SOUTH AFRICAN MATHEMATICS OLYMPIAD

Organised by the
SOUTH AFRICAN MATHEMATICS FOUNDATION

2019 THIRD ROUND
JUNIOR SECTION: GRADES 8 AND 9

25 July 2019 **Time: 4 Hours** **Number of questions: 15**

TOTAL: 100

Instructions

- Answer all the questions.
- All working details and explanations must be shown. Answers alone will not be awarded full marks.
- The neatness in your presentation of the solutions may be taken into account.
- Diagrams are not necessarily drawn to scale.
- No calculator or geometric instruments of any form may be used.
- Use your time wisely and do not spend all your time on only a few questions.
- Questions are not necessarily arranged in order of difficulty.
- Answers and solutions will be made available at: www.samf.ac.za

Do not turn the page until you are told to do so.
Draai die boekie om vir die Afrikaanse vraestel.

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Organisations involved: AMESA, SA Mathematical Society,
SA Akademie vir Wetenskap en Kuns

1. Determine all integer solutions to $1 < (x-2)^2 < 25$.

2. In the multiplication table on the right, the input factors (in the first row and first column) are all missing, and only some of the products in the table are given. For example, $m \times n = 40$.

All the numbers in the table (including those not shown) are positive integers.

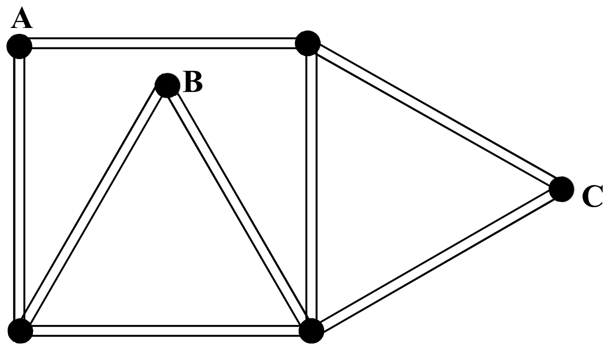
Find the value of:

A + B + C + D + E

X			n		
	A	10		20	
m	15	**B**	40		
	18		**C**	60	
		20		**D**	24
			56		**E**

3. Matchsticks are arranged as shown.

Prove that the matchstick heads A, B and C lie on a straight line.



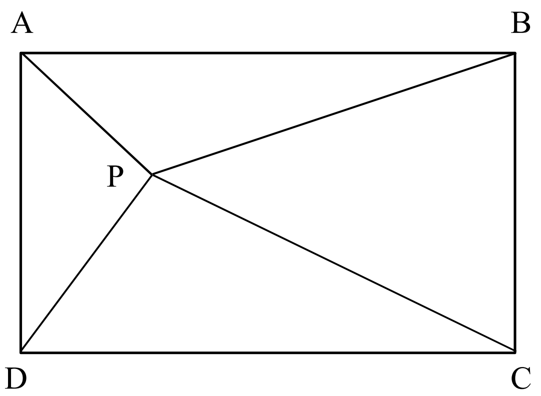
4. This alphametic puzzle shows an arithmetic problem with letters in place of the digits, you need to deduce which digit corresponds to each letter. Solve this addition sum.

$$\begin{array}{rrrrr} & T & H & I & S \\ & & & I & S \\ & & H & I & S \\ \hline C & L & A & I & M \end{array}$$

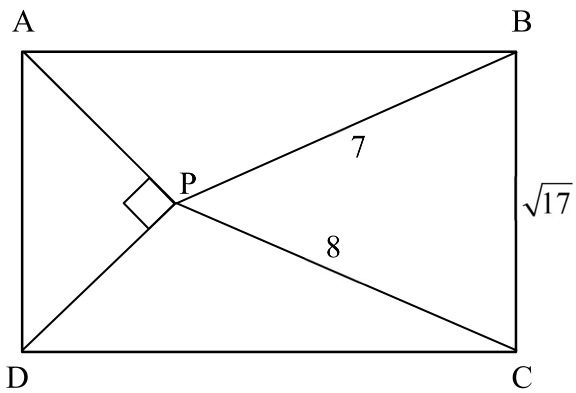
Did you know?

If ABCD is a rectangle and P is a point in its interior,

then $AP^2 + CP^2 = BP^2 + DP^2$.

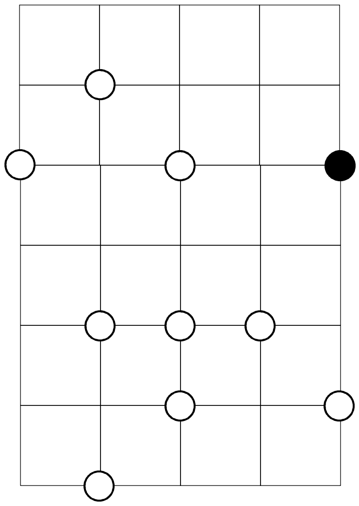


5. Given rectangle ABCD with interior point P such that $A\hat{P}D$ is a right angle, find the area of triangle APD.



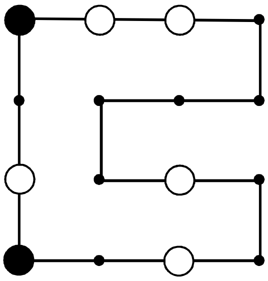
6. Draw a loop through some points on the grid such that:

- the loop doesn't cross itself;
- it goes through all the black and white circles;
- it turns in a black circle, and continues straight for at least one segment before and after the turn;
- it passes straight through a white circle.

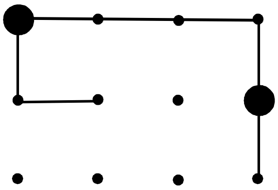


E.g.

Permitted:



Not permitted:



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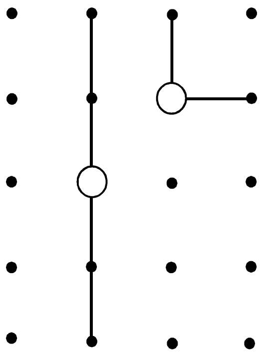


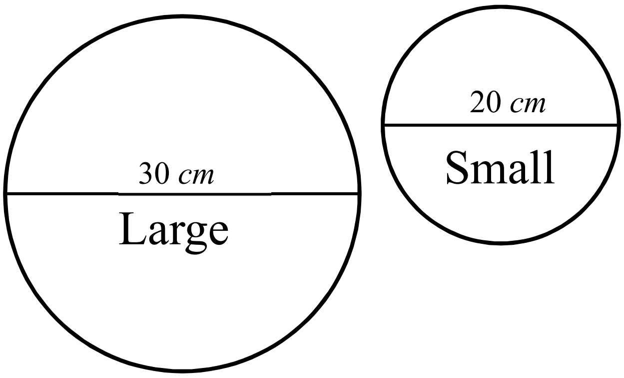
7. Let N be the smallest multiple of 84 which consists of only the digits 6 and 7.

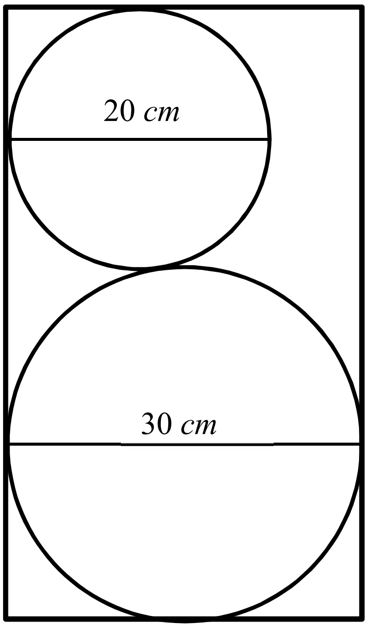
a) Show that N ends with 76.

b) Prove that the number of 7s in N is divisible by 3.

c) Find N.

8. PiThagoras Pizza Parlour offers the following Pepperoni pizzas:





a) Your waiter brings your pizzas on a tray.
The pizzas fit on the tray as shown in the figure.

(Figure not drawn to scale)

What is the length of the tray?
(Leave your answer in square root form.)

b) Debonairs Pizza across the road sells square pizzas with side length equal to the diameter of a regular round pizza.

They claim that the square pizza gives you 30% more base and toppings.

Is this 30% figure exactly correct, an exaggeration or an understatement?

Prove your answer.

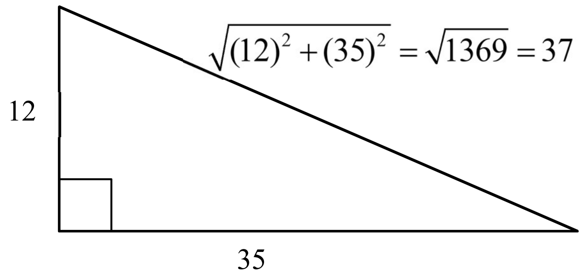
9. Sipho writes down all the 7 digit numbers that have their digits increasing from left to right. 1234568 would be one such number, but 3254780 and 1223456 are not allowed. Leading 0's are not allowed.

a) How many numbers does Sipho write down?

b) One of these numbers is chosen at random, what is the probability that its hundreds digit is a 5?

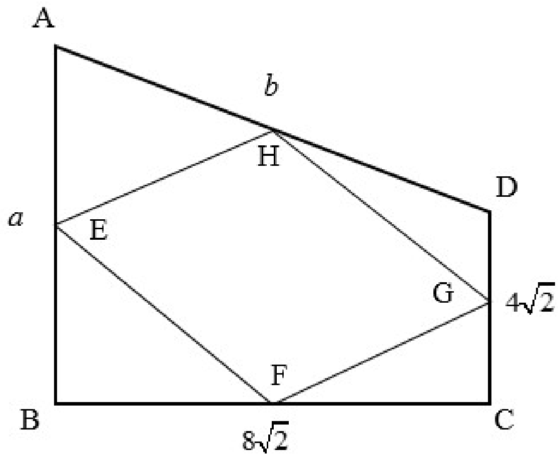
10. If you add the reciprocals of any two consecutive odd numbers, the numerator and denominator of the resulting fraction will be the shorter sides of a right-angled triangle of which the hypotenuse is an integer.

Example: $\frac{1}{5}+\frac{1}{7}=\frac{12}{35}$



Prove that this is always true.

11. The midpoints of the sides of trapezium ABCD are joined to form EFGH.



a) Prove that EFGH is a parallelogram

b) Prove that $\text{area EFGH}=\frac{1}{2}\text{area ABCD}$.

c) If $\text{area EFGH}=64$, find the length of *b*.

12. The **floor function** $\lfloor x \rfloor$ is defined as the largest integer smaller than or equal to *x*, e.g.. $\lfloor 7,3 \rfloor = 7$, $\left\lfloor \frac{10}{4} \right\rfloor = 2$. In the equations below, *x* and *m* are non-negative integers.

a) If $\left\lfloor \frac{x}{20} \right\rfloor = 5$, show that $100 \le x \le 119$.

b) Show that the equation $\left\lfloor \frac{x}{20} \right\rfloor = \left\lfloor \frac{x}{19} \right\rfloor = m$ has exactly $19 - m$ solutions if $0 \le m \le 18$ and no solutions otherwise.

c) If *x* is a non-negative integer, how many solutions does the equation $\left\lfloor \frac{x}{20} \right\rfloor = \left\lfloor \frac{x}{19} \right\rfloor$ have?

13. In a school you normally calculate your average using the arithmetic mean. In other words, you add up all your marks and divide by the total number of marks. However, there are also other ways of calculating averages. For example, for two numbers *a* and *b*:

- The arithmetic mean $= \dfrac{a+b}{2}$
- The geometric mean $= \sqrt{ab}$.
- The harmonic mean $= \dfrac{2}{\frac{1}{a} + \frac{1}{b}}$.

a) Find the arithmetic mean, geometric mean and harmonic mean of 8 and 18.

b) Show that if *a* and *b* are positive numbers, then the arithmetic mean is greater than or equal to the geometric mean.

c) Use part (b) to show that the geometric mean is greater than or equal to the harmonic mean.

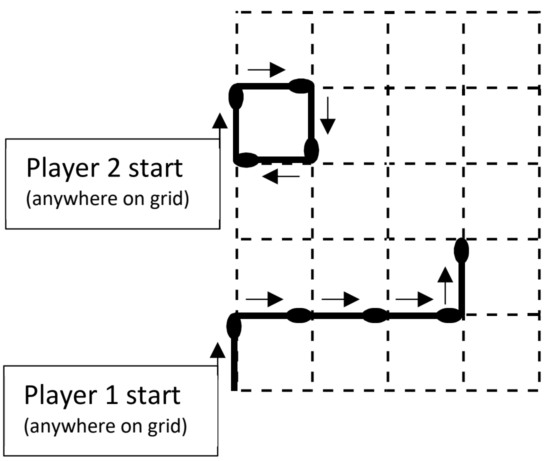
14. Two players play a game on a 20 x 19 grid. Matches are placed according to the following rules:

- At least one match must be placed.
- More matches can be placed in a player's turn as long as every subsequent match is placed in such a way that its base touches the head of the previous match placed. In other words the placed matches must form a continuous path on the grid.
- Players can start each turn anywhere on the grid.
- At the end the grid must be filled.

The player who places the last match wins.

Which player can always win? Describe the strategy used.

Example:

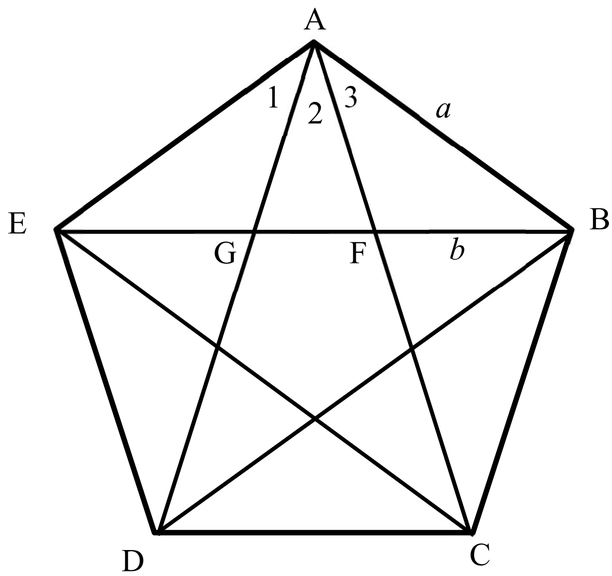


15. ABCDE is a regular pentagon.

AB = a and FB = b

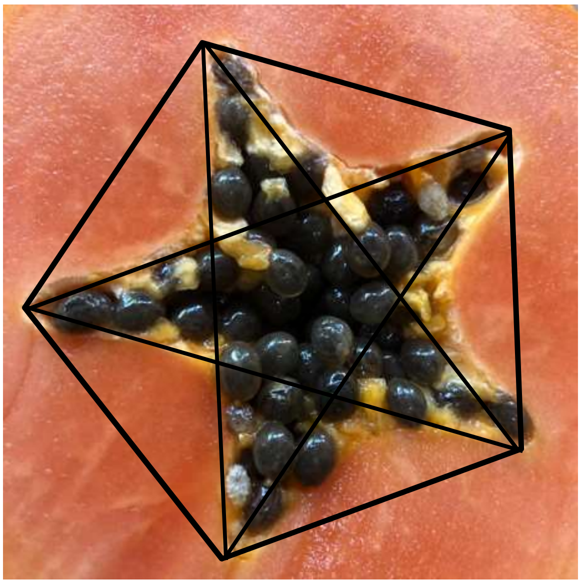
a) If $\phi = \dfrac{1+\sqrt{5}}{2}$ (the **Golden Ratio**), prove that $\phi^2 = \phi + 1$

b) Prove that $\dfrac{a}{b} = \dfrac{1+\sqrt{5}}{2}$.



Did you know?

The Golden Ratio

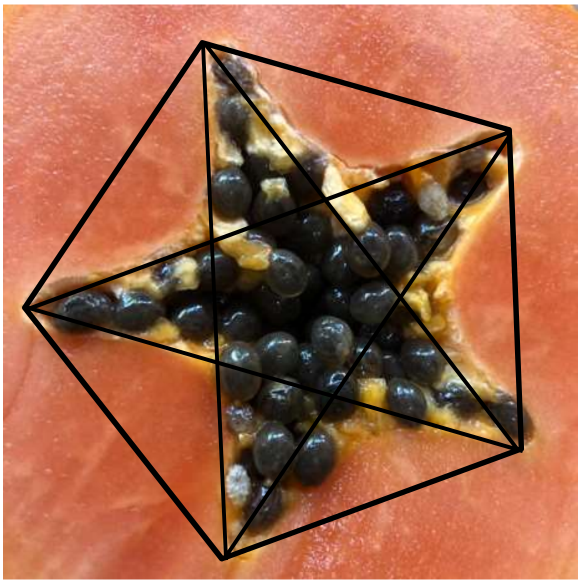


The number $\phi = \frac{1+\sqrt{5}}{2} = 1{,}61803398\ldots$ is called the golden ratio. Ancient Greek mathematicians first studied the ratio because of its frequent appearance in geometry; the division of a line into "extreme and mean ratio" (the golden section) is important in the geometry of regular pentagrams and pentagons. One finds the occurrence or the golden ratio in nature, for example in the cross-section of a pawpaw (photograph on the right). It is also known as the golden mean or the golden section (this appears the first time in the 1500's and was used until the 19th century). Martin Ohm (1792 – 1872), is believed to be the first to use the term golden ratio. Mathematician Mark Barr proposed using the first letter in the name of Greek sculptor Phidias, phi, (ϕ, lower case) to symbolize the golden ratio.

THE END

Did you know?

The Golden Ratio



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DIE EINDE

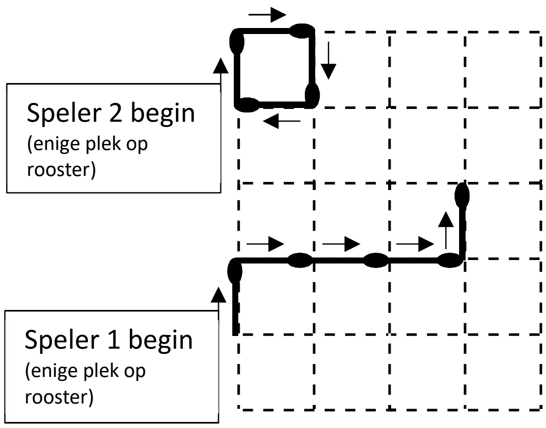
14. Twee spelers speel ’n spel op ’n 20 x 19 rooster. Vuurhoutjies word geplaas volgens die volgende reëls:

- Ten minste een vuurhoutjie moet geplaas word.
- Meer vuurhoutjies kan geplaas word gedurende ’n speler se beurt mits elke volgende vuurhoutjie geplaas word op so ’n manier dat sy basis die kop van die vorige vuurhoutjie raak. Die geplaasde vuurhoutjies moet met ander woorde ’n kontinue pad op die rooster vorm.
- Spelers kan elke beurt enige plek op die rooster begin.
- Aan die einde moet die rooster gevul wees.

Die speler wat die laaste vuurhoutjie plaas wen.

Watter speler kan altyd wen? Beskryf die strategie wat gevolg word.

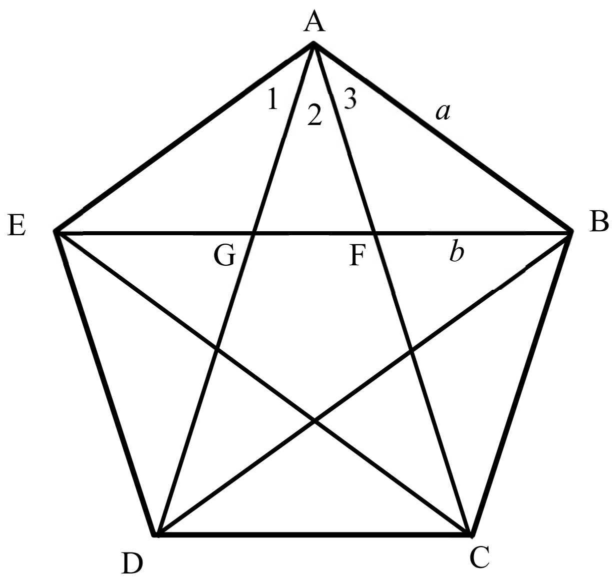
Byvoorbeeld:



15. ABCDE is ’n reëlmatige vyfhoek.
AB = a en FB = b

a) As $\phi = \dfrac{1+\sqrt{5}}{2}$ (die **Goue Verhouding**), bewys dat $\phi^2 = \phi + 1$

b) Bewys dat $\dfrac{a}{b} = \dfrac{1+\sqrt{5}}{2}$.



12. Die **vloerfunksie** $\lfloor x \rfloor$ word gedefinieer as die grootste heelgetal kleiner of gelyk aan x, bv. $\lfloor 7{,}3 \rfloor = 7$, $\left\lfloor \frac{10}{4} \right\rfloor = 2$. In die vergelykings hieronder is x en m nie-negatiewe heelgetalle.

a) As $\left\lfloor \frac{x}{20} \right\rfloor = 5$, toon dat $100 \le x \le 119$.

b) Toon dat die vergelyking $\left\lfloor \frac{x}{20} \right\rfloor = \left\lfloor \frac{x}{19} \right\rfloor = m$ presies $19 - m$ oplossings het as $0 \le m \le 18$ en andersins geen oplossing nie.

c) As x 'n nie-negatiewe heelgetal is, hoeveel oplossings het die vergelyking $\left\lfloor \frac{x}{20} \right\rfloor = \left\lfloor \frac{x}{19} \right\rfloor$?

13. In 'n skool bereken jy normaalweg jou gemiddeld deur die rekenkundige gemiddeld te gebruik. Met ander woorde jy tel al jou punte bymekaar en deel deur die totale getal punte. Daar is egter ook ander manière om gemiddeldes te bereken.
Byvoorbeeld, vir twee getalle a en b:

- Die rekenkundige gemiddeld $= \frac{a+b}{2}$
- Die meetkundige gemiddeld $= \sqrt{ab}$.
- Die harmoniese gemiddeld $= \dfrac{2}{\frac{1}{a} + \frac{1}{b}}$.

a) Bereken die rekenkundige gemiddeld, meetkundige gemiddeld en harmoniese gemiddeld van 8 en 18.

b) Toon aan dat as a en b positiewe getalle is, dan is die rekenkundige gemiddeld groter of gelyk aan die meetkundige gemiddeld.

c) Gebruik deel (b) om te toon dat die meetkundige gemiddeld groter of gelyk is aan die harmoniese gemiddeld.

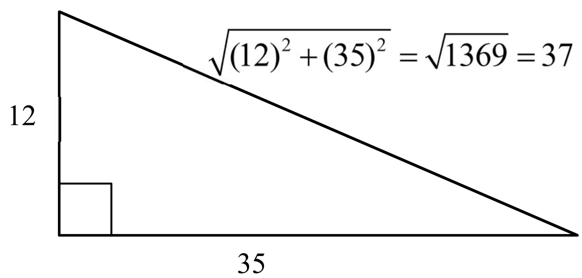
9. Sipho skryf al die 7-syfer getalle neer waarvan hul syfers stygend is van links na regs. 1234568 is een so ’n getal, maar 3254780 en 1223456 word nie toegelaat nie. 0'e aan die begin word nie toegelaat nie.

a) Hoeveel getalle skryf Sipho neer?

b) Een van hierdie getalle word willekeurig gekies. Wat is die waarskynlikheid dat die honderdesyfer ’n 5 is?

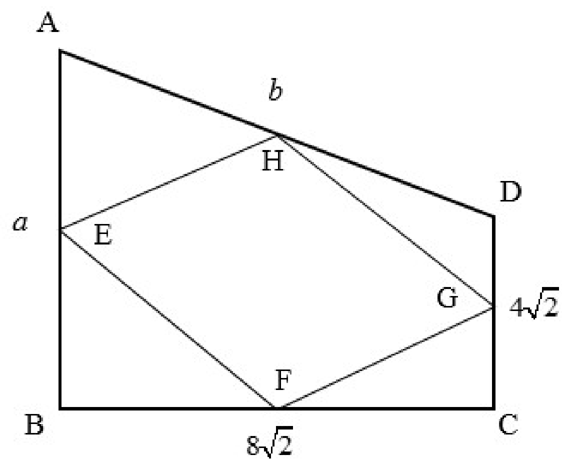
10. As jy die resiproke van enige twee opeenvolgende onewe getalle bymekaar tel, dan sal die teller en noemer van die resultaat die korter sye van ’n reghoekige driehoek wees, waarvan die skuinssy ’n heelgetal is.

Voorbeeld: $\frac{1}{5}+\frac{1}{7}=\frac{12}{35}$



Bewys dat hierdie altyd waar is.

11. Die middelpunte van die sye van trapesium ABCD word verbind om EFGH te vorm.

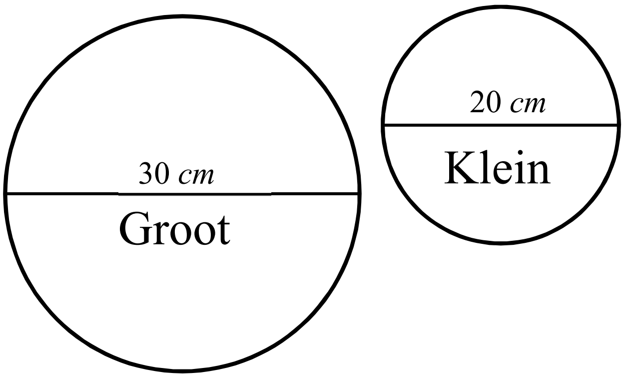


a) Bewys dat EFGH ’n parallelogram is.

b) Bewys dat area EFGH $=\frac{1}{2}$area ABCD .

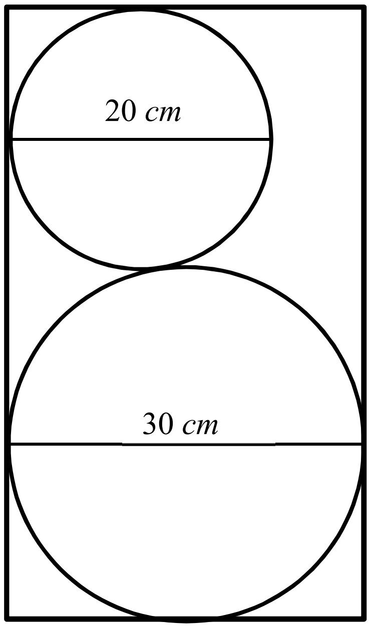
c) As area EFGH $= 64$, bepaal die lengte van b.

8. PiThagoras Pizza Parlour het die volgende Pepperoni pizzas:



a) Jou kelner bring jou pizzas op 'n skinkbord. Die pizzas pas op die skinkbord soos aangetoon in die figuur. (Figuur nie op skaal geteken nie.)

Wat is die lengte van die skinkbord?
(Los jou antwoord in vierkantswortelvorm.)

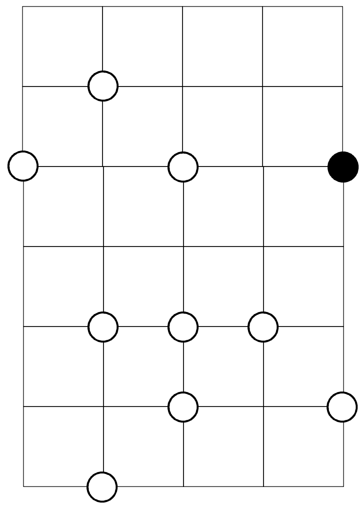


b) Debonairs Pizza aan die oorkant van die pad verkoop vierkantige pizzas met sylengte gelyk aan die deursnee van 'n normale ronde pizza. Hulle beweer dat die vierkantige pizza jou 30% meer basis en vulsels gee.

Is hierdie 30% syfer presies korrek, 'n oordrywing of 'n onderskatting?

Bewys jou antwoord.

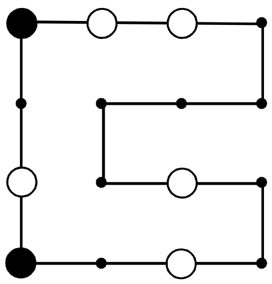
6. Trek 'n lus deur sommige punte op die rooster sodat:



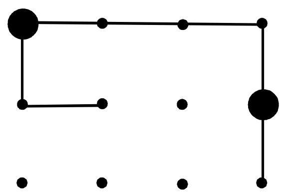
- die lus homself nie sny nie;
- dit deur al die swart en wit sirkels gaan;
- dit in 'n swart sirkel draai en reguit aan gaan vir ten minste een segment voor en na die draai;
- dit reguit deur 'n wit sirkel gaan.

Bv.

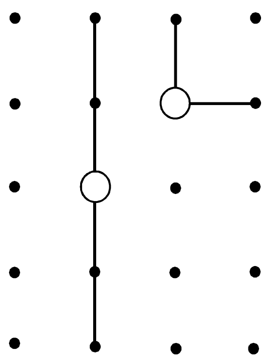
Toelaatbaar:



Nie toelaatbaar nie.



Nie toelaatbaar nie.



7. Laat N die kleinste veelvoud van 84 wees wat slegs uit die syfers 6 en 7 bestaan.

a) Toon dat N met 76 eindig.

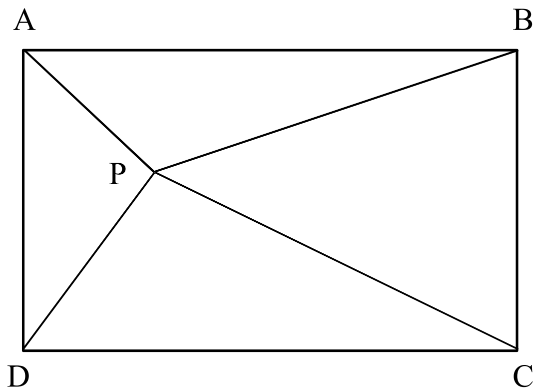
b) Bewys dat die aantal 7's in N deelbaar deur 3 is.

c) Vind N.

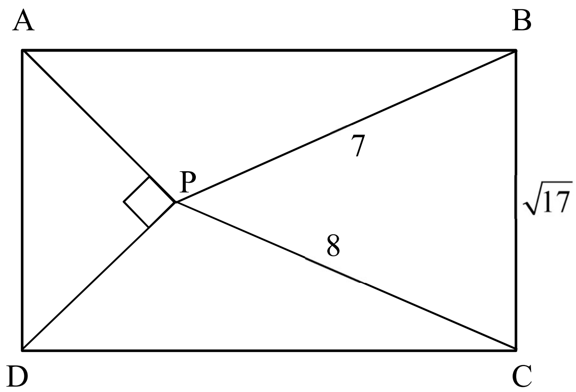
Het jy geweet?

As ABCD ’n reghoek is en P is ’n punt in die binnekant,

dan is $AP^2 + CP^2 = BP^2 + DP^2$.



5. Gegee reghoek ABCD met punt P in die binnekant sodat $A\hat{P}D$ ’n regtehoek is, bepaal die oppervlakte van driehoek APD.



1. Bepaal alle heeltallige oplossings van $1 < (x-2)^2 < 25$.

2. In die vermenigvuldigingstabel regs is die inset faktore (in die eerste ry en eerste kolom) almal uitgelaat en slegs van die produkte in die tabel word gegee. Byvoorbeeld, $m \times n = 40$.

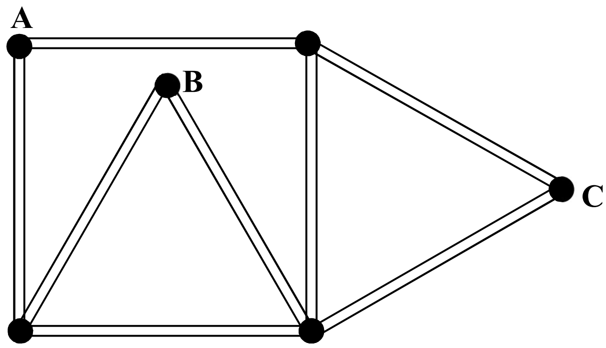
Al die getalle in die tabel (insluitend die wat nie getoon word nie) is positiewe heelgetalle.

Vind die waarde van:

A + B + C + D + E

X			n		
	A	10		20	
m	15	**B**	40		
	18		**C**	60	
		20		**D**	24
			56		**E**

3. Vuurhoutjies word gerangskik soos getoon. Bewys dat die vuurhoutjiekoppe A, B en C in 'n reguit lyn lê.



4. Hierdie alfametiese raaisel toon 'n rekenkunde probleem met letters in die plek van syfers en jy moet aflei watter syfer ooreenstem met watter letter. Los hierdie optelsom op.

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  T H I S
      I S
    H I S
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C L A I M
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SUID-AFRIKAANSE WISKUNDE-OLIMPIADE

Georganiseer deur die
SOUTH AFRICAN MATHEMATICS FOUNDATION

2019 DERDE RONDE
JUNIOR AFDELING: GRAAD 8 EN 9

25 Julie 2019 | **Tyd: 4 Ure** | **Aantal vrae: 15**

TOTAAL: 100

Instruksies

- Beantwoord al die vrae.
- Alle berekeninge en motiverings moet getoon word. Antwoorde sonder motivering sal nie volpunte verdien nie.
- Die netheid van jou oplossings mag in ag geneem word.
- Diagramme is nie noodwendig volgens skaal geteken nie.
- Geen sakrekenaar, in welke vorm ook al, of enige meetkunde instrumente, mag gebruik word nie.
- Gebruik jou tyd oordeelkundig en moenie al jou tyd op slegs 'n paar vrae spandeer nie.
- Vrae is nie noodwendig in volgorde van maklik na moeilik gerangskik nie.
- Die antwoorde en oplossings sal beskikbaar wees by: www.samf.ac.za

Moenie omblaai voordat daar vir jou gesê word om dit te doen nie.
Turn the booklet over for the English paper.

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SA Akademie vir Wetenskap en Kuns