THE HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

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in collaboration with HARMONY GOLD MINING, AMESA and SAMS

SECOND ROUND 2003

SENIOR SECTION: GRADES 10, 11 AND 12

20 May 2003

TIME: 120 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. C
2. D
3. D
4. A
5. B
6. A
7. D
8. D
9. D
10. C
11. A
12. A
13. B
14. A
15. A
16. E
17. A
18. E
19. B
20. A

SOLUTIONS

1. **Answer C.** One way is to square each number given and see which square is closest to 2003. Alternatively, you can make an estimate as follows: since $2003 = 10^2 \times 20.03$, it follows that $\sqrt{2003} = 10 \times \sqrt{20.03}$. Now $\sqrt{16} = 4$ and $\sqrt{25} = 5$, and 20.03 is roughly half-way between 16 and 25. Thus $\sqrt{20.03} \approx 4.5$, so $\sqrt{2003} \approx 45$.

2. **Answer D.** Remember that Odd plus Odd and Even plus Even are both Even, and Odd plus Even and Even plus Odd are both Odd.

3. **Answer D.** The smallest cube he can build has sides of length 6, since his block must fit into it. He can build the cube with six layers of thickness 1, each layer consisting of three blocks side by side, making a total of 18 blocks. Alternatively, the volume of the cube is $6^3 = 216$, and the volume of each block is 12, so the number of blocks required is $216 \div 12 = 18$.

4. **Answer A.** If the photograph is enlarged three times, then all lengths are multiplied by 3, and all areas are multiplied by $3^2 = 9$. However, the percentages of black and white are unchanged, since all areas are multiplied by the same factor.

5. **Answer B.** If $5.123456789 \times 10^{20}$ is written without using scientific notation, then the decimal point moves 20 places to the right, giving 512 345 678 900 000 000 000. If now 30 is added to this number, then the last three digits on the right become 030, but the significant digits (which are on the left) are unchanged. Thus the display on the calculator is also unchanged.

6. **Answer A.** Each item is exactly one cent short of a whole number of rands. Since the total cost is 29 cents short of a whole number of rands, he must have bought 29 items. (The other possible answers, 129 and 229 and so on, are ruled out because then the total cost would be much higher.)

7. **Answer D.** This can only done by trial and error. For example, if A is correct, then the correct answer is "one", which has three letters, not one. Thus A cannot be correct. A similar argument rules out B, C, and E. However, if D is correct, then the correct answer is "four", which has four letters, as required.

8. **Answer D.** Suppose $n = k^2$, so $k = \sqrt{n}$. The next perfect square is $(k+1)^2 = k^2 + 2k + 1 = n + 2\sqrt{n} + 1$.

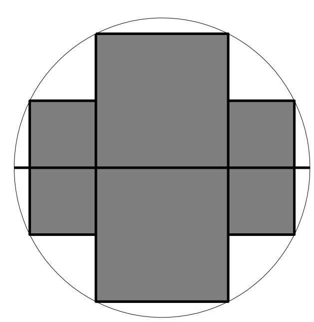
9. **Answer D.** Suppose the distance to Gamkaskloof (also called "Die Hel") is d. Since Time = Distance/Speed, the time to cycle there is d/x and the time to cycle back is d/y. The total time is therefore
$$\frac{d}{x} + \frac{d}{y} = \frac{d(x+y)}{xy}.$$
The total distance is $2d$, and the average speed is Total Distance divided by Total Time, which gives $2xy/(x+y)$.

10. **Answer C.** At half-time, Chiefs' score could be anything from 0 to m, giving $m+1$ possible scores. Similarly, there are $n+1$ possible half-time scores for Dynamos. Each possible Chiefs' score can be combined with each possible Dynamos' score, making a total number of $(m+1)(n+1)$ possible half-time scores for the match.

11. **Answer A.** Draw a square through the centres of the four squares. Each side of the square has length 1 (two radii of a cable, which is one diameter). The diameter of the pipe containing the cables is equal to the diagonal of the square plus two radii of the cables, which is $\sqrt{2} + 1$.

12. **Answer A.**

If the figure is combined with its mirror image in the diameter, then it is easy to see that four small squares make up one big square. More prosaically, if the large square has side of length $2a$, then the co-ordinates of its top right-hand corner are $(a; 2a)$ and the co-ordinates of the top right-hand corner of the smaller square are $(a + \sqrt{7}; \sqrt{7})$. Since both corners lie on the circle, it follows that $a^2 + (2a)^2 = (a + \sqrt{7})^2 + 7$. This simplifies to $2a^2 - a\sqrt{7} + 7 = 0$, for which the only positive solution is $a = \sqrt{7}$. Finally, the area of the large square is $4a^2 = 28$.



13. **Answer B.** Take the origin to be at the centre of the chessboard, and suppose the squares on the board have sides of unit length. It is sufficient to count the squares in the first quadrant that are covered by the circle, and multiply the number by four. A square in the first quadrant will be completely covered if its top right-hand corner is covered. These corners have co-ordinates $(i; j)$, where $1 \le i \le 4$ and $1 \le j \le 4$. The corner $(i; j)$ will be covered if it lies inside the circle of radius 4, that is, if $i^2 + j^2 \le 16$. It is now easy to count the pairs $(i; j)$ that satisfy the three inequalities: they are $(1; 1)$, $(1; 2)$, $(1; 3)$, $(2; 1)$, $(2; 2)$, $(2; 3)$, $(3; 1)$, and $(3; 2)$. Thus eight squares in the first quadrant are completely covered, making 32 squares on the whole board.

14. **Answer A.** In 50 minutes, Ellie runs 10 000 m and Michael runs 9 500 m, so Ellie's speed is 200 m/min and Michael's speed is 190 m/min. In the second race, Ellie has to run 10 500 m, which takes her 52.5 min. In that time, Michael runs $190 \times 52.5 = 9\,975$ m, so he is still 25 m behind the finishing line when Ellie finishes.

15. **Answer A.** If you reflect the picture in sides BC, AB, and AD, then the path of the ball from the reflection of X to the reflection of Y is a straight line, which is the hypotenuse of a right-angled triangle. The total distance travelled parallel to AB is 1.40 (to BC) plus 2.00 (to AD) plus 0.20 (to Y), making 3.60 m. Similarly, the total distance travelled parallel to AD is 0.80 (to AB) plus 0.25 (to Y), making 1.05 m. In centimetres, the sides have length $360 = 15 \times 24$ and $105 = 15 \times 7$, so the length of the hypotenuse is

$$15 \times \sqrt{24^2 + 7^2} = 15 \times \sqrt{576 + 49} = 15 \times \sqrt{625} = 15 \times 25 = 375,$$

and the total distance travelled is 3.75 m.

16. **Answer E.** If O lies on the diagonal BD, then triangles BAO and BCO are congruent. (Side BO is common, the sides BC and BA are equal, and the included angles are both equal to 45°.) Thus $B\hat{O}C = y$, and BOD is a straight line, so $x + y = x + B\hat{O}C = 180°$, as required. A similar argument applies if O lies on the other diagonal, so answer E gives at least part of the solution. It follows that answers A, C and D, which do not include the diagonals, cannot be correct. It is not quite so easy to eliminate answer B, but suppose O lies very near one of the sides, say AB. Then angle y is very nearly 180°, but angle x is not near zero, so $x + y > 180°$. It follows that O cannot lie at every point in the interior, so answer B is also not correct.

17. **Answer A.** Suppose the first bricklayer lays x bricks per hour on his own, and the second one lays y bricks per hour. Then the number of bricks in the wall is $9x$ which is equal to $10y$. Working together, they lay $x + y - 10$ bricks per hour, so the number of bricks in the wall is also $5(x + y - 10)$. Solving the equations gives $x = 100$ and $y = 90$, so the number of bricks in the wall is 900.

18. **Answer E.** Assume $x = y = z$, and apply the Sine Rule to triangles BAD and BDE. Since $x = y$ and $AD = DE$, it follows that

$$\frac{BA}{\sin(B\widehat{D}A)} = \frac{BE}{\sin(B\widehat{D}E)}.$$

But $\sin(B\widehat{D}A) = \sin(B\widehat{D}E)$, since the angles are supplementary, so $BA = BE$. Thus triangle BAE is isosceles, and the angle bisector BD is perpendicular to the base line AC. By a similar argument applied to triangles BDE and BEC, we see that BE is also perpendicular to AC, and therefore parallel to BD. This is a contradiction, because two parallel lines never intersect, whereas BD and BE intersect at B.

19. **Answer B.** Suppose p is the probability that either marksman hits the target with a given shot, and q is the probability that he misses. (In this case, $p = q = \frac{1}{2}$.) When they take turns at shooting, the probability that the first hit occurs with the n-th shot is $q^{n-1}p$, because all previous shots must have been misses. The one who has the first shot will also have the third, fifth, seventh shots, and so on until one shot is a hit. The probability that the first hit occurs with one of his shots is

$$p + q^2p + q^4p + q^6p + \cdots = p(1 + q^2 + q^4 + q^6 + \cdots) = \frac{p}{1 - q^2} = \frac{1/2}{3/4} = \frac{2}{3}.$$

(You can check this by calculating the probability that the one who has the second shot wins. The probability is $qp + q^3p + q^5p + \cdots$, which simplifies to $pq/(1-q^2)$ and equals $\frac{1}{3}$, as expected.)

20. **Answer A.** This can be done by trial and error, and by contradiction. Take each statement in turn, suppose it is true, and see if the assumption leads to a contradiction. If it does, then the person who made that statement must be lying.
Suppose first that Anna's statement is true, so Charles and Dumisani are lying. Since Charles is lying, at least one of Ben and Dumisani is telling the truth. But Dumisani is lying, so it must be Ben who is telling the truth. Therefore Anna and Ellie are lying. This contradicts the assumption that Anna is telling the truth.
We have been lucky first time, and have got a contradiction from assuming that Anna is telling the truth. Thus Anna is definitely lying.
(It is a good exercise to try to show that assuming any of the other statements is true does *not* lead to a contradiction. In fact, there are two possibilities: either Ben and Dumisani are the only ones telling the truth, or Charles and Ellie are the only truthful ones.)