THE SOUTH AFRICAN
MATHEMATICS OLYMPIAD

organised by the SOUTH AFRICAN ACADEMY OF SCIENCE AND ARTS
in collaboration with OLD MUTUAL, AMESA and SAMS

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SECOND ROUND 1999

SENIOR SECTION: GRADES 10, 11 AND 12
(STANDARDS 8, 9 AND 10)

22 JUNE 1999

TIME: 120 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. B
2. B
3. C
4. C
5. A
6. C
7. A
8. C
9. C
10. C
11. B
12. A
13. D
14. E
15. C
16. B
17. E
18. E
19. B
20. A

SOLUTIONS

Part A: 3 marks each.

1. The winning party needs to have only one more vote than each of the other parties. If the number of votes cast for the three parties are 33 334, 33 333 and 33 333 then the party with 33 334 votes wins.

2. Each term in the numerator is 8 times the corresponding term in the denominator.

3. The sides of the two triangles are in the ratio $\sqrt{3} : 1$. But areas depend on the product of the length of two sides. Hence the areas are in the ratio $(\sqrt{3})^2 : 1$.

4. The sum of the twenty numbers is $20 \times 20 = 20^2$. Therefore the average of the remaining 11 is

$$\frac{20^2 - 9^2}{11} = \frac{29 \times 11}{11} = 29.$$

5. In $\triangle BDC$, the sum of the angles at B and C is $180^o - w$ It follows that in $\triangle ABC$, $x = 180^o - (y + z + 180^o - w) = w - y - z$.

Part B: 5 marks each

6. The given equations can be written $c = d + 7$, $c = a + 2$ and $a = b + 3$. These show us that $c > d$, $c > a$ and $a > b$. Hence $c > b$ also. Therefore c is the largest.

7.

$$\begin{aligned} 100^2 - 99^2 &= (100 + 99)(100 - 99) = 199 \\ 98^2 - 97^2 &= (98 + 97)(98 - 97) = 195 \\ &\vdots \\ 4^2 - 3^2 &= (4 + 3)(4 - 3) = 7 \\ 2^2 - 1^2 &= (2 + 1)(2 - 1) = 3 \end{aligned}$$

The first and last of these add up to 202. So does the second and second last, etc. The sum is $25 \times 202 = 5050$.

8. We are given that $n = 2m - 3$. Hence $m - 2n = m - 2(2m - 3) = 3(2 - m)$. The only thing that we can say for certain is that this is a multiple of 3.

9. For notation see question 11. 21! is the product of 10 even numbers, each of which has a factor 2; 5 multiples of 4, each of which has an extra factor 2; 2 multiples of 8, each of which has another factor of 2; and one multiple of 16 which has another factor of 2. Hence $x = 10 + 5 + 2 + 1 = 18$.

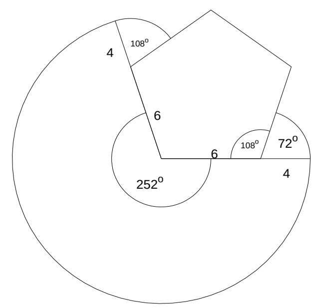
10. m and $m + 1$ are successive integers. Therefore one of them is even and the other odd. For the product to be a power of 2, m and $m + 1$ must be 1 and 2, or -1 and -2. In each case $n = 1$.

11. 1! ends in 1, 2! ends in 2, 3! ends in 6, 4! ends in 4, and from 5! onwards $n!$ ends in 0. Therefore the sum ends in 3, because $1 + 2 + 6 + 4 = 13$.

12. The internal angle of a pentagon is $\frac{3\times 180^o}{5} = 108^o$. The goat can graze part of a disc of radius 10m and two parts of a disc of radius 4m. The total is

$$\pi \times 10^2 \times \frac{360^o - 108^o}{360^o} + 2 \times \pi \times 4^2 \times \frac{72^o}{360^o} = (70 + \frac{32}{5})\pi = \frac{382}{5}\pi.$$

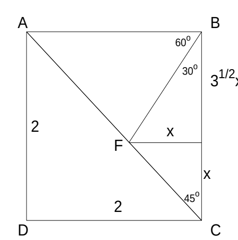
This is approximately $\frac{382}{5} \times \frac{22}{7}$, which in turn is approximately $11 \times 22 = 242$.



13. There are 5-digit and 4-digit numbers. For the 5-digit numbers we must choose a digit for each of the five positions, representing the units, tens, hundreds, thousands and ten thousands. This can be done in $5! = 120$ ways. For the 4-digit numbers we must choose a digit for each of the four positions where there are 3 choices for the left hand position, 4 for the next position, 3 for the third, and 2 for the last position. Therefore there are $3 \times 4 \times 3 \times 2 = 72$ possible 4-digit numbers. A total of 120+72= 192.

14. In January every day is relatively prime. Similarly November has 28 (all but the two multiples of 11). October has 13 (only the days that are odd and not divisible by 5). Clearly we need a month whose number is divisible by as many, different, small primes as possible. The two candidates are June and December. December has one more relatively prime day(the 31st) than June.

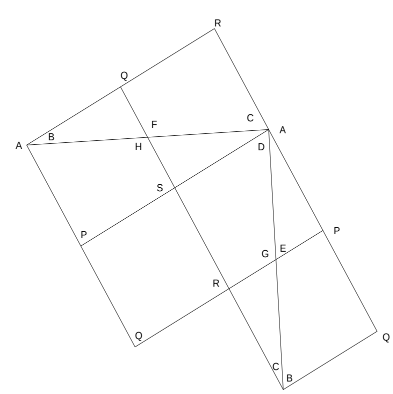
15. Let $FE = x$. Then the area of $\triangle FBC$ is $\frac{1}{2}BC.FE = x$, since $BC = 2$. But $BC = BE + EC = \sqrt{3}x + x = x(1 + \sqrt{3})$. Therefore $x = \frac{2}{(\sqrt{3}+1)}\frac{(\sqrt{3}-1)}{(\sqrt{3}-1)} = \sqrt{3} - 1$.



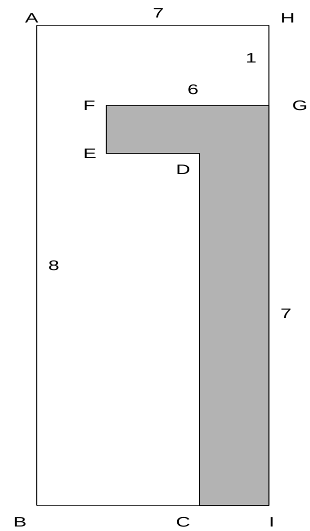
Part C: 7 marks each

16. Petra must be lying. Therefore at least one learner attended. If more than one learner attended then these would all give the same answer. But all the answers are different. It follows that exactly one learner (Peter) attended.

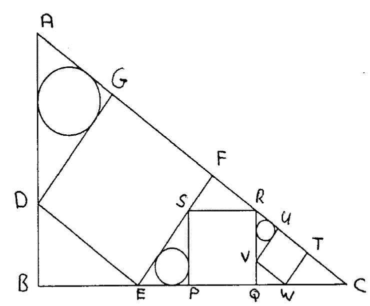
17. In the original figure, triangles AEP and ABF are similar. Therefore $AP = PQ$ since $AE = EB$. Similarly $CR = RS = AP = PQ$, and $PS = SD = BQ = QR$. Now move triangle BCR upwards so that points B, F and C coincide with points A, H and D, as in the new figure. Similarly move triangle ABQ to the right so that points A, E and B coincide with points D, G and C. Then the original $PQRS$ is one of 5 equal parallelograms.



18. AB must be 8 units long. DC cannot be 7 (then $EF + GH = 1$). Similarly FG cannot be 7 (then AH is also 8). Therefore $AH = 7$. DC cannot be 6 (then $FE = HG = 1$). Therefore $FG = 6$. To maximize the area of the hexagon put $HG = 1$. Therefore $GI = 7$. Now the area of the hexagon is $FG.GI - ED.DC = 6 \times 7 - ED.DC$. Hence we must make $ED.DC$ as small as possible. Therefore $ED = 3$ and $DC = 2$. Then $BC = 4$ and $EF = 5$, so we have used each of the integers 1, 2, 3, 4, 5, 6, 7, 8 exactly once. Note: If we put $ED = 2$ and $DC = 3$ then we must have $EF = BC = 4$, and no side has length 5. Hence the maximum area of the hexagon is $6 \times 7 - 2 \times 3 = 36$.



19. All triangles in the figure are similar to the original triangle ABC. the lengths of the sides of triangle ADG are shrunk by a factor to obtain ΔESP, and by the same factor again to obtain ΔRVU. The same can be said about the radii of the inscribed circles of these three triangles. Therefore if the radius of the middle circle is r then $\frac{r}{99} = \frac{19}{r}$, and $r^2 = 19 \times 99$.



20. We may as well assume that strings 1 and 2, 3 and 4, and 5 and 6 are tied together at the top. Then we tie the lower ends and we have the 15 possibilities shown in the figure. Two numbers joined by a line indicates that those ends are tied together. Those making a ring are labelled 'Y'. There are 8 of these.