THE HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

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SECOND ROUND 2004

SENIOR SECTION: GRADES 10, 11 AND 12

13 MAY 2004

TIME: 120 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. E
2. B
3. B
4. B
5. D
6. B
7. D
8. B
9. C
10. A
11. B
12. B
13. D
14. B
15. A
16. C
17. D
18. E
19. E
20. B

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Solutions

1. **Answer E.** Half of 2^{20} equals $\frac{1}{2} \times 2^{20} = 2^{-1} \times 2^{20} = 2^{-1+20} = 2^{19}$.

2. **Answer B.** Each corner of each rectangle can be cut off by the other rectangle, giving eight triangular regions, together with one octagonal region in the middle. See diagram alongside.

3. **Answer B.** Since the numbers are all positive, it is sufficient to determine how many of their squares are greater than 100. Their squares are, respectively,

$$9 \times 11 = 99, \quad 16 \times 7 = 112, \quad 25 \times 5 = 125, \quad 36 \times 3 = 108, \quad 49 \times 2 = 98,$$

of which the three middle numbers are greater than 100.

4. **Answer B.** We can draw up a table of the first few remainders and hope to find a pattern.

n	1	2	3	4	5	6	7	...
2^n	2	4	8	16	32	64	128	...
Rem.	2	4	1	2	4	1	2	...

It certainly appears that there are only three distinct remainders (1, 2, and 4), which repeat cyclically, and in fact modular arithmetic (the arithmetic of remainders) guarantees that the pattern does continue unchanged. However, if you would like to prove that (say) 4 is always followed by 1, suppose 2^n has remainder 4 after dividing by 7. Then $2^n = 7q + 4$, where q is the quotient. It follows that

$$2^{n+1} = 2 \times 2^n = 2 \times (7q + 4) = 14q + 8 = 7(2q + 1) + 1,$$

so the next remainder is indeed 1, as we expected.

5. **Answer D.** We obtain two equations: $x + y = 10$ and $x^2 - y^2 = 40$. The second one gives $(x+y)(x-y) = 40$, that is, $10(x-y) = 40$, so $x-y = 4$. Now add and subtract the equations to get $x = 7$ and $y = 3$, so $x^2 + y^2 = 49 + 9 = 58$.

6. **Answer B.** Since x and z are consecutive integers, it follows that $z = x + 1$. (We cannot have $z = x - 1$, because the hypotenuse is the longest side.) If the third side is y, then by Pythagoras' theorem

$$y^2 = z^2 - x^2 = (x + 1)^2 - x^2 = 2x + 1 = x + (x + 1) = x + z.$$

7. **Answer D.** The first number is 2357111317192329, which has 16 digits, of which we must cross out eight. The leftmost digit is the most significant, since it is multiplied by the highest power of 10, so we must make it as large as possible. The best we can do is to cross out the first three digits (235) so that the leftmost digit is 7. We must now make the second digit as large as possible by crossing out at most five digits after the 7. It turns out we can cross out exactly five digits (11131) and make the second digit 7 again. We are now left with the eight-digit number 77192329, whose second digit is 7.

8. **Answer B.** Consider any one of the four squares forming the big square. The shaded triangle clearly occupies half of half of the small square, that is, one quarter of it. The same fraction is then true for the whole figure. (Alternatively, it is easy to cut the shaded region into four triangles that can be pasted to fill one quarter of the big square without overlapping.)

9. **Answer C.** Triangle ABC is isosceles, so angles $\widehat{A}$ and $\widehat{B}$ are equal. Also $Q\widehat{P}B = \widehat{A}$ and $M\widehat{P}A = \widehat{B}$ (corresponding angles), so triangles APM and PBQ are also isosceles. Thus $MP = MA$ and $PQ = BQ$ so

$$CM + MP + PQ + QC = CM + MA + BQ + QC = CA + BC = 15 + 15 = 30.$$

Alternatively, since the position of P is unspecified, you can make it coincide with B (say). The perimeter is then $2BC = 30$.

10. **Answer A.** The hexagon is made up of six equilateral triangles with side 1, each of which has one-ninth of the area of the equilateral triangle with side 3. (The ratio of the areas of similar figures is equal to the square of the ratio of their sides.) Thus the required ratio is $6 : 9 = 2 : 3$.

11. **Answer B.** Let the sum of all the numbers be S, so the middle number is $\frac{1}{9}S$. The sum of the five largest numbers is $5 \times 68 = 340$, and the sum of the five smallest numbers is $5 \times 44 = 220$. If we add these sums together, then we have counted the middle number twice, so $S + \frac{1}{9}S = 340 + 220$, that is $\frac{10}{9}S = 560$. Thus $S = \frac{9}{10} \times 560 = 504$.

12. **Answer B.** The sum of the interior angles of any n-sided polygon is $(n-2)180°$, so each interior angle of a regular n-sided polygon is $(1-\frac{2}{n})180°$. In particular, for a regular pentagon (with $n = 5$) each interior angle is $108°$. From the portion of the unknown n-gon that is shown in the diagram, we see that its interior angle is $360° - 2 \times 108° = 144°$. Now solve $(1 - \frac{2}{n})180 = 144$ to get $n = 10$.

13. **Answer D.** Suppose that the areas of the three white regions are x, y, z (from left to right). Then the lightly shaded region has area $(121 - x) + (49 - y - z) = 170 - (x + y + z)$, and the black region has area $(81 - x - y) + (25 - z) = 106 - (x + y + z)$. The difference between the areas of the two regions is $170 - 106 = 64$.

14. **Answer B.** We have to consider all sets of three positive integers whose product is 36. They are:
$$\{1,1,36\} \quad \{1,2,18\} \quad \{1,3,12\} \quad \{1,4,9\} \quad \{1,6,6\} \quad \{2,2,9\} \quad \{2,3,6\}.$$
The sums of the numbers in these sets are all different, except for $1+6+6 = 13 = 2+2+9$. Since Johann's set was not uniquely determined by its sum, it must have been one of the sets whose sum is 13.

15. **Answer A.** Suppose there are n marbles in each jar. Then the number of white marbles is $\frac{1}{8}n + \frac{1}{10}n = \frac{9}{40}n$, which equals 90, so $n = 400$. The number of red marbles in the second jar is $\frac{9}{10}n = 360$.

16. **Answer C.** If the tens and units digits are t and u respectively, then $10t + u = q(t + u)$. Now suppose $10u + t = r(u + t)$, where r is to be determined. Adding the equations gives $11(t + u) = (q + r)(t + u)$, so $r = 11 - q$.

17. **Answer D.** First notice that the sum of all the digits from 1 to 9 is 45, which is divisible by 3. Thus the sum of seven chosen numbers is divisible by 3 if and only if the sum of the two remaining numbers is also divisible by 3. It is much simpler to count how many ways to choose the two remaining numbers: either they are both divisible by 3, or one leaves remainder 1 and the other leaves remainder 2 after dividing by 3. Thus either both come from the set $\{3, 6, 9\}$ (giving three possibilities), or one comes from the set $\{1, 4, 7\}$ and the other from the set $\{2, 5, 8\}$ (giving nine more possibilities), for a total of 12.

18. **Answer E.** Set up axes with A at the origin, the x-axis along AD and the y-axis along AB. If B is $(0; b)$, then the line AC has equation $\dfrac{x}{12} = \dfrac{y}{b}$. The lines BD, BE, and so on all have equations of the form $\dfrac{x}{h} + \dfrac{y}{b} = 1$, where h is the intercept on the x-axis. If we solve this with the equation of BD to find the next intercept on the x-axis, then we obtain
$$\frac{x}{h} + \frac{x}{12} = 1, \quad \text{so} \quad x = \frac{12h}{12 + h}.$$
Starting at D we have $h = 12$, so $x = 6$ which is (obviously) the intercept of E. Now put $h = 6$ to get $x = 4$, which is the intercept of F (as can also be seen geometrically). Now put $h = 4$ to get $x = 3$, which is the intercept of G. (In fact, if you continue the process indefinitely, then the intercept of the n-th point (counting D as the first) is $\dfrac{12}{n}$.)

19. **Answer E.** Choose a player at random. He (or she) can be paired with any one of the seven children remaining. Now choose another player at random from the six left over. He can be paired with one of the five others. Similarly, there are three choices for the third pair, and there is one pair left over. Thus the total number of pairings is $7 \times 5 \times 3 \times 1 = 105$.

20. **Answer B.** Write down the previous equation, in which n is replaced by $n-1$:

$$f(1)+f(2)+\cdots+f(n-1)=(n-1)^2 f(n-1).$$

Subtracting this from the given equation gives $f(n)=n^2 f(n)-(n-1)^2 f(n-1)$, so

$$f(n)=\frac{(n-1)^2}{n^2-1}f(n-1)=\frac{n-1}{n+1}f(n-1).$$

Now apply this result repeatedly for $n=2004, 2003, \ldots$. We obtain

$$f(2004)=\frac{2003}{2005}f(2003)=\frac{2003}{2005}\cdot\frac{2002}{2004}f(2002)=\frac{2003}{2005}\cdot\frac{2002}{2004}\cdot\frac{2001}{2003}f(2001)\ldots$$

This finally becomes

$$f(2004)=\frac{(2003)(2002)(2001)(2000)\ldots(3)(2)(1)}{(2005)(2004)(2003)(2002)\ldots(5)(4)(3)}f(1)=\frac{(2)(1)}{(2005)(2004)}f(1),$$

since everything else cancels. Since $f(1)=2005$, it follows that $f(2004)=2/2004=1/1002$.
