SENIOR SECOND ROUND 2014

1. **Answer 049**
   Since $\sqrt{2+\sqrt{x}}=3$, it follows that $2+\sqrt{x}=3^2=9$, so $\sqrt{x}=9-2=7$, and $x=7^2=49$.

2. **Answer 012**
   A perfect square must be positive or zero, so if the sum of perfect squares is zero, then each perfect square must itself be zero. Thus from

   $$(x-3)^2+(y-4)^2+(z-5)^2=0$$

   it follows that $x-3=y-4=z-5=0$. Therefore $x=3$ and $y=4$ and $z=5$, so $x+y+z=3+4+5=12$.

3. **Answer 015**
   $\sqrt{24}\approx\sqrt{25}=5$, so $\sqrt{24}-1\approx 5-1=4$ and $\sqrt{\sqrt{24}-1}\approx\sqrt{4}=2$. Therefore $30/\sqrt{\sqrt{24}-1}\approx 30/2=15$.

4. **Answer 025**
   $Q=\frac{100}{30}\times M=10$ and $P=\frac{100}{20}\times Q=50$ and $N=\frac{50}{100}\times P=25$.

5. **Answer 008**
   In general, the median of nine numbers is the middle one, which is greater than four of the other numbers and less than the remaining four. We already have (in order) $3,5,5,7,8,9$, and 8 is greater than the first four of these. If the extra three numbers are all greater than 8, then 8 is also less than four numbers (9 and the extra three), so the median is 8. This is the largest possible value of the median, because any number greater than 8 is larger than at least five out of the nine numbers.

6. **Answer 004**
   Angle DAD' is equal to $60°$, since the quadrilateral is rotated through $60°$. Also $AD'=AD=4$, so triangle ADD' is isosceles, with the two base angles each equal to $\frac{1}{2}(180°-60°)=60°$. Thus the triangle is in fact equilateral, so the third side is equal to the other two, that is, $DD'=4$.

7. **Answer 139**
Clearly $2709 = 9 \times 301$, and with a bit more effort $301 = 7 \times 43$, so $2709 = 3 \times 3 \times 7 \times 43$ in prime factors. To write this as a product of three distinct numbers we need to combine one 3 with one of the other three factors. The largest sum comes from the expression $2709 = 3 \times 7 \times 129$, for which $3 + 7 + 129 = 139$.

8. **Answer 720**
There are six choices for first place, leaving five choices for second place, four for third place, and so on, with finally only one choice for sixth place. Thus the total number of rankings or orderings is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$. [The expression $6 \times 5 \times 4 \times 3 \times 2 \times 1$ is denoted 6! and read as 6 factorial.]

9. **Answer 048**
We need two blocks to complete the solid shown up to height 1 cm, then another six blocks to complete up to height 5 cm,and finally four blocks to complete to height 6 cm, making a total of 12 blocks so far. However, the solid then has dimensions 6 cm by 6 cm by 4 cm, so to make a cube we need to add a section 6 cm by 6 cm by 2 cm, which requires another 36 blocks. Thus the total number of blocks required to complete the cube is $12 + 36 = 48$.

10. **Answer 002**
By adding the two equations we see that $8(2^a) = 16\sqrt{2}$, so $2^a = 2\sqrt{2} = 2^{3/2}$ and $a = \frac{3}{2}$. By substituting for 2^a in either equation we see that $2^b = \sqrt{2}$, so $b = \frac{1}{2}$. Thus $a + b = 2$.

11. **Answer 025**
If 4 shows up on exactly one die out of the three, then there are three choices for the die showing the 4. There are five choices for the number appearing on each of the other two dice, so the total number of successes is $3 \times 5 \times 5 = 75$. There are in total $6 \times 6 \times 6 = 216$ throws, which are all equally probable, so the probability of exactly one 4 is $\frac{75}{216} = \frac{25}{72}$.

12. **Answer 024**
For a number to be divisible by 9, the sum of its digits must also be divisible by 9. Of the five ways of choosing four out of the five odd numbers $1, 3, 5, 7, 9$, the only choice whose sum is divisible by 9 is

$1, 3, 5, 9$.These four digits can be arranged in $4! = 24$ ways to give 24 four-digit numbers divisible by 9.

13. **Answer 729**
There are various ways of tackling this problem:
(a) By inspection, the sums of the numbers in the four groups shown are $1, 8, 27, 64$, that is, $1^3, 2^3, 3^3, 4^3$, so a good guess ("multiple-choice technique") is that the sum of the numbers in the ninth group is $9^3 = 729$.
(b) By brute force, writing out all the groups in turn, you can discover that the ninth group consists of the nine odd numbers from 73 to 89, whose sum is 729.
(c) More mathematically, the first n groups contain $1 + 2 + \cdots + n$ odd numbers, that is, $\frac{1}{2}n(n+1)$ odd numbers (arithmetic series). It is also well known that the sum of the first k odd numbers is k^2 (another arithmetic series). Thus the sum of all numbers in the first n groups is $[\frac{1}{2}n(n+1)]^2$. The sum of the numbers in the n-th group alone is the difference between the sum of the first n groups and the first $(n-1)$ groups, that is,
$$[\frac{1}{2}n(n+1)]^2 - [\frac{1}{2}(n-1)n]^2,$$
which indeed simplifies to n^3. (Another method of proof is to show that the average of the numbers in the n-th group is n^2.)

14. **Answer 008**
Extend AD and BC to intersect at E. If angle $B = x$, then angle $D = 2x$ (given), so angle $A = 180 - 2x$ (internal angles) and angle $DCE = x$ (corresponding angles). Also angle $E = x$, using the sum of the angles in triangle ACE. Thus triangles ACE and DCE are both isosceles, with base angles equal to x. It follows that $DE = DC = 3$ and $AB = AE = AD + DE = 5 + 3 = 8$.

Alternatively, draw DE parallel to CB to intersect AB at E. If angle B is x, then angle EC is also x since $EDCB$ is a parallelogram. Since angle $ADC = 2x$ (given), it follows that angle $ADE = x$. But angle $AED = x$ as it corresponds with angle B. Hence, triangle AED is isosceles with $AE = AD = 5$. Since $EB = DC = 3$ ($EDCB$ is a parallelogram), it follows that $AB = AE + EB = 5 + 3 = 8$.

15. **Answer 012**
The diagonal entries (in the same row and column) are $1, 3, 7, 13, 21, \ldots$, in which the differences between successive terms are $2, 4, 6, 8, \ldots$, an arithmetic sequence. The n-th diagonal entry is therefore $1 + (2 + 4 + 6 + 8 + \cdots)$, where the arithmetic series in brackets has $(n - 1)$ terms. Thus the n-th diagonal entry is equal to $1+(n-1)n = n^2-n+1$. The last diagonal entry less than 2014 occurs when $n = 45$ and is equal to 1981. Since 45 is odd, the numbers following 1981 are found higher up in column 45. Thus 1982 is in row 44, and 1983 is in row 43, and so on. Note that the sum of the entry and the row number is 2026 each time. Thus the row number for entry 2014 is $2026 - 2014 = 12$.

16. **Answer 017**
Let O be the centre of the circle, join OA and OP, and let Q be the foot of the perpendicular from P to OA. Then $AQ = 9$ and $PQ = 15$, so if r is the radius of the circle, then $OP = r$ and $OQ = r - 9$. By Pythagoras' theorem in triangle OPQ we have $15^2 = r^2 - (r - 9)^2$, giving $225 = 9(2r - 9)$, from which it follows that $r = 17$.

17. **Answer 036**
Suppose Dr Richards is guilty of the theft: then by the first statement Lady Windermere is also guilty, which contradicts the second statement. Thus Dr Richards is innocent (i.e., not guilty), so by the third statement Miss Carlyle is also innocent. Since at least one of the three is guilty, the only possibility is that Lady Windermere (in room 36) is the thief.

18. **Answer 011**
The prime factorisation of $14! = 1 \times 2 \times 3 \times \cdots \times 14$ is $2^{11}3^5 5^2 7^2 11^1 13^1$. Since in a perfect square all prime factors must occur to an even power, we must reduce all the odd powers by one. That means we must divide by $2 \times 3 \times 11 \times 13$, which can be done by removing the factors $6, 11, 13$ from the expression for $14!$, leaving us with the product of 11 numbers.

19. **Answer 480**
If the circumference of the track is x m, then when they first meet B has travelled 100 m and A has travelled $\frac{1}{2}x - 100$ m. When they next meet, A has travelled $x - 60$ m and B has travelled $\frac{1}{2}x + 60$ m.Since A and B travel at constant speeds, these distances must be in the same

ratio, that is,

$$\frac{\frac{1}{2}x - 100}{100} = \frac{x - 60}{\frac{1}{2}x + 60}.$$

This gives $\frac{1}{4}x^2 - 20x - 6000 = 100x - 6000$, which simplifies to $x(x - 480) = 0$. Since $x = 0$ is obviously not possible, the circumference is 480 m.

20. **Answer 210**
The number of balls in each box is

$$\frac{n^2 + 290n - 2490}{n + 300} = n - 10 + \frac{510}{n + 300},$$

which must be an integer. It follows that $510/(n + 300)$ is an integer, but it is obviously positive and is also less than $510/300$, which is less than 2.The only possibility is that $510/(n+300) = 1$, so $n+300 = 510$, giving $n = 210$.