The South African Mathematical Olympiad
Junior Third Round 2013
Solutions

1. Note that $20^{13} = 2^{13} \cdot 10^{13} = 8192 \times 10^{13}$. This number equals 8192 followed by 13 zeroes, which gives 17 digits in total.

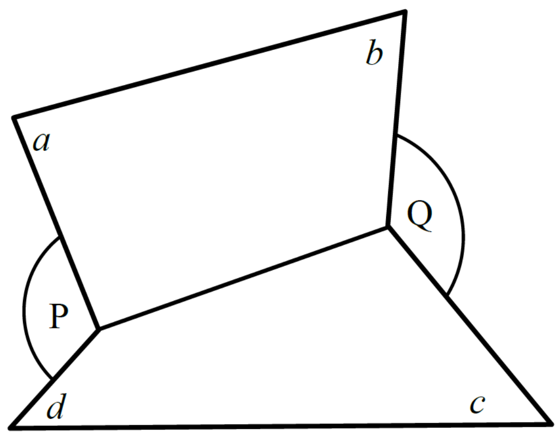
2. A number is divisible by $45 = 9 \times 5$ if and only if it is divisible by both 5 and 9. If our palindromic number is divisible by 5, it must end in a 5, and hence also starts with a 5.

   Next, for it to be divisible by 9, the sum of the digits of the number must be divisible by 9. The largest palindromic 8-digit number that starts with a 5 and has digit sum divisible by 9 equals 59944995.

3. Join the points P and Q to divide the figure up into two quadrilaterals. The sum of the interior angles of a quadrilateral equals $360°$, so computing the total angle sum of the two quadrilaterals in two ways, we obtain

$$2 \times 360° = (360° - P) + (360° - Q) + a + b + c + d,$$

   which simplifies to $P + Q = a + b + c + d$.



4. Let's compute the height of twenty trillion bills of thickness 0.1mm stacked on top of each other:

$$\begin{aligned} \text{height} &= 20,000,000,000,000 \times 0.1 \text{ mm} \\ &= 2,000,000,000,000 \text{ mm} \\ &= 2,000,000 \text{ km}. \end{aligned}$$

   This distance is *much* greater than the height that aeroplanes fly. To put it in perspective, the distance between the earth and the moon is on average 384,000km, so the stack of bills will reach higher up than 5 times the distance to the moon.

5. The first few terms in the Fibonacci sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.$$

   We can write $120 = 89 + 21 + 8 + 2$, so using the described method, 120 km converts to

$$144 + 34 + 13 + 3 = 194 \text{ miles}.$$

6. We use two important observations to solve this problem:

   - Every 2×2 subtable includes the centre square;
   - Every 2×2 subtable includes exactly one corner square.

   This means that every time we increase the numbers in a 2×2 subtable by 1, we increase the centre square by one, and increase exactly one of the corner squares by one. Hence, the sum of the numbers in the corner squares increases by 1 every time. This means that the sum of the numbers in the corner squares must equal the number in the centre square after each step. However, the sum of the numbers in the corner squares equals $4+5+6+7=22$, which is not equal to 18, the number in the centre square. Hence it is impossible to obtain the table given.

   Alternative Solution

   Note, as above, that every 2×2 subtable includes the centre square. Since the number in the centre square is 18, there must have been 18 moves. Next, every move increases the total sum of the numbers in the table by 4. Since we started with a table with only zeroes (sum 0) and there were 18 moves, the total sum of the numbers in the table after 18 moves must be $18 \times 4 = 72$. However, the sum of the entries in the given table is 84, which implies that this configuration is not possible.

7. (a) Multiplying out the left-hand side and grouping terms, we need to solve the equation

   $$a^2 + 15b^2 + 2ab\sqrt{15} = 31 - 8\sqrt{15}$$

   over the integers. This means that $a^2 + 15b^2 = 31$ and $2ab = -8$. The first equation immediately implies that $b^2 = 1$ and so $a^2 = 16$. So $a = \pm 4$ and $b = \pm 1$, but $ab = -4$, so there are two solutions: $a = 4, b = -1$ or $a = -4, b = 1$.

   Alternative Solution

   From the second equation above, we find that $b = -\frac{4}{a}$. Substitute this into the first equation to obtain

   $$\begin{aligned} a^2 + 15b^2 &= 31 \\ \implies a^2 + 15\left(-\frac{4}{a}\right)^2 &= 31 \\ \implies a^4 + 15(16) &= 31a^2 \\ \implies a^4 - 31a^2 + 15(16) &= 0 \\ \implies (a^2 - 15)(a^2 - 16) &= 0. \end{aligned}$$

   Since a is an integer, only $a^2 = 16$ applies, and we find that $a = \pm 4$, and substituting this into $b = -\frac{4}{a}$ we find that $b = \mp 1$.

   (b) From part (a), $31 - 8\sqrt{15} = (4 - \sqrt{15})^2$, and similarly, $31 + 8\sqrt{15} = (4 + \sqrt{15})^2$, so we have that

   $$\sqrt{31 + 8\sqrt{15}} + \sqrt{31 - 8\sqrt{15}} = (4 - \sqrt{15}) + (4 + \sqrt{15}) = 8.$$

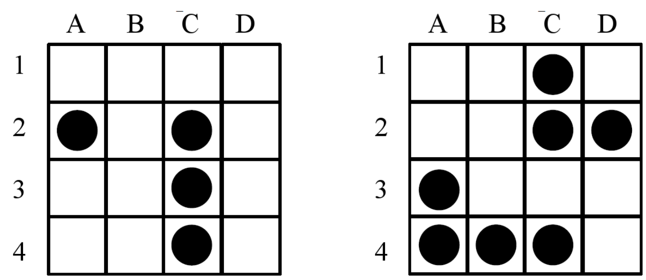
8. (a) The only winning move will be to remove any two adjacent stones in column C. If player 1 makes that move, player 2 is forced to remove one of the two remaining stones, which enables player 1 to remove the last stone.

   No other initial move will guarantee a win:

   - if player 1 removes the stone $A2$, player 2 wins by removing the three remaining stones;

- if player 1 removes any one stone in column C, player 2 removes any 1 of the remaining stones in column C;
- if player 1 removes the three stones in column C, player 2 wins in his very next move.

(b) Remove the stone in square $C4$. If that is the case, what remains is two identical L-shaped configurations of stones. Whatever player 2 does, player 1 copies on the other configuration (for example, if player 2 removes the stone $C2$, player 1 removes the stone $A4$). This guarantees that after player 2 has moved, player 1 can also move, which means that player 2 cannot remove the last stone.



9. Label the columns A to E from left to right, and the rows 1 to 5 from top to bottom.

Note that if 5 buildings can be seen from a particular vantage point, then the 5 buildings must appear in the order $1, 2, 3, 4, 5$. Also, if only 1 building can be seen from a particular vantage point, then the tallest building (with 5 storeys) must be the frontmost one. Using these two observations, we immediately see that row 5 contain the numbers from 1 to 5 in increasing order, and $A4 = B1 = 5$.

Now look at column A, viewed from the north. Since Jonathan can see four buildings from the north, the three buildings closest to him must be from shortest to tallest, that is, 2, 3 and 4, in that order.

Now look at column B, viewed from the south. The building in front of Jonathan has 2 storeys, so he won't be able to see the building with height 1. Since he can see the building with height 2 and the building with height 5, he can only see one other building - this means that the building with height 3 hides behind the building with height 4. Since row 2 already contains a building with height 3, the only possible configuration is $B2 = 1, B3 = 3, B4 = 4$.

Now looking at row 3 from the east, both the buildings with height 3 and 4 will be hidden behind the building with height 5. Since Jonathan can see three buildings from this vantage point, it means that $C3 = 5$, $D3 = 2$ and $E3 = 1$.

Now look at column C viewed from the north. Jonathan can see two buildings from this point: the frontmost building and the building with height 5. This means that the frontmost building can't have height 1 (otherwise he would also be able to see the building in position $C2$), and the building with height 1 cannot be in position $C2$, since there is already a building of height 1 in row 2. Hence $C4 = 1$. It then follows that $C1 = 4$ and $C2 = 2$.

Now look at row 1 from the east. Jonathan can see three buildings from this vantage point: the buildings with heights 4 and 5, and the frontmost building. Since the building in $D1$ must hide behind the building in $E1$, it follows that $D1 = 1$ and $E1 = 3$.

The rest of the table is now easy to complete.

		A	B	C	D	E	
		4	1	2	2	3	
		↓	↓	↓	↓	↓	
1	2 →	2	5	4	1	3	← 3
2	2 →	3	1	2	5	4	← 2
3	2 →	4	3	5	2	1	← 3
4	1 →	5	4	1	3	2	← 4
5	5 →	1	2	3	4	5	← 1
		↑	↑	↑	↑	↑	
		2	3	2	2	1	

10. (a) The fifth term equals $2(5) - 2(5-1)(5-2)(5-3) = -38$.

    (b) The first three terms of the sequence hold for any value of k, and the fourth term is then equal to $2(4) - k(4-1)(4-2)(4-3) = 8 - 6k$. Setting this equal to 38 and solving for k, we find that $k = -5$.

11. Since p is odd, we can write $p = 2q + 1$. Suppose n is the sum of p consecutive integers:

$$n = (a-q)+(a-q+1)+(a-q+2)+\cdots+(a-1)+a+(a+1)+\cdots+(a+q-2)+(a+q-1)+(a+q).$$

This sum simplifies to $n = pa$, which immediately shows that n is a multiple of p.

(Alternatively, note that the sum of the p consecutive integers

$$b+(b+1)+(b+2)+\cdots+(b+p-1) = pb+1+2+\cdots+p-1 = pb+\frac{p(p-1)}{2} = p\left(b+\frac{p-1}{2}\right)$$

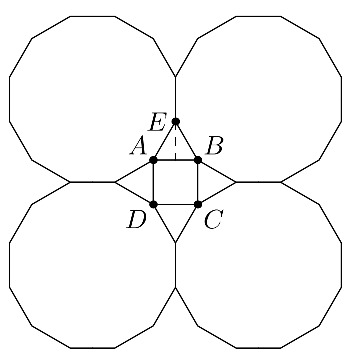
is a multiple of p, since $p-1$ is even.)

The number in question (a) must then be divisible by 3, 5 and 7. The smallest such number is $3 \times 5 \times 7 = 105$, which we can write as the sums

$$\begin{aligned} 105 &= 34 + 35 + 36 \\ &= 19 + 20 + 21 + 22 + 23 \\ &= 12 + 13 + 14 + 15 + 16 + 17 + 18. \end{aligned}$$

12. Join the points A, B, C and D as indicated. Note that if one rotates the diagram by $90°$, one obtains exactly the same diagram. This means that the segment AB rotated through 90 degrees is the segment BC, and so the quadrilateral $ABCD$ is a square.

Next, the external angle of a regular dodecagon is equal to $\frac{360°}{12} = 30°$. Hence the angle $\angle AEB$ equals $2 \times 30° = 60°$. Since $AE = EB = 1$, the triangle AEB is isosceles with one $60°$ angle, which implies that triangle AEB is an equilateral triangle and so $AB = 1$.



The total area of the shaded region is thus the area of a square with side length 1 plus four times the area of an equilateral triangle with side length 1, which equals $1+4\cdot\frac{\sqrt{3}}{4}=1+\sqrt{3}$.

13. Group the numbers in the following way:

$$\{4,100\},\ \{7,97\},\ \{10,94\},\ \ldots,\ \{49,55\},\{1\},\{52\}.$$

If both numbers from any of the first 16 pairs of numbers are in the set A, then their sum will be 104. Now, to avoid this, the best one can do is select one number from each of the 16 pairs, the number 1 and the number 52. This results in a set of 18 numbers. Hence if A contains 19 numbers, A *must* contain both numbers of one of the first 16 pairs.

14. Suppose that x is the number in the circle common to all three lines, and suppose that the numbers on each line add up to S. Adding the three lines together, we see that we add the numbers from 1 to 12, but the number x is added three times. This means that

$$3S=1+2+\cdots+12+2x=78+2x.$$

This means that $2x=3S-78=3(S-26)$, which is divisible by 3. Hence x is divisible by 3, and so the only possible values for x are 3, 6, 9 and 12. The sums S corresponding to these x values are 28, 30, 32 and 34.

We still need to show that these four sums *are* actually attainable by giving an example that works. This is easy enough and is left as an exercise.

15. (a) In this case, $s=\frac{13+14+15}{2}=21$ and so

$$K=\sqrt{21\cdot 8\cdot 7\cdot 6}=\sqrt{2^4\cdot 3^2\cdot 7^2}=84.$$

The perimeter equals $13+14+15=42=\frac{84}{2}$, as required.

(b) Suppose that the middle even number is n; then the triangle has side lengths $n-2$, n and $n+2$. In this case $s=\frac{(n-2)+n+(n+2)}{2}=\frac{3n}{2}$. Hence

$$K^2=\frac{3n}{2}\cdot\frac{n}{2}\cdot\left(\frac{n}{2}-2\right)\left(\frac{n}{2}+2\right)=\frac{3n^2}{4}\left(\frac{n^2-16}{4}\right)=\frac{3n^2(n^2-16)}{16}.$$

The perimeter of the triangle equals $(n-2)+n+(n+2)=3n$, so equating the squares of the area and the perimeter, we obtain

$$\begin{aligned}(3n)^2 &= K^2=\frac{3n^2(n^2-16)}{16}\\ \implies 9n^2 &= \frac{3n^2(n^2-16)}{16}\\ \implies 16\cdot 9 &= 3(n^2-16)\\ \implies 48 &= n^2-16\\ \implies n^2 &= 64\\ \implies n=8,\end{aligned}$$

since n must be positive. Hence the (right-angled) triangle with side lengths 6, 8 and 10 is the only triangle with this property.