THE HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

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FIRST ROUND 2004
SENIOR SECTION: GRADES 10, 11 AND 12
18 MARCH 2004
TIME: 60 MINUTES
NUMBER OF QUESTIONS: 20

ANSWERS

1. C
2. A
3. B
4. E
5. A
6. B
7. D
8. C
9. B
10. B
11. A
12. A
13. A
14. C
15. E
16. B
17. D
18. B
19. D
20. E

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SOLUTIONS

1. **Answer C.** $0.2^4 = (\frac{2}{10})^4 = \frac{2^4}{10^4} = 16 \times 10^{-4} = 16 \times 0.0001 = 0.0016$.

2. **Answer A.** The tallest skyscraper has over 100 storeys, and each storey requires 3–4 m of ceiling height, so the total height must be around 400 m. Even if you don't know that, it is clear that 4 m and 40 m are much too small, and 4 km and 40 km are much too large.

3. **Answer B.** $\frac{37^2+111}{37} = \frac{37 \times 37 + 37 \times 3}{37} = \frac{37(37+3)}{37} = 37 + 3 = 40$.

4. **Answer E.** $\frac{4444^4}{2222^4} = (\frac{4444}{2222})^4 = 2^4 = 16$.

5. **Answer A.** The small square has area 4 and therefore side length 2, which is two radii. The large square has side length four radii, which is 4, so its area is $4^2 = 16$.

6. **Answer B.** The expression equals $1 - \dfrac{1 - \frac{1/2}{3/2}}{1 + \frac{3/2}{1/2}} = 1 - \dfrac{1 - \frac{1}{3}}{1+3} = 1 - \dfrac{2/3}{4} = 1 - \dfrac{1}{6} = \dfrac{5}{6}$.

7. **Answer D.** We are given $17x + 51y = 85$. Now divide by 17 to get $x + 3y = 5$, and multiply by 13 to get $13x + 39y = 65$.

8. **Answer C.** Working from right to left we have $p + 0 = 1$, or $p = 1$. Then $n + p = 0$, giving $n + 1 = 0$, so $n = -1$. Next $m + n = p$, so $m - 1 = 1$, and $m = 2$. Finally, $k + m = n$, so $k + 2 = -1$, and we have $k = -3$. (This sequence of numbers is called the Fibonacci sequence.)

9. **Answer B.** Putting $x = 2004$, we have $x^2 - (x-1)(x+1) = x^2 - (x^2 - 1) = 1$.

10. **Answer B.** $\frac{n+3}{n-1} = 1 + \frac{4}{n-1}$. For this to be an integer, $n - 1$ must be one of the (positive or negative) divisors of 4, which are ± 1, ± 2, and ± 4. This gives six possible integer values for $n - 1$, and therefore six possible integer values for n.

11. **Answer A.** First, $BG = BF = AB - AF = \sqrt{3} - 1$. Next, $HE = GC = BC - BG = 1 - (\sqrt{3} - 1) = 2 - \sqrt{3}$.

12. **Answer A.** If $p + 1$ is a square, say k^2, then $p = k^2 - 1 = (k+1)(k-1)$. Since p is prime, it has no factors other than 1 and itself, so $k - 1 = 1$ and $p = k + 1$. This gives $p = 3$ as the only solution.

13. **Answer A.** $2^2 \times 3^3 \times 4^4 \times 5^11 = 2^{2+8} \times 3^3 \times 5^11 = 27 \times 5 \times 10^{10} = 135 \times 10^{10}$, which is 135 followed by ten zeros. Thus the sum of the digits is $1 + 3 + 5 = 9$.

14. **Answer C.** Suppose the parallelogram has base b ($= AB$) and height h, so its area is bh. Then the area of $\triangle AFN = \frac{1}{2}(\frac{1}{2}b)(\frac{1}{2}h) = \frac{1}{8}bh$, and the area of $\triangle NBC = \frac{1}{2}(\frac{1}{2}b)(h) = \frac{1}{4}bh$. Thus the area of quadrilateral $FNCD = bh - \frac{1}{8}bh - \frac{1}{4}bh = \frac{5}{8}bh$. Finally the ratio of the required areas is $\frac{1}{8}bh \div \frac{5}{8}bh = \frac{1}{5}$.

15. **Answer E.** The only one-digit number starting with 9 is 9 itself. The two-digit numbers starting with 9 are 90–99, of which there are 10. The three-digit numbers starting with 9 are 900–999, of which there are $100 = 10^2$. Similarly, there are 10^3 four-digit numbers starting with 9, and 10^4 five-digit numbers. The total is therefore $1+10+100+1000+10000 = 11111$. (Alternatively, since there are only nine possibilities for the first digit, you can say that one-ninth of the numbers from 1 to 99999 start with 9. This gives the correct answer quickly in this case, but sometimes a quick argument like this is not exact.)

16. **Answer B.** Each of the nine equal sides subtends an angle of $360/9 = 40$ degrees at the centre, so the angle subtended at the circumference (including any of the other vertices) is half of that, namely $20°$.

17. **Answer D.** Suppose the sides of the square have length x. Then, since the smaller white triangle is similar to the whole triangle, it follows that $\dfrac{b-x}{x} = \dfrac{b}{a}$, so $x = \dfrac{ab}{a+b}$. The required ratio is $\dfrac{x^2}{\frac{1}{2}ab} = \dfrac{(ab)^2}{(a+b)^2}\dfrac{2}{ab} = \dfrac{2ab}{(a+b)^2}$.

18. **Answer B.** Let $p = \sqrt{2+\sqrt{3}}$ and $q = \sqrt{2-\sqrt{3}}$, so $pq = \sqrt{2^2-3} = 1$. Then $(p-q)^2 = p^2 - 2pq + q^2 = (2+\sqrt{3}) - 2 + (2-\sqrt{3}) = 2$. Since $p - q$ is clearly positive, it follows that $p - q = \sqrt{2}$.

19. **Answer D.** By similar triangles, $\dfrac{MP}{MC} = \dfrac{AB}{AC}$, so $MP = \dfrac{MC.AB}{AC} = \dfrac{a}{\sqrt{(2a)^2-1}}$.

20. **Answer E.**

Let the vertexes of the triangle be A, B and C as shown in the figure. Let O denote the centre of the semicircle, and P denote the point of tangency, and join OP and OC. Then by similar triangles $AP = OP$ and $PC = BC$. But $BC = 1$ and $AP + PC = AC = \sqrt{2}$ (Pythagoras), so $OP = AP = AC - PC = AC - BC = \sqrt{2} - 1$.