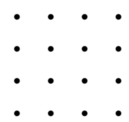
The Old Mutual South African Mathematics Olympiad
Third Round 2022
Senior Division (Grades 10 to 12)

1. Consider 16 points arranged as shown, with horizontal and vertical distances of 1 between consecutive rows and columns. In how many ways can one choose four of these points such that the distance between every two of those four points is strictly greater than 2?



2. Find all pairs of real numbers x and y which satisfy the following equations:

$$x^2 + y^2 - 48x - 29y + 714 = 0$$
$$2xy - 29x - 48y + 756 = 0$$

3. Let a, b, and c be nonzero integers. Show that there exists an integer k such that

$$\gcd(a + kb, c) = \gcd(a, b, c).$$

(Note: '$\gcd$' stands for 'greatest common divisor')

4. Let ABC be a triangle with $AB < AC$. A point P on the circumcircle of ABC (on the same side of BC as A) is chosen in such a way that $BP = CP$. Let BP and the angle bisector of $\angle BAC$ intersect at Q, and let the line through Q and parallel to BC intersect AC at R. Prove that $BR = CR$.

5. Let $n \geqslant 3$ be an integer, and consider a set of n points in three-dimensional space such that:

   (i) every two distinct points are connected by a string which is either red, green, blue, or yellow;

   (ii) for every three distinct points, if the three strings between them are not all of the same colour, then they are of three different colours;

   (iii) not all the strings have the same colour.

   Find the maximum possible value of n.

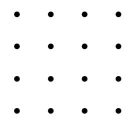
6. Show that there are infinitely many polynomials P with real coefficients such that if x, y, and z are real numbers such that $x^2 + y^2 + z^2 + 2xyz = 1$, then

$$P(x)^2 + P(y)^2 + P(z)^2 + 2P(x)P(y)P(z) = 1.$$

Each problem is worth 7 points.

Die Old Mutual Suid-Afrikaanse Wiskunde Olimpiade
Derde Ronde 2022
Senior Afdeling (Grade 10 tot 12)

1. Beskou 16 punte gerangskik soos aangetoon, met horisontale and vertikale afstand van 1 tussen opeenvolgende rye en kolomme. Op hoeveel maniere kan 'n mens vier van hierdie punte kies sodat die afstand tussen elke twee van hierdie vier punte streng groter as 2 is?



2. Bepaal alle pare reële getalle x en y wat die volgende vergelykings waar maak:

$$\begin{aligned} x^2 + y^2 - 48x - 29y + 714 &= 0 \\ 2xy - 29x - 48y + 756 &= 0 \end{aligned}$$

3. Laat a, b en c nie-nul heelgetalle wees. Bewys dat daar 'n heelgetal k bestaan sodanig dat

$$\operatorname{ggd}(a + kb, c) = \operatorname{ggd}(a, b, c).$$

(Nota: 'ggd' staan vir 'grootste gemene deler')

4. Laat ABC 'n driehoek wees met $AB < AC$. 'n Punt P op die omgeskrewe sirkel van ABC (aan dieselfde kant van BC as A) is gekies op só 'n wyse dat $BP = CP$. Laat BP en die halveerlyn van $\angle BAC$ in Q ontmoet, en laat die lyn deur Q en parallel aan BC, AC in R sny. Bewys dat $BR = CR$.

5. Laat $n \geqslant 3$ 'n heelgetal wees, en beskou 'n versameling met n punte in die drie-dimensionele ruimte, sodanig dat:

   (i) elke twee verskillende punte verbind is deur 'n string wat óf rooi, óf groen, óf blou, óf geel is;

   (ii) vir elke drie verskillende punte, as die drie stringe tussen hulle nie almal dieselfde kleur het nie, dan is hulle van drie verskillende kleure;

   (iii) nie al die stringe het dieselfde kleur nie.

   Bepaal die grootste moontlike waarde van n.

6. Bewys dat daar oneindig veel polinome P met reële koëffisiënte is sodat as x, y en z reële getalle is sodanig dat $x^2 + y^2 + z^2 + 2xyz = 1$, dan is

$$P(x)^2 + P(y)^2 + P(z)^2 + 2P(x)P(y)P(z) = 1.$$

Elke probleem is 7 punte werd.