THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

organised by the SOUTH AFRICAN ACADEMY OF SCIENCE AND ARTS
in collaboration with OLD MUTUAL, AMESA and SAMS

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SECOND ROUND 2000

SENIOR SECTION: GRADES 10, 11 AND 12
(STANDARDS 8, 9 AND 10)

6 JUNE 2000

TIME: 120 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. C
2. A
3. C
4. E
5. B
6. B
7. E
8. B
9. A
10. B
11. D
12. C
13. E
14. B
15. D
16. A
17. A
18. E
19. B
20. D

SOLUTIONS

1. Answer C. $\sqrt{24} \approx \sqrt{25} = 5$, so

$$\frac{3}{\sqrt{\sqrt{24}-1}} \approx \frac{3}{\sqrt{5-1}} = \frac{3}{\sqrt{4}} = \frac{3}{2} = 1,5.$$

Answer: C

2. Answer A. If the sides of the square have length s, then $2s^2 = d^2$ by Pythagoras, and the area of the square $s^2 = \frac{1}{2}d^2$.

Answer: A

3. Answer C. $\sqrt{2000^{2000}} = (2000^{2000})^{1/2} = 2000^{(2000\times 1/2)} = 2000^{(1/2\times 2000)} = (2000^{1/2})^{2000} = (\sqrt{2000})^{2000} = (20\sqrt{5})^{2000}$.

Answer: C

4. Answer E. The cube roots of (A) to (E) are 7 (approximately), 33, $3^3 = 27$, 3^9, and 3^{11}, and it is clear that 3^{11} is the greatest.

Answer: E

5. Answer B. The distance from one vertex of the cube (say top left front) to the diametrically opposite vertex (say bottom right back) must be $9\sqrt{3}$ cm. If the sides of the cube have length s cm and the diagonal of one square face has length d cm, then by Pythagoras' theorem $2s^2 = d^2$ and $d^2 + s^2 = (9\sqrt{3})^2$, so $3s^2 = 243$, giving $s^2 = 81$ and $s = 9$. The volume of the cube in cm^3 is $s^3 = 9^3 = 729$.

Answer: B

6. Answer B. If the right hand side is brought to the common denominator $x^2 - 1$, then the numerator is

$$A(x+1) + B(x-1) = (A+B)x + (A-B).$$

Since this is equal to $3x - 5$ for all values of x, it follows that $A + B = 3$ and $A - B = -5$. Then either solve to get $A = -1$ and $B = 4$, or observe that $A^2 + B^2 = \frac{1}{2}((A+B)^2 + (A-B)^2)$.

Answer: B

7. Answer E. $DE = AD = \sqrt{2}$ (radii of the same circle), and by Pythagoras' theorem so $CE^2 = DE^2 - DC^2 = (\sqrt{2})^2 - 1^2 = 2 - 1 = 1$.

Answer: E

8. Answer B. The sequence repeats in blocks of length 9, so every letter is the same as the letter nine places further along. When 2000 is divided by 9, the remainder is 2, so the 2000th letter is the same as the second letter.

Answer: B

9. Answer A. (Remember that 1 is not regarded as a prime number, and that the only even prime number is 2, which is also the smallest prime number.) If a prime p is written as the sum of two primes q and r, then p cannot be the smallest prime, so p must be odd. Therefore either q or r must be even, and the other must be odd. This means that either q or r is equal to 2, and the other must be equal to $p - 2$. It follows that there is at most one possible way to express a prime number as the sum of two other prime numbers. (Prime numbers of the form $p-2$ and p are called twin primes: it is unknown whether or not there are infinitely many twin primes.)

Answer: A

10. Answer B. The sum of the interior angles in a polygon with n sides is $(180n - 360)^\circ$, and a hexagon has six sides, so the angle sum is 720°. For a convex polygon, each interior angle must be less than 180°. If there are m acute angles, then their sum is less than $90m^\circ$. The sum of the remaining $6 - m$ angles is then greater than $(720 - 90m)^\circ$, but is also less than $180(6 - m)^\circ$. Thus $720 - 90m < 180(6 - m)$, giving $m < 4$.

Answer: B

11. Answer D. The inequalities give $5(n + 16) < 9n$ and $7n < 4(n + 16)$, that is, $80 < 4n$ and $3n < 64$, so $20 < n < 21\frac{1}{3}$.

Answer: D

12. Answer C. By experiment (or by modular arithmetic), the sequence of remainders when n^2 is divided by 7 starts $1, 4, 2, 2, 4, 1, 0$ and then repeats in blocks of length 7. The distinct values are $1, 4, 2, 0$.

Answer: C

13. Answer E. The simplest method is to try out each of the given remainders to see which of them gives the right remainders after division by $x - 1$ and $x - 2$. More rigorously, after division by $(x - 1)(x - 2)$ the remainder must be of the form $ax + b$, so if the quotient is $q(x)$, then the polynomial is $(x - 1)(x - 2)q(x) + ax + b$. This can be written as $(x - 1)\{(x - 2)q(x) + a\} + a + b$, or as $(x - 2)\{(x - 1)q(x) + a\} + 2a + b$, so $a + b = 2$ and $2a + b = 1$, giving $a = -1$ and $b = 3$. (The remainders can also be found by the Remainder Theorem.)

Answer: E

14. Answer B. If there are s sheep and c chickens, then the number of legs is $4s + 2c$ and the number of animals is $s + c$. Thus $\ell = (4s + 2c)/(s + c)$, giving $(4 - \ell)s = (\ell - 2)c$, from which s/c can be found.

Answer: B

15. Answer D. The first 9 pages need 1 digit each, the next 90 pages need two digits each, the next 900 pages need three digits each, and so on. The total number of digits required is thus of the form k up to 9, then $9 + 2k$ up to 189, then $189 + 3k$ up to 2889. More simply, the total number of digits can be any number up to 9, then any odd number up to 189, then any multiple of 3 up to 2889.

Answer: D

16. Answer A. Join AE and ED. Then $C\widehat{E}D = \theta$ (isosceles triangle), $E\widehat{D}A = 2\theta$ (exterior angle), $D\widehat{A}E = 2\theta$ (isosceles triangle), and $E\widehat{A}B = 63^\circ$ (isosceles triangle). From the angle sum of triangle ABC we then have $126^\circ + 3\theta = 180^\circ$.

Answer: A

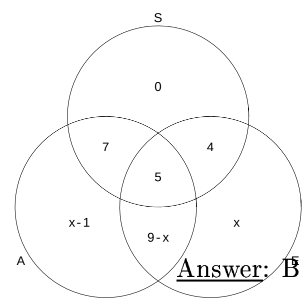
17. Answer A. Point C moves on a circle, since point B and length BC are fixed. If M denotes the intersection of the diagonals, then M is also the midpoint between A and C. Let O be the midpoint between A and B; then by the midpoint theorem $OM \parallel BC$ and $|OM| = \frac{1}{2}|BC|$. Since point O is fixed, this means that M moves on a circle with centre at point O and radius equal to $\frac{1}{2}|BC|$.

Answer: A

18. Answer E. The equation gives $mn - 15(m+n) = 0$, and by adding 15^2 to both sides (a useful trick to remember) we obtain $(m-15)(n-15) = 225$. Thus $m - 15$ can be any one of the divisors of 225, and $n - 15 = 225/(m-15)$. There are nine divisors of 225, namely $1, 3, 5, 9, 15, 25, 45, 75, 225$, and therefore nine solutions.

Answer: E

19. Answer B.
In the Venn diagram above, the overlapping discs represent the sets of people speaking Sotho, Afrikaans, and English respectively. The numbers in each region can be entered in the following order from the information given. First 5, then 4 ($= 9 - 5$), then 7 ($= 12 - 5$), then 0. Now put in x for the unknown number required, then $9 - x$ ($= 18 - 5 - 4 - x$), then $x - 1 = (20 - 7 - 5 - (9 - x))$. Finally, the sum of all the numbers is 30, which gives $x = 6$.



Answer: B

20. Answer D. The equation is symmetrical in all variables except u, so it is easiest to consider the possible values of u separately.

$u = 0$. The sum of the other five variables is 3, so there there are three cases. (a) One variable equal to 3: there are 5 possible solutions. (b) One variable equal to 2 and another equal to 1: there are $5 \times 4 = 20$ possible solutions. (c) Three different variables each equal to 1: there are $5 \times 4 \times 3 \div 6 = 10$ possible solutions. (We have to divide by 6 because the order in which the non-zero variables are chosen doesn't matter.)

$u = 1$. The sum of the other variables is 1, so one variable must be equal to 1, giving 5 possible solutions.

Thus the total number of solutions is $5 + 20 + 10 + 5 = 40$.

Answer: D