South African Mathematics Olympiad
Third Round 1999

Answer all questions. No calculators or other technological accessories are allowed except the usual geometric drawing instruments.
Time: 4 hours.

1. How many non-congruent triangles with integer sides and perimeter 1999 can be constructed?

2. A, B, C and D are points on a given straight line, in that order. Construct a square PQRS, with all of P, Q, R and S on the same side of AD, such that A, B, C and D lie on PQ, SR, QR and PS produced respectively.

3. The bisector of angle BAD in the parallelogram ABCD intersects the lines BC and CD at the points K and L respectively. Prove that the centre of the circle passing through the points C, K and L lies on the circle passing through the points B, C and D.

4. The sequence $L_1, L_2, L_3, \ldots$ is defined by
$$L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2} \text{ for } n > 2,$$
so the first six terms are $1, 3, 4, 7, 11, 18$. Prove that $L_p - 1$ is divisible by p if p is prime.

5. Let S be the set of all rational numbers whose denominators are powers of 3. Let a, b and c be given non-zero real numbers. Determine all real-valued functions f that are defined for $x \in S$, satisfy
$$f(x) = af(3x) + bf(3x-1) + cf(3x-2)$$
if $0 \leq x \leq 1$, and are zero elsewhere.

6. You are at a point (a, b) and need to reach another point (c, d). Both points are below the line $x = y$ and have integer coordinates. You can move in steps of length 1, either upwards or to the right, but you may not move to a point on the line $x = y$. How many different paths are there?