

# State Fidelity

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## 1 Introduction

Assume we are working with pure states, and we have two density matrices:  $\rho$  and  $\sigma$ .

$$\rho = |\psi_\rho\rangle \langle \psi_\rho|, \sigma = |\psi_\sigma\rangle \langle \psi_\sigma|.$$

$\rho$  is a rank 1 matrix or projector. The eigenvalues of  $\rho$  are one +1 with eigenvector  $|\psi\rangle$  and the rest are 0 with eigenvectors not  $|\psi\rangle$ . The algebraic multiplicity of  $\rho$  is  $n$ , and the geometric multiplicity of  $\rho$  is also  $n$ .

$$m_{a,\rho}(1) = 1, m_{g,\rho}(1) = 1, m_{a,\rho}(0) = n - 1, m_{g,\rho}(0) = n - 1.$$

$$\begin{aligned} \rho\sigma\rho &= |\langle \psi_\rho | \psi_\sigma \rangle|^2 |\psi_\rho\rangle \langle \psi_\rho| \\ &= \sum_i \alpha_i |\alpha_i\rangle \langle \alpha_i| \end{aligned}$$

Another way to see this,  $\alpha$  is the eigenvalue of  $\rho\sigma\rho$ , since  $\rho\sigma\rho$  is Hermitian, we can use Schur decomposition on  $\rho\sigma\rho$ .

$$\rho\sigma\rho = QDQ^\dagger$$

Diagonal elements of  $D$  contain the eigenvalues of  $\rho\sigma\rho$ , the columns of  $Q$  are  $|\alpha\rangle$ , and the rows of  $Q^\dagger$  are  $\langle \alpha|$ .

$$\sqrt{\rho\sigma\rho} = \sum_i \sqrt{\alpha_i} |\alpha_i\rangle \langle \alpha_i|$$

$\sqrt{\alpha}$  is basically the overlap between  $|\psi_\rho\rangle$  and  $|\psi_\sigma\rangle$ .  $|\psi\rangle \langle \psi|$  is an rank 1 projector. If we want to find out the overlap(diagonal elements of  $D$ ) between two density matrices  $\rho$  and  $\sigma$ , then we have to sum over the overlap  $|\langle \psi_\rho | \psi_\sigma \rangle|$ .

$$\text{trace}(\sqrt{\rho\sigma\rho}) = \sum_i \sqrt{\alpha_i}.$$