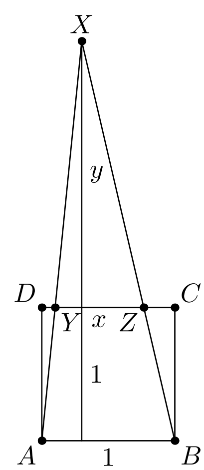
The South African Mathematical Olympiad
Third Round 2012
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Given that
$$\frac{1+3+5+\cdots+(2n-1)}{2+4+6+\cdots+(2n)}=\frac{2011}{2012},$$
determine n.

Solution: Using the sum formula for arithmetic progressions, we obtain
$$\frac{1+3+5+\cdots+(2n-1)}{2+4+6+\cdots+(2n)}=\frac{n\cdot\frac{1+2n-1}{2}}{n\cdot\frac{2+2n}{2}}=\frac{n}{n+1}=\frac{2011}{2012},$$
from which it follows that $n=2011$.

2. Let $ABCD$ be a square and X a point such that A and X are on opposite sides of CD. The lines AX and BX intersect CD in Y and Z respectively. If the area of $ABCD$ is 1 and the area of XYZ is $\frac{2}{3}$, determine the length of YZ.



Solution: Let the length of YZ be x, and let y be the associated height of triangle XYZ. Note that triangles XYZ and XAB are similar. Since corresponding sides YZ and AB have lengths x and 1 respectively, the heights y and $1+y$ have to satisfy
$$\frac{y}{1+y}=\frac{x}{1}=x.$$
Hence the area of XYZ is
$$\frac{2}{3}=\frac{xy}{2}=\frac{y^2}{2(1+y)}.$$

This yields the quadratic equation

$$y^2 - \frac{4}{3}y - \frac{4}{3} = \left(y + \frac{2}{3}\right)(y-2) = 0,$$

from which it follows that $y = 2$ and thus

$$x = \frac{y}{1+y} = \frac{2}{3}.$$

3. Sixty points, of which thirty are coloured red, twenty are coloured blue, and ten are coloured green, are marked on a circle. These points divide the circle into sixty arcs. Each of these arcs is assigned a number according to the colours of its endpoints: an arc between a red and a green point is assigned a number 1, an arc between a red and a blue point is assigned a number 2, and an arc between a blue and a green point is assigned a number 3. The arcs between two points of the same colour are assigned a number 0. What is the greatest possible sum of all the numbers assigned to the arcs?

   Solution: Let the *score* of a red point be 0, the score of a green point be 1, and the score of a blue point be 2. Note that the number assigned to an arc is at most the sum of the scores of the endpoints. This means that the sum of all the numbers assigned to the arcs is at most twice the sum of all the sixty scores, which is

$$2\,(30 \cdot 0 + 20 \cdot 2 + 10 \cdot 1) = 100.$$

   Equality holds if there are no arcs with two green or two blue endpoints. This can be achieved, for instance, by letting red and non-red points alternate. Hence the greatest possible sum is 100.

4. Let p and k be positive integers such that p is prime and $k > 1$. Prove that there is at most one pair (x, y) of positive integers such that

$$x^k + px = y^k.$$

   Solution 1: We distinguish two different cases:

Case 1: $\gcd(x, p) = 1$. In this case, x and $x^{k-1} + p$ do not have a common divisor (other than 1) either, and it follows from the factorisation

$$x(x^{k-1} + p) = y^k$$

that both x and $x^{k-1} + p$ have to be kth powers, say $x = u^k$ and $x^{k-1} + p = v^k$. Then it follows that

$$p = v^k - u^{k(k-1)} = (v - u^{k-1})\left(v^{k-1} + v^{k-2}u^{k-1} + \cdots + u^{(k-1)^2}\right).$$

Both factors have to be positive, and it is clear that

$$v - u^{k-1} < v \le v^{k-1} \le v^{k-1} + v^{k-2}u^{k-1} + \cdots + u^{(k-1)^2},$$

so since p is given to be a prime number, we must have $v-u^{k-1}=1$ and thus $v=u^{k-1}+1$. Then

$$p=\left(u^{k-1}+1\right)^{k-1}+\left(u^{k-1}+1\right)^{k-2}u^{k-1}+\cdots+u^{(k-1)^2}.$$

The right hand side is an increasing function of u, so there is at most one value of u which satisfies the equation. If there is such an integer u, then there is only one corresponding v and thus only one solution (x,y).

Case 2: $\gcd(x,p)=p$. Then x^k+px has to be divisible by p, which implies that y^k, and thus y, is divisible by p as well. It follows that x^k and y^k are divisible by p^k, hence this has to be the case for px as well, so $x=p^{k-1}u$ for some integer u. Since y is divisible by p, we can also set $y=pv$ to obtain

$$p^{k(k-1)}u^k+p^ku=p^kv^k$$

and thus

$$p^{k(k-2)}u^k+u=v^k.$$

However, this leads to a contradiction:

$$\left(p^{k-2}u\right)^k<p^{k(k-2)}u^k+u<p^{k(k-2)}u^k+kp^{(k-1)(k-2)}u^{k-1}<\left(p^{k-2}u+1\right)^k,$$

so that v would have to lie between the two consecutive integers $p^{k-2}u$ and $p^{k-2}u+1$, an obvious contradiction.

We conclude that there is always at most one solution (x,y).

Solution 2: Since $x^k+px=y^k$, we define α as the difference between x and y. Then

$$x^k+px=(x+\alpha)^k>x^k+\alpha x \quad\Rightarrow\quad \alpha<p.$$

In addition

$$\alpha|(x+\alpha)^k-x^k=px \quad\Rightarrow\quad \alpha|x.$$

On the other hand, by binomially expanding $(x+\alpha)^k$, every term (except α^k) in the equation is divisible by x, therefore $x|\alpha^k$. Let $\beta=\alpha^k/x$ – we will show that $\beta=1$.

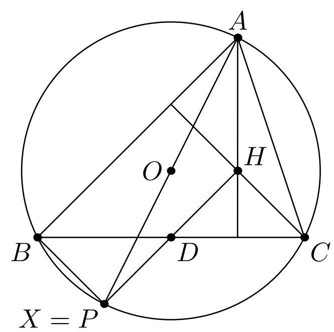
$$\begin{aligned}px&=(x+\alpha)^k-x^k\\ \beta^kpx&=(\beta x+\beta\alpha)^k-(\beta x)^k\\ \beta^{k-1}p\alpha^k&=(\alpha^k+\beta\alpha)^k-(\alpha^k)^k\\ \beta^{k-1}p&=(\alpha^{k-1}+\beta)^k-(\alpha^{k-1})^k\end{aligned}$$

However, since $\alpha|x$, $x|\alpha^k$ and $x\beta=\alpha^k$, we have $\beta|\alpha^{k-1}$. This means that every term on the right hand side is divisible by β^k, and the left is only $p\beta^{k-1}$. So either $p=\beta$, in which case we require

$$y^k=(x+\alpha)^k=x^k+px=x^k+\alpha^k,$$

which has no solutions for $x>0$, or $\beta=1$, in which case $x=\alpha^k$. The equation can then be written as $p=(\alpha^{k-1}+1)^k-(\alpha^{k-1})^k$, and as the right hand side is a strictly increasing function of α, for a given p and k there can be at most one solution.

5. Let ABC be a triangle such that $AB \neq AC$. We denote its orthocentre by H, its circumcentre by O and the midpoint of BC by D. The extensions of HD and AO meet in P. Prove that triangles AHP and ABC have the same centroid.



Solution 1: Let AO intersect the circumcircle at X. Since AX is a diameter, ABX is a right angle, hence $BX \| CH$ (both are perpendicular to AB). For the same reason, $CX \| BH$, so $BHCX$ is a parallelogram. This means that XH passes through D, the midpoint of BC. Hence X coincides with P (it lies on both AO and HD), and D is also the midpoint of $HX = HP$.

But this implies that AD is a median in both triangles ABC and AHP. Since the centroid always divides the median in a $2:1$ ratio, we conclude that the centroids of ABC and AHP coincide as well.

Solution 2: It is well known that the reflection H' of H about the side BC lies on the circumcircle. If we reflect H' again about the perpendicular bisector of BC, we obtain another point H'' on the circumcircle. The double reflection amounts to a single reflection with centre D, since D is the intersection of BC and its perpendicular bisector. Hence H'' lies on HD. Moreover, since $AH'H''$ is a right angle by construction, AH'' is a diameter, so H'' lies on AO and must thus coincide with P. One could now conclude as in the previous solution, but let us show a different approach instead: the centroid of ABC is known to lie on the Euler line, which also passes through H and O. On the other hand, OH is also a median in AHP by our observations. AD is a common median in both triangles, hence the intersection of OH and AD is the centroid of both triangles, which completes the proof.

6. Find all functions $f : \mathbb{N} \to \mathbb{R}$ ($\mathbb{N}$ denotes the set of all positive integers, $\mathbb{R}$ the set of all real numbers) such that

$$f(km) + f(kn) - f(k)f(nm) \geq 1$$

for all $k, m, n \in \mathbb{N}$.

Solution 1: Plugging in $k = n = m = 1$ yields

$$f(1)^2 - 2f(1) + 1 = (f(1) - 1)^2 \leq 0,$$

which implies $f(1) = 1$. Plugging in $k = 1$, $n = m$ and $k = n$, $m = 1$, respectively, we obtain the two inequalities

$$2f(n) - f(n^2) \geq 1, \tag{1}$$

$$f(n^2) + f(n) - f(n)^2 \geq 1. \tag{2}$$

We add the two to get

$$-f(n)^2 + 3f(n) \geq 2$$

or

$$(f(n)-1)(f(n)-2) \leq 0.$$

This means that $1 \leq f(n) \leq 2$ for all n. Now assume that there is a positive integer n such that $f(n) = a > 1$. The function $g(x) = \frac{x^2+1}{x} = x + \frac{1}{x}$ is increasing on $(1, \infty)$: for $x > y > 1$, we have

$$g(x) - g(y) = x - y + \frac{1}{x} - \frac{1}{y} = (x-y)\left(1 - \frac{1}{xy}\right) > 0.$$

Hence, for any $x \geq a$, we have

$$x^2 + 1 \geq \frac{a^2+1}{a} x$$

and thus

$$x^2 - x + 1 \geq \frac{a^2 - a + 1}{a} \cdot x.$$

Returning to (2), we now find, by our assumption that $f(n) = a$,

$$f(n^2) \geq f(n)^2 - f(n) + 1 \geq \frac{a^2 - a + 1}{a} \cdot f(n),$$

and since

$$b = \frac{a^2 - a + 1}{a} = 1 + \frac{(a-1)^2}{a} > 1,$$

we get

$$f(n^2) \geq bf(n) \geq a.$$

Iterating this inequality yields $f(n^4) \geq b^2 f(n) = b^2 a$, $f(n^8) \geq b^3 f(n) = b^3 a$, etc., and generally (by induction) $f(n^{2^k}) \geq b^k a$. Since $a > 1$ and $b > 1$, this implies $f(n^{2^k}) > 2$ for sufficiently large k, which contradicts the inequality $1 \leq f(n^{2^k}) \leq 2$ that was obtained earlier.

It follows that there is no n such that $f(n) > 1$, which means that the constant function $f(n) \equiv 1$ is the only solution (and it is easy to see that this function satisfies the condition, since the left hand side of the inequality is always equal to 1 in this case).

Solution 2: It is given that

$$f(km) + f(kn) - f(k)f(mn) \geqslant 1 \tag{3}$$

for all $k, m, n \in \mathbb{N}$.

Put $m = n = k = 1$ into (3) to get $(f(1)-1)^2 \leqslant 0$, giving

$$f(1) = 1. \tag{4}$$

Put $m = n = 1$ into (3) to get $2f(k) - f(k) \geqslant 1$, giving

$$f(k) \geqslant 1 \tag{5}$$

for all $k \in \mathbb{N}$. Put $k = m = n$ into (3) to get $2f(k^2) - f(k)f(k^2) \geqslant 1$, giving

$$f(k^2)(2 - f(k)) \geqslant 1, \tag{6}$$

again for all $k \in \mathbb{N}$. From (5) and (6) it follows that

$$1 \leqslant f(k) < 2 \tag{7}$$

for all $k \in \mathbb{N}$. We now prove the statement

$$S(M) : 1 \leqslant f(k) < 1 + \frac{1}{M} \text{ for all } k \in \mathbb{N}$$

to be true for all $M \geqslant 1$, using induction on M. The case $M = 1$ is just (7).

Assume $S(M)$ to be true. Suppose there is a $k_0 \in \mathbb{N}$ such that $f(k_0) \geqslant 1 + \frac{1}{M+1}$. Then, using (6), we obtain

$$1 \leqslant f(k_0^2)(2 - f(k_0)) \leqslant f(k_0^2)\left(2 - 1 - \frac{1}{M+1}\right) = f(k_0^2)\left(\frac{M}{M+1}\right),$$

giving $f(k_0^2) \geqslant 1 + \frac{1}{M}$, which violates $S(M)$. So $S(M+1)$ follows.

It is now clear that the constant function $f(n) \equiv 1$ is the only solution.