THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior First Round 2017
Solutions

1. **Answer A**
   $1009^2 - 1008^2 = (1009 - 1008)(1009 + 1008) = 1 \times 2017 = 2017$.

2. **Answer D**
   Exactly half the biggest circle is shaded (as you can see if you rotate the center ring through 90°), so the shaded area is $\frac{1}{2}\pi 6^2 = 18\pi$.

3. **Answer B**
   $3^n \cdot 3^3 = 3^{n+3}$, which will be a perfect square if $n + 3$ is even (and non-negative), that is, if n is an odd number and $n \geq -3$. The smallest value of n with these properties in the given set is -1.

4. **Answer C**
   The bar graph shows that 5 students travel for between 11 and 15 minutes, and 2 students travel for between 16 and 20 minutes, and $5 + 2 = 7$.

5. **Answer A**
   The gradient of the line is $\frac{7-3}{4-2} = \frac{4}{2} = 2$, so the equation of the line is $(y-3) = 2(x-2)$ [or $(y - 7) = 2(x - 4)$], both of which simplify to $y = 2x - 1$. The only given point satisfying this equation is $(3, 5)$.

6. **Answer C**
   The three sides must be different numbers from the list $\{2, 4, 6, \ldots\}$ and the longest side must be less than the sum of the other two by the Triangle Inequality. That means that the shortest side must be greater than 2. The three smallest possible sides are $4, 6, 8$, giving perimeter 18.

7. **Answer B**
   If the numbers are a, b, c, d, e, f, then $N = a+b+c$ and $M = d+e+f$, But $d = a+3$, and similarly $e = b+3$ and $f = c+3$. Thus $M = (a+3)+(b+3)+(c+3) = N+9$.

8. **Answer C**
   Since $a + (2 - a) = 2$, it follows that the point must lie on the line $x + y = 2$, which cuts the axes at $(0, 2)$ and $(2, 0)$. The line passes through the second, first and fourth quadrants only.
   Alternatively, if $a < 0$, then $2 - a > 2 > 0$, which means that if the point is to the left of the y-axis, then it is also above the x-axis. Thus the point cannot lie in the third quadrant.

9. **Answer B**
   From the theorem that the exterior angle of a triangle is equal to the sum of the two opposite interior angles, it follows that $n = a + b$ and also $n = c + d$. Thus $a + b + c + d = 2n$.

10. **Answer D**
The box contains $8+7+6=21$ balls, of which the only ones that are neither red nor green are the 7 blue balls. Thus the required probability is $\frac{7}{21}=\frac{1}{3}$.

11. **Answer E**
The base of the triangle is 2 (the side of the small square) and the height of the triangle is $6-2=4$ (the difference between the sides of the two squares). Thus the area of the triangle is $\frac{1}{2}\times 2\times 4=4$.

12. **Answer E**
The information given can be put into a table:

	Red	Yellow	Total
Black	14		
Grey			18
Total		31	50

Then by subtraction it is easy to complete the table:

	Red	Yellow	Total
Black	14	18	32
Grey	5	13	18
Total	19	31	50

The number of girls wearing yellow shirts and gray pants is 13.

13. **Answer A**
Suppose the mass of the container (also called the tare mass) is x kg and the mass of the contents (also called the nett mass) is y kg. Then $x+\frac{1}{4}y=36$ and $x+\frac{1}{3}y=40$. Subtracting the equations gives $\frac{1}{12}y=4$, so $y=48$ and $x=36-\frac{1}{4}\times 48=24$.

14. **Answer E**
Join AD and draw EM perpendicular to AD. Then AME and DME are congruent 30°, 60°, 90° triangles, which together form an equilateral triangle ADE. Thus $AE=ED=AD=12$, so the perimeter of the figure is $5\times 12=60$.

15. **Answer C**
The procedure is to eliminate one variable (say a) from the last two equations, and then eliminate b from the last equation. We will then be left with the value of c. The given equations are

$$a+b+c=12, \qquad 3a+2b+c=6, \qquad 5a+3b+2c=9.$$

After subtracting 3 times the first equation from the second, and 5 times the first equation from the third, the last two equations become

$$-b-2c=-30 \text{ and } -2b-3c=-51, \text{ or } b+2c=30 \text{ and } 2b+3c=51.$$

Now subtract twice the first equation from the second to obtain $-c=-9$, so $c=9$.

16. **Answer B**
Let G be the foot of the perpendicular from P to YT. Then $GTAP$ is a rectangle, so $GT = 12$ and $YG = YT - GT = 18 - 12 = 6$. By Pythagoras' theorem, $PG = \sqrt{PY^2 - YG^2} = \sqrt{10^2 - 6^2} = 8$. Thus $AT = 8$ also, and again by Pythagoras' theorem, $TH = \sqrt{AH^2 - AT^2} = \sqrt{17^2 - 8^2} = 15$. The total area of the trapezium $PYHA$ is average width times height $= \frac{1}{2}(PA + YH) \times AT = \frac{1}{2}(12 + 33) \times 8 = 180$. Alternatively, the area can be found by adding the areas of triangle PYG, rectangle $PGTA$, and triangle ATH.

17. **Answer C**
$\sqrt{23}$ lies between $\sqrt{16}$ and $\sqrt{25}$, that is, between 4 and 5, so its ceiling is 5. Next, -3.2 lies between -4 and -3, so its floor is -4. Thus $\lceil\sqrt{23}\,\rceil + \lfloor -3.2 \rfloor = 5 - 4 = 1$.

18. **Answer D**
Suppose the digits of n are a and b. Then $n = 10a + b$ and $f(n) = a + b$. The inequality becomes $10a + b > 8(a + b)$, which simplifies to $2a > 7b$. Since a and b are digits, they both lie between 0 and 9, so $7b < 2a \leq 18$. This shows that $b = 0$ or 1 or 2. With $b = 0$ we have $2a > 0$, so $1 \leq a \leq 9$ (nine values). With $b = 1$ we have $2a > 7$, so $4 \leq a \leq 9$ (six values), and with $b = 2$ we have $2a > 14$ or $a > 7$, so $a = 8$ or 9 (two values). The required total is then $9 + 6 + 2 = 17$.

19. **Answer B**
30 litres at 6% contains $30 \times 6 \div 100 = 1.8$ litres of concentrate. Next, 20 litres at 5% contains $20 \times 5 \div 100 = 1.0$ litres of concentrate. After they are mixed, there are 50 litres containing 2.8 litres of concentrate, so the final concentration is $2.8 \div 50 \times 100 = 5.6\%$.

20. **Answer D**
In this sort of problem, often a good method is to work out the first few cases and try to see a pattern. The first term is $\frac{1}{3}$; the sum of the first two terms is $\frac{1}{3} + \frac{1}{15} = \frac{5+1}{15} = \frac{2}{5}$; and the sum of the first three terms is $\frac{2}{5} + \frac{1}{35} = \frac{14+1}{35} = \frac{3}{7}$. The pattern looks clear: $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \dots, \frac{n}{2n+1}, \dots$. The individual terms also form a regular pattern, and the n-th term is $\frac{1}{(2n-1)(2n+1)}$. It follows that the required sum contains 50 terms, so its sum will be $\frac{50}{101}$. Of course, this is not a proof, though it is good enough for present purposes, but a proof by induction is not difficult if you feel the need.
Alternatively, this result can be proved by the method of telescoping series. The n-th term $\frac{1}{(2n-1)(2n+1)}$ can be written as $\frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$, so the sum of the first n terms is
$$\frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{2n-1} - \frac{1}{2n+1}\right).$$
If you look at this carefully, you can see that, except for the first and last terms, each term cancels out the term either before or after it. Thus the series collapses like a telescope down to just two terms
$$\frac{1}{2}\left(\frac{1}{1} - \frac{1}{2n+1}\right) = \frac{1}{2}\,\frac{2n}{2n+1} = \frac{n}{2n+1}.$$