THE HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

Organised by the SOUTH AFRICAN MATHEMATICS FOUNDATION
Sponsored by HARMONY GOLD MINING

SECOND ROUND 2005
SENIOR SECTION: GRADES 10, 11 AND 12
10 MAY 2005
TIME: 60 MINUTES
NUMBER OF QUESTIONS: 20

ANSWERS

1. D
2. D
3. E
4. A
5. D
6. B
7. A
8. A
9. C
10. E
11. C
12. B
13. A
14. A
15. B
16. A
17. D
18. E
19. E
20. D

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SOLUTIONS

1. **Answer D.** The values are (A) $\frac{1}{4}$, (B) $\frac{1}{16}$, (C) $\frac{1}{8}$, (D) 4, (E) $\frac{1}{2}$.

2. **Answer D.** Write the factors out: $1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108$, or note that from the factorization $108 = 2^2 3^3$ the number of factors can be found by adding one to each exponent and multiplying the results: $(2+1) \times (3+1) = 12$.

3. **Answer E.** If R is the radius of the larger circle, then $2\pi R - 8\pi = 10$, so $R = \dfrac{8\pi + 10}{2\pi}$.

4. **Answer A.** $4 = 2^2$ and $9 = 3^2$, so we have $2^a > 2^{2c}$ and $3^b > 3^{2a}$, giving $a > 2c$ and $b > 2a$. Thus $\frac{1}{2}b > a > 2c$, so (since all numbers are positive) it follows that $b > a > c$.

5. **Answer D.** Remember Odd + Odd = Even, Odd $\times$ Odd = Odd, Odd + Even = Odd, Odd $\times$ Even = Even.

6. **Answer B.** Factorize as the difference of two squares:

$$2^{28} - 1 = (2^{14} - 1)(2^{14} + 1) = (2^7 - 1)(2^7 + 1)(2^{14} + 1).$$

It may or may not factorize further, but since $2^7 = 128$ it follows that the first two factors both lie between 120 and 130, and their sum $= 2 \times 2^7 = 256$.

7. **Answer A.** The expression on the right-hand side is called a continued fraction. To find it, break up each improper (top-heavy) fraction into an integer plus a proper fraction and then invert the proper fraction, thus: $\dfrac{97}{19} = 5 + \dfrac{2}{19}$, then $\dfrac{19}{2} = 9 + \dfrac{1}{2}$, then $\dfrac{2}{1} = 2$. It follows that $w + x + y = 5 + 9 + 2 = 16$.

8. **Answer A.** Let $h = ST$ and $k = AT$. Then by using similar triangles we have $\dfrac{h}{4.5} = \dfrac{k}{8}$ and $\dfrac{h}{6} = \dfrac{8-k}{8}$. Adding the equations gives $1 = h(\dfrac{2}{9} + \dfrac{1}{6}) = h \times \dfrac{7}{18}$. Alternatively, one can use co-ordinate geometry, taking A as the origin.

9. **Answer C.** Let $a = 2^{125} \times 3^{81} \times 5^{128}$. Then $w = a \times 2^4 = 16a$, while $x = a \times 2^2 = 4a$, and $y = a \times 2 \times 3 = 6a$ and $z = a \times 3 \times 5 = 15a$. Since $4 < 6 < 12 < 15$, it follows that $x < y < z < w$.

10. **Answer E.** The ratio of the speeds of adjacent wheels is the reciprocal of the ratio of their radii, so the speeds are: first 60, then $60 \times \dfrac{6}{3} = 120$, then $120 \times \dfrac{3}{2} = 180$, then $180 \times \dfrac{2}{4} = 90$. (Notice that the two intermediate wheels can actually be ignored, since their contributions cancel out.)

11. **Answer C.** The first digit can be anything from 1 to 9, while the other digits can have any value from 0 to 9. Thus the smallest sum of digits is 1 and the largest is 72, each of which occurs for one number only. The frequency of occurrence increases for values in between, and the two most commonly occurring digit sums are the two middle values, namely 36 and 37.

12. **Answer B.** There are 20 ways of choosing three vertices out of the six (the formula is $(6 \times 5 \times 4) \div (3 \times 2 \times 1)$), and two of these choices give equilateral triangles, so the probability is $\dfrac{2}{20}$.

13. **Answer A.** Let A, B, C denote the numbers of students taking respectively Art only, Biology only, and Commerce only, and let AB, AC, BC, ABC have their obvious meaning. We are given that $A + B + C + AB + AC + BC + ABC = 65$ and $ABC = 0$. Next $AB = 2BC = AC$. Finally, $A = 12$ and $B + AB + BC \geq C + AC + BC + 3$. By simplifying these, we obtain $B + C + 5BC = 53$ and $B \geq C + 3$. By inspection, there is a solution with $BC = 10$, $C = 0$, and $B = 3$, which gives the minimum value for B, since obviously C cannot be negative.

14. **Answer A.** The units digit in every power of 5 is 5, so the units digit in each of the brackets is 6. The product of two numbers with units digit 6 again has units digit 6, and we can apply this fact as many times as we need.

15. **Answer B.** If $x = QR$, then the diameters of the circles are 2 and $x+2$ and $x+3$. Since the areas are proportional to the squares of the diameters, it follows that $(x+2)^2 = \frac{1}{2}(2^2 + (x+3)^2)$. This simplifies to give $x^2 + 2x - 5 = 0$, for which the only positive solution is $x = \sqrt{6} - 1$.

16. **Answer A.**

$$\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} = \frac{x^2+y^2+z^2}{xyz} = \frac{(x+y+z)^2 - 2(xy+xz+yz)}{xyz} = \frac{36-22}{6}.$$

17. **Answer D.** If a number is divisible by 3, then the sum of its digits is divisible by 3. In this case, since the sum of digits is also divisible by 4, it must in fact be divisible by 12, and obviously cannot be more than 27. The possibilities are $X3Z$ with digit sum 12 and $X6Z$ with digit sum 24. The first case gives nine numbers from 138 to 930, while the second case gives only 969.

18. **Answer E.** The co-ordinates of T are $(\frac{p}{2}, 5)$ and the slope of AB is $\frac{2}{p}$. Thus the slope of ST is $-\frac{p}{2}$, so $\frac{-5}{x - p/2} = -\frac{p}{2}$, giving $p(2x-p) = 20$. The integers p and $2x - p$ are both even, since their sum $(2x)$ and product (20) are both even. It follows that p must be an even divisor of 20 (not necessarily positive) such that $20/p$ is also even, so the only possibilities are $p = \pm 2$ and $p = \pm 10$.

19. **Answer E.** Partition the sequence into groups of length $1, 2, 3, \ldots$, and so on. Then the fractions in the k-th group are $\frac{k}{1}, \ldots, \frac{1}{k}$ and the last member of the group is in position $1 + 2 + \cdots + k = \frac{1}{2}k(k+1)$. The largest value of k for which $\frac{1}{2}k(k+1) \leq 2005$ is (by trial and error) $k = 62$, so the fraction $\frac{1}{62}$ occurs in position $\frac{1}{2}(62)(63) = 1953$. The fraction in position 2005 is the 52nd member of the next group, which is $\frac{12}{52}$.

20. **Answer D.** Since the king has only seven moves to get seven columns to the right, at each move it must go one column to the right. It can stay in the same row, go one row up, or go one row down (unless it is in the bottom row). The number of paths to each square is the sum of the numbers of paths to all squares from which it can be reached in one move. Using this principle, and working from left to right, we can build up the following table of the numbers of paths to all squares of interest. (Squares that can be reached from A, but do not lead to B, can be ignored.)

			1	4			
		1	3	9	25		
	1	2	5	12	30	76	
1	1	2	4	9	21	51	127