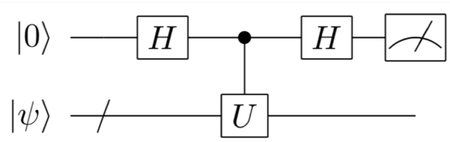
Hadamard Test

Henry Lin

June 2025

1 Introduction

Hadamard test can be used to find the real or imaginary part of a quantum state. The circuit looks like the following:



1.1 Case 1: Eigenstate of U

Suppose $|\psi\rangle$ is the eigenstate of U with eigenvalue of $e^{i\lambda}$. We can write this as the following:

$$U|\psi\rangle = e^{i\lambda}|\psi\rangle .$$

Now we will analyze the circuit from left to right. At the first stage, we have a Hadamard gate, this will evolve our state as the following:

$$H(|0\rangle \otimes |\psi\rangle) = \frac{1}{\sqrt{2}}((|0\rangle + |1\rangle) \otimes |\psi\rangle).$$

After that we have control-unitary:

$$\begin{aligned} CU(\frac{1}{\sqrt{2}}((|0\rangle + |1\rangle) \otimes |\psi\rangle)) &= \frac{1}{\sqrt{2}} \cdot (|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle) \\ &= \frac{1}{\sqrt{2}} \cdot (|0\rangle|\psi\rangle + e^{i\lambda}|1\rangle|\psi\rangle) \end{aligned}$$

Lastly, we have a Hadamard gate again:

$$H(\frac{1}{\sqrt{2}} \cdot (|0\rangle|\psi\rangle + e^{i\lambda}|1\rangle|\psi\rangle)) = (\frac{1+e^{i\lambda}}{2}|0\rangle + \frac{1-e^{i\lambda}}{2}|1\rangle) \otimes \psi$$

Basically, we have to measured the first ancilla qubit and from there we can estimate what is the value of λ, thus, figure out the eigenvalue of U. We can write the probability of measuring $|0\rangle$ and $|1\rangle$ as the following:

$$p_m = \frac{1 + (-1)^m \cos\lambda}{2}$$

with measurement results 0 or 1. If we want to If we want to estimate $\cos\lambda$ with a certain error ϵ, then we only need to sample about a polynomial number of times of the inverse of ϵ by using Chernoff bound.

1.2 Case 2: General Input State

First Hadamard:

$$H(|0\rangle \otimes |\psi\rangle) = \frac{1}{\sqrt{2}}((|0\rangle + |1\rangle) \otimes |\psi\rangle).$$

CU:

$$CU(\frac{1}{\sqrt{2}}((|0\rangle + |1\rangle) \otimes |\psi\rangle) = \frac{1}{\sqrt{2}} \cdot (|0\rangle |\psi\rangle + |1\rangle U |\psi\rangle)$$

Second Hadamard:

$$\begin{aligned} H(\frac{1}{\sqrt{2}} \cdot (|0\rangle |\psi\rangle + |1\rangle U |\psi\rangle)) &= \frac{(|0\rangle + |1\rangle) |\psi\rangle}{2} + \frac{(|0\rangle - |1\rangle) U |\psi\rangle}{2} \\ &= |0\rangle \frac{I+U}{2} |\psi\rangle + |1\rangle \frac{I-U}{2} |\psi\rangle \end{aligned}$$

Now, let's calculate the probability of measuring $|0\rangle$ for the first ancilla qubit. Let define a projective operator that does this:

$$\begin{aligned} \hat{p}_0 &= |0\rangle \langle 0| \otimes I \\ \hat{p}_1 &= |1\rangle \langle 1| \otimes I \end{aligned}$$

Now, let's use this projective operator to obtain the probabilities:

$$
\begin{aligned}
p_0 &= |(|0\rangle\langle 0| \otimes I)[|0\rangle \frac{I+U}{2}|\psi\rangle + |1\rangle \frac{I-U}{2}|\psi\rangle]|^2 \\
&= |\,|0\rangle \frac{I+U}{2}|\psi\rangle + 0|^2 \\
&= |\,|0\rangle\,|^2 |\frac{I+U}{2}|\psi\rangle\,|^2 \\
&= \frac{\langle\psi|\,(I+U)(I+U^\dagger)\,|\psi\rangle}{4} \\
&= \frac{\langle\psi|\,(2I+U+U^\dagger)\,|\psi\rangle}{4} \\
&= \frac{1+\mathrm{Re}\langle U\rangle}{2} \\
p_1 &= \frac{1-\mathrm{Re}\langle U\rangle}{2}
\end{aligned}
$$

This means we can use Hadamard Test to find the real and imaginary part of a quantum state. To obtain the imaginary part, we just have to make sure that the input state to the CU gate is $(|0\rangle - i\,|1\rangle) \otimes |\psi\rangle$. This can be achieved by applying a $S^\dagger$ after the Hadamard gate.