THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior First Round 2015
Solutions

1. **Answer D.**

$$\frac{2015 \times 2 + 4 \times 4030}{4030} = \frac{4030 + 4 \times 4030}{4030} = \frac{5 \times 4030}{4030} = 5.$$

2. **Answer D.**
After simplification, the numbers are (A) 8, (B) 7.5, (C) 9, (D) 16, (E) 7.75.

3. **Answer D.**
Suppose the numbers are x, y, z, with $0 < x < y < z$ and $x + y + z = 7$. If $x \geq 2$, then $y \geq 3$ and $z \geq 4$, so $x + y + z \geq 9$, which is too large. Therefore $x = 1$, which leaves $y + z = 6$. Since $2 \leq y < z$, the only possibility is $y = 2$ and $z = 4$. The product $xyz = 1 \times 2 \times 4 = 8$.

4. **Answer D.**
Clearly 4350 is divisible by 50, since the last two digits are 50. It is also divisible by 3, since the sum of the digits is divisible by 3. Dividing by 150 gives a quotient of 29, which is prime, so the prime factorization is $4350 = 2 \times 3 \times 5^2 \times 29$, and the largest prime factor is 29.

5. **Answer C.**
Since $2x + 2y = 14$, we have $x + y = 7$. Also since $x^2 - y^2 = (x + y)(x - y)$, we have $7(x - y) = 21$, so $x - y = 21/7 = 3$.

6. **Answer B.**
Since $9^3 = 729 < 900$ and $10^3 = 1000 > 900$, it follows that $9 < \sqrt[3]{900} < 10$.

7. **Answer D.**
There are 8 possibilities for first place, each of which can be combined with 7 possibilities for second place and 6 possibilities for third place. Thus the number of possible results for all three places is $8 \times 7 \times 6 = 336$. (This assumes that no two athletes finish exactly together.)

8. **Answer C.**
Multiplying out the equation gives $x^2 + 5x - 84 = -48$, so $x^2 + 5x - 36 = 0$. This factorizes to $(x + 9)(x - 4) = 0$, so $x = -9$ or $x = +4$.

9. **Answer E.**
It is easier to work with exterior angles, since the sum of the exterior angles of a polygon is $360°$. There are 180 exterior angles, which are all equal since the polygon is regular. Thus each exterior angle is equal to $360°/180 = 2°$, and each interior angle is equal to $180° - 2° = 178°$.

10. **Answer A.**
The median is the middle number, which is 30, so the mean is also 30. The sum of all nine numbers is therefore $9 \times 30 = 270$, which means that
$x = 270 - (16 + 19 + 24 + 25 + 30 + 31 + 32 + 46) = 270 - 223 = 47$.

11. **Answer E.**
Time = Distance/Speed, so the time taken is equal to $(72 \times 6)/288 = 1.5$ hours.

12. **Answer A.**
A median of a triangle (that is, a line from a vertex to the midpoint of the opposite side) divides the triangle into two smaller triangles of equal area, since they have equal bases and the same height. Thus area $\triangle ADC = \frac{1}{2}$area $\triangle ABC = \frac{1}{2}(24) = 12$. Similarly, area $\triangle ADE = \frac{1}{2}$area $\triangle ADC = 6$, and area $\triangle DEF = \frac{1}{2}$area $\triangle ADE = 3$.

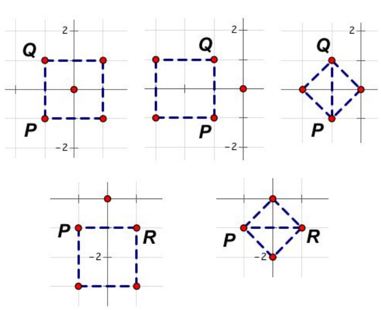
13. **Answer B.**
The last digits of the first few powers of 2 are $2, 4, 8, 6, 2, 4, \ldots$, so the digits repeat in cycles of length 4. Since the remainder after dividing 2015 by 4 is 3, it follows that the last digit of 2^{2015} is the same as the last digit of 2^3, which is 8. Finally, the last digit of any power of 5 is 5, so the last digit of $2^{2015} + 5^{2015}$ is equal to the last digit of $8 + 5$, which is 3.

14. **Answer C.**
If we call the three numbers $n-1$, n, $n+1$, then $(n-1)^2 + n^2 + (n+1)^2 = 770$, which simplifies to $3n^2 + 2 = 770$. Therefore $n^2 = 768/3 = 256 = 16^2$, so $n = 16$ (since $n > 0$), and the largest number is $n + 1 = 17$.
Without doing any algebra, it is clear that $n^2 \approx 770/3 = 256\frac{2}{3}$, and the nearest perfect square is 256.

15. **Answer D.**
It is easy to see that for $n = 1, 2, \ldots, 20$ there are n equally probable ways to obtain a total of $n + 1$. (One die shows any number x between 1 and n, and the other die shows $n + 1 - x$.) In particular, a total of 21 can be obtained with 20 different throws. Beyond that, the number of possibilities decreases again from 19 down to 1. Thus the most probable total is 21.

16. **Answer D.**
We are given $\dfrac{RA}{AT} = \dfrac{1}{3}$ and $\dfrac{AT}{TE} = \dfrac{5}{2}$, so $\dfrac{RA}{TE} = \dfrac{RA}{AT} \times \dfrac{AT}{TE} = \dfrac{1}{3} \times \dfrac{5}{2} = \dfrac{5}{6}$.

17. **Answer E.**
If we reflect $P(-1;-1)$ in the x-axis to map to $Q(-1;1)$ we obtain 3 squares with the x-axis as a line of symmetry, as shown in the diagram. Two of the cases have PQ as an edge/side and in one case PQ is a diagonal of the square. Similarly, if we reflect P in the y-axis to R, then there are three possible squares. However, one of these is the same as before, so in total there are exactly five squares possible.



18. **Answer D.**
There is only one way to get to each of the points adjacent to A, so label each of these points with 1. From then on, label each point successively with the number of ways to get to it, which is the sum of the labels on the (one or two) adjacent points below or to the left of it. Finally, the label on point B is 15.

19. **Answer C.**
A rhombus is a parallelogram with all its sides equal, so $AB = BC$. For $FECG$ to be a kite, we must have $EC = CG$, which equals GA, since the diagonals of any parallelogram bisect each other. Therefore $CA = 2CG = 2CE = CB$, which equals BA. It follows that triangle ABC must be equilateral, so all its angles must be $60°$, and x must equal $30°$.

20. **Answer A.**
Let ℓ, b, h denote the numbers of cubes in the length, breadth and height respectively. Then $\ell bh = 120$. To find the unpainted cubes we must remove a layer at both ends of the length, breadth and height, giving $(\ell - 2)(b - 2)(h - 2) = 24$. By trying various factorizations of 120, we soon find that ℓ, b, h are equal to $4, 5, 6$ in some order. The surface area is $2(\ell b + \ell h + bh) = 2(20 + 30 + 24) = 148$.
(Trying to solve the equations algebraically is harder than using trial and error.)
