The South African Mathematical Olympiad
Third Round 2015
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Points E and F lie inside a square $ABCD$ such that the two triangles ABF and BCE are equilateral. Show that DEF is an equilateral triangle.

2. Determine all pairs of real numbers a and x that satisfy the simultaneous equations
$$5x^3 + ax^2 + 8 = 0$$
and
$$5x^3 + 8x^2 + a = 0.$$

3. We call a divisor d of a positive integer n *special* if $d+1$ is also a divisor of n. Prove: at most half the positive divisors of a positive integer can be special. Determine all positive integers for which exactly half the positive divisors are special.

4. Let ABC be an acute-angled triangle with $AB < AC$, and let points D and E be chosen on the sides AC and BC respectively in such a way that $AD = AE = AB$. The circumcircle of ABE intersects the line AC at A and F and the line DE at E and P. Prove that P is the circumcentre of BDF.

5. Several small villages are situated on the banks of a straight river. On one side, there are 20 villages in a row, and on the other there are 15 villages in a row. We would like to build bridges, each of which connects a village on the one side with a village on the other side. The bridges must be straight, must not cross, and it should be possible to get from any village to any other village using only those bridges (and not any roads that might exist between villages on the same side of the river). How many different ways are there to build the bridges?

6. Suppose that a is an integer and that $n! + a$ divides $(2n)!$ for infinitely many positive integers n. Prove that $a = 0$.

Each problem is worth 7 points.