. Note that $C\hat{E}D = C\hat{D}E$. We will use the fact that $AC = BC$ if and only if $ED \| AB$.



Let $AM = MF$. Then $AM^2 = MF^2 = MQ \cdot ME$, using the Power of the Point Theorem. Hence $\frac{AM}{MQ} = \frac{ME}{AM}$, implying that $\triangle AMQ ||| \triangle EMA$. But then $M\hat{A}D = A\hat{E}M = E\hat{D}A$, showing that $ED \| AB$, and hence $AC = BC$.

Conversely, if $AC = BC$ (so that $ED \| AB$), then $M\hat{A}D = E\hat{D}A = A\hat{E}M$. So, $\triangle AMQ ||| \triangle EMA$, implying that $\frac{AM}{MQ} = \frac{ME}{AM}$. But then $AM^2 = MQ \cdot ME = MF^2$, and it follows that $AM = MF$.

5. Consider

$$\begin{aligned}
&\left(\frac{n}{n+x_1}\right)\left(\frac{n+x_1}{n+x_1+x_2}\right)\left(\frac{n+x_1+x_2}{n+x_1+x_2+x_3}\right)\cdots\left(\frac{n+x_1+\cdots+x_{n-1}}{n+x_1+\cdots+x_n}\right)\\
= \quad & \frac{n}{n+x_1+\cdots+x_n}\\
\le \quad & \frac{n}{2n}, \text{ since } x_1+\cdots+x_n \ge n(x_1\cdots x_n)^{1/n} = n, \text{ by the AM-GM inequality}\\
= \quad & \frac{1}{2}.
\end{aligned}$$

Hence there exists $k \in \{1,2,\ldots,n\}$ such that

$$\frac{n+x_1+\cdots+x_{k-1}}{n+x_1+\cdots+x_k} \le \left(\frac{1}{2}\right)^{1/n}.$$

This translates to

$$\frac{x_k}{n+x_1+\cdots+x_k} \ge 1-\left(\frac{1}{2}\right)^{1/n},$$

and, since $k \le n$, it follows that

$$\frac{x_k}{k+x_1+\cdots+x_k} \ge 1-\left(\frac{1}{2}\right)^{1/n}.$$

6. Let a_n denote the n-th element of the sequence.

Our first observation is that the last member of the n-th block is n^2. In fact, if the last number in the n-th block is n^2, then the elements of the $(n+1)$-st block are given by $n^2+1, n^2+3, \ldots, n^2+2(n+1)-1 = (n+1)^2$, so this observation is easily verified by induction.

Since the lengths of the blocks are successively $1, 2, 3, \ldots$, it follows that $a_{t_k} = k^2$, where $t_k = k(k+1)/2$ denotes the k-th triangular number.

Therefore, for any $n \ge 1$, a_n belongs to the $(k+1)$-st block for some $k \ge 0$, whereupon

$$n = \frac{k(k+1)}{2} + q \quad \text{for some } q,\ 1 \le q \le k+1, \tag{2}$$

and

$$a_n = k^2 + 2q - 1. \tag{3}$$

If we solve for q in (2) and substitute this into (3), we find that

$$a_n = 2n - k - 1. \tag{4}$$

By (2), $8n = 4k^2 + 4k + 8q$, i.e., $8n - 7 = (2k+1)^2 + 8(q-1)$. Using the fact that $0 \le q - 1 \le k$, it follows that

$$(2k+1)^2 \le 8n - 7 \le (2k+1)^2 + 8k < (2k+3)^2.$$

But then

$$\begin{array}{rrcll} & 2k+1 & \le & \sqrt{8n-7} & < & 2k+3 \\ \Rightarrow & 2k+2 & \le & 1+\sqrt{8n-7} & < & 2k+4 \\ \Rightarrow & k+1 & \le & (1+\sqrt{8n-7})/2 & < & k+2, \end{array}$$

and we see that

$$k + 1 = \left\lfloor \frac{1+\sqrt{8n-7}}{2} \right\rfloor.$$

Plug this into (4) and the problem is solved.