THE SOUTH AFRICAN
MATHEMATICS OLYMPIAD

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SECOND ROUND 2001

SENIOR SECTION: GRADES 10, 11 AND 12
(STANDARDS 8, 9 AND 10)

29 May 2001

TIME: 120 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. B
2. A
3. E
4. A
5. E
6. E
7. B
8. A
9. C
10. B
11. A
12. E
13. C
14. E
15. E
16. B
17. E
18. B
19. A
20. D

SOLUTIONS

1. Answer B. The unshaded area consists of a square and two right-angled triangles, with total area $3^2 + 2 \times (\frac{1}{2} \times 1 \times 4) = 13$. Thus the shaded region has area $16 - 13 = 3$.

2. Answer A. This is an estimation question, where only an approximate answer is required. The time in hours is $3 \times 7 \times 24 = 504 \approx 500$, but we must estimate the distance. It is much more than 1000 km (which is approximately the distance from Cape Town to Bloemfontein), but less than 20 000 km (which is half-way around the earth). Thus the speed is much more than $1000/500 = 2$ km/h, but less than $20000/500 = 40$ km/h, and only one answer fits.

3. Answer E. Suppose there are b boys and g girls in the family. Each boy has $b-1$ brothers and g sisters, so $b - 1 = g$. Each girl has b brothers and $g - 1$ sisters, so $b = 2(g - 1)$. Substituting for g gives $b = 2(b - 2)$, so $b = 4$ and $g = 3$, and the total number of children is $b + g = 7$.

4. Answer A. $x = \frac{x}{100} \times y$ and $y = \frac{y}{100} \times z$. Since $x \neq 0$, we can cancel x in the first equation to get $y = 100$, then substitute for y in the second equation to get $z = 100$.

5. Answer E. In Fig. 1 there is 1 circle with diameter 1, so its area is $\frac{\pi}{4}$. In Fig. 2 there are 4 circles with diameter $\frac{1}{2}$, and the total area is $4 \times \frac{\pi}{4}(\frac{1}{2})^2 = \frac{\pi}{4}$. In Fig. 3 the total area is $9 \times \frac{\pi}{4}(\frac{1}{3})^2 = \frac{\pi}{4}$, and in Fig. 4 it is $16 \times \frac{\pi}{4}(\frac{1}{4})^2 = \frac{\pi}{4}$ as well.

6. Answer E. All the triangles have a side of length 5 and one of length 12. Suppose they are all drawn with the side 12 as base, and the side of length 5 rotating about one end. For answers A and B the angle between these two sides is acute, for answers C and D it is obtuse, but for answer E it is a right angle, so the height (and also the area) of the triangle is a maximum.

7. Answer B. Suppose the number is x. Putting a 2 at the end changes it to $10x + 2$, and putting a 2 at the beginning adds 2×10^k (where k depends on the number of digits in x.) It follows that $2 \times 10^k + 10x + 2 = x + 2317 = x + 2 \times 10^4 + 317$, so $k = 4$ and we have $10x + 2 = x + 317$. Thus $x = 315/9 = 35$ and the sum of its digits is 8.

8. Answer A. If a natural number is both a perfect square and a perfect cube, then it must be a sixth power. There are three sixth powers up to 2001, namely $1^6 = 1$, $2^6 = 64$, and $3^6 = 729$, because $4^6 = 4096$, which is too big. (Alternatively, you could list the cubes from $1^3 = 1$ to $12^3 = 1728$, and eliminate those that are not also squares.)

9. Answer C. Since each of the two dice has six faces, there are $6 \times 6 = 36$ equally likely ways they can fall. The possible sums range from 2 to 12, of which $2, 3, 5, 7, 11$ are prime. There is only one way to have sum 2 $(1 + 1)$, there are two ways to have sum 3 $(1 + 2 = 2 + 1)$, four ways to have sum 5 $(1 + 4 = 2 + 3 = 3 + 2 = 4 + 1)$, six ways to have sum 7 $(1 + 6 = 2 + 5 = 3 + 4 = 4 + 3 = 5 + 2 = 6 + 1)$, and two ways to have sum 11 $(5 + 6 = 6 + 5)$. Thus out of the 36 possibilities, there are $1 + 2 + 4 + 6 + 2 = 15$ ways for which the sum is prime, so the probability is $\frac{15}{36} = \frac{5}{12}$.

10. Answer B. $f(9) = (9 \times 8)^{2001} = (3^2 \times 2^3)^{2001}$, while $f(3) = (3 \times 2)^{2001}$ and $f(4) = (4 \times 3)^{2001} = (3 \times 2^2)^{2001}$. Thus $f(3)f(4) = (3^{1+1} \times 2^{1+2})^{2001} = (3^2 \times 2^3)^{2001} = f(9)$.

11. Answer A. The divisors of 24 are $1, 2, 3, 4, 6, 8, 12, 24$ (note that it does not say prime divisors), which add up to 60, as given. Then
$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{12} + \frac{1}{24} = \frac{24 + 12 + 8 + 6 + 4 + 3 + 2 + 1}{24} = \frac{60}{24} = \frac{5}{2}.$$
(If you think a bit, you can see that for any natural number, the sum of the reciprocals of the divisors is equal to the sum of the divisors divided by the number itself.)

12. Answer E. By clearing fractions, the first equation becomes

$$x = 3 + \frac{x}{3x+1} = \frac{10x+3}{3x+1},$$

so $3x^2 + x = 10x + 3$, giving $x^2 - 3x - 1 = 0$, so x can be either of the two roots of this equation. Similarly,

$$y = 3 + \cfrac{1}{3 + \cfrac{y}{3y+1}} = 3 + \frac{3y+1}{10y+3} = \frac{33y+10}{10y+3},$$

so $10y^2 + 3y = 33y + 10$, giving $y^2 - 3y - 1 = 0$ (the same equation as for x). Thus y can also be either of the two roots of the equation, and $x - y$ cannot be determined uniquely.

13. Answer C. Put $m = 2000$, to simplify the working. Then $n+1 = m^2+(m+1)^2 = 2m^2+2m+1$, so $n = 2m^2+2m$. Therefore $2n+1 = 4m^2+4m+1 = (2m+1)^2$, so $\sqrt{2n+1} = 2m+1 = 4001$.

14. Answer E.

$$\begin{aligned} \text{10th term} &= \text{sum of first 10 terms} - \text{sum of first 9 terms} \\ &= (10)(11)(12) - (9)(10)(11) \\ &= 10 \times 11 \times (12 - 9) = 330. \end{aligned}$$

15. Answer E. If the radius of the circle is r cm, then the centre O of the circle is r cm from each line and also r cm from T. This gives a right-angled triangle with sides $r-8$, $r-9$, and r, so $(r-8)^2 + (r-9)^2 = r^2$. The equation simplifies to $r^2 - 34r + 145 = 0$, which factorizes easily to $(r-5)(r-29) = 0$, so $r = 5$ or $r = 29$. The solution $r = 5$ does not apply since $r > 9$ from the information given.

16. Answer B. Partition the numbers from 1 to 500 into seven arithmetic sequences with difference 7 as follows:
$S_1 = 1, 8, \ldots, 498$, $\quad S_2 = 2, 9, \ldots, 499$, $\quad S_3 = 3, 10, \ldots, 500$,
$S_4 = 4, 11, \ldots, 494$, $\quad S_5 = 5, 12, \ldots, 495$, $\quad S_6 = 6, 13, \ldots, 496$,
$S_7 = 7, 14, \ldots, 497$.
S_1 to S_3 have 72 terms each, and S_4 to S_7 have 71 terms each. The sum of any term in S_1 and any term in S_6 is divisible by 7, so the required sequence cannot contain terms from both S_1 and S_6. Similarly, it cannot contain terms from both S_2 and S_5, or from both S_3 and S_4. It can also contain at most one term from S_7. Thus the longest sequence is obtained by taking all terms from S_1, S_2, and S_3, and one term from S_7, for a total of 217 terms.

17. Answer E. On the first circuit, the numbers $1, 7, \ldots, 1999$ are marked. The 6th number after 1999 is $1999 + 6 - 2001 = 4$, so on the second circuit the numbers $4, 10, \ldots, 1996$ are marked. The 6th number after 1996 is $1996 + 6 - 2001 = 1$, which has already been marked. Now every 3rd number has been marked $(1, 4, 7, \ldots, 1996, 1999)$, so there are $2001/3 = 667$ marked numbers, and $2001 - 667 = 1334$ numbers unmarked.

18. Answer B. For $n = 1$ the expression is $x + x^{-1}$, which equals 2 when $x = 1$. In general, $x + x^{-1} - 2 = x^{-1}(x^2 - 2x + 1) = x^{-1}(x-1)^2 \geq 0$, so $x + x^{-1} \geq 2$ for all $x > 0$. Thus $A = 2$ when $n = 1$. Now try $n = 2$: the expression is $x^2 + 1 + x^{-2}$, and $x^2 + x^{-2} \geq 2$ as before, so $x^2 + 1 + x^{-2} \geq 3$. Thus $A = 3$ when $n = 2$. The only answer tht fits these two examples is B, but if you want to prove it in general, pair off the terms from the outside inwards: first $x^n + x^{-n}$, then $x^{n-2} + x^{2-n}$, and so on. Each pair is greater than or equal to 2 by the argument above, with equality when $x = 1$. If n is odd, then there are $\frac{1}{2}(n+1)$ pairs, so $A = 2 \times \frac{1}{2}(n+1) = n+1$. If n is even, then there are $\frac{1}{2}n$ pairs, and a term 1 left over in the middle. Thus $A = 2 \times \frac{1}{2}n + 1 = n + 1$.

19. Answer A. If m is an odd integer, say $m = 2k+1$, then $m^2 = 4k^2+4k+1$, so $m^2-1 = 4k(k+1)$, which is divisible by 8 since either k or $k+1$ must be even. Thus if n is the sum of six odd squares, then $n-6$ must be divisible by 8, and $n = 1998$ is the only answer that fits. (One such expression is $1998 = 5^2 + 19^2 + 19^2 + 19^2 + 19^2 + 23^2$.)

20. Answer D. Suppose the time period is t days, during which it rains on x mornings and y afternoons. It never rains on both morning and afternoon of the same day. Therefore
Number of rainy days $= x + y = 11$.
Number of clear mornings $= t - x = 9$.
Number of clear afternoons $= t - y = 12$.
Adding these gives $2t = 32$, so $t = 16$ and the number of rain-free days is $t - 11 = 5$.