1. Longest side 999, other sides (999,1) to (500,500): total 500. Longest side 998, other sides (998,3) to (501,500): total 498. Longest side 997, other sides (997,5) to (501,501): total 497. Longest side 996, other sides (996,7) to (502,501): total 495. Go on ... Longest side 667, other sides (667,665) to (666,666): total 2. There are 333 possible longest sides, and the total in each case decreases in a patten 2,1,2,1, ... The required total is therefore

$$\begin{aligned} & 2+3+5+6+\cdots+497+498+500 \\ &= (2+498)+(3+497)+(5+495)+\cdots+(249+251)+500 \\ &= 167\times 500 = 83500. \end{aligned}$$

2. Let s be the side of the square and $\theta = \angle PAD$. Then

$$\sin\theta = \frac{s}{AB}, \quad \cos\theta = \frac{s}{CD},$$

giving

$$\tan\theta = \frac{CD}{AB}.$$

So construct BE and CF perpendicular to AD so that $BE = CD$ and $CF = AB$, then P is at the intersection of AE and DF (produced if necessary) and draw parallels to PD through C and PA through B.

3. The case $AB = AD$ is trivial since C, K and L then coincide, so assume wlg $AB > AD$. Then K is between C and D and C is between B and L. Do some angle-chasing to find $\angle BAK = \angle DAK = \angle DKA = \angle CKL = \angle CLK$. Hence $DK = AD = BC$ and $BL = AB = DC$. Let the circle through C, L and K have centre O and radius r. By the power of a point theorem, $DO^2 - r^2 = DC\cdot DK = BL\cdot BC = BO^2 - r^2$. So $DO = BO$. Triangles DKO and BCO are congruent (three sides), giving $\angle OBC = \angle ODK = \angle ODC$. Therefore OCBD is a cyclic quadrilateral as required.

4. Note first that the result is true for $p = 2$. Using standard techniques as in the case of Fibonacci numbers, we find $L_n = a^n + b^n$ where a and b are the roots of $x^2 - x - 1 = 0$, so that $a+b = 1$ and $ab = -1$. Now write $L_n - 1$ in the form $L_n - 1 = a^n + b^n - (a+b)^n$. Expanding the last term, obtain in the case of $n = 2k+1$ (which includes all primes $n = p > 2$)

$$\begin{aligned} L_n - 1 &= -\binom{n}{1}ab^{n-1} - \binom{n}{2}a^2b^{n-2} - \cdots - \binom{n}{n-1}a^{n-1}b \\ &= -\binom{n}{1}\left(ab^{n-1} + a^{n-1}b\right) - \binom{n}{2}\left(a^2b^{n-2} + a^{n-2}n^2\right) - \cdots \\ &\quad - \binom{n}{k}\left(a^kb^{n-k} + a^{n-k}b^k\right) \end{aligned}$$

where

$$\binom{n}{j} = \frac{n(n-1)(n-2)\cdots(n-j+1)}{1\cdot 2\cdot 3\cdot \cdots \cdot j}.$$

It is therefore sufficient to show that $a^jb^{n-j} + a^{n-j}b^j$ is an integer for $j = 1, 2, \ldots, k$, since when $n = p$ the factor p in $\binom{p}{j}$ cannot cancel. This follows because

$$a^jb^{n-j} + a^{n-j}b^j = (ab)^j\left(a^{n-2j} + b^{n-2j}\right) = (-1)^jL_{n-2j}.$$

5. We examine first the case $x = 0$. Then $f(0) = af(0)$, which gives $f(0) = 0$ except when $a = 1$, in which case $f(0)$ is arbitrary. Similarly, when $x = 1$ we get $f(1) = 0$ except when $c = 1$, in which case $f(1)$ is arbitrary. We now express $f(x)$ when $0 < x < 1$ in terms of these values (zero or arbitrary) of $f(0)$ and $f(1)$.

   Let the base 3 representation of x be

$$x = 0.u_1u_2u_3\dots u_k$$

   with $u_k \neq 0$. Then

$$3x = u_1.u_2u_3\dots u_k.$$

   When $k = 1$, so that $x = 0.u$, only the arguments $3x - u = 0$ and $3x - (u - 1) = 1$ fall in $[0, 1]$, and therefore

$$f(0.u) = (d_u f(0) + d_{u-1} f(1)),$$

   where $d_0 = a$, $d_1 = b$ and $d_2 = c$.

   When $k > 1$, only the argument $3x - u_1$ falls in $[0, 1]$, and we get

$$\begin{aligned}
f(0.u_1u_2u_3\dots u_k) &= d_{u_1} f(0.u_2u_3\dots u_k) \\
&= d_{u_1} d_{u_2} f(0.u_3u_4\dots u_k) \\
&\dots \\
&= d_{u_1} d_{u_2} \dots d_{u_{k-1}} f(0.u_k) \\
&= d_{u_1} d_{u_2} \dots d_{u_{k-1}} (d_{u_k} f(0) + d_{u_k-1} f(1)).
\end{aligned}$$

6. By embedding a rectangle with opposite corners at $(0, 0)$ and (m, n) in Pascal's triangle, it is obvious that there are $\binom{m+n}{n}$ paths to (m, n) from $(0, 0)$. Hence without the restriction on the line $x = y$ (hereafter called the *river*) there would be $\binom{c-a+b-d}{b-d}$ paths. We now count the number of forbidden paths.

   Take any forbidden path P and let (t, t) be the last point at which it crosses the river, so that the subpath from (t, t) to (c, d) lies entirely below the river. Now consider a path obtained from P by reflecting only that last subpath in the river. This gives us a path P' from (a, b) to (d, c) and it is clear that two forbidden paths P and Q are different if and only if P' and Q' are different. Moreover, there is no path from (a, b) to (d, c) that cannot be obtained in this way, since (a, b) and (d, c) lie on opposite sides of the river and any path between them must cross the river.

   The number of forbidden paths therefore equals the number of all paths between (a, b) and (d, c). By subtracting this number from the preceding total, we find that there are $\binom{c-a+b-d}{b-d} - \binom{d-a+b-c}{b-c}$ different paths.