Solution 4 It should be noted that this problem was severly underestimated by the problems committee, whose solutions all required tacit additional hypotheses and would not have scored more than 4 out of 10. In fact, the problem is harder than Problem 6.

Let A and B be any pair of friends. Together with their mutual friend they form a 'friendship triangle'. None of these three friendships can occur in another triangle without violating uniqueness of mutual friendship. The total number of friendships equals $3t$, where the number of triangles is t. Each person X has two friends per triangle, and therefore an even number of friends.

Now take any triangle ABC. The people other than A, B and C can be divided into four mutually exclusive sets: other friends of A, B and C; and 'strangers'. Let these sets contain respectively a, b, c and s people. Now the mutual friend of any pair of other friends of A and B must be a stranger, otherwise uniqueness of mutual friendship is violated. (To see this, it is easiest to refer to a diagram.) On the other hand, every stranger can be identified in a unique way from the pair of his mutual friends with A and B respectively. There are ab such pairs. Therefore

$$s = ab = bc = ca,$$

where the last two equations follow from symmetry.

If $s = 0$, two of a, b and c must also be zero. The person corresponding to the nonzero value is then a friend of everyone else. This will turn out to be the only case possible.

If $s \neq 0$, then $a = b = c$, which implies that everybody has the same number of friends d. We can count the total number n of people in terms of d : other friends of A, B and C tally $(d-2)$ each, and strangers $(d-2)^2$. This gives

$$n = 3 + 3(d-2) + (d-2)^2 = d^2 - d + 1.$$

The last and most difficult part of the proof is to show that for $d > 2$, this count for n is incompatible with the other conditions, which eliminates the case $s \neq 0$, thereby proving the theorem. Remember that d must be even, and that the total number of friendships is $3t$. Note that therefore $3t = \frac{1}{2}nd$. We start with the case $d = 4$. This gives $n = 13$ and $\frac{1}{2}nd = 26$, which is not a multiple of 3.

For d not of the form $6j-2$ the same argument gives no contradiction. We need to find a generalization of a triangle which can be counted in two ways, such that the two counts cannot possibly be equal. A suitable object to count is a 'friendship circle' $A_1A_2\ldots A_k$, in which each person is a friend of the two people adjacent to him, with A_1 adjacent to A_k, and where it is not required that all A_i be distinct. A friendship circle reduces to a triangle when $k = 3$. Let m_k be the number of such circles.

Now A_1 can be any of n people, and each A_i from A_2 to A_{k-1} any of the d friends of his predecessor, giving nd^{k-2} possibilities. At that point there are two cases:

1. If $A_{k-1} = A_1$, then $A_1A_2\ldots A_{k-2}$ is a friendship circle, which can happen in m_{k-2} ways. A_k can be any of the d friends of A_1, which gives us dm_{k-2} possibilities.
2. If $A_{k-1} \neq A_1$, then there is no further freedom — A_k must be the unique friend of A_{k-1} and A_1. This gives us $nd^{m-2} - m_{k-2}$ possibilities.

We therefore get:

$$m_k = dm_{k-2} + nd^{m-2} - m_{k-2} = (d-1)m_{k-2} + nd^{m-2}.$$

Since $n = d(d-1)+1$, it follows that $m_k \equiv 1 \mod (d-1)$ when $k > 2$.

On the other hand, whenever k is a prime, the friendship circles $A_1A_2\ldots A_k$, $A_2\ldots A_kA_1$, $\ldots$, $A_kA_1\ldots A_{k-1}$, are all different. (To see this, imagine people moving up a fixed number of places around a circular table.) Therefore friendship circles come in mutually exclusive classes of k items each, which implies $m_k \equiv 0 \mod k$.

To complete the proof, note that $d-1$ is odd and must therefore have a prime factor $k>2$ whenever $d \geq 4$. This leads to $m_k \equiv 1 \mod k$, which gives the required contradiction.

Solution 6 The required result is a special case of the following:

n squares of total area Q can be packed into a rectangle of area $2Q$ provided that the largest of the squares can fit into the rectangle.

We prove this result by induction. The case $n=1$ is trivial.

In the proof, we shall repeatedly use the easily proved fact that if two squares have total area s^2, their sides cannot sum to more than $\sqrt{2}s$, which is attained when the squares are equal.

In the general case, denote the sides of the rectangle by a and b with $a \leq \sqrt{2Q} \leq b$, the k-th largest square by S_k and its side by s_k.

If $s_1 > a/2$, we can cut the rectangle into two pieces: one of dimension $s_1 \times a$ containing S_1, and the other of dimension $(b-s_1)\times a$ into which the other $n-1$ squares of total area $Q-s_1^2$ are to be packed. Since $a(b-s_1) < ab-2s_1^2 = 2(Q-s_1^2)$, this is possible by the induction hypothesis provided that $s_1+s_2 \leq b$. This is always the case because $s_1^2+s_2^2 \leq Q$ whereas $b \geq \sqrt{2Q}$.

We are left with the case $s_1 \leq a/2$. It is then possible to pack S_1, S_2, S_3, S_4 into the rectangle, one into each corner, so that S_1 and S_2 adjoin one of the shorter sides of the rectangle, and S_1 and S_3 adjoin one of the longer sides. We now argue that it is possible to fit S_5 along that side between them, i.e. that $s_1+s_3+s_5 \leq b$.

Clearly the hardest case occurs when $s_2 = s_3$ and $s_4 = s_5$. Then $s_3+s_5 \leq \sqrt{2(s_3^2+s_5^2)} \leq \sqrt{Q-s_1^2}$ since the five squares have total area not exceeding Q. But the largest possible value of $s_1+\sqrt{Q-s_1^2}$ is $\sqrt{2Q} \leq b$.

We now choose a set of squares, starting with S_1 and S_2 and continuing with S_5 and smaller squares, so that their total area Q_1 is at least $as_1/2$, but removing any square from the set reduces the total area to less than $as_1/2$. The idea is that $c = 2Q_1/a$ should exceed s_1 by as little as possible. In fact, $c-s_1 \leq s_5^2/(a/2) \leq s_5^2/s_1 \leq s_5$.

Cut the rectangle into two pieces, one of dimension $a \times c$ where $c = 2Q_1/a$, into which this set of squares can be packed by the induction hypothesis since by construction $s_1 \leq 2Q_1/a$; and the other of dimension $2(Q-Q_1)/a$, into which the remaining squares can be packed by the induction hypothesis, provided that $s_3 \leq b-c$.

But we showed earlier that $s_1+s_3+s_5 \leq b$, hence $s_3 \leq b-(s_1+s_5) \leq b-c$ as required.