South African Mathematics Olympiad
Junior Third Round 2015
Solutions

1. Label the 5×5 grid as shown:

	A	B	C	D	E
1	M	A	T	H	S
2					
3					
4		M	A	T	
5					

Consider the diagonal A5 - E1: the letter T must appear in the diagonal, but it cannot appear in D2 or C3, since columns C and D already contain a T. Hence A5 = T. The letter A must also appear in the diagonal, but cannot be in C3, since column C already contains an A, so D2 = A and hence C3 = H.

	A	B	C	D	E
1	M	A	T	H	S
2				**A**	
3			**H**		
4		M	A	T	
5	**T**				

Considering the other diagonal, B2 cannot be an A since B1 = A, hence E5 = A which forces B2 = S. Looking at row 2 and letter T, columns A and C already contain Ts, which implies that E2 = T. This then forces C2 = M and A2 = H.

	A	B	C	D	E
1	M	A	T	H	S
2	**H**	**S**	**M**	A	**T**
3			H		
4		M	A	T	
5	T				**A**

The only place in row 3 where A can occur is in A3 and T can only occur in B3. The rest of the grid can now be easily completed in a similar fashion.

	A	B	C	D	E
1	M	A	T	H	S
2	H	S	M	A	T
3	**A**	**T**	H	**S**	**M**
4	**S**	M	A	T	**H**
5	T	**H**	**S**	**M**	A

2. If the average of four numbers is 8, their sum must be $8 \times 4 = 32$. To maximize the largest one of these numbers, we choose the smallest possible values for the other three, which are 1, 2 and 3 (since the integers must be positive and different). The remaining number is then $32 - (1 + 2 + 3) = 26$.

3. $\frac{1110222222003333333330004444444444440000}{111} = 10002002000030030030000040040040040000$, which contains 28 zeroes.

4. There are $7 \times 24 = 168$ hours in a week and there are $4 \times 2.5 = 10$ hours during which load shedding may occur. Since there is a 60% chance of load shedding, the probability that there is load shedding at a particular moment during the week is

$$\frac{4 \times 2.5}{7 \times 24} \times 60\% = \frac{10}{168} \times \frac{60}{100} = \frac{1}{28}.$$

5. Placing a 3 at both ends of the number increases it by 3372, a four-digit number. Hence the original number is a two-digit number. Suppose the original number is x. Then the number formed by placing a 3 at both ends is equal to $3003 + 10x$. This is 3372 more than x, that is,

$$3003 + 10x - x = 3372 \implies 9x = 369 \implies x = 41.$$

6. (a) Suppose the side length of the cube is n; we wish to find the largest n such that $n^3 \leq 500$. Now, $7^3 = 343$ and $8^3 = 512$, so the largest value of n is 7.

   (b) Suppose the side length of the hollow cube is n. The hollow cube is obtained by removing the smaller interior cube from the solid cube, so the shell contains $n^3 - (n-2)^3$ cubes, so we want to find the largest n such that $n^3 - (n-2)^3 \leq 500$. Now

$$\begin{aligned} & n^3 - (n-2)^3 \leq 500 \\ \implies & 2(n^2 + n(n-2) + (n-2)^2) \leq 500 \\ \implies & n^2 + n^2 - 2n + n^2 - 4n + 4 \leq 250 \\ \implies & 3n^2 - 6n \leq 246 \\ \implies & n(n-2) \leq 82. \end{aligned}$$

   Now, $10 \times 8 = 80 < 82$ while $11 \times 9 = 99 > 82$, so the largest value for n is 10.

7. We will compute the angle sizes of quadrilateral $DEGB$ and use the fact that the sum of the interior angles of a quadrilateral is equal to $360°$.

   Since AFC is an equilateral triangle, $\angle FAC = 60°$ and so $\angle EAF = 90° - 60° = 30°$, which yields $\angle BAE = 90° + 30° = 120°$. Since $AB = AC = AF$, triangle BAF is isosceles, and so

$$\angle GBC = \angle FBA = \frac{1}{2}(180° - \angle BAF) = \frac{1}{2}(180° - 120°) = 30°.$$

   Next, $AE = AB = AC$, and so triangle EAC is a right-angled isosceles triangle, which means that $\angle BCG = \angle ACE = 45°$. Hence,

$$\angle BGE = \angle GBC + \angle GCB = 30° + 45° = 75°.$$

8. Suppose that the rectangular sheet of paper has dimensions a and b, with b being the longer side. We calculate the volume of the two cylinders formed by joining the long sides and short sides, respectively.

   - Suppose the short sides are glued together. Then the height of the cylinder is a and the circumference of the cylinder is b. If r is the radius of the cylinder, it means that $2\pi r = b$, or $r = \frac{b}{2\pi}$. Hence the cylinder's volume is $\pi r^2 h = \pi \left(\frac{b}{2\pi}\right)^2 \cdot a = \frac{ab^2}{4\pi}$.

- Suppose the long sides are glued together. Then the height of the cylinder is b and the circumference of the cylinder is a. If r is the radius of the cylinder, it means that $2\pi r = a$, or $r = \frac{a}{2\pi}$. Hence the cylinder's volume is $\pi r^2 h = \pi \left(\frac{a}{2\pi}\right)^2 \cdot b = \frac{a^2 b}{4\pi}$.

Comparing these two numbers, we see that the first volume can be written as $b\left(\frac{ab}{4\pi}\right)$, while the second volume is $a\left(\frac{ab}{4\pi}\right)$. Since the number in brackets is the same for both and $b > a$, it means that the first volume $b\left(\frac{ab}{4\pi}\right)$ is the largest. Hence the largest volume is obtained when the short sides of the sheet of paper are glued together.

9. In any arrangement of cubes, each edge is aligned in one of three directions, and note that for each direction, the number of edges is the same. Hence to count the total number of visible edges, we only need to count the number of edges in a particular direction, and multiply it by 3. For argument's sake, let's consider the vertical edges only.

   Note that in the top layer, there are 3 vertical edges, in the second layer there are 5 and in the third there are 7. Each next layer can be formed by copying the last layer, and adding one extra cube, which adds two extra vertical edges. Hence the sequence of vertical edges in the layers is $3, 5, 7, 9, \ldots$, the sequence of odd integers and so the n^{th} layer has $2n+1$ vertical edges.

   Now, we know that $1 + 3 + 5 + \cdots + (2n-1) = n^2$, so

$$3 + 5 + 7 + \cdots + (2n+1) = [1 + 3 + 5 + \cdots + (2(n+1) - 1)] - 1 = (n+1)^2 - 1.$$

   Hence in total there are $3((n+1)^2 - 1)$ visible edges.

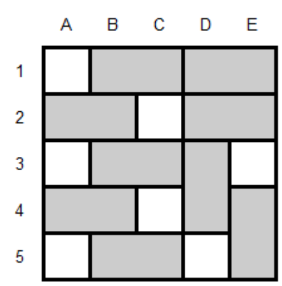
   (a) If $n = 3$, this is equal to $3(4^2 - 1) = 45$.

   (b) If $n = 20$, this is equal to $3(21^2 - 1) = 1320$.

10. (a) The chessboard contains an odd number of squares and every time a 2×1 tile is placed, the number of open squares decrease by 2, an even number. Hence the number of remaining squares is always an odd number (odd minus even is odd).

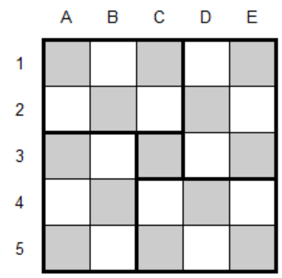
   (b) Before any tile is placed, there are 13 black squares and 12 white squares. Every time a tile is placed, the tile covers one black square and one white square, so the number of open black squares and open white squares both decrease by 1. Since there is one more black square than white squares at the start, there will be one more black square than white squares at the end.

   (c) The following arrangement shows 7 empty 1×1 squares.



   Assume for a contradiction that it is possible to have more than 7 empty squares. Since there must be an odd number, there are at least 9 empty squares.

Divide the board into four 2×3 rectangles and one 1×1 centre square:



If the top left 2×3 rectangle had 3 empty squares, then these three empty squares must be touching diagonally, and hence so must the three covered squares. This means that either A1 is covered and B1 and A2 are empty (impossible with 1×2 tiles) or B1 is covered and A1, B2 and C1 are empty (again impossible). This means that this rectangle can contain at most 2 empty squares. A similar argument shows that all of the 2×3 rectangles have maximum two empty squares. However, this means that to get 9 empty squares in total, all four rectangles must have exactly two empty squares, and in addition the central square must be empty. It follows that the white squares C2, B3, D3 and C4 must all be covered.

Now, from part (b) we know that of the 9 empty squares, 5 must be black and 4 white. The only remaining white squares are A2, A4, B1, B5, D1, D5, E2, E4, of which four must be empty. None of the pairs (A2,B1), (A4,B5), (D1,E2) and (D5,E4) can be empty, since that would force the corner square also empty, a contradiction. Hence exactly one in each of the above mentioned pairs is empty, and hence the three black squares surrounding it must be covered. For the first white square, this forces three black squares to be covered, and for each of the remaining three white squares, at least two additional black squares are covered, leaving at most $13 - (3 + 2 + 2 + 2) = 4$ black squares empty, which contradicts the fact that 5 black squares must be empty.

11. Group the numbers into 7 groups according to their remainders upon division by 7: the groups are

$$\{1, 8, 15, \ldots, 498\},\ \{2, 9, 16, \ldots, 499\},\ \{3, 10, 17, \ldots, 500\},$$
$$\{4, 11, 18, \ldots, 494\},\ \{5, 12, 19, \ldots, 495\},\ \{6, 13, 20, \ldots, 496\},\ \{7, 14, 21, \ldots, 497\}.$$

Note that if a number from a group that leaves remainder x is not deleted, then *all* the numbers from the group that leaves remainder $7 - x$ must be deleted, otherwise two numbers will sum to a multiple of 7. In particular, only one number from the last group above may be selected.

Note also that if one number from a group (besides the last one) is selected, we may select *all* numbers in that group, since they all leave the same remainder when divided by 7. The first three groups contain 72 numbers each, while the last four contain 71 numbers each. If we select the first three groups, we may not select any number in the next three groups, and finally we can select one number from the last group. This yields $72 \times 3 + 1 = 217$ numbers.

12. The number

$$5^{2015} + 5^{2016} + 5^{2017} + 31^{2015}$$

is the sum of four odd numbers, which is even. Thus 2 is a prime factor.

Next,

$$5^{2015} + 5^{2016} + 5^{2017} + 31^{2015} = 5^{2015}(1 + 5 + 5^2) + 31^{2015} = 31 \times 5^{2015} + 31^{2015},$$

which shows that 31 is also a prime factor.

Finally,

$$5^{2015} + 5^{2016} + 5^{2017} + 31^{2015} = (5^{2016} + 5^{2017}) + (5^{2015} + 31^{2015}) = 5^{2016}(5 + 1) + (5^{2015} + 31^{2015}).$$

The first term is divisible by 3, and the second term is the sum of two 2015^{th} powers, which means it can factorise as $(5+31)$(terms of the second bracket), which is divisible by 3 as well. Hence 3 divides the original number.

Alternatively, observe that 5 leaves remainder 2 when divisible by 3, $5^2 = 25$ leaves remainder 1, $5^3 = 125$ leaves remainder 2, and so on. Hence 5^k, when divided by 3, leaves remainder 2 if k is odd and remainder 1 if k is even. Similarly, 31^k always leaves remainder 1 when divided by 3. Hence when dividing the original number by 3, the remainder is $2+1+2+1 = 6$, which is divisible by 3, and so the original number is divisible by 3.

13. Suppose that the alphanumeric does have a solution. Twice the 6-digit number TWENTY is a 7-digit number, which means that $C = 1$ and $T \geq 5$.

    Next note that $Y + Y = T$ or $Y + Y = T + 10$; in both cases the left hand side is even, so the right hand side must also be even, and so T must be even. Together with $T \geq 5$, this means $T = 6$ or $T = 8$.

    Next, note that since $C = 1$, the column $E + E$ yields 1, but since $E + E$ itself is even, a one must have carried from the $N + N$ calculation. This means that $E + E + 1 = 1$ or $E + E + 1 = 11$, which yields $E = 0$ or $E = 5$.

    Finally, consider the $T + T$ part of the sum. We know that $T = 6$ or $T = 8$, and there might be a 1 carrying from the $Y + Y$ part of the sum, so the possible values for E are 2 ($T + T = 6 + 6 = 12$), 3 ($T + T + 1 = 6 + 6 + 1 = 13$), 6 ($T + T = 8 + 8 = 16$) or 7 ($T + T + 1 = 8 + 8 + 1 = 17$). But we know that $E = 0$ or $E = 5$, which is a contradiction. Hence the alphanumeric does not have a solution.

14. If x is negative, both $\lfloor \frac{7}{x} \rfloor$ and $\lfloor \frac{x}{7} \rfloor$ are negative, so there are no solutions.

    If x is positive, both terms are non-negative integers, and the only way for them to sum to 2 is if both are equal to 1, or one is equal to 0 and the other equal to 2. We consider these cases:

    - If both terms are equal to 1, we must have that

    $$\left\lfloor \frac{7}{x} \right\rfloor = 1 \implies 1 \leq \frac{7}{x} < 2 \text{ and } \left\lfloor \frac{x}{7} \right\rfloor = 1 \implies 1 \leq \frac{x}{7} < 2 \implies \frac{1}{2} < \frac{7}{x} \leq 1.$$

      The only way both inequalities are satisfied is if $\frac{7}{x} = 1 \implies x = 7$.

    - If $\lfloor \frac{7}{x} \rfloor = 2$ and $\lfloor \frac{x}{7} \rfloor = 0$, it means that $2 \leq \frac{7}{x} < 3$ and $0 \leq \frac{x}{7} < 1$. This leads to $\frac{1}{3} < \frac{x}{7} \leq \frac{1}{2}$ and $0 \leq x < 7$, which simplifies to $\frac{7}{3} < x \leq \frac{7}{2}$.

    - If $\lfloor \frac{7}{x} \rfloor = 0$ and $\lfloor \frac{x}{7} \rfloor = 2$, it means that $0 \leq \frac{7}{x} < 1$ and $2 \leq \frac{x}{7} < 3$. This leads to $1 < \frac{x}{7}$ and $14 \leq x < 21$, which simplifies to $14 \leq x < 21$.

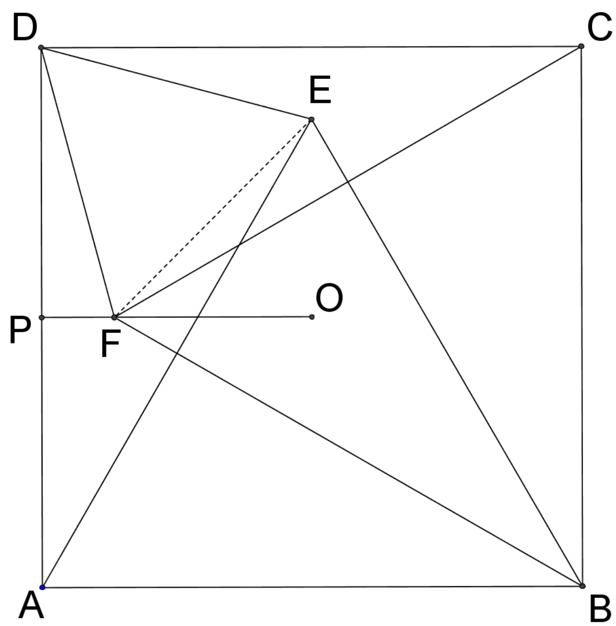
    The possible values for x are $\frac{7}{3} < x \leq \frac{7}{2}$, $x = 7$ and $14 \leq x < 21$.

15. First note that reflecting the figure around the line BD doesn't change the figure (the equilateral triangle AEB reflects to the triangle CBF). This means that DE and DF are symmetric about the line BD, which means that $DF = DE$. If we can show that $\angle FDE = 60°$, then triangle DEF is an isosceles triangle with one $60°$ angle, which means it's equilateral.

    Now, since $DA = AB = AE$, triangle ADE is isosceles. Next,

    $$\angle DAE = 90° - \angle EAB = 90° - 60° = 30° \text{ and so } \angle ADE = \frac{1}{2}(180° - \angle DAE) = \frac{1}{2}(180° - 30°) = 75°.$$

    Similarly, $\angle DFC = 75°$.



Let O be the centre of the square and let P be the midpoint of AD. Since the triangle BFC is isosceles, it means that O, F and P are on a straight line. Moreover, the triangle CFB is symmetric about the line OP, which means that $\angle OFC = \frac{1}{2}\angle BFC = 30°$. Finally,

$$\angle DFP = 180° - \angle DFC - \angle CFO = 180° - 75° - 30° = 75°.$$

Hence

$$\angle PDF = 90° - \angle PFD = 90° - 75° = 15° \text{ and } \angle FDE = \angle PDE - \angle PDF = 75° - 15° = 60°,$$

as required.