THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

SENIOR SECOND ROUND 2020

Solutions

1. **Answer 000**
   Since each digit is equal to 1 or more, a sum of less than 7 means that at least one of the digits is zero. Therefore the product of the digits is zero.

2. **Answer 001**
   The LCM of $2, 3, 4, 5$ is 60, and 60 is the only *positive* integer less than 100 and divisible by 60.

3. **Answer 170**
   Suppose Kerstin is on page n of her book. Then $n+(n+1) = 341$, so $n = (341-1)/2 = 170$.

4. **Answer 505**
   The prime factorization of 2020 is $2^2 \times 5 \times 101$. In a perfect square, the exponent of each prime factor must be even, so the smallest number is 5×101.

5. **Answer 040**
   In minutes after 06:00, the times are -30, -15, 90, 115, and their average or mean is $(-30 - 15 + 90 + 115)/4 = 160/4 = 40$.

6. **Answer 057**
   The number who failed at least one subject is $30 + 25 - 12 = 43$, so the number who passed both subjects is $100 - 43 = 57$.
   Alternatively, the number who failed Physics only is $30 - 12 = 18$ and the number who failed Mathematics only is $25 - 12 = 13$. Therefore the number who passed both subjects is $100 - (18 + 13 + 12) = 57$.

7. **Answer 363**
   The number of entries in each row is three times the number in the previous row, so the numbers at the ends of the rows increase by powers of 3: 1, $1 + 3 = 4$, $4 + 3^2 = 13$, and so on. The last number in the sixth row is therefore
   $$1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 = 1 + 3 + 9 + 27 + 81 + 243 = 364,$$
   and the second-last number is 363.
   Alternatively, using the geometric series formula, the last number in the sixth row is $(3^6 - 1)/(3 - 1) = 728/2 = 364$.

8. **Answer 060**
   Herman travels 120 km in the first two hours, stays still for an hour, travels 60 km back the next hour, then 60 km away the following hour, and finally 120 km home again in the last hour. (These are *distances*, not displacements.) The total distance is $120 + 60 + 60 + 120 = 360$ km in six hours, for an average speed of $360/6 = 60$ km/h.

9. **Answer 256**
   Let x be the required number and let $y = \sqrt{x}$. Then $y - \sqrt{y} = 12$, so $\sqrt{y} = y - 12$. By squaring, it follows that $y = y^2 - 24y + 144$, giving $y^2 - 25y + 144 = 0$. This factorizes to $(y - 16)(y - 9) = 0$, so $y = 16$ or $y = 9$. The second answer is not valid, because $9 - \sqrt{9} = 6$. Therefore $y = 16$ and $x = y^2 = 256$.

10. **Answer 016**
If $x = 0.\dot{7}$, then $10x = 7.\dot{7} = 7 + x$, so $9x = 7$ and $x = \frac{7}{9}$, which is in lowest terms, and $7 + 9 = 16$.

11. **Answer 001**
There are four pairs of diagonally opposite places, and one green counter can be put into each pair. The green counters should be located to separate the reds, as far as possible. This leads to the following cyclic arrangement, which uses only one blue: RGRGGRGB.

12. **Answer 090**
By Pythagoras' theorem, the right-angled triangles in the square have sides in the ratio $1 : 3 : \sqrt{10}$. The shaded square has side length 3, so if the large square has side length $3x$, then $3 : x = 3 : \sqrt{10}$, so $x = \sqrt{10}$ and the total area is $9x^2 = 9 \times 10 = 90$.
Alternatively, inside the triangle with sides x, $3x$, $\sqrt{10}x$, it can be seen by similar triangles that $\sqrt{10}x = 9 + x/\sqrt{10}$, which again gives $x = \sqrt{10}$.

13. **Answer 018**
Suppose heads appears h times and tails appears $30 - h$ times. Then Ellie gives away $2h$ sweets and receives back $3(30 - h)$ sweets. It follows that $2h = 3(30 - h)$, so $5h = 90$ and $h = 18$.

14. **Answer 729**
Every day after the first day, the number of learners knowing the rules is multiplied by 3, since twice as many new people are added to those who knew the rules before. After six days (since $20 - 15 + 1 = 6$), the number will be $3^6 = 729$.

15. **Answer 180**
Let the required distance be h. Then triangle DCE has altitude $h - 60$, so its area is $\frac{1}{2}(60)(h - 60)$, which is equal to 60^2, the area of square $ABCD$. Therefore $\frac{1}{2}h - 30 = 60$, so $h = 180$.

16. **Answer 023**
The prime numbers up to 31 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and there are only eleven of them. Therefore Priscilla must be lying, so the day must be one of the above prime days. It cannot be 2 January, because then Odo would have to tell the truth, whereas his statement about the number of days he has lied could not be true. Therefore the day must be one of the odd primes, so Odo must also be lying. Suppose the day of the month is $2k - 1$, which is the k-th odd day. Then the statements "$k \geq 13$" and "$k < 11$" are both false, so $k = 11$ or 12, and the day of the month is 21 or 23. Since the day must also be prime, it must be 23.

17. **Answer 040**
Draw lines from each vertex to the centre of the circle. This divides the quadrilateral into four triangles, each with base on one side of the quadrilateral and with altitude equal to the radius. The area of each triangle is half the radius time its base, so by summing the areas of the triangles it follows that the area of the quadrilateral is equal to half the radius times the perimeter P. Therefore $60 = \frac{1}{2} \times 3 \times P$, so $P = \frac{2}{3} \times 60 = 40$.
Remark. This result holds for any polygon with an inscribed circle, and, in particular, for every triangle.

18. **Answer 091**
If there are m male students and f female students who pass, then $0.52(m+f) = 0.48m + 0.55f$ and the ratio of $m : f = 3 : 4 = 30 : 40 = 300 : 400$. $0.48 \times 300 = 144$ and $0.55 \times 400 = 220$ so that $\frac{m}{f} = \frac{144}{220} = \frac{36}{55}$ in lowest terms and we then have $m + f = 36 + 55 = 91$.

19. **Answer 052**
We are given $f(n+1) = f(n) + \frac{1}{2}$, so each term is obtained by adding $\frac{1}{2}$ to the previous term. It follows that $f(101) = f(1) + 100 \times \frac{1}{2} = 2 + 50 = 52$.
Remark. This is, of course, an arithmetic sequence.

20. **Answer 090**
Harry's choice of of a blue ball can arise in the following ways, with their associated probabilities:

Dick	Harry	Probability	
Blue Blue	Blue	$\frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} =$	$\frac{6}{210}$
Blue Red	Blue	$\frac{3}{7} \times \frac{4}{6} \times \frac{2}{5} =$	$\frac{24}{210}$
Red Blue	Blue	$\frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} =$	$\frac{24}{210}$
Red Red	Blue	$\frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} =$	$\frac{36}{210}$
		Total =	$\frac{90}{210}$

21. **Answer 006**
Arrange the crates into 25 piles corresponding to the numbers of apples inside, from 120 to 144. The average number of crates per pile is $128/25 = 5.12 > 5$, so the largest pile must contain at least 6 crates.
Remark. This is an extension of the Pigeonhole Principle, which says that if m objects are placed in n boxes, where $m > n$, then at least one box contains more than one object. In this case $m > kn$, where $m = 128$, $k = 5$, and $n = 25$, and the conclusion is that at least one box contains more than k objects. It can also be stated as "The maximum of a finite set of numbers is greater than or equal to their mean."

22. **Answer 006**
Let $x = a + b\sqrt{2}$, where a and b are rational. Then $x^2 = a^2 + 2b^2 + 2ab\sqrt{2} = 38 - 12\sqrt{2}$, so $a^2 + 2b^2 = 38$ and $2ab = -12$ by equating rational and irrational parts. Substituting $b = -6a^{-1}$ into the first equation gives

$$a^2 + 72a^{-2} = 38, \text{ or } (a^2)^2 - 38a^2 + 72 = 0, \text{ or } (a^2 - 36)(a^2 - 2) = 0.$$

The only rational solutions are $a = \pm 6$, so $b = \mp 1$ and $x = \pm(6 - \sqrt{2})$. Since x is a square root, it must be positive, so $x = 6 - \sqrt{2}$. Therefore

$$x^2 - 12x + 40 = (38 - 12\sqrt{2}) - 12(6 - \sqrt{2}) + 40 = 38 - 72 + 40 = 6.$$

Remark. The values of a and b can also be found by inspection, especially bearing in mind that, since the answer is an integer, it follows that $x^2 - 12x$ is an integer, which implies that $x + \sqrt{2}$ is at least rational.

23. **Answer 516**

There are 16 points, and each triangle (possibly with zero area) corresponds to a choice of three of the points as its vertices. This gives a total of $\frac{16\times15\times14}{3\times2\times1} = 560$ possible triangles. We must remove those with zero area, which arise when the three chosen points lie on a line. There are ten lines containing four points (four vertical lines, four horizontal, and two diagonal), and each line gives four triangles with zero area. There are also four diagonal lines with three points, each line giving one triangle with zero area. This gives a total of $10 \times 4 + 4 \times 1 = 44$ triangles to be removed, so the number of triangles with non-zero area is $560 - 44 = 516$.

24. **Answer 007**

The points $B_1(-8;2)$ and $B_2(2;2)$ lie on the original circle and are collinear with its centre and with A. The midpoints of AB_1 and AB_2 are $Q_1(\frac{1}{2}x - 4;2)$ and $Q_2(\frac{1}{2}x + 1;2)$, respectively. They are extreme points of the smaller circle, so the centre of the smaller circle is the midpoint of Q_1Q_2, which is the point $(\frac{1}{2}(x-3);2)$. This point lies on the original circle if it coincides with B_2, that is, if $\frac{1}{2}(x-3) = 2$, giving $x = 7$.

25. **Answer 273**

Suppose that when going from A to B there are x km uphill, y km level, and z km downhill, for a total of $x+y+z$ km. Then $\frac{x}{56}+\frac{y}{63}+\frac{z}{72} = 4$, or $9x+8y+7z = 4\times7\times8\times9 = 2016$. On the return trip, uphill and downhill are interchanged, so $7x+8y+9z = \frac{14}{3}\times7\times8\times9 = 2352$. Adding the two equations gives $16(x+y+z) = 4368$, so $x+y+z = 4368/16 = 273$.