South African Mathematics Olympiad
Third Round 2001: Solutions

1. For the upper bound, use the triangle inequality in triangles ABC and ADC. Then $AB + BC > AC$ and $AD + DC > AC$, giving $p > 2AC$. Similarly, $p > 2BD$. Therefore $2p > 2AC + 2BD$.

   For the lower bound, let AC and BD meet in E. Since $ABCD$ is convex, E is inside the quadrilateral, and therefore $AE + EC = AC$ and $BE + ED = BD$. Now use the triangle inequality in each of the four small triangles, giving

$$\begin{aligned} p &= AB + BC + CD + DA \\ &< (AE + EB) + (BE + EC) + (CE + ED) + (DE + EA) \\ &= 2AE + 2BE + 2CE + 2DE = 2(AC + DB). \end{aligned}$$

2. Let $x + y + z = xyz = a$, $xy + yz + zx = b$. Multiplying out the left-hand side, we find

$$\begin{aligned} & x - xy^2 - xz^2 + xy^2z^2 + y - yz^2 - yx^2 + yz^2x^2 + +z - zx^2 - zy^2 + zx^2y^2 \\ &= (x + y + z) - (xy^2 + zy^2 + xyz) - (yx^2 + zx^2 + xyz) - \\ &\quad - (yz^2 + yx^2 + xyz) + 3xyz + xyz(yz + zx + xy) \\ &= 4a - xb - yb - zb + ab \\ &= 4a - (x + y + z)b + ab \\ &= 4a. \end{aligned}$$

   Therefore the first equation is redundant. From the second equation we obtain $z = (x + y)/(xy - 1)$. The required triples are therefore $(x, y, (x + y)/(xy - 1))$ where x is any real number and y any real number except $1/x$.

3. If $x = 1$ there is nothing to prove, so let $x^{1919} = k + t$, $x^{1960} = m + t$ and $x^{2001} = n + t$, where k, m and n are distinct integers and $0 \leqslant t < 1$. Then

$$x^{41} = \frac{m + t}{k + t} = \frac{n + t}{m + t}.$$

   Therefore

$$0 = (m + t)^2 - (k + t)(n + t) = (m^2 - kn) - (k - 2m + n)t.$$

   If $k-2m+n = 0$, then also $m^2-kn = 0$ and it follows that $0 = m^2-k(2m-k) = (m-k)^2$ which contradicts $k \neq m$. Therefore $t = (m^2 - kn)/(k - 2m + n)$ is rational, and x^{41} and x^{1919} are both rational. Since 41 is prime and $1919 = 19 \times 101$ is not divisible by 41, it follows that 41 and 1919 are mutually prime, and we can find integers a and b such that $41a + 1919b = 1$. Therefore

$$x = x^{41a+1919b} = (x^{41})^a + (x^{1919})^b$$

   is rational, say $x = u/v$ with $v > 0$ and u and v mutually prime.

   Then the denominators of x^{1960} and x^{2001}, expressed in lowest terms, are v^{1960} and v^{2001}. But since $x^{1960} - x^{2001}$ is an integer, these denominators must be equal. It follows that $v = 1$.

4. Let R and B be the sets of red and blue points respectively. For each one-to-one function $f : R \to B$, define the length $L(f)$ of the pairing f to be the sum of the distances between corresponding points. More precisely, if $d(r, f(r))$ is the distance beween corresponding points, $L(f) = \sum_{r \in R} d(r, f(r))$. Note that there are only finitely many such functions, so amongst all the $L(f)$ there is a least one. Call it $L(g)$. We claim that g has no intersecting segments.

   If not, we have a pair of offending initial points r_1 and r_2, whose segments cross. Now let $\bar{g}$ be the same as g except that the end points of r_1 and r_2 are switched. Therefore $(r_1, \bar{g}(r_1))$ and $(r_2, \bar{g}(r_2))$ are opposite sides of the convex quadrilateral with diagonals $(r_1, g(r_1))$ and $(r_2, g(r_2))$. As shown in Problem 1,

   $$d(r_1, \bar{g}(r_1)) + d(r_2, \bar{g}(r_2)) < d(r_1, g(r_1)) + d(r_2, g(r_2)).$$

   Hence $L(\bar{g}) < L(g)$, a contradiction, a contradiction, since $L(g)$ was supposedly least.

   Second solution: Call a set of points *balanced* if it has as many red as blue points. The proof is by strong induction on n. The pairing is trivially possible when $n = 1$. The idea is to find a straight line that divides S into balanced subsets of $2k$ and $2n-2k$ points, where $1 < k < n$. These subsets can be paired by the induction hypothesis, and the line segments cannot cross the dividing line, so neither subset can interfere with the other.

   Consider the smallest convex polygon C that contains all the given points. Each of its vertices obviously coincides with some given point. If any two vertices A and B of C are of different colour, match them. None of the other points can lie on the line AB (that would give three collinear points) and therefore they all lie to the same side of the line AB, giving a balanced set of $2n - 2$ points which can be paired.

   In the case that all the vertices of C are of the same colour, say red, draw a line not passing through any given point, with slope different from any possible line connecting two given points, separating the set into two non-empty subsets, each with an even number of points. This is possible because there are only finitely many slopes.

   If these subsets are not balanced, one of them has an excess of blue points. Move the line parallel to itself so that two points are transferred from that subset to the other. Continue doing so until the subsets are balanced.

   This must happen because when only two points are left, one of them is a vertex of C and therefore red. So that last pair cannot have a blue excess.

5. Draw the diagonals of $\mathbf{Q}_0 = ABCD$ and let $C\widehat{A}D = \alpha$, $D\widehat{B}A = \beta$, $A\widehat{C}B = \gamma$ and $B\widehat{D}C = \delta$. Label the circumscribed quadrilateral $\mathbf{Q}_1 = A_1B_1C_1D_1$ so that A_1BB_1, B_1CC_1, C_1DD_1 and D_1AA_1 are its sides. Repeated use of the tan-chord theorem gives that the angles of $A_1B_1C_1D_1$ are 2α, 2β, 2γ and 2δ respectively. If $A_1B_1C_1D_1$ is also cyclic, it follows that

   $$\alpha + \gamma = 90^\circ = \beta + \delta.$$

   Draw the diagonals of $A_1B_1C_1D_1$ and let α_1, β_1 etc. be the analogous angles to α, β etc. If the sequence is to continue to a third cyclic quadrilateral, by the above argument applied to $A_1B_1C_1D_1$we require that

   $$\alpha_1 + \gamma_1 = 90^\circ = \beta_1 + \delta_1.$$

By consideration of the angles in $\triangle A_1B_1D_1$ we find that

$$\begin{aligned}180^\circ &= 2\alpha + \beta_1 + \gamma_1 \\ &= 2\alpha + (2\beta - \alpha_1) + (90^\circ - \alpha_1) \\ \Rightarrow \alpha_1 &= \alpha + \beta - 45^\circ; \\ \beta_1 &= \beta - \alpha + 45^\circ.\end{aligned}$$

Rewrite these equations as

$$\begin{aligned}(\alpha_1 - 45^\circ) &= (\alpha - 45^\circ) + (\beta - 45^\circ); \\ (\beta_1 - 45^\circ) &= (\beta - 45^\circ) - (\alpha - 45^\circ).\end{aligned}$$

Continue to $\mathbf{Q}_2$:

$$\begin{aligned}(\alpha_2 - 45^\circ) &= (\alpha_1 - 45^\circ) + (\beta_1 - 45^\circ) = 2(\beta - 45^\circ); \\ (\beta_2 - 45^\circ) &= (\beta_1 - 45^\circ) - (\alpha_1 - 45^\circ) = -2(\alpha - 45^\circ);\end{aligned}$$

and to $\mathbf{Q}_4$:

$$(\alpha_4 - 45^\circ) = 2(\beta_2 - 45^\circ) = -4(\alpha - 45^\circ).$$

Now move in steps of four to find

$$(\alpha_{4n} - 45^\circ) = (-4)^n(\alpha - 45^\circ).$$

If $\mathbf{Q}_{4n}$ is convex, we must have $0 < \alpha_{4n} < 180^\circ$. But unless $\alpha = 45^\circ$ (when $ABCD$ is a square) the quantity $\alpha_{4n} - 45^\circ$ is a divergent geometric sequence and will eventually move outside the range -45° to 135°, and therefore the sequence is finite.

6. Let $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ be another sequence of numbers that satisfies the equations. We shall show:

   when n is even, the $\hat{x}_i$ cannot satisfy the order relations;

   when n is odd, it is possible that the x_i are such that the $\hat{x}_i$ can satisfy the order relations.

   Therefore the answer to the problem is: all even $n \geqslant 4$.
   We first consider the case where $n = 2m$. The conditions for d_i and s imply that there is a number h such that

$$\hat{x}_i - x_i = \begin{cases} h, & \text{for odd } i; \\ -h, & \text{for even } i.\end{cases}$$

The condition for t implies that

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} \hat{x}_i^2 - x_i^2 \\
&= \sum_{i=1}^{n} (\hat{x}_i - x_i)(\hat{x}_i + x_i) \\
&= h\sum_{i=1}^{m} (2x_{2i-1} + h) - h\sum_{i=1}^{m} (2x_{2i} - h) \\
&= mh^2 + 2h\sum_{i=1}^{m} x_{2i-1} + mh^2 - 2h\sum_{i=1}^{m} x_{2i} \\
&= 2mh^2 - 2h\sum_{i=1}^{m} (x_{2i} - x_{2i-1})
\end{aligned}
$$

The last equation has two solutions: $h = 0$, the "true" solution, and

$$\hat{h} = \frac{1}{m}\sum_{i=1}^{m} (x_{2i} - x_{2i-1}),$$

the "spurious" solution. From the order relations, it follows that $\hat{h} > 0$, i.e. $\hat{x}_1 > x_1$. If the $\hat{x}_i$ satisfy the order relations, we can start with $\hat{x}_i$ as the "true" solution and repeat the process to obtain a third solution in which

$$\tilde{x}_1 > \hat{x}_1 > x_1.$$

But there are only two solutions. This contradiction shows that the assumption that the $\hat{x}_i$ satisfy the order relations is untenable.
On the other hand, when $n = 2m - 1$ ($m \geqslant 2$) one can easily find a counterexample. Let $x_i = m - i + h_i$ where

$$h_i = \begin{cases} \frac{h}{m-1} \text{ for even } i; \\ \frac{-h}{m} \text{ for odd } i; \end{cases}$$

where h is any nonzero real number such that $x_i < x_{i+1}$ holds, e.g. $h = \frac{1}{2}$. Here $s = 0$ and all $d_i = 1$. Now take $\hat{x}_i = -x_{2m-i}$. Clearly the $\hat{x}_i$ satisfy the order conditions, all $\hat{x}_{i+2} - \hat{x}_i = 1$, $\sum_{i=1}^{2m-1} \hat{x}_i = -s = 0$ and $\sum_{i=1}^{2m-1} \hat{x}_i^2 = \sum_{i=1}^{2m-1} x_i^2 = t$. But $\hat{x}_m = -x_m$ and x_m is nonzero, therefore $\hat{x}_m \neq x_m$ and the $\hat{x}_i$ form a different valid solution.

Suid-Afrikaanse Wiskunde-olimpiade
Derde Ronde 2001: Oplossings

1. Vir die bogrens, gebruik die driehoeksongelykheid in driehoeke ABC en ADC. Dan $AB + BC > AC$ en $AD + DC > AC$, dus $p > 2AC$. Net so, $p > 2BD$. Daarom $2p > 2AC + 2BD$.

   Vir die ondergrens, laat AC en BD in E sny. Omdat $ABCD$ konveks is, is E binne die vierhoek, en daarom $AE + EC = AC$ en $BE + ED = BD$. Gebruik die driehoeksongelykheid in elk van die vier klein driehoeke, waaruit

$$\begin{aligned} p &= AB + BC + CD + DA \\ &< (AE + EB) + (BE + EC) + (CE + ED) + (DE + EA) \\ &= 2AE + 2BE + 2CE + 2DE \\ &= 2(AC + DB). \end{aligned}$$

2. Laat $x + y + z = xyz = a$, $xy + yz + zx = b$. As ons die linkerkant uitmaal, kry ons

$$\begin{aligned} & x - xy^2 - xz^2 + xy^2z^2 + y - yz^2 - yx^2 + yz^2x^2 + z - zx^2 - zy^2 + zx^2y^2 \\ =\ & (x + y + z) - (xy^2 + zy^2 + xyz) - (yx^2 + zx^2 + xyz) - \\ & - (yz^2 + yx^2 + xyz) + 3xyz + xyz(yz + zx + xy) \\ =\ & 4a - xb - yb - zb + ab \\ =\ & 4a - (x + y + z)b + ab \\ =\ & 4a. \end{aligned}$$

   Die eerste vergelyking is dus oorbodig.. Uit die tweede vergelyking verkry ons $z = (x + y)/(xy - 1)$. Die vereiste triplette is dus $(x, y, (x + y)/(xy - 1))$ waar x enige reële getal is, en y enige reële getal behalwe $1/x$.

3. As $x = 1$ is daar niks om te bewys nie; laat dus $x^{1919} = k + t$, $x^{1960} = m + t$ en $x^{2001} = n + t$, waar k, m en n almal verskillende heeltalle is, en $0 \leqslant t < 1$. Dan

$$x^{41} = \frac{m + t}{k + t} = \frac{n + t}{m + t}.$$

   Dus

$$0 = (m + t)^2 - (k + t)(n + t) = (m^2 - kn) - (k - 2m + n)t.$$

   As $k - 2m + n = 0$, dan ook $m^2 - kn = 0$ en dit volg dat $0 = m^2 - k(2m - k) = (m - k)^2$ wat $k \neq m$ weerspreek. Dus is $t = (m^2 - kn)/(k - 2m + n)$ rasionaal, en x^{41} en x^{1919} is albei rasionaal. Omdat 41 priem is en $1919 = 19 \times 101$ nie deelbaar deur 41 is nie, volg dat 41 en 1919 onderling priem is. Ons kan dus heelgetalle a en b vind sodat $41a + 1919b = 1$. Dus is

$$x = x^{41a+1919b} = (x^{41})^a + (x^{1919})^b$$

   rasionaal, sê $x = u/v$ met $v > 0$ en u en v onderling priem.

   Dan is die noemers van x^{1960} en x^{2001}, uitgedruk in kleinste terme, v^{1960} en v^{2001}. Maar omdat $x^{1960} - x^{2001}$ 'n heeltal is, moet hierdie noemers gelyk wees. Dit volg dat $v = 1$.

4. Laat R en B onderskeidelik die versamelings rooi en blou punte wees. Vir elke een-tot-een funksie $f : R \to B$, definieer die lengte $L(f)$ van die afparing f as die som van die afstande tussen ooreenstemmende punte. Meer presies, as $d(r, f(r))$ die afstand tussen ooreenstemmende punte is, dan is $L(f) = \sum_{r \in R} d(r, f(r))$. Let op: daar is net 'n eindige aantal sulke funksies, dus is daar tussen al die $L(f)$ 'n kleinste een. Noem dit $L(g)$. Ons beweer dat g geen segmente het wat mekaar sny nie.

   Anders het ons 'n paar hinderlike beginpunte r_1 en r_2, wie se segmente kruis. Laat nou $\bar{g}$ dieselfde as g wees, behalwe dat die eindpunte r_1 en r_2 omgeruil is. Dus is $(r_1, \bar{g}(r_1))$ en $(r_2, \bar{g}(r_2))$ teenoorstaande sye van dieselfde konvekse vierhoek met diagonale $(r_1, g(r_1))$ en $(r_2, g(r_2))$. Soos in Probleem 1 aangetoon,

$$d(r_1, \bar{g}(r_1)) + d(r_2, \bar{g}(r_2)) < d(r_1, g(r_1)) + d(r_2, g(r_2)).$$

   Dus is $L(\bar{g}) < L(g)$, 'n teenspraak, want $L(g)$ was tog die kleinste.

   Tweede oplossing: Noem 'n versameling *gebalanseerd* as dit ewe veel rooi en blou punte bevat. Die bewys werk met sterk induksie op n. Die afparing is triviaal moontlik wanneer $n = 1$. Die idee is om 'n reguit lyn te vind wat S in gebalanseerde subversamelings met $2k$ en $2n - 2k$ punte opdeel, waar $1 < k < n$. Hierdie subversamelings kan volgens die induksiehipotese afgepaar word, en die lynsegmente kan nie die skeidslyn kruis nie, dus kan die twee subversamelings nie mekaar pla nie.

   Beskou die kleinste konvekse veelhoek C wat al die gegewe punte omsluit. Elke hoekpunt daarvan val klaarblyklik saam met 'n gegewe punt. As twee hoekpunte A en B van C verskillende kleure het, paar hulle af. Nie een van die ander punte kan op die lyn AB lê nie (dit sou drie saamlynige punte gee) en daarom lê hulle almal aan dieselfde kant van die lyn AB en vorm 'n gebalanseerde versameling van $2n - 2$ punte wat afgepaar kan word,

   Ingeval al die hoekpunte van C dieselfde kleur het, sê rooi, trek 'n lyn wat nie deur enige gegewe punt gaan nie, met helling verskillend van enige moontlike lyn tussen twee gegewe punte, sodat dit die versameling in twee nie-leë versamelings opdeel, elk met 'n ewe aantal punte. Dit is moontlik, want daar is net 'n eindige aantal hellings om te vermy.

   As die twee subversamelings nie gebalanseerd is nie, het een van hulle 'n oormaat blou punte. Skuif die lyn parallel aan homself sodat twee punte van daardie versameling na die ander oorgedra word. Hou aan om dit te doen totdat die subversamelings gebalanseerd is.

   Dit moet gebeur, want as daar net twee punte oor is, is een van hulle 'n hoekpunt van C en dus rooi. Daardie laaste paar kan nie 'n blou oormaat hê nie.

5. Trek die hoeklyne van $\mathbf{Q}_0 = ABCD$ en laat $C\widehat{A}D = \alpha$, $D\widehat{B}A = \beta$, $A\widehat{C}B = \gamma$ en $B\widehat{D}C = \delta$. Merk die omgeskrewe vierhoek $\mathbf{Q}_1 = A_1B_1C_1D_1$ op so 'n manier dat A_1BB_1, B_1CC_1, C_1DD_1 en D_1AA_1 sy sye is. Herhaalde gebruik van die raaklyn-koord-stelling gee dat die hoeke van $A_1B_1C_1D_1$ onderskeidelik 2α, 2β, 2γ en 2δ is. As $A_1B_1C_1D_1$ ook siklies is, volg dit dat

$$\alpha + \gamma = 90^\circ = \beta + \delta.$$

   Trek die hoeklyne van $A_1B_1C_1D_1$ en laat α_1, β_1 ens. die hoeke wees wat met α, β ens. ooreenstem. As die ry na 'n volgende vierhoek moet voortgaan, gee hierdie argument, toegepas op $A_1B_1C_1D_1$ dat

$$\alpha_1 + \gamma_1 = 90^\circ = \beta_1 + \delta_1.$$

Deur die hoeke in $\triangle A_1B_1D_1$ te beskou, vind ons dat

$$\begin{aligned} 180^\circ &= 2\alpha + \beta_1 + \gamma_1 \\ &= 2\alpha + (2\beta - \alpha_1) + (90^\circ - \alpha_1) \\ \Rightarrow \alpha_1 &= \alpha + \beta - 45^\circ; \\ \beta_1 &= \beta - \alpha + 45^\circ. \end{aligned}$$

Herskryf hierdie vergelykings as

$$\begin{aligned} (\alpha_1 - 45^\circ) &= (\alpha - 45^\circ) + (\beta - 45^\circ); \\ (\beta_1 - 45^\circ) &= (\beta - 45^\circ) - (\alpha - 45^\circ). \end{aligned}$$

Gaan aan na $\mathbf{Q}_2$:

$$\begin{aligned} (\alpha_2 - 45^\circ) &= (\alpha_1 - 45^\circ) + (\beta_1 - 45^\circ) = 2(\beta - 45^\circ); \\ (\beta_2 - 45^\circ) &= (\beta_1 - 45^\circ) - (\alpha_1 - 45^\circ) = -2(\alpha - 45^\circ); \end{aligned}$$

en na $\mathbf{Q}_4$:

$$(\alpha_4 - 45^\circ) = 2(\beta_2 - 45^\circ) = -4(\alpha - 45^\circ).$$

Beweeg nou in stappe van vier om te bevind dat

$$(\alpha_{4n} - 45^\circ) = (-4)^n(\alpha - 45^\circ).$$

As $\mathbf{Q}_{4n}$ konveks is, moet ons $0 < \alpha_{4n} < 180^\circ$ hê. Maar tensy $\alpha = 45^\circ$ (wanneer $ABCD$ 'n vierkant is) vorm $\alpha_{4n} - 45^\circ$ 'n divergente meetkundige ry, wat op die ou ent buite die interval tussen -45° moet 135° val, en dus is die ry eindig.

6. Laat $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ nog 'n ry getalle wees wat die vergelykings bevredig. Ons sal aantoon dat:

as n ewe is, kan die $\hat{x}_i$ nie die ordening bevredig nie;

as n onewe is, is dit moontlik om x_i te kry sodat $\hat{x}_i$ die ordening bevredig.

Dus is die antwoord op die probleem: alle ewe $n \geqslant 4$.
Ons beskou eers die geval $n = 2m$. Die voorwaardes vir d_i en s bring mee dat daar 'n getal h bestaan sodat

$$\hat{x}_i - x_i = \begin{cases} h, & \text{vir onewe } i; \\ -h, & \text{vir ewe } i. \end{cases}$$

Die voorwaarde vir t bring mee dat

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \hat{x}_i^2 - x_i^2 \\ &= \sum_{i=1}^{n} (\hat{x}_i - x_i)(\hat{x}_i + x_i) \\ &= h\sum_{i=1}^{m}(2x_{2i-1} + h) - h\sum_{i=1}^{m}(2x_{2i} - h) \\ &= mh^2 + 2h\sum_{i=1}^{m} x_{2i-1} + mh^2 - 2h\sum_{i=1}^{m} x_{2i} \\ &= 2mh^2 - 2h\sum_{i=1}^{m}(x_{2i} - x_{2i-1}) \end{aligned}$$

Die laaste vergelyking het twee oplossings: $h = 0$, die "ware" oplossing, en

$$\hat{h} = \frac{1}{m}\sum_{i=1}^{m}(x_{2i} - x_{2i-1}),$$

die "vals" oplossing. Uit die ordening volg dat $\hat{h} > 0$, d.w.s. $\hat{x}_1 > x_1$. As al die $\hat{x}_i$ die ordening bevredig, kan ons met $\hat{x}_i$ as die "ware" oplossing begin en die hele proses herhaal om 'n derde oplossing met

$$\tilde{x}_1 > \hat{x}_1 > x_1$$

te verkry. Maar daar is net twee oplossings. Hierdie teenspraak toon aan dat die aanname dat $\hat{x}_i$ die ordening bevredig, onhoudbaar is.

Aan die ander kant, as $n = 2m - 1$ ($m \geqslant 2$) kan 'n mens maklik 'n teenvoorbeeld kry. Laat $x_i = m - i + h_i$ waar

$$h_i = \begin{cases} \frac{h}{m-1} \text{ vir ewe } i; \\ \frac{-h}{m} \text{ vir onewe } i; \end{cases}$$

waar h enige nie-nul reële getal is waarvoor $x_i < x_{i+1}$ geld, bv. $h = \frac{1}{2}$. Hier is $s = 0$ en alle $d_i = 1$. Neem nou $\hat{x}_i = -x_{2m-i}$. Duidelik bevredig $\hat{x}_i$ die ordening, is alle $\hat{x}_{i+2} - \hat{x}_i = 1$, $\sum_{i=1}^{2m-1} \hat{x}_i = -s = 0$ en $\sum_{i=1}^{2m-1} \hat{x}_i^2 = \sum_{i=1}^{2m-1} x_i^2 = t$. Maar $\hat{x}_m = -x_m$ en x_m is nie-nul, dus is $\hat{x}_m \neq x_m$ en $\hat{x}_i$ vorm 'n tweede geldige oplossing.