The South African Mathematical Olympiad
Third Round 2019
Senior Division (Grades 10 to 12)
Solutions

1. Determine all positive integers a for which a^a is divisible by 20^{19}.

   Solution: If $20^{19} \mid a^a$, then $10 \mid a$, hence $a \in \{10, 20, 30, \ldots\}$. Clearly, $a = 10$ is not possible (10^{10} is smaller than 20^{19}), while $a = 20$ is possible ($20^{20} = 20 \cdot 20^{19}$). Also, all $a = 10k,\ k \geqslant 4$, are possible (for $q = 5^{10k-19} \cdot 2^{10k-38} \cdot k^{10k}$ we have $(10k)^{10k} = q \cdot 20^{19}$). What about $a = 30$? If $30^{30} = q \cdot 20^{19}$ for some $q \in \mathbb{Z}$, then $3^{30} \cdot 5^{11} = q \cdot 2^8$, with the left-hand side odd and the right-hand side even, an impossibility. Thus $a = 30$ is not possible. The solution is therefore all $a \in \{20\} \cup \{10k : k \geqslant 4\}$.

2. We have a deck of 90 cards that are numbered from 10 to 99 (all two-digit numbers). How many sets of three or more different cards in this deck are there such that the number on one of them is the sum of the other numbers, and those other numbers are consecutive?

   Solution: We are looking at sums of the form

   $$s(10+\ell, k) = \sum_{i=0}^{k-1}(10+\ell+i) = k(10+\ell+\tfrac{1}{2}(k-1)),$$

   where $\ell \geqslant 0$ ($10+\ell$ represents the starting number of the consecutive numbers in a set) and $k \geqslant 2$ represents the number of consecutive numbers in a set. Each of these sums (the $(k+1)$-th number in the set) must be less than, or equal to, 99. By taking $\ell = 0$, we see that the maximum possible value for k is 7.

   For the case $k = 2$ (i.e., where three cards are used), we must have $2 \cdot (10+\ell+\frac{1}{2}(2-1)) \leqslant 99$, so that $\ell \leqslant 39$. The $39+1 = 40$ possibilities of three cards are: $10+11 = 21, 11+12 = 23, \ldots, 49+50 = 99$.

   For the case $k = 3$ (i.e., where four cards are used), we get $3 \cdot (10+\ell+\frac{1}{2}(3-1)) \leqslant 99$, so that $\ell \leqslant 22$, representing the $22+1 = 23$ groups of four cards: $10+11+12 = 33, \ldots, 32+33+34 = 99$.

   Continuing in this way, we find that $\ell \leqslant 13$ when $k = 4$, $\ell \leqslant 7$ when $k = 5$, $\ell \leqslant 4$ when $k = 6$, and $\ell \leqslant 1$ when $k = 7$. All in all, there are $40+23+14+8+5+2 = 92$ sets of cards satisfying the conditions.

3. Let A, B, C be points on a circle whose centre is O and whose radius is 1, such that $\angle BAC = 45°$. Lines AC and BO (possibly extended) intersect at D, and lines AB and CO (possibly extended) intersect at E. Prove that $BD \cdot CE = 2$.

   Solution 1: Consider the figure:

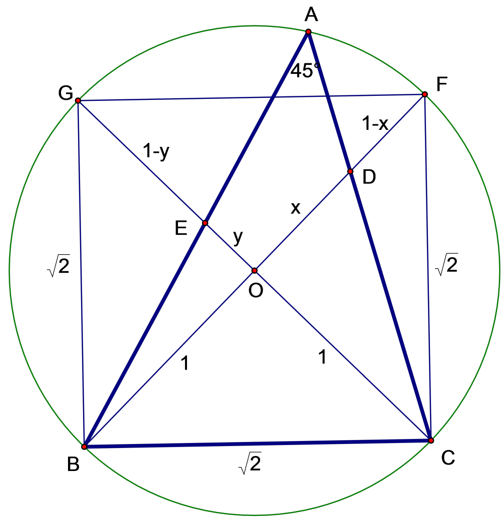

Figure 1

Put $x = OD$ and $y = OE$. Let BO extended meet the circle again in F (so that $DF = 1-x$). Since $\angle BFC = 45^\circ$ and BF is a diameter of the circle, we have $\angle BCF = 90^\circ$ and $BC = CF = \sqrt{2}$. Similarly, if CE extended meets the circle again in G, we have $GE = 1-y$, and CG and BF are the diagonals of square $BCFG$ that intersect perpendicularly at O. Therefore, from the right triangles BOE and COD, we have $CD = \sqrt{1+x^2}$ and $BE = \sqrt{1+y^2}$.

By the similarity of triangles BDA and CDF, we have $\frac{AD}{DF} = \frac{BD}{CD}$, i.e., $\frac{AD}{1-x} = \frac{1+x}{\sqrt{1+x^2}}$, so that $AD = \frac{1-x^2}{\sqrt{1+x^2}}$.

Then, by the similarity of triangles BEG and CEA, $\frac{AD+DC}{GB} = \frac{CE}{BE}$, i.e.,

$$\frac{\frac{1-x^2}{\sqrt{1+x^2}} + \sqrt{1+x^2}}{\sqrt{2}} = \frac{1+y}{\sqrt{1+y^2}}.$$

Simplifying and solving for x gives $x = \frac{1-y}{1+y}$, so that

$$BD \cdot CE = (1+x)(1+y) = \left(1 + \frac{1-y}{1+y}\right)(1+y) = 2.$$

Note that we have solved here the case where A lies strictly between G and F on the short arc $\overset{\frown}{FG}$. The case where $A = F$ or $A = G$ follows trivially, because here $BD \cdot CE$ reduces to either $BF \cdot CO = 2 \cdot 1 = 2$ or $CG \cdot BO = 2$. The case where A lies between C and F on the short arc $\overset{\frown}{CF}$ is shown in Figure 2.

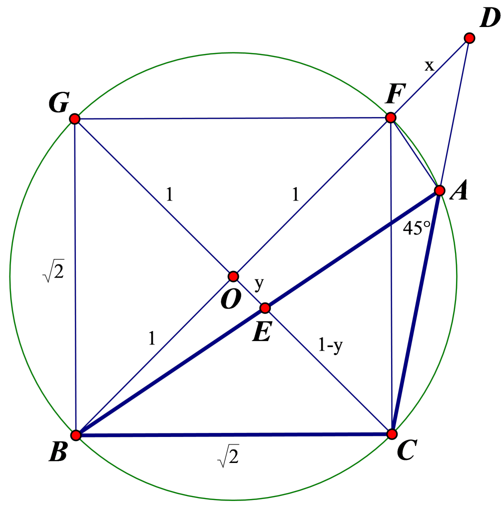

Figure 2

In a way similar to the first case, using the similarity between triangles BDA and CDF, we see that $\frac{CD}{2+x} = \frac{\sqrt{1+(1+x)^2}}{2+x} = \frac{x}{AD}$, so that $AD = \frac{x(2+x)}{\sqrt{1+(1+x)^2}}$. The similarity between triangles BEG and CEA gives

$$\frac{1-y}{\sqrt{1+y^2}} = \frac{CA}{\sqrt{2}} = \frac{CD-AD}{\sqrt{2}} = \frac{\sqrt{1+(1+x)^2} - \frac{x(2+x)}{\sqrt{1+(1+x)^2}}}{\sqrt{2}}.$$

From this equation we solve $x = \frac{2y}{1-y}$ and it follows that $BD \cdot CE = (2+x)(1-y) = \left(2 + \frac{2y}{1-y}\right)(1-y) = 2.$

Finally, the case where A lies between B and G on the short arc $\overset{\frown}{BG}$ follows symmetrically from the above.

Solution 2: We can also follow a trigonometric approach: We first consider the case where O is inside triangle ABC. See Figure 3.

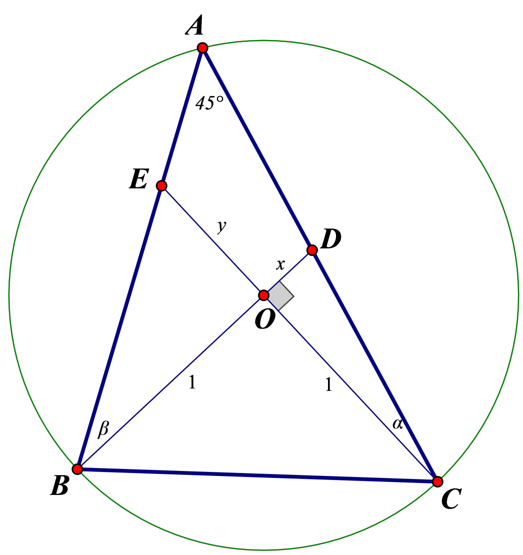

Figure 3

We have $\angle BOC = 90°$ (twice $\angle BAC = 45°$). Put $x = OD, y = OE, \alpha = \angle ACE, \beta = \angle ABD$. Then $\tan\alpha = x$ and $\tan\beta = y$. Furthermore, $\alpha + \beta = 45°$ (using $180° = \angle BAD + \angle ABD + \angle BDA = 45° + \beta + (90° + \alpha)$), so that

$$1 = \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{x + y}{1 - xy}.$$

Hence, $x + y = 1 - xy$, and we have $BD \cdot CE = (1 + x)(1 + y) = 1 + x + y + xy = 1 + 1 = 2$.

In Figure 4 we have O outside triangle ABC.

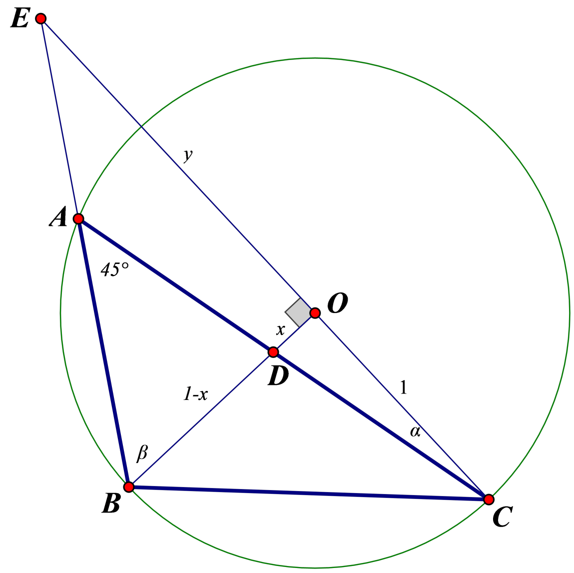

Figure 4

In this case we have $\beta - \alpha = 45°$ (using $180° = \angle BAC + \angle ABC + \angle BCA = 45° + (\beta + 45°) + (45° - \alpha)$), so that

$$1 = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta} = \frac{y - x}{1 + xy}.$$

Hence, $y - x = 1 + xy$, and we have $BD \cdot CE = (1 - x)(1 + y) = 1 - x + y - xy = 1 + 1 = 2$.

Finally, there are the cases where O is either on AB or on AC. In the former case we have $D = A$ and $E = O$, so that $BD \cdot CE = BA \cdot CO = 2 \cdot 1 = 2$, and in the latter case, $D = O$ and $E = A$, so that $BD \cdot CE = BO \cdot CA = 1 \cdot 2 = 2$.

4. The squares of an 8×8 board are coloured alternatingly black and white. A rectangle consisting of some of the squares of the board is called *important* if its sides are parallel to the sides of the board and all its corner squares are coloured black. The side lengths can be anything from 1 to 8 squares. On each of the 64 squares of the board, we write the number of important rectangles in which it is contained. The sum of the numbers on black squares is B, and the sum of the numbers on white squares is W. Determine the difference $B - W$.

   Solution: In each important rectangle, the number of black squares is one more than the number of white squares. Hence, each important rectangle contributes $+1$ to the difference

$B-W$. The value of $B-W$ is thus the same is the number of important rectangles on the board.

Let us number the rows on the board $1,2,\ldots,8$ from the top downwards and the columns $1,2,\ldots,8$ from the left to the right. So $(1,1)$ is the upper left square and $(8,8)$ denotes the lower right square. Assume $(1,1)$ is a black square. Then all (i,j) with both i and j odd, as well as all those with both i and j even, are black squares. All other squares are white. By focusing only on the four odd-numbered rows and the four odd-numbered columns, we find that they determine $(4+\binom{4}{2})^2=100$ important rectangles. Similarly, the four even-numbered rows and the four even-numbered columns determine another 100 important rectangles, giving a total of 200 important rectangles on the board. It follows that $B-W=200$.

5. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that

$$f(a^3)+f(b^3)+f(c^3)+3f(a+b)f(b+c)f(c+a)=(f(a+b+c))^3$$

for all $a,b,c\in\mathbb{Z}$.

Solution: Suppose f satisfies the condition.

By taking $(a,b,c)=(0,0,0)$, we get $3f(0)+3f(0)^3=f(0)^3$, so that either $f(0)=0$, or $3=-2f(0)^2$. The latter is not possible in $\mathbb{Z}$, so we must have $f(0)=0$.

By taking $(a,b,c)=(n,-n,0)$, we get $f(n^3)+f(-n^3)=0$, resulting in

$$f(-n^3)=-f(n^3) \text{ for all } n\in\mathbb{Z}. \tag{1}$$

By taking $(a,b,c)=(n,0,0)$, we get

$$f(n^3)=f(n)^3 \text{ for all } n\in\mathbb{Z}. \tag{2}$$

By combining (1) and (2), we see that, for any $n\in\mathbb{Z}$,

$$f(-n)^3=f((-n)^3)=f(-n^3)=-f(n^3)=-f(n)^3=(-f(n))^3,$$

so that $f(-n)=-f(n)$, i.e., f is an odd function.

Now take $(a,b,c)=(k,1-k,0)$, so that

$$f(k)^3+f(1-k)^3+3f(k)f(1-k)f(1)=f(1)^3 \text{ for all } k\in\mathbb{Z}. \tag{3}$$

Also, from (2), $f(1)=f(1)^3$, so that $f(1)\in\{-1,0,1\}$.

First, if $f(1)=0$, then from (3), $f(k)=-f(1-k)=f(k-1)$ for all $k\in\mathbb{Z}$, so that $f(n)=0$ for all $n\in\mathbb{Z}$ (using induction and the fact that f is odd).

Second, if $f(1)=1$, then from (3), $f(k)^3+f(1-k)^3+3f(k)f(1-k)=1$, for all $k\in\mathbb{Z}$, i.e., the Diophantine equation $X^3+Y^3+3XY=1$ is satisfied by $(X,Y)=(f(k),f(1-k))$. This equation can be rewritten as $(X+Y-1)(X^2-XY+Y^2+X+Y+1)=0$. Note that $X^2-XY+Y^2+X+Y+1=0$ is only solvable in $\mathbb{Z}$ if $X=Y=-1$ (otherwise the discriminant is negative when considered as a quadratic in X). So there are two options here:

1. $f(k) = 1 - f(1-k) = 1 + f(k-1)$ for all $k \in \mathbb{Z}$. Then it follows immediately by induction and the fact that f is odd that $f(n) = n$ for all $n \in \mathbb{Z}$.

2. There is some $k_0 \in \mathbb{Z}$ such that $f(k_0) = f(1-k_0) = -1$. Then $f(-k_0) = 1 = 1 - f(1+k_0)$ implies that $f(1+k_0) = 0$. Hence, by using $(a,b,c) = (1+k_0, 1-k_0, -1)$ in the functional equation, we get that $0 - 1 - 1 + 3f(2)(-1)(1) = 1$, so that $f(2) = -1$. Thus $k_0 = 2$ is the smallest positive value of k_0 with the property that $f(k_0) = -1 = f(1-k_0)$. We have therefore established the base case of the proof by induction that $(f(3k), f(3k+1), f(3k+2)) = (0, 1, -1)$ for all $k \geqslant 0$. Assume now that this statement is true for some $k \geqslant 0$. Then, from $f(-3k-2) = 1 = 1 - f(1+3k+2)$, we find that $f(3+3k) = f(3(k+1)) = 0$. Using $(a,b,c) = (3k, 1, 3)$ in the functional equation (and recalling that $f(3) = f(1+2) = f(1+k_0) = 0$), we get $0 + 1 + 0 + 3(0)(1)(0) = f(3k+4)^3$, so that $f(3k+4) = f(3(k+1)+1) = 1$. Similarly, with $(a,b,c) = (3k, 2, 3)$, we see that $f(3k+5) = f(3(k+1)+2) = -1$, and the induction is complete. It follows now from the fact that f is odd that $(f(3k), f(3k+1), f(3k+2)) = (0, 1, -1)$ for all $k \in \mathbb{Z}$, i.e.,

$$f(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \\ 1 & \text{if } n \equiv 1 \pmod 3 \\ -1 & \text{if } n \equiv 2 \pmod 3 \end{cases}, \quad \text{for all } n \in \mathbb{Z}. \tag{4}$$

Finally, the observation that whenever f satisfies the equation then $-f$ also satisfies the equation, takes care of the case $f(1) = -1$.

It is straightforward to check that the three functions $f(n) = 0$, $f(n) = n$ and $f(n) = -n$ (for all $n \in \mathbb{Z}$) satisfy the given functional equation. The function (4) requires a little more effort (and time) to check, but offers no difficulty. Its negative then follows automatically. We conclude that there are five functions that solve the equation, as described above.

6. Determine all pairs (m,n) of non-negative integers that satisfy the equation

$$20^m - 10m^2 + 1 = 19^n.$$

Solution: Let the pair (m,n) satisfy the equation. $m = 0$ implies that $2 = 19^n$, an impossibility. So we must have $m > 0$. Taking both sides modulo 10 gives $1 \equiv (-1)^n \pmod{10}$, implying that n must be even. Taking both sides modulo 20 gives $-10m^2 + 1 \equiv (-1)^n = 1 \pmod{20}$, so that $2 \mid m^2$, whence m is even too. So let us put $m = 2k$ and $n = 2l$ for a positive integer k and a non-negative integer l. The original equation can now be written as

$$10m^2 - 1 = 20^{2k} - 19^{2l} = (20^k - 19^l)(20^k + 19^l).$$

As both $20^k + 19^l$ and $10m^2 - 1$ are positive (recall that $m \geqslant 1$), we must have that the integer $20^k - 19^l \geqslant 1$. It follows that

$$10m^2 - 1 = (20^k - 19^l)(20^k + 19^l) \geqslant 20^k + 19^l \geqslant 20^k + 1,$$

and we see that $20^k \leqslant 10(2k)^2 - 2 = 40k^2 - 2$. This inequality is not true for $k = 2$, and it can be easily verified by induction that it also fails to hold for any $k > 2$. The inequality holds for $k = 1$, and it follows (and is easily checked) that the only pair that solves the original equation, is $(m,n) = (2,2)$.