South African Mathematics Olympiad
Third Round 2000

Answer all questions. No calculators or other technological accessories are allowed except the usual geometric drawing instruments.
Time: 4 hours.

1. A number x_n of the form $1010101 \ldots 1$ has n ones. Find all n such that x_n is prime.

2. Solve for x, given $36x^4 + 36x^3 - 7x^2 - 6x + 1 = 0$.

3. Let $c \geq 1$ be an integer, and define the sequence $a_1, a_2, a_3, \ldots$, by

$$\begin{aligned} a_1 &= 2; \\ a_{n+1} &= ca_n + \sqrt{(c^2-1)(a_n^2-4)}, \text{ for } n = 1, 2, 3, \ldots . \end{aligned}$$

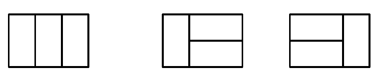
   Prove that a_n is an integer for all n.

4. $ABCD$ is a square of side 1. P and Q are points on AB and BC such that $\widehat{PDQ} = 45^\circ$. Find the perimeter of $\triangle PBQ$.

5. Find all $f : \mathbb{Z} \to \mathbb{Z}$ (where $\mathbb{Z}$ is the set of all integers) such that

$$2000f(f(x)) - 3999f(x) + 1999x = 0 \text{ for all } x \in \mathbb{Z}.$$

6. There are three ways to tile a 2×3 rectangle using 2×1 tiles:



   Let A_n be the number of ways to tile a $4 \times n$ rectangle using 2×1 tiles. Prove that A_n is divisible by 2 if and only if A_n is divisible by 3.