THE OLD MUTUAL
SOUTH AFRICAN MATHEMATICS OLYMPIAD
SENIOR SECOND ROUND 2021
Solutions

1. **Answer 001**
If the mass of each cake is x kg, then $8x = 6x + 0.125$, so $2x = 0.125$, and therefore $16x = 8 \times 0.125 = 1$ kg.

2. **Answer 036**
The total area of the four strips is $4 \times 10 \times 1 = 40$ cm^2, but the four 1×1 cm^2 squares where the strips cross are each covered by two strips, so the area of the table covered is $40 - 4 = 36$ cm^2.

3. **Answer 110**
There are no numbers of the form $15X51$ greater than 15951, so the next number must be of the form $16X61$, and the smallest value for X is 0. The distance travelled is $16061 - 15951 = 110$.

4. **Answer 027**
Since $365 = 7 \times 52 + 1$ and $366 = 7 \times 52 + 2$, it follows that the day of the week for 14 February advances by one except in the years following a leap year, when it advances by two. (Note that the extra day in a leap year comes after 14 February.) The days of the week for Valentine's Day in the next few years will therefore be:

2021	2022	2023	2024	2025	2026	2027
Sunday	Monday	Tuesday	Wednesday	Friday	Saturday	Sunday

5. **Answer 080**
Since $AB = BC = AC = 30$ and $DE = DB = BE = 10$ (equilateral triangles), it follows that $AD = EC = 20$. The perimeter of quadrilateral $ADEC$ is therefore $20 + 10 + 20 + 30 = 80$.

6. **Answer 012**
A number is divisible by 3 if and only if the sum of its digits is divisible by 3. From the set $\{2, 3, 7, 9\}$ the only three-digit subsets with sum divisible by 3 are $\{2, 3, 7\}$ and $\{2, 7, 9\}$. The digits in each subset can be arranged in $3 \times 2 \times 1 = 6$ ways to give different numbers divisible by 3, so the total is $6 + 6 = 12$.

7. **Answer 059**
The largest perfect cube less than 100 is 64. The sum of two prime numbers must be 63, so the one must be even and the other must be odd. The only even prime number is 2. That will be $61 + 2$ and their difference is $61 - 2 = 59$.

8. **Answer 100**
The largest difference occurs when one prime is the smallest possible and the other is the largest. Since 126 is divisible by 2 and 3 and 7, these three primes need not be considered. Trying the other primes in ascending order, we obtain $126 = 5 + 11^2$ and $126 = 11 + 5 \times 23$ and $126 = 13 + 113$. Both 13 and 113 are prime, and $113 - 13 = 100$.

9. **Answer 017**
   Let the scores of the four candidates be P, N, G, F respectively. We are given $P + N + G + F = 4 \times 16 = 64$, and similarly $P + N = 32$, $P + F = 26$, and $N + F = 36$. Adding the last three equations gives $2(P + N + F) = 94$, so $P + N + F = 47$ and $G = 64 - 47 = 17$.

10. **Answer 090**
   The two successive primes must be close to $\sqrt{2021}$, which lies between 40 and 50. To obtain the last digit 1, as a product of two distinct numbers, the two primes must end in 3 and 7. The only possibilities are 43 and 47, and indeed $43 \times 47 = 2021$, and $43 + 47 = 90$.
   Alternatively, note that $2021 = 2025 - 4 = 45^2 - 2^2 = 43 \times 47$.

11. **Answer 077**
   The longest side of the triangle must be less than the sum of the other two sides (Triangle Inequality). We know that $3^2 + 4^2 = 5^2$ (Pythagorean triple), so the first possibility is $4^2, 5^2, 6^2$, where $4^2 + 5^2 = 41 > 6^2$ and $4^2 + 5^2 + 6^2 = 77$.
   Alternatively, the inequality $(n - 1)^2 + n^2 > (n + 1)^2$ simplifies to $n(n - 4) > 0$, so $n > 4$, since we know $n > 0$.

12. **Answer 200**
   Water starts flowing into the two containers at the same time. The time t is the same for both P and Q. So, for P we have $t = \frac{v-60}{4}$ and for Q we have $t = \frac{v+10}{6}$. From this we get $v = 200$ litres.

13. **Answer 009**
   The sum of the numbers in the 3×3 square shown is $1 + 2 + \cdots + 8 + 9 = 45$, so $20 + 16 + X = 45$, giving $X = 9$.

14. **Answer 016**
   The important fact is that the man never travels by train twice on the same day. Suppose he travels x times by train in the morning. We can draw up a contingency table:

pm\am	Bus	Train	Total
Bus	$15 - x$	x	15
Train	$9 - x$	0	$9 - x$
Total	8	x	

   It follows from the first column that $(15 - x) + (9 - x) = 8$, so $x = 8$ and the total number of days worked is 16.
   Alternatively, one can use a Venn diagram, using the two events "Bus morning" and "Bus afternoon", and observing that the intersection of their complements is empty.

15. **Answer 006**
   Suppose the committee has n members. In each meeting of 3 members there are three pairs, each of which can appear at most once in the four meetings. Thus the number of pairs of committee members must be at least $4 \times 3 = 12$, that is, $\frac{1}{2}n(n - 1) \geq 12$, which simplifies to $(2n - 1)^2 \geq 97$. Thus $2n - 1 \geq 11$, giving $n \geq 6$.
   With committee members A, B, C, D, E, F, a possible solution for the four meetings is $\{A, B, C\}$, $\{A, D, E\}$, $\{B, D, F\}$, and $\{C, E, F\}$, which can also be found by trial and error.

16. **Answer 020**
Each exterior angle of a regular nonagon is equal to $\frac{1}{9} \times 360° = 40°$. A line through the top vertex parallel to the bottom side bisects the exterior angle at the top vertex. The required angle is equal to half the exterior angle (alternate angles between parallel lines), so is equal to $\frac{1}{2} \times 40° = 20°$.

17. **Answer 512**
By ignoring the first digit in 1000, and inserting zeros for blanks in the numbers from 1 to 99, the 1000 ticket numbers can be regarded as the 10^3 arrangements (with repetition) of the ten digits $\{0, 1, \ldots, 9\}$, or as the 10^3 outcomes of three selections (with replacement) of one digit at a time. The tickets not using 7 or 9 involve only the remaining eight digits, so the number of these tickets is $8^3 = 512$, which is also 1000 times the probability of drawing one of these tickets first.

18. **Answer 392**
In the left-hand figure, the two smaller triangles are each half the area of the square, so the area of triangle ABC is twice the area of the square, that is, $2 \times 441 = 882$. In the second figure, if the square has area S, then the two congruent triangles each have area $\frac{1}{2}S$ and the smallest triangle has area $\frac{1}{4}S$. It follows that $882 = \frac{9}{4}S$, so $S = \frac{4}{9} \times 882 = 392$.

19. **Answer 003**
There are $\frac{1}{2}(8 \times 7) = 28$ matches, with one point awarded per match, for a total of 28 points. If the participants' scores, in descending order, are $P_1, P_2, \ldots, P_8$, then we are given that $P_2 = P_5 + P_6 + P_7 + P_8$ and $P_8 = \frac{1}{2}$. For P_5 to be a maximum, the lowest scores must be as low as possible, so $P_7 = 1$ and $P_6 = 1\frac{1}{2}$, giving $P_2 = P_5 + 3$. Similarly, P_3 and P_4 must be as low as possible, so $P_3 = P_5 + 1$ and $P_4 = P_5 + \frac{1}{2}$. We now have

$$\begin{aligned} 28 &= P_1 + P_2 + P_3 + P_4 + P_5 + (P_6 + P_7 + P_8) \\ &= P_1 + (P_5 + 3) + (P_5 + 1) + (P_5 + \tfrac{1}{2}) + P_5 + 1\tfrac{1}{2} + 1 + \tfrac{1}{2} \qquad (1) \\ &= P_1 + 4P_5 + 7\tfrac{1}{2}, \end{aligned}$$

so $P_1 + 4P_5 = 20\frac{1}{2}$. Also $P_1 > P_2 = P_5 + 3$, so $5P_5 + 3 < 20\frac{1}{2}$. This gives $P_5 < 17\frac{1}{2}/5 = 3\frac{1}{2}$, so the maximum value of P_5 is 3.

20. **Answer 126**

For cases 1, 2, 3 and 4 the partial sums are $\frac{1}{1}$; $\frac{4}{3}$; $\frac{9}{6}$; $\frac{16}{10}$, etc. The numerators, $1, 4, 9, 16 \cdots$, are square numbers, which in general are given by the formula n^2. The denominators are triangular numbers, $1, 3, 6, 10, \cdots$, which in general are given by the formula $\frac{n(n+1)}{2}$. Hence, combining the two formulae, we get the formula for the given series as $\frac{2n^2}{n(n+1)} = \frac{2n}{n+1}$. So, $66 \times 2 \times \frac{21}{22} = 126$.

Alternatively, For any natural number n, it can easily be shown that $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ and that the reciprocal

$$\frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1}$$

(use partial fractions on the left-hand side, or bring the right-hand side to a common denominator). The expression therefore becomes

$$66\Big(\frac{2}{1} - \frac{2}{2} + \frac{2}{2} - \frac{2}{3} + \frac{2}{3} - \frac{2}{4} + \cdots + \frac{2}{21} - \frac{2}{22}\Big),$$

in which all terms inside the bracket disappear, except for the first and the last, to give

$$66\Big(\frac{2}{1} - \frac{2}{22}\Big) = 66 \times \frac{21}{11} = 126.$$

21. **Answer 014**
Drop perpendiculars from the centre A of the circle to the legs of the right-angled triangle. If the circle has radius r, then it follows by similar triangles that

$$\frac{r}{63-r}=\frac{18}{63} \text{ and } \frac{r}{18-r}=\frac{63}{18},$$

either of which gives

$$r=\frac{18\times 63}{18+63}=\frac{9\times 2\times 9\times 7}{9\times 9}=14.$$

22. **Answer 007**
Summarise the information in a table:

	A	B	New mixture
concentration	20%	45%	30%
litres	10	x	$10+x$
calculation	$20\% \times 10$ $=2$	$45\% \times x$ $=\frac{9}{20}x$	$30\% \times (10+x)$ $=\frac{30+3x}{10}$

From the equation $2+\frac{9}{20}x=3+\frac{3}{10}x$ it follows that $x=\frac{20}{3}$. Rounded to the nearest integer, the answer is 7.

23. **Answer 192**
Draw the line through the centres of the circles, which bisects the angle between the lines L_1 and L_2, and drop perpendiculars from the centres to both lines. If the radii of the circles are r_1, r_2, r_3, r_4, r_5 then by similar triangles it follows that

$$\frac{r_{n+1}-r_n}{r_{n+1}+r_n}=k, \text{ say (a constant)},$$

for $n=1,2,3,4$. It follows that the ratio

$$\frac{r_{n+1}}{r_n}=\frac{1+k}{1-k}=m, \text{ say},$$

so the radii form a geometric sequence with common ratio m. Therefore $r_5=r_1m^4$ and

$$r_3=r_1m^2=\sqrt{r_1r_5}=\sqrt{288\times 128}=\sqrt{2^5\times 3^2\times 2^7}=2^6\times 3=192.$$

24. **Answer 270**
Join DE, and let G be the intersection of AE and DF. Then triangles ADG and EFG have equal area, since the first is triangle ABE minus quadrilateral $DBEG$, and the second is quadrilateral $DBEF$ minus the same quadrilateral $DBEG$. It follows, by adjoining triangle DEG to triangles ADG and EFG respectively, that triangles DEA and DEF have equal area. These two triangles also have the same base DE, which implies that they have equal heights, and therefore that $DE||AF$ (or AC). Thus $\triangle DBE|||\triangle ABC$, and since $\frac{DB}{AB}=\frac{3}{5}$, it follows that the area of triangle DBE equals $(\frac{3}{5})^2\times 450=162$. Finally, since $\frac{AB}{DB}=\frac{5}{3}$, it follows that $\triangle ABE=\frac{5}{3}\triangle DBE=\frac{5}{3}\times 162=270$.

25. **Answer 250**

Since $10 = 2 \times 5$, we need to find the lesser of the exponents of 2 and 5 in the prime factorization of 1005!. It is sufficient to find the exponent of 5, since the exponent of 2 is clearly greater. In multiplying out 1005!, each multiple of 5 contributes one factor 5 to the product, to begin with. Furthermore, each multiple of $5^2 = 25$ contributes one extra factor 5, then each multiple of $5^3 = 125$ contributes one more factor 5, and each multiple of $5^4 = 625$ contributes a final factor 5. The process stops there, because $5^5 > 1005$. The number of multiples of a natural number d up to any natural number k is equal to the quotient after dividing k by d and ignoring the remainder. This quotient is often denoted $\lfloor \frac{k}{d} \rfloor$. Thus the exponent of 5 in 1005! is equal to

$$\lfloor \frac{1005}{5} \rfloor + \lfloor \frac{1005}{25} \rfloor + \lfloor \frac{1005}{125} \rfloor + \lfloor \frac{1005}{625} \rfloor = 201 + 40 + 8 + 1 = 250.$$