2018 Junior Third Round - Solutions

1. The average of the 5 numbers is $\frac{19+31+29+17+59}{5} = \frac{155}{5} = 31$ and since 31 is one of these numbers, it means that 31 is the average of the other four numbers 19, 29, 17 and 59.

2. $20^{18} = (2^2 \times 5)^{18} = 2^{36} \times 5^{18}$ and hence the largest power of 2 which divides into 20^{18} is 36.

3. The table below shows the solution. You can deduce this by finding the numbers in the right most column and the numbers in the row containing the given 3 first. The rest of the grid is then filled in easily thereafter.

3	1	2	4
4	2	1	3
1	4	3	2
2	3	4	1

4. Notice that every 4th box has to be filled in with the same number, since if 5 boxes are in a row, then the first and fifth have to be the same for the product of 4 consecutive boxes to be 120. The two boxes between the 4 and the x have to be filled with 2 and 3 respectively. Hence we have $4 \times 2 \times 3 \times x = 120$. Hence the value of x is 5.

5. Notice that $BDEF$ and $AEFG$ are parallelograms. Hence

$$AD = AG + GD = EF + GD = BD + GD = BG,$$

since opposite sides of parallelograms are equal in length.

6. Molly should look in treasure chest D, the unlabelled chest. If it contains the gold nugget, she is done. If it does not contain the gold nugget, but contains sand, she knows the gold nugget is in chest A, which was labelled "Gold or Sand". If it contains gravel, she knows the gold nugget is in chest B, which was labelled "Gold or Gravel". If it contains pebbles, she knows chest C, labelled as "Pebbles or Gravel" contains gravel, so chest B, labelled as "Gold or Gravel", contains the gold nugget. Hence, Molly is indeed able to find the gold nugget with certainty.

7. The total number of grains is given by doubling the number and subtracting one. So the answer is 18 446 744 073 709 551 615 or just $2^{64} - 1$.

8. Since C is the centre of a circle, $CA = CB$ and hence angles x and y are equal. Hence $\angle FCE = 2y$.
Since B is the centre of a circle $BD = BE$ and thus $\angle FEC = \angle BED = \angle BDE = \frac{180^\circ - y}{2}$.
So in triangle FCE we have angles $12^\circ + 2y + \frac{180^\circ - y}{2} = 180^\circ$.
Solving for y we get $y = 52^\circ$ and then $x = 52^\circ$ and finally $z = 76^\circ$.

9. (a) The volume of the rain gauge is given by $\frac{100\times100\times300}{3} = 1\ 000\ 000$.
Hence the height in the square-based right prism will be $\frac{1000000}{100\times100} = 100$mm.
   (b) If the height in the rain gauge is 150 mm, then the base surface area will be 50mm by 50mm, since the height is half and hence the side lengths of the surface area will also be half. Therefore the volume of the rain gauge is given by $\frac{50\times50\times150}{3} = 125000$. Hence the height in the square-based right prism will be 12.5mm.

10. (a) If you roll the die 2 upwards, the red will be at the bottom. Then one to the right and the red will be on the left side of the die. You roll it down twice and then onto the original square and the red face will now, after 6 moves, be at the bottom of the original square. You can now just make pairs of moves: roll it off and back onto the original square. Clearly after all even moves greater or equal to 6, the red face is at the bottom. So it is possible for 2018.
   (b) If you colour the grid black and white like a chessboard, notice that after every two moves, you are back on the same colour. Hence, to get the die back on the original square will take an even number of moves, so 2019 is not possible.

11. In every two consecutive rows with n and $n+1$ students, one row contains and even number of seats and the other an odd number of seats. These two rows together can seat a maximum of $n+1$ students. Since the first row contains p seats and there are 20 rows, we have

$$\begin{aligned}
\text{seats} &= (p+1) + (p+3) + (p+5) + \cdots + (p+19) \\
&= 10p + 1 + 3 + 5 + \cdots + 19 \\
&= 10p + (1+19) + (3+17) + (5+15) + (7+13) + (9+11) \\
&= 10p + 100.
\end{aligned}$$

12. Clearly $S = 1$, $H = 0$ and $D = 9$ to make the 4-digit and 3-digit numbers sum to a 5-digit number. Since $S = 1$, we have that $T = O + 1$ and therefore O is less than T (since $T \neq 0$). Looking in the middle column $A + T = O$, which is only possible if $A = 8$ and there is a carry of one from the $Y + O$ column, since T and O differ by 1 and $T > O$. The only pairs possible for (O, T) are (2,3), (3,4), (4,5), (5,6), (6,7). However, the maximum value of Y is 7 and therefore to be able to create a carry in $Y + O$ column, O has to be bigger than 4, since $R \neq 0$ and $R \neq 1$. However, the pair (6,7) implies that $Y \neq 7$ and $Y \neq 6$, but $Y \leq 5$ leads to either $R = 1$, or $R = 0$ or no carry, which is a contradiction. Therefore, the only valid pair for (O, T) is (5,6). Using this we see that $Y = 7$ and $R = 2$. In conclusion we have $9871 + 655 = 10526$.

13. (a) 77 goes to 49, which goes to 36, which goes to 18, which goes to 8.

    (b) Note that the two digit number $10x + y \geq 10x > (y)x = xy$, which means that the digit product of a two digit number is strictly smaller than the number. Similarly for the three digit number $100x + 10y + z \geq 100x = (10)(10)x > (y)(z)x = xyz$ and therefore, the digit product of a three digit number is strictly less than the number. Similarly, the digit product of any number will be strictly less than the number. Since the numbers in the digit product chain become strictly less at each step, it must eventually become and stop at a single digit.

14. For the game on a board of any finite size, note that the game must end and that only one player can close a 1×1 square leading to a loss. Thus there must be a winner and hence a winning strategy. However, the winner might depend on the size of the board. The winning strategies and winners for the specific boards are:

    (a) On the 2×7 grid, Alice draws the vertical line in the centre on the first move. It cuts the board into two symmetric parts. Wherever Bongile plays, Alice can just mirror the move onto the other side. The only way Alice can complete a square, is if Bongile already completed a square. Hence Alice can never lose and since there must be a winner and a loser, it means Bongile loses and Alice wins.

    (b) On the 5×5 grid, the second player on every moves plays the $180°$ rotation around the centre. Similarly to the 2×7 board, it means player 2 can never lose and wins. Thus Bongile wins.

    (c) On the 2018×2019 grid, Alice draws the line in the very centre of the grid. Alice on every other move plays the $180°$ rotation around the centre. Similarly to the 2×7 board, it means Alice can never lose and wins.

15. We have the expression $(n + 1) \times 2^n$ for $n = 1, 2, \ldots$

    $n = 1 : (n + 1) \times 2^n = (1 + 1) \times 2^1 = 2 \times 2 = 4 = 2^2.$
    $n = 2 : (n + 1) \times 2^n = (2 + 1) \times 2^2 = 3 \times 4 = 12.$
    $n = 3 : (n + 1) \times 2^n = (3 + 1) \times 2^3 = 4 \times 8 = 32.$
    $n = 4 : (n + 1) \times 2^n = (4 + 1) \times 2^4 = 5 \times 16 = 80.$
    $n = 5 : (n + 1) \times 2^n = (5 + 1) \times 2^5 = 6 \times 32 = 192.$
    $n = 6 : (n + 1) \times 2^n = (6 + 1) \times 2^6 = 7 \times 64 = 448.$
    $n = 7 : (n + 1) \times 2^n = (7 + 1) \times 2^7 = 8 \times 128 = 1024 = 32^2.$
    $n = 8 : (n + 1) \times 2^n = (8 + 1) \times 2^8 = 9 \times 256 = 2304 = 48^2.$
    $n = 9 : (n + 1) \times 2^n = (9 + 1) \times 2^9 = 10 \times 515 = 5150.$

    Notice that we do have two consecutive squares for $n = 7$ and $n = 8$. We will now prove that we cannot have more than two consecutive squares by proving that two consecutive odd values for n cannot both be squares and two consecutive even values for n cannot both be squares.

Let $n = 2k - 1$, then we have $(2k - 1 + 1) \times 2^{2k-1} = 2k \times 2^{2k-1} = k \times 2^{2k}$.
Let $n = 2k + 1$, then we have $(2k + 1 + 1) \times 2^{2k+1} = (2k + 2) \times 2^{2k+1} = (k + 1) \times 2^{2k+2}$.

Since 2^{2k} and 2^{2k+2} are squares, we must have that both k and $k + 1$ are squares, if both expressions are to be squares. But k and $k+1$ are consecutive, and the only consecutive squares are 0 and 1. Hence we cannot have two consecutive odd values for n giving squares.

Let $n = 2k$, then we have $(2k + 1) \times 2^{2k} = (2k + 1) \times 2^{2k}$.
Let $n = 2k + 2$, then we have $(2k + 2 + 1) \times 2^{2k+2} = (2k + 3) \times 2^{2k+2}$.

Since 2^{2k} and 2^{2k+2} are squares, we must have that both $(2k + 1)$ and $(2k + 3)$ are squares, if both expressions are to be squares. But $(2k + 1)$ and $(2k + 3)$ are consecutive odd numbers, but there are no consecutive odd numbers which are both squares. Hence we cannot have two consecutive even values for n giving squares.

Thus, the largest number of consecutive values that are perfect squares is 2.