THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

organised by the SOUTH AFRICAN ACADEMY OF SCIENCE AND ARTS
in collaboration with OLD MUTUAL, AMESA and SAMS

SPONSORED BY OLD MUTUAL

FIRST ROUND 1999

SENIOR SECTION: GRADES 10, 11 AND 12
(STANDARDS 8, 9 AND 10)

29 APRIL 1999

TIME: 60 MINUTES

NUMBER OF QUESTIONS: 20

ANSWERS

1. B
2. E
3. B
4. E
5. D
6. E
7. B
8. D
9. E
10. A
11. A
12. A
13. B
14. C
15. E
16. A
17. C
18. B
19. A
20. B

SOLUTIONS

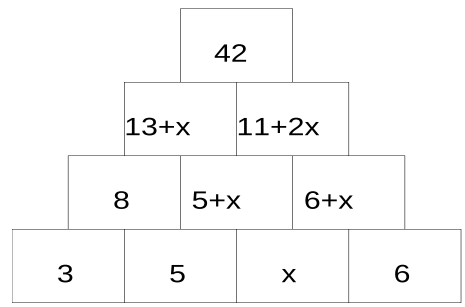
Part A: Three marks each.

1. $\frac{0,1+0,01}{1+0,1} = \frac{0,11}{1,1} = 0,1.$

2. Party C won 45% of the seats, i.e. $\frac{45}{100} \times 400 = 180.$

3. A 3 minute call will cost you $\frac{30,9}{13,6} \times 3 \times 60$ cents. This is approximately $\frac{31\times3\times60}{13,6} = \frac{5580}{13,6}$ cents. It is now easy to check that $13,6 \times 4 = 54,4$ cents is the closest.

4. From the figure we note that $42 = 24 + 3x$, or $x = 6$.



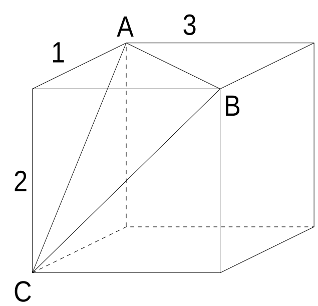
5. 2^n grows faster that any polynomial power, i.e. for large n, 2^n is the largest.

Part B: 5 marks each

6. If x is the side length, then $x^3 = 6x^2$, or $x = 6$cm.

7.

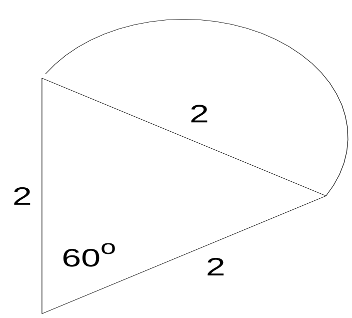
$$\begin{aligned}AB &= \sqrt{1^2+3^2} = \sqrt{10}\\ AC &= \sqrt{1^2+2^2} = \sqrt{5}\\ BC &= \sqrt{2^2+3^2} = \sqrt{13}.\end{aligned}$$



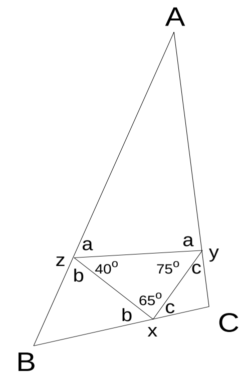
8. The trick is to spot a pattern: We start at the inside. If $x_0 = 1$ then $x_1 = 1 + 2x_0$; $x_2 = 1 + 2x_1 = 1 + 2(1+2) = 1 + 2 + 2^2$; $x_3 = 1 + 2x_2 = 1 + 2 + 2^2 + 2^3$. The value we are looking for is $x_{11} = 1 + 2 + \cdots + 2^{11} = 2^{12} - 1$.

9. Note that $0,5^2 = 0,25$; $0,5^3 = 0,125$; $0,5^4 = 0,0625$. Therefore $x = 0,5$; $y < 0,5$ and $z > 0,5$, or $y < x < z$.

10. Area of a single 'ice cream cone' : $\frac{1}{2}\pi + \frac{1}{2} \times 2 \times 2\sin(60^o) = \frac{1}{2}(\pi + 2\sqrt{3})$. The total area consists of six 'cones' and is given by $\frac{1}{3}(\pi + 2\sqrt{3})$.



11.



$$\begin{aligned} a+c &= 180^o - 75^o &= 105^o \\ a+b &= 180^o - 40^o &= 140^o \\ c+b &= 180^o - 65^o &= 115^o \end{aligned}$$

From the second and third equations follow that $a - c = 140^o - 115^o = 25^o$.
Using the first equation: $2a = 105^o + 25^o = 130^o$.
Therefore the angle at A is $180^o - 2a = 50^o$.

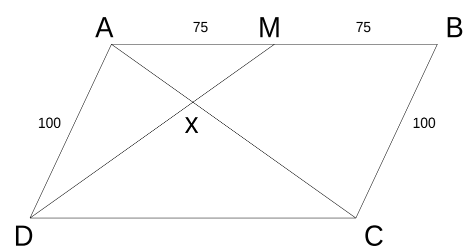
12. Note that $17 \times 23 = 391$ The natural numbers less than 400 and divisible by 17 are: $17 \times 1; 17 \times 2, \cdots, 17 \times 23$, i.e. 23 numbers. Those divisible by 23 are: $23 \times 1, 23 \times 2, \cdots, 23 \times 17$, i.e. 17 numbers. Only the last ones coincide, i.e. a total of $23 + 17 - 1 = 39$ numbers are divisible by 23 or 17. Thus $399 - 39 = 360$ numbers *less* than 400 are not divisible by 23 or 17.

13.

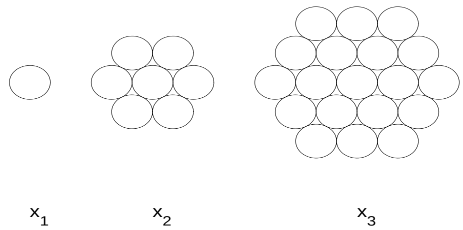
In 60 minutes the short hand moves	30^o
In 1 minute the short hand moves	$\frac{1}{2}^o$
In 60 minutes the long hand moves	360^o
In 1 minute the long hand moves	6^o

If the two hands are at right angles after time t_1, we have $6t_1 - \frac{1}{2}t_1 = 90$, or $t_1 = \frac{90 \times 2}{11}$.
If they are again at right angles after time t_2, we have $6t_2 - \frac{1}{2}t_2 = 270$, or $t_2 = \frac{270 \times 2}{11}$.
The time between these two events is: $t_2 - t_1 = \frac{180 \times 2}{11} = 32\frac{8}{11}$ minutes.

14. It is easy to show that triangles AMX and CDX are similar. Since the length of DC is twice that of AM, it follows that the height of triangle AMX is also twice that of triangle CDX. Therefore the area of triangle CDX is four times the area of triangle AMX.



15.



number of coins in the first pattern: $x_1 = 1$
number of coins in the second pattern: $x_2 = x_1 + 6(2-1) = 1 + 6 \times 1$
number of coins in the third pattern: $x_3 = x_2 + 6(3-1) = 1 + 6(1+2)$

In general: $x_{n+1} = x_n + 6n$ with $x_1 = 1$. Repeated substitution gives, $x_{n+1} = x_{n-1} + 6(n+n-1) = x_{n-2} + 6(n + (n-1) + (n-2))$, etc. We are looking for

$$\begin{aligned} x_{21} &= 1 + 6(20 + 19 + \cdots + 2 + 1) \\ &= 1 + \frac{6}{2} 20 \times 21 \\ &= 1261 \end{aligned}$$

Part C: 7 marks each

16. Calculate the remainder for a few powers of 2:

Number:	1	2	2^2	2^3	2^4
Remainder:	1	2	4	3	1

We now note that Remainder$(1 + 2 + 2^2 + 2^3)$ = Remainder$(1 + 2 + 4 + 3)$ = Remainder$(10) = 0$. The pattern repeats. Since we are adding 2 000 numbers and 2 000 is divisible by 4, the remainder is zero.

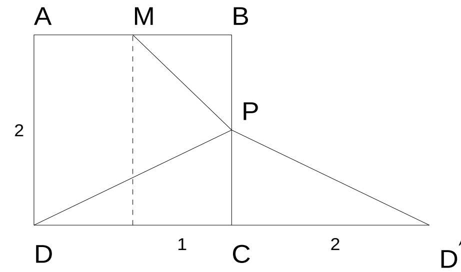
17. We are looking for the remainder of

$$\frac{x + 10y}{x + y}.$$

If $x = y = 9$ the remainder is 9.
If $x + y = 17$, the maximum remainder (upon division by 17) is 16. However, investigating the possibilities yield a maximum remainder of 13 in this particular case.
If$x + y = 16$, the maximum remainder (upon division by 16) is 15. This is attained for $x = 7, y = 9$. More specifically, $79 = 4 \times 16 + 15$.

18. For 14 tosses, 2^{14} different sequences are possible. 13 losses can be attained in 14 different ways. The probability is therefore $14/2^{14} = 7/2^{13}$.

19. This problem makes use of the reflection principle: Choose D' such that $DC = CD'$. Then $MP + PD$ equals $MP + PD'$ which is clearly the shortest if MPD' is a straight line. In this case $MP + PD = \sqrt{(1+2)^2 + 2^2} = \sqrt{13}$.



20. Denote the original population by a^2. We know that $a^2 + 100 = b^2 + 1$, or $b^2 - a^2 = 99$, or $(b+a)(b-a) = 99$.
We can factorize 99 in two different ways:

1. $b + a = 99$ and $b - a = 1$ in which case $a = 49$.
2. $b + a = 11$ and $b - a = 9$, in which case $a = 1$.

We also know that $a^2 + 200 = c^2$ which is only possible for $a = 49$.