The South African Mathematical Olympiad
Third Round 2017
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

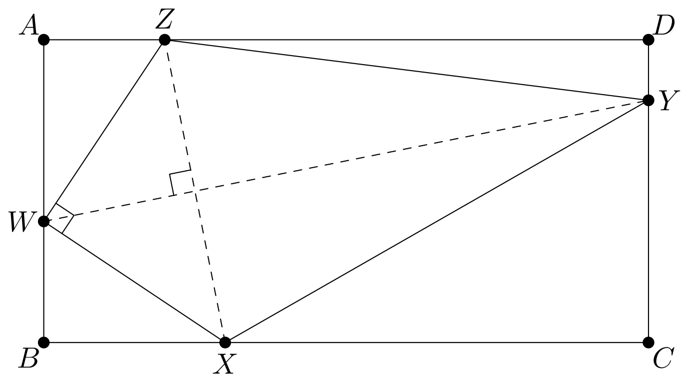
1. Together, the two positive integers a and b have 9 digits and contain each of the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once. For which possible values of a and b is the fraction a/b closest to 1?

   Solution: If $a > b$, then a has at least five digits, so $a \geq 12345$, and b has at most four digits, so $b \leq 9876$. In this case, we have
   $$\frac{a}{b} \geq \frac{12345}{9876} > 1,$$
   so the value of a/b that is closest to 1 is
   $$\frac{12345}{9876} = 1 + \frac{2469}{9876}$$
   in this case.

   On the other hand, if $a < b$ ($a = b$ is impossible, since a and b cannot have the same number of digits), then a has at most four digits and b at least five, so $a \leq 9876$ and $b \geq 12345$, which means that
   $$\frac{a}{b} \leq \frac{9876}{12345} < 1.$$
   Hence in this case the value that is closest to 1 is
   $$\frac{9876}{12345} = 1 - \frac{2469}{12345}.$$
   Since the denominator 12345 is greater than the denominator 9876, we see that the value of $9876/12345$ is closer to 1 than that of $12345/9876$ (in fact, we have $9876/12345 = 4/5$ and $12345/9876 = 5/4$, so the distances are $1/5$ and $1/4$ respectively). Thus the values of a and b for which a/b is closest to 1 are $a = 9876$ and $b = 12345$.

2. Let $ABCD$ be a rectangle with side lengths $AB = CD = 5$ and $BC = AD = 10$. W, X, Y, Z are points on AB, BC, CD and DA respectively chosen in such a way that $WXYZ$ is a kite, where $\angle ZWX$ is a right angle. Given that $WX = WZ = \sqrt{13}$ and $XY = ZY$, determine the length of XY.

Solution:



Note that $\angle AZW = 90^\circ - \angle AWZ = \angle BWX$ and $\angle AWZ = 90^\circ - \angle BWX = \angle BXW$. Moreover, $WX = WZ$, so the two triangles AWZ and BXW are congruent. Let $BW = AZ = x$, so that $BX = AW = 5 - x$. Pythagoras' theorem gives us

$$x^2 + (5-x)^2 = 13,$$

which simplifies to $x^2 - 5x + 6 = 0$. The two solutions are $x = 2$ and $x = 3$, and by symmetry we can assume that $x = 2$. So $BX = 3$, $XC = 7$, $AZ = 2$ and $ZD = 8$. Now let $CY = y$, so that $YD = 5 - y$. Applying Pythagoras' theorem again (and making use of the fact that $XY = YZ$), we get

$$7^2 + y^2 = 8^2 + (5-y)^2.$$

This simplifies to $10y = 40$, so $y = 4$. Now we finally find that $XY = \sqrt{7^2 + 4^2} = \sqrt{65}$.

3. A representation of $\dfrac{17}{20}$ as a sum of reciprocals

$$\frac{17}{20} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}$$

is called a *calm representation* with k terms if the a_i are distinct positive integers and at most one of them is not a power of two.

(a) Find the smallest value of k for which $\dfrac{17}{20}$ has a calm representation with k terms.

(b) Prove that there are infinitely many calm representations of $\dfrac{17}{20}$.

Solution: Note first that there is no calm representation with 2 terms: if either a_1 or a_2 is 1 or $a_1 = a_2 = \frac{1}{2}$, then the sum is greater than $\frac{17}{20}$. Otherwise, the sum is at most $\frac{1}{2} + \frac{1}{3} = \frac{5}{6} < \frac{17}{20}$, thus too small.

On the other hand, there is a representation with 3 terms, namely

$$\frac{17}{20} = \frac{1}{2} + \frac{1}{4} + \frac{1}{10},$$

showing that the smallest possible value of k is 3.

Now we show that there are infinitely many calm representations: take $a_1 = 5 \cdot 2^{4n+1}$ and consider the difference

$$\frac{17}{20} - \frac{1}{5 \cdot 2^{4n+1}} = \frac{17 \cdot 2^{4n-1} - 1}{5 \cdot 2^{4n+1}}.$$

Since $2^4 = 16 \equiv 1 \mod 5$, the numerator is $17 \cdot 2^{4n-1} - 1 \equiv 17 \cdot 8 - 1 = 135 \equiv 0 \mod 5$. Thus the factor 5 cancels, and we have

$$\frac{17}{20} - \frac{1}{5 \cdot 2^{4n+1}} = \frac{A}{2^{4n+1}}$$

for some positive integer $A < 2^{4n+1}$. Since A has a binary representation as $A = 2^{b_1} + 2^{b_2} + \cdots + 2^{b_r}$ with distinct nonnegative integers $b_1, b_2, \ldots, b_r$, we get

$$\frac{17}{20} = \frac{1}{5 \cdot 2^{4n+1}} + \frac{1}{2^{4n+1-b_1}} + \frac{1}{2^{4n+1-b_2}} + \cdots + \frac{1}{2^{4n+1-b_r}},$$

which is a calm representation for every n. This completes the proof.

4. Andile and Zandre play a game on a 2017×2017 board. At the beginning, Andile declares some of the squares *forbidden*, meaning that nothing may be placed on such a square. After that, they take turns to place coins on the board, with Zandre placing the first coin. It is not allowed to place a coin on a forbidden square or in the same row or column where another coin has already been placed. The player who places the last coin wins the game.

   What is the least number of squares Andile needs to declare as forbidden at the beginning to ensure a win? (Assume that both players use an optimal strategy.)

   Solution: The minimum number is 2017. For example, Andile can achieve a win by declaring all squares of the last row forbidden, so that 2016 rows remain. After that, there will be exactly 2016 moves possible, no matter how the two play, since placing a coin always eliminates exactly one row and one column from further use. This means that Andile gets the last move.

   On the other hand, we prove that 2016 or fewer forbidden squares are not sufficient, no matter how they are placed. Generally, we show by induction that Zandre has a winning strategy on a $(2n-1) \times (2n-1)$ board if Zandre gets to place a coin first and no more than $2n-2$ squares have been forbidden. This is trivial for $n = 1$: Zandre can simply place a coin on the only square.

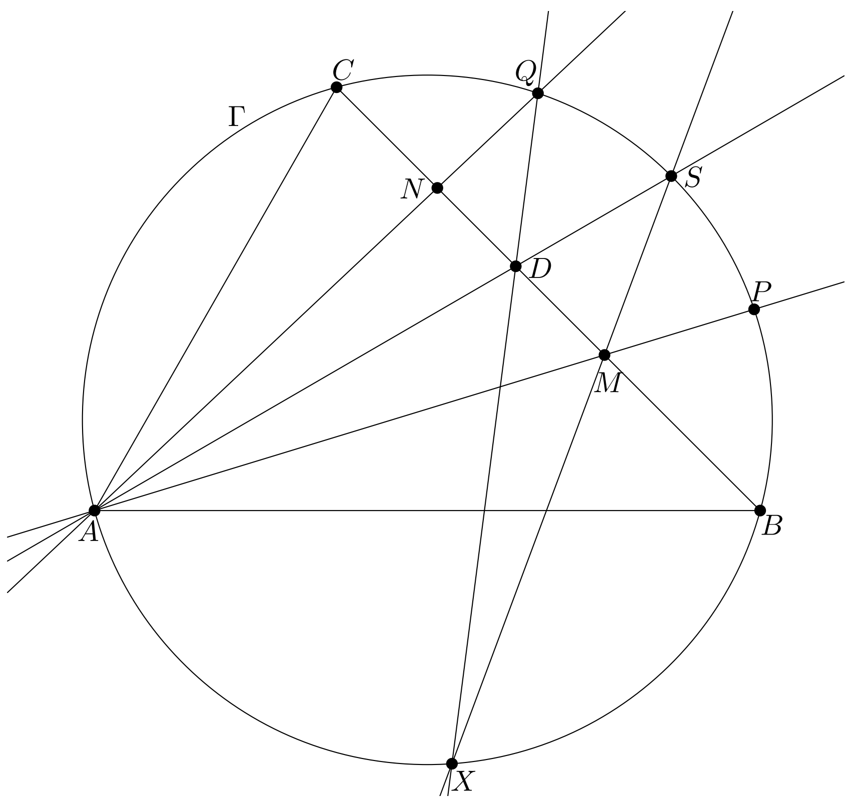
   For the induction step, consider a $(2n+1) \times (2n+1)$ board with at most $2n$ forbidden squares. If there are two forbidden squares in the same row or column somewhere, then Zandre places a coin in this row/column. This is possible since there are fewer forbidden squares than squares in a row or column. If there are at least two forbidden squares but no two of them in the same row or column, Zandre chooses any two of them, then places a coin on the intersection of the row of the first forbidden square and the column of the second forbidden square. This is possible because of the assumption that there are no two forbidden squares in the same row or column. Finally, if there is only one forbidden square, Zandre places a coin anywhere in the same row or column, and if there are no forbidden squares, Zandre just places it on an arbitrary square.

   After Andile's move, the rows and columns where Andile and Zandre placed their coins can be removed, since no further coins can be placed there anymore. This leaves us with a

$(2n-1) \times (2n-1)$ board, and by the choice of Zandre's move there are at most $2n-2$ forbidden squares on it. So by the induction hypothesis, Zandre has a winning strategy for the remaining position, which completes the proof.

5. Let ABC be a triangle with circumcircle Γ. Let D be a point on segment BC such that $\angle BAD = \angle DAC$, and let M and N be points on segments BD and CD, respectively, such that $\angle MAD = \angle DAN$. Let S, P and Q (all different from A) be the intersections of the rays AD, AM and AN with Γ, respectively. Show that the intersection of SM and QD lies on Γ.

   Solution:



   Let X be the intersection of SM with the circumcircle. Note that $\angle MXA = \angle SXA = \angle SBA = \angle SBC + \angle CBA$. Since AS bisects the angle $\angle BAC$, we have $\angle SBC = \angle SAC = \angle BAS$. It follows that

$$\angle MXA = \angle BAS + \angle CBA = \angle BAD + \angle DBA = 180^\circ - \angle ADB = 180^\circ - \angle ADM,$$

   so $AXMD$ is a cyclic quadrilateral. Consequently, $\angle SXD = \angle MXD = \angle MAD = \angle DAN = \angle SAQ = \angle SXQ$, which means that X, D and Q lie on a common straight line. Hence X is also the intersection of SM and QD, which completes the proof.

6. Determine all pairs (P, d) of a polynomial P with integer coefficients and an integer d such that the equation $P(x) - P(y) = d$, where x and y are integers and $x \neq y$, has infinitely many solutions.

Solution: Note first that $x-y$ divides $P(x)-P(y)$. So if $d \neq 0$, there are only finitely many possibilities for $x-y$. For one of these possible values, let us denote it by a, there must be infinitely many pairs (x, y) such that $x-y=a$ and $P(x)-P(y)=d$. Note that

$$P(x)-P(x-a)-d$$

is still a polynomial. The only way it can have infinitely many zeros is that it is identically zero. But then $P(x)=dx/a+b$ for some number b.

This gives us the first set of solutions, consisting of an arbitrary integer d and a polynomial $P(x)=dx/a+b$, where a is a (positive or negative) divisor of d and b an arbitrary integer. This includes the solution where $d=0$ and the polynomial $P(x)$ is constant.

For the rest of this solution, we can assume that $d=0$. If the degree of the polynomial is odd, then there exist integers A and B such that $P(x)$ is either increasing for $x \geq A$ and for $x \leq B$, or decreasing for $x \geq A$ and $x \leq B$. If $P(x)$ is increasing for $x \geq A$, then $P(A), P(A+1), P(A+2), \ldots$ are a strictly increasing sequence of integers, thus distinct. Moreover, from some point on these numbers are all greater than the maximum of $P(x)$ for $x \leq A$. Likewise, $P(B), P(B-1), P(B-2), \ldots$ are a strictly decreasing sequence of integers, and from some point on they are all less than the minimum of $P(x)$ for $x \geq B$. This means that there cannot be infinitely many pairs x, y with $x \neq y$ such that $P(x)-P(y)=0$. The same argument applies if $P(x)$ is decreasing for $x \geq A$ and $x \leq B$.

Thus $P(x)$ must be a poynomial of even degree. Now there exist integers A and B such that $P(x)$ is increasing for $x \geq A$ and decreasing for $x \leq B$, or vice versa. Thus there can only be infinitely many pairs x, y with $x \neq y$ such that $P(x)-P(y)=0$ if one of the two (x, say) is $\geq A$ while the other is $\leq B$. We show that $x+y$ has to be constant for these solutions. Let the first terms of the polynomial be as follows:

$$P(x)=ax^n+bx^{n-1}+\cdots$$

Assume that a is positive, for otherwise one can replace $P(x)$ by $-P(x)$. If $an(x+y) \geq -2b+1$, then we have

$$P(x)-P(y) \geq P(x)-P\Big(\frac{-2b+1}{an}-x\Big)=x^{n-1}+\cdots,$$

where the dots stand for terms involving lower powers of x. For large enough x, this will be strictly positive, so $P(x)-P(y) \neq 0$. Likewise, if $an(x+y) \leq -2b-1$, then we have

$$P(x)-P(y) \leq P(x)-P\Big(\frac{-2b-1}{an}-x\Big)=-x^{n-1}+\cdots,$$

which is negative for large enough x. Hence $P(x)-P(y) \neq 0$ in this case as well. Thus there must be infinitely many solutions of $P(x)-P(y)=0$ with $an(x+y)=-2b$. In this case,

$$P(x)-P\Big(\frac{-2b}{an}-x\Big)$$

is a polynomial with infinitely many zeros, hence it is constant. Thus

$$P(x)=P\Big(\frac{-2b}{an}-x\Big)$$

for all x, which gives

$$Q(x) = P\Big(\frac{-b}{an} + x\Big) = P\Big(\frac{-b}{an} - x\Big) = Q(-x)$$

for all x. It follows that the polynomial $Q(x)$ can only contain even powers of x, i.e. $Q(x) = R(x^2)$. This gives us the second set of solutions, consisting of a polynomial of the form $P(x) = R((x-c)^2)$, where $2c$ is an integer, and $d = 0$.