THE SOUTH AFRICAN MATHEMATICS OLYMPIAD
SENIOR SECOND ROUND 2016
Solutions

1. **Answer 010**
   25% of $40 = \frac{25}{100} \times 40 = 10$.

2. **Answer 014**
   The prime factorisation of 2016 is $2^5 \times 3^2 \times 7$. [Divide by 2 as many times as you can, then by 3, and so on.] In a perfect square, each prime factor has an even exponent (or power), so we need one more power of 2 and of 7. The answer is $2 \times 7 = 14$.

3. **Answer 002**
   It is easy to see that if $a = b$, then the expression is equal to 2, and a few trials will suggest that 2 is the smallest value. A proof is that
   $$\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = 2 + \frac{a^2 - 2ab + b^2}{ab} = 2 + \frac{(a-b)^2}{ab} \geq 2,$$
   since $(a - b)^2$ is a perfect square and $ab > 0$.

4. **Answer 045**
   The sides of the four shaded squares are the square roots of their areas, that is, $3, 4, 5, 7$. The two biggest of these squares fill up one side of the large square, so its side is $7 + 5 = 12$, and its area is $12^2 = 144$. Thus the unshaded area is $144 - (9 + 16 + 25 + 49) = 144 - 99 = 45$.

5. **Answer 011**
   The length of the third side of a triangle cannot exceed the sum of the lengths of the first two sides, which is $5 + 7 = 12$. The largest prime number less than 12 is 11.

6. **Answer 140**
   A temperature of 80° C is 80% of the distance from freezing point to boiling point, so on the SAMO scale the reading is $20 + \frac{80}{100}(170 - 20) = 20 + 120 = 140°$.

7. **Answer 700**
   It is easy to find a route that uses all nine paths except for the two paths on the line XY.

[Since the number of paths meeting at every junction is even, it follows from a result of Euler that there is a route that covers every path once only, and starts and ends at the same point. To go from X to Y we can then remove the two direct paths from such a route.]

8. **Answer 008**
Let $N = 10x+y$. Then $10x+y+10y+x = 11(x+y)$ must be a perfect square. Since $1 \leq x+y \leq 18$, it follows that $x+y = 11$. There are eight such numbers: 29, 38, 47, 56, 65, 74, 83 and 92.

9. **Answer 006**
The first two faces glued to each other are 6 and 4; the next two are 5 and 5, and the last two are 7 and 3. The visible face marked $*$ is then $9-3=6$.

10. **Answer 003**
For any three consecutive odd integers, exactly one is divisible by 3, so 3 is a common factor of all such numbers P. Even the first two non-overlapping values $P = 1 \times 3 \times 5$ and $P = 7 \times 9 \times 11$ have highest common factor 3, which is therefore the highest common factor of all values of P.

11. **Answer 030**
If the points A and B are joined to the centre of the circle, say O, then triangle OAB is equilateral, so $A\widehat{O}B = 60°$. The angle subtended by a chord at the centre of the circle is double the angle subtended at the circumference, so $A\widehat{C}B = \frac{1}{2}A\widehat{O}B = 30°$.

12. **Answer 003**

$$\frac{f(x+1)+f(x)}{f(x)} = \frac{f(x+1)}{f(x)} + 1 = \frac{2^{x+1}}{2^x} + 1 = 2^1 + 1 = 3.$$

13. **Answer 012**
Suppose tails appears t times, which means heads appears $30-t$ times. Then Harry gives $3t$ sweets to Simon and receives $2(30-t)$ sweets in return. These numbers are equal, so $3t = 2(30-t)$. Therefore $3t = 60-2t$, so $5t = 60$ and $t = 12$.

14. **Answer 006**
The first 50 even positive integers are $2, 4, 6, \ldots, 96, 98, 100$, and their sum is $50 \times \frac{1}{2}(2+100) = 50 \times 51 = 2550$. To reduce the sum to 2016 we must subtract 534. The average of the numbers removed must be less than 100, so we need to remove at least six. To use as few numbers as possible, we first remove the five largest numbers from 100 down. Now $100 + 98 + 96 + 94 + 92 = 480$, which is nearly there, so to bring the total subtracted to 534 we finally need to remove 54.

[There are, of course, many other combinations of five integers with a sum of 534 that can be removed.]

15. **Answer 090**
By Pythagoras' theorem, $BC = \sqrt{1+3} = 2$ and $AD = \sqrt{3+1} = 2$ also, while $AC = \sqrt{3+4} = \sqrt{7}$. Thus $AD^2 + DC^2 = 4 + 3 = 7 = AC^2$. Again by Pythagoras' theorem it follows that triangle ADC is right-angled, so $A\widehat{D}C = 90°$.

16. **Answer 008**
If two triangles have the same height, then the ratio of their areas is equal to the ratio of their bases. It follows that $\dfrac{DE}{EC} = \dfrac{6}{10} = \dfrac{3}{5}$, so $\dfrac{DE}{DC} = \dfrac{3}{3+5} = \dfrac{3}{8}$. Similarly, $\dfrac{DF}{FC} = \dfrac{17}{7}$, so $\dfrac{DF}{DC} = \dfrac{17}{17+7} = \dfrac{17}{24}$. Therefore $\dfrac{EF}{DC} = \dfrac{DF - DE}{DC} = \dfrac{17}{24} - \dfrac{3}{8} = \dfrac{17-9}{24} = \dfrac{8}{24} = \dfrac{1}{3}$. It follows that $\triangle BEF = \dfrac{1}{3}\triangle BDC = \dfrac{1}{3}(17+7) = 8$.

17. **Answer 018**
Clear fractions to get $c^2x + 2a^2y = 2a^2c^2$ and $2b^2x + c^2y = 2b^2c^2$. Then subtract the equations to give $(c^2 - 2b^2)x + (2a^2 - c^2)y = 2c^2(a^2 - b^2)$. Since $c^2 = a^2 + b^2$ (Pythagoras), it follows that

$$c^2 - 2b^2 = 2a^2 - c^2 = a^2 - b^2,$$

which we are given is non-zero. We can therefore cancel $a^2 - b^2$ to obtain $x + y = 2c^2 = 18$.

18. **Answer: Any integer $x \geq 5$.**
Any integer $x \geq 5$ is a solution. To see this, let $n \geq 2$ be a natural number. Let $m = 2n^2 + 13n - 24$. Suppose each of the learners

pays $n+3$ rands. The total amount paid is $(n+3)(2m+2n+20) = 4n^3+40n^2+56n-84 = 2mn+7m+13n+84$, as can be easily verified.

Since more than one answer (i.e. any integer greater or equal to five) is correct, as shown above, this question was disregarded in the computation of the final scores. Unfortunately an unnoticed error had crept into the formulation of the question somewhere during the course of various edits. The original question was supposed to have been as follows:

> There are $2m$ boys and 13 girls in grade 10, and 7 boys and $2n$ girls in grade 11 where m and n are positive integers. Each learner pays the same integral number of rands into a fund, and the total amount of money raised by each grade is $2mn+7m+13n+84$ rands. What is the number of rands paid by each learner?

The solution to *this* problem is unique and can be found as follows:

Since each learner contributes the same amount (say k rand) and each grade collects the same total, it follows that the number of learners in each grade is the same, that is, $2m+13 = 2n+7$, so $n = m+3$. Then

$$k = \frac{2mn+7m+13n+84}{2m+13} = n + \frac{7m+84}{2m+13} = n+a, \text{ say,}$$

where a, which is positive, must be an integer. This gives $7m+84 = a(2m+13)$, which implies that $(2a-7)m = 84-13a$. Again both sides must be positive, so $a \geq 4$ and $a \leq 6$. By inspection, the only value of a that gives an integer value for m is $a=4$, for which $m=32$. Finally $k = n+a = m+3+a = 32+3+4 = 39$.

19. **Answer 001**
(We do this by inspection, by trying out some values until we can see the pattern.) Draw up a list of the remainders left by the powers of 2 after division by 13:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
2^n mod 13	1	2	4	8	3	6	12	11	9	5	10	7	1

[Note that a mod b denotes the remainder left over when a is divided by b. Also note that it is not necessary to calculate each power of 2 in full

before dividing by 13: it is enough to multiply the previous remainder by 2 and then subtract 13 as many times as required.]
Since 2^0 and 2^{12} have the same remainder after division by 13, it follows that the pattern of remainders repeats in cycles of length 12. Now divide 2016 by 12 to get $2016 = 12 \times 168 + 0$, so 2^{2016} has the same remainder as 2^0, which is 1.

[Note: 12 is the maximum cycle length for the powers of any number mod 13, because it includes all possible non-zero remainders. It was proved by Fermat that if 13 is replaced by any prime number p and 2 is replaced by any number a not divisible by p, then the cycle length of powers of a will always be a factor of $p - 1$. It is also always possible to find a value of a such that the cycle length is exactly $p - 1$, as in this case.]

20. **Answer 028**
Let $AB = a$ and $BC = c$. The points A, D, F lie on a line (the bisector of $B\widehat{A}C$), so by similar triangles $\dfrac{a-3}{3} = \dfrac{a+21}{21}$. This gives $a + 21 = 7a - 21$, so $6a = 42$ and $a = 7$. Similarly, from the points C, D, E, we have $\dfrac{c+4}{4} = \dfrac{c-3}{3}$, so $4c - 12 = 3c + 12$ and $c = 24$. Now let T and U respectively be the points of tangency of the largest circle with the lines AC and BC, and let V be the foot of the perpendicular from D to BC. Quadrilateral $GTCU$ is cyclic, because the right angles at T and U show that opposite angles are supplementary. Therefore $T\widehat{G}U = T\widehat{C}B$ (external angle of a cyclic quadrilateral). Also GC is the bisector of $T\widehat{G}U$, just as DC is the bisector of $T\widehat{C}B$, so triangles GCU and CDV are similar. If g denotes the radius of the largest circle, then $GU = g$ and $UC = g - c = g - 24$.Therefore, again by similar triangles, $\dfrac{g}{g-24} = \dfrac{c-3}{3} = \dfrac{21}{3} = 7$. Therefore $g = 7g - 168$, so $6g = 168$ and $g = 28$.

The problem can also quite easily be solved from more general theorems that are perhaps not that well-known. For example, in general, for any triangle, the reciprocal of the radius of the incircle equals the sum of the reciprocals of the radii of the 3 excircles, from which one can easily deduce the result. Specifically, for right triangles, the sum of the radii of the three smaller circles equals the radius of the escribed circle on the hypotenuse (e.g. specifically for the problem $3+4+21 = 28$, and this general result can be deduced directly from the previous theorem). Lastly, another theorem from which the problem can easily be solved is that for right triangles, in general, the product of the radii of the incircle and the excircle on the hypotenuse equals the product of the radii of the excircles on the rectangular sides (e.g. specifically for the problem $3 \times 28 = 4 \times 21$).

Lastly, it is also possible to solve the problem by setting up co-ordinate axes with origin at B.