Die Suid-Afrikaanse Wiskunde-Olimpiade
Derde Ronde 2005
Senior Afdeling (Grade 10 tot 12)
Tyd: 4 uur

1. Vyf getalle word uit die diagram hieronder gekies, sodat daar nie twee getalle uit dieselfde ry of uit dieselfde kolom gekies word nie. Toon aan dat hulle som altyd dieselfde is.

1	4	7	10	13
16	19	22	25	28
31	34	37	40	43
46	49	52	55	58
61	64	67	70	73

2. Laat F die versameling van alle breuke m/n wees, waar m en n positiewe heeltalle is met $m + n \leqslant 2005$. Vind die grootste getal a in F sodat $a < 16/23$.

3. ’n Pakhuis bevat 175 nommer-8 stewels, 175 nommer-9 stewels, en 200 nommer-10 stewels. Van hierdie 550 stewels is 250 linkerstewels en 300 regterstewels. Laat n staan vir die totale aantal bruikbare pare stewels in die pakhuis. (’n Bruikbare paar bestaan uit ’n linker- en regterstewel van dieselfde grootte.)

   (a) Is $n = 50$ moontlik? (b) Is $n = 51$ moontlik?

4. Die ingeskrewe sirkel van driehoek ABC raak die sye BC, CA en AB by D, E en F onderskeidelik. Laat Q staan vir die ander snypunt van AD en die ingeskrewe sirkel. Bewys dat die verlenging van EQ deur die middelpunt van AF gaan as en slegs as $AC = BC$.

5. Laat $x_1, x_2, \ldots, x_n$ positiewe getalle wees met produk gelyk aan 1. Bewys dat daar ’n $k \in \{1, 2 \ldots, n\}$ bestaan sodat

$$\frac{x_k}{k + x_1 + x_2 \cdots + x_k} \geqslant 1 - \frac{1}{\sqrt[n]{2}}.$$

6. Beskou die stygende ry $1, 2, 4, 5, 7, 9, 10, 12, 14, 16, 17, 19, \ldots$ van positiewe heeltalle, verkry deur om die beurt blokke $\{1\}, \{2, 4\}, \{5, 7, 9\}, \{10, 12, 14, 16\}, \ldots$ van onewe en ewe getalle aanmekaar te las. Elke blok bevat een element meer as die vorige een, en die eerste element van elke blok is een meer as die laaste element van die vorige blok. Bewys dat die n-de element van die ry gegee word deur

$$2n - \left\lfloor \frac{1 + \sqrt{8n - 7}}{2} \right\rfloor.$$

($\lfloor x \rfloor$ dui die grootste heeltal kleiner as of gelyk aan x aan.)

The South African Mathematical Olympiad
Third Round 2005
Senior Division (Grades 10 to 12)
Time : 4 hours

1. Five numbers are chosen from the diagram below, such that no two numbers are chosen from the same row or from the same column. Prove that their sum is always the same.

1	4	7	10	13
16	19	22	25	28
31	34	37	40	43
46	49	52	55	58
61	64	67	70	73

2. Let F be the set of all fractions m/n where m and n are positive integers with $m+n \leqslant 2005$. Find the largest number a in F such that $a < 16/23$.

3. A warehouse contains 175 boots of size 8, 175 boots of size 9 and 200 boots of size 10. Of these 550 boots, 250 are for the left foot and 300 for the right foot. Let n denote the total number of usable pairs of boots in the warehouse. (A usable pair consists of a left and a right boot of the same size.)

   (a) Is $n = 50$ possible? (b) Is $n = 51$ possible?

4. The inscribed circle of triangle ABC touches the sides BC, CA and AB at D, E and F respectively. Let Q denote the other point of intersection of AD and the inscribed circle. Prove that EQ extended passes through the midpoint of AF if and only if $AC = BC$.

5. Let $x_1, x_2, \ldots, x_n$ be positive numbers with product equal to 1. Prove that there exists a $k \in \{1, 2 \ldots, n\}$ such that

$$\frac{x_k}{k + x_1 + x_2 \cdots + x_k} \geqslant 1 - \frac{1}{\sqrt[n]{2}}.$$

6. Consider the increasing sequence $1, 2, 4, 5, 7, 9, 10, 12, 14, 16, 17, 19, \ldots$ of positive integers, obtained by concatenating alternating blocks $\{1\}, \{2, 4\}, \{5, 7, 9\}, \{10, 12, 14, 16\}, \ldots$ of odd and even numbers. Each block contains one more element than the previous one and the first element in each block is one more than the last element of the previous one. Prove that the n-th element of the sequence is given by

$$2n - \left\lfloor \frac{1 + \sqrt{8n - 7}}{2} \right\rfloor.$$

   ($\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)