Quantum Phase Estimation

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1 Introduction

Suppose we are given a unitary matrix U and its eigenstate $|\psi\rangle$, then we can write this as

$$U|\psi\rangle = e^{i\lambda}|\phi\rangle .$$

ϕ is the phase of the eigenvalue $e^{i\phi}$. Our goal is to estimate ϕ, and this is the job of quantum phase estimation algorithm.

1.1 Quantum Fourier Transform

This is the goal of QFT:

$$\mathrm{QFT}|\theta\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0} e^{i2\pi\theta k}|k\rangle$$

θ here is in the binary representation, $\theta \in \{0,1\}^n$. k is in the decimal representation, $k \in [1, 2^n]$.

Quantum Fourier Transform(QFT) is one of the most important building blocks in quantum algorithms. The QFT is quantum analog of discrete Fourier transform - the main tool of digital signal processing - which is used to analyze periodic function by mapping between time and frequency representation.

Phase Gate is defined as the following:

$$S(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^k} \end{pmatrix}$$

Example:

$$S(\theta)(|0\rangle + |1\rangle) = |0\rangle + e^{i2\pi/2^\theta}|1\rangle$$

,and control-phase gate is the following:

$$\begin{aligned} CS(\theta) &= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes S(\theta) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i2\pi/2^\theta} \end{pmatrix} \end{aligned}$$

Defined $\omega_N = e^{i2\pi/N}$ and $\omega_N^N = 1$, this basically divides the unit circle into N equal sections. Example, if N = 3, then $\omega = e^{i2\pi/3}$ because $(e^{i2\pi/3})^3 = 1$, the phase ϕ of ω is $2\pi/3$. Now, we can define the discrete Fourier Transform (DFT) matrix:

$$F_N := \begin{pmatrix} \omega_N^0 & \omega_N^0 & \dots & \omega_N^0 \\ \omega_N^0 & \omega_N^1 & \dots & \omega_N^{N-1} \\ \omega_N^0 & \omega_N^2 & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_N^0 & \omega_N^{N-1} & \dots & \omega_N^{(N-1)(N-1)} \end{pmatrix}$$

QFT results in a uniform superposition, where each basis state has an additional phase. If we can prepare that uniform distribution state, then apply the inverse QFT this will give $|\theta\rangle$ in the estimation register. However, the phase now is like θk. This can be solved by applying the Unitary U, k number of times.

$$U^k |\psi\rangle = e^{i2\pi\theta k} |\psi\rangle$$

1.2 Quantum Phase Estimation(QPE)

1. Start with the state $|\psi\rangle |0\rangle$. Apply a Hadamard gate to all estimation qubits to implement the transformation

   $$|\psi\rangle |0\rangle^{\otimes K} \to |\psi\rangle \frac{1}{\sqrt{2^K}} \sum_{k=0}^{K} |k\rangle$$

   We have K number of estimation qubits.

2. Apply a ControlledSequence operation, i.e., U^{2^m} controlled on the m-th estimation qubit. This gives

   $$|\psi\rangle \frac{1}{\sqrt{2^K}} \sum_{k=0}^{K} |k\rangle \to |\psi\rangle \frac{1}{\sqrt{2^K}} \sum_{k=0}^{K} e^{i2\pi k\theta} |k\rangle$$

   k is in decimal representation, this tells us how many time that U should be apply to $|\psi\rangle$. For example, suppose K = 2, we have this state:

   $$|\psi\rangle |1\rangle |1\rangle .$$

   Since the estimation qubit is in 11, this translates to 3 in the decimal representation. We have to apply U^3 on $|\psi\rangle$, this can be model as control-U gate. If estimation qubit 0 is in 1, then apply U. If estimation qubit 1 is in 1, then apply U^2. Following this logical, if mth qubit is in 1, then we apply U^{2^m} on $|\psi\rangle$.

3. Apply the inverse quantum Fourier transform to the estimation qubits

$$|\psi\rangle \frac{1}{\sqrt{2^K}} \sum_{k=0}^{K} e^{i2\pi k\theta} |k\rangle \rightarrow |\psi\rangle |\theta\rangle$$

   $\theta \in \{0, 1\}^K$.

4. Measure the estimation qubits to recover θ.

U is the unitary that we are interested in finding the eigenvalues. From knowing the value of θ, then we can reconstruct the eigenvalue by the following step:
We know that $e^{i2\pi\theta}$ is the eigenvalue of U, and θ has to be in between $[0, 1]$, then bitstring that we measured for θ can be used to find the eigenvalue as the following:

$$e^{i2\pi\phi} = e^{i2\pi\cdot 0.\theta}$$

$0.\theta = \theta_0 \cdot 2^{-1} + \theta_1 \cdot 2^{-2} + \cdots + \theta_{n-1} \cdot 2^{-n}$.

Note, QPE founded the eigenvalue of U without diagonalizing U. However, there exists quantum algorithm by diagonalize U.

QPE algorithm also works, if the input state $|\psi\rangle$ is in the eigenstate of U. Suppose $|\psi\rangle$ is in the representation of the eigenbasis of U:

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle$$

then QPE outputs the eigenphase θ_i with probability $|c_i|^2$.