Dynamic Programming and Applications

Discrete Time Dynamics and Optimization

Lecture 2

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Outline

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Part 1: difference equations

1. Stochastic processes

- Let X_t be a random variable that is time t adapted
- Discrete time: We index time discretely $t = 0, 1, 2, \ldots, T \leq \infty$
- Stochastic process in discrete time: a sequence of random variables indexed by t, $\{X_t\}_{t=0}^{T}$
- Continuous time: We index time continuously $t \in [0, T]$ with $T \leq \infty$
- Stochastic process in continuous time: a sequence of random variables indexed by t, $\{X_t\}_{t \geq 0}$

2. Markov chains

- A stochastic process $\{X_t\}$ has the *Markov property* if for all $k \geq 1$ and all t:

$$\mathbb{P}(X_{t+1} = x \mid X_t, X_{t-1}, \ldots, X_{t-k}) = \mathbb{P}(X_{t+1} = x \mid X_t)$$

- *State space* of the Markov process = set of events or states that it visits
- A Markov chain is a Markov process (stochastic process with Markov property) that visits a finite number of states (*discrete state space*)
- Simplest example: Individual i is randomly hit by earnings (employment) shocks and switches between $X_t \in \{X^L, X^H\}$

- Markov chains have a *transition matrix* P that describes the probability of transitioning from state i to state j

- Simplest example with state space $\{X^L, X^H\}$

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}$$

- This says: P of staying in employment state = 0.8, P of switching = 0.2

- P_{ij} is the probability of switching from state i to state j (one period)

- P^2 characterizes transitions over two periods: $(P^2)_{ij}$ is prob of going from i to j in two periods

- The rows of the transition matrix have to sum to 1 (definition of probability measure)

3. Difference equations

- We start with deterministic (non-random) dynamics and then conclude with stochastic (random) dynamics

- The *first-order linear difference equation* is defined by

$$x_{t+1} = bx_t + cz_t \tag{1}$$

  where $\{z_t\}$ is an exogenously given, bounded sequence

- For now, all objects are (real) scalars (easy to extend to vectors and matrices)

- Suppose we have an *initial condition* (i.e., given initial value) x_0

- When $c = 0$, (1) is a *time-homogeneous* difference equation

- When cz_t is constant for all t, (1) is an *autonomous* difference equation

Autonomous equations

- Consider the autonomous equation with $z_t = 1$
- A particular solution is the constant solution with $x_t = \frac{c}{1-b}$ when $b \neq 1$
- Such a point is called a *stationary point* or *steady state*
- General solution of the autonomous equation (for some constant x):

$$x_t = (x_0 - x)b^t + x \tag{2}$$

- Important question is long-run behavior (stability / convergence)
- When $|b| < 1$, (2) converges asymptotically to steady state x for any initial value x_0 (steady state x is globally stable)
- If $|b| > 1$, (2) explodes and is not stable (except when $x_0 = x$)

Examples in macro

Capital accumulation:

$$K_{t+1} = (1-\delta)K_t + I_t$$

- δ is depreciation and I_t is investment
- This is a *forward equation* and requires an initial condition K_0
- If $I_t = 0$ and $0 < \delta < 1$, $K_t \to 0$
- If $I_t = c$ constant, then K_t converges to $\frac{c}{\delta}$: $K_{t+1} = (1-\delta)\frac{c}{\delta} + c = \frac{c}{\delta}$

Wealth dynamics:

$$a_{t+1} = R_t a_t + y_t - c_t$$

- R_t is the gross real interest rate, y_t is income, c_t is consumption
- This is a *forward equation* and requires an initial condition a_0
- We will study this as a *controlled* process because c_t will be chosen optimally
- Work out the following: $R_t = R$ and $y_t = y$ constant, and

$$c_t = \left(1 - \frac{1}{R}\right)\left(a_t + \sum_{s=t}^{\infty} R^{-(s-t)} y\right)$$

  What are the dynamics of a_t?

Consumption Euler equation:

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- $\frac{1}{C_t} = u'(C_t)$ is marginal utility with log preferences
- This is a *backward equation* and requires a terminal condition or transversality condition, i.e., c_T must converge to something
- Suppose there exists time T s.t. for all $t \geq T$, $C_t = C$
- Then solve *backwards* from: $\frac{1}{C_{T-1}} = \beta R_{T-1} \frac{1}{C_T}$ or expressed as *time-homogeneous first-order linear difference equation*

$$C_{T-1} = \frac{1}{\beta R_{T-1}} C_T$$

- Difference between *forward* and *backward* equations is critical! This is closely related to the idea of *boundary conditions* (much more to come)

4. Stochastic difference equations

- Consider the process $\{X_t\}$ with

$$X_{t+1} = AX_t + Cw_{t+1} \tag{3}$$

  where w_{t+1} is an iid. process with $w_{t+1} \sim \mathcal{N}(0, 1)$

- Equation (3) is a *first-order, linear stochastic difference equation*

- Let $\mathbb{E}_t$ the *conditional expectation* operator (conditional on time t information)

- For example:

$$\begin{aligned} \mathbb{E}_t(X_{t+1}) &= \mathbb{E}(X_{t+1} \mid X_t) = \mathbb{E}(AX_t + Cw_{t+1} \mid X_t) \\ &= AX_t + C\mathbb{E}(w_{t+1} \mid X_t) = AX_t + C\mathbb{E}(w_{t+1}) = AX_t \end{aligned}$$

- Rational expectations: agents' beliefs about stochastic processes are consistent with the true distribution of the process

- Key equation: wealth dynamics with income fluctuations:

$$a_{t+1} = R_t a_t + y_t - c_t,$$

  where y_t is a stochastic process

- Consumption Euler equation with uncertainty (e.g., stochastic income):

$$u'(C_t) = \beta R \mathbb{E}_t \Big[u'(C_{t+1}) \Big]$$

Part 2: stochastic dynamic programming

1. Stochastic dynamic programming

- Follow Ljungqvist-Sargent notation, Chapter 3.2
- Under uncertainty, household problem takes the form

$$\max_{\{c_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to $k_{t+1} = g(k_t, c_t, \epsilon_{t+1})$ (*first-order stochastic difference equation*)

- $\{\epsilon_t\}_{t=0}^{\infty}$ is sequence of iid random variables (*stochastic process*)
- Initial condition x_0 given

- Dynamic programming approach: we again look for recursive representation on state space $k_t \in \mathcal{X}$

- The problem is to look for a *policy function* $c(k)$ that solves

$$V(k) = \max_c \left\{ u(c) + \beta \mathbb{E}\left[V\Big(g(k,c,\epsilon)\Big) \,\Big|\, k \right] \right\}$$

  where $\mathbb{E}[V(\cdot) \mid k] = \int V(\cdot) dF(\epsilon)$

- $V(k)$ is the (lifetime) value that an agent obtains from solving this problem starting from k

- FOC that characterizes the consumption policy function $c(k)$ is

$$0 = u'(c(k)) + \beta \mathbb{E}\left\{ \partial_k V\Big(g(k,c(k),\epsilon)\Big) \cdot \partial_c g(k,c(k),\epsilon) \,\Big|\, k \right\} = 0$$

2. History notation

- A very popular approach to deal with uncertainty in macro is to use history notation (Ljungqvist-Sarget, e.g., chapters 8, 12)
- Economy populated by individuals, indexed by $i \in I$
- Time is discrete and indexed by $t = 0, 1, \ldots$
- At every t, there is a realization of a stochastic event $s_t \in \mathcal{S}$
- We denote the **history** of such events up to t by $s^t = \{s_0, s_1, \ldots, s_t\}$
- The unconditional probability of history s^t is given by $\pi_t(s^t \mid s_0)$
- If Markov, $\pi_t\left(s^t \mid s_0\right) = \pi(s_t \mid s_{t-1})\pi(s_{t-1} \mid s_{t-2}) \ldots \pi(s_0)$
- Single consumption good (dollars) as numeraire

- The **lifetime value** of individual i is then defined as

$$V_i(s_0) = \sum_{t=0}^{T} \left(\beta_i\right)^t \sum_{s^t} \pi_t\left(s^t \mid s_0\right) u_i\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right)\right)$$

- *Generalizations*: heterogeneous beliefs, general preferences (Epstein-Zin), recursive formulation, multiple commodities, intergenerational considerations
- Suppose $c_t^i(\cdot)$ and $l_t^i(\cdot)$ are functions of some primitive (policy) θ

- Consider policy experiment $d\theta$, then i's **private welfare assessment** is

$$\frac{dV_i(s_0)}{d\theta} = \sum_{t=0}^{T} \left(\beta_i\right)^t \sum_{s^t} \pi_t\left(s^t \mid s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i} \frac{du_{i|c}(s^t)}{d\theta}$$

Instantaneous consumption-equivalent effect of policy $d\theta$ at date t, history s^t on individual i:

$$\frac{du_{i|c}(s^t)}{d\theta} \equiv \frac{\frac{du_i(c_t^i(s^t), n_t^i(s^t))}{d\theta}}{\frac{\partial u_i(s^t)}{\partial c_t^i}} = \underbrace{\frac{dc_t^i(s^t)}{d\theta} + \frac{\frac{\partial u_i(s^t)}{\partial n_t^i}}{\frac{\partial u_i(s^t)}{\partial c_t^i}} \frac{dn_t^i(s^t)}{d\theta}}_{\text{in } \textit{consumption} \text{ units}}$$

- Example: $d\theta$ gives i marginal dollar at s^t, then

$$\frac{dV_i(s_0)}{d\theta} = (\beta_i)^t \pi_t(s^t \mid s_0) \frac{\partial u_i(s^t)}{\partial c_t^i}$$

Conventional benchmarks:

- Policy $d\theta$ is Pareto improving if $\frac{dV_i(s_0)}{d\theta} \geq 0$ for all i, strictly for some i
- Welfarist planners: SWF given by $\mathcal{W}\left(\{V_i\left(s_0\right)\}_{i\in I}\right)$
  Utilitarian, Isoelastic, Rawlsian, Nash, Dictator, ...
- Policy $d\theta$ is desirable for a welfarist planner if

$$\int \lambda_i\left(s_0\right)\frac{dV_i(s_0)}{d\theta}di > 0\,, \qquad \text{where } \lambda_i\left(s_0\right) = \frac{\partial\mathcal{W}\left(\{V_i\left(s_0\right)\}_{i\in I}\right)}{\partial V_i}$$

Remarks:

- All welfarist planners agree when individuals ex-ante (s_0) homogeneous
- How to make welfare assessments with *heterogeneous* individuals?
- If you're interested, see Dávila-Schaab (2022)

3. Stochastic Growth Model

- Discrete time: $t \in \{0, 1, \ldots, T\}$, where $T \leq \infty$
- At t, event $s_t \in \mathcal{S}$ is realized; history $s^t = (s_0, \ldots, s_t)$ has probability $\pi_t(s^t)$
- Representative household has preferences of paths of consumption $c_t(s^t)$ and labor $l_t(s^t)$

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t\left(s^t\right) u\left(c_t(s^t), l_t(s^t)\right)$$

- Inada conditions $\lim_{c \to 0} u_c(c, l) = \lim_{l \to 0} u_l(c, l) = \infty$
- At $t = 0$, household endowed with k_0

- Technology, capital accumulation, and budget / resource constraint:

$$c_t(s^t) + \iota_t(s^t) \leq A_t(s^t)F(k_t(s^{t-1}), l_t(s^t))$$
$$k_{t+1}(s^t) = (1-\delta)k_t(s^{t-1}) + \iota_t(s^t)$$

- $F(\cdot)$ is twice continuously differentiable and constant returns to scale
- Source of uncertainty is stochastic process for TFP $A_t(s^t)$
- Standard regularity conditions on $F(\cdot)$ (see LS)

Lagrangian approach to sequence problem

- Form Lagrangian

$$
\begin{aligned}
L = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \Big\{ & u(c_t(s^t), l_t(s^t)) \\
& + \lambda_t(s^t) \Big[A_t(s^t) F(k_t(s^{t-1}), l_t(s^t)) - c_t(s^t) + (1-\delta) k_t(s^{t-1}) - k_{t+1}(s^t) \Big] \Big\}
\end{aligned}
$$

- FOCs for $c_t(s^t)$, $l_t(s^t)$ and $k_{t+1}(s^t)$ are given by

$$
\begin{aligned}
u_c(s^t) &= \lambda_t(s^t) \\
u_l(s^t) &= u_c(s^t) A_t(s^t) F_n(s^t) \\
u_c(s^t) \pi_t(s^t) &= \beta \sum_{s^{t+1}|s^t} u_c(s^{t+1}) \pi_{t+1}(s^{t+1}) \Big[A_{t+1}(s^{t+1}) F_k(s^{t+1}) + (1-\delta) \Big]
\end{aligned}
$$

- Summation over $(s^{t+1} \mid s^t)$ is like conditional expectation (summing over histories that branch out from s^t)

Recursive representation: dynamic programming

- Assume time-homogeneous Markov process:

$$\mathbb{E}_t(A_{t+1}) = \mathbb{E}\Big[A(s^{t+1}) \mid s^t\Big] = \mathbb{E}\Big[A(s_{t+1}) \mid s_t\Big] = \sum_{s'} \pi(s' \mid s_t) A(s')$$

- Drop t subscripts: s is current state, s' denotes next period's draw

- Denote by X_t the *endogenous state* of the problem: for now, assume there is such a representation

- Intuitively: s is the exogenous state and X is the endogenous state

- Bellman equation can be written:

$$V(X,s) = \max_{c,l} \left\{ u(c,l) + \beta \sum_{s'} \pi(s' \mid s) V(X', s') \right\}$$

subject to $X' = g(X, c, l, s, s')$

4. Application: optimal stopping problem

Problem: Every period t, an agent draws an offer x from the unit interval $[0, 1]$. The agent can accept the offer, in which case her payoff is x, and the game ends. Draws are independent. The agent discounts the future at β. The game continues until the agent receives an offer she accepts.

- Agent's dynamic optimization problem given recursively by Bellman equation
$$V(x) = \max \left\{ x, \, \beta \mathbb{E} V(x') \right\}$$
where the expectation (operator) $\mathbb{E}$ is taken over the next draw x'
- There will be a threshold $x^* \in [0, 1]$ such that agent accepts for $x \geq x^*$
- This is also called a **free boundary problem** because we have to look for the endogenous boundary of the problem x^*
- Many applications (problems in life) look like this:
buying a house, searching for a partner, closing a production plant, exercising an option, adopting a new technology, ...

- The value function will look like:

$$V(x) = \begin{cases} x & \text{if} \quad x \geq x^* \\ x^* & \text{if} \quad x < x^* \end{cases}$$

- Find the value x^* such that this function satisfies the Bellman equation
- At $x = x^*$, indifferent between accepting and stopping:

$$\begin{aligned} V(x^*) &= x^* \\ &= \beta \mathbb{E} V(x') \\ &= \beta \int_0^{x^*} x^* f(x) dx + \beta \int_{x^*}^1 x f(x) dx \\ &= \beta x^* [x]_0^{x^*} + \beta \frac{1}{2} [x^2]_{x^*}^1 \end{aligned}$$

where $f(x) = \frac{1}{1-0}$ is the uniform density
- Solution: $x^* = \beta (x^*)^2 + \beta \frac{1}{2} [1 - (x^*)^2]$ or $x^* = \frac{1}{\beta}(1 - \sqrt{1 - \beta^2})$