Dynamic Programming and Applications

Applications

Lecture 7

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Outline

Part 1: Stochastic processes, Brownian motion, and stochastic differential equations

1. Stochastic processes in continuous time
2. Continuous time Markov chains
3. Brownian motion
4. Diffusion processes
5. Ito's Lemma
6. Poisson processes
7. The generator of a stochastic process

Outline

Part 2: Optimization with stochastic dynamics

1. Stochastic neoclassical growth model
2. Stochastic neoclassical growth with diffusion process
3. Stochastic neoclassical growth with Poisson process

Real Business Cycles

- Next semester, Yuriy will teach the Real Business Cycle model. This is basically the stochastic neoclassical growth model, estimated to match business cycle moments

- Recall that this model is efficient so we can look at the planning problem

- Preferences are:
$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

- And combining technologies and resource constraints yields:
$$dk_t = \left[f(k_t, z_t) - \delta k_t - c_t \right] dt$$

- Now suppose TFP follows a Feller process: $dz_t = \theta(\bar{z} - z_t)dt + \sigma \sqrt{z_t} dB_t$

- With $dz_t = \theta(\bar{z} - z_t)dt + \sigma\sqrt{z_t}dB_t$, the HJB is then given by:

$$\rho V(k,z) = \max_c \left\{ u(c) + \Big[f(k,z) - \delta k - c \Big] V_k(k,z) + \theta(\bar{z} - z)V_z(k,z) + \frac{1}{2}\sigma^2 z V_{zz}(k,z) \right\}$$

- We have now seen 3 variants of this model with 3 different assumptions for the process dz_t

- This model is the foundation for business cycle macro

AK Technology and Log Utility

- Assume that $u(c_t) = \log c_t$ and $f(k_t, z_t) = z_t k_t$
- Assuming that dz_t follows a stationary diffusion process:

$$\rho V(k,z) = \max_c \left\{ \log c + \left[f(k,z) - \delta k - c \right] V_k(k,z) + \mu(z) V_z(k,z) + \frac{1}{2} \sigma(z)^2 V_{zz}(k,z) \right\}$$

- Show that the consumption policy function is:

$$c(k,z) = \rho k$$

- As a result, model solution characterized by the two *forward* equations:

$$\begin{aligned} dk_t &= (z_t - \rho - \delta) k_t dt \\ dz_t &= \mu(z_t) dt + \sigma(z_t) dB_t \end{aligned}$$

Proof:

- Guess and verify:
$$V(k,z) = v(z) + A \log k$$

- FOC:
$$u'(c(k,z)) = V_k(k,z) \quad \Longrightarrow \quad \frac{1}{c(k,z)} = A\frac{1}{k} \quad \Longrightarrow \quad c(k,z) = \frac{1}{A}k$$

- Plug back into HJB:
$$\rho v(z) + \rho A \log k = \log \frac{1}{A} + \log k + (z - \frac{1}{A} - \delta)kA\frac{1}{k} + \mu(z)v'(z) + \frac{1}{2}\sigma(z)^2 v''(z)$$

- Collect terms in $\log k$ and confirm $\rho A = 1$. Solve ODE for $v(z)$!

- Economics: log preferences $\Longrightarrow$ income and substitution effects of future z_t changes cancel $\Longrightarrow$ constant savings rate ρ