Introduction to Lie Algebra

Henry Lin

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1 Introduction

Two important sub-classes of linear operators in quantum mechanics are unitary and non-Hermitian operators. **Hermitian** and **symmetric** operator both satisfy the property that the eigenvalues are real. **Unitary** and **orthogonal** operator are norm-preserving operator. In the context of quantum mechanics, Hermitian operator is important because they are corresponds to the real-observable. Unitary operator is important because it preserves the probabilities of the quantum state that we are trying to evolve which means we are working with a closed system.

$$H = H^\dagger \quad \text{Hermitian} \tag{1}$$
$$H^\dagger = H^{-1} \quad \text{Unitary} \tag{2}$$
$$H^T = H \quad \text{Symmetric} \tag{3}$$
$$H^T = H^{-1} \quad \text{Orthogonal} \tag{4}$$
$$H^\dagger H = HH^\dagger \quad \text{Normal} \tag{5}$$

A unitary operator can always be written as:

$$U = e^{iH} \tag{6}$$

We say that H is a generator for U.
The space of all such unitary operators forms the so-called special unitary group $SU(N)$, where for qubit systems we have $N = 2^n$ with N equals to the dimension of the group and n the number of qubits.

1.1 Lie Algebras

An algebra is a vector space equipped with a bilinear operation. A Lie algebra g is a special case where the bilinear operation behaves like a commutator. In particular, the bilinear operation: $g \times g \to g$ needs to satisfy the following conditions:

1. $[x, x] = 0 \quad \forall x \in g \quad$ (alternativity)

2. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \quad \forall x,y,z \in g$ (Jacobi Identity)

3. $[x,y]=-[y,x] \quad \forall x,y \in g$ anti-commutativity

The bilinear operation that we defined here is the commutator relation $[\cdot,\cdot]$. Note also that we are talking about a vector space in the mathematical sense, and the elements ("vectors") in g are actually operators (matrices) in our case looking at quantum physics.

One very relevant Lie algebra for us is the special unitary algebra , the space of $N \times N$ skew-Hermitian matrices with trace zero. The fact that we look at skew-Hermitian ($H^\dagger = -H$) instead of Hermitian ($H^\dagger = H$) matrices is a technical detail.

The commutator of two Hermitian operators H_1 and H_2 is always skew-Hermitian.

$$\begin{aligned}
[H_1,H_2] &= H_1H_2 - H_2H_1 \\
([H_1,H_2])^\dagger &= (H_1H_2 - H_2H_1)^\dagger \\
&= H_2H_1 - H_1H2 \\
([H_1,H_2])^\dagger &= -[H_1,H_2]
\end{aligned}$$

This shows that the commutator of two Hermitian operators are always skew-Hermitian. This implies that Hermitian operators are not closed under commutation(commutation mapped Hermitian matrices outside the Herimitian matrcies). However, skew-Hermitian is closed under commutation, thus, skew-Hermitian mactrices form a unitary algebra $u(N)$. If the skew-hermitian also satisfy the additional property of traceless, then it is $su(N)$. They generate the unitary group $U(N)$ and the special unitary group $SU(N)$ with determinant of 1.

The Pauli Matrcies are $\{iX, iY, iZ\}$ span the $su(2)$ algebra that we can associate with single qubit dynamic.

$$\begin{aligned}
(iX)^\dagger &= -iX \\
(iY)^\dagger &= -iY \\
(iZ)^\dagger &= -iZ
\end{aligned}$$

We can see each member is skew-hermitian, and they form special unitary algebra group. In the case of n-qubit dynamic, then we have the following:

$$su(2^n) = \text{span}(\{iX_0,\ldots,iY_0,\ldots,iX_0,\ldots,iZ_0Z_1,\ldots\})$$

Lie algebra elements "live" in the exponent of a unitary operator, and having that exponent become Hermitian instead of skew-Hermitian destroys the unitary property.

2 Relation to Lie Group

For every Unitary matrix $U \in SU(N)$, there is a real linear combination of elements $iP_j \in su(N)$ such that:

$$U = e^{i \sum_{j=1}^{N} \lambda_j P_j}$$

In other words, every unitary matrix can be written in this form. In unitary matrix $U \in SU(2^n)$ can be decomposed in a finite product of elements from a universal gate set U.

$$U = \prod_j U_j$$

for $U_j \in U$. A univeral gate set is formed exactly when the generators of its element form $su(2^n)$.

3 Dynamical Lie Algebras

Suppose we have a set of generator $\{iG_j\}$ and one natural question to ask is what of unitary evolutions they can generate. Dynamical lie algebras(DLA) ig is given by all possible nested commutators between the generators $\{iG_j\}$, until no new and linearly independent skew-Hermitian operator is generated. This is called the Lie-Closure and it is written like

$$ig = \langle iG_1, iG_2, \cdots \rangle_{Lie}$$

3.1 Example

Suppose we have $\{iX, iY\}$ and let's compute the Lie closure of $\{iX, iY\}$. We can take the commutator of $[iX, iY] \propto iZ$, so $\langle(\{iX, iY\})\rangle_{Lie} = \{iX, iY, iZ\}$. We view this like the generator of $\{iX, iY\}$ is not able to describe the full dynamics.

However, Lie closure enables to describe the full dynamics. Lie closure ensures the set that we have is closed under commutation. Lie closure basically filling the missing operator to describe all the possible dynamics.

4 Hamiltonian Symmetries

Suppose we have an observable $\hat{O}$, and $[\hat{O}, H] = 0$. This means there are some symmetries presented in the system. One of the immediate consequences of this is $[e^{i\hat{O}}, H] = 0$ and $[\hat{O}, e^{iH}] = 0$. Hence, H is invariant under any action of $e^{i\hat{O}} \in SU(2)$.

$$e^{i\hat{O}} H e^{-i\hat{O}} = H$$

Thus, H is said to be $SU(2)$ symmetric.
Another perspective on the inherent $SU(2)$ symmetry of H is that the expectation value of $\hat{O}$ with respect to any state $|\psi\rangle$ is invariant under the evolution of H.

$$\langle\psi(t)|\,\hat{O}\,|\psi(t)\rangle = \langle\psi|\,e^{itH}\hat{O}e^{-itH}\,|\psi\rangle = \langle\psi|\,e^{itH}e^{-itH}\hat{O}\,|\psi\rangle = \langle\psi|\,\hat{O}\,|\psi\rangle$$

The expected value of $\hat{O}$ is invariant under the evolution of H, $\langle\psi(t)|\,\hat{O}\,|\psi(t)\rangle = \langle\psi|\,\hat{O}\,|\psi\rangle$.
If there are some symmetries presented in the system, then this means there are some conserved quantities presented in the system as well.