The South African Mathematical Olympiad
Third Round 2021
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Find the smallest and largest integers with decimal representation of the form $ababa$ ($a \neq 0$) that are divisible by 11.

2. Let PAB and PBC be two similar right-angled triangles (in the same plane) with $\angle PAB = \angle PBC = 90^\circ$ such that A and C lie on opposite sides of the line PB. If $PC = AC$, calculate the ratio $\frac{PA}{AB}$.

3. Determine the smallest integer $k > 1$ such that there exist k distinct primes whose squares sum to a power of 2.

4. Let ABC be a triangle with $\angle ABC \neq 90^\circ$ and AB its shortest side. Denote by H the intersection of the altitudes of triangle ABC. Let K be the circle through A with centre B. Let D be the other intersection of K and AC. Let K intersect the circumcircle of BCD again at E. If F is the intersection of DE and BH, show that BD is tangent to the circle through D, F and H.

5. Determine all polynomials $a(x), b(x), c(x), d(x)$ with real coefficients satisfying the simultaneous equations:

$$b(x)c(x) + a(x)d(x) = 0$$
$$a(x)c(x) + (1 - x^2)b(x)d(x) = x + 1$$

6. Jacob and Laban take turns playing a game. Each of them starts with the list of square numbers $1, 4, 9, \ldots, 2021^2$, and there is a whiteboard in front of them with the number 0 on it. Jacob chooses a number x^2 from his list, removes it from his list, and replaces the number W on the whiteboard with $W + x^2$. Laban then does the same with a number from his list, and they repeat back and forth until both of them have no more numbers in their list. Now every time that the number on the whiteboard is divisible by 4 after a player has taken his turn, Jacob gets a sheep. Jacob wants to have as many sheep as possible by the end of the game, whereas Laban wants Jacob to have as few sheep as possible. What is the greatest number K such that Jacob can guarantee to get at least K sheep by the end of the game, no matter how Laban plays?

Each problem is worth 7 points.

Die Suid-Afrikaanse Wiskunde Olimpiade
Derde Ronde 2021
Senior Afdeling (Grade 10 tot 12)
Tyd : 4 ure
(Geen rekenapparaat word toegelaat nie)

1. Vind die kleinste en grootste heelgetalle van die (desimale) vorm $ababa$ ($a \neq 0$) wat deelbaar is deur 11.

2. Laat PAB en PBC twee gelykvormige reghoekige driehoeke (in dieselfde vlak) wees, met $\angle PAB = \angle PBC = 90^\circ$ en sodanig dat A en C aan weerskante van die lyn PB geleë is. Indien $PC = AC$, bepaal die verhouding $\frac{PA}{AB}$.

3. Bepaal die kleinste heelgetal $k > 1$ sodanig dat daar k verskillende priemgetalle bestaan waarvan die som van hulle kwadrate 'n mag van 2 is.

4. Laat ABC 'n driehoek wees met $\angle ABC \neq 90^\circ$ en AB die kortste sy. Die snypunt van die hoogtelyne van driehoek ABC word deur H aangedui. Laat K die sirkel wees wat A bevat, en met middelpunt B. Laat D die ander snypunt van K en AC wees. Verder, laat K die omgeskrewe sirkel van BCD weer by E sny. Indien F die snypunt is van DE en BH, wys dat BD 'n raaklyn is aan die sirkel deur D, F en H.

5. Vind alle polinome $a(x), b(x), c(x), d(x)$ met reële koëffisiënte wat die volgende twee vergelykings gelyktydig bevredig:

$$b(x)c(x) + a(x)d(x) = 0$$
$$a(x)c(x) + (1 - x^2)b(x)d(x) = x + 1$$

6. Jakob en Laban neem beurte om 'n spel te speel. Elkeen van hulle begin met die lys van kwadrate $1, 4, 9, \ldots, 2021^2$, en daar is 'n witbord voor hulle met die getal 0 daarop. Jakob kies 'n getal x^2 van sy lys, verwyder dit uit sy lys, en vervang die getal W op die witbord met $W + x^2$. Laban doen dan dieselfde met 'n getal uit sy lys, en hulle herhaal hierdie proses om die beurt totdat beide hulle lyste leeg is. Elke keer as die getal op die witbord deelbaar is deur 4 nadat 'n speler sy beurt geneem het, kry Jakob 'n skaap. Jakob wil so veel as moontlik skape hê aan die einde van die spel terwyl Laban wil hê dat Jakob so min as moontlik skape aan die einde van die spel moet hê. Wat is die grootste getal K sodat Jakob kan verseker om minstens K skape teen die einde van die spel te hê, ongeag hoe Laban speel?

Elke probleem is 7 punte werd.