The South African Mathematical Olympiad
Third Round 2018
Senior Division (Grades 10 to 12)
Solutions

1. One hundred glasses are arranged in a 10×10 array. Now we pick a of the rows and pour blue liquid into all glasses in these rows, so that they are half full. The remaining rows are filled halfway with yellow liquid. Afterwards, we pick b of the columns and fill them up with blue liquid. The remaining columns are filled up with yellow liquid. The mixture of blue and yellow liquid turns green. If both halves in a glass have the same colour, then that colour remains as it is.

   (a) Determine all possible combinations of values for a and b so that exactly half of the glasses contain green liquid at the end.

   (b) Is it possible that precisely one quarter of the glasses contain green liquid at the end?

   Solution: The total number of glasses that are green at the end of the procedure is

$$a(10-b)+b(10-a)=10a+10b-2ab=2(5a+5b-ab).$$

   We immediately observe that this number is always even, so the number of green glasses cannot be 25 (i.e., one quarter). Hence the answer to the second question is no. For the first question, we have to solve the equation

$$10a+10b-2ab=50,$$

   which is equivalent to

$$2ab-10a-10b+50=2(a-5)(b-5)=0.$$

   Thus exactly half of the glasses contain green liquid if either $a=5$ or $b=5$ (and the other is arbitrary).

2. In triangle ABC, $AB=AC$, and D is on BC. A point E is chosen on AC, and a point F is chosen on AB, such that $DE=DC$ and $DF=DB$. It is given that $\dfrac{DC}{BD}=2$ and $\dfrac{AF}{AE}=5$. Determine the value of $\dfrac{AB}{BC}$.

   Solution: Note that triangles ABC and DFB are both isosceles, and they share the angle at B. Therefore, they must be similar. Likewise, triangle DCE is similar to these two. Let $BF=x$ and $BD=FD=y$. Then we have $DC=2BD=2y$, and since DFB and DCE are similar, it also follows that $CE=2x$. Next, we let $AB=AC=a$, so $AF=AB-BF=a-x$ and $AE=AC-CE=a-2x$. We are given that $AF=5AE$, so

$$a-x=5(a-2x),$$

which is equivalent to $4a = 9x$, i.e. $a = \frac{9x}{4}$. Finally, since ABC and DFB are similar, we have

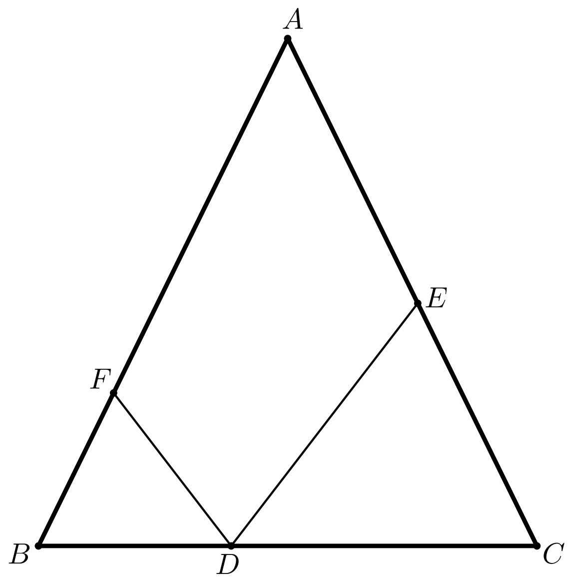
$$\frac{\frac{9x}{4}}{3y} = \frac{AB}{BC} = \frac{BD}{BF} = \frac{y}{x}.$$

Thus

$$\left(\frac{y}{x}\right)^2 = \frac{3}{4},$$

and finally

$$\frac{AB}{BC} = \frac{y}{x} = \frac{\sqrt{3}}{2}.$$



3. Determine the smallest positive integer n whose prime factors are all greater than 18, and that can be expressed as $n = a^3 + b^3$ with positive integers a and b.

Solution: We can factorise n as

$$n = a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

The first factor $a + b$ has to be at least 19, since n would otherwise contain a prime factor that is smaller than 18. Setting $a + b = s$, we obtain

$$a^2 - ab + b^2 = a^2 - a(s - a) + (s - a)^2 = 3a^2 - 3as + s^2 = 3\big(a - \tfrac{s}{2}\big)^2 + \tfrac{s^2}{4}$$

by completing the square. Hence the second factor is greater or equal to $\frac{s^2}{4}$, and becomes smaller the closer a is to $\frac{s}{2}$. If $s = 19$, then for $a = 9$ or $a = 10$, the second factor is $91 = 7 \cdot 13$, which contains a prime factor smaller than 18. For $a = 8$ or $a = 11$, however, it is equal to 97, which is prime. In this case, $n = 19 \cdot 97 = 1843 = 11^3 + 8^3$ satisfies the conditions.

If $s = 19$ and $a < 8$ or $a > 11$, then

$$a^2 - ab + b^2 = 3\big(a - \tfrac{19}{2}\big)^2 + \tfrac{\cdot 19^2}{4} > 3\big(\tfrac{3}{2}\big)^2 + \tfrac{19^2}{4} = 97,$$

thus $n > 19 \cdot 97 = 1843$. If $s > 19$, then s must be at least 20 (in fact at least 23, so that it does not contain prime factors smaller than 18), so

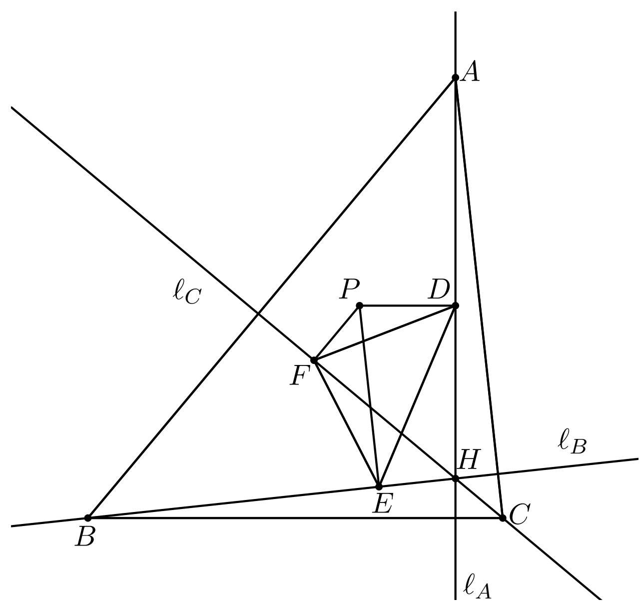
$$n = s\Big(3\big(a - \tfrac{s}{2}\big)^2 + \tfrac{s^2}{4}\Big) \geq s \cdot \tfrac{s^2}{4} \geq \tfrac{20^3}{4} = 2000.$$

This means that 1843 is indeed the smallest number with the desired properties.

4. Let ABC be a triangle with circumradius R, and let ℓ_A, ℓ_B, ℓ_C be the altitudes through A, B, C respectively. The altitudes meet at H. Let P be an arbitrary point in the same plane as ABC. The feet of the perpendicular lines through P onto ℓ_A, ℓ_B, ℓ_C are D, E, F respectively. Prove that the areas of DEF and ABC satisfy the following equation:

$$\mathrm{area}(DEF) = \frac{PH^2}{4R^2}\,\mathrm{area}(ABC).$$

Solution:



Note that $\angle PDH = \angle PFH = \angle PEH = 90^\circ$ by construction, so by Thales's Theorem, D, E and F lie on a circle whose diameter is PH. Therefore, the angle between lines DE and FE is the same as the angle between lines DH and FH (angles subtended by the chord DF), which in turn is the same as the angle between lines AB and BC (angles between pairwise perpendicular lines). Repeating the argument, we find that the angles between lines DE, DF, EF coincide with those between lines AB, AC, BC, so triangles ABC and DEF are similar. The diameter of the circumcircle of ABC is $2R$, the diameter of the circumcircle of DEF is PH as observed before. Since areas of similar triangles are proportional to squared lengths, we obtain

$$\mathrm{area}(DEF) = \frac{PH^2}{4R^2}\,\mathrm{area}(ABC)$$

as required.

5. Determine all sequences $a_1, a_2, a_3, \ldots$ of nonnegative integers such that $a_1 < a_2 < a_3 < \cdots$ and a_n divides $a_{n-1} + n$ for all $n \geq 2$.

Solution: We claim that the only possible sequences are the following:

- $a_n = n - 1$ for all n, or
- $a_n = \dfrac{n^2+n}{2} + k$ for all n, where k is a fixed nonnegative integer, or
- $a_n = \begin{cases} n-1 & n \leq N, \\ \frac{n^2+n}{2} - \frac{N^2-N+2}{2} & n > N, \end{cases}$ where N is a fixed nonnegative integer.

Let us first verify that each of these sequences satifies the conditions:

- If $a_n = n - 1$ for all n, then $a_{n-1} + n = 2n - 2 = 2a_n$ is indeed divisible by a_n.
- If $a_n = \frac{n^2+n}{2} + k$ for all n, then $a_{n-1} + n = \frac{n^2-n}{2} + k + n = \frac{n^2+n}{2} + k = a_n$ is also divisible by a_n.
- In the third case, a_n divides $a_{n-1} + n$ for $n \leq N$ as in the first case. Next note that $a_{N+1} = \frac{(N+1)^2+(N+1)}{2} - \frac{N^2-N+2}{2} = 2N$ divides $a_N + (N+1) = 2N$. Finally, for $n > N+1$, we have $a_n = a_{n-1} + n$ as in the second case, so a_n again divides $a_{n-1} + n$.

Now we prove that these are the only such sequences. First, let a_k be an element of the sequence such that $a_k \geq k$ (if such an element exists). Recall that $a_k + k + 1$ has to be a multiple of a_{k+1}. However, since $a_{k+1} > a_k$, we have

$$2a_{k+1} \geq 2(a_k + 1) > 2a_k + 1 \geq a_k + k + 1.$$

So the only possible multiple of a_{k+1} that $a_k + k + 1$ could be is $1 \cdot a_{k+1}$, and it follows that $a_{k+1} = a_k + k + 1$. But then $a_{k+1} \geq k + k + 1 \geq k + 1$, so we can repeat the argument with $k+1$ instead of k to show that $a_{k+2} = a_{k+1} + k + 2$, etc. Generally, we get $a_{n+1} = a_n + n + 1$ for all $n \geq k$.

If $a_1 \geq 1$, then we can invoke this observation immediately: $a_{n+1} = a_n + n + 1$ for all $n \geq 1$, so

$$a_n = a_{n-1} + n = a_{n-2} + (n-1) + n = \cdots = a_1 + 2 + 3 + \cdots + (n-1) + n = \frac{n^2+n}{2} + (a_1 - 1),$$

which is exactly our second solution.

Suppose finally that $a_1 = 0$, and let N be the largest index for which $a_N = N - 1$; if there is no largest index, then $a_n = n - 1$ for all n, and we obtain the first solution. Next note that a_{N+1} has to divide $a_N + N + 1 = 2N$. By our choice of N, we have $a_{N+1} \neq N$, and since $a_{N+1} > a_N = N - 1$, the only possible value (the only divisor of $2N$) for a_{N+1} is $2N$. But then $a_{N+1} = 2N \geq N + 1$, and we can apply the same observation as before: $a_{m+1} = a_m + m + 1$ for all $m \geq N$, thus

$$\begin{aligned} a_n = a_{n-1} + n = \cdots = a_N + (N+1) + (N+2) + \cdots + (n-1) + n = \\ = (N-1) + \frac{n^2+n}{2} - \frac{N^2+N}{2} = \frac{n^2+n}{2} - \frac{N^2-N+2}{2} \end{aligned}$$

for all $n > N$, which is indeed the third solution.

6. Let n be a positive integer, and let $x_1, x_2, \ldots, x_n$ be distinct positive integers with $x_1 = 1$. Construct an $n \times 3$ table where the entries of the k-th row are $x_k, 2x_k, 3x_k$ for $k = 1, 2, \ldots, n$. Now follow a procedure where, in each step, two identical entries are removed from the table. This continues until there are no more identical entries in the table.

   (a) Prove that at least three entries remain at the end of the procedure.

   (b) Prove that there are infinitely many possible triples that can remain at the end of the procedure for suitable choices of n and $x_1, x_2, \ldots, x_n$.

Solution:

(a) We write X as a shorthand for the set $\{x_1, x_2, \ldots, x_n\}$. Note that the number 1 can only appear in the first row, so it remains at the end of the procedure. Next, let x_k be the largest power of 2 that occurs in X. This is possible since $x_1 = 1$ is a power of 2. The number $2x_k$ in the second column of row k is a power of two, and it does not occur in any of the other rows: it is not one of the numbers in the first column (i.e., one of the other x_j) by our choice of x_k, it is not anywhere else in the second column (i.e., equal to $2x_j$ for some other j) since the x_j are all distinct, and it is not in the third column (i.e., of the form $3x_j$) since the numbers there are multiples of 3. Hence $2x_k$ remains in the table until the end of the procedure.

By the same argument, if we choose x_ℓ to be the largest power of 3 that occurs among the numbers $x_1, x_2, \ldots, x_n$, then $3x_\ell$ remains in the table at the end. Thus there are at least three numbers left at the end, namely $1, 2x_k, 3x_\ell$.

(b) We construct a sequence of sets X_r in the following way: $X_0 = \{1\}$, and

$$X_{r+1} = X_r \cup \{2^{2^r} x \,:\, x \in X_r\} \cup \{3^{2^r} x \,:\, x \in X_r\}$$

for all $r \geq 0$. In words: X_{r+1} consists of X_r, all elements of X_r multiplied by 2^{2^r}, and all elements of X_r multiplied by 3^{2^r}. We claim that the final table, if $x_1, x_2, \ldots, x_n$ are chosen to be the elements of X_r, consists of the three numbers $1, 2^{2^r}$ and 3^{2^r}. Clearly, this is true for $r = 0$: there is only one row, and the initial numbers $1, 2, 3$ remain in the table.

For the induction step, suppose the statement is true for X_r; we prove it for X_{r+1}. By the induction hypotesis, we can perform the reduction procedure in such a way that the rows corresponding to X_r are reduced to $1, 2^{2^r}$ and 3^{2^r}. The rows corresponding to $\{2^{2^r} x \,:\, x \in X_r\}$ can be reduced to $2^{2^r}, 2^{2^r} \cdot 2^{2^r} = 2^{2^{r+1}}$ and $2^{2^r} \cdot 3^{2^r} = 6^{2^r}$, and the rows corresponding to $\{3^{2^r} x \,:\, x \in X_r\}$ can be reduced to $3^{2^r}, 3^{2^r} \cdot 2^{2^r} = 6^{2^r}$ and $3^{2^r} \cdot 3^{2^r} = 3^{2^{r+1}}$. Applying the reduction procedure three more times to 2^{2^r}, 3^{2^r} and 6^{2^r}, we end up with the three numbers $1, 2^{2^{r+1}}$ and $3^{2^{r+1}}$. This completes the induction and thus the proof of the second part.