THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior First Round 2021
Solutions

1. **Answer D**
One half of 2^{22} is equal to $\frac{1}{2} \times 2^{22} = 2^{-1} \times 2^{22} = 2^{22-1} = 2^{21}$.

2. **Answer B**
Let the numbers be x and y. Then $80 = x^2 - y^2 = (x + y)(x - y)$ and $16 = x + y$. Therefore $80 = 16(x - y)$, so $x - y = 80/16 = 5$. ($y - x = -5$, but only the positive difference is required.)

3. **Answer A**
$3\sqrt{3} + 2\sqrt{11} - (3\sqrt{11} - \sqrt{3}) = 3\sqrt{3} + 2\sqrt{11} - 3\sqrt{11} + \sqrt{3} = 4\sqrt{3} - \sqrt{11}$, so $a = -1$.

4. **Answer D**
Let $2023 = x$ and $2021 = y$. Then
$$\frac{2023^2 - 2021^2}{2} - \frac{2023^2 - 2021^2}{4044} = \frac{x^2 - y^2}{x - y} - \frac{x^2 - y^2}{x + y} = (x+y)-(x-y) = 2y = 4042.$$
Alternatively, the expression is equal to
$$(2023^2 - 2021^2)(\frac{1}{2} - \frac{1}{4044}) = (2023+2021)(2023-2021)\frac{2022 - 1}{4044} = (2)(2021) = 4042.$$

5. **Answer A**
$250 = 2^1 \times 5^3$, so n will divide exactly into 250 only if $n = 2^a \times 5^b$, where $0 \le a \le 1$ and $0 \le b \le 3$. This gives two choices for a and four choices for b, so the number of possible values of n is $2 \times 4 = 8$.

6. **Answer C**
Suppose there are m marbles in each bag. Then the total number of marbles is $3m$ and the number of red marbles is
$$50\% \times m + 25\% \times m + 21\% \times m = (50 + 25 + 21)m/100 = 96m/100.$$
The combined percentage of red marbles is $(96m/100) \div (3m) \times 100 = 32$.

7. **Answer C**
Suppose the fish has mass m kg and the man has mass M kg. Then $m = 75 + \frac{1}{4}m$, so $\frac{3}{4}m = 75$, giving $m = \frac{4}{3} \times 75 = 100$. Similarly, $M = 100 + \frac{1}{5}M$, so $\frac{4}{5}M = 100$ and $M = \frac{5}{4} \times 100 = 125$. Therefore $M - m = 125 - 100 = 25$.

8. **Answer E**
Substituting $x = 8$ gives $8\diamondsuit y = 3(8) - 8y + (8)y = 24$. Thus $8\diamondsuit y = 24$ for all real numbers y, and there are infinitely many solutions.

9. **Answer B**
Reminder: Percentage change = 100×(New value−Old Value)/(Old Value).
If each side of the cube is increased by 50%, then the new side length is $\frac{100+50}{100} = \frac{3}{2}$ times the original length. The volume is therefore multiplied by $(\frac{3}{2})^3 = \frac{27}{8} = 3.375$, and the percentage change in volume is $100(3.375 - 1)/1 = 237.5\%$.

10. **Answer E**
The fraction of the circle forming the monster is $\frac{360-60}{360} = \frac{5}{6}$, so the length of the curved part of the perimeter is $\frac{5}{6}(2\pi) = \frac{5}{3}\pi$. The two radii are each of length 1 , so the total perimeter is $\frac{5}{3}\pi + 2$.

11. **Answer D**
By Pythagoras' theorem, $AB^2 = 12^2 + 5^2 = 169$, so $AB = \sqrt{169} = 13$. Furthermore, $AM = AC = 12$ and $BN = BC = 5$. Finally, $MN = AM + BN - AB = 12 + 5 - 13 = 4$.

12. **Answer D**

$$\sqrt[3]{p\sqrt[3]{p\sqrt[3]{p}}} = (p(p.p^{1/3})^{1/3})^{1/3} = (p(p^{4/3})^{1/3})^{1/3} = (p(p^{4/9}))^{1/3} = (p^{13/9})^{1/3} = p^{13/27}.$$

Alternatively, if $x = \sqrt[3]{p\sqrt[3]{p\sqrt[3]{p}}}$, then $x^3 = p\sqrt[3]{p\sqrt[3]{p}}$, $x^9 = p^3p\sqrt[3]{p}$, and $x^{27} = p^9p^3p = p^{9+3+1} = p^{13}$.

13. **Answer D**
By inspection (or by calculation) it is easy to see that $\frac{2}{3} \times 54 = 36$, $\frac{2}{3} \times 36 = 24$, and so on. For the tenth term we have to multiply 54 by $\frac{2}{3}$ nine times. This gives

$$\frac{2^a}{3^b} = 54(\frac{2}{3})^9 = 2.3^3\frac{2^9}{3^9} = \frac{2^{10}}{3^6},$$

so $a - b = 10 - 6 = 4$.

14. **Answer C**
Since $\triangle BCD$ is isosceles, it follows that the external angle at C is equal to $2 \times \angle BDC = 20^\circ$. Next, the sum of the external angles of any polygon is 360°, and the given polygon is regular, so the number of its sides is $360/20 = 18$, and its perimeter is 18 .

15. **Answer C**
If Zoliswa runs x minutes till she catches Abbey, then Abbey has run for $x + 2$ minutes. Since the distances they have travelled are equal, it follows that $13x = 11(x + 2)$, so $2x = 22$ and $x = 11$.

16. **Answer B**
It is sufficient to consider only the last (or units, or ones) digits. Any positive power of 6 ends in 6. The last digits of the powers of 7 cycle in groups of length four: $7, 9, 3, 1, 7, 9, 3, 1, \ldots$, so 7^{82} has the same last digit as 7^2, that is, 9. The required last digit is the same as that of 6×9, which is 4.

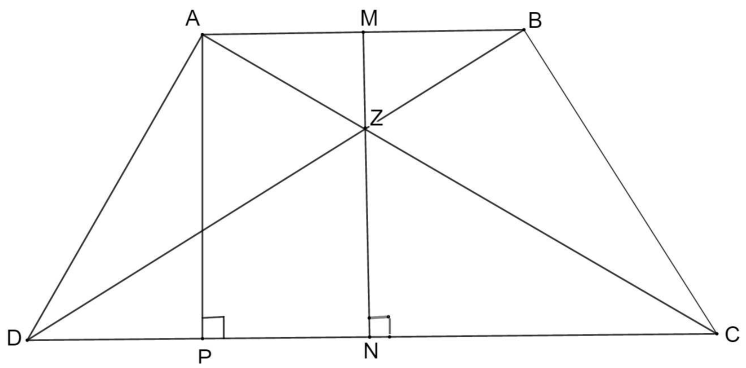
Alternatively, since the last digit of 6×7 is 2, the required answer is the last digit of 2^{82}, which is the same as the last digit of 2^2, since the last digits of the positive powers of 2 also cycle in groups of length four.

17. **Answer C**
There are $3^3 = 27$ equally likely ways of making three drawings, with replacement, from the bag. There are six possible arrangements to get a sum of 6, which is by drawing 1 and 2 and 3. And one more arrangement is by drawing 2 and 2 and 2. The probability is therefore $\dfrac{7}{27}$.

18. **Answer D**
Since $\angle QSR = \angle PQS$ (alternate angles), it follows that $\triangle PQS$ is similar to $\triangle QSR$ (in that order), so $SR : QS = QS : PQ = 7 : 5$. It follows that $SR = \frac{7}{5}QS = \frac{49}{5}$, and similarly $QR = \frac{7}{5}PS = \frac{28}{5}$. Thus the perimeter of $PQRS$ is equal to
$$5 + \frac{28}{5} + \frac{49}{5} + 4 = \frac{25 + 28 + 49 + 20}{5} = \frac{122}{5}.$$

19. **Answer A**
Assuming all 26 letters of the alphabet are used, as well as all ten digits, the present system allows $26^2 10^4$ licence plates, while the new system will allow $26^4 10^3$. The ratio is therefore $26^{4-2} : 10^{3-4} = 26^2/10$.

20. **Answer B**



By definition of a trapezium, and from the diagram, we have $AB \parallel DC$, and since the diagonals are equal, it follows that the diagram is symmetrical, so $\triangle ABZ$ and $\triangle CDZ$ are isosceles and similar to each other. Let M and N be the midpoints of AB and DC respectively (so MZN is perpendicular to both AB and CD), and let P be the foot of the perpendicular from A to CD. Then $DP = \frac{1}{2}(54 - 30) = 12$, so $PC = 42$. By Pythagoras' theorem in $\triangle APC$ we have $AP^2 = 56^2 - 42^2 = 14^2(4^2 - 3^2) = 14^2(7)$, so $AP = 14\sqrt{7}$. By similar triangles,
$$\frac{MZ}{ZN} = \frac{AB}{DC} = \frac{30}{54} = \frac{5}{9}, \text{ so } \frac{MZ}{MN} = \frac{MZ}{MZ + MN} = \frac{5}{14}.$$
Therefore $MZ = \frac{5}{14}MN = \frac{5}{14}AP = \frac{5}{14}(14\sqrt{7}) = 5\sqrt{7}$. Finally, the area of $\triangle ABZ$ is equal to$\frac{1}{2}AB \cdot MZ = \frac{1}{2} \cdot 30 \cdot 5\sqrt{7} = 75\sqrt{7}$.