Genus Zero Surface Conformal Mapping*

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1 Basic Idea

It is well known that any genus zero surface can be mapped conformally onto the sphere and any local portion thereof onto a disk. This mapping, a conformal equivalence, is one-to-one, onto, and angle-preserving. Moreover, the elements of the first fundamental form remain unchanged, except for a scaling factor (the so-called *Conformal Factor*). For this reason, conformal mappings are often described as being similarities in the small. Since the cortical surface of the brain is a genus zero surface, conformal mapping offers a convenient method to retain local geometric information, when mapping data between surfaces. In this paper, we propose a new genus zero surface conformal mapping algorithm and demonstrate its use in computing conformal mappings between brain surfaces. Our algorithm depends only on the surface geometry and is invariant to changes in image resolution and the specifics of the data triangulation. Our experimental results show that our algorithm has advantageous properties for cortical surface matching.

Suppose K is a simplicial complex, and $f : |K| \rightarrow R^3$, which embeds $|K|$ in R^3; then (K, f) is called a mesh. Given two genus zero meshes M_1, M_2, there are many conformal mappings between them. Our algorithm for computing conformal mappings is based on the fact that for genus zero surfaces S_1, S_2, $f : S_1 \rightarrow S_2$ is conformal if and only if f is harmonic. All conformal mappings between S_1, S_2 form a group, the so-called Möbius group. Figure 1 show some examples of Möbius transformations. We can conformally map the surface of the head of Michelangelo's David to a sphere. When we draw the longitude and latitude lines on the sphere, we can induce corresponding circles on the original surface (a) and (b). We apply a Möbius transformation to the sphere and make the two eyes become north and south poles. When we draw the longitude and latitude lines again (c), we get an interesting result shown in (d). Note all the right angles between the lines are well preserved in (b) and (d). This example demonstrates that all the conformal mapping results form a Möbius group.

Our method is as follows: we first find a homeomorphism h between M_1 and M_2, then deform h such that h minimizes the harmonic energy. To ensure the convergence of the algorithm, constraints are added; this also ensures that there is a unique conformal map.

*The original paper is Gu et al. "Genus Zero Surface Conformal Mapping and Its Application to Brain Surface Mapping", IEEE Transactions on Medical Imaging, 23(8), 2004, pp. 949-958

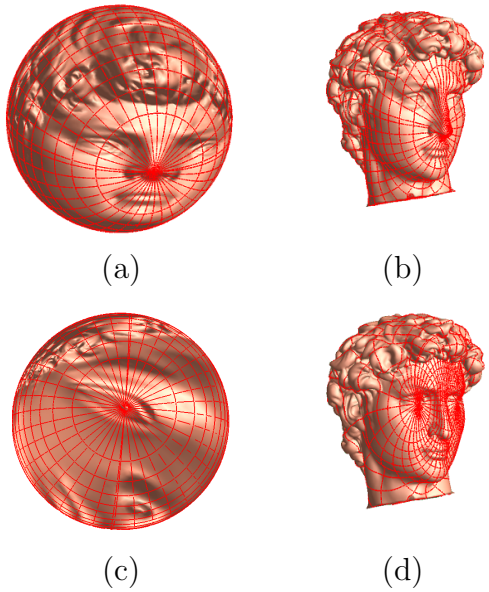

Figure 1: Möbius transformation example. We conformally map the surface of the head of Michelangelo's David to a sphere. In (a), we select the nose tip as the north pole and draw longitude and latitude lines on the sphere. (b) shows the results on the original David head model. We apply a Möbius transformation on the sphere in (a) and make the two eyes become the north and south poles. When drawing the longitude and latitude lines on the sphere (c), we get an interesting configuration for the lines on the original surface (d).

2 Piecewise Linear Function Space, Inner Product and Laplacian

For the diffeomorphisms between genus zero surfaces, if the map minimizes the harmonic energy, then it is conformal. Based on this fact, the algorithm is designed as a steepest descent method.

This section formulates the mathematical concepts in a rigorous way. The major concepts, the harmonic energy of a map and its derivative, are defined. Because all the calculation is carried out on surfaces, we use the absolute derivative. Furthermore, for the purpose of implementation, we introduce the definitions in discrete form.

We use K to represent the simplicial complex, u, v to denote the vertices, and $\{u, v\}$ to denote the edge spanned by u, v. We use f, g to represent the piecewise linear functions defined on K, use $\vec{f}$ to represent vector value functions. We use Δ_{PL} to represent the discrete Laplacian operator.

Definition 1 *All piecewise linear functions defined on K form a linear space, denoted by $C^{PL}(K)$.*

In practice, we use $C^{PL}(K)$ to approximate all functions defined on K. So the final result is an approximation to the conformal mapping. The higher the resolution of the mesh is, the more accurate the approximated conformal mapping is.

Definition 2 *Suppose a set of string constants $k_{u,v}$ are assigned for each edge $\{u, v\}$, the inner product on C^{PL} is defined as the quadratic form*

$$< f, g >= \frac{1}{2} \sum_{\{u,v\} \in K} k_{u,v}(f(u) - f(v))(g(u) - g(v)) \tag{1}$$

The energy is defined as the norm on C^{PL}.

Definition 3 *Suppose $f \in C^{PL}$, the string energy is defined as:*

$$E(f) =< f, f >= \sum_{\{u,v\} \in K} k_{u,v}||f(u) - f(v)||^2 \tag{2}$$

By changing the string constants $k_{u,v}$ in the energy formula, we can define different string energies.

Definition 4 *If string constants $k_{u,v} \equiv 1$, the string energy is known as the Tuette energy.*

Definition 5 *Suppose edge $\{u, v\}$ has two adjacent faces T_α, T_β, with $T_\alpha = \{v_1, v_2, v_3\}$, as shown in Figure 2, define the parameters*

$$a^{\alpha}_{v_1,v_2} = \frac{1}{2} \frac{(v_1 - v_3) \cdot (v_2 - v_3)}{|(v_1 - v_3) \times (v_2 - v_3)|} \tag{3}$$

$$a^{\alpha}_{v_2,v_3} = \frac{1}{2} \frac{(v_2 - v_1) \cdot (v_3 - v_1)}{|(v_2 - v_1) \times (v_3 - v_1)|} \tag{4}$$

$$a^{\alpha}_{v_3,v_1} = \frac{1}{2} \frac{(v_3 - v_2) \cdot (v_1 - v_2)}{|(v_3 - v_2) \times (v_1 - v_2)|} \tag{5}$$

$$\tag{6}$$

T_β is defined similarly. If $k_{u,v} = a^\alpha_{u,v} + a^\beta_{u,v}$, the string energy obtained is called the harmonic energy.

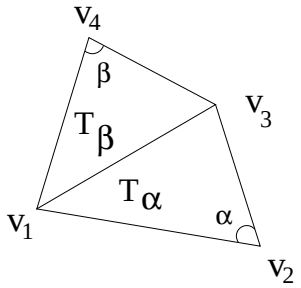

Figure 2: Discrete Laplace-Beltrami operator. Edge $\{v_1, v_3\}$ has two corners against it α, β. The edge weight is defined as the summation of the cotangents of these corner angles.

The string energy is always a quadratic form. By carefully choosing the string coefficients, we make sure the quadratic form is positive definite. This will guarantee the convergence of the steepest descent method.

Definition 6 *The piecewise Laplacian is the linear operator $\Delta_{PL} : C^{PL} \to C^{PL}$ on the space of piecewise linear functions on K, defined by the formula*

$$\Delta_{PL}(f)|_u = \sum_{\{u,v\} \in K} k_{u,v}(f(v) - f(u)) \tag{7}$$

If f minimizes the string energy, then f satisfies the condition $\Delta_{PL}(f) = 0$. Suppose M_1, M_2 are two meshes and the map $\vec{f} : M_1 \to M_2$ is a map between them, $\vec{f}$ can be treated as a map from M_1 to R^3 also.

Definition 7 *For a map $\vec{f} : M_1 \to R^3$, $\vec{f} = (f_0, f_1, f_2)$, $f_i \in C^{PL}, i = 0, 1, 2$, we define the energy as the norm of $\vec{f}$:*

$$E(\vec{f}) = \sum_{i=0}^{2} E(f_i) \tag{8}$$

The Laplacian is defined in a similar way,

Definition 8 *For a map $\vec{f} : M_1 \to R^3$, the piecewise Laplacian of $\vec{f}$ is*

$$\Delta_{PL}\vec{f} = (\Delta_{PL}f_0, \Delta_{PL}f_1, \Delta_{PL}f_2) \tag{9}$$

A map $\vec{f} : M_1 \to M_2$ is harmonic, if and only if $\Delta_{PL}\vec{f}$ only has a normal component, and its tangential component is zero.

$$\Delta_{PL}(\vec{f}) \;\; = \;\; (\Delta_{PL}\vec{f})^\perp \tag{10}$$

A decomposition of Laplacian Δf is shown in Figure 3.

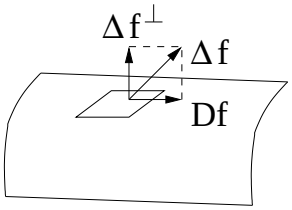

Figure 3: Projected Laplacian. The Laplacian Δf is a vector in R^3, which can be decomposed into a normal component, $(\Delta_{PL}\vec{f})^{\perp}$, and tangential component, $D\vec{f}$. The normal component is collinear with the normal to the target surface, and the tangential component is in the tangent space of the target surface.

3 Steepest Descent Algorithm

Suppose we would like to compute a mapping $\vec{f} : M_1 \to M_2$ such that $\vec{f}$ minimizes a string energy $E(\vec{f})$. This can be solved easily by the steepest descent algorithm:

$$\frac{d\vec{f}(t)}{dt} = -\Delta\vec{f}(t) \tag{11}$$

$\vec{f}(M_1)$ is constrained to be on M_2, so $-\Delta\vec{f}$ is a tangent vector field of M_2.
Specifically, suppose $\vec{f} : M_1 \to M_2$, and denote the image of each vertex $v \in K_1$ as $\vec{f}(v)$. The normal on M_2 at $\vec{f}(v)$ is $\vec{n}(\vec{f}(v))$. Define the normal component as

Definition 9 *The normal component*

$$(\Delta\vec{f}(v))^{\perp} =< \Delta\vec{f}(v), \vec{n}(\vec{f}(v)) > \vec{n}(\vec{f}(v)), \tag{12}$$

where $<,>$ is the inner product in R^3.

Definition 10 *The absolute derivative is defined as*

$$D\vec{f}(v) = \Delta\vec{f}(v) - (\Delta\vec{f}(v))^{\perp} \tag{13}$$

Then equation 11 is $\delta\vec{f} = -D\vec{f} \times \delta t$.

4 Conformal Spherical Mapping

Suppose M_2 is S^2, then a conformal mapping $\vec{f} : M_1 \to S^2$ can be constructed by using the steepest descent method. The major difficulty is that the solution is not unique but forms a Möbius group.

Definition 11 *Mapping $f : \mathcal{C} \to \mathcal{C}$ is a Möbius transformation if and only if*

$$f(z) = \frac{az+b}{cz+d},\ a, b, c, d \in \mathcal{C}, ad - bc = 1.0, \tag{14}$$

where $\mathcal{C}$ is the complex plane.

All Möbius transformations form the Möbius transformation group. In order to determine a unique solution we can add different constraints. In practice we use the following two constraints: the zero mass-center constraint and a landmark constraint.

Definition 12 *Mapping $\vec{f} : M_1 \to M_2$ satisfies the zero mass-center condition if and only if*

$$\int_{M_1} \vec{f} d\sigma_{M_1} = 0, \tag{15}$$

where $d\sigma_{M_1}$ is the area element on M_1.

All conformal maps from M_1 to S^2 satisfying the zero mass-center constraint are unique up to the Euclidean rotation group (which is 3 dimensional). We use the Gauss map as the initial condition.

Definition 13 *A Gauss map $N : M_1 \to S^2$ is defined as*

$$N(v) = \vec{n}(v), v \in M_1, \tag{16}$$

where $\vec{n}(v)$ is the normal at v.

Algorithm 1 *Spherical Tuette Mapping*

Input (mesh M,step length δt, energy difference threshold δE), output($\vec{t} : M \to S^2$) where $\vec{t}$ minimizes the Tuette energy.

1. *Compute Gauss map $N : M \to S^2$. Let $\vec{t} = N$, compute Tuette energy E_0.*
2. *For each vertex $v \in M$, compute Absolute derivative $D\vec{t}$.*
3. *Update $\vec{t}(v)$ by $\delta\vec{t}(v) = -D\vec{t}(v)\delta t$.*
4. *Compute Tuette energy E.*
5. *If $E_{n+1} - E_n < \delta E$, return $\vec{t}$. Otherwise, assign E to E_0 and repeat steps 2 through to 5.*

Because the Tuette energy has a unique minimum, the algorithm converges rapidly and is stable. We use it as the initial condition for the conformal mapping.

Algorithm 2 *Spherical Conformal Mapping*

Input (mesh M,step length δt, energy difference threshold δE), output($\vec{h} : M \to S^2$). Here $\vec{h}$ minimizes the harmonic energy and satisfies the zero mass-center constraint.

1. *Compute Tuette embedding $\vec{t}$. Let $\vec{h} = \vec{t}$, compute Tuette energy E_0.*
2. *For each vertex $v \in M$, compute the absolute derivative $D\vec{h}$.*
3. *Update $\vec{h}(v)$ by $\delta\vec{h}(v) = -D\vec{h}(v)\delta t$.*

4. *Compute Möbius transformation* $\vec{\varphi}_0 : S^2 \to S^2$, *such that*

$$\Gamma(\vec{\varphi}) = \int_{M_1} \vec{\varphi} \circ \vec{h} d\sigma_{M_1}, \vec{\varphi} \in Mobius(S^2) \quad (17)$$

$$\vec{\varphi}_0 = \min_{\vec{\varphi}} ||\Gamma(\vec{\varphi})||^2 \quad (18)$$

where $d\sigma_{M_1}$ *is the area element on* M_1. $\Gamma(\vec{\varphi})$ *is the mass center, and* $\vec{\varphi}$ *minimizes the norm in the mass center condition.* $Mobius(S^2)$ *is the conformal automorphism group of* S^2, *and it can be analytically represented as* $\tau^{-1} \circ \theta \circ \tau$, *where* $\tau : S^2 \to C$ *is the stereographic projection,*

$$\tau(p) = (\frac{x}{1-z}, \frac{y}{1-z}), p = (x, y, z) \in S^2$$

and $\theta : C \to C$ *is a Möbius transformation as defined in Definition 11.*

5. *compute the harmonic energy* E.

6. *If* $E_{n+1} - E_n < \delta E$, *return* $\vec{t}$. *Otherwise, assign* E *to* E_0 *and repeat step* 2 *through to step* 6.

Step 4 is non-linear and expensive to compute. In practice we use the following procedure to replace it:

4'-1 Compute the mass center $\vec{c} = \int_{S^2} \vec{h} d\sigma_{M_1}$;

4'-2 For all $v \in M$, $\vec{h}(v) = \vec{h}(v) - \vec{c}$;

4'-3 For all $v \in M$, $\vec{h}(v) = \frac{\vec{h}(v)}{||\vec{h}(v)||}$.

This approximation method is good enough for our purpose. The resulting angle distortion is proportional to the square of the distance between the mass center and the origin. When the deviation is small, this provides a very accurate approximation to a Möbius transformation. By choosing the step length carefully, the energy can be decreased monotonically at each iteration.