MEMO (SOLUTIONS)

Question 1

We write both numbers as the square root of an integer:

- $2\sqrt{2021} = \sqrt{4 \times 2021} = \sqrt{8084}$
- $20\sqrt{21} = \sqrt{400 \times 21} = \sqrt{8400}$

Hence $20\sqrt{21}$ is larger.

Question 2

If the two temperature scales give the same reading, then $F = C$, and so we need to solve the equation

$$\mathrm{C} = \frac{9}{5}C + 32 \Rightarrow -\frac{4}{5}\mathrm{C} = 32 \Rightarrow \mathrm{C} = -40$$

.

Question 3

a) Area of triangle APB $= \frac{1}{2}AB \cdot DP = \frac{1}{2} \cdot 6\sqrt{3} \cdot 1 = 3\sqrt{3}$

Area of triangle BPC $= \frac{1}{2}BC \cdot EP = \frac{1}{2} \cdot 6\sqrt{3} \cdot 3 = 9\sqrt{3}$

b) Let AB = BC = CA = s, and let $|XYZ|$ denote the area of triangle XYZ. Then

$$|ABC| = |APB| + |BPC| + |CPA|$$
$$\Rightarrow \frac{1}{2}AB \cdot h = \frac{1}{2}AB \cdot DP + \frac{1}{2}BC \cdot EP + \frac{1}{2}CA \cdot FP$$
$$\Rightarrow \frac{1}{2}s \cdot h = \frac{1}{2}s \cdot DP + \frac{1}{2}s \cdot EP + \frac{1}{2}s \cdot FP$$
$$\Rightarrow h = DP + EP + FP$$

c) Using Pythagoras in triangle AHC (DIAGRAM NEEDED), we see that

$$AH^2 + HC^2 = AC^2$$
$$\Rightarrow h^2 + \left(3\sqrt{3}\right)^2 = \left(6\sqrt{3}\right)^2$$
$$\Rightarrow h^2 = 108 - 27 = 81$$
$$\Rightarrow h = 9$$

The equation in part (b) then implies that PF = 9 – 1 – 3 = 5.

Question 4

a) The sum of the first k odd numbers is equal to k^2, a perfect square. Since 2021 is not a perfect square, Steve must have made a mistake.

b) Suppose that Steve added up the first k odd numbers and accidentally skipped the number x. In that case

$$1 + 3 + \cdots + (2k - 1) - x = 2021$$
$$\Rightarrow k^2 - x = 2021$$
$$\Rightarrow k^2 = 2021 + x$$

This means that we need to find a number x such that $2021 + x$ is a perfect square. (Note that since x is an odd number, it follows that k is also an odd number.) Now, $45^2 = 2025 = 2021 + 4$, so $x = 4$ is a possibility. It's possible that Steve accidentally skipped 4 when he added the numbers.

c) Suppose that Steve added up the first k odd numbers and accidentally added the number x twice. In that case

$$1 + 3 + \cdots + (2k - 1) + x = 2021$$
$$\Rightarrow k^2 + x = 2021$$
$$\Rightarrow k^2 = 2021 - x$$

Similar to part (b), we need to find a number x such that $2021 - x$ is a perfect square. (Note that since x is an odd number, it follows that k is also an odd number). From (b) we know that $45^2 = 2025$ is the first square number greater than 2021, so $43^2 = 1849$ is the largest odd square number less than 2021. In that case $x = 2021 - 1849 = 172$. However, in this case $k = 43$ and so Steve added all the odd numbers up to $2k - 1 = 85 < 172$, so Steve couldn't have added 172 twice.
If we consider k less than 43, it means then that $x > 172$ and the same argument shows that it's impossible for Steve to have added x twice, since it's bigger than the largest number he added.

Question 5

a) $x^4 + 4y^4 = x^4 + 2mx^2y^2 + m^2y^4 - n^2x^2y^2 = x^4 + (2m - n^2)x^2y^2 + m^2y^4$

$$\Rightarrow m^2 = 4 \quad \text{and} \quad 2m - n^2 = 0$$
$$\Rightarrow m = 2 \quad \text{and} \quad n = \pm 2$$

b) $x^4 + 4y^4 = (x^2 + my^2)^2 - (nxy)^2$

$$= (x^2 + 2y^2)^2 - (2xy)^2$$

or $$= (x^2 + 2y^2)^2 - (-2xy)^2$$

$$= (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)$$

c) $5^4 + 2^{14} = 5^4 + 2^2(2^3)^4$

$$= 5^4 + 4 \times 8^4$$
$$= (5^2 + 2 \times 8^2 - 2 \times 5 \times 8)(5^2 + 2 \times 8^2 + 2 \times 5 \times 8) \text{ (from part (b))}$$
$$= (25 + 128 - 80)(25 + 128 + 80)$$
$$= (73)(233)$$

Question 6

Note that for the game to end, the losing player's counter must be trapped at one end of the strip with the other player's counter right up against it. If we colour the squares of the strip alternatingly black and white, it means that at the end of the game, the two players' counters are on different coloured squares – if the counters are on the same coloured square, it means that there is at least one square open between them, and it's possible for either player to still make a move.

a) If n is even, then at the start of the game the two players' counters are on different coloured squares. If the first player makes a move, then both players' counters are on the same coloured square. If the second player then moves, the counters end up on different coloured squares again, and so on. Since the game can only end when the two counters are on different coloured squares, the game can only end after the second player has made a move. This means that only the second player can win.

b) If n is odd, then at the start of the game the two players' counters are on the same colour squares. If the first player makes a move, then the players' counters are on different colour squares. If the second player then moves, the counters end up on the same coloured squares again, and so on. Since the game can only end when the two counters are on different colour squares, the game can only end after the first player has made a move. This means that only the first player can win.

Question 7 (DIAGRAM NEEDED)

a) Note that BC = AC = AD = CD, and so triangle ADC is equilateral and triangle BCA is isosceles. Hence

$$60° = \widehat{ACD} = \widehat{ABC} + \widehat{BAC} = 2\widehat{ABC} \Rightarrow \widehat{ABC} = 30°$$

b) Since $\widehat{ABC} = 30°$and $\widehat{ADC} = 60°$it follows that $\widehat{BAD} = 90°$. By Pythagoras, it follows that

$$BA^2 = BD^2 - DA^2 = 2^2 - 1^2 = 3 \Rightarrow BA = \sqrt{3}$$

Hence the area of triangle ABD equals $\frac{1}{2}AB \cdot AD = \frac{\sqrt{3}}{2}$. The shaded area is the area of triangle ABD minus the segment ADC of the circle with radius D. But since $\widehat{ADC} = 60°$, this segment has area $\frac{60°}{360°} = \frac{1}{6}$ of the area of the circle, which equals $\frac{1}{6} \cdot \pi 1^2 = \frac{\pi}{6}$.

The area of the shaded region is thus equal to $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$.

Question 8

Let the shared sum of the numbers in the four triangles ABC, XYZ, KLM and PQR equal S, and let x be the number in the central triangle XQC. If we add all the numbers in triangles ABC, PQR and XYZ, note that we add the number in every small triangle exactly once, except for the centre triangle, which is added 3 times. Hence we get the equation $3S = 1 + 2 + \cdots + 10 + 2x = 55 + 2x$. Since the left hand side is a multiple of 3, the right hand side is also. The only possibilities are $x = 1, 4, 7$ or 10. However, since the numbers 4 and 7 are already placed in the grid, this only leaves us with $x = 1$ or 10.

If x = *1*, 3S = 57, which gives S = 19. Since the sum of the numbers in triangle PQR = 19, the number in triangle CMX must equal 19 – 2 – 7 – 1 = 9.

The remaining numbers are now 3, 6, 8 and 10, and we must arrange them in the remaining squares.

- From triangle XYZ we get that QYL + QCL = 19 – 5 – 1 = 13. The only two remaining numbers that sum to 13 are 3 and 10, so those two squares must be 3 and 10 in some order.
- From triangle KLM we get that XKQ + QCL = 19 – 1 – 9 = 9. The only two remaining numbers that sum to 9 are 3 and 6, so those two squares must be 3 and 6 in some order.
- This means that the common triangle QCL in those two equations must equal the common number 3. The rest of the diagram is now easily completed to show that this solution works.

If x = *10,* 3S = 75, which gives S = 25. Since the sum of the numbers in triangle PQR = 25, the number in triangle CMX must equal 25 – 2 – 7 – 10 = 6.

The remaining digits are now 1, 3, 8 and 9, and we must arrange them in the remaining squares.

- From triangle XYZ we get that QYL + QCL = 25 – 5 – 10 = 10. The only two remaining numbers that sum to 10 are 1 and 9, so those two squares must be 1 and 9 in some order.
- From triangle KLM we get that XKQ + QCL = 25 – 10 – 6 = 9. The only two remaining numbers that sum to 9 are 1 and 8, so those two squares must be 1 and 8 in some order.
- This means that the common triangle QCL in those two equations must equal the common number 1. The rest of the diagram is now easily completed to show that this solution works.

The possible values of the number in triangle QCL are thus 1 and 3.

Question 9

First solution

Multiplying both sides of the equation by 5*mn* we get

$$5mn\left(\frac{1}{m}+\frac{1}{n}\right) = \frac{5mn}{5} \Rightarrow 5m + 5n = mn$$

$$\Rightarrow mn - 5m - 5n = 0$$

$$\Rightarrow (m-5)(n-5) = 25$$

Since the factors of 25 are ±1, ±5 and ±25, it follows that one of the following cases must hold (remembering that $m \geq n$):

$$m-5 = n-5 = 5$$

$$m-5 = n-5 = -5$$

$$m-5 = 25, n-5 = 1$$

$$m-5 = -1, n-5 = -25$$

The first set of equations leads to *m = n = 10.*

The second set of equations leads to *m = n = 0*, which doesn't work.

The third set of equations leads to *m = 30, n = 6.*

The fourth set of equations leads to *m = 4, n = -20.*

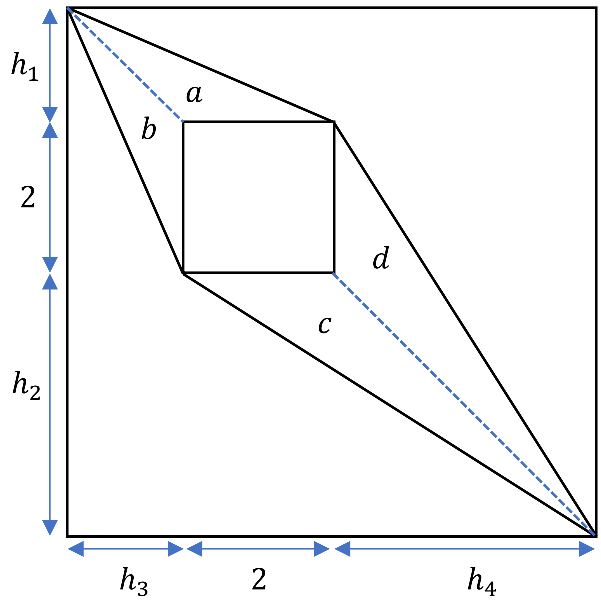
(This method proves that these three are the only solutions.)

Second solution

Let's ignore the fact that $m \geq n$, since if we find a solution where $n < m$, we can just swap m and n. Solving the given equation for m, we get $m = \frac{5n}{n-5}$. Since m is an integer, we need for find values for n such that the expression on the right hand side is an integer. The obvious choices for n such that $n - 5$ divides into $5n$ are $n - 5 = \pm 1$ and if $n - 5 = 5$, which lead to the same three solutions found above.

(This method does *not* prove that these are the only solutions, since we still need to show that there aren't other values of n for which the fraction $\frac{5n}{n-5}$ is an integer.)

Question 10



Split the shaded area into four triangles a, b, c, d.

Calculate the area of each one separately and add up the corresponding areas.

The area is thus:

$a + c + b + d$

$= \frac{1}{2} \cdot 2 \cdot h_1 + \frac{1}{2} \cdot 2 \cdot h_2 + \frac{1}{2} \cdot 2 \cdot h_3 + \frac{1}{2} \cdot 2 \cdot h_4$

$= \frac{1}{2} \cdot 2 \cdot (h_1 + h_2) + \frac{1}{2} \cdot 2 \cdot (h_3 + h_4)$

But $h_1 + h_2 = h_3 + h_4 = 7 - 2 = 5$

$= \frac{1}{2} \cdot 2 \cdot (h_1 + h_2) + \frac{1}{2} \cdot 2 \cdot (h_3 + h_4)$

$= \frac{1}{2} \cdot 2 \cdot (5) + \frac{1}{2} \cdot 2 \cdot (5)$

$= 10$

Similarly, you can join the vertices of the two squares, calculate the areas of the outside triangles and subtract them and the small square from the big square for the shaded area.

Therefore, it follows that the final result is not dependent on h_1 and h_2

$= \frac{1}{2} \cdot 2 \cdot (h_1 + h_2) + \frac{1}{2} \cdot 2 \cdot (h_3 + h_4)$

$= \frac{1}{2} \cdot 2 \cdot (5) + \frac{1}{2} \cdot 2 \cdot (5)$

Question 11

a) Write $7^{2021} = 7 \cdot 7^{2020} = (6+1)7^{2020} = 6 \cdot 7^{2020} + 7^{2020}$, which leaves the same remainder as 7^{2020} when divided by 6. Repeating the argument, we see that $7^{2020}, 7^{2019}, 7^{2018}, \dots 7^1$ all leave the same remainder when divided by 6, which is thus equal to 1.

b) $6n = 7n - n = 7(7 + 7^2 + \cdots + 7^{2021}) - (7 + 7^2 + \cdots + 7^{2021})$
$= (7^2 + 7^3 + \cdots + 7^{2022}) - (7 + 7^2 + \cdots + 7^{2021})$
$= 7^{2022} - 7$

Now, observe the following:
The last two digits of $7^2 = 49$ is 49.
The last two digits of $7^3 = 343$ is 43.
The last two digits of $7^4 = 2401$ is 01.
The last two digits of 7^5 is thus $7 \times 01 = 07$, and the pattern will repeat $07, 49, 43, 01$ from here on.
Note that by the pattern, if *n* is a multiple of 4, then 7^n ends with 01. Hence 7^{2022} ends with 43, and so $7^{2022} - 7$ ends with 37.

Question 12 (DIAGRAMS NEEDED)

a) Consider any 2x2 square in the grid. Since black squares may touch diagonally, but not share an edge, any such 2x2 square can have a maximum of two squares painted black. So if we divide the 20x20 square into a hundred 2x2 squares, we see that at most half of the squares can be painted black. Painting the 20x20 grid in checkerboard style shows that it's possible to colour exactly half the squares black, so the maximum value is 200 squares.

b) Consider any 2x2 square in the grid. Since every unit square in the 2x2 square touches all three other squares, at most one of the four unit square can be painted black. So, if we divide the 20x20 grid into a hundred 2x2 squares, we see that at most a quarter of the squares can be painted black. A possible such colouring is to paint every top left square in each 2x2 square black. The maximum value is thus 100 squares.

Question 13

We make the following observations:

- The number VIRUS doubled gives a 6-digit even number, it forces **V > 4** and **C = 1**. It also implies that **A is even**.
- Since the first and third columns from the left both result in O, it means that V and R differ by exactly 5. Since V > 4 from the previous point, it means that **R = V – 5**.
- The above also implies that there is a carry into both the first and third columns, or neither has a carry, which means that **I and U are either both less than 5, or both greater than or equal to 5**.
- Since R < 5, there is no carry into the second column from the left. This means that R is even; i.e. **R = 2 or R = 4.**
- The second column from the left implies that 2I = R or 2I = R + 10, so **I = R/2 or I = R/2+5**.

- If $R = 2$:
    - Then from the above, $V = 7$ and $I = 1$ or 6.
    - Since $C = 1$, it forces $I = 6$, and so $O = 5$ and $U > 4$.
    - The remaining possible values of A are 0, 4 or 8:
        - If $A = 0$, $S = 5$, which is impossible since $O = 5$.
        - If $A = 4$, $S = 2$ or $S = 7$, both impossible (since $R = 2$, $V = 7$)
    - Hence, $A = 8$, and $S = 4$ or $S = 9$.
    - If $S = 9$, then N is odd and so $N = 3$ (the only remaining odd number). But then $U = 6$ (since $U > 4$), which is impossible since $I = 6$.
    - If $S = 4$, then N is even and hence must be 0 (the only remaining even number). But then $U = 5$, which is impossible since $O = 5$.
    - Hence, $R=2$ is not a solution

- Hence **$R = 4$** and **$V = 9$.**
- It follows from the first column that **$O = 8$** and **U and I both < 5.**
- Hence, **$I = R/2 = 2$.**
- Since U cannot be 0 (otherwise N is either 0 or 1, both of which are used) it must be that **$U = 3$** (the only remaining number less than 5), and so $N = 6$ (when $S < 5$) or $N = 7$ (when $S > 4$).
- If $N = 6$, it forces $A = 0$ (the only remaining even number) and so $S = 5$, contradicting $S < 5$, hence **$N = 7$** and **$S > 5$.**
- $S = 6$ leads to $A = 2$ (impossible since $I = 2$) and so **$S = 5$** (the only remaining number greater than or equal to 5) and **$A = 0$**.

$$\begin{array}{r} \mathbf{92435} \\ \mathbf{92435} \\ \text{---------} \\ \mathbf{184870} \end{array}$$

Question 14

Alice < Christine
Brad > Christine
Christine > David
David < Erin
ALL > Farah
But, ?? > Brad
Therefore Erin > Brad

Therefore, order must be Erin > Brad > Christine > Alice, David > Farah

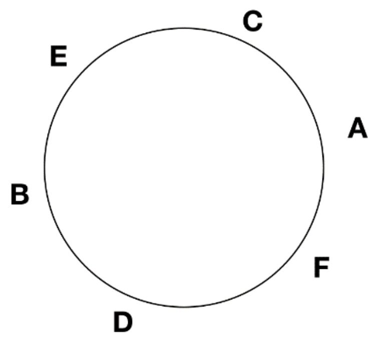
Alice > person to her right
Either Alice David or Alice Farah

Brad < person to his right
Brad Erin

Christine < person to her left
Either Brad or Christine therefore Brad Erin Christine

David > person to his left
Either Alice David or Farah David

Therefore, final arrangement must be Brad Erin Christine Alice Farah David



Question 15

If we roll out the cylinder into a flat rectangle the ribbon will be wrapped around as shown below with width equal to the circumference of the circle radius 2 at 4π.

Because the angle of the ribbon is 45° it makes the shape below a square, and thus the length of the square is also 4π. This makes the diagonal of the square $4\sqrt{2}\pi$ using Pythagoras.

The diagonal drawn in from bottom left to top right is shown. The ribbon and the space in between have an equal width and thus the ribbon width is just $\frac{4\sqrt{2}\pi}{4} = \sqrt{2}\pi$.