THE HARMONY GOLD SOUTH AFRICAN MATHEMATICS OLYMPIAD

organised by the SOUTH AFRICAN ACADEMY OF SCIENCE AND ARTS
in collaboration with Harmony Gold Mining, AMESA and SAMS

FIRST ROUND 2002

SENIOR SECTION: GRADES 10, 11 AND 12

19 March 2002

TIME: 60 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. A
2. D
3. B
4. B
5. E
6. A
7. B
8. C
9. A
10. D
11. A
12. C
13. C
14. E
15. B
16. A
17. B
18. A
19. D
20. A

SOLUTIONS

1. Answer A. The first runner is the fastest (approximately 75) and the third runner is the slowest (approximately 90), so the difference is 15.

2. Answer D. Bracket the numbers two by two into 1001 pairs:

$$(2002 - 2001) + (2000 - 1999) + \cdots + (2 - 1).$$

Each bracket is equal to 1, so the value of the whole expression is 1001.

3. Answer B. If you cannot see immediately that the average increases by 20, suppose the original numbers were $n_1, n_2, \ldots, n_{10}$, so the old average was $\frac{1}{10}(n_1+n_2+\cdots+n_{10})$. The new average is

$$\frac{1}{10}((n_1+20) + (n_2+20) \cdots + (n_{10} + 20)) = \frac{1}{10}(n_1 + n_2 + \cdots + n_{10}) + \frac{200}{10},$$

which is 20 more than the old average.

4. Answer B. The cost of two 50 mℓ tubes is R 5,98, which is R 0,89 more than the cost of one 100 mℓ tube. The percentage increase is $\dfrac{0,89 \times 100}{5,09} \approx \dfrac{90}{5} = 18$.

5. Answer E. By substituting $a = 4$ we get $f(4) = 4 - 2 = 2$. Then by substituting $p = 3$ and $q = f(4) = 2$ we get $g(3, f(4)) = 3^2 + 2 = 11$.

6. Answer A. By adding the units column $(4 + 5 + 7 = 16)$, we see that 1 is carried into the tens column. Now by adding the tens column we see that $1 + P + Q + R$ ends with 0 and must start with 2, since 2 is carried into the hundreds column. Thus $1 + P + Q + R = 20$, giving $P + Q + R = 19$. If $R = 1$, then $P + Q = 18$. But $P \leq 9$ and $Q \leq 9$, giving the only possibility $P = Q = 9$. However, this is not allowed, since P and Q are different. Thus R cannot be equal to 1.

7. Answer B. The interior angles of a regular n-sided polygon are equal to $180 - \frac{360}{n}$ degrees, so angle $B\widehat{C}D = 180 - \frac{360}{5} = 108°$. Triangle CBD is isosceles, so $C\widehat{B}F = C\widehat{D}F = \frac{1}{2}(180 - 108) = 36°$. Similarly, $D\widehat{C}F = 36°$, so $D\widehat{F}C = 180 - 2 \times 36 = 108°$.

8. Answer C. Calculations involving average speeds must be made using the formula Speed=Distance/Time. It does not matter that some distances and times are not given. Suppose the distance for each half of the race is D and the times for the two halves are T_1 and T_2. Then $D/T_1 = 30$ and $D/T_2 = S$, the unknown speed for the second half. By looking at the race as a whole, we see that $2D/(T_1 + T_2) = 40$, or $D/(T_1 + T_2) = 20$. Then, taking reciprocals, we have

$$\frac{1}{20} = \frac{T_1}{D} + \frac{T_2}{D} = \frac{1}{30} + \frac{1}{S}, \quad \text{so} \quad \frac{1}{S} = \frac{1}{20} - \frac{1}{30} = \frac{1}{60}.$$

Thus $S = 60$.

9. Answer A. Suppose the tens and units digits are T and U, so the number is equal to $10T + U$, and the sum of its digits is $T + U$. We must therefore solve $10T + U = 7(T + U)$, which gives $T = 2U$. The possible numbers are therefore 21, 42, 63, and 84, giving four possibilities.

10. Answer D. The last digit of any number is its remainder when divided by 10, so we must look at the last digits of the powers of 7. These are 7 (from 7 itself), then 9 (from 49), then 3, then 1, then 7 again. Thus the pattern repeats itself after every four powers. Since $2002 = 4 \times 50 + 2$, the last digit of 7^{2002} will be the same as the last digit of 7^2, which is 9.

11. Answer A. Only odd rows have a middle number, and the middle numbers in rows $1, 3, 5$ are $1, 9, 25$, that is, $1^2, 3^2, 5^2$. Thus the middle number in row 51 will be $51^2 = 2601$.

12. Answer C. The product $xy = x(-1-x) = -x^2 - x = -(x+\frac{1}{2})^2 + \frac{1}{4} \leq \frac{1}{4}$, with equality when $x = -\frac{1}{2}$.

13. Answer C. The number q must be odd (or else the left hand side will be even), and $3q = 25 - 2p \leq 23$, so the only possibilities for q are $1, 3, 5, 7$. The corresponding values of p are $11, 8, 5, 2$, respectively.

14. Answer E. It is given that $AD = 7\sqrt{2}$. Since the sides of a right-angled isosceles triangle are in the ratio $1 : 1 : \sqrt{2}$, we see that $EF = FC = 2$. Triangle EFC has base and height equal to 2, so its area is $\frac{1}{2}(2)(2) = 2$. Triangle ACD has base and height equal to $7\sqrt{2}$, so its area is $\frac{1}{2}(7\sqrt{2})(7\sqrt{2}) = 49$. The area of $AFED$ is therefore $49 - 2 = 47$.

15. Answer B. The first few values of $n!$ are $1, 2, 6, 24, 120, 720, \ldots$ (notice that from here on they all have last digit zero). The given sequence therefore starts with 1, then $3\,(= 1+2)$, then $9\,(= 3+6)$, then $33\,(= 9+24)$, then $153\,(= 33+120)$. (Notice that the n-th term is obtained by adding $n!$ to the previous term.) From here on, all terms have last digit 3, but a perfect square cannot have last digit 3, so there are no more perfect squares in the sequence. Thus the only perfect squares in the sequence are 1 and 9.

16. Answer A. The height of the triangle is $\sqrt{12^2 - 3^2} = \sqrt{135} = 3\sqrt{15}$. If the radius of the circle is r, then its centre is at a distance r from each vertex, so the centre is at a distance $3\sqrt{15} - r$ from the midpoint of the base. This gives a right-angled triangle with hypotenuse r and other two sides 3 and $3\sqrt{15} - r$, so $r^2 = 3^2 + (3\sqrt{15} - r)^2$. This simplifies to $6r\sqrt{15} = 144$, giving $r = \dfrac{24}{\sqrt{15}} = \dfrac{8\sqrt{15}}{5}$.

17. Answer B. If the digits are H, T, U, then $H \times T \times U = p$, a prime number. However, since p has no factors other than 1 and itself, two of the digits must be 1 and the third must be p. Thus p must be one of the single-digit primes $2, 3, 5, 7$ (four possibilities), and the digits $1, 1, p$ can be arranged in three different ways, so the total number of possibilities is 12.

18. Answer A. Since $a < b < c$, it follows that $\frac{1}{a} > \frac{1}{b} > \frac{1}{c}$. Then since the sum of these reciprocals is 1, their average value is $\frac{1}{3}$. Thus $\frac{1}{a} > \frac{1}{3}$, and obviously $\frac{1}{a} < 1$, so, by taking reciprocals again, we see that $1 < a < 3$. This gives $a = 2$, so $\frac{1}{b} + \frac{1}{c} = 1 - \frac{1}{2} = \frac{1}{2}$. A similar argument shows that $2 < b < 4$, so $b = 3$, whence $c = 6$.

19. Answer D. Factorize $n^5 - n = n(n^2-1)(n^2+1) = (n-1)(n)(n+1)(n^2+1)$. This is divisible by the product of three successive integers $(n-1)(n)(n+1)$. Any such product is divisible by 6, since one of the three integers must be divisible by 3 and at least one must be divisible by 2. Thus $n^5 - n$ is divisible by at least 6 for every natural number $n \geq 2$. (In fact, it can be shown that $n^5 - n$ is always divisible by 30. Try it, using an induction argument!)

20. **Answer A.** If circle M has radius r, then its centre is at a distance $1 - r$ above the centre of the large semicircle, and at a distance $\frac{1}{2} + r$ from the centre of either small semicircle. This gives a right-angled triangle with hypotenuse $\frac{1}{2} + r$, and other two sides $1 - r$ and $\frac{1}{2}$, so $(\frac{1}{2} + r)^2 = (1 - r)^2 + \frac{1}{4}$. This simplifies to $r = 1 - 2r$, so $r = \frac{1}{3}$.
