THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior First Round 2012
Solutions

1. **Answer E.**
   $\frac{1}{4} + 0.025 = 0.25 + 0.025 = 0.275 = \frac{275}{1000} = \frac{25\times11}{25\times40} = \frac{11}{40}$.
   Alternatively, $\frac{1}{4} + 0.025 = \frac{1}{4} + \frac{1}{40} = \frac{10+1}{40} = \frac{11}{40}$.

2. **Answer B.**
   1% of 60 000 000 $= \frac{1}{100} \times 60\,000\,000 = 600\,000$.

3. **Answer E.**
   $2012^2 - 2011^2 = (2012 - 2011)(2012 + 2011) = (1)(4023) = 4023$.

4. **Answer B.**
   The sum of the original five numbers is five times their mean: $5 \times 12 = 60$. The sum of the remaining four numbers is four times their mean: $4 \times 14 = 56$. The number that was removed is the difference between these sums, which is $60 - 56 = 4$.

5. **Answer E.**
   The large circle has radius twice that of the small circle, that is, $2 \times 2 = 4$. The shaded region is obtained by removing two small semicircles from a large semicircle, so its area is $\frac{1}{2}\pi 4^2 - 2\frac{1}{2}\pi 2^2 = 8\pi - 4\pi = 4\pi$.

6. **Answer D.**
   From each figure to the next, the side of the square increases by 2. We can therefore draw up the following table of values:

Figure	1	2	3	...	n
Side	1	3	5	...	$2n-1$
Area	1	9	25	...	$(2n-1)^2$

   Putting $n = 10$, we see that in Fig 10 side of the square is 19 and its area is 361.

7. **Answer C.**
   The gradient of a line is the ratio Vertical Distance : Horizontal Distance. By Pythagoras' theorem a triangle with vertical and horizontal sides of length 3 and 4 has hypotenuse of length 5, so we have

   $$\text{Vertical Distance} : \text{Horizontal Distance} : \text{Slant Distance} = 3 : 4 : 5.$$

   If I walk a slant distance of 15 m uphill, then my height or vertical distance increases by $\frac{3}{5} \times 15 = 9$ m.

8. **Answer B.**
   If the watch loses 5 minutes every hour, then the time on the watch increases by 55 minutes for every true hour that passes. On the watch, from 07:00 to 09:45 is $2 \times 60 + 45 = 165 = 3 \times 55$ minutes, so the true time that has passed is 3×60 minutes, or three hours. The true time is therefore 10:00.

9. **Answer B.**
Firstly, suppose the people are all in a line. The cards are dealt successively to every third person, so card n goes to person $3n + c$, where c is a constant to be determined. By checking with $n = 1, 2, 3, 4$ (any one will do), we see that $c = -2$. Next, since the card after person 10 goes to person 3 instead of person 13, it follows that there are 10 people around the table. In this case we know that there are 10 people.
For the n-th card, the person's number around the table will therefore be the remainder after $3n - 2$ has been divided by 10, which can be written $3n - 2 \pmod{10}$. With $n = 52$ we have $3n - 2 = 154$ and $3n - 2 \pmod{10} = 4$.

10. **Answer D.**
The gradient of the line is $\frac{7-3}{4-2} = 2$, so the line has equation $y = 2x + c$. By checking with either of the given points, we see that $c = -1$, so the equation is $y = 2x - 1$. The point $(3, 5)$ satisfies the equation, and the others do not.
Alternatively, the general equation of a line is $y = mx + c$. Substituting the two given points gives $3 = 2m + c$ and $7 = 4m + c$. These can be solved to give $m = 2$ and $c = -1$, and we proceed as before.

11. **Answer D.**
Let S be the required sum. Every fourth term has a negative sign, so let T be the sum obtained if we change all negative signs to plus signs. Then $T - S = 2(4 + 8 + 12 + \cdots + 100)$. From a formula for the sum of an arithmetic series we have

$$T = 1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100 = \frac{100}{2}(1 + 100) = 5050 \text{ and}$$

$$4 + 8 + 12 + \cdots + 100 = \frac{25}{2}(4 + 100) = 1300.$$

Thus $S = 5050 - 2 \times 1300 = 2450$.
Alternatively, if we bracket off the terms four at a time and add up each of the 25 brackets, we see that $1+2+3-4 = 2$, then $5+6+7-8$=10, then $9+10+11-12 = 18$, and so on, ending up with $97 + 98 + 99 - 100 = 194$. This is again an arithmetic series and its sum is $\frac{25}{2}(2 + 194) = 2450$.

12. **Answer A.**
In each of three perpendicular directions (length, breadth, and height) the box has four parallel edges. Taking them two at a time gives $\frac{1}{2} \times 4 \times 3 = 6$ pairs of parallel edges in each direction. Thus the total number of pairs of parallel edges is $3 \times 6 = 18$.

13. **Answer D.**
The height of the ball above the ground decreases slowly at first, then more quickly as the ball accelerates due to gravity. This matches the graph in (D). (Graph (A) is impossible, since it shows time standing still as the ball drops. Graph (B) shows the ball remaining still at the top of the tower, graph (C) shows the ball descending at constant speed, and graph (E) shows the ball rising from the ground at increasing speed.)

14. **Answer A.**
The number of ways of choosing two numbers from a set of nine is $\frac{1}{2} \times 9 \times 8 = 36$. For the sum of the two chosen numbers to be even, they must both be even or both be odd. The number of ways of choosing two even numbers is $\frac{1}{2} \times 5 \times 4 = 10$ and of choosing two odd numbers is $\frac{1}{2} \times 4 \times 3 = 6$. Thus the probability of obtaining an even sum is $\frac{10+6}{36} = \frac{4}{9}$.
Alternatively, the probability of choosing two even numbers is $\frac{5}{9} \times \frac{4}{8} = \frac{5}{18}$ and of choosing two odd numbers is $\frac{4}{9} \times \frac{3}{8} = \frac{1}{6}$. The required probability is therefore $\frac{5}{18} + \frac{1}{6} = \frac{5+3}{18} = \frac{4}{9}$.

15. **Answer D.**
To run 100 m in 10 s is $\frac{1}{10}$ km in $\frac{10}{3600}$ h, which is a speed of $\frac{1}{10} \div \frac{10}{3600} = \frac{1}{10} \times \frac{3600}{10} =$ 36 km/h. Usain Bolt runs about 3% faster than this (since his time is about 3% less than 10 s), so his speed is approximately $1.03 \times 36 \approx 37$ km/h.

16. **Answer B.**
If you wrap a piece of paper around the cylinder and then unwrap it after Andre's journey, then you will see that his path is the hypotenuse of a triangle with vertical side 8 (the height of the cylinder) and horizontal side 4π (half the circumference of the cylinder). The distance he travels is therefore $\sqrt{8^2 + (4\pi)^2} = 4\sqrt{4 + \pi^2}$.

17. **Answer E.**
The number of learners who take History OR Geography is $54 + 48 - 16 = 86$. (We must subtract the number who take both History AND Geography, because in adding 54 and 48 we have counted them twice.) If we then add in the 12 learners who do not take either subject, then we obtain a total of 98 learners in the group.

18. **Answer C.**
The interior angles of an equilateral triangle is 60° each. Then we have the larger angle at a vertex of the triangle equal to $360^\circ - 60^\circ = 300^\circ$. For the regular polygon each interior angle is 150°. The sum of the interior angles of a polygon is $180^\circ \times (n - 2)$. Now we have $150^\circ \times n = 180^\circ \times (n - 2)$, so that $n = 12$.

19. **Answer A.**
A chain with n links will have $n - 1$ overlaps, and its length will be the total length of the links minus the total length of the overlaps. If the length of each link is x cm and the length of each overlap is y cm, then we are given that $2x - y = 12$ and $5x - 4y = 27$. Solving these equations gives $x = 7$ and $y = 2$. The length of a chain with 40 links is therefore $40x - 39y = 40 \times 7 - 39 \times 2 = 280 - 78 = 202$ cm.

20. **Answer A.**
The area of $ABCD$ is the sum of the areas of the triangles PAB, PBC, PCD, PDA. Suppose the angles subtended at P by the sides AB, BC, CD, DA are θ_1, θ_2, θ_3, θ_4, respectively, where $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 360^\circ$. Then the area of $ABCD$ equals $\frac{1}{2}(2)(3)\sin\theta_1 + \frac{1}{2}(3)(5)\sin\theta_2 + \frac{1}{2}(5)(6)\sin\theta_3 + \frac{1}{2}(6)(2)\sin\theta_4$. Now $\sin\theta = 1$ if $\theta = 90^\circ$ and $\sin\theta < 1$ if θ is any other angle between 0° and 180°. Thus the area of $ABCD$ will be a maximum if $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 90^\circ$, and the maximum area is $\frac{1}{2}(2 \times 3 + 3 \times 5 + 5 \times 6 + 6 \times 2) = \frac{1}{2}(6 + 15 + 30 + 12) = 31\frac{1}{2}$.

(Note that in the quadrilateral of maximum area the diagonals AC and BD intersect at right-angles at P.)
or
The area of $ABCD$ is the sum of the areas of the triangles $PAB,\ PBC,\ PCD,\ PDA$. Maximise the area of each triangle. E.g. for triangle PAB we use the formula: Area $= \frac{1}{2} \times AP \times\ h$. $PB = 3$ will be the maximum height if the angle between AP and BP is a right triangle. Similar for the other three triangles. Area of the triangles are 3, $7\frac{1}{2}$, 15 and 6, respectively. The area of $ABCD = 31\frac{1}{2}$.
