HARMONY SOUTH AFRICAN MATHEMATICS OLYMPIAD

Organised by the SOUTH AFRICAN MATHEMATICS FOUNDATION in collaboration with, AMESA, SAMS and the SUID-AFRIKAANSE AKADEMIE VIR WETENSKAP EN KUNS
Sponsored by HARMONY GOLD MINING

Third Round 2005
Junior Section: Grades 8 and 9
Solutions

1. The area of one side of a rectangular box is 126 cm^2. The area of another side of the rectangular box is 153 cm^2. The area of the top of the rectangular box is 238 cm^2.

What is the volume of the box?

Solution:

$xz = 126 \qquad (1)$

$zy = 153 \qquad (2)$

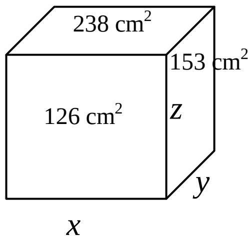
$xy = 238 \qquad (3)$

$(1)\times(2)\times(3) = (xyz)^2 = 126\times153\times238$

$= (2.3^2.7)\times(3^2.17)\times(2.7.17)$

$= 2^2.3^4.7^2.17^2$

$\therefore$ Volume $= xyz = 2.3^2.7.17 = 2142$ cm^3



2. In a Hockey tournament, each team played each other team once.
The final league table was:

	WINS	DRAWS	LOSSES	POINTS
BULLS	1	2	0	4
CHEETAHS	1	1	1	3
LIONS	1	1	1	3
SHARKS	1	0	2	2

If Sharks beat Cheetahs, then which of the statements are true?

(i) Lions defeated Cheetahs, but lost to Sharks.
(ii) Bulls won against either Cheetahs or Lions.
(iii) In matches against Sharks, Cheetahs were more successful than Lions.
(iv) In matches against Lions, Bulls were more successful than Cheetahs.
(v) Lions were undefeated, except against Cheetahs.

Solution:

(i) No. Lions did not loose against Sharks, since Sharks only win was against Cheetahs.

(ii) No. Let B = Bulls; C = Cheetahs; L = Lions; S = Sharks
Then (S, C) = (2, 0), meaning that in the game between S and C, S earned 2 points and C 0 points. S has lost its other 2 games, so (S, L) = (0, 2) and (S, B) = (0, 2). But then B drew against C and L, so (B, C) = (B, L) = (1, 1). But C has also won a game, so (C, L) = (2, 0). From (B, C) = (B, L) = (1, 1), the answer to (ii) is No.

(iii) No. Follows from (S, C) = (2, 0) and (S, L) = (0, 2).

(iv) No. Follows from (B, L) = (1, 1) and (C, L) = (2, 0).

(v) Yes. Follows from (S, L) = (0, 2) and (B, L) = (1, 1) and (C, L) = (2, 0).

Or

	B	C	L	S
B				
C	D D			
L	D D	L W		
S	L W	W L	L W	

Only (v)

3. A sequence has first term 12, after which every term is the sum of the squares of the digits of the preceding term. Thus the second term is $1^2+2^2=5$, the third term $5^2=25$, the fourth term $2^2+5^2=29$, and so on.

Find the 2005th term of the sequence.

Solution:

The first 22 terms are: 12; 5; 25; 29; 85; 89; 145; 42; 20; 4; 16; 37; 58;
89; 145; 42; 20; 4; 16; 37; 58;
89; …

Note that 89; 145; 42; 20; 4; 16; 37; 58 form a cycle which continue to repeat after the first 5 terms. Therefore the 6th, 14th, 22nd, 30th, 38th, 1998th, terms are all 89.
Hence the 2005th term is 58.

4. A 4 by 4 “Antimagic square” is an arrangement of the numbers 1 to 16 in a square grid such that the totals of each of the four rows and columns and the main diagonals are 10 consecutive numbers in some order. The diagram shows an incomplete antimagic square. Complete the square.

4	**5**	**7**	**14**
6	**13**	**3**	
11	**12**	**9**	
10			

Solution:

The following numbers need to be placed:
1; 2; 8; 15 & 16.

Known: Total of column A: 31
Total of row 1: 30
Diagonal from A4 to D1: 39
Therefore the 10 numbers are 30, 31, … 39
Working: D4 can only be 8, because 15 or 16 will make the diagonal sum too big

	A	B	C	D	39
1	**4**	**5**	**7**	**14**	30
2	**6**	**13**	**3**	15	37
3	**11**	**12**	**9**	1	33
4	**10**	2	16	8	36
	31	32	35	38	34

Diagonal from A1 to D4: 34
D3 can only be 1
Total of row 3: 33
B4 can only be 2
Total of column B: 32
D4 can only be 15
Total of row 2: 37
C4 can only be 16
Total of column C: 35

5. Each letter in the addition sum shown below stands for a different digit, with *S* standing for 3.

$$\begin{array}{r} SO \\ +\ MANY \\ \hline SUMS \\ \hline \end{array}$$

What is the value of $Y \times O$?

Solution:
The addition sum is:

$$\begin{array}{r} 3O \\ +\ MANY \\ \hline 3UM3 \\ \hline \end{array}$$

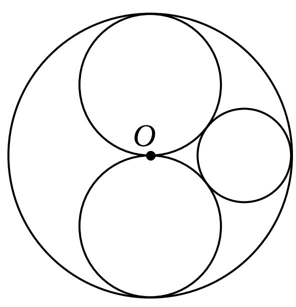
Clearly $M \neq 3$. *M* cannot be greater than 3 since we do not have a 10 thousandth term. So $M < 3$. *U* must follow from the sum of *A* and the carry over of at least one, which can't be more than 1 either. Therefore it has to be 1. M has to be 2. 3 + *N* must give 2 and carry over 1, with a carrying over of 1 from *O* + *Y*, that gives *N* = 8. *A* + 1 = *U* carry over 1, which means *A* = 9 and *U* = 0. *O* + *Y* must give 13 and because neither 6 nor 7 have been used, *O* = 6, then *Y* = 7 or *O* = 7, then *Y* = 6, but the answer will stay 42.

Answer:

$$\begin{array}{r} 37 \\ +\ 2986 \\ \hline 3023 \\ \hline \end{array} \quad or \quad \begin{array}{r} 36 \\ +\ 2987 \\ \hline 3023 \\ \hline \end{array}$$

$Y \times O = 42$

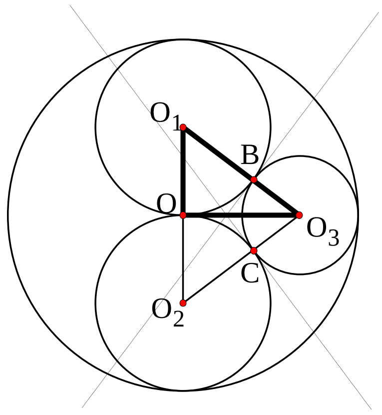
6. A circle of radius 2 cm with centre O, contains three smaller circles as shown in the diagram; two of them touch the outer circle, and touch each other at *O*, and the third touches each of the other circles.



Determine the radius of the third circle, in centimeters.

Solution:

Let O_1, O_2 and O_3 be the centers of the smaller (inner) circles respectively.
Let O, B and C be the points of intersection, respectively, between these circles. Then
O_1BO_3 and O_2CO_3 are straight lines. (radii $\perp$ tangents)
Let r be the radius of third circle. Then
$O_1O = 1$, $OO_3 = 2 - r$ and $O_1O_3 = 1 + r$
ΔO_1OO_3 is a right-angled triangle.
By the theorem of Pythagaros



$$(1+r)^2 = 1^2 + (2-r)^2$$

$$6r = 4$$

$$r = \frac{2}{3}\text{cm}$$

7. Find the smallest natural number which when multiplied by 123 yields a product that ends in 2005.

Solution:

Table 1

	1	2	3
			5
	6	1	5
		9	0
2	0	0	5

Table 2

	1	2	3
		3	5
	6	1	5
3	6	9	0
	7	0	0
2	0	0	5

Table 3

	1	2	3
	9	3	5
	6	1	5
3	6	9	0
0	7	0	0
7	0	0	0
2	0	0	5

Table 4

	1	2	3
9	9	3	5
	6	1	5
3	6	9	0
0	7	0	0
7	0	0	0
2	0	0	5

To get a 5 in the end column the required number must end in a 5 (see Table 1). This yields a 1 in the second column and a 6 in the third column. To get a 0 in the second column, we need a 9 to be added to the 1 in the second column, hence our number ends in 35 (see Table 2). This yields a 6 in the second column and a 3 in the fourth. Also note that a 1 is carried from the second the third column. To get a 0 in the third column, we need $1+6+6+7 \Rightarrow 0$, hence our number ends in 935 (see Table 3). This yields a 0 in the fourth column. Again notice that we carry a 2 from the third column. To get a 2 in the fourth column we need $2+3+7 \Rightarrow 2$ (see Table 4), hence our number is 9935.

Or

Question: What is the smallest number (between 1 and 9) times 3, that gives a unit digit of 5? Only 5. Therefore 123*5=615. Now we need a 90 to get the Tenth digit 0.

Question: What is the smallest number (between 10 and 90) times 3, that gives a tenth digit of 9? Only 30.Therefore 123*35=4305. Now we need a 700 to get the Hundredth digit 0.

Question: What is the smallest number (between 100 and 900) times 3, that gives a hundredth digit of 7? Only 900. Therefore 123*935=115005. Now we need a 7000 to get the Thousandth digit 2. Question: What is the smallest number (between 1000 and 9000) times 3, that gives a thousandth digit of 7? Only 9000.Therefore 123*9935=1222005.

This is the answer!

8. Let A = 200420042004 $\times$ 2005200520052005

and B = 200520052005 $\times$ 20042004

Find the value of $\frac{A}{B}$.

Solution 1:

$$\frac{A}{B}=\frac{2004\times100010001\times2005\times1000100010001}{2005\times100010001\times2004\times10001}$$
$$=\frac{1000100010001}{10001}$$
$$=\frac{10001\times100000001}{10001}$$
$$=100000001$$

Or:

Solution 2:

Let $T = 10\,000$, so A = 2004 × (T^2+T+1) × 2005 × (T^3 +T^2+T+1), etc.

Then follows:

$$\frac{A}{B}=\frac{[2004\times(T^2+T+1)\times2005\times(T^3+T^2+T+1)]}{[2005\times(T^2+T+1)\times2004\times(T+1)]}$$
$$=\frac{[(T+1)\times(T^2+1)]}{(T+1)}$$
$$=T^2+1$$
$$=10^8+1$$
$$=100\,000\,001$$

9. Alfred, Brigitte, Carodene, Dolly, and Effie play a game in which each is either a dog or a mouse. A dog's statement is always false while a mouse's statement is always true.

(a) Alfred says that Brigitte is a mouse.
(b) Carodene says that Dolly is a dog.
(c) Effie says that Alfred is not a dog.
(d) Brigitte says that Carodene is not a mouse.

(e) Dolly says that Effie and Alfred are different kinds of animals.

How many dogs are there?

Solution:

First try:

From (a) If Alfred is a mouse, then Brigitte is a mouse too,

From (d) Carodene is a dog,

From (b) Dolly is a mouse,

From (e) Effie is a dog,

From (c) Alfred is a dog.

NOT True!

Second try:

From (a) If Alfred is a dog, then Brigitte is a dog too,

From (d) Carodene is a mouse,

From (b) Dolly is a dog,

From (e) Effie is a dog,

From (c) Alfred is a dog.

TRUE!

There are 4 dogs.

10. We have assigned different positive integers to different letters and then multiplied their values together to make the values of words.

For example, if C = 4; A = 8 and T = 12, then CAT = $4\times8\times12=384$.

Given that HILL = 15

PHOTO = 8470

HILLS = 195

HIPHOP = 3300

Find the value of PITSTOP.

Solution:

$\text{HILL} = 15 = 5 \times 3 \times 1 \times 1$

Obviously $\text{L} = 1$

$\text{PHOTO} = 8470 = 7 \times 5 \times 11 \times 2 \times 11$

$\Rightarrow \text{O} = 11$ and $\text{H} = 5$ and $\text{I} = 3$

$\text{HILLS} = 195 = 5 \times 3 \times 1 \times 1 \times 13$

$\Rightarrow \text{S} = 13$

$\text{HIPHOP} = 3300 = 5 \times 3 \times 2 \times 5 \times 11 \times 2$

$\Rightarrow \text{P} = 2$

Hence, $\text{T} = 7$

P	I	T	S	T	O	P
2	3	7	13	7	11	2

$\therefore \text{PITSTOP} = 84084$

11. Consider the following number pattern using only the odd natural numbers:

			1					(Row 1)
		3		5				(Row 2)
	7		9		11			(Row 3)
13		15		17		19		(Row 4)

....

A B
C is taken from the pattern, where A and B are two adjacent numbers in (say) the *i*-th row, and C is the number in the (*i*+1)-st row, directly below, and between, A and B. If A+B+C=2093, find the value of C.

Solution 1:

2093/3 = 697,667, therefore C > 698 with A< 697 and B < 697
If one looks at the middle number of the odd lines, 1, 9, 25, 49 you get squares of odd numbers. The number is the square number of the row number and it contains the same number of odd numbers as the row number (row 1 contains 1 number, row 5, 5 numbers, row 27, 27 numbers etc.). Looking at the square numbers 1, 9, 25, ….625, 729, … The number C has to be in row 27. Row 27th middle number is 729 and it contains 27 odd numbers, therefore from 703 to 755. From A to C one has to add the (row number x 2) to make provision for the even numbers.
Therefore n + (n + 2) + (n + 54) = 2093. That is: n = 679, which means A = 679, B = 681 and C = 679 + 54 = 733.

651 653 655 657 659 661 663 665 667 669 671 673 675 677 **679 681** 683 685 687 689 691 693 695 697 699 701
703 705 707 709 711 713 715 717 719 721 723 725 727 729 731 **733** 735 737 739 741 743 745 747 749 751 753 755

OR

Solution 2:

The first number in the n^{th} row is $n^2 - n + 1$. (Easily obtained by observing that the 2nd row of differences is constant)
Hence, the k^{th} number ($0 \le k \le n-1$) in the n^{th} row is $n^2 - n + 1 + 2k$
For the given triangle of numbers A, B, C, we therefore have n and k such that
A $= n^2 - n + 1 + 2k$
B $= n^2 - n + 1 + 2(k+1)$
C = $(n+1)^2 - (n+1) + 1 + 2(k+1)$.
Adding gives 2093 = $3n^2 - n + 7 + 6k$ (*)
Therefore, $0 \le 6k = -3n^2 + n + 2086 \le +6(n-1)$.
From the first inequality, $n \le 26$.
From the second inequality, $n \ge 26$.
So n = 26, giving k = 14 (from (*).
Consequently, C = $27^2 - 27 + 1 + 2 \times 15 = 733$.

The terms in the n-th row could have been counted from k = 1 to k = n instead of k = 0 to k = n-1.

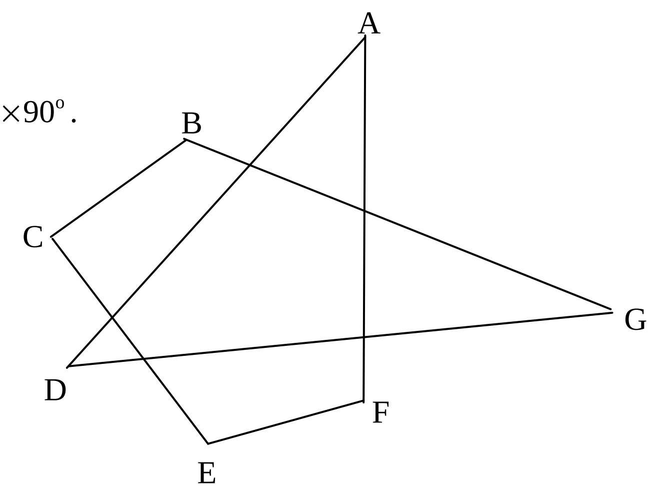
OR

Solution 3:

The first number in the n^{th} row is $n^2 - n + 1$. (Easily obtained by observing that the 2nd row of differences is constant)
Hence, the k^{th} number ($1 \le k \le n$) in the n^{th} row is $n^2 - n + 2(k-1) + 1 = n^2 - n + 2k - 1$
So, for the given triangle of numbers A, B, C, there exist n and k ($1 \le k \le n-1$), such that
A $= n^2 - n + 2k - 1$
B $= n^2 - n + 2(k+1) - 1 = n^2 - n + 2k + 1$
C = $(n+1)^2 - (n+1) + 2k + 1 = n^2 + n + 2k + 1$.
Adding gives 2093 = $3n^2 - n + 6k + 1$ (*)
Therefore, $0 \le 6k = -3n^2 + n + 2092 \le +6(n-1)$.
From the first inequality, $n \le 26$.
From the second inequality, $n \ge 26$.
So n = 26, giving k = 15 (from (*).
Consequently, C = $27^2 - 27 + 2 \times 16 - 1 = 733$.

12. In the diagram, $\hat{A} + \hat{B} + \hat{C} + \hat{D} + \hat{E} + \hat{F} + \hat{G} = p \times 90^{\circ}$.

Find p.



Solution 1:

$\hat{A}+\hat{D}+x=180^\circ$ and $\hat{B}+\hat{C}+\hat{G}+y=360^\circ$

Therefore $\hat{A}+\hat{B}+\hat{C}+\hat{D}+\hat{G}+(x+y)=6\times 90^\circ$

But $(x+y)=\hat{E}+\hat{F}$

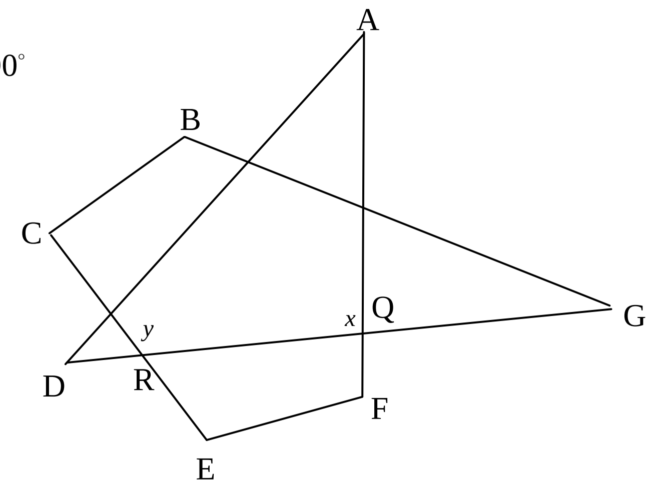
$[y+\hat{R}_2+x+\hat{Q}_2=360^\circ$,

but $\hat{R}_2+\hat{E}+\hat{F}+\hat{Q}_2=360^\circ$

$\therefore (x+y)=\hat{E}+\hat{F}]$

$\therefore \hat{A}+\hat{B}+\hat{C}+\hat{D}+\hat{E}+\hat{F}+\hat{G}=6\times 90^\circ$

$\therefore p=6$



Solution 2:

The figure consists of three triangles and two quadrilaterals surrounding a pentagon. The sum of the interior angles of the outer five polygons is $3 \times 180^o + 2 \times 360^o$.

It is also equal to $\hat{A}+\hat{B}+\hat{C}+\hat{D}+\hat{E}+\hat{F}+\hat{G}$ plus twice the sum of the exterior angles of the pentagon (because there is a clockwise and an anticlockwise set), that is,

$\hat{A}+\hat{B}+\hat{C}+\hat{D}+\hat{E}+\hat{F}+\hat{G}+2\times 360^\circ$. This leads to the answer:

$$\hat{A}+\hat{B}+\hat{C}+\hat{D}+\hat{E}+\hat{F}+\hat{G}+2\times 360^\circ=3\times 180^\circ+2\times 360^\circ$$

$$\therefore \hat{A}+\hat{B}+\hat{C}+\hat{D}+\hat{E}+\hat{F}+\hat{G}=6\times 90^\circ$$

Solution 3:

If you trace the edges in the order ADGBCEFA, then you encircle any point in the inner pentagon twice, so the sum of the exterior angles is 2×360^o, not 360^o. The sum of the exterior angles is also equal to

$$(180^\circ-\hat{A})+(180^\circ-\hat{B})+\ldots+(180^\circ-\hat{G})=7\times 180^\circ-(\hat{A}+\hat{B}+\hat{C}+\hat{D}+\hat{E}+\hat{F}+\hat{G}).$$

$$7\times 180^\circ-(\hat{A}+\hat{B}+\hat{C}+\hat{D}+\hat{E}+\hat{F}+\hat{G})=2\times 360^\circ \therefore \hat{A}+\hat{B}+\hat{C}+\hat{D}+\hat{E}+\hat{F}+\hat{G}=3\times 180^\circ=6\times 90^\circ$$

13. Paul and James went out for a cycle and were 16 km from home when Paul ran into a tree damaging his bicycle beyond repair. They decide to return home and that Paul will start on foot and James will start on his bicycle. After some time, James will leave his bicycle beside the road and continue on foot, so that when Paul reaches the bicycle he can mount it and cycle the rest of the distance. Paul walks at 4 km per hour and cycles at 10 km per hour, while James walks at 5 km per hour and cycles at 12 km per hour. For what length of time should James ride the bicycle, if they are both to arrive home at the same time?

Solution:

A to B: Paul on foot at 4 km/h

James on bicycle at 12 km/h

B to C: Paul on bicycle at 10 km/h

James on at foot 5 km/h

$\therefore$ Total time for Paul = Total time for James

$$\Rightarrow \quad \frac{x}{4}+\frac{16-x}{10}=\frac{x}{12}+\frac{16-x}{5}$$

$$\Rightarrow \quad x = 6\ km$$

James on bicycle (from A to B): $\frac{6}{12}=\frac{1}{2}$ hour.

Or

	Speed	Time	Distance
Paul (foot)	4 km/h	*12x/4*	*12x*
Paul (bike)	10 km/h	*(16 - 12x)/10*	*16 - 12x*
James (foot)	5 km/h	*(16 - 12x)/5*	*16 - 12x*
James (bike)	12 km/h	*x*	*12x*

Let's say James' time is x hours, then:

$$\frac{12x}{4}+\frac{(16-12x)}{10}=\frac{(16-12x)}{5}+x$$

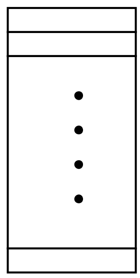
$$\therefore 60x+2(16-12x)=4(16-12x)+20x$$

$$\therefore 36x+32=-28x+64$$

$$\therefore 64x=32$$

$$\therefore x=0.5 \text{ hours or 30 min}$$

14. The diagram shows a large rectangle whose perimeter is 300 cm. It is divided up as shown into a number of identical rectangles, each of perimeter 58 cm. Each side of these rectangles is a whole number of centimeters. Show that there are exactly two ways of splitting up the rectangle as described above.

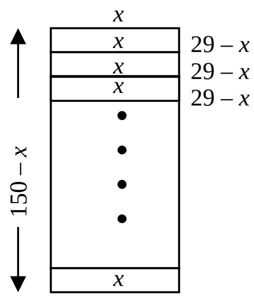


Let the larger rectangle have dimensions:

x cm x $(150 - x)$ cm

Then the smaller ones have dimensions:

x cm x $(29 - x)$ cm



This implies that:

$$1 \le x < 29$$

Let there be k smaller rectangles. Then:

$$k(29 - x) = 150 - x$$

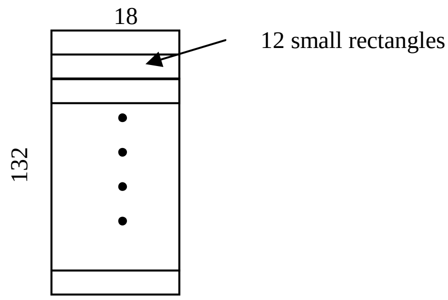
$$\Rightarrow k = \frac{150 - x}{29 - x} = 1 + \frac{121}{29 - x}$$

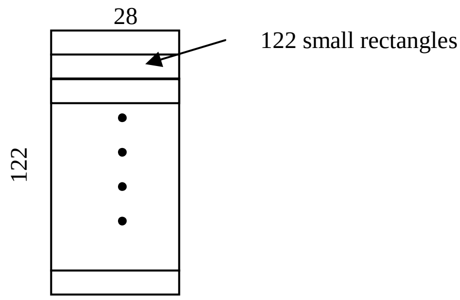
Since k is an integer, $29 - x \in \{\pm 1,\ \pm 11,\ \pm 121\}$

Only two of these choices, namely:

$$(29 - x) = 1 \text{ and } (29 - x) = 11, \text{ give } 1 \le x < 29$$

These two values give x = 28 and x = 18, which forces k = 122 and k = 12. respectively.
Consequently, two solutions:



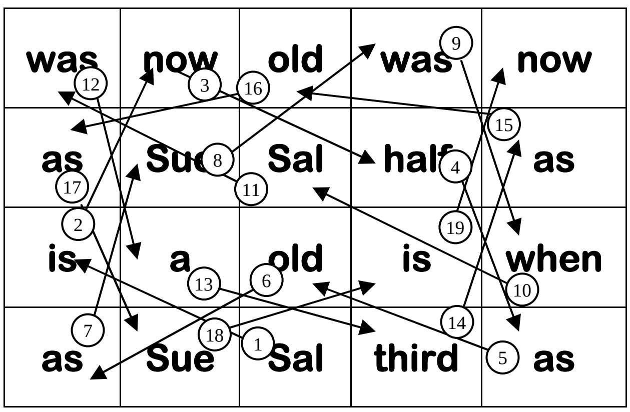


15. A complete sentence of 20 words is concealed here. If you start with Sal and then follow the jumps of a chess knight from word to word along the correct route, a simple problem will be revealed. Sue is in her teens, so how old is Sal?

was	now	old	was	now
as	Sue	Sal	half	as
is	a	old	is	when
as	Sue	Sal	third	as

Solution 1:

Pattern



Answer: Sal is now half as old as Sue was when Sal was a third as old as Sue is now (20 words)

Sal now: x
Sue now: y
Time: z

$$x = \frac{1}{2}(y - z) \quad (1) \qquad and \qquad x - z = \frac{1}{3}y$$

$$\therefore x = \frac{1}{3}y - z \quad (2)$$

$$\text{In } 1: \quad \frac{1}{3}y + z = \frac{1}{2}(y - z)$$

$$\therefore y = 9z \text{ and } x = 4z$$

Because Sue is in her teens, z can only be 2, thus Sue is 18 years old and Sal is 8 years old.

Test: Sal is now 8, which is half of (18 – 2) = 16 (the age of Sue 2 years ago).

Sue is now 18, which is three time the current age of Sal less 2 years.

OR

Solution 2:
Sal is now half as old as Sue was when Sal was a third as old as Sue is now.'
Let Sue be x years old and let n be the number of year between then and now.

	Sal	Sue
Now	$\frac{1}{2}(x - n)$	x
Then	$\frac{1}{3}x$	$x - n$

Hence

$\frac{1}{2}(x - n) - \frac{1}{3}x = n$

$x = 9n$

If $n = 1$, then $x = 9$, but Sue is in her teens.

If $n = 3$, then $x = 27$, again contradicting the fact that Sue is a teenager.

$\therefore n = 2$, and so $x = 18$

Hence Sal is 8 years old.