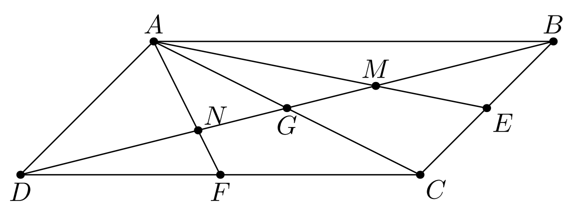
South African Mathematical Olympiad
Third Round 2003
Solutions

1. **Solution by Marietjie Venter (Grade 12, Hoërskool Stellenbosch).**

   Every time a piece of paper is cut into 5 smaller pieces, the total number of pieces of paper first decreases by 1, then increases by 5. Therefore it increases by 4.

   After n pieces of paper have been cut, there are $5 + 4n$ pieces of paper. But the equation $5 + 4n = 2003$ does not have an integral solution, since 4 does not divide $2003 - 5 = 1998$.

2. **Solution by Tamara von Glehn (Grade 11, St Stithians Girls' College).**



   Let G be the intersection of the diagonals AC and BD.

   In $\triangle ABC$, E is the midpoint of BC and G is the midpoint of AC (diagonals of a parallelogram bisect each other). Thus M is the centroid of $\triangle ABC$. Since the centroid of a triangle divides the medians in the ratio $2:1$, $BM = \frac{2}{3}BG$. Similarly, $DN = \frac{2}{3}DG$.

   But $BG = DG = \frac{1}{2}BD$, so $BM = \frac{2}{3}(\frac{1}{2}BD) = \frac{1}{3}BD$, and similarly $DN = \frac{1}{3}BD$. Also, $MN = BD - BM - DN = BD - \frac{1}{3}BD - \frac{1}{3}BD = \frac{1}{3}BD$. It follows that $BM = MN = ND$, and M and N divide BD into three equal parts.

3. **Solution by Ingrid von Glehn (Grade 11, St Stithians Girls' College).**

   Remove the 1137 from the front of n, and multiply the remaining number by 10 000. We now have the remaining digits of n with four 0's at the end. Let this number be m, and suppose m leaves remainder x when divided by 7.

   We can now add a four-digit number formed from the digits 1, 1, 3 and 7. Now, the numbers

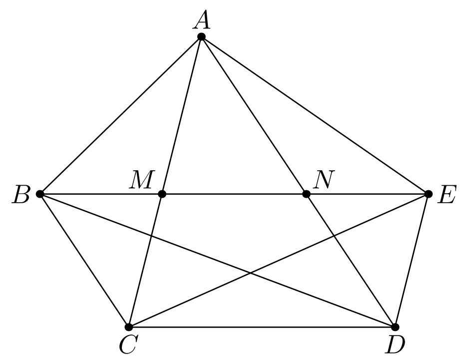
   $$3171,\ 1317,\ 1731,\ 1137,\ 1173,\ 1713,\ 1371$$

   leave remainders of $0,\ 1,\ 2,\ 3,\ 4,\ 5$ and 6 respectively when divided by 7. Thus it is possible to form a four digit number y from 1, 1, 3 and 7 such that y leaves remainder $7 - x$ when divided by 7.

   Now, $m + y$ is divisible by 7, since it leaves remainder $x + (7 - x) = 7$ when divided by 7. Also, the digits of $m + y$ is clearly a rearrangement of the digits of n. It is thus possible to rearrange the digits of n such that the new number is divisible by 7.

4. **Solution by Shaun Hammond (Grade 12, Hoërskool Swartland).**

It is given that $|\triangle BCD| = |\triangle CDE|$. These triangles have the same base, so they must have equal altitudes, so $CD \parallel BE$. Similarly, $BC \parallel AD$ and $DE \parallel AC$.



These parallel lines now give two parallelograms, $BCDN$ and $MCDE$. In $BCDN$ we have $BN = CD$ and in $MCDE$ we have $ME = CD$ (equal opposite sides in both cases). Thus

$$\begin{aligned} BN &= ME \\ BM + NM &= NM + ME \\ \therefore BM &= NE. \end{aligned}$$

5. **Solution by Dirk Basson (Grade 11, Hoërskool Diamantveld).**

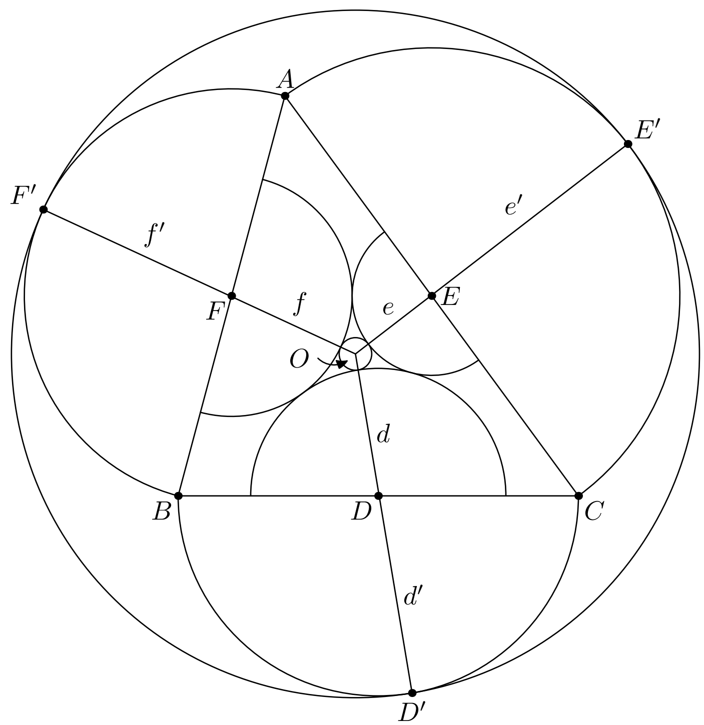
Suppose that the sum of the squares of two consecutive positive integers equals the sum of the fourth powers of two consecutive positive integers. Thus, for some $m,\ n \in \mathbb{N}$,

$$\begin{aligned} n^2 + (n+1)^2 &= m^4 + (m+1)^4 \\ \Rightarrow 2n^2 + 2n + 1 &= 2m^4 + 4m^3 + 6m^2 + 4m + 1 \\ \Rightarrow 2n^2 + 2n + 2 &= 2m^4 + 4m^3 + 6m^2 + 4m + 2 \\ \Rightarrow n^2 + n + 1 &= m^4 + 2m^3 + 3m^2 + 2m + 1 \\ &= (m^2 + m + 1)^2 \end{aligned}$$

which is certainly the square of an integer.

On the other hand, since $n > 0$, $n^2 < n^2 + n + 1 < n^2 + 2n + 1 = (n+1)^2$, which shows that $n^2 + n + 1$ is definitely not the square of an integer. Hence, no solution exist.

6. **Solution by Prof. Dirk Laurie (University of Stellenbosch).**



Let O denote the centre of the circle and let D, E, F be the midpoints of BC, CA, AB, respectively. Denote by D', E', F' the points at which the circle is tangent to the semicircles. Let d', e', f' be the radii of the semicircles. Then all of DD', EE', FF' pass through O, and $s = d' + e' + f'$.

Put

$$d = \frac{s}{2} - d' = \frac{-d' + e' + f'}{2}, \quad e = \frac{s}{2} - e' = \frac{d' - e' + f'}{2}, \quad f = \frac{s}{2} - f' = \frac{d' + e' - f'}{2}.$$

Note that $d + e + f = s/2$. Construct smaller semicircles inside the triangle ABC with radii d, e, f and centres D, E, F. Then the smaller semicircles touch each other, since $d+e = f' = DE$, $e+f = d' = EF$, $f+d = e' = FD$. In fact, the points of tangency are the points where the incircle of the triangle DEF touches its sides.

Suppose that the smaller semicircles cut DD', EE', FF' at D'', E'', F'', respectively. Since these semicircles do not overlap, the point O is outside the semicircles. Therefore $DO' > DD''$, and so $t > s/2$. Put $g = t - s/2$.

Clearly, $OD'' = OE'' = OF'' = g$. Therefore the circle with centre O and radius g touches all of the three mutually tangent semicircles.

Claim. We have

$$\frac{1}{d^2} + \frac{1}{e^2} + \frac{1}{f^2} + \frac{1}{g^2} = \frac{1}{2}\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f} + \frac{1}{g}\right)^2.$$

Proof. Consider a triangle PQR and let $p = QR$, $q = RP$, $r = PQ$. Then

$$\cos Q\widehat{P}R = \frac{-p^2 + q^2 + r^2}{2qr}$$

and

$$\sin Q\widehat{P}R = \frac{\sqrt{(p+q+r)(-p+q+r)(p-q+r)(p+q-r)}}{2qr}.$$

Since

$$\cos E\widehat{D}F = \cos(O\widehat{D}E + O\widehat{D}F) = \cos O\widehat{D}E \cos O\widehat{D}F - \sin O\widehat{D}E \sin O\widehat{D}F,$$

we have

$$\frac{d^2 + de + df - ef}{(d+e)(d+f)} = \frac{(d^2 + de + dg - eg)(d^2 + df + dg - fg)}{(d+g)^2(d+e)(d+f)} - \frac{4dg\sqrt{(d+e+g)(d+f+g)ef}}{(d+g)^2(d+e)(d+f)},$$

which simplifies to

$$(d+g)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f} + \frac{1}{g}\right) - 2\left(\frac{d}{g} + 1 + \frac{g}{d}\right) = -2\sqrt{\frac{(d+e+g)(d+f+g)}{ef}}.$$

Squaring and simplifying, we obtain

$$\begin{aligned}
\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f} + \frac{1}{g}\right)^2 &= 4\left(\frac{1}{de} + \frac{1}{df} + \frac{1}{dg} + \frac{1}{ef} + \frac{1}{eg} + \frac{1}{fg}\right) \\
&= 2\left(\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f} + \frac{1}{g}\right)^2 - \left(\frac{1}{d^2} + \frac{1}{e^2} + \frac{1}{f^2} + \frac{1}{g^2}\right)\right),
\end{aligned}$$

from which the conclusion follows. □

Solving for the smaller value of g, i.e. the larger value of $1/g$, we obtain

$$\begin{aligned}
\frac{1}{g} &= \frac{1}{d} + \frac{1}{e} + \frac{1}{f} + \sqrt{2\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right)^2 - 2\left(\frac{1}{d^2} + \frac{1}{e^2} + \frac{1}{f^2}\right)} \\
&= \frac{1}{d} + \frac{1}{e} + \frac{1}{f} + 2\sqrt{\frac{d+e+f}{def}}.
\end{aligned}$$

Comparing the formulas $\text{area}(\Delta DEF) = \text{area}(\Delta ABC)/4 = rs/4$ and $\text{area}(\Delta DEF) = \sqrt{(d+e+f)def}$, we have

$$\frac{r}{2} = \frac{2}{s}\sqrt{(d+e+f)def} = \sqrt{\frac{def}{d+e+f}}$$

All we have to prove is that

$$\frac{r}{2g} \geqslant \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$

Since

$$\frac{r}{2g} = \sqrt{\frac{def}{d+e+f}}\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}+2\sqrt{\frac{d+e+f}{def}}\right) = \frac{x+y+z}{\sqrt{xy+yz+zx}}+2,$$

where $x = 1/d$, $y = 1/e$, $z = 1/f$, it suffices to prove that

$$\frac{(x+y+z)^2}{xy+yz+zx} \geqslant 3$$

which is true from the rearrangement inequality.