The South African Mathematical Olympiad
Third Round 2014
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Determine the last two digits of the product of the squares of all positive odd integers less than 2014.

2. Given that
$$\frac{a-b}{c-d} = 2 \qquad \text{and} \qquad \frac{a-c}{b-d} = 3$$
for certain real numbers a, b, c, d, determine the value of
$$\frac{a-d}{b-c}.$$

3. In obtuse triangle ABC, with the obtuse angle at A, let D, E, F be the feet of the altitudes through A, B, C respectively. DE is parallel to CF, and DF is parallel to the angle bisector of $\angle BAC$. Find the angles of the triangle.

4. (a) Let a, x, y be positive integers. Prove: if $x \neq y$, then also
$$ax + \gcd(a, x) + \operatorname{lcm}(a, x) \neq ay + \gcd(a, y) + \operatorname{lcm}(a, y).$$
   (b) Show that there are no two positive integers a and b such that
$$ab + \gcd(a, b) + \operatorname{lcm}(a, b) = 2014.$$

5. Let $n > 1$ be an integer. An $n \times n$-square is divided into n^2 unit squares. Of these unit squares, n are coloured green and n are coloured blue, and all remaining ones are coloured white. Are there more such colourings for which there is exactly one green square in each row and exactly one blue square in each column; or colourings for which there is exactly one green square and exactly one blue square in each row?

6. Let O be the centre of a two-dimensional coordinate system, and let $A_1, A_2, \ldots, A_n$ be points in the first quadrant and $B_1, B_2, \ldots, B_m$ points in the second quadrant. We associate numbers $a_1, a_2, \ldots, a_n$ to the points $A_1, A_2, \ldots, A_n$ and numbers $b_1, b_2, \ldots, b_m$ to the points $B_1, B_2, \ldots, B_m$, respectively. It turns out that the area of triangle OA_jB_k is always equal to the product a_jb_k, for any j and k. Show that either all the A_j or all the B_k lie on a single line through O.