South African Mathematics Olympiad
Third Round 2001

Answer all questions. No calculators or other technological accessories are allowed except the usual geometric drawing instruments.
Time: 4 hours.

1. $ABCD$ is a convex quadrilateral with perimeter p. Prove that
$$\tfrac{1}{2}p < AC + BD < p.$$
(A polygon is convex if all of its interior angles are less than $180°$.)

2. Find all triples (x, y, z) of real numbers that satisfy
$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2) = 4xyz = 4(x+y+z).$$

3. For a certain real number x, the differences between x^{1919}, x^{1960} and x^{2001} are all integers. Prove that x is an integer.

4. n red and n blue points on a plane are given so that no three of the $2n$ points are collinear. Prove that it is always possible to split up the points into n pairs, with one red and one blue point in each pair, so that no two of the n line segments which connect the two members of a pair intersect.

5. Starting from a given cyclic quadrilateral $\mathcal{Q}_0$, a sequence of quadrilaterals is constructed so that $\mathcal{Q}_{k+1}$ is the circumscribed quadrilateral of $\mathcal{Q}_k$ for $k = 0, 1, \ldots$. The sequence terminates when a quadrilateral is reached that is not cyclic. (The circumscribed quadrilateral of a cyclic quadrilateral $ABCD$ has sides that are tangent to the circumcircle of $ABCD$ at A, B, C and D.)

   Prove that the sequence always terminates, except when $\mathcal{Q}_0$ is a square.

6. The unknown real numbers $x_1, x_2, \ldots, x_n$ satisfy $x_1 < x_2 < \ldots < x_n$, where $n \geqslant 3$. The numbers s, t and $d_1, d_2, \ldots, d_{n-2}$ are given, such that
$$s = \sum_{i=1}^{n} x_i;$$
$$t = \sum_{i=1}^{n} x_i^2;$$
$$d_i = x_{i+2} - x_i, \; i = 1, 2, \ldots, n-2.$$
For which n is this information always sufficient to determine $x_1, x_2, \ldots, x_n$ uniquely?

Suid-Afrikaanse Wiskunde-olimpiade
Derde Ronde 2001

Beantwoord al die vrae. Geen sakrekenaars of ander tegnologiese hulpmiddels behalwe die gewone meetkundige tekengereedskap word toegelaat nie.
Tyd: 4 uur.

1. $ABCD$ is 'n konvekse vierhoek met omtrek p. Bewys dat
$$\tfrac{1}{2}p < AC + BD < p.$$
('n Veelhoek is konveks as al sy binnehoeke kleiner as 180° is.)

2. Vind alle triplette reële getalle (x, y, z) wat
$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2) = 4xyz = 4(x+y+z)$$
bevredig.

3. Vir 'n sekere reële getal x is die verskille tussen x^{1919}, x^{1960} en x^{2001} almal heelgetalle. Bewys dat x 'n heelgetal is.

4. n rooi en n blou punte in 'n platvlak word gegee sodat geen drie van die $2n$ punte saamlynig is nie. Bewys dat dit altyd moontlik is om die punte in n pare op te deel, met een rooi en een blou punt in elke paar, sodat geen twee van die n lynsegmente wat die twee lede van 'n paar verbind, mekaar sny nie.

5. Vanaf 'n gegewe koordevierhoek $\mathcal{Q}_0$, word 'n ry vierhoeke gekonstrueer sodat $\mathcal{Q}_{k+1}$ die omgeskrewe vierhoek van $\mathcal{Q}_k$ is vir $k = 0, 1, \ldots$. Die ry eindig wanneer 'n vierhoek bereik word wat nie 'n koordevierhoek is nie. ('n Koordevierhoek $ABCD$ se omgeskrewe vierhoek het sye wat die omgeskrewe sirkel van $ABCD$ by A, B, C en D raak.)
Bewys dat die ry altyd eindig, behalwe wanneer $\mathcal{Q}_0$ 'n vierkant is.

6. Die onbekende reële getalle $x_1, x_2, \ldots, x_n$ bevredig $x_1 < x_2 < \ldots < x_n$, waar $n \geqslant 3$. Die getalle s, t en $d_1, d_2, \ldots, d_{n-2}$ word gegee, sodat
$$s = \sum_{i=1}^{n} x_i;$$
$$t = \sum_{i=1}^{n} x_i^2;$$
$$d_i = x_{i+2} - x_i,\ i = 1, 2, \ldots, n-2.$$
Vir watter n is hierdie inligting altyd genoeg om $x_1, x_2, \ldots, x_n$ uniek te bepaal?