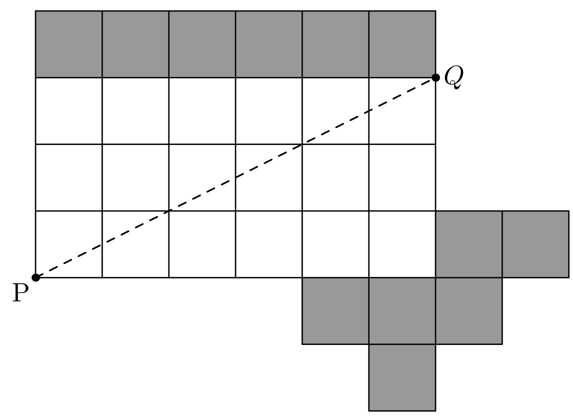
The South African Mathematical Olympiad
Junior Third Round 2012
Solutions

1. Note that the shaded regions are equal in area. Dividing the remaining rectangle in half would thus divide the given figure in half. The diagonal PQ of the remaining rectangle does the trick.



2. The pattern in which the beads are arranged is as follows: 1 white bead, 1 black bead, 1 white bead, 2 black beads, 1 white bead, 3 black beads, 1 white bead, and so on: the number of black beads increase by one every time.

   The last section of black beads visible before the string enters the box is a sequence of 4 black beads and a white bead. The first full section of black beads we see once the string exits the box is a sequence of 8 black beads. Hence the number of beads inside the box equals

$$5 \text{ (black) } + 1 \text{ (white) } + 6 \text{ (black) } + 1 \text{ (white) } + (7-2) \text{ (black) } = 18.$$

3. The hypotenuse AB of the large triangle has length 13 (by Pythagoras and the equality $13^2 = 12^2 + 5^2$). We calculate the area of the triangle in two ways:

$$\begin{aligned} \frac{1}{2} \cdot AC \cdot BC &= \tfrac{1}{2} \cdot AB \cdot CD \\ \implies \frac{1}{2} \cdot 5 \cdot 12 &= \frac{1}{2} \cdot 13 \cdot x \\ \implies x &= \frac{60}{13}. \end{aligned}$$

Alternative Solution

Let $BD = y$. Then $AD = 13 - y$. Using Pythagoras in triangle BCD, we obtain $12^2 = x^2 + y^2$. Using Pythagoras in triangle ACD we obtain $5^2 = x^2 + (13-y)^2$. Subtracting the two equations from each other, we obtain

$$\begin{aligned} 12^2 - 5^2 &= y^2 - (13-y)^2 \\ \implies 119 &= 26y - 169 \\ \implies y &= \frac{288}{26} = \frac{144}{13}. \end{aligned}$$

Then from $5^2 = x^2 + (13-y)^2$ we obtain $x^2 = 25 - \left(13 - \frac{144}{13}\right)^2$ which simplifies to $x = \frac{60}{13}$.

Alternative Solution

Note that the triangles BCD and BAC are similar, so

$$\frac{CD}{CB} = \frac{AC}{AB} \implies \frac{x}{12} = \frac{5}{13} \implies x = \frac{60}{13}.$$

4. (a) $((1 \div 2) \div 3) \div 4 = \frac{1}{24}$.
   (b) $(1 \div 2) \div (3 \div 4) = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$.
   (c) $1 \div ((2 \div 3) \div 4) = 1 \div \frac{2}{12} = \frac{12}{2} = 6$.
   (d) $(1 \div (2 \div 3)) \div 4 = \frac{3}{2} \div 4 = \frac{3}{8}$.
   (e) $1 \div (2 \div (3 \div 4)) = 1 \div \left(2 \times \frac{4}{3}\right) = 1 \div \frac{8}{3} = \frac{3}{8}$.

   Only the third expression is an integer.

5. Let D be the diameter of the large semi-circle, and let $d_1, d_2, \ldots, d_n$ be the diameters of the smaller semi-circles. Then we have that

$$D = d_1 + d_2 + \cdots + d_n.$$

The distance of the route along the big circle equals $\frac{1}{2}\pi D$. Taking the route along the smaller circles, one would travel the distance

$$\frac{1}{2}\pi d_1 + \frac{1}{2}\pi d_2 + \cdots + \frac{1}{2}\pi d_n = \frac{1}{2}\pi(d_1 + d_2 + \cdots + d_n) = \frac{1}{2}\pi D,$$

which shows that the two routes have exactly the same length.

6. (a) We have that $R = 3$. For the given credit card number to be valid, the quantity

$$(4+5+6+5+3+3+2+8+9+1+3+2+6+6+2+X)+(4+6+3+2+9+3+6+2)+3 = 113+X$$

   must be divisible by 10. The only digit value of X for which this works is $X = 7$.

   (b) In this case, the value of R is dependent on the value of Y. If $Y < 5$, then $R = 4$ and if $Y \geq 5$, then $R = 5$. For the given credit card number to be valid, the quantity

$$(5+2+3+2+7+1+9+8+3+4+0+2+Y+7+8+1)+(5+3+7+9+3+0+Y+8)+R = 97+2Y+R$$

   must be divisible by 10. Hence $97 + 2Y + R$ must be even. Thus R must be odd, so $R = 5$ (and so $Y \geq 5$). Thus $102 + 2Y$ must be divisible by 10. The only two possible values of Y for which this is true are $Y = 4$ and $Y = 9$. However, since $Y \geq 5$, it follows that $Y = 9$.

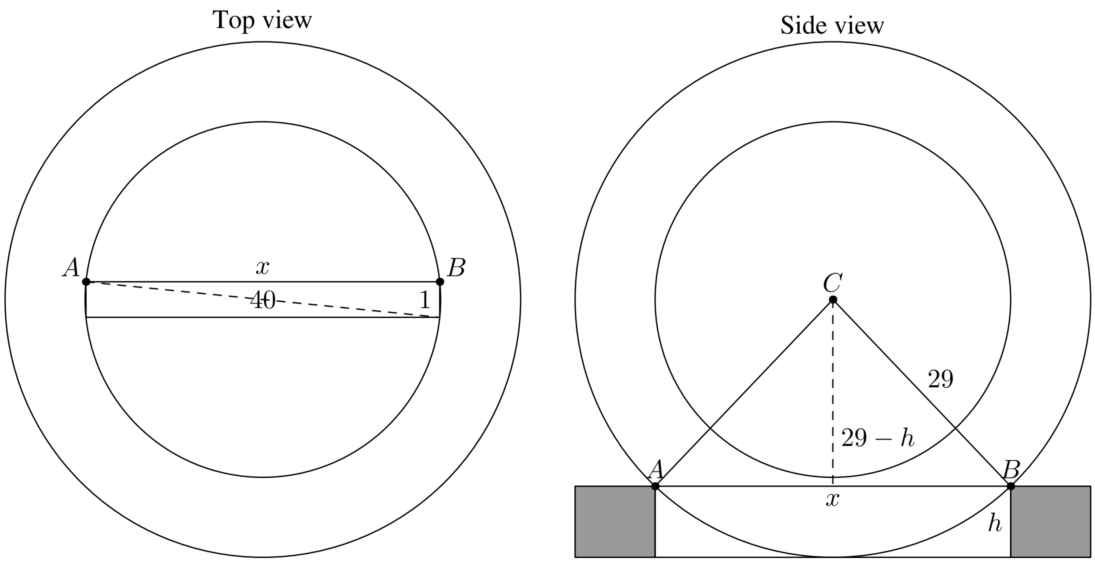
7. (a) Fred's average speed is equal to $\frac{\text{total distance}}{\text{total time}}$. Suppose that the distance up the hill is d. Then, since his uphill speed is 5 km/h, his uphill time is $t_1 = \frac{d}{5}$. Similarly, his downhill time is $t_2 = \frac{d}{45}$, so his total time is $t_1 + t_2 = \frac{d}{5} + \frac{d}{45} = \frac{10d}{45} = \frac{2d}{9}$.
   Hence his average speed is $\frac{\text{total distance}}{\text{total time}} = 2d \div \frac{2d}{9} = 9$ km/h.

   (b) Fred's uphill time is $\frac{d}{5}$, and by increasing his downward speed Fred can decrease his downhill time. However, his total time (up and down) will always be at least $\frac{d}{5}$. Hence his average speed is at most $2d \div \frac{d}{5} = 10$ km/h.
   (Note: Fred can never actually achieve 10 km/h, unless he teleports downhill instantaneously. The faster he cycles downhill, the closer his average speed gets to 10 km/h.)

8. The display must accommodate for the 11$^{\text{th}}$ and the 22$^{\text{nd}}$, so each cube must have a 1 and a 2. Also, since each of the days $01, 02, \ldots, 09$ can be displayed, both cubes must contain a 0 as well (if only one cube has a 0, there can be a maximum of 6 combinations of 0 with another digit). The rest of the digits can be arranged in any way, since no two of them need to appear together.

   However... there are 12 spaces on the two cubes, and 6 of the spaces are taken by the two 0s, 1s and 2s. This leaves 6 spaces for the remaining 7 digits. However, a 9 is not needed, since an upside down 6 will do perfectly well, as can be seen on the given picture.

9.



   The diagram above depicts the Top View and the Side View of the arrangement. We want to find the height h for which the vertical ring just touches the horizontal surface.

   Using Pythagoras in the first diagram, we obtain $x^2 = 40^2 - 1$, and then using Pythagoras in the second diagram, we have that

$$\begin{aligned}
(29-h)^2 + \left(\frac{x}{2}\right)^2 &= 29^2 \\
\implies h &= 29 - \sqrt{29^2 - \frac{x^2}{4}} \\
h &= 29 - \sqrt{29^2 - \frac{40^2-1}{4}} \\
&= 29 - \sqrt{29^2 - 20^2 + \frac{1}{4}} \\
&< 29 - \sqrt{29^2 - 20^2} \\
&= 29 - \sqrt{9 \cdot 49} \\
&= 29 - 3 \cdot 7 = 8.
\end{aligned}$$

   Hence 8 rings will suffice.

10. Let x be the value in the top left corner of the rectangle. Then the bottom left of the rectangle equals $x - 3h$ where $h = 1$ or $h = 2$. The bottom right corner's value is $x - 3h + k$ where $k = 1$ or $k = 2$, and the top right corner's value is $x + k$. Let $a = x$, $b = x - 3h$, $c = x - 3h + k$ and $d = x + k$.

These numbers then forms the number $abcd$ (where $abcd$ represents the four-digit number $abcd$, not the product of the four numbers). If the calculator is rotated through $90°$, $180°$ and $270°$ the numbers $bcda$, $cdab$ and $dabc$ are obtained. In all four cases, the number is divisible by 11 if an only if the difference $(a + c) - (b + d)$ is.

Now, $(a + c) - (b + d) = (x + x - 3h + k) - (x - 3h + x + k) = 0$, which is always divisible by 11.

11. Let us first examine the case when N is a power of 2. If $N = 2$, the first player shoots the second and automatically wins. If $N = 4$, the first player shoots the second player, the third player shoots the fourth player, and then the first player shoots the third player, and player 1 wins again. We now prove, using induction, that if N is a power of 2, then player 1 always win. Suppose this is the case when $N = 2^{k-1}$.

    Now, let $N = 2^k$. Then player 1 shoots player 2, player 3 shoots player 4, and so on, until player $2^k - 1$ shoots player 2^k and hands the paintball gun to player 1. At that point, all the even-numbered players have been eliminated, so there are $\frac{1}{2} \cdot 2^k = 2^{k-1}$ players remaining. Thus the situation is exactly the same as when $N = 2^{k-1}$, and so player 1 wins.

    Now let's investigate the given case $N = 100$. The greatest power of 2 smaller than 100 is $64 = 2^6$, and $64 = 100 - 36$. So, at the moment when 36 players are eliminated, there will be 64 players remaining, and whoever "starts" from that position wins.

    During the first round, all the even-numbered players are eliminated: player 1 shoots player 2, 3 shoots 4, and so on. So when player $2 \times 36 = 72$ is shot, 36 players have been eliminated, and player 73 has the paintball gun in hand. Now, since the number of players remaining is a power of two, and player 73 effectively starts from this position, player 73 wins the game.

12. Let the product be

    $$ABC \times DE = (100A + 10B + C)(10D + E) = 1000AD + 100(AE + BD) + 10(BE + CD) + CE.$$

    We wish to maximise this product. The numbers A to E come from the set 5 to 9, and so they're all less than 10. Hence the product of any two of them is less than 100. Clearly, we want the factor AD with coefficient 1000 to be the largest, so $\{A, D\} = \{8, 9\}$. Next, we wish to maximize $AE + BD$, which has coefficient 100. So either this value equals $9E + 8B$ or $8E + 9B$. This will be maximized when $\{B, E\} = \{6, 7\}$. The two possible values are thus $9(6) + 8(7) = 110$ and $9(7) + 8(6) = 111$, of which the second one is largest. This means that either $A = 9$, $E = 7$, $B = 6$, $D = 8$ or $A = 8$, $E = 6$, $B = 7$, $D = 9$. In both cases, $C = 5$.

    Finally, we wish to maximize $BE + CD$. Since $\{B, E\} = \{6, 7\}$ and $C = 5$, this product equals $6 \cdot 7 + 5D$, so the second case (with $D = 9$) maximises the product, and we have $A = 8$, $B = 7$, $C = 5$. $D = 9$ and $E = 6$, i.e. 875×96.

13. The first player has a winning strategy. We break down the possible cases as follows:

Situation A: Only one pile remains with at least 2 rings. In this case, the first player removes all but one ring from the pile, and wins.

Situation B: Two piles remain, each with the same number of rings. In this case, whatever move the first player makes, the second player can make using the *other* pile, resulting in two piles with the same number of rings. The second player keeps mimicking the first player, until the first player either leaves one ring remaining in a pile, or removes all of the rings. In the first case, player two then removes the other pile completely, leaving one ring and winning. The second case reduces to Situation A with player 2 being the first to move, and hence wins.

Situation C: Two of the three piles have the same number of rings. In this case, the first player removes the third pile entirely, reducing the game to Situation B but with the second player now acting as player 1. Hence the first player wins in this case.

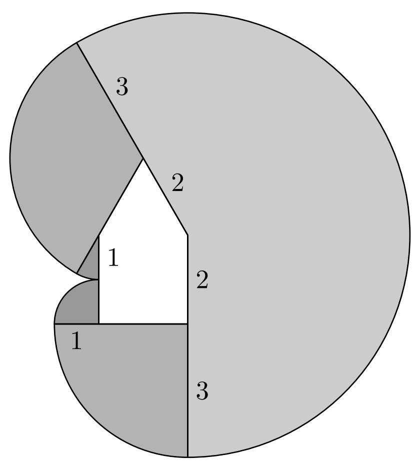
Situation D: Two piles remain, each with a different number of rings (and at least two rings). In this case, the first player removes enough rings from the tallest pile to make the two piles contain the same number of rings, reducing the game to situation B, but with the first and second player interchanged. Hence the first player wins in this case.

Situation E: Two piles contain only 1 ring, and the third pile contain more than 1 ring. In this case, the first player removes all but one ring from the third pile, leaving three piles with one ring each. This scenario results in a win for player 1.

Situation F: The three piles have 1, 2 and 3 rings, respectively. In this case, *any* move that player 1 makes will reduce the game to either Situation C, D or E. In all three cases, player 2 wins.

We now turn to the given problem. The first player removes two rings from the pile with three rings, leaving piles containing 1, 4 or 5 rings. Now, if player 2 removes the pile with only one ring, the game is reduced to Situation D, which results in the first player winning. If player 2 removes 1 ring from the pile of 5 rings, the game is reduced to situation C, in which case the first player wins. If the second player removes 3 rings from the 4-pile or 4 rings from the 5-pile, the game is reduced to situation E, which results in the first player wining. Any other move player 2 makes will result in either the 4-pile or the 5-pile reduced to a 3-pile or a 2-pile. Thus the number of rings in the three piles will be either 1, 2, x or 1, 3, x where $x = 4$ or $x = 5$. Player 1 then reduced the pile with x rings to 3 rings in the first case, and to two rings in the second, resulting in Situation F with the second player to move first. This results in a win for player 1 as well.

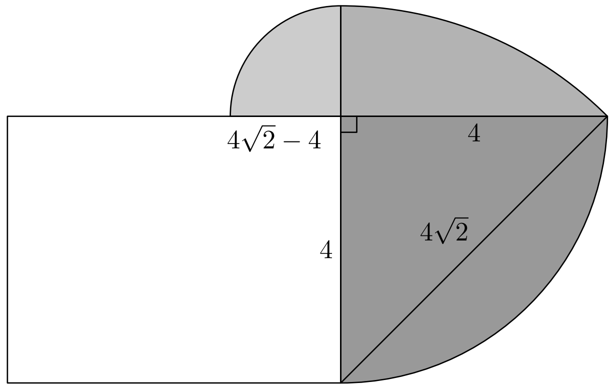
14. (a)



The region that the goat can reach consists of the parts of 5 circles, as shown above. The total area is thus equal to

$$\frac{210^\circ}{360^\circ}\pi(5)^2 + \frac{90^\circ + 120^\circ}{360^\circ}\pi(3)^2 + \frac{90^\circ + 31^\circ}{360^\circ}\pi(1)^2 = \frac{121\pi}{6}.$$

(b)



The region that the goat can reach consists of the parts of 3 circles, as shown above. To calculate the area, we add a quarter of the circle with radius 4, a quarter of the circle with radius $4\sqrt{2}-4$ and an eighth of the circle with radius $4\sqrt{2}$. However, this will count the area of the right-angled triangle in the picture twice, which we thus need to subtract. The total area is thus

$$\frac{1}{4}\pi(4)^2 + \frac{1}{4}\pi(4\sqrt{2}-4)^2 + \frac{1}{8}\pi(4\sqrt{2})^2 - \frac{1}{2}\cdot 4 \cdot 4 = (20-8\sqrt{2})\pi - 8.$$

15. Let the next year with this property be CY, where C is the century, and Y is the year. We wish to find the next year such that $C+Y$ and $|C-Y|$ are both powers of 2.

    Note that if $Y < C$, then $2C = (C+Y)+(C-Y)$, so C is the average of two powers of 2. If $Y > C$, then $2C = (C+Y)-(Y-C)$ is the average difference between two powers of 2. Suppose that $C+Y = 2^k$.

    In the first case, $2C = 2^k + (C-Y) > 2^k$ and so $C > 2^k$. In the second case, $2C = 2^k - (Y-C) > 2^k - 2^{k-1} = 2^{k-1}$, so $C > 2^{k-2}$.

    We wish to minimize C, so in the first case we need to choose k as small as possible while still ensuring that $C \geq 20$. The value $k=5$ suffices, since $2^5 = 32 > 20$. In this case, $2C = 32 +$ smaller power of 2, so $2C = 32+16$ or $2C = 32+8$ (the other powers of 2 lead to values of $C < 20$, which leads to $C = 24$ or $C = 20$. In the first case, $Y = 32-24 = 08$ and in the second case $Y = 32-20 = 12$. The first case yields 2408, while the second case yields 2012.

    Now, if $Y > C$, we need to choose k as small as possible while still ensuring that $C > 20$. In this case, $k = 6$ is a possibility. Then $2C = 64 -$ smaller power of 2, and the smallest such value of C with $C \geq 20$ is $2C = 64-16 \implies C = 24$. This leads to $Y = 64-24 = 40$, which also works, but the year 2440 comes after 2408.

    The answer is thus 2408.