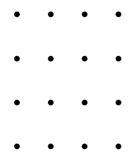
The South African Mathematical Olympiad
Third Round 2022
Senior Division (Grades 10 to 12)

Solutions

1. Consider 16 points arranged as shown, with horizontal and vertical distances of 1 between consecutive rows and columns. In how many ways can one choose four of these points such that the distance between every two of those four points is strictly greater than 2?



Solution: Label the sixteen points $A, B, C, \ldots, P$, as shown below in Figure 1:

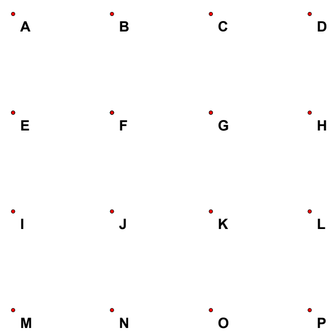


Figure 1

A selection of four points satisfying the condition that the distance between every two of the four points is greater than 2, will be called a *valid* selection.

Let us first consider a valid selection which includes one of the points of the inner square $FGKJ$. Clearly, only one of these points (F, G, K, J) can be used; say we select F. The only points from the outer square $ABCDHLPONMIE$ at distance more than 2 from F, are D, L, P, O and M. If we choose either L or O, then there are not enough points left among the remaining ones on the outer square to form a valid selection. We are therefore forced to select D, P and M, together with F, to obtain a valid selection. Similarly, by symmetry, there are three further valid selections that contain points from the inner square: $\{G, P, M, A\}$, $\{K, M, A, D\}$ and $\{J, A, D, P\}$.

All that remains, is to consider valid selections using only points from the outer square. We note that the only way to choose two points in a valid selection from the same side of the outer square, is to choose two corner points, such as A and D. But then the only option for the other two points in the valid selection would be to choose the other two corner points of the outer square, P and M, giving the valid selection $\{A, D, P, M\}$. Moving away from corner points, leaves us with the two remaining valid selections (where only one point from each of the sides of the outer square is selected), namely $\{B, H, O, I\}$ and $\{C, L, N, E\}$.

Hence, there are seven possible valid selections.

2. Find all pairs of real numbers x and y which satisfy the following equations:

$$x^2 + y^2 - 48x - 29y + 714 = 0$$
$$2xy - 29x - 48y + 756 = 0$$

Solution 1: The two equations can be rewritten as

$$\begin{aligned}(x-24)^2 + (y - \tfrac{29}{2})^2 &= \tfrac{289}{4} \\ (x-24)(y - \tfrac{29}{2}) &= -30.\end{aligned}$$

By putting $X = x - 24$ and $Y = y - \frac{29}{2}$, we obtain

$$X^2 + Y^2 = \tfrac{289}{4} \qquad (1)$$
$$XY = -30. \qquad (2)$$

From (2) we have $Y = -30/X$. Plug this into (1) to get $X^2 + \frac{900}{X^2} = \frac{289}{4}$, which simplifies to

$$X^4 - \tfrac{289}{4}X^2 + 900 = 0.$$

We can now solve for X^2, using the quadratic formula:

$$\begin{aligned}X^2 &= \tfrac{1}{2}\left(\tfrac{289}{4} \pm \sqrt{\left(\tfrac{289}{4}\right)^2 - 60^2}\right) \\ &= \tfrac{1}{2}\left(\tfrac{289}{4} \pm \sqrt{\left(\tfrac{289}{4} - 60\right)\left(\tfrac{289}{4} + 60\right)}\right) \\ &= \tfrac{1}{2}\left(\tfrac{289}{4} \pm \sqrt{\tfrac{49}{4} \cdot \tfrac{529}{4}}\right) \\ &= \tfrac{1}{2}\left(\tfrac{289 \pm 161}{4}\right),\end{aligned}$$

so that $X^2 = \frac{225}{4}$ or $X^2 = 16$. Thus, $X = \pm\frac{15}{2}$ or $X = \pm 4$, and correspondingly, $Y = \mp 4$ or $Y = \mp\frac{15}{2}$. Using $x = X + 24$ and $y = Y + \frac{29}{2}$, we find that (x, y) solves the original set of equations if and only if $(x, y) \in \{(28, 7), (20, 22), (\frac{33}{2}, \frac{37}{2}), (\frac{63}{2}, \frac{21}{2})\}$.

Solution 2: Proceed as in Solution 1 up to the two equations

$$X^2 + Y^2 = \tfrac{289}{4} \qquad (1)$$
$$XY = -30. \qquad (2)$$

By either adding $2XY = -60$ to (1), or subtracting it from (1), we get, respectively,

$$\begin{aligned}X^2 + 2XY + Y^2 = (X+Y)^2 &= \tfrac{49}{4} = \left(\tfrac{7}{2}\right)^2 \\ X^2 - 2XY + Y^2 = (X-Y)^2 &= \tfrac{529}{4} = \left(\tfrac{23}{2}\right)^2.\end{aligned}$$

From this, we have to solve four systems of equations:

$$\begin{aligned}X + Y &= \pm\tfrac{7}{2} \\ X - Y &= \pm\tfrac{23}{2}\end{aligned}$$

In the following table, solutions to all four cases are shown, and we also add two columns for $x = X + 24$ and $y = Y + \frac{29}{2}$:

$X+Y$	$X-Y$	X	Y	x	y
$\frac{7}{2}$	$\frac{23}{2}$	$\frac{15}{2}$	-4	$\frac{63}{2}$	$\frac{21}{2}$
$\frac{7}{2}$	$-\frac{23}{2}$	-4	$\frac{15}{2}$	20	22
$-\frac{7}{2}$	$\frac{23}{2}$	4	$-\frac{15}{2}$	28	7
$-\frac{7}{2}$	$-\frac{23}{2}$	$-\frac{15}{2}$	4	$\frac{33}{2}$	$\frac{37}{2}$

We conclude that there are exactly four pairs (x, y) of real numbers that solve the original two equations, namely $(\frac{63}{2}, \frac{21}{2})$, $(20, 22)$, $(28, 7)$ and $(\frac{33}{2}, \frac{37}{2})$.

3. Let a, b, and c be nonzero integers. Show that there exists an integer k such that

$$\gcd(a + kb, c) = \gcd(a, b, c).$$

(Note: '$\gcd$' stands for 'greatest common divisor')

Solution 1: We may assume that a, b, c are all positive, since if a, b, c are all positive, and $\gcd(a + kb, c) = \gcd(a, b, c)$ for some integer k, then we immediately have $\gcd(-a+(-k)b, \pm c) = \gcd(-a+k(-b), \pm c) = \gcd(a+(-k)(-b), \pm c) = \gcd(\pm a, \pm b, \pm c)$. Moreover, if at least one of a, b and c is equal to 1, then the results follows immediately: if $a = 1$ or $c = 1$, choose $k = 0$; otherwise, if $b = 1$, choose $k = 1 - a$. In all these cases, $\gcd(a + kb, c) = \gcd(a, b, c) = 1$.

Henceforth, assume that all of a, b and c are greater than 1. This implies that there is a list of prime numbers $p_1, p_2, \ldots, p_n$ and non-negative integers $\alpha_i, \beta_i, \gamma_i$, $i = 1, 2, \ldots, n$, such that

$$\begin{aligned} a &= p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \\ b &= p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n} \\ c &= p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_n^{\gamma_n}. \end{aligned}$$

Let us first assume that $\gcd(a, b, c) = 1$. This implies that, for each $i \in \{1, 2, \ldots, n\}$, not all three of α_i, β_i and γ_i are positive. We may also assume that, for each $i \in \{1, 2, \ldots, n\}$, not all three of α_i, β_i and γ_i are 0 (otherwise we could simply discard the primes p_i for which this happen). We now call a prime p_i a *one-prime* if exactly one of α_i, β_i and γ_i is positive. Likewise, we call a prime p_i a *two-prime* if exactly two of α_i, β_i and γ_i are positive. Let $E = \{p_{i_1}, p_{i_2}, \ldots, p_{i_t}\}$ be the complete list of one-primes among $p_1, p_2, \ldots, p_n$. If there are no one-primes in this list, put $E = \varnothing$. Let

$$k = \begin{cases} p_{i_1} p_{i_2} \cdots p_{i_t} & \text{if } E \neq \varnothing \\ 1 & \text{if } E = \varnothing. \end{cases}$$

We show that, for this k, $\gcd(a + kb, c) = \gcd(a, b, c) = 1$.

Since $\gcd(a + kb, c)$ is a divisor of c, the only possible way that $\gcd(a + kb, c) > 1$ is that it is divisible by at least one of $p_1, p_2, \ldots, p_n$. We show that this is not the case.

Firstly, consider any $p_{i_j} \in E$ (if $E \neq \varnothing$). For the triple $(\alpha_{i_j}, \beta_{i_j}, \gamma_{i_j})$ there are three possibilities:

I. $(\alpha_{i_j}, \beta_{i_j}, \gamma_{i_j}) = (\alpha_{i_j}, 0, 0)$, with $\alpha_{i_j} > 0$. Here, p_{i_j} does not divide c.

II. $(\alpha_{i_j}, \beta_{i_j}, \gamma_{i_j}) = (0, \beta_{i_j}, 0)$, with $\beta_{i_j} > 0$. Here, again, p_{i_j} does not divide c.

III. $(\alpha_{i_j}, \beta_{i_j}, \gamma_{i_j}) = (0, 0, \gamma_{i_j})$, with $\gamma_{i_j} > 0$. Here, p_{i_j} does not divide $a + kb$ (where $k = p_{i_1} p_{i_2} \cdots p_{i_t}$).

So $\gcd(a + kb, c)$ is not divisible by any of the one-primes. Secondly, let p_r denote any of the two-primes. For the triple $(\alpha_r, \beta_r, \gamma_r)$ there are three possibilities:

I'. $(\alpha_r, \beta_r, \gamma_r) = (\alpha_r, \beta_r, 0)$, with $\alpha_r, \beta_r > 0$. Here, p_r does not divide c.

II'. $(\alpha_r, \beta_r, \gamma_r) = (\alpha_r, 0, \gamma_r)$, with $\alpha_r, \gamma_r > 0$. Here, p_r does not divide $a + kb$ (recall that k is not divisible by p_r).

III'. $(\alpha_r, \beta_r, \gamma_r) = (0, \beta_r, \gamma_r)$, with $\beta_r, \gamma_r > 0$. Here, again, p_r does not divide $a + kb$.

It follows that $\gcd(a + kb, c) = 1 = \gcd(a, b, c)$.

Finally, if $\gcd(a, b, c) = d > 1$, then $\gcd(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}) = 1$, and from the above, there exists an integer k such that $\gcd(\frac{a}{d} + k \cdot \frac{b}{d}, \frac{c}{d}) = 1 = \gcd(\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$. Multiplying both sides by d gives $\gcd(a + kb, c) = d = \gcd(a, b, c)$, and we are done.

Solution 2: As in Solution 1, we may assume that $\gcd(a, b, c) = 1$. By Dirichlet's Theorem, the sequence $s_n = a/\gcd(a, b) + n \cdot b/\gcd(a, b)$, $n \in \mathbb{N}$, contains infinitely many primes. Let $p = a/\gcd(a, b) + k \cdot b/\gcd(a, b), k \in \mathbb{N}$, be prime, with $p > c$. Then, as $\gcd(a, b)$ and c are relatively prime,

$$1 = \gcd(p, c) = \gcd(p \cdot \gcd(a, b), c) = \gcd(a + kb, c).$$

4. Let ABC be a triangle with $AB < AC$. A point P on the circumcircle of ABC (on the same side of BC as A) is chosen in such a way that $BP = CP$. Let BP and the angle bisector of $\angle BAC$ intersect at Q, and let the line through Q and parallel to BC intersect AC at R. Prove that $BR = CR$.

Solution 1: Let AQ extended meet the circumcircle of triangle ABC in T, and let the line through Q and R intersect PC in D. See Figure 2:

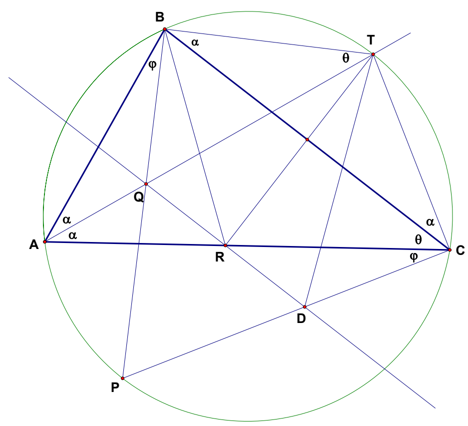


Figure 2

Put $\angle BAQ = \angle CAQ = \alpha$, so that also $\angle BCT = \alpha$ and $\angle TBC = \alpha$. Put $\angle BCA = \theta$, so that also $\angle BTA = \theta$. Put $\angle ACP = \phi$, so that also $\angle ABP = \phi$.

Then $\angle PBC = \angle PCB$ (since $BP = CP$) $= \theta + \phi$. From $\angle PBT + \angle TCP = 180^\circ$ (since BPCT is a cyclic quadrilateral), we therefore have $2(\alpha + \theta + \phi) = 180^\circ$, giving $\alpha + \theta + \phi = 90^\circ$. Since triangle QPD is isosceles ($QD \| BC$, so that $\angle PQD = \angle PBC = \angle PCB = \angle PDQ$), we have $PQ = PD$, forcing $BQ = CD$. Furthermore, chords BT and CT both subtend the same angle α at A, hence are equal in length. It follows that $\triangle TBQ \equiv \triangle TCD$, hence $\angle DTC = \angle QTB = \theta$. But since $\angle DRC = \angle BCR = \theta$ (recall that $QD \| BC$), we see that RDCT is a cyclic quadrilateral. This implies that $\angle TRD = 90^\circ$ (because $\angle TCD = \alpha + \theta + \phi = 90^\circ$). So $\angle TRQ = 90^\circ$, and we also have that RQBT is a cyclic quadrilateral. (Recall that $\angle QBT = 90^\circ$.) It follows that $\angle QRB = \angle QTB = \theta$, and we thus have $\angle RBC = \theta$, since $QD \| BC$. Finally, $BR = RC$ follows from the fact that $\angle RBC = \angle RCB = \theta$.

Solution 2: Let the line through R and Q intersect AB and PC in S and T, respectively. See Figure 3:

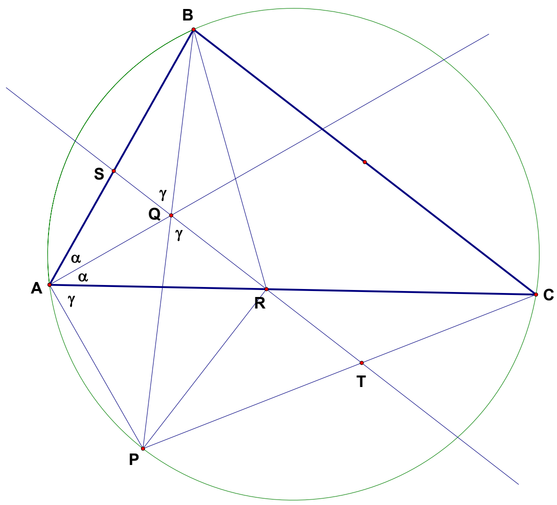


Figure 3

Put $\gamma = \angle PBC = \angle PCB$. Then also $\angle PAC = \gamma$. From the fact that $ST \| BC$, it follows that $\angle BQS = \angle QBC = \gamma$, so that $\angle PQR = \gamma$. Thus PAQR is a cyclic quadrilateral. Hence $\angle RPQ = \angle RAQ = \alpha$, and since $\angle CPB = \angle CAP = 2\alpha$, we have $\angle RPT = \alpha$. So $QR = TR$.

We see that $\triangle BQR \equiv \triangle CTR$ ($\angle BQR = \angle CTR = 180^\circ - \gamma$, $QR = RT$ and $BQ = CT$ (from $PB = PC$ and $PQ = PT$)). Consequently, $BR = CR$.

5. Let $n \geqslant 3$ be an integer, and consider a set of n points in three-dimensional space such that:

   (i) every two distinct points are connected by a string which is either red, green, blue, or yellow;

   (ii) for every three distinct points, if the three strings between them are not all of the same colour, then they are of three different colours;

   (iii) not all the strings have the same colour.

   Find the maximum possible value of n.

   Solution: Let us fix a notation: Number the points $1, 2, \ldots, n$; The colour of the string between points i and j is denoted by $c(i,j)$, so that $c(i,j) \in \{B, G, R, Y\}$ for all $1 \leqslant i, j \leqslant n$, $i \neq j$, and where $B =$ Blue, $G =$ Green, $R =$ Red, and $Y =$ Yellow.

For $n = 3$, there is a trivial solution, by putting, say, $c(1,2) = B,\ c(2,3) = G,\ c(3,1) = R$. Also, for $n = 4$ it is fairly straightforward to see a solution, say $c(1,2) = c(3,4) = B,\ c(1,3) = c(2,4) = G$, and $c(1,4) = c(2,3) = R$.

Henceforth, assume that $n > 4$.

We first observe that no point can have three strings of the same colour attached to it. For suppose (without loss of generality) that $c(1,2) = c(1,3) = c(1,4) = B$. Then $c(2,3) = c(2,4) = c(3,4) = B$. This implies that there must be a fifth point, 5, say, such that $c(1,5) \neq B$ (otherwise all strings will have the same colour B). Assume $c(1,5) = G$. Then $c(2,5), c(3,5), c(4,5) \notin \{B, G\}$. Assume that $c(2,5) = R$, say. Then $c(3,5) \neq R$, forcing $c(3,5) = Y$. But then $c(4,5) \notin \{R, Y\}$, by condition (ii) applied to the two sets $\{2,4,5\}$ and $\{3,4,5\}$ of three points each. The contradiction $c(4,5) \notin \{B, G, R, Y\}$ proves that our claim is valid. Consequently, there can be at most two strings of each colour attached to each point, implying that there can be no more than nine points in total.

We now show that it is indeed possible that nine points can be mutually connected by strings such that all conditions (i) to (iii) are satisfied.

Partition the points $1, 2, \ldots, 9$ into three subsets of three points each: $X = \{1,2,3\}$, $Y = \{4,5,6\}$, $Z = \{7,8,9\}$. First, consider the three 'internal' monochromatic triangles (all sides Blue) where the points in each of these subsets are connected with Blue strings. Then we construct nine further monochromatic triangles KLM, where $K \in X$, $L \in Y$, and $M \in Z$ (three of each of the colours Green, Red and Yellow). This can be done in several ways (six ways, to be precise). We show here a nice symmetrical one: Three Green triangles $\{1,4,7\}$, $\{2,5,8\}$, $\{3,6,9\}$; Three Red triangles $\{1,5,9\}$, $\{2,6,7\}$, $\{3,4,8\}$; Three Yellow triangles $\{1,6,8\}$, $\{2,4,9\}$, $\{3,5,7\}$. See Figure 4.

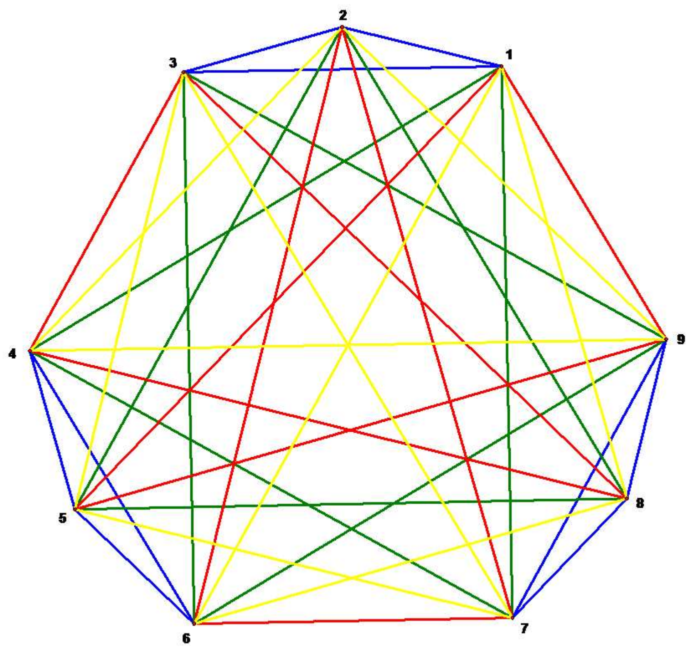


Figure 4

The motivation behind the method used here was to write a 3×3 table:

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$$

The rows of T ($\{1,2,3\}$, $\{4,5,6\}, \{7,8,9\}$) are used to determine the three Blue triangles. The columns ($\{1,4,7\}$, $\{2,5,8\}, \{3,6,9\}$) are used to determine the three Green triangles. For the final two colours we use one element from each row and one element from each column, in two possible ways: For the Red triangles: $\{1,5,9\}$, $\{2,6,7\}, \{3,4,8\}$; and for the Yellow triangles: $\{3,5,7\}$, $\{2,4,9\}, \{1,6,8\}$.

Now we have a string between each two points, and if a triangle contains two sides/strings of the same colour, the third side must also be of that colour.

We conclude that the maximum possible value of n is $n = 9$.

6. Show that there are infinitely many polynomials P with real coefficients such that if x, y, and z are real numbers such that $x^2 + y^2 + z^2 + 2xyz = 1$, then

$$P(x)^2 + P(y)^2 + P(z)^2 + 2P(x)P(y)P(z) = 1.$$

Solution: Let us call a triple (x, y, z) of real numbers a $*$-*triple* if it satisfies

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

Let us call a polynomial $p(x)$ with real coefficients a $*$-*polynomial* if

$$(x, y, z) \text{ a } *\text{-triple implies } (p(x), p(y), p(z)) \text{ a } *\text{-triple.}$$

We first investigate polynomials of degree at most 1. Hence, assume that $p(x) = ax+b$ $(a, b \in \mathbb{R})$, and suppose that $p(x)$ is a $*$-polynomial. Since $(x, -x, 1)$ and $(x, x, -1)$ are $*$-triples for all $x \in \mathbb{R}$, we have

$$p(x)^2+p(-x)^2+p(1)^2+2p(x)p(-x)p(1) = 1 \text{ and } p(x)^2+p(x)^2+p(-1)^2+2p(x)p(x)p(-1) = 1$$

for all $x \in \mathbb{R}$. This simplifies, respectively, to

$$\begin{aligned} 2a^2(1-a-b)x^2 + 2b^2(1+a+b) + (a+b)^2 &= 1 \qquad (1) \\ \text{and} \quad 2a^2(1-a+b)x^2 + 4ab(1-a+b)x + 2b^2(1-a+b) + (b-a)^2 &= 1. \qquad (2) \end{aligned}$$

Since these equations are valid for all $x \in \mathbb{R}$, we must have both the coefficients $2a^2(1-a-b)$ and $2a^2(1-a+b)$ equal to 0. Therefore, $a = 0$, or $1 - a - b = 0 = 1 - a + b$.

If $a = 0$, then, from (1), $2b^2(1+b) + b^2 = 1$. It is clear that $b = -1$ is a solution to this equation, and it follows readily that $b = \frac{1}{2}$ is the only other solution. It is straightforward to check that the constant polynomials $p(x) = -1$ and $p(x) = \frac{1}{2}$ are indeed $*$-polynomials.

In case $a \neq 0$, then $1 - a - b = 0 = 1 - a + b$, so that $b = 0$. From (1), the coefficient $2a^2(1-a-b) = 2a^2(1-a) = 0$, giving $a = 1$. From this we get the trivial $*$-polynomial $p(x) = x$.

Our next observation is that when $p(x)$ is a $*$-polynomial, then $p(p(x)) = p^2(x)$ is also a $*$-polynomial — and, in fact, it follows by an easy induction that $p^n(x)$ are $*$-polynomials for all $n \geqslant 1$, where $p^n(x)$ means the composition of $p(x)$ with itself, n times. So if we can find a $*$-polynomial $p(x)$ such that infinitely many polynomials in the sequence $p(x), p^2(x), p^3(x), \ldots$ are different from each other, the problem will be solved. Unfortunately, none of the three $*$-polynomials we have found so far has this property. We therefore look for a possible second degree $*$-polynomial $p(x) = ax^2 + bx + c,\ a, b, c \in \mathbb{R},\ a \neq 0$. In this case, assuming that $p(x)$ is a $*$-polynomial, and again using the $*$-triples $(x, -x, 1)$ and $(x, x, -1)$ (for all $x \in \mathbb{R}$), we obtain, after simplification:

$$\begin{aligned} 2a^2(1+a+b+c)x^4 + 2[2ac + b^2 + (2ac - b^2)(a+b+c)]x^2 + & \\ 2c^2(1+a+b+c) + (a+b+c)^2 &= 1 \quad (3) \end{aligned}$$

and

$$\begin{aligned} 2a^2(1+a-b+c)x^4 + 4ab(1+a-b+c)x^3 + 2(2ac+b^2)(1+a-b+c)x^2 & \\ +4bc(1+a-b+c)x + 2c^2(1+a-b+c) + (a-b+c)^2 &= 1 \quad (4) \end{aligned}$$

Since the coefficients of x^4 in (3) and (4) must be 0, and we have $a \neq 0$, we must have $a + b + c = -1 = a - b + c$, so that $b = 0$. Thus $a + c = -1$ and we conclude that

$$p(x) = ax^2 + c = (-1-c)x^2 + c$$

for some $c \in \mathbb{R}$. But $p(x)$ is assumed to be a $*$-polynomial, and since $(x, \sqrt{1-x^2}, 0)$ are $*$-triples for all $-1 \leqslant x \leqslant 1$, we get that

$$((-1-c)x^2+c)^2 + ((-1-c)(1-x^2)+c)^2 + c^2 + 2((-1-c)x^2+c)((-1-c)(1-x^2)+c)c = 1,$$

which simplifies to

$$2(1+c)^2(1-c)x^4 - 2(1+c)^2(1-c)x^2 + 1 = 1.$$

As before, the coefficients of x^4 and x^2 must be 0, and we see that $c \in \{-1, 1\}$. The case $c = -1$ gives $p(x) = -1$, which we have already dealt with. Hence, the only possibe candidate at this stage for a second degree $*$-polynomial, is $p(x) = -2x^2 + 1$. We now verify that $p(x) = -2x^2 + 1$ is indeed a $*$-polynomial:

Let (x, y, z) be an arbitrary $*$-triple. Then

$$\begin{aligned} & (-2x^2+1)^2 + (-2y^2+1)^2 + (-2z^2+1)^2 + 2(-2x^2+1)(-2y^2+1)(-2z^2+1) \\ = \ & -16x^2y^2z^2 + 4(x^4+y^4+z^4) + 8(x^2y^2+y^2z^2+z^2x^2) - 8(x^2+y^2+z^2) + 5 \\ = \ & -4(1-(x^2+y^2+z^2))^2 + 4(x^2+y^2+z^2)^2 - 8(x^2+y^2+z^2) + 5 \\ = \ & 1, \end{aligned}$$

and we conclude that $p(x) = -2x^2 + 1$ is indeed a $*$-polynomial.

This solves the problem, since we now have an infinite sequence $p(x), p^2(x), p^3(x), \ldots$ of $*$-polynomials, and they are all different, since $\deg(p^n(x)) = 2^n$ for each $n \geqslant 1$, a fact that can easily be verified by induction.

Each problem is worth 7 points.