The South African Mathematical Olympiad
Third Round 2016
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

Solutions

1. At the start of the Mighty Mathematicians Football Team's first game of the season, their coach noticed that the jersey numbers of the 22 players on the field were all the numbers from 1 to 22. At halftime, the coach substituted her goal-keeper, with jersey number 1, for a reserve player. No other substitutions were made by either team at or before halftime. The coach noticed that after the substitution, no two players on the field had the same jersey number and that the sums of the jersey numbers of each of the teams were exactly equal. Determine

   (a) the greatest possible jersey number of the reserve player,

   (b) the smallest possible (positive) jersey number of the reserve player.

   Solution: If we leave out the reserve player, the greatest possible difference between the jersey numbers of the two teams is obtained when the reserve player's team has jersey numbers 2–11, while the other team has numbers 12–22. The difference in this case is

$$(12 + 13 + \cdots + 22) - (2 + 3 + \cdots + 11) = 122,$$

   which is therefore the greatest possible jersey number of the reserve player. On the other hand, since

$$2 + 3 + 4 + \cdots + 21 + 22 = 252$$

   and the total sum of all jersey numbers must be even for the two teams to have the same sum, the reserve player must have an even jersey number. The smallest positive even number that is not already taken by another player is 24, and indeed it is possible that the sums of the two teams are the same in this case, for example:

$$2, 5, 6, 9, 10, 13, 14, 17, 18, 20, 24 \qquad \text{vs.} \qquad 3, 4, 7, 8, 11, 12, 15, 16, 19, 21, 22.$$

2. Determine all pairs of real numbers a and b, $b > 0$, such that the solutions to the two equations

$$x^2 + ax + a = b \qquad \text{and} \qquad x^2 + ax + a = -b$$

   are four consecutive integers.

   Solution 1: The quadratic formula gives us

$$\frac{-a \pm \sqrt{a^2 - 4a + 4b}}{2} \qquad \text{and} \qquad \frac{-a \pm \sqrt{a^2 - 4a - 4b}}{2}$$

respectively. It follows that the average of the four numbers is $-a/2$ (since the average is $-a/2$ for each of the two pairs). Since we know that the solutions are four consecutive numbers, it follows that they are $-a/2-3/2, -a/2-1/2, -a/2+1/2, -a/2+3/2$. Thus we find that

$$\sqrt{a^2-4a+4b}=3 \qquad \text{and} \qquad \sqrt{a^2-4a-4b}=1.$$

If we subtract the two resulting equations

$$\begin{aligned} a^2-4a+4b&=9,\\ a^2-4a-4b&=1, \end{aligned}$$

we obtain $8b=8$ and thus $b=1$. It follows that $a^2-4a=5$, which gives the two possibilities $a=-1$ (giving the four consecutive numbers $-1,0,1,2$ as solutions) and $a=5$ (giving the four consecutive numbers $-4,-3,-2,-1$ as solutions).

In conclusion, we either have $a=-1$ and $b=1$ or $a=5$ and $b=1$.

Solution 2: Suppose that the four consecutive numbers are $n-1, n, n+1, n+2$. The parabola $y=x^2+ax+a$ reaches its minimum at $x=-a/2$, and the line $x=-a/2$ is its axis of symmetry. The two pairs of solutions both have to have this axis of symmetry, and the solutions to the second equation have to lie closer to the minimum at $x=-a/2$. Thus the only possibility is that $n-1$ and $n+2$ are the solutions to the first equation, while n and $n+1$ are the solutions to the second equation. This gives us the equations

$$(n-1)^2+a(n-1)+a=b, \tag{1}$$
$$n^2+an+a=-b, \tag{2}$$
$$(n+1)^2+a(n+1)+a=-b. \tag{3}$$

Subtract (1) from (3) to obtain

$$4n+2a=-2b. \tag{4}$$

Subtract (2) from (3) to obtain

$$2n+1+a=0. \tag{5}$$

Multiply (5) by 2 and subtract from (4):

$$-2=-2b,$$

so $b=1$. Now we know that the two roots of $x^2+ax+a=1$ differ by 3:

$$\frac{-a+\sqrt{a^2-4a+4}}{2}=\frac{-a-\sqrt{a^2-4a+4}}{2}+3,$$

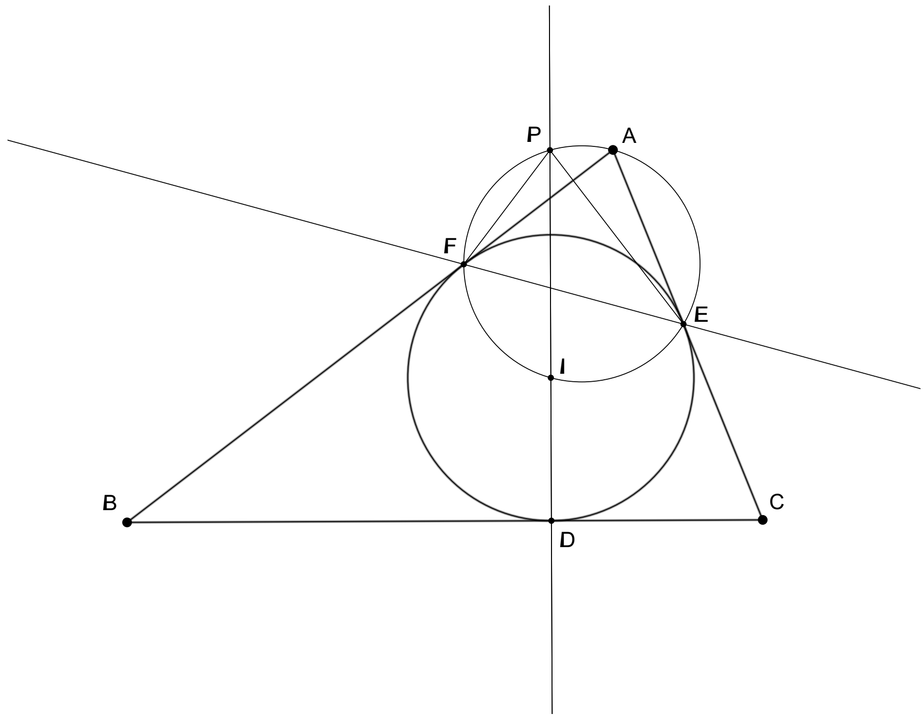
so

$$\sqrt{a^2-4a+4}=\sqrt{(a-2)^2}=3,$$

which means that $a-2=\pm 3$. Once again, we find that the two possibilities are $a=-1$, $b=1$ and $a=5$, $b=1$.

3. The inscribed circle of triangle ABC, with centre I, touches sides BC, CA and AB at D, E and F, respectively. Let P be a point, on the same side of FE as A, for which $\angle PFE=\angle BCA$ and $\angle PEF=\angle ABC$. Prove that P, I and D lie on a straight line.

Solution:



Since $\angle AEI = \angle AFI = 90°$, the points A, E, F, I lie on a circle with diameter AI. Moreover, since $\angle PFE = \angle BCA$ and $\angle PEF = \angle ABC$ by our assumptions on P, triangles ABC and PEF are similar, so $\angle EPF = \angle BAC = \angle EAF$. P was assumed to lie on the same side of EF as A, thus it follows that P also lies on the same circle as A, E, F and I.

We also know that $BDIF$ is a cyclic quadrilateral (using the same reasoning as before, namely that $\angle BDI = \angle BFI = 90°$), so $\angle FID + \angle FBD = 180°$.

Now we can conclude that $\angle PIF = \angle PEF = \angle ABC = \angle FBD = 180° - \angle FID$, so $\angle PIF + \angle FID = 180°$, which means that P, I, D lie on a straight line.

4. For which integers $n \geq 2$ is it possible to draw n distinct straight lines in the plane in such a way that there are at least $n-2$ points where exactly three of the lines intersect?

   Solution: For $n = 2$, any two lines satisfy the condition, and for $n = 3$, we can take any three lines passing through a common point.

   For $n = 4$, there is no feasible choice of four lines: suppose there are two points where exactly three lines meet. At most one of the lines can pass through both, so we need at least $1 + 2 \times 2 = 5$ lines.

   The same argument shows that it is impossible for $n = 5$: suppose there are three points where exactly three lines meet. If one line passes through all of them, we still require two further lines through each of the three, and no two of them can coincide. This already gives us $1 + 3 \times 2 = 7$ lines. If the three points do not lie on a line, then there can be at most one line passing through any two of them, leaving us with at least one more line through each of the points that does not pass through any of the others. This gives us a total of at least $3+3=6$ lines.

There is a possible configuration for every $n \geq 6$: for $n=6$, we can take the (extended) sides and diagonals of any (non-degenerate) quadrilateral. For larger values of n, we use an inductive construction: if we start with the sides and diagonals of a quadrilateral for which opposite sides are not parallel, then there are also two intersections of exactly two lines (namely the opposite sides). In each further step, we add a line through one of the intersections of exactly two lines, chosen in such a way that it does not pass through any of the other intersections that were obtained previously. This ensures that we get new intersections of exactly two lines with each step, and the number of points where exactly three lines meet increases by one. Thus the number of points where three lines meet will be (exactly) $n-2$ when the n-th line is drawn.

We conclude that it is possible to draw n lines in a suitable way for all $n \neq 4, 5$.

5. For every positive integer n, determine the greatest possible value of the quotient

$$\frac{1-x^n-(1-x)^n}{x(1-x)^n+(1-x)x^n}$$

where $0<x<1$.

Solution 1: It is convenient to write $y=1-x$, so that $x+y=1$. We show that the maximum of the resulting expression

$$\frac{1-x^n-y^n}{xy^n+yx^n}$$

is attained when $x=y=\frac{1}{2}$. The value in this case is 2^n-2. We first rewrite the expression as follows:

$$\begin{aligned}\frac{1-x^n-y^n}{xy^n+yx^n} &= \frac{x+y-x^n-y^n}{xy^n+yx^n} = \frac{x(1-x^{n-1})+y(1-y^{n-1})}{xy(x^{n-1}+y^{n-1})} \\ &= \frac{x(1-x)(1+x+\cdots+x^{n-2})+y(1-y)(1+y+\cdots+y^{n-2})}{xy(x^{n-1}+y^{n-1})} \\ &= \frac{xy(1+x+\cdots+x^{n-2}+1+y+\cdots+y^{n-2})}{xy(x^{n-1}+y^{n-1})} \\ &= \frac{1+1}{x^{n-1}+y^{n-1}}+\frac{x+y}{x^{n-1}+y^{n-1}}+\cdots+\frac{x^{n-2}+y^{n-2}}{x^{n-1}+y^{n-1}}.\end{aligned}$$

We will therefore be done if we can show that

$$\frac{x^a+y^a}{x^b+y^b}$$

attains its maximum for $x=y=\frac{1}{2}$ whenever $0 \leq a < b$. This can be achieved in many ways, for example as follows: the general mean inequality yields

$$\Big(\frac{x^a+y^a}{2}\Big) \leq \Big(\frac{x^b+y^b}{2}\Big)^{a/b}$$

and

$$\frac{1}{2}=\frac{x+y}{2} \leq \Big(\frac{x^b+y^b}{2}\Big)^{1/b},$$

with equality in both cases when $x = y = \frac{1}{2}$. Thus

$$\left(\frac{1}{2}\right)^{b-a} \cdot \left(\frac{x^a + y^a}{2}\right) \le \left(\frac{x^b + y^b}{2}\right)^{(b-a)/b} \left(\frac{x^b + y^b}{2}\right)^{a/b} = \frac{x^b + y^b}{2},$$

and it follows that

$$\frac{x^a + y^a}{x^b + y^b} \le 2^{b-a},$$

with equality when $x = y = \frac{1}{2}$.

Solution 2: Using the same notation as in the previous solution, the binomial theorem gives us

$$\frac{1 - x^n - y^n}{xy^n + yx^n} = \frac{(x+y)^n - x^n - y^n}{xy^n + yx^n} = \frac{\sum_{k=1}^{n-1} \binom{n}{k} x^k y^{n-k}}{xy(x^{n-1} + y^{n-1})} = \frac{\sum_{k=1}^{n-1} \binom{n}{k} x^{k-1} y^{n-k-1}}{x^{n-1} + y^{n-1}}.$$

We group terms pairwise (k and $n-k$ forming a pair):

$$\frac{1 - x^n - y^n}{xy^n + yx^n} = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k} \frac{x^{k-1}y^{n-k-1} + x^{n-k-1}y^{k-1}}{x^{n-1} + y^{n-1}} + \begin{cases} \binom{n}{n/2} \frac{x^{n/2-1}y^{n/2-1}}{x^{n-1}+y^{n-1}} & n \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$

Now we only need to show that

$$\frac{x^{k-1}y^{n-k-1} + x^{n-k-1}y^{k-1}}{x^{n-1} + y^{n-1}} = \frac{(xy)^{k-1}(x^{n-2k} + y^{n-2k})}{x^{n-1} + y^{n-1}}$$

attains its maximum when $x = y = \frac{1}{2}$. Note that the potential extra term for even n is also if this form, up to a factor 2:

$$\binom{n}{n/2} \frac{x^{n/2-1}y^{n/2-1}}{x^{n-1} + y^{n-1}} = \frac{1}{2} \cdot \frac{(xy)^{n/2-1}(x^0 + y^0)}{x^{n-1} + y^{n-1}}.$$

Since $xy \le \left(\frac{x+y}{2}\right)^2 = \frac{1}{4}$ by the inequality between the arithmetic and geometric mean (with equality for $x = y = \frac{1}{2}$), this can again be achieved by showing that

$$\frac{x^a + y^a}{x^b + y^b}$$

attains its maximum for $x = y = \frac{1}{2}$ whenever $0 \le a < b$, as in the first solution.

6. Let k and m be integers with $1 < k < m$. For a positive integer i, let L_i be the least common multiple of $1, 2, \ldots, i$. Prove that k is a divisor of $L_i \cdot \left[\binom{m}{i} - \binom{m-k}{i}\right]$ for all $i \ge 1$. [Here, $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ denotes a binomial coefficient. Note that $\binom{n}{i} = 0$ if $n < i$.]

   Solution: We prove the statement by induction on m. When $m = k$, we have

$$L_i \left[\binom{m}{i} - \binom{m-k}{i}\right] = L_i \binom{k}{i} = L_i \cdot \frac{k!}{i!(k-i)!} = k \cdot \frac{L_i}{i} \cdot \frac{(k-1)!}{(i-1)!(k-i)!} = k \cdot \frac{L_i}{i} \cdot \binom{k-1}{i-1}.$$

   Since $\frac{L_i}{i}$ is an integer (by definition of L_i), and $\binom{k-1}{i-1}$ is a binomial coefficient and thus also an integer, we see that k is indeed a divisor.

For the induction step, assume that the statement holds for a specific value of m. We use the recursion $\binom{m+1}{i} = \binom{m}{i} + \binom{m}{i-1}$ to show that it holds for $m+1$ as well:

$$\begin{aligned}
L_i\left[\binom{m+1}{i} - \binom{m+1-k}{i}\right] &= L_i\left[\binom{m}{i} - \binom{m-k}{i}\right] + L_i\left[\binom{m}{i-1} - \binom{m-k}{i-1}\right] \\
&= L_i\left[\binom{m}{i} - \binom{m-k}{i}\right] + \frac{L_i}{L_{i-1}} \cdot L_{i-1}\left[\binom{m}{i-1} - \binom{m-k}{i-1}\right].
\end{aligned}$$

Note that k divides both $L_i[\binom{m}{i} - \binom{m-k}{i}]$ and $L_{i-1}[\binom{m}{i-1} - \binom{m-k}{i-1}]$ by the induction hypothesis (if $i = 1$, the latter term is simply zero), and L_{i-1} (the least common multiple of $1, 2, \ldots, i-1$) divides L_i (the least common multiple of $1, 2, \ldots, i$, or equivalently the least common multiple of L_{i-1} and i). It follows that k is also a divisor of $L_i[\binom{m+1}{i} - \binom{m+1-k}{i}]$, which completes the proof.