South African Mathematics Olympiad
Third Round 2019
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. Determine all positive integers a for which a^a is divisible by 20^{19}.

2. We have a deck of 90 cards that are numbered from 10 to 99 (all two-digit numbers). How many sets of three or more different cards in this deck are there such that the number on one of them is the sum of the other numbers, and those other numbers are consecutive?

3. Let A, B, C be points on a circle whose centre is O and whose radius is 1, such that $\angle BAC = 45^\circ$. Lines AC and BO (possibly extended) intersect at D, and lines AB and CO (possibly extended) intersect at E. Prove that $BD \cdot CE = 2$.

4. The squares of an 8×8 board are coloured alternatingly black and white. A rectangle consisting of some of the squares of the board is called *important* if its sides are parallel to the sides of the board and all its corner squares are coloured black. The side lengths can be anything from 1 to 8 squares. On each of the 64 squares of the board, we write the number of important rectangles in which it is contained. The sum of the numbers on the black squares is B, and the sum of the numbers on the white squares is W. Determine the difference $B - W$.

5. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a^3) + f(b^3) + f(c^3) + 3f(a+b)f(b+c)f(c+a) = (f(a+b+c))^3$$

for all integers a, b, c.

6. Determine all pairs (m, n) of non-negative integers that satisfy the equation

$$20^m - 10m^2 + 1 = 19^n.$$

Each problem is worth 7 points

Die Suid-Afrikaanse Wiskunde-Olimpiade
Derde Ronde 2019
Senior Afdeling (Grade 10 tot 12)
Tyd : 4 uur
(Geen rekenapparaat word toegelaat nie)

1. Vind alle positiewe heelgetalle a sodanig dat a^a deelbaar is deur 20^{19}.

2. Beskou 'n pak van 90 kaarte wat genommer is van 10 tot 99 (al die twee-syfer getalle). Hoeveel versamelings van drie of meer verskillende kaarte is daar in hierdie pak sodanig dat die nommer op een van hulle gelyk is aan die som van die nommers op die ander, en hierdie ander kaarte se nommers is opeenvolgend?

3. Laat A, B, C punte op 'n sirkel, met middelpunt O en radius 1 wees, sodanig dat $\angle BAC = 45^\circ$. Lyne AC en BO (moontlik verleng) sny by D, en lyne AB en CO (moontlik verleng) sny by E. Bewys dat $BD \cdot CE = 2$.

4. Die vierkante op 'n 8×8 bord word alternatiewelik swart en wit gekleur. 'n Reghoek, bestaande uit sommige van die vierkante op die bord, word *belangrik* genoem indien sy sye ewewydig is aan die sye van die bord, en al die vierkante wat die hoeke van die reghoek vorm, is swart. Die sylengtes kan wissel van 1 tot 8 vierkante. Op elkeen van die 64 vierkante van die bord skryf ons die aantal belangrike reghoeke neer waarin dit bevat is. Die som van die getalle op die swart vierkante is B, en die som van die getalle op die wit vierkante is W. Bepaal die verskil $B - W$.

5. Vind alle funksies $f : \mathbb{Z} \to \mathbb{Z}$ sodanig dat

$$f(a^3) + f(b^3) + f(c^3) + 3f(a+b)f(b+c)f(c+a) = (f(a+b+c))^3$$

vir alle heelgetalle a, b, c.

6. Bepaal alle pare (m, n) van nie-negatiewe heelgetalle wat die volgende vergelyking bevredig:

$$20^m - 10m^2 + 1 = 19^n$$

Elke probleem is 7 punte werd.