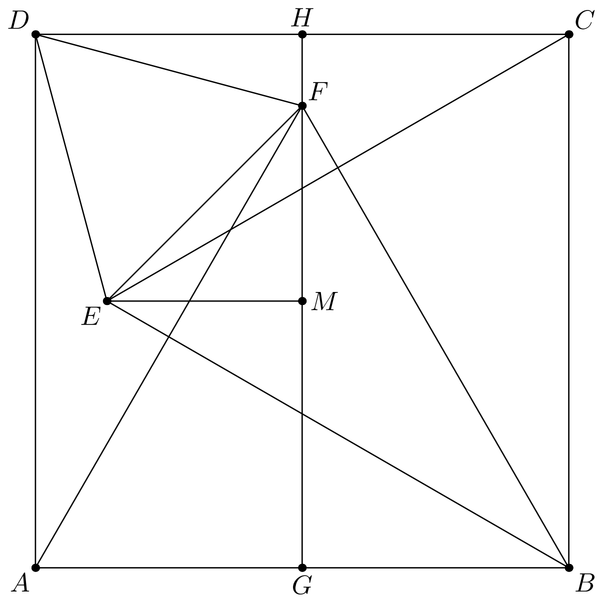
The South African Mathematical Olympiad
Third Round 2015
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Points E and F lie inside a square $ABCD$ such that the two triangles ABF and BCE are equilateral. Show that DEF is an equilateral triangle.

   Solution 1: We have $\angle FAD = \angle BAD - \angle BAF = 90^\circ - 60^\circ = 30^\circ$. Since $AF = AB = AD$, triangle AFD is isosceles, which means that $\angle ADF = \angle AFD = \frac{180^\circ - \angle FAD}{2} = 75^\circ$ and $\angle CDF = \angle CDA - \angle ADF = 90^\circ - 75^\circ = 15^\circ$. By symmetry, we also have $\angle ADE = 15^\circ$, thus $\angle EDF = 90^\circ - \angle ADE - \angle CDF = 60^\circ$.

   Again by symmetry (with respect to the diagonal BD), $DE = DF$, so DEF is an isosceles triangle with an angle of 60°. Therefore, DEF is indeed an equilateral triangle.



   Solution 2: Let G and H be the midpoints of AB and CD respectively, and let M be the centre of the square. We denote the side length of the square and the two equilateral triangles by a. By Pythagoras' Theorem,

$$GF^2 = AF^2 - AG^2 = a^2 - \left(\frac{a}{2}\right)^2 = \frac{3a^2}{4},$$

   so $GF = \frac{\sqrt{3}a}{2}$. Next we find $FH = GH - GF = a - \frac{\sqrt{3}a}{2} = \frac{(2-\sqrt{3})a}{2}$, $FM = GF - GM = \frac{\sqrt{3}a}{2} - \frac{a}{2} = \frac{(\sqrt{3}-1)a}{2}$ and by symmetry $EM = FM = \frac{(\sqrt{3}-1)a}{2}$.

Applying Pythagoras' Theorem again, we obtain

$$DF^2 = DH^2 + FH^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{(2-\sqrt{3})a}{2}\right)^2 = a^2\left(\frac{1}{4} + \frac{4-4\sqrt{3}+3}{4}\right) = a^2(2-\sqrt{3})$$

and

$$EF^2 = EM^2 + FM^2 = 2\left(\frac{(\sqrt{3}-1)a}{2}\right)^2 = 2a^2 \cdot \frac{3-2\sqrt{3}+1}{4} = a^2(2-\sqrt{3}).$$

Thus $DF = EF$, and by symmetry $DE = EF$. This means that DEF is an equilateral triangle.

2. Determine all pairs of real numbers a and x that satisfy the simultaneous equations

$$5x^3 + ax^2 + 8 = 0$$

and

$$5x^3 + 8x^2 + a = 0.$$

Solution 1: If we subtract the two equations, we obtain

$$ax^2 + 8 - 8x^2 - a = (a-8)(x^2-1) = (a-8)(x+1)(x-1) = 0,$$

thus either $a = 8$ or $x = -1$ or $x = 1$. If $a = 8$, we are left with

$$5x^3 + 8x^2 + 8 = (x+2)(5x^2 - 2x + 4) = 0.$$

The second factor has no real roots, since its discriminant $2^2 - 4 \cdot 5 \cdot 4 = -76$ is negative. Thus $x = -2$ in this case.

If $x = -1$, we get $a = -5x^3 - 8x^2 = -3$, and if $x = 1$, we get $a = -5x^3 - 8x^2 = -13$.

In summary, there are three possible pairs: $(a, x) = (8, -2)$, $(a, x) = (-3, -1)$ and $(a, x) = (-13, 1)$.

Solution 2: Solving the second equation for a gives us $a = -5x^3 - 8x^2$, thus

$$5x^3 + (-5x^3 - 8x^2)x^2 + 8 = -5x^5 - 8x^4 + 5x^3 + 8 = 0.$$

The polynomial can be factorised:

$$\begin{aligned} -5x^5 - 8x^4 + 5x^3 + 8 &= 5x^3(1-x^2) + 8(1-x^4) = 5x^3(1-x^2) + 8(1+x^2)(1-x^2) \\ &= (1-x^2)(5x^3 + 8x^2 + 8) = (1-x)(1+x)(x+2)(5x^2 - 2x + 4). \end{aligned}$$

Again, we find that $x = 1$, $x = -1$ or $x = -2$, since the quadratic factor has no real roots. The value of a is obtained from the equation $a = -5x^3 - 8x^2$, so we end up with the three possibilities $(a, x) = (8, -2)$, $(a, x) = (-3, -1)$ and $(a, x) = (-13, 1)$ again.

3. We call a divisor d of a positive integer n *special* if $d+1$ is also a divisor of n. Prove: at most half the positive divisors of a positive integer can be special. Determine all positive integers for which exactly half the positive divisors are special.

Solution: We prove that no positive divisor d of n that is greater or equal to $\sqrt{n}$ can be special: if d is special, then $d+1$ is also a divisor, so n/d and $n/(d+1)$ are both integers, which means that their difference is at least 1. Thus

$$\frac{n}{d} \geq \frac{n}{d+1} + 1,$$

which is equivalent to $n \geq d(d+1)$. But since $d(d+1) > d^2 \geq n$, this is a contradiction. Thus only divisors less than $\sqrt{n}$ can be special. Since divisors come in pairs (a and n/a) such that one of them is less than $\sqrt{n}$ and one greater than $\sqrt{n}$ (when n is a square, $\sqrt{n}$ is paired with itself), this means that at most half the divisors can be special.

If precisely half the divisors are special, then n cannot be a square, and every divisor less than $\sqrt{n}$ has to be special. Thus 1 has to be a special divisor, meaning that 2 is a divisor (and thus also special), so 3 is a divisor, and so on, up to the greatest integer k that is less than $\sqrt{n}$. Finally, k is special, so $k+1$ has to be a divisor as well. Since k is the greatest divisor less than $\sqrt{n}$ and $k+1$ the least divisor greater than $\sqrt{n}$, their product must be n, so $n = k(k+1) = k^2 + k$. Moreover, $k-1$ is also a divisor of $n = k^2 + k$ (unless $k = 1$), so it also divides $n - (k-1)(k+2) = k^2 + k - (k^2 + k - 2) = 2$.

This leaves us with $k = 1$, $k = 2$ and $k = 3$ as the only possibilities, giving us $n = 2$, $n = 6$ or $n = 12$. In all these cases, exactly half the divisors are special.

4. Let ABC be an acute-angled triangle with $AB < AC$, and let points D and E be chosen on the sides AC and BC respectively in such a way that $AD = AE = AB$. The circumcircle of ABE intersects the line AC at A and F and the line DE at E and P. Prove that P is the circumcentre of BDF.

Solution: Since $AD = AE$, AED is isosceles, so we also have $\angle ADE = \angle AED$. Combining this with the fact that $ABPE$ and $ABPF$ are cyclic, we obtain

$$\angle ABP = 180^\circ - \angle AEP = \angle AED = \angle ADE = \angle ADP$$

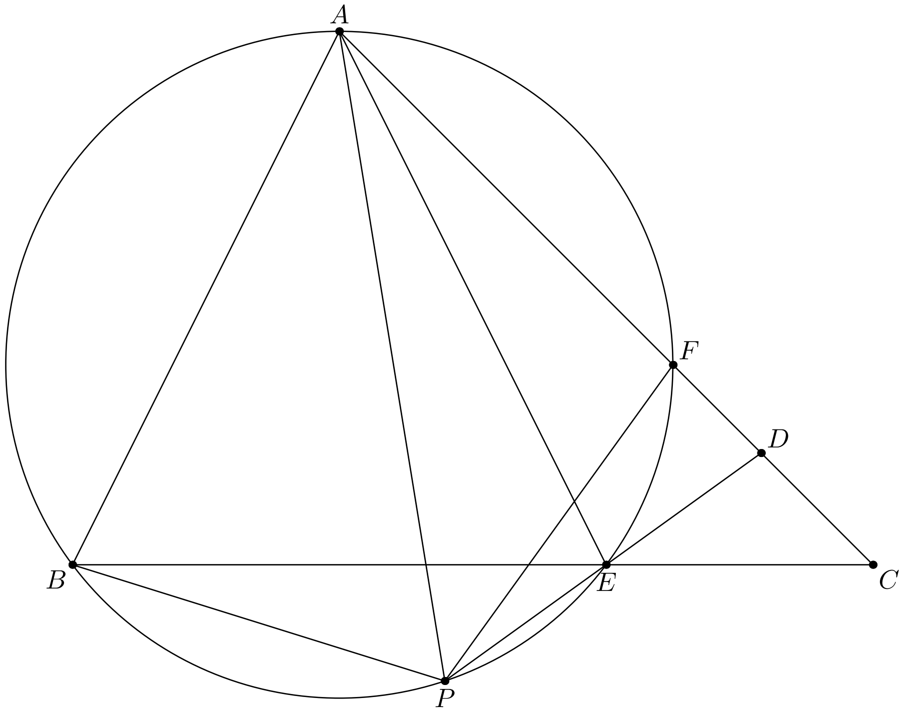
as well as

$$\angle ABP = 180^\circ - \angle AFP = \angle DFP,$$

so $\angle FDP = \angle DFP$, which implies that $PD = PF$. Likewise, $\angle ABE = \angle AEB$ since $AB = AE$, which means that

$$\angle APB = \angle AEB = \angle ABE = \angle APE = \angle APD.$$

Now we see that triangles ABP and ADP have the same angles ($\angle APB = \angle APD$, $\angle ABP = \angle ADP$) and the common side AP, so they are congruent. Thus $PB = PD = PF$, which means that P is the circumcentre of BDF.



5. Several small villages are situated on the banks of a straight river. On one side, there are 20 villages in a row, and on the other there are 15 villages in a row. We would like to build bridges, each of which connects a village on the one side with a village on the other side. The bridges must be straight, must not cross, and it should be possible to get from any village to any other village using only those bridges (and not any roads that might exist between villages on the same side of the river). How many different ways are there to build the bridges?

   Solution: We show that the answer is generally $\binom{a+b-2}{a-1} = \frac{(a+b-2)!}{(a-1)!(b-1)!}$ if there are a towns on one side and b on the other. In our particular instance, there are thus $\binom{33}{14} = 818\,809\,200$ ways to build the bridges. We prove our general formula by induction on the total number of villages. Note that the formula always holds if either $a = 1$ or $b = 1$. In this case, $\binom{a+b-2}{a-1} = 1$, and indeed there is only one possibility: to build bridges from the single village on the one side of the river to all other villages. This forms the base of our induction.

   We call the villages one the one side $A_1, A_2, \ldots, A_a$, and the villages on the other side $B_1, B_2, \ldots, B_b$ (in this order). Note first that A_1 and B_1 cannot both be connected by a bridge to villages other than each other: if there is a bridge between A_1 and B_k and a bridge between B_1 and A_l, where $k, l > 1$, then these bridges cross, which is impossible.

   This leaves us with two possibilities:

   - A_1 is directly connected to B_1 only, while B_1 is connected to the rest. In this case, we can ignore A_1 and only count the possibilities to build bridges between $A_2, A_3, \ldots, A_a$ and $B_1, B_2, \ldots, B_b$. By the induction hypothesis, this can be done in $\binom{a+b-3}{a-2}$ ways.
   - B_1 is directly connected to A_1 only, while A_1 is connected to the rest. In this case, we can ignore B_1, and the induction hypothesis shows that there are $\binom{a+b-3}{a-1}$ possibilities.

Altogether, this gives us

$$\binom{a+b-3}{a-2}+\binom{a+b-3}{a-1}=\binom{a+b-2}{a-1}$$

possibilities to build the bridges, which completes our induction.

6. Suppose that a is an integer, and that $n!+a$ divides $(2n)!$ for infinitely many positive integers n. Prove that $a=0$.

Solution 1: Note that

$$(2n)!=\binom{2n}{n}\cdot n!^2\equiv\binom{2n}{n}\cdot(-a)^2 \bmod (n!+a),$$

so if $n!+a$ divides $(2n)!$, then it also divides $a^2\binom{2n}{n}$. We will show that when n is large, $n!+a$ is greater than $a^2\binom{2n}{n}$ and therefore does not divide it (unless $a=0$). Assume in the following that $a\neq 0$ and $n>\frac{4^5}{12}a^2$. First of all, by the binomial theorem,

$$\binom{2n}{n}\leq\sum_{k=0}^{2n}\binom{2n}{k}=2^{2n}=4^n.$$

If $a>0$, we have

$$0<\frac{a^2}{n!+a}\binom{2n}{n}<\frac{a^2}{n!}\binom{2n}{n}\leqslant\frac{a^2}{n!}4^n=\left(\frac{4^{n-5}}{5\cdot 6\cdots(n-1)}\right)\frac{4^5a^2}{24n}<1.$$

If $a<0$, then we observe that $n!>n>2|a|$, so that

$$0<\frac{a^2}{n!+a}\binom{2n}{n}=\frac{a^2}{n!-|a|}\binom{2n}{n}<\frac{a^2}{\frac{1}{2}n!}\binom{2n}{n}\leqslant\frac{2a^2}{n!}4^n=\left(\frac{4^{n-5}}{5\cdot 6\cdots(n-1)}\right)\frac{4^5a^2}{12n}<1.$$

Thus $\frac{a^2\binom{2n}{n}}{n!+a}$ is not an integer for all such n, which contradicts our assumption that $n!+a$ divides $(2n)!$ (and thus $a^2\binom{2n}{n}$) for infinitely many n.

So $a=0$ is the only possibility (and indeed $n!$ divides $(2n)!$ for all n).

Solution 2: We show that $n!+a$ cannot divide $(2n)!$ if $a\neq 0$, $n\geq 2|a|$ and $n\geq 9$. In this case, $\frac{n!}{a}$ is an integer. If $n!+a$ divides $(2n)!$, then so does $\frac{n!}{a}+1$. Next note that every prime $p\leq n$ divides $\frac{n!}{a}$: if $p\nmid a$, this is obvious (since $p\mid n!$), and if $p\mid a$, then we see that p divides $2|a|$, which occurs as a factor of $\frac{n!}{a}$. In either case, we find that $\frac{n!}{a}+1$ is not divisible by p. So $\frac{n!}{a}+1$ does not have any prime factors $\leq n$, which means that it is coprime to all numbers $\leq n$ as well as all the even numbers $\leq 2n$.

Therefore $\frac{n!}{a}+1$ must divide the product off all odd numbers greater than n and less than $2n$. By our assumption, $\left|\frac{n!}{a}+1\right|\geq\frac{n}{a}\cdot(n-1)!-1\geq 2(n-1)!-1$. We show that the product of the odd numbers between n and $2n$ is less than $2(n-1)!-1$ for $n\geq 9$, which yields the desired contradiction.

For $n=9$ and $n=10$, we have the two inequalities

$$2\cdot 8!-1=80639>36465=11\cdot 13\cdot 15\cdot 17$$

and

$$2 \cdot 9! - 1 = 725759 > 692835 = 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19$$

respectively. Now we proceed by induction. Assume that

$$2(n-1)! - 1 > \prod_{\substack{n<k<2n \\ k \text{ odd}}} k.$$

If $n > 10$ is even, we use the induction hypothesis to obtain

$$\begin{aligned}\prod_{\substack{n<k<2n \\ k \text{ odd}}} k &= \frac{(2n-3)(2n-1)}{n-1} \prod_{\substack{n-2<k<2n-4 \\ k \text{ odd}}} k < 2(2n-1) \prod_{\substack{n-2<k<2n-4 \\ k \text{ odd}}} k < (4n-2)(2(n-3)!-1) \\ &< (n-1)(n-2)(2(n-3)!-1) = 2(n-1)! - (n-1)(n-2) < 2(n-1)! - 1,\end{aligned}$$

and if $n > 10$ is odd, we get analogously

$$\begin{aligned}\prod_{\substack{n<k<2n \\ k \text{ odd}}} k &= \frac{(2n-3)(2n-1)}{n} \prod_{\substack{n-2<k<2n-4 \\ k \text{ odd}}} k < 2(2n-3) \prod_{\substack{n-2<k<2n-4 \\ k \text{ odd}}} k < (4n-6)(2(n-3)!-1) \\ &< (n-1)(n-2)(2(n-3)!-1) = 2(n-1)! - (n-1)(n-2) < 2(n-1)! - 1.\end{aligned}$$

This completes the induction and thus the proof.

Remark: In both proofs, we need that an inequality holds when n is large enough, and there are plenty of different methods to achieve this. The estimates above are rather crude and can be modified in many ways.