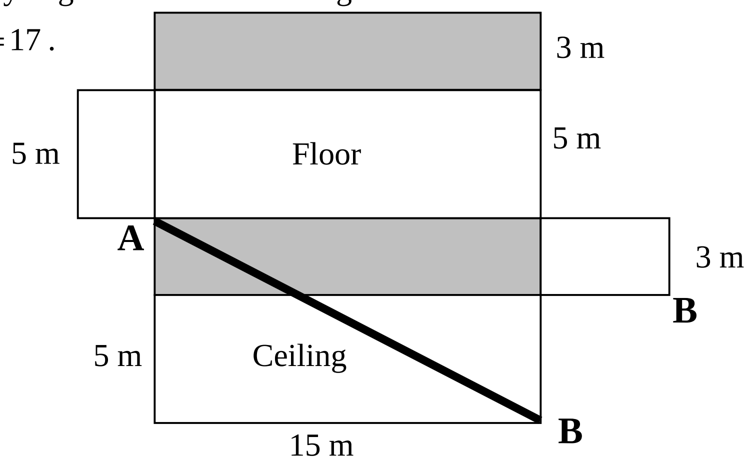
The South African Mathematics Olympiad
Third Round 2004
Junior Section: Grades 8 and 9
Solutions

Question 1

Rectangles A and B have the same area.
The diagonal PR divides the whole rectangle into two equal halves, and also divides each of the two unshaded rectangles in half. Thus (above PR) the area of B equals half of the difference between the whole rectangle and the two smaller ones, and (below PR) the area of A is the same.

Question 2

The shortest distance is 17 m.
Imagine the ceiling (15m by 5m) lifted up until it is parallel with the front wall (15m by 3m). Then the shortest distance from A to B is the diagonal of a rectangle with sides of 15m and (5+3)m, so by Pythagoras' theorem its length in metres is $\sqrt{15^2+8^2}=\sqrt{225+64}=\sqrt{289}=17$.



Question 3

The correct total is 1953, and the number Fred entered twice is 51.
The sum of the integers from 1 to p is $\frac{1}{2}p(p+1)$, and is called a triangular number.
By experiment, or otherwise, we find that the triangular numbers nearest to 2004 are 1953 (for p = 62) and 2016 (for p = 63). Since Fred included one number twice, his total of 2004 was too high, so the correct answer is 1953, and the number he added twice is the difference 2004 – 1953 = 51.

Question 4

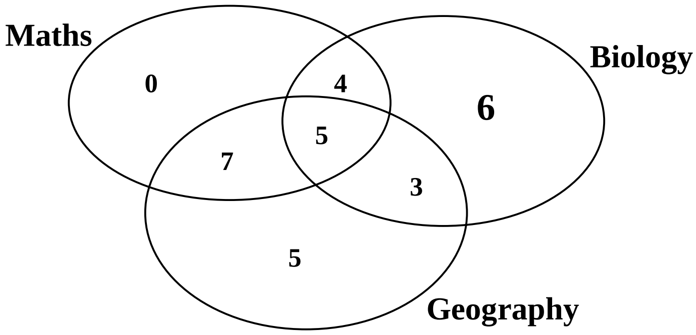
The numbers are equal for n = 8.
Every cube has 8 vertices, 12 edges, and 6 faces. The small cubes with three faces painted are at the vertices of the large cube, so there are eight of them. Those with two faces painted are on the edges of the large cube, but not at the vertices, so there are 12(n-2) of them. Those with one face painted are on the faces of the large cube, but not on the edges, so there are $6(n-2)^2$ of them. Finally, the rest of the small cubes, with no faces painted, are in the interior of the large cube, so there are $(n-2)^3$ of them. It follows from the information given that $(n-2)^3=6(n-2)^2$, so $n-2=0$ or $n-2=6$, giving n = 2 or n = 8. But if n = 2, then there are no small cubes except for the eight at the vertices, which contradicts the information given. The only solution is therefore n = 8.

Question 5

There are 6 learners taking only Biology.

The possible combinations of Maths (M), Geography (G), and Biology (B), since every learner takes at least one subject, are M, G, B, MG, MB, GB, MGB.
We are given that MGB = 5, and MB+MGB=9. (Note that those who take Maths and Biology include those who also take Geography.) Thus MB = 4. Next G+MG+GB+MGB=20 and MG+MGB=12, so G+GB=8 and MG=7. Next B+MB+GB+MGB=18 (which we do not need), and we are also given that M=0.
Finally, since the total number of learners is 30, we have

B = 30-M-(G+GB)-MG-MB-MGB = 30-0-8-7-4-5 = 6.



Question 6

		8	3	5
			1	2
	9	3	1	2
1	0	1	5	9

The only way that T... in the third row can give ID… in the total is by "carrying one", so T=9, I=1, and D=0. (A little extra checking shows that "carrying two" is impossible.) Also H+2=O or O+10, but H cannot be 9 (since T=9), so H+2=O, and there is no carrying to the next column. Thus W+H=11, and, assuming no other "carrying", also O+2S=9. Since O = H+2 = 9-2S, we see that O is odd, O>2, and O<7. If O=3, then S=3 also, which is not permitted, so O=5, which gives S=2 and H=3, so W=8.

Question 7
The area is $\pi - 2$.
The two sectors CAD and BAE cover the whole triangle ABC, and overlap in the shaded region ADE. Thus the required area is the sum of the two sectors minus the triangle. Since each sector has an angle of 45° at the centre, it is one-eighth of a circle of radius 2, so the sum of the areas of the two sectors is $2\times\frac{1}{8}\times\pi\times 2^2 = \pi$. The triangle has base and height of length 2, so its area is $\frac{1}{2}\times 2\times 2 = 2$. The answer is the difference between these values, that is, $\pi - 2$.

Question 8
The speeds are 135 km/h and 81 km/h or 37.5 m/s and 22.5 m/s.
Suppose the speeds are *x* km/h and *y* km/h. When the cars travel in opposite directions, their relative speed is *x*+*y* km/h, and when they travel in the same direction, their relative speed is *x*-*y* km/h. Then, using the formula
Speed = Distance/Time, we have (in units of kilometres and hours)

$$x + y = 1.8/(30/3600) = 1.8\times 120 = 216 \text{ and } x - y = 1.8/(2/60) = 1.8\times 30 = 54.$$

Add and subtract the equations and divide by 2 to get x = 135 and y = 81.

Question 9

Sakhile is the goalkeeper, Danny is the coach, Nelly is the captain, and Pravin is the manager.

Draw up a table of people and positions, then mark all the impossible combinations, as explained below, until there is only one blank cell in each row and column.

	Sakhile	Danny	Nelly	Pravin
Manager	(a)	(d)	(a)	
Coach	(a)		(e)	(c)
Captain	(b)	(b)		(d)
Goalkeeper		(b)	(b)	(b)

(a) From 1 - 3 we see that Sakhile is not the manager or the coach, and that Nelly is not the manager.

(b) From 4 - 6 we have that Danny is not the captain, neither Danny nor Pravin is the goalkeeper, and also Nelly is not the goalkeeper. This leaves only Sakhile to be the goalkeeper, so he is not the captain.

(c) From 8 we see that Pravin is not the coach.

(d) Suppose Danny is the manager: then he is not the coach, so only Nelly can be the coach, which contradicts 7 and 2. Therefore Danny is not the manager, and Pravin is the only other possibility for the manager, so he is not the captain.

(e) Now Nelly is the only one left to be captain, and Danny must be the coach.

Question 10

There are 24 numbers divisible by 3.

First note that the remainder when a number is divided by 3 is equal to the remainder when the sum of its digits is divided by 3, so it is sufficient to consider the sum of the digits. For two-digit numbers formed from 1, 2, and 4, we have the following table of remainders when the sum of the digits is divided by 3:

	1	**2**	**4**
1	2	0	2
2	0	1	0
4	2	0	2

We can summarize this in the following table for the nine two-digit numbers formed from 1, 2, and 4:

Remainder of sum of digits:	0	1	2
Number of possibilities:	4	1	4

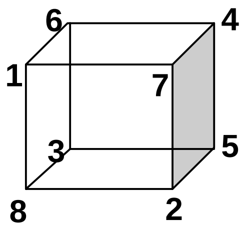
To make a four-digit number that is divisible by 3, we must combine two two-digit numbers so that the sum of their digits is divisible by 3. Reading off from the table above, we obtain the following possibilities:

Remainder of digit sum of first two-digit number	Remainder of digit sum of second 2-digit number	No of possibilities
0	0	$4 \times 4 = 16$
1	2	$1 \times 4 = 4$
2	1	$4 \times 1 = 4$

The total number of possibilities is 16 + 4 + 4 = 24 (out of 81 numbers overall).

Question 11

(a) This is one of many possible arrangements.



Any pair of parallel faces contains all eight vertices numbered from 1 to 8, whose total is 36. Thus the sum of the four vertices on each face must be 18. Now experiment to find one of the many possible arrangements. Using the result in (b) can save time. [3 marks]

(b) Let XY be one of the edges parallel to AB and CD. Then A+B+X+Y = C+D+X+Y, so A+B = C+D.

Question 12

There are 20 numbers in the list.

Arrange the numbers from 1 to 30 in a table, so that every number appears above its double, where possible:

1	3	5	7	9	11	13	15	17	19	21	23	25	27	29
2	6	10	14	18	22	26	30							
4	12	20	28											
8	24													
16														

We are not allowed to choose two successive numbers in any column, but to get the longest list we must choose one out of every such pair. One way is to choose all the numbers in the first, third, and fifth rows, giving a total of 15 + 4 + 1 = 20 numbers.

Question 13

The largest number is 25.

The sum of k consecutive numbers, starting with n, is $\frac{1}{2}k(2n+k-1)$, so we need to find the largest number k such that $k(2n+k-1)=2000$. Note that if k is even then $2n+k-1$ is odd, and *vice versa*. Furthermore, $2n+k-1>k$, since $n > 0$. By inspection, we obtain the following factorisations of 2000 as a product of an odd and an even factor, with the second factor greater than the first:

k	$2n+k-1$	n
5	400	198
16	125	55
25	80	28

From the table, we see that the greatest possible value of k is 25.

Question 14

(a) No.

There are 49 squares on the board, and each piece covers 3 squares. Since 3 is not a divisor of 49, the pieces cannot cover the board exactly.

(b) No.

There are 100 squares on the board, and each T-piece covers four squares, so we need 25 pieces to cover the board. If the squares are coloured alternately black and white, as on a chessboard, then each T-piece covers an odd number of black squares, no matter how it is placed on the board. Since the sum of an odd number of odd numbers is always odd (this is called the Principle of Parity), it follows that the 25 T-pieces will together always cover an odd number of black squares. However, there are 50 black squares on the board, so the T-pieces cannot cover the board completely.

Question 15

The smallest value is $10\frac{1}{2}$.

Since X is the hundreds digit and Y is the tens digit, the algebraic value of the number written XYZ is actually100X+10Y+Z. The expression to be minimised is therefore

$$\frac{100X+10Y+Z}{X+Y+Z}=10+\frac{90X-9Z}{X+Y+Z}=10+9\frac{10X-Z}{X+Y+Z},$$

so we must find the smallest value of the numerator 10X-Z and the largest value of the denominator X+Y+Z. The smallest value of the numerator occurs when X = 1 and Z = 9. The largest value of the denominator then occurs with Y = 8 (since Y cannot equal 9 as well), and the value of the whole expression is $10+9\times\frac{1}{18}=10\frac{1}{2}$.