Exercise 2

a)

$x = az + b$ for (z_1, x_1) / (z_2, x_2)

$$x_1 = a z_1 + b \quad \Leftrightarrow \quad x_1 - a z_1 = b$$
$$x_2 = a z_2 + b \qquad\qquad x_2 - a z_2 = b$$

$$\Leftrightarrow \quad x_1 - a z_1 = x_2 - a z_2$$
$$\Leftrightarrow \quad x_1 - x_2 = a(z_1 - z_2)$$

$$\Rightarrow \quad \underline{a = \frac{x_1 - x_2}{z_1 - z_2}}$$

$$\Rightarrow \quad \underline{b = x_1 - \frac{x_1 - x_2}{z_1 - z_2} \cdot z_1}$$

Fehler über Gaußsche Fehlerfortpflanzung:

$$\Rightarrow \quad \sigma_{f(x)} = \sqrt{\sum_i \left(\frac{\partial f(x)}{\partial x_i} \sigma_{x_i}\right)^2} \quad \leftarrow \text{ if } x_i \text{ is uncorrelated}$$

Also für a, b:

$$\sigma_a = \sqrt{\left(\frac{1}{z_1 - z_2}\sigma_{x_1}\right)^2 + \left(\frac{1}{z_1 - z_2}\sigma_{x_2}\right)^2}$$

$$\sigma_b = \sqrt{\left(1 - \frac{z_1}{z_1 - z_2}\sigma_{x_1}\right)^2 + \left(\frac{z_1}{z_1 - z_2}\sigma_{x_2}\right)^2}$$

covariance matrix: $\vec{a} = M \vec{x}$

with $\vec{a} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow M = \begin{pmatrix} -\frac{1}{z_2 - z_1} & \frac{1}{z_2 - z_1} \\ \frac{z_2}{z_2 - z_1} & \frac{-z_1}{z_2 - z_1} \end{pmatrix}$$

script: $\vec{y} = M \vec{x} \Rightarrow Var[\vec{y}] = M \, Var[\vec{x}] \cdot M^T$

where $Var[\vec{x}] = \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix}$, $M = \frac{1}{z_2 - z_1} \begin{pmatrix} -1 & 1 \\ z_2 & -z_1 \end{pmatrix}$

$$\Rightarrow Var[\vec{a}] = \left(\frac{1}{z_2 - z_1} \right)^2 \begin{pmatrix} -1 & 1 \\ z_2 & -z_1 \end{pmatrix} \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix} \begin{pmatrix} -1 & z_2 \\ 1 & -z_1 \end{pmatrix}$$

$$= \left(\frac{1}{z_2 - z_1} \right)^2 \begin{pmatrix} -\sigma_{x_1}^2 & \sigma_{x_2}^2 \\ z_2 \sigma_{x_1}^2 & -z_1 \sigma_{x_2}^2 \end{pmatrix} \begin{pmatrix} -1 & z_2 \\ 1 & -z_1 \end{pmatrix}$$

$$= \left(\frac{1}{z_2 - z_1} \right)^2 \begin{pmatrix} \sigma_{x_1}^2 + \sigma_{x_2}^2 & z_2 (\sigma_{x_2}^2 - \sigma_{x_1}^2) \\ -(z_2 \sigma_{x_1}^2 + z_1 \sigma_{x_2}^2) & z_2^2 \sigma_{x_1}^2 + z_1^2 \sigma_{x_2}^2 \end{pmatrix}$$

correlation coefficient

$$\rho_{ab} = \frac{cov(a,b)}{\sigma_a \, \sigma_b}$$

$$cov(a,b) = -(z_2 \sigma_{x_1}^2 + z_1 \sigma_{x_2}^2) / (z_2 - z_1)^2$$

$$\Rightarrow \rho_{ab} = \cancel{\frac{(z_2 - z_1)^2}{(z_2 - z_1)^2}} \cdot \frac{-z_2 \sigma_{x_1}^2 - z_1 \sigma_{x_2}^2}{\sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2} \cdot \sqrt{z_2^2 \sigma_{x_1}^2 + z_1^2 \sigma_{x_2}^2}}$$

b) $z = z_3$ -> third detector plane

$$x_3 = a z_3 + b \qquad \text{with} \begin{cases} \Rightarrow a = \dfrac{x_1 - x_2}{z_1 - z_2} \\ \Rightarrow b = x_1 - \dfrac{x_1 - x_2}{z_1 - z_2} \cdot z_1 \end{cases}$$

$$\Rightarrow x_3 = \frac{x_1 - x_2}{z_1 - z_2} z_3 + x_1 - \frac{x_1 - x_2}{z_1 - z_2} \cdot z_1$$

with $\sigma_y = \sqrt{\sum_{i=1}^{m} \left| \frac{\partial y}{\partial x_i} \sigma_{x_i} \right|^2 + 2 \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} \left| \frac{\partial y}{\partial x_i} \right| \left| \frac{\partial y}{\partial x_k} \right| \cdot cov(a,b)}$

$$\sigma_{x_3} = \sqrt{(z_3 \cdot \sigma_a)^2 + \sigma_b^2 + 2 z_3 \cdot cov(a,b)}$$

mit $cov(a,b)$, σ_a, σ_b aus Teil a)

c)

It can lead to under - or overestimate the uncertainty. The more correlated the variables are the more likely it is that the actual error is greater than the value calculated without considering correlations!

$$\Rightarrow \sigma_{x_3} = \sqrt{(z_3 \cdot \sigma_a)^2 + \sigma_b^2}$$