Dynamic Programming and Applications

Stochastic Dynamic Programming in Continuous Time

Lectures 5 – 6

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Outline

Part 1: Stochastic processes, Brownian motion, and stochastic differential equations

Outline

Part 2: Optimization with stochastic dynamics

1. Stochastic neoclassical growth model
2. Stochastic neoclassical growth with diffusion process
3. Stochastic neoclassical growth with Poisson process

Outline

Part 3: Applications

1. Consumption-savings with stochastic income fluctuations
2. Portfolio choice
3. Firm profit maximization
4. XXX

Part 1: Stochastic Processes

1. Stochastic processes in continuous time

Definition. A **stochastic process** is a time-indexed sequence of random variables.

- A random variable maps an "event" into a scalar, a stochastic process maps an event into a path
- Formally, an event is $\omega \in \Omega$ and a random variable $X : \Omega \to \mathbb{R}$ (learn some basic measure theory!)
- A stochastic process is a sequence $\{X_t\}_{t \geq 0}$ such that each X_t is a random variable. In discrete time $t \in \{0, 1, \ldots\}$ and in continuous time $t \in [0, \infty)$
- We can also think of a stochastic process as a random variable, but one that maps into a different space—a function space!

Definition. Let $X = \{X_t\}_{t \geq 0}$ be a sequence of random variables taking values in a finite or countable state space $\mathcal{X}$. Then X is a *continuous-time Markov chain* if it satisfies the *Markov property*: For any sequence $0 \leq t_1 < t_2 < \ldots < t_n$ of times

$$\mathbb{P}(X_{t_n} = x \mid X_{t_1}, \ldots, X_{t_{n-1}}) = \mathbb{P}(X_{t_n} = x \mid X_{t_{n-1}})$$

Definition. Process X is **time-homogeneous** or **stationary** if the conditional probability does not depend on the current time, i.e., for $x, y \in \mathcal{X}$:

$$\mathbb{P}(X_{t+s} = x \mid X_s = y) = \mathbb{P}(X_t = x \mid X_0 = y)$$

- Alternatively: If all possible joint distributions are independent of time t.

- The *transition density* of process X is denoted $p(t, x \mid s, y)$ and is defined as

$$\mathbb{P}(X_t \in A \mid Y_s = y) = \int_A p(t, x \mid s, y) dx$$

  for any (Borel) set $A \subset \mathcal{X}$. In words: $p(t, x \mid s, y)$ is the probability (density) that process X_t ends up at $X_t = x$ at time t if it started at $X_s = y$ at time s

- *Conditional expectation* can be written as: $\mathbb{E}[f(X_t) \mid X_0 = y] = \int p(t, x \mid 0, y) f(x) dx$

Definition. A process X is a **martingale** if it satisfies

$$\mathbb{E}\Big[X_{t+s} \mid \text{ information available at } t\Big] = X_t$$

- The most important martingale in macroeconomics: Suppose $\beta R_t = 1$ then

$$\mathbb{E}_t\Big[u'(C_{t+s})\Big] = u'(C_t).$$

- Efficient markets hypothesis in finance $\approx \mathbb{E}_t(P_{t+s}) = P_t$
- A random walk is a martingale (but not every martingale is a random walk)

Example:

- Consider the two-state employment process $z_t \in \{z^L, z^H\}$ with transition rates λ^{LH} (from L to H) and λ^{HL} (from H to L)

- The associated transition matrix (*generator*) is

$$\mathcal{A}^z = \begin{pmatrix} -\lambda^{LH} & \lambda^{LH} \\ \lambda^{HL} & -\lambda^{HL} \end{pmatrix}$$

- Interpretation: households transition *out of* state i at rate λ^{ij}

- Notice: In discrete time, Markov transition matrix rows sum to 1. Here, rows sum to 0 (*mass preservation*)

2. Brownian motion

- Brownian motion is the most
- Einstein (1905) uses Brownian motion to model motion of particles
- We will introduce / discuss Brownian motion from 3 perspectives

Brownian motion: perspective #1

- Consider a process $X(t)$. Every Δ time step, the process either goes up or down by h

$$\Delta X \equiv X(t+\Delta) - X(t) = \begin{cases} +h & \text{with probability } p \\ -h & \text{with probability } q = 1-p \end{cases}$$

- Then we have:

$$\begin{aligned} \mathbb{E}(\Delta X) &= ph - qh = (p-q)h \\ \mathbb{E}((\Delta X)^2) &= ph^2 + qh^2 = h^2 \\ Var(\Delta X) &= \mathbb{E}[\Delta X - \mathbb{E}(\Delta X)]^2 = 4pqh^2 \end{aligned}$$

- Notice that $X(t) - X(0)$ is a binomial random variable with

$$\mathbb{E}[X(t) - X(0)] = n(p-q)h = t(p-q)\frac{h}{\Delta}$$

$$Var[X(t) - X(0)] = n4pqh^2 = t4pq\frac{h^2}{\Delta}$$

  where $n = t/\Delta$ is the number of jumps in interval $[0, t]$

- Next, let

$$h = \sigma\sqrt{\Delta} \qquad \text{and} \qquad p = \frac{1}{2}\left[1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right]$$

- This implies

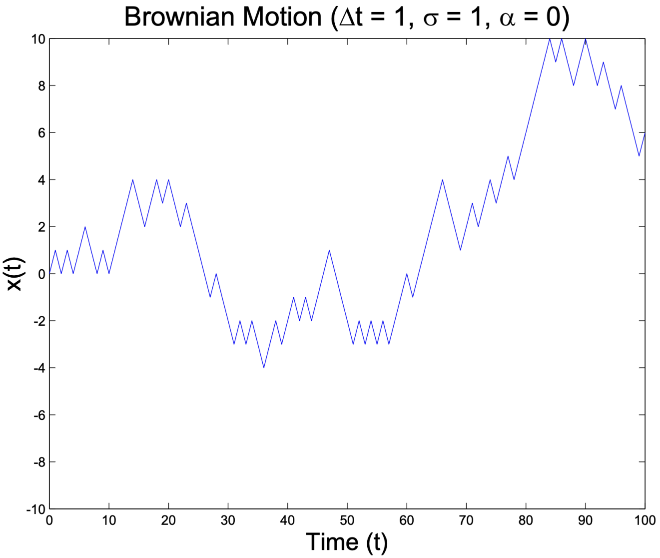
$$(p-q) = \frac{\mu}{\sigma}\sqrt{\Delta}$$

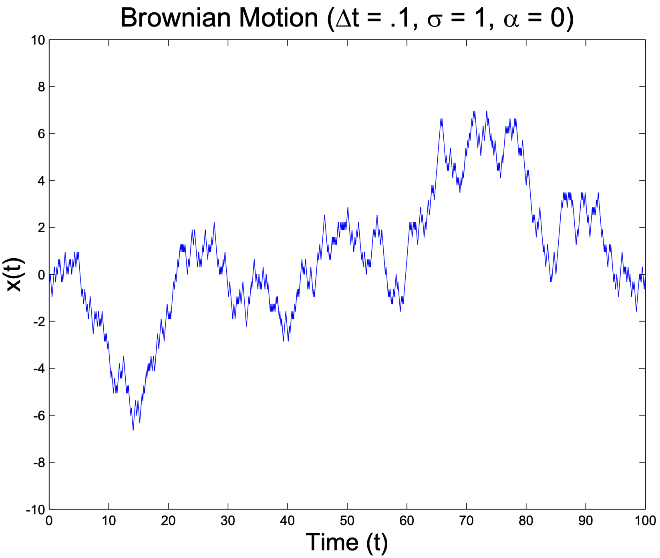
$$\mathbb{E}[X(t) - X(0)] = t\frac{\mu}{\sigma}\sqrt{\Delta}\sigma\frac{\sqrt{\Delta}}{\Delta} = \mu t$$

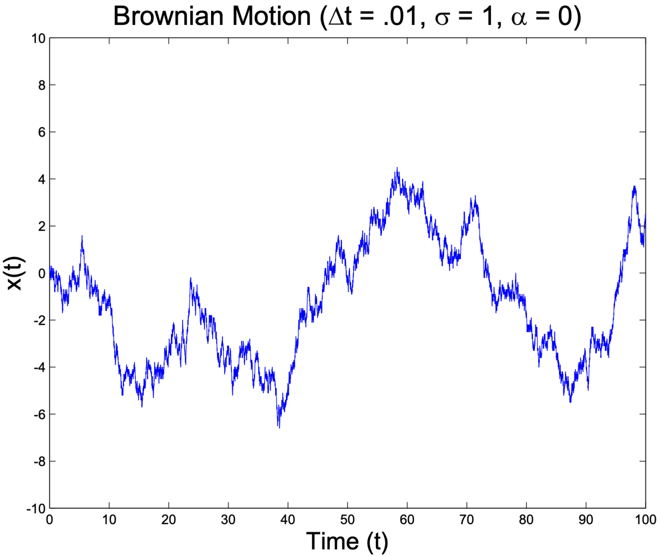
$$Var[X(t) - X(0)] \longrightarrow_{\text{as}\Delta\to 0} \sigma^2 t$$

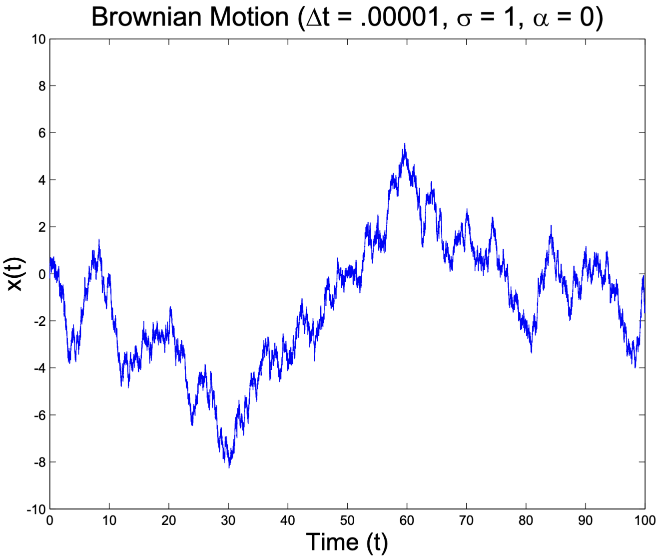
Implications:

- Vertical movements proportional to $\sqrt{\Delta}$ not Δ
- Convergence in distribution: $X(t) - X(0) \to^D \mathcal{N}(\mu t, \sigma t)$ since Binomial $\to^D$ Normal
- Distance traveled (length of curve) during $t \in [0, 1]$ is $= nh = \frac{1}{\Delta}\sigma\sqrt{\Delta} = \sigma\frac{1}{\sqrt{\Delta}} \to \infty$
- Time derivative $\mathbb{E}\frac{dX}{dt}$ doesn't exist: $\frac{\Delta X}{\Delta} = \frac{\pm\sigma\sqrt{\Delta}}{\Delta} = \frac{\pm\sigma}{\sqrt{\Delta}} \to \infty$
- $\frac{\mathbb{E}\Delta X}{\Delta} = \frac{\mu h^2/\sigma^2}{\Delta} = \mu$ so we can write $\mathbb{E}(dX) = \mu dt$
- $\frac{Var(\Delta X)}{\Delta} \to \sigma^2$ so we can write $Var(dX) = \sigma^2 dt$


Brownian Motion (Δt = 1, σ = 1, α = 0)
10
8
6
4
2
0
-2
-4
-6
-8
-10
x(t)
0
10
20
30
40
50
60
70
80
90
100
Time (t)


Brownian Motion (Δt = .1, σ = 1, α = 0)
10
8
6
4
2
0
-2
-4
-6
-8
-10
x(t)
0
10
20
30
40
50
60
70
80
90
100
Time (t)


Brownian Motion ($\Delta t = .01$, $\sigma = 1$, $\alpha = 0$)
x(t)
Time (t)
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
10
20
30
40
50
60
70
80
90
100


Brownian Motion (Δt = .00001, σ = 1, α = 0)
10
8
6
4
2
0
-2
-4
-6
-8
-10
x(t)
0
10
20
30
40
50
60
70
80
90
100
Time (t)

Brownian motion: perspective #2

- Consider a random walk process in discrete time:

$$W_{t+1} = W_t + \epsilon_t, \qquad \text{where } W_0 = 0 \text{ and } \epsilon_t \sim \mathcal{N}(0,1)$$

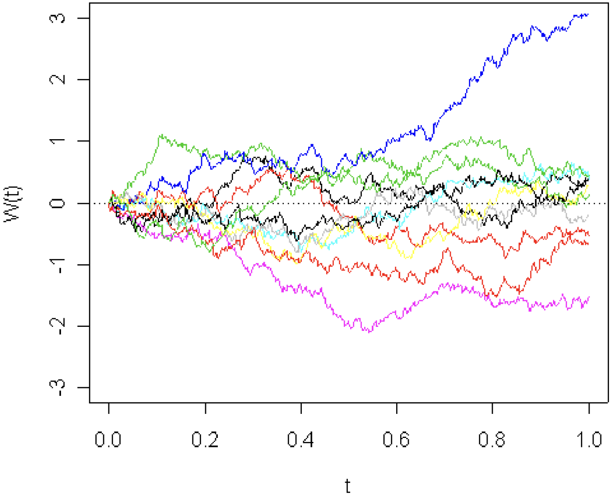
- We can extend this to a generalized time step:

$$W_{t+\Delta} = W_t + \sqrt{\Delta}\epsilon_t, \qquad \text{where } W_0 = 0 \text{ and } \epsilon_t \sim \mathcal{N}(0,1)$$

- Why the square root?

$$Var(W_{t+\Delta} - W_t) = Var(\sqrt{\Delta}\epsilon_t) = \Delta Var(\epsilon_t) = \Delta \quad \Longrightarrow \quad Var(W_t) = t$$

- You get Brownian motion by taking limit $\Delta \to 0$: Brownian motion = continuous time random walk



W(t)
t
3
2
1
0
-1
-2
-3
0.0
0.2
0.4
0.6
0.8
1.0

Brownian motion: perspective #3

Definition. Brownian motion $\{B_t\}_{t\geq 0}$ is a stochastic process with properties:

(i) $B_0 = 0$

(ii) (*Independent increments*) For non-overlapping $0 \leq t_1 < t_2 < t_3 < t_4$, we have $B_{t_2} - B_{t_1}$ independent from $B_{t_4} - B_{t_3}$

(iii) (*Normal, stationary increments*) $B_t - B_s \sim \mathcal{N}(0, t-s)$ for any $0 \leq s < t$

(iv) (*Continuity of paths*) The sample paths of B_t are continuous

- Brownian motion is the only stochastic process with stationary and independent increments that's also continuous
- Brownian motion is a Markov process
- Brownian motion is nowhere differentiable
- Brownian motion is a martingale
- Summary: Brownian motion is a continuous time random walk with zero drift and unit variance

Other properties of Brownian motion

- $B_t \sim B_t - B_0 \sim \mathcal{N}(0, t)$
- Important: $dB_t \sim \mathcal{N}(0, dt)$ because

$$dB_t \approx B_{t+\Delta} - B_t \sim \mathcal{N}(0, t + \Delta - t = \mathcal{N}(0, \Delta)$$

  and now take $\Delta \to dt$ (continuous-time limit)

- Alternatively: $B_{t+\Delta} - B_t \sim \mathcal{N}(0, \Delta) \sim \epsilon_t \sqrt{\Delta}$ where $\epsilon_t \sim \mathcal{N}(0, 1)$. So as $\Delta \to dt$,

$$\mathbb{E}(dB_t) = \mathbb{E}(\epsilon_t \sqrt{dt}) = 0$$
$$\mathbb{E}[(dB_t)^2] = \mathbb{E}[(\epsilon_t \sqrt{dt})^2] = dt$$

4. Ito's Lemma

- **Q**: How do functions / transformations of stochastic processes evolve?

- Warm-up: Consider the process x_t characterized by the ODE

$$\dot{x}_t = \frac{dx_t}{dt} = \mu_t \qquad \Longrightarrow \qquad dx_t = \mu_t dt$$

- Let $y_t = f(x_t)$. What do we know about $\dot{y}_t$ or dy_t? Using calculus, we simply have:

$$\frac{dy_t}{dt} = f'(x_t)\frac{dx_t}{dt} \qquad \Longrightarrow \qquad dy_t = f'(x_t)dx_t = f'(x_t)\mu_t dt$$

- Next, suppose $y_t = f(t, x_t)$. Then using $f_x = \frac{\partial f}{\partial x}$ we have:

$$\frac{dy_t}{dt} = f_x(t, x_t)\frac{dx_t}{dt} + f_t(t, x_t) \qquad \Longrightarrow \qquad dy_t = f_x(t, x_t)\mu_t dt + f_t(t, x_t)dt$$

- Now what about functions of Brownian motion or other diffusion processes?

- What to remember for Ito's lemma: (i) take a 2^{nd}-order Taylor expansion and (ii) remember that

$$(dt)^2 = 0 \quad \text{and} \quad (dt)(dB) = 0 \quad \text{and} \quad (dB)^2 = 0$$

- Suppose $Y_t = f(B_t)$ where B_t is Brownian motion. Show that:

$$dY_t = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$$

- Suppose $Y_t = f(t, B_t)$. Show that:

$$dY_t = f_t(t, B_t)dt + f_b(t, B_t)dB_t + \frac{1}{2}f_{bb}(t, B_t)dt$$

- This is **Ito's lemma**, a core building block of **stochastic calculus**

5. Diffusion processes

- Consider the class of stochastic processes $\{X_t\}$ characterized by

$$dX_t = \mu_t(X_t)dt + \sigma_t(X_t)dB_t$$

- These are called **diffusion processes** (continuous sample paths)

- For any twice differentiable function $Y_t = f(t, X_t)$, we have

$$\begin{aligned} dY &= f_t dt + f_x dX + \frac{1}{2} f_{xx}(dX)^2 \\ &= f_t dt + f_x(\mu(t,X)dt + \sigma(t,X)dB) + \frac{1}{2} f_{xx}\sigma(t,X)^2 dt \\ &= \left(f_t + f_x\mu(t,X) + \frac{1}{2} f_{xx}\sigma(t,X)^2\right)dt + \sigma(t,X)dB \end{aligned}$$

- Powerful result: Any (twice differentiable) function of a diffusion is also a diffusion

6. Stochastic differential equations

Geometric Brownian motion:

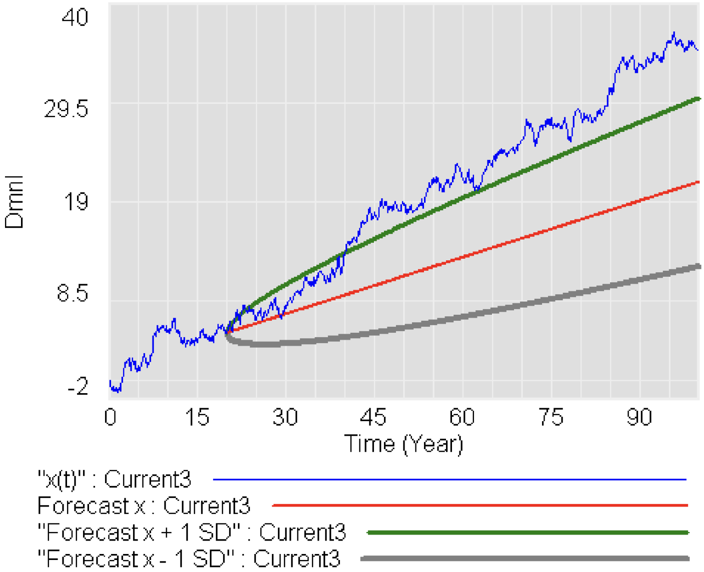
- Stochastic differential equations (SDEs) add noise / uncertainty to ordinary differential equations (ODEs)
- Start with $\dot{X}_t = \mu X_t$ with solution $X_t = X_0 e^{\mu t}$
- Rewrite as $dX_t = \mu X_t dt$ and "add noise" (using Brownian motion):

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

- Solution to geometric Brownian motion (work this out on homework):

$$X_t = X_0 e^{\mu t - \frac{\sigma^2}{2} t + \sigma B_t}$$

Brownian Motion with Drift



Ornstein-Uhlenbeck (OU) process:

- Continuous time analog of the AR(1) process

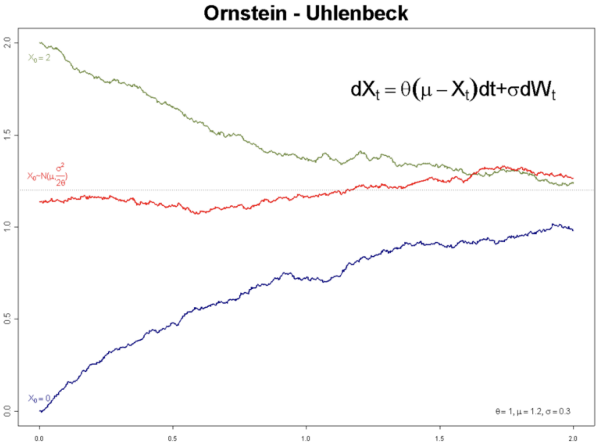
$$dz_t = \theta(\bar{z} - z_t)dt + \sigma dB_t$$

- Popular model for earnings risk and income fluctuations

- Auto-correlation of $e^{-\theta} \approx 1 - \theta$. Compare to:

$$x_{t+1} = \theta\bar{x} + (1-\theta)x_t + \sigma\epsilon_t$$

- Stationary distribution is $\mathcal{N}(\bar{z}, \frac{\sigma^2}{2\theta})$


Ornstein - Uhlenbeck
$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$
$X_0 = 2$
$X_0 \sim N(\mu, \frac{\sigma^2}{2\theta})$
$X_0 = 0$
$\theta = 1, \mu = 1.2, \sigma = 0.3$
0.0
0.5
1.0
1.5
2.0

- Diffusion processes allow us to construct many other processes with desired properties by simply choosing $\mu(t, X)$ and $\sigma(t, X)$

- Process that stays in interval $[0, 1]$ and mean-reverts around $1/2$:

$$dX = \theta(\frac{1}{2} - X)dt + \sigma X(1 - X)dB$$

- Feller square root process (finance: "Cox-Ingersoll-Ross"):

$$dX = \theta(\bar{X} - X)dt + \sigma\sqrt{X}dB$$

with stationary distribution $\sim \Gamma\left(\frac{2\theta\bar{X}}{\sigma^2}, \frac{\sigma^2}{2\theta\bar{X}}\right)$

Part 2: Optimization with stochastic dynamics

1. The generator of a stochastic process

Definition. The **generator** of a stochastic process X is defined (for any test function f) as

$$\mathcal{A}f = \lim_{\Delta t \to 0} \mathbb{E}_t \frac{f(t + \Delta t, X(t + \Delta t)) - f(t, X(t))}{\Delta t}$$

- The generator $\mathcal{A}$ tells us how the stochastic process is *expected* to evolve
- The generator $\mathcal{A}$ is a functional operator

Generator for diffusion processes

- We start with the diffusion process

$$dX = \mu(t, X)dt + \sigma(t, X)dB$$

- On homework, you will show that:

$$\mathcal{A}f = \partial_t f(t, X) + \mu(t, X)\partial_X f(t, X) + \frac{1}{2}\sigma(t, X)^2\partial_{XX} f(t, X)$$

- For the general / multi-dimensional version see Oksendal

Generator for Poisson processes

- Next, we consider the poisson process $\{Y_t\}$ where $Y_t \in \{Y^1, Y^2\}$. This is a two-state Markov chain in continuous time.

- We assume that the Poisson intensity / arrival rate / hazard rate is λ

- The generator is now given by

$$\mathcal{A}f(Y^j) = \lambda \Big[f(Y^{-j}) - f(Y^j) \Big]$$

- Intuition: at rate λ you transition, so you lose the value of your current state, $f(Y^j)$, and obtain the value of the new state, $f(Y^{-j})$

- Again see Oksendal for general version of this and more details

2. Stochastic Neoclassical Growth Model

Preferences. Lifetime utility of representative household

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

Technology. Final consumption good produced using technology

$$y_t = F(k_t, z_t) \qquad \text{where } F(0, z) = 0, \text{ and } F_k, F_z > 0, \text{ and } F_{kk} < 0$$

where z_t is exogenous productivity, and capital accumulation technology is

$$dk_t = (i_t - \delta k_t) dt.$$

Resource constraint for the final good is: $y_t = c_t + i_t$

Endowment. At time $t = 0$, economy's initial state is (k_0, z_0)

Planning problem. Taking as given (k_0, z_0), choose an allocation to maximize lifetime utility of representative household subject to technologies and resource constraints:

$$V(k_0, z_0) = \max \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

subject to

$$\begin{aligned} y_t &= F(k_t, z_t) \\ y_t &= c_t + i_t \\ dk_t &= (i_t - \delta k_t) dt \\ dz_t &\text{ exogenous} \end{aligned}$$

and taking as given (k_0, z_0)

Combining: $$dk_t = [F(k_t, z_t) - c_t - \delta k_t] dt$$

3. Productivity as a diffusion process

- We start with diffusion process:

$$dz_t = -\theta z_t dt + \sigma dB_t,$$

  where θ and σ are constants

- This is a continuous-time, mean-reverting AR(1) process called the Ornstein-Uhlenbeck process

- State space is now given by

$$\left\{(k, z) \mid k \in [0, \bar{k}] \text{ and } z \in [\underline{z}, \bar{z}]\right\}$$

- In discrete time, we would have

$$V(k_t, z_t) = \max_c \left\{ u(c)\Delta t + \frac{1}{1+\rho\Delta t}\mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) \right\}$$

- Difference from previous lecture: $\mathbb{E}$ because there is uncertainty

$$\begin{aligned}(1+\rho\Delta t)V(k_t, z_t) &= \max_c \left\{ (1+\rho\Delta t)u(c)\Delta t + \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) \right\} \\ \rho\Delta t V(k_t, z_t) &= \max_c \left\{ u(c)\Delta t + \mathbb{E}_t V(k_{t+\Delta t}, z_{t+\Delta t}) - V(k_t, z_t) \right\} \\ \rho V(k_t, z_t) &= \max_c \left\{ u(c) + \mathbb{E}_t \frac{V(k_{t+\Delta t}, z_{t+\Delta t}) - V(k_t, z_t)}{\Delta t} \right\}\end{aligned}$$

- Take limit $\Delta t \to 0$ and drop time subscripts:

$$\rho V(k, z) = \max_c \left\{ u(c) + \mathbb{E}\frac{dV(k, z)}{dt} \right\}$$

- What remains? Characterizing continuation value $\frac{d}{dt}V(k, z)$ (i.e., characterizing how process dV evolves)

- The generator $\mathcal{A}$ is exactly the answer to this question! I.e.,

$$\begin{aligned}\mathbb{E}\frac{dV(k,z)}{dt} &= \mathcal{A}V(k,z) \\ &= \Big(F(k,z) - \delta z - c\Big)\partial_k V(k,z) - \theta z \partial_z V(k,z) + \frac{\sigma^2}{2}\partial_{zz}V(k,z)\end{aligned}$$

- Therefore, we arrive at the Hamilton-Jacobi-Bellman equation

$$\begin{aligned}\rho V(k,z) = \max_c \Big\{ & u(c) + \Big(F(k,z) - \delta z - c\Big)\partial_k V(k,z) \\ & - \theta z \partial_z V(k,z) + \frac{\sigma^2}{2}\partial_{zz}V(k,z)\Big\}\end{aligned}$$

with first-order condition

$$u'(c(k,z)) = \partial_k V(k,z)$$

4. Productivity as a Poisson process

- Next, consider Poisson process for $\{z_t\}$ with $z_t \in \{z^L, z^H\}$

- Generator now given by

$$\mathcal{A}V(k,z^j) = \Big(F(k,z) - \delta z - c\Big)\partial_k V(k,z) + \lambda\Big[V(k,z^{-j}) - V(k,z^j)\Big]$$

- Note: derivation of HJB exactly as before *up to* characterizing $\mathbb{E}[dV]$

- With Poisson process, HJB becomes

$$\rho V(k,z^j) = \max_c \Big\{u(c) + \Big(F(k,z) - \delta z - c\Big)\partial_k V(k,z) \\ + \lambda\Big[V(k,z^{-j}) - V(k,z^j)\Big]\Big\}$$

with first-order condition

$$u'(c(k,z^j)) = \partial_k V(k,z^j)$$

Part 3: Applications

1. Consumption-savings with income fluctuations

- Economy is populated by representative household that faces income risk
- Household accumulates wealth according to

$$\dot{a}_t = ra_t + e^{z_t} - c_t$$

subject to borrowing constraint $a_t \geq 0$

- Preferences again: $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$
- Income follows diffusion process: $dy_t = -\theta y_t dt + \sigma dB_t$
- Away from borrowing constraint, HJB given by

$$\rho V = \max_c \left\{ u(c) + (ra + e^z - c)V_a - \theta z V_z + \frac{\sigma^2}{2} V_{zz} \right\}$$

with $V_a = \partial_a V(a, z)$ (you'll see this often)

2. Firm profit maximization

- Firm maximizes NPV of profit: $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-rt} \pi_t dt$

- For now, profit given by: $\pi_t = A_t n_t^\alpha - w_t n_t$ where firm chooses labor n_t
  Assume $\alpha < 1$, so this is a decreasing-returns production function

- Firm is small and takes wage $\{w_t\}$ as given (wages determined in general equilibrium)

- Productivity follows two-state high-low process, with $A_t \in \{A^{\text{rec}}, A^{\text{boom}}\}$

- Recursive representation: A is only state variable, $w_t = w(A_t)$

$$rV(A^{\text{boom}}) = \max_n \left\{ A^{\text{boom}} n^\alpha - w(A^{\text{boom}})n + \lambda \left[V(A^{\text{rec}}) - V(A^{\text{boom}}) \right] \right\}$$

  with first-order condition

$$n = \left(\frac{\alpha A^j}{w(A^j)} \right)^{\frac{1}{1-\alpha}}$$

3. Capital investment with adjustment cost

- Firm again maximizes NPV of profit: $V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-rt} \pi_t dt$

- Now: let $\psi(\cdot)$ denote an adjustment cost

$$
\begin{aligned}
\pi_t &= e^{A_t} k_t^\alpha - Q_t \iota_t - \psi(\iota_t, k_t) \\
dk_t &= (\iota_t - \delta k_t) dt \\
dA_t &= -\theta A_t dt + \sigma dB_t
\end{aligned}
$$

- Firm is small and takes capital price as given

- Recursive representation in terms of (k, A), i.e., $Q_t = Q(k_t, A_t)$

$$
\begin{aligned}
rV(k, A) = \max_\iota \Big\{ & e^{A_t} k_t^\alpha - Q(A)\iota_t - \psi(\iota_t, k_t) + (\iota - \delta k)\partial_k V(k, A) \\
& - \theta A \partial_A V(k, A) + \frac{\sigma^2}{2} \partial_{AA} V(k, A) \Big\}
\end{aligned}
$$

with first-order condition: $Q(k, A) + \partial_\iota \psi(\iota(k, A), k) = \partial_k V(k, A)$

4. Investing in stocks

- Suppose you optimize lifetime utility $V_0 = \mathbb{E}_0 \int_0^\infty u(c_t)dt$
- You can trade two assets: riskfree bond (return rdt), and risky stock

$$dR = (r + \pi)dt + \sigma dB, \text{ where } \pi \text{ is the equity premium}$$

- You have wealth a_t and invest a share θ_t in stocks, thus,

$$da_t = \theta_t a_t dR_t + (1 - \theta_t)a_t r_t dt + y - c_t$$

or, rearranging, and dropping t subscripts

$$da = ra + \theta a \pi dt + y - c + \theta a \sigma dB$$

- HJB becomes:

$$\rho V(a) = \max_{c,\theta} \left\{ u(c) + (ra + \theta a \pi dt + y - c)V'(a) + \frac{1}{2}(\sigma \theta a)^2 V''(a) \right\}$$

with FOCs: (i) $u'(c) = V'(a)$ and (ii) $\theta = -\frac{\pi}{\sigma^2}\frac{V'(a)}{aV''(a)}$

5. Real Business Cycles

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