THE SOUTH AFRICAN MATHEMATICS OLIMPIAD

in conjunction with THE SOUTH AFRICAN MATHEMATICAL SOCIETY and THE ASSOCIATION FOR MATHEMATICS EDUCATION OF SOUTH AFRICA

THE SOUTH AFRICAN MATHEMATICS OLIMPIAD

SECOND ROUND 1997

SENIOR SECTION: GRADES 10, 11 AND 12

(STANDARDS 8, 9 AND 10)

10 JUNE 1997

14:00

TIME: 120 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

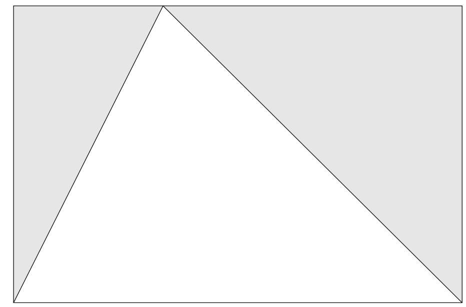
1. E
2. C
3. B
4. E
5. A
6. D
7. D
8. E
9. D
10. A
11. D
12. A
13. B
14. E
15. C
16. B
17. D
18. B
19. A
20. E

SOLUTIONS

1. $25 + 8$ is half the number. Hence the number is $2(25 + 8) = 66$.

2. If we draw 3 socks they could be one of each color. But if we draw one more there must be a repetition.

3. The area of the rectangle is $7 \times 12 = 84$ and the area of the large unshaded triangle is $\frac{1}{2} \times 7 \times 12 = 42$. Therefore the area of the shaded part is $84 - 42 = 42$.

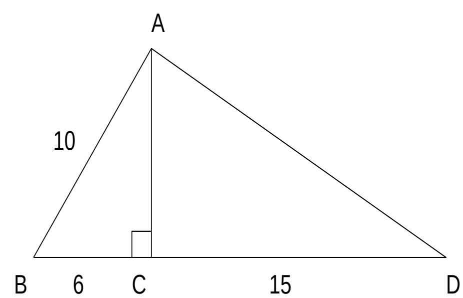


4. 2c + 5c + 10c + 50c + R1 = R1,67. Since 10,02/1,67= 6, there are 6 of each of the 5 coins. A total of $6 \times 5 = 30$ coins.

5. By using Pythagoras' theorem in $\triangle ABC$,

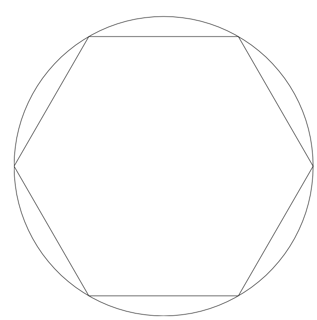
$$\begin{aligned} AC^2 &= AB^2 - BC^2 \\ &= 100 - 36 = 64. \end{aligned}$$

Therefore $AC = 8$. Similarly in $\triangle ACD$, $AD = \sqrt{AC^2 + CD^2} = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$.



6. The third side of the box has length between 50 cm and 90 cm. Therefore the smallest possible volume of the box is $50 \times 50 \times 90 = 225\ 000$ cm^3, and the largest possible volume is $50 \times 90 \times 90 = 405\ 000$ cm^3. Only 360 000 lies between these numbers.

7. Angle XYZ is 120^o and $OX = OY = OZ$. Therefore the angles at X and Y are each 60^o. Hence each triangle is equilateral, with sides equal to the radius r. It follows that the perimeter of the hexagon is $6r$ and the circumference of the circle is $2\pi r$. The required ratio is $2\pi r : 6r$.



8. Suppose that the two-digit number is mn, which means $10m + n$. The sum of the digits is $m + n$. So we are given that $(10m + n) - (m + n) = 9m = 45$. Hence, $m = 5$. n can be any of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. The ten possible two-digit numbers are $50, 51, 52, \ldots, 59$.

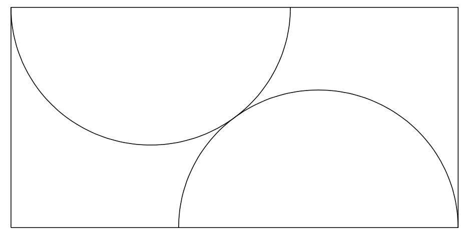
9. Let the 4 consecutive numbers be $n-1, n, n+1$ and $n+2$. The sum of their squares is

$$(n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 = 4n^2 + 4n + 6.$$

We are given that $4n^2 + 4n + 6 = 5334$. Simplifying we get, $n^2 + n - 1332 = 0$. By factorizing 1332, or by guessing that its square root is about 36, we see that $(n+37)(n-36) = 0$. Therefore $n = 36$ and the smallest of the 4 numbers, $n-1$, is 35.

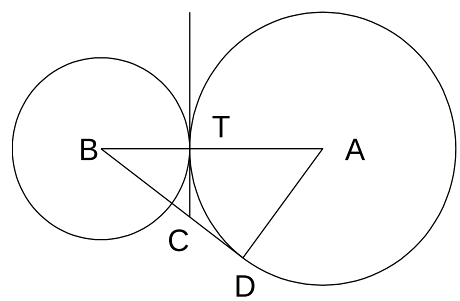
10. The natural numbers that leave a remainder of 41 when divided into 1997 are those numbers that divide exactly into $1997 - 41 = 1956$. So we are looking for all positive integers greater than 41 which are factors of 1956. Since $1956 = 2^2 \times 3 \times 163$ where 163 is a prime number, the 6 possible number are $163, 2 \times 163, 4 \times 163, 3 \times 163, 6 \times 163$ and 12×163.

11. Let R be the radius of the largest semicircle and let C and O be the centres. Then $OC = 2R, CX = 80$, and $OX = 160 - 2R$. Now apply Pythagoras' theorem to $\triangle CXO$: $(2R)^2 = 80^2 + (160-2R)^2$. Therefore, $0 = 5 \times 80^2 - 640R$, from which it follows that $R = 50$ cm. Therefore $2R = 100$ cm.

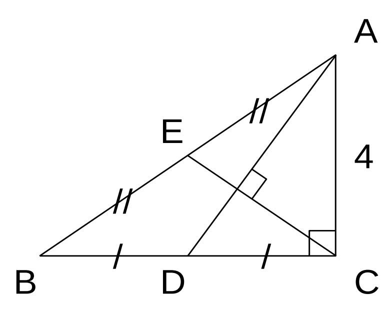


12. Since $\sqrt{x^4+16} > \sqrt{x^4} = x^2 > x^2 - 4$, it follows that $\sqrt{x^4+16}$ can never be equal to $x^2 - 4$. **Note**: If we square both sides of the given equation we get, $x^4 + 16 = x^4 - 8x^2 + 16$ which leads to $8x^2 = 0$ and $x = 0$. But that is not a solution to the given equation; it is solution to the equation $\sqrt{x^4+16} = -(x^2-4)$.

13. Tangents and radii meet at right angles. Hence the angles at T and D are 90^o. Triangles BTC and BDA are similar. Therefore $BT/BD = TC/DA$. But $BT = 2, BD = \sqrt{BA^2 - AD^2} = \sqrt{25-9} = 4, TC = CD$ and $DA = 3$. Thus $CD = 2 \times 3/4 = 3/2$.



14. $EA = EB = EC$, since $A\hat{C}B = 90^o$ (think of AB as the diameter of a circle). Let $D\hat{A}C = \alpha$. Then $D\hat{C}O = \alpha$, and also $A\hat{B}C = \alpha$ (since $EB = EC$). Triangles ACD and BCA are similar, therefore $\frac{AC}{BC} = \frac{CD}{AC}$. But $CD = \frac{1}{2}BC$ and $AC = 4$, hence $BC^2 = 32$. Finally apply Pythagoras' theorem to triangle ABC to obtain $AB^2 = BC^2 + CA^2 = 32 + 16 = 48$. Therefore $AB = 4\sqrt{3}$.



15. A unitary fraction is a fraction of the form $\frac{1}{n}$ where n is a natural number. We look for the largest unitary fraction less than $\frac{23}{30}$. So we look for the smallest natural number m such that $\frac{1}{m} \leq \frac{23}{30}$. Clearly $m = 2$. So $\frac{23}{30} = \frac{1}{2} + (\frac{23}{30} - \frac{1}{2}) = \frac{1}{2} + \frac{8}{30} = \frac{1}{2} + \frac{4}{15}$. Now we repeat the argument above for $\frac{4}{15}$ instead of $\frac{23}{30}$. Notice that $\frac{1}{4} = \frac{15}{60} < \frac{16}{60} = \frac{4}{15} < \frac{5}{15} = \frac{1}{3}$. Thus the next unitary fraction we need is $\frac{1}{4}$, not $\frac{1}{3}$. So far we have $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. Since $\frac{23}{30} - \frac{3}{4} = \frac{46-45}{60} = \frac{1}{60}$, one way of writing $\frac{23}{30}$ as a sum of unitary fractions is $\frac{23}{30} = \frac{1}{2}+\frac{1}{4}+\frac{1}{60}$. This requires 3 terms. Two other ways of writing $\frac{23}{30}$ as the sum of 3 unitary fractions are $\frac{1}{3} + \frac{1}{3} + \frac{1}{10}$ and $\frac{1}{2} + \frac{1}{5} + \frac{1}{15}$. There are others.

Is it possible to use fewer than 3 unitary fractions?

Clearly we need only to consider the sum of two such fractions. Thus, suppose that there are 2 positive integers p ad q such that $\frac{1}{p} + \frac{1}{q} = \frac{23}{30}$ and let us suppose that $p \geq q$. Obviously $q \geq 2$. Then $\frac{1}{p} \leq \frac{1}{q}$, and $\frac{23}{30} = \frac{1}{p} + \frac{1}{q} \leq \frac{2}{q}$. Taking reciprocals we get $\frac{q}{2} \leq \frac{30}{23}$ so that $q \leq \frac{60}{23} = 2\frac{14}{23}$. Since, $2 \leq q$, it follows that q can only be 2. Then $\frac{1}{p} = \frac{23}{30} - \frac{1}{q} = \frac{23}{30} - \frac{1}{2} = \frac{8}{30} = \frac{4}{15}$. This means that p is not an integer. Hence, two unitary fractions cannot add up to $\frac{23}{30}$.

16.

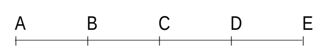
$$\frac{b}{c} = \frac{1997^2}{1996 \times 1998} = \frac{1997^2}{(1997-1)(1997+1)} = \frac{1997^2}{1997^2 - 1} > 1.$$

Therefore, $b > c$.

$$\frac{a}{b} = \frac{1996^2 \times 1998}{1995 \times 1997^2} = \frac{1996^2}{1996^2 - 1} \times \frac{1998}{1997} > 1$$

because each term in the last product is greater than 1. Therefore $a > b$ and we have $a > b > c$.

17. B, C and D are points such that $AB = BC = CD = DE$. Only

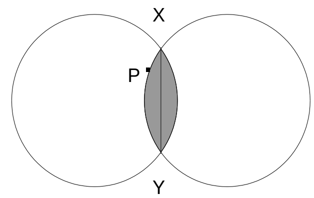


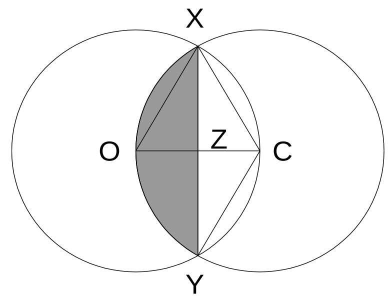
if we cut the string between A and B or between D and E will the longest piece be at least 3 times as long as the shortest piece. $AB + DE$ makes up $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ of the length of the string.

18. We can rewrite the given equation as $97y - 97 = 1900 - 19x$. When we take out common factors this becomes $97(y-1) = 19(100-x)$. Both 19 and 97 are primes, therefore 97 must be a multiple of $100 - x$. Thus, $x = 100$ or $x = 3$. The choice $x = 100$ gives $y = 1$ and $x + y = 101$. If we choose $x = 3$, then $97(y-1) = 19 \times 97$. Hence $y - 1 = 19$, or $y = 20$, with $x + y = 23$.

19. From $(a+b)^2 = 25$ follows that $a^2 + b^2 = (a+b)^2 - 2ab = 25 - 2 \times 2 = 21$. Then $(a^2+b^2)^2 = 21^2 = 441$. Therefore, $a^4 + b^4 = (a^2+b^2)^2 - 2a^2b^2 = 441 - 2 \times 2^2 = 433$.

20. Let the centres of the two circles be at O and C and let the radius of each circle be R. The chord XY of the circle centred at C will subtend the angle at any point P on the arc XY, and twice that angle at the centre C. Therefore the reflex angle at C is 240^o, and the angle XCY is $360^o - 240^o = 120^o$.





This means that the two circles pass through each other's centres, as shown in the second picture. Note that $\triangle XOC$ is an equilateral triangle with each side equal to R. Since the altitude ZX is given to be $3/2$ and $Z\hat{X}C = 30^o$, it follows that $R = \sqrt{3}$.

The required area is twice the shaded area $XOYZ$, which is the area of the sector CXOY minus the area of the triangle CXY. The sector $CXOY$ is one third the area of the circle because the angle XCY is 120^o. Therefore the area of the sector is $\frac{1}{3}\pi R^2 = \frac{1}{3}\pi(\sqrt{3})^2 = \pi$. The area of triangle CXY is $\frac{1}{2} \times 3 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}/4$. Therefore the answer is $2(\pi - 3\sqrt{3}/4)$.