THE SOUTH AFRICAN MATHEMATICS OLYMPIAD
SENIOR SECOND ROUND 2017
Solutions

1. **Answer 020**
   In 60 minutes the train travels 100 km, so in 12 minutes it travels $\frac{12}{60} \times 100 = 20$ km.

2. **Answer 003**
   Since there are only two different colours, it follows that if you take out three socks, then you will have two of the same colour.

3. **Answer 144**
   If each side is scaled by a factor, then the area is scaled by the square of that factor. Here each side is scaled by 3, so new area $= 3^2 \times$ old area $= 9 \times 16 = 144$.

4. **Answer 101**
   The average of an arithmetic sequence is equal to the average of its first and last terms. The 100th even positive integer is 200, so the average is $\frac{1}{2}(2 + 200) = 101$.

5. **Answer 417**
   $(x - 40)(x + 40) = x^2 - 40^2 = 2017 - 1600 = 417$.

6. **Answer 010**
   $2^{12} \times 5^8 = 2^4 \times 10^8$, which is written as 16 followed by eight 0s, so the total number of digits is $2 + 8 = 10$.

7. **Answer 080**
   The interior angles at A and C are $180 - 42 = 138$ and $180 - 38 = 142$ respectively. The angle-sum of a quadrilateral is $360°$, so $\widehat{P} + \widehat{Q} = 360 - (138 + 142) = 80°$.
   Alternatively, join PQ. Then $D\widehat{C}Q = C\widehat{P}Q + C\widehat{Q}P$ (exterior angle of a triangle), and similarly $B\widehat{A}Q = A\widehat{P}Q + A\widehat{Q}P$. Finally,
   $$B\widehat{P}D + A\widehat{Q}C = (A\widehat{P}Q + A\widehat{Q}P) + (C\widehat{P}Q + C\widehat{Q}P) = B\widehat{A}Q + D\widehat{C}Q = 42 + 38 = 80°.$$

8. **Answer 004**
   The shaded arrowhead lies between two triangles with base 4, one with height 4 and the other with height 2. The shaded area is therefore $\frac{1}{2} \times 4 \times (4 - 2) = 4$.

9. **Answer 067**
   All that matters is whether the terms are even (E) or odd (O). The five terms given are O, O, E, O, O. Then, since $O + O = E$ and $O + E = O$, we see that the sequence continues E, O, O, E, O, O, and so on. It therefore repeats in cycles of length three, starting O, O, E. The first 99 terms form 33 cycles, in which O occurs 66 times. The 100th term is O, making a total of 67 odd terms.

10. **Answer 020**
    $\sqrt{5} \oplus \sqrt{5} = (2\sqrt{5})^2 - 0^2 = 20$.
    Alternatively, using the difference of two squares, $x \oplus y = (2x)(2y) = 4xy$, so $\sqrt{5} \oplus \sqrt{5} = 4(\sqrt{5})^2 = 20$ again.

11. **Answer 420**
If the dimensions of the cube in cm are x, y, z, then the areas in cm^2 of the faces shown are xy, yz, zx. The volume of the cube in cm^3 is $xyz = \sqrt{xy \times yz \times zx} = \sqrt{84 \times 30 \times 70} = 10 \times \sqrt{12 \times 7 \times 3 \times 7} = 10 \times 7 \times 6 = 420$.

12. **Answer 020**
The diagram has a rotational symmetry about the midpoint, so the shaded region is congruent to the similar region above the diagonal. Therefore the area of the shaded region is half of the area remaining after removing two circles from the rectangle. If the shorter side of the rectangle has length x cm, then the shaded area in cm^2 is $\frac{1}{2}(2x^2 - 2\frac{\pi}{4}x^2) = x^2 - \frac{\pi}{4}x^2 = 100 - 25\pi$. By inspection it is clear that $x^2 = 100$, so $x = 10$. Therefore, the length $2x = 20$.

13. **Answer 012**
We are given $r = (2 \times 3)^{2b} = (6^2)^b$ and $r = 3^b \times x^b = (3x)^b$. Thus $36 = 3x$ and $x = 12$.

14. **Answer 090**
Suppose there were n children at the party. Then $n = \frac{1}{2}n + \frac{1}{3}n + 15$, so $\frac{1}{6}n = 15$ and $n = 90$.

15. **Answer 002**
If a positive integer n has a factorization $n = d \times q$, then both d and q are factors of n. Thus the factors occur in pairs unless there is a factorization in which $d = q$, in which case $n = d^2$. Thus the number of factors of n is even unless n is a perfect square.This cuts the number of possibilities with five factors down to the numbers $\{1^2, 2^2, \ldots, 10^2\}$, for which it is easy to proceed by trial and error. However, it can also be shown that if the prime factorization of n is $n = p^a q^b ...$, then the number of factors is $(a+1)(b+1)...$. Since 5 is prime, the only solution of $(a+1)(b+1)\ldots = 5$ is $a = 4$, so the only two numbers with five factors are $2^4 = 16$ and $3^4 = 81$.

16. **Answer 015**
If we set up axes with the origin at M, then the parabola passes through the points $A(-20; 0)$, $B(20; 0)$ and $C(0; 16)$. From the roots at A and B the equation is $y = a(x+20)(x-20) = a(x^2-400)$. Then solving for C gives $16 = -400a$, so $a = -\frac{1}{25}$ and $y = -\frac{1}{25}(x^2 - 400) = 16 - \frac{1}{25}x^2$. At $x = \pm 5$ the height $y = 16 - \frac{1}{25}5^2 = 15$.

17. **Answer 040**
Suppose the string of length 1 is cut at a random point x, giving pieces of lengths x and $1 - x$. There is success if $x \geq 2(1 - x)$ or $(1 - x) \geq 2x$, that is, if $x \geq \frac{2}{3}$ or $x \leq \frac{1}{3}$. The probability of success is therefore $\frac{2}{3}$, and the expected number of successes in 60 trials is 40.

18. **Answer 032**
Using B for "boy" and G for "girl", we know that BBB does not occur, so there are seven equally likely possibilities: $GGG, GGB, GBG, BGG, GBB, BGB, BBG$. Of these, the first four have at least two girls, so the probability of at least two girls is $\frac{4}{7} = \frac{x}{56}$. Thus $x = 32$.

19. **Answer 063**
Suppose the number of learners who left school at the end of 2016 is n, which is 5% or $\frac{1}{20}$ of the 2016 total, which is therefore $20n$. In 2017 the total becomes $20n - n + 189$. This is 10% or $\frac{1}{10}$ more than the 2016 total, so it is equal to $20n(1 + \frac{1}{10}) = 22n$. Therefore $19n + 189 = 22n$, so $3n = 189$ and $n = 63$.

20. **Answer 384**
The ratio of the areas of triangles ADE and ABC is $\frac{2}{3}$, so the ratio of their sides is $\dfrac{\sqrt{2}}{\sqrt{3}}$. Thus $DE = \dfrac{\sqrt{2}}{\sqrt{3}} BC = \dfrac{\sqrt{2}}{\sqrt{3}} 192\sqrt{6} = 192 \times 2 = 384$.

21. **Answer 120**
There are 24 vertices on the solid, each vertex being shared by two octagons and a triangle. From any given vertex, lines are on the surface if they go to one of the other vertices on these three faces. Thus the number of vertices giving lines on the surface of the solid is $(7 + 7 + 2) - 3 = 13$. (We have to subtract 3 because three vertices are counted twice, where the lines are actually edges of the solid.) That leaves $23 - 13 = 10$ vertices where the lines pass through the interior of the solid. The total number of lines through the interior is therefore $\frac{1}{2}(24 \times 10) = 120$ lines passing through the interior. (We have to divide by 2 because each line is counted twice: once for each of the two vertices it joins.)

22. **Answer 007**
Suppose the population in 2015 is x^2. In 2016 it is $x^2 + 100 = y^2 + 1$ say, and in 2017 it is $y^2 + 101 = z^2$ say. It follows that $(z + y)(z - y) = 101$, which is prime. Therefore $z + y = 101$ and $z - y = 1$, so $z = 51$ and $y = 50$. Then the 2015 population is $x^2 = y^2 - 99 = 2500 - 99 = 2401$, and the sum of the digits is $2 + 4 + 0 + 1 = 7$.

23. **Answer 336**
The formula for an infinite geometric series gives $S(r) = 12/(1 - r)$ and $S(-r) = 12/(1 + r)$. It follows that $2016 = 144/(1 - r^2)$, which simplifies to $1 - r^2 = \frac{1}{14}$. Finally,
$$S(r) + S(-r) = \frac{12}{1 - r} + \frac{12}{1 + r} = \frac{24}{1 - r^2} = 24 \times 14 = 336.$$

24. **Answer 042**
Here is the diagram for one shuffle:

$$\begin{array}{ccccccccccccc} A & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & J & Q & K \\ 6 & A & 7 & 4 & 2 & 9 & K & J & 10 & Q & 8 & 5 & 3 \end{array}$$

The best way is to imagine that in one shuffle each card in the top line changes into the one below it, then follow its progress through several shuffles until it gets back to where it started. We shall indicate these changes by arrows. For example, $A \to 6 \to 9 \to 10 \to Q \to 5 \to 2 \to A$. We can abbreviate this as $(A\,6\,9\,10\,Q\,5\,2)$, which is called a cycle of length 7. It follows that A returns to its original place after seven shuffles, and so do all seven cards in that cycle. We can show similarly that the cycles of the remaining cards are $(3\,7\,K)$, (4) and $(8\,J)$. Finally, the least number of shuffles required for all cards to return to their original places is the LCM of all the cycle lengths, and the LCM of $\{7, 3, 1, 2\}$ is 42.

25. **Answer 045**
Draw line segment AM. Then angle $M\widehat{A}C = 15°$ from the exterior angle theorem for a triangle. Choose a point X on AC so that angle $M\widehat{X}C = 30°$. Hence, triangle XMC would be isosceles, and we have $XM = MC = BM$. This implies that M is the centre of the semi-circle through points B, X and C. Therefore, angle $B\widehat{X}C = 90°$ (angle in semi-circle). In addition, since angle $A\widehat{M}X = 15°$ from the construction, that angle $B\widehat{M}X = 60$. But since $XM = BM$ as already shown, it follows that triangle BMX is equilateral, and therefore that $BX = XM$, and therefore $BX = AX$. Since $B\widehat{X}C = 90°$ as shown, this implies that the required $B\widehat{A}X$ (= angle $B\widehat{A}C$) $= 45°$ in the isosceles right triangle AXB.

Alternatively, the result can be proved by using the Sine and Cosine Rules in triangles AMC and AMB, and the fact that $2\sqrt{2}\sin 15° = \sqrt{3}-1$,which can be obtained from double angle formulae.

Other different solutions to this problem can be found in the November 2016 Newsletter of the International Group for Mathematical Creaitivity and Giftedness at: `https://drive.google.com/file/d/0BzbVsIIa4AYVNXhsNkNRN2gwV28/view` from pp. 11-13.