THE SOUTH AFRICAN MATHEMATICS OLYMPIAD

Senior First Round 2019
Solutions

1. **Answer A**
$2^0 + 3^1 + 1^{2019} = 1 + 3 + 1 = 5.$

2. **Answer C**
There are 60 minutes in an hour, and there are 60 seconds in a minute, so there are $60 \times 60 = 3600$ seconds in an hour. Therefore there are $3600/40 = 90$ seconds in one-fortieth of an hour.

3. **Answer A**
$3^2 = 9$ and $2^3 = 8$, and $9 > 8$.

4. **Answer E**
$27\,000 = 3^3 \times 10^3$, so $\sqrt[3]{27\,000} = 3 \times 10 = 30$.

5. **Answer B**
In four weeks there are four Mondays. If Harry eats fish on a Monday, he will not eat fish on the following Monday (and vice versa), because the number of days in a week is seven, which is odd. Therefore Harry will eat fish on two of the four Mondays.

6. **Answer C**
Since digits can be repeated, there are ten choices for each of the three digits, so the number of codes is $10 \times 10 \times 10 = 10^3 = 1000$.

7. **Answer E**
If the edges of the cube are x m long, then the total area of the six faces is $6x^2$, which equals $216\,\mathrm{m}^2$. Therefore $x^2 = 216/6 = 36$, so $x = 6$. Finally, the volume of the cube is $x^3 = 6^3 = 216$ m^3.

8. **Answer B**
In two hours at 280 km/h, Steve will travel $280 \times 2 = 560$ km. This distance is 70 laps, so the length of each lap is $560/70 = 8$ km.

9. **Answer D**
Suppose the squares have sides of length ℓ. Since point B is four squares to the right of point A, it follows from the x-co-ordinates that $6 + 4\ell = 38$, so $\ell = 8$. Next, point C is two squares to the left of and below B, so the co-ordinates of C are $(38 - 2\ell,\ 36 - 2\ell) = (38 - 16,\ 36 - 16) = (22,\ 20)$.

10. **Answer D**
The angles in an equilateral triangle are all $60°$, so the other angles between the straight line and the triangles are $180° - 60° - 75° = 45°$ and $180° - 60° - 65° = 55°$. The third angle in the bottom triangle is therefore $180° - 45° - 55° = 80°$. Finally, $x + 60° + 80° = 180°$, so $x = 40°$.
(Alternatively, using exterior angles, we have $45° + 55° = x + 60°$, giving $x = 40°$ again.)

11. **Answer B**
$(x+y)^5 = 32 = 2^5$, so $x+y=2$. (This is the only real solution.) Next, $x-y = 2^4 = 16$, and subtracting the equations gives $2y = 2-16 = -14$, so $y=-7$.

12. **Answer C**
There are ten multiples of 28 less than 300. The LCM (least common multiple) of 28 and 12 is $84 = 3\times 28$. It follows that three of the multiples of 28, 3×28, 6×28 and 9×28, are also multiples of 12, leaving seven multiples of 28.

13. **Answer D**
The number of sweets in bag 1 is the difference between the total in all five bags and the number in bags 2, 3, 4 and 5, which is $100-43-30=27$.

14. **Answer C**
For $n\geq 5$, the units digit of $n!$ is zero, because $n!$ is divisible by both 2 and 5. Thus $1!+2!+3!+4! = 1+2+6+24 = 33$, which has units digit 3.

15. **Answer E**
If x is the fourth root of the number, then $x^2-x=12$, so $(x-4)(x+3)=0$, giving $x=4$ or $x=-3$. However, square roots and fourth roots must be positive by definition, so $x=4$ and the required number is $4^4=256$.
(Alternatively, use trial and error: 16 and 256 are the only options that have integer fourth roots.)

16. **Answer E**
The lower shaded area can be cut in two and fitted exactly into a 2×2 square, so it has area 4. The upper shaded area is the difference between a 4×4 square and a quarter-circle with radius 4, so its area is $16-\frac{1}{4}\pi 4^2 = 16-4\pi$. The total shaded area is $20-4\pi$.

17. **Answer D**
Since $EH = ET = ER$, it follows that triangle HTR lies in a semicircle with centre E. Thus $H\widehat{T}R = 90^\circ$, and by Pythagoras' therem $TR^2 = HR^2 - HT^2 = 17^2 - 15^2 = 289-225 = 64$, so $TR = 8$.

18. **Answer D**
Let each child receive one banana; we must then count the possible ways of distributing the remaining three:

Distribution	Number
One child receives all three	4
One child receives two and another child receives one	$4\times 3 = 12$
Three children receive one each	4

This gives a total of $4+12+4=20$ ways of distributing the bananas.

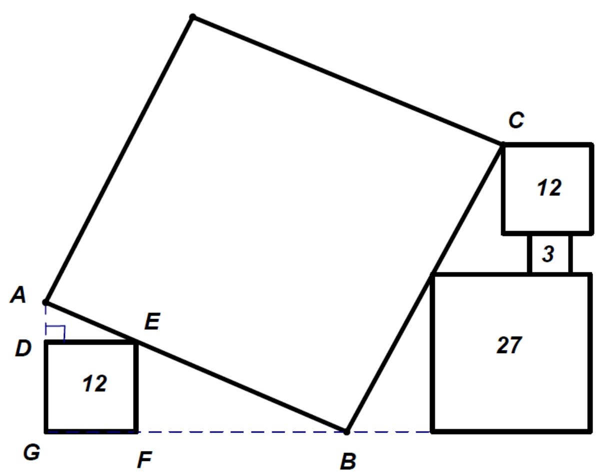
19. **Answer C**
(We need the inequality $(1+x)^n > 1+nx$ for $n>1$ and $x>0$, which simply says that compound interest is greater than simple interest.)

Since $D = (2^6)^{1663} = 2^{9978}$, $A = 2^{10029}$ and $B = (2^5)^{2020} = 2^{10100}$, it is clear that $D < A < B$. Next,

$$\frac{C}{B} = \frac{1}{32}(1 + \frac{1}{32})^{2019} > \frac{1}{32}(1 + \frac{2019}{32}) = \frac{2051}{1024} > 1, \text{ so } C > B.$$

(Clearly, E is close to D and is not a candidate for the greatest number. In fact, $D < E < A$.)

20. **Answer B**
First, a geometric solution:



Drop a perpendicular from C to the extension of line GB and label it H. Then, determining the side lengths of the three squares adjacent to BC, from their given areas, it follows that $CH = 3\sqrt{3} + \sqrt{3} + 2\sqrt{3} = 6\sqrt{3}$. Triangles AGB and BHC are congruent ($\angle$, $\angle$, S). Hence, $GB = 6\sqrt{3}$. This implies that $FB = 6\sqrt{3} - 2\sqrt{3} = 4\sqrt{3}$, since $GB = GF + FB$. From the similarity between triangles ADE and EFB, we now have $AD = \sqrt{3}$. Thus, the area of the tilted square $= AB^2 = GB^2 + AG^2 = 3^2 \times 3 + 6^2 \times 3 = 135$ square units.

Now, an alternative trigonometric solution: Note that all the right-angled triangles in the figure are similar to one another. If α denotes the smallest angle in each triangle, then the bottom right side of the tilted square is $(\sqrt{27} + \sqrt{3} + \sqrt{12}) \sec\alpha = 6\sqrt{3}\sec\alpha$. The bottom left side is $\sqrt{12}\sec\alpha + \sqrt{12}\operatorname{cosec}\alpha = 2\sqrt{3}(\sec\alpha + \operatorname{cosec}\alpha)$. Thus $2\sec\alpha = \operatorname{cosec}\alpha$, giving $\tan\alpha = \frac{1}{2}$ and $\sec^2\alpha = \frac{5}{4}$. The area of the tilted square is $(6\sqrt{3}\sec\alpha)^2 = 108 \times \frac{5}{4} = 135$.

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