South African Mathematical Olympiad
Third Round 2002
Time: 4 hours

1. Given a quadrilateral $ABCD$ such that $AB^2+CD^2 = BC^2+AD^2$, prove that $AC \perp BD$.

2. Find all triples of natural numbers (a, b, c) such that a, b and c are in geometric progession, and $a + b + c = 111$.

3. A small square PQRS is contained in a big square. Produce PQ to A, QR to B, RS to C and SP to D so that A, B, C and D lie on the four sides of the large square in order, produced if necessary.

   Prove that $AC = BD$ and $AC \perp BD$.

4. How many ways are there to express 1 000 000 as a product of exactly three integers greater than 1? (For the purpose of this problem, abc is not considered different from bac, etc.)

5. In acute-angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides. r_b and r_c are defined similarly. r is the radius of the incircle of ABC.

   Show that $\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$.

6. Find all rational numbers a, b, c and d such that

$$8a^2 - 3b^2 + 5c^2 + 16d^2 - 10ab + 42cd + 18a + 22b - 2c - 54d = 42;$$
$$15a^2 - 3b^2 + 21c^2 - 5d^2 + 4ab + 32cd - 28a + 14b - 54c - 52d = -22.$$

Suid-Afrikaanse Wiskunde-Olimpiade
Derde Ronde 2002
Tyd: 4 uur

1. Gegee 'n vierhoek $ABCD$ sodat $AB^2 + CD^2 = BC^2 + AD^2$, bewys dat $AC \perp BD$.

2. Vind alle triplette van natuurlike getalle (a, b, c) sodat a, b en c 'n meetkundige reeks is en $a + b + c = 111$.

3. 'n Klein vierkant $PQRS$ is bevat in 'n groot vierkant. Verleng PQ na A, QR na B, RS na C en SP na D sodat A, B, C en D op die vier sye van die groot vierkant lê, in volgorde en verleng indien nodig,

   Bewys dat $AC = BD$ en $AC \perp BD$.

4. Op hoeveel maniere kan 'n mens 1 000 000 uitdruk as 'n produk van presies drie heelgetalle groter as 1? (Vir die doel van hierdie probleem word abc nie as verskillend van bac beskou nie, ens.)

5. In die skerphoekige driehoek ABC word 'n halfsirkel met straal r_a getrek sodat sy basis op BC lê en hy die ander twee sye raak. r_b en r_c word op soortgelyke wyse gedefinieer. r is die straal van die ingeskrewe sirkel van ABC.

   Bewys dat $\dfrac{2}{r} = \dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c}$.

6. Vind alle rasionale getalle a, b, c en d sodat

$$8a^2 - 3b^2 + 5c^2 + 16d^2 - 10ab + 42cd + 18a + 22b - 2c - 54d = 42;$$
$$15a^2 - 3b^2 + 21c^2 - 5d^2 + 4ab + 32cd - 28a + 14b - 54c - 52d = -22.$$