THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior First Round 2014

Solutions

1. **Answer E.**
   Of the 16 smaller squares, seven are shaded and two others are half-shaded. This gives $7 + 2 \times \frac{1}{2} = 8$ shaded small squares. The fraction of the large square that is shaded is therefore $\frac{8}{16} = \frac{1}{2}$.

2. **Answer C.**
   In each of the five squads there are $15 - 2 = 13$ more players than coaches. In the whole tournament there are $5 \times 13 = 65$ more players than coaches.

3. **Answer B.**
   A simple trial shows that the cards can be arranged in increasing order by first swapping cards 5 and 2, then swapping cards 3 and 2. Since three cards are wrong at the start, and one swap corrects at most two cards, it is impossible to correct all three with only one swap.
   (Other numbers of swaps can be used, but it can be shown that it will always be an even number. That is another reason why one swap is not possible.)

4. **Answer C.**
   Since three batteries run the torch for six hours, Betsie needs $5 \times 3 = 15$ batteries to run the torch for 30 hours. However, since the batteries come in packs of four, she will have to buy the next bigger multiple of 4, which is 16 batteries or four packs.

5. **Answer D.**
   Let the tens digit be t and the units digit be u, so $t - u = \pm 3$. Also $1 \leq t \leq 9$ and $0 \leq u \leq 9$. (Note that the tens digit cannot be zero.) If $t = u + 3$, then $0 \leq u \leq 6$, giving seven numbers. If $t = u - 3$, then $4 \leq u \leq 9$, giving another six numbers, for a total of 13.

6. **Answer A.**
   Since x is a real number, it follows that $x^2 + 1$ cannot be zero, because a perfect square cannot be negative. We may therefore divide both sides of the equation by $(x^2 + 1)$ to give $2 - x = 0$, for which the only solution is $x = 2$.

7. **Answer E.**
   If $v = 30$ m/s and $a = 5$ m/s^2, then $\dfrac{v^2}{2a} = \dfrac{30^2}{2 \times 5} = \dfrac{900}{10} = 90$ m.

8. **Answer E.**
   In the 4×4-grid shown, the 40 matches occur in five horizontal lines of four matches each, and the same arrangement vertically. For a 10×10-grid, there will be eleven lines of ten matches each, horizontally and vertically. The total is therefore
   $11 \times 10 \times 2 = 220$.

9. **Answer D.**
We are given that $2x + 2y = xy$, so $xy - 2x - 2y = 0$. By adding 4 to both sides, we see that $xy - 2x - 2y + 4 = 4$, or $(x-2)(y-2) = 4$. Since $x - 2$ and $y - 2$ are integers, they must be factors of 4, that is, $x - 2 = y - 2 = \pm 2$, or $x - 2 = \pm 1$ and $y - 2 = \pm 4$, or *vice versa*. The only solutions with x and y both positive are $x = y = 4$ or $x = 3$ and $y = 6$, or *vice versa*. Thus $x + y = 4 + 4 = 8$ (not given) or $x + y = 3 + 6 = 9$.

10. **Answer B.**
Since he has taken 16 wickets at 21.625 runs each, a total of $16 \times 21.625 = 346$ runs have already been scored. When he has taken 14 more wickets, he will have taken a total of 30 wickets. For his average to be 20 at that time, the total number of runs will be $30 \times 20 = 600$, so $600 - 346 = 254$ more runs will have been scored.

11. **Answer D.**
Triangle ACC' is isosceles, with angle $C\hat{A}C' = 130°$, so $A\hat{C}C' = \frac{1}{2} \times (180 - 130) = 25°$. Next, $B\hat{C}A = 60°$, because triangle ABC is equilateral, so $B\hat{C}C' = 60° - 25° = 35°$.

12. **Answer A.**
If he averages 270 km/h for one minute, which is 1/60 of an hour, then the distance travelled will be $270/60 = 4.5$ km.

13. **Answer D.**
A $3 \times 3 \times 3$ cube is made up of 27 unit cubes (i.e., with sides of length 1), each of which weighs $810/27 = 30$ g. Making each of the holes removes three unit cubes, but that counts the centre cube at the centre three times, so the number of unit cubes removed is seven, not nine. (Alternatively, making the first hole removes three unit cubes, and making each of the other two holes removes two unit cubes, for a total of $3 + 2 \times 2 = 7$.) There are thus $27 - 7 = 20$ unit cubes left, and the remaining solid weighs $20 \times 30 = 600$ g.

14. **Answer A.**
The first digit A can only be 1. (If $A \geq 2$, then $\overline{ABCD} \geq 2100$ because $B \geq 1$,and the number will be too large.) The last digit D must be one of $2, 4, 6$ because $\overline{ABCD}$ is even, giving three possibilities for D. Then B can be any one of the remaining five given digits, and C any one of the remaining four. This gives a total of $1 \times 3 \times 5 \times 4 = 60$ different numbers that can be formed.

15. **Answer D.**
With three dice, there are $6 \times 6 \times 6 = 216$ equally probable throws that can be made. The combinations with a sum of 15 are $6, 6, 3$ or $6, 5, 4$ or $5, 5, 5$ if we ignore the order. But $6, 6, 3$ occurs with three different throws, $6, 5, 4$ occurs with six different throws, and $5, 5, 5$ occurs with only one throw. Thus the total number of throws with the correct sum is $3 + 6 + 1 = 10$. The probability of obtaining a sum of 15 is therefore $\frac{10}{216} = \frac{5}{108}$.

16. **Answer C.**
The area of triangle ABC is half the area of the rectangle, that is, $15/2$. If we regard AC and EF as the bases of triangles ABC and EBF respectively, then triangle EBF has the same altitude as triangle ABC, but only one-third of the base, since $EF = \frac{1}{3}AC$. Thus the area of triangle EBF is equal to $\frac{1}{3} \times \frac{15}{2} = \frac{5}{2}$.

17. **Answer A.**
The numerical order of the numbers will be the same as that of their cube roots, which are:

$$2^7 = 128, \qquad 3^5 = 243, \qquad 5^3 = 125, \qquad 6^3 = 216, \qquad 11^2 = 121.$$

Since $121 < 125 < 128 < 216 < 243$, we see that the middle one of the original numbers is 2^{21}.

18. **Answer E.**
If the sides have lengths x, y, z, with z the hypotenuse, then $\frac{1}{2} \times 2z = 5$ and $\frac{1}{2}xy = 5$ since the area of the triangle is 5. Thus $z = 5$ and $xy = 10$. Next, $x^2 + y^2 = z^2 = 25$ by Pythagoras' theorem. Finally, $(x+y)^2 = (x^2+y^2) + 2xy = 25 + 2 \times 10 = 45$, so $x + y = 3\sqrt{5}$. The perimeter $x + y + z$ is therefore equal to $3\sqrt{5} + 5$.
(Alternatively, since the altitude divides the triangle into two triangles that are similar to each other, it can be shown that the sides x and y are equal to $\sqrt{5}$ and $2\sqrt{5}$ in some order.)

19. **Answer C.**
Saying P divides AB in the ratio $2:3$ means $AP : PB = 2:3$, so $AP : AB = 2 : (2+3) = 2:5$, or $AP = \frac{2}{5}AB$. Similarly, $AQ = \frac{3}{7}AB$. Therefore $PQ = AQ - AP = (\frac{3}{7} - \frac{2}{5})AB = \frac{1}{35}AB$. Since $PQ = 2$, it follows that $AB = 2 \times 35 = 70$.

20. **Answer D.**
The three-digit number $\overline{abc}$ is equal to $100a + 10b + c$ in algebraic notation, and similarly $\overline{ac}$ is equal to $10a + c$. The given equation therefore becomes

$$100a + 10b + c = 9(10a + c) + 4c,$$

which simplifies to $5(a+b) = 6c$. Since a, b, c are integers between 0 and 9, and $a \neq 0$, the only possibility is $a + b = 6$ and $c = 5$. The six possible numbers then arise from taking $a = 1, 2, \ldots, 6$ and $b = 6 - a$.
