Time	Group	Submission in Moodle; Mails with subject: [SMD2023]
Th. 12:05–13:05	A	tristan.gradetzke@udo.edu and samuel.haefs@udo.edu
Fr. 09:00–10:00	B	lucas.witthaus@udo.edu and david.venker@udo.edu

Exercise 3 *Sample Variance* **5 p.**

For all calculations, $x_1, \dots, x_n$ are the expressions of the square-integrable, pairwise uncorrelated, real-valued random variables $X_1, \dots, X_n$ with variance σ^2 and mean μ.

(a) Test whether the formula (arithmetic mean)

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1}$$

is an unbiased estimator of the mean μ of the population. If the estimator is not unbiased, look for an appropriate correction.

(b) The standard error of the arithmetic mean (1) is defined as the square root of the variance of $\bar{X}$. Show that

$$\mathrm{E}\left((\bar{X}-\mu)^2\right) = \mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n} \tag{2}$$

holds. *Hint:* Look at calculation rules for calculating with variances.

(c) Test whether the formula

$$S_0^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 \tag{3}$$

is an unbiased estimator of the variance σ^2 of the population. If the estimator is not unbiased, look for an appropriate correction.

(d) Usually the variance σ^2 of the population is unknown and the estimator (1) is used instead of μ. Then (3) becomes:

$$S_1'^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2 . \tag{4}$$

Test whether (4) is an unbiased estimator of the variance σ^2 of the population. If the estimator is not unbiased, look for an appropriate correction.

Hint: Expand the summand with $-\mu+\mu$ and use the given relation (2).

Exercise 4 *Lab Experiment* **5 p.**

In a lab experiment the following values are measured:

Ψ / °	Asymmetry	Ψ / °	Asymmetry	Ψ / °	Asymmetry
0	−0.032	30	0.010	60	0.057
90	0.068	120	0.076	150	0.080
180	0.031	210	0.005	240	−0.041
270	−0.090	300	−0.088	330	−0.074

The asymmetry values have a measurement error of ± 0.011. The theory says that the asymmetry is described by an ansatz of the form:

$$f(\Psi) = A_0 \cos(\Psi + \delta).$$

(a) Start with the ansatz

$$f(\Psi) = a_1 f_1(\Psi) + a_2 f_2(\Psi)$$

with

$$f_1(\Psi) = \cos(\Psi) \quad \text{und} \quad f_2(\Psi) = \sin(\Psi)$$

and write down the design matrix $\mathbf{A}$.

(b) Calculate the solution vector $\hat{\mathbf{a}}$ for the parameters using the method of least squares.

(c) Calculate the covariance matrix $\mathbf{V}[\hat{\mathbf{a}}]$ as well as the errors of a_1 and a_2 and the correlation coefficient.

(d) Calculate A_0 and δ, their error, and the correlation of a_1 and a_2.