THE HARMONY GOLD SOUTH AFRICAN MATHEMATICS OLYMPIAD

organised by the SOUTH AFRICAN ACADEMY OF SCIENCE AND ARTS
in collaboration with HARMONY GOLD MINING, AMESA and SAMS

SECOND ROUND 2002

SENIOR SECTION: GRADES 10, 11 AND 12

21 May 2002

TIME: 120 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. C
2. C
3. E
4. B
5. E
6. D
7. D
8. B
9. B
10. A
11. D
12. A
13. B
14. D
15. B
16. E
17. C
18. E
19. E
20. D

SOLUTIONS

1. **Answer C.** The volume of water is equal to the area of the field times the depth of the rain, that is, $50 \text{ m} \times 100 \text{ m} \times 10 \text{ mm} = (5000 \times 10000 \times 1) \text{ cm}^3$. The mass is therefore $(5000 \times 10000 \div 1000) \text{ kg} = 50000 \text{ kg} = 50$ ton.

2. **Answer C.** Shifting the decimal point of a number one place to the right is the same as multiplying the number by 10. Therefore, shifting the decimal point four places to the right is the same as multiplying the number by 10^4. If the given number is x, then we are told that $10^4 x = 9\frac{1}{x}$, so $x^2 = 9 \times 10^{-4}$. Since x is positive, the only answer is $x = 3 \times 10^{-2} = 0.03$.

3. **Answer E.** For every full revolution of the wheel with radius 6 cm, the wheel with radius 15 cm will turn $\frac{6}{15} = \frac{2}{5}$ of a revolution, and the wheel with radius 20 cm will turn $\frac{6}{20} = \frac{3}{10}$ of a revolution. If the smallest wheel completes n revolutions before all arrows point down again, then the other two wheels will complete $\frac{2n}{5}$ and $\frac{3n}{10}$ revolutions. Since the arrows all point down again, $\frac{2n}{5}$ and $\frac{3n}{10}$ must be whole numbers. The smallest value is $n = 10$. (The answer can be obtained directly from the fact that $n = \frac{\text{LCM}(6, 15, 20)}{6}$.)

4. **Answer B.** There are five terms in the first bracket, each of which is multiplied by all five terms in the second bracket. This makes a total of 25 terms before simplification. However, the expression now includes $cd+dc+ce+ec+de+ed$, which simplifies to $2cd+2ce+2de$. The number of terms is therefore reduced by three, giving a total of 22 terms after simplification.

5. **Answer E.** If we group the terms into 1001 pairs, and factorize each pair as the difference of two squares, we see that the sum is equal to

$$(2002 + 2001) + (2000 + 1999) + \cdots + (4 + 3) + (2 + 1),$$

that is, the sum of all the natural numbers from 1 to 2002. This is the sum of an arithmetic series and is equal to $1001 \times 2003 = 2\,005\,003$.

6. **Answer D.** An equation of this kind is called a *functional equation*, and can often be solved by choosing particular values for the variables. In this case, by choosing $x = 1$, we see that $f(y) = \frac{f(1)}{y}$ for all y. Therefore $20 = f(30) = \frac{f(1)}{30}$, so $f(1) = 600$. Finally, $f(40) = \frac{600}{40} = 15$.

Alternatively, note that $f(40) = f(30 \times \frac{4}{3}) = \frac{f(30)}{4/3} = \frac{20}{4/3} = 15$.

7. **Answer D.** We can factorize $2^{32} - 1 = (2^{16}+1)(2^{16}-1)$, and then do the same with $2^{16} - 1$ and $2^8 - 1$, and so on. This gives

$$2^{32} - 1 = (2^{16} + 1)(2^8 + 1)(2^4 + 1)(2^2 + 1)(2^1 + 1)(2^1 - 1).$$

The first five factors are known to be prime, and the last factor is 1, which is not prime, so there are five prime factors.

8. **Answer B.** The solid is obtained by removing a cube with side $(a-1)$ cm from a cube with side a cm, so its volume in cm^3 is equal to $a^3 - (a-1)^3 = 3a^2 - 3a + 1$. We therefore have $3a^2 - 3a + 1 = 1801$, so $a^2 - a - 600 = 0$, giving $a = 25$.

9. **Answer B.** Rewrite the equation as $m^2 + 2n - n^2 - 2m = 5$, which can be factorized as $(m-n)(m+n-2) = 5$. Both factors on the left hand side are integers, and the only integer factors of 5 are ± 5 and ± 1. Thus $m - n = \pm 5$ or ± 1, and $m + n - 2 = 5/(m-n)$. This gives four possibilities, but the only one for which m and n are positive is $m - n = 1$ and $m + n - 2 = 5$, giving $m = 4$ and $n = 3$.

10. **Answer A.** Let $BD = x$, and let T be the point of tangency on AD. Then $DT = x$, and $A\hat{B}D$ and $A\hat{T}P$ are right angles. Therefore $AT^2 = AP^2 - PT^2 = 6^2 - 2^2 = 32$, so $AT = 4\sqrt{2}$. Also $AD^2 = AB^2 + BD^2$, so $(4\sqrt{2} + x)^2 = 8^2 + x^2$. This gives $32 + 8x\sqrt{2} + x^2 = 64 + x^2$, so $x = \frac{32}{8\sqrt{2}} = 2\sqrt{2}$.

11. **Answer D.** The man had exactly half the numbers, so the chance of his being unlucky in one draw was $\frac{1}{2}$ or 1 in 2. Since the draws are independent, his chances of being unlucky in twenty separate draws are $(\frac{1}{2})^{20}$ or 1 in 2^{20}. Now $2^{10} = 1024 \approx 10^3$, so $2^{20} \approx 10^6 = 1\,000\,000$.

12. **Answer A.** If the three numbers are x, y, z, then the first calculation gives $\frac{1}{2}x + \frac{1}{2}y + z = 23$. The other two calculations give $\frac{1}{2}x + y + \frac{1}{2}z = 31$ and $x + \frac{1}{2}y + \frac{1}{2}z = 32$. If we add the three equations, then we obtain $2x + 2y + 2z = 86$, so $x + y + z = 43$. Double each of the first three equations and subtract the last one to get $z = 2 \times 23 - 43 = 3$ and $y = 2 \times 31 - 43 = 19$ and $x = 2 \times 32 - 43 = 21$.

13. **Answer B.** From the defining formula we see that

$$t_2 = \frac{t_1 + 2}{t_1 + 1} = \frac{1 + 2}{1 + 1} = \frac{3}{2} \text{ and } t_3 = \frac{t_2 + 2}{t_2 + 1} = \frac{\frac{3}{2} + 2}{\frac{3}{2} + 1} = \frac{7}{5}.$$

If t_n approaches a fixed (or limiting) value t, then t_{n-1} also approaches the value t, so from the defining formula we must have $t = \dfrac{t+2}{t+1}$. Thus $t^2 + t = t + 2$, so $t^2 = 2$. By the defining formula, t is obviously positive, so $t = \sqrt{2}$.

14. **Answer D.** Let the radius of the circle be r, and consider the distances from the vertices of the quadrilateral to the points of tangency. On the left, all four distances are equal to r; from the top right vertex both distances are $a - r$; from the bottom right vertex both distances are $b - r$. Thus the lengths of the left and right sides are $2r$ and $(a + b - 2r)$. If we drop a perpendicular from the top right vertex to the base, then we obtain a right-angled triangle with hypotenuse $a+b-2r$ and other two sides $2r$ and $b-a$. Thus $(a+b-2r)^2 = (2r)^2+(b-a)^2$, which simplifies to $r = \dfrac{ab}{a+b}$. The area of the quadrilateral is the sum of the areas of the above triangle and the remaining $2r \times a$ rectangle, so

$$\text{Area} = \frac{1}{2}(2r)(b - a) + (2r)(a) = r(b + a) = ab.$$

15. **Answer B.** If we denote the second sum by t and subtract t from s, then the odd terms cancel out. We are left with

$$s - t = \frac{2}{2^2} + \frac{2}{4^2} + \frac{2}{6^2} + \cdots = \frac{2}{2^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) = \frac{1}{2}s,$$

so $t = \frac{1}{2}s$. (Incidentally, it is known that $s = \pi^2/6$.)

16. **Answer E.** Let the three points of tangency be P (on AB produced), Q (on AC produced), and R (on BC). Let $P\widehat{O}B = \alpha$ and $Q\widehat{O}C = \beta$. Then $R\widehat{O}B = \alpha$, because the right-angled triangles POB and ROB are congruent, and similarly $R\widehat{O}C = \beta$. We can now obtain the result in at least two different ways.

    (a) It follows that $A\widehat{B}C = 2\alpha$ and $A\widehat{C}B = 2\beta$, so $2(\alpha + \beta) = 180° - \widehat{A} = 150°$. Finally, $B\widehat{O}C = \alpha + \beta = 75°$.

    (b) Note that $2B\widehat{O}C = 2\alpha + 2\beta = P\widehat{O}Q$ regardless of the positions of B and C. This means that $B\widehat{O}C = A\widehat{O}Q$ (just move B to coincide with A in which case C coincides with Q). Now it is straightforward to find that $A\widehat{O}Q = 90° - 15° = 75°$.

17. **Answer C.** Let x be the total area of the three unshaded regions. Then, by considering the large circle only, we see that $v + x = \frac{\pi}{4} \times 6^2 = 9\pi$, and by considering the three smaller circles, we see that $w + x = \frac{\pi}{4}(4^2 + 4^2 + 2^2) = 9\pi$ also. Thus $v - w = 0$, so $v = w$.

18. **Answer E.** Choose any point inside the polygon, and join it to the n vertices to make n triangles, each with base length s. The shortest distances from the interior point to the n sides are the heights of these triangles, which we call $h_1, h_2, \ldots, h_n$. The areas of the triangles are $\frac{1}{2}sh_1, \frac{1}{2}sh_2, \ldots, \frac{1}{2}sh_n$. The area of the polygon is equal to the sum of the areas of the triangles, so $A = \frac{1}{2}s(h_1 + h_2 + \cdots + h_n)$, giving $h_1 + h_2 + \cdots + h_n = \dfrac{2A}{s}$.

19. **Answer E.** Suppose the first 121 numbers occupy n complete rows (with perhaps some coming from row $n+1$). Then n is the largest natural number such that $1+2+\cdots+n \leq 121$, that is, $\frac{1}{2}n(n+1) \leq 121$. It is easy to see that $n = 15$ and that $1+2+\cdots+15 = 120$, so the sum will be the sum of the first 15 rows plus the first number from row 16 (which is equal to 1). The sums of the rows are 1, then $1+1 = 2$, then $1+2+1 = 4$, and so on, doubling each time. Thus the sum of the first 15 rows is $1+2+\cdots+2^{14}$, which equals $2^{15}-1$. For the final answer we must add the first 1 from row 16 to obtain 2^{15}.

20. **Answer D.** Each player draws one of 10 cards, so there are 100 equally likely outcomes, each of which will, on average, occur once in every 100 games. The 90 outcomes in which A and B are different cancel one other out, since for each game in which Bob pays Alice something, there will be another game in which Alice pays Bob the same amount. We are left with the 10 outcomes in which $A = B$. Taking the odd cases together, Bob pays Alice $1^2+3^2+5^2+7^2+9^2$ rands, that is, R165,00. Taking the even cases together, Alice pays Bob $0^2+2^2+4^2+6^2+8^2$ rands, that is, R120,00. Thus on average, Alice wins R45,00 every 100 games, which is R0,45 per game.