# Quantum State Tomography

Henry Lin

# Background

- Determine the quantum state
- how good the measurements are

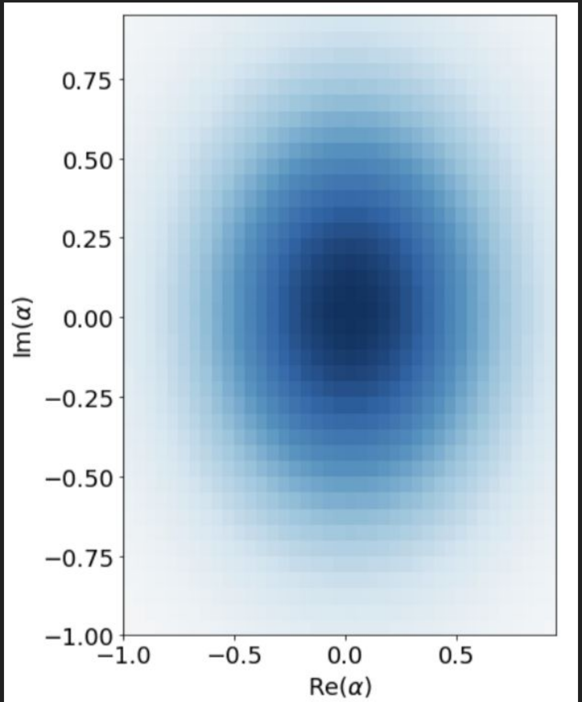
# Mp=x

$$x = \mathrm{Tr}\{\boldsymbol{M}\boldsymbol{\rho}\} = \sum_{ij} \boldsymbol{M}_{ij}\boldsymbol{\rho}_{ji}.$$

$$x = \langle\!\langle \boldsymbol{M} | \boldsymbol{\rho} \rangle\!\rangle$$

$$\vec{x} = \left( \sum_{i}^{N_{\text{exp}}} |i)\langle\!\langle \boldsymbol{M}_i| \right) |\boldsymbol{\rho}\rangle\!\rangle$$

$$\equiv \mathcal{M} |\boldsymbol{\rho}\rangle\!\rangle$$



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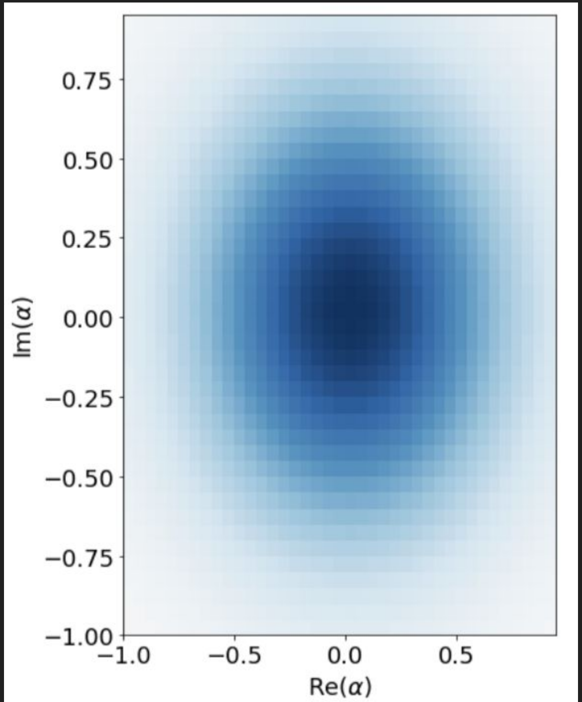

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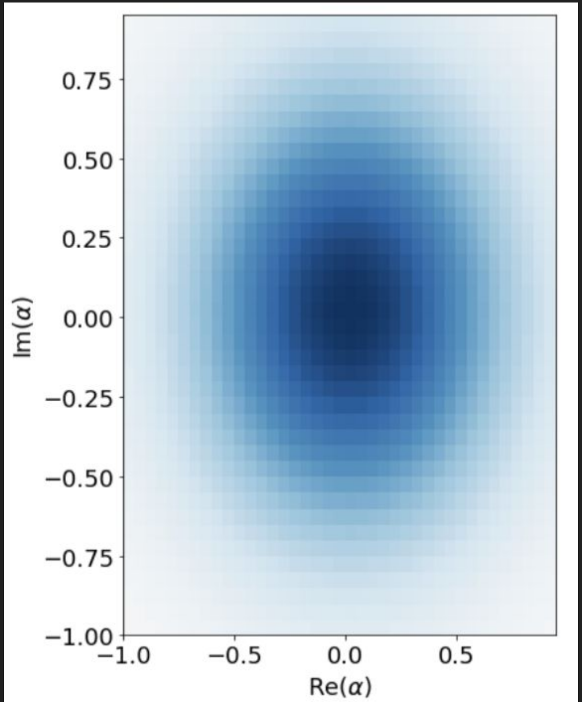

# How the curly M is constructed?

We have a grid of wigner points. Each point consists of two complex numbers. This make up our grid, then we apply the wigner operator to each of the points.

Wigner Operator:

$$\begin{aligned}\frac{2}{\pi} W_\alpha(|n\rangle\langle m|) &= \langle m| \boldsymbol{D}_\alpha \boldsymbol{\Pi} \boldsymbol{D}_{-\alpha} |n\rangle \\ &= e^{-2|\alpha|^2} \sqrt{\frac{m!}{n!}} (-1)^{m-1} (2\alpha)^{n-m} L_m^{n-m}(4|\alpha|^2),\end{aligned}$$

N and M are in the fock basis. D is the displacement operator and PI is the parity operator. The first line the Wigner operator in analytic form, and the second line is in the numerical form.

# Reconstruct the density matrix

1. Minimizing the L2 norm

$$\underset{\boldsymbol{\rho}}{\text{minimize}}\, \| \mathcal{M} | \boldsymbol{\rho} \rangle\!\rangle - \vec{x} \|$$

$$| \boldsymbol{\rho}_{\text{least-sq}} \rangle\!\rangle = \left( \mathcal{M}\mathcal{M}^T \right)^{-1} \mathcal{M}^T \vec{x}$$

$$\langle\!\langle \boldsymbol{I} | \boldsymbol{\rho} \rangle\!\rangle = 1$$

$$\begin{bmatrix} \mathcal{M}^T\mathcal{M} & \langle\!\langle I|^T \\ \langle\!\langle I| & 0 \end{bmatrix} \begin{bmatrix} \vec{\rho} \\ \vec{\lambda} \end{bmatrix} = \begin{bmatrix} \mathcal{M}\vec{b} \\ 1 \end{bmatrix}$$

# Reconstruct the density matrix

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$$\downarrow$$

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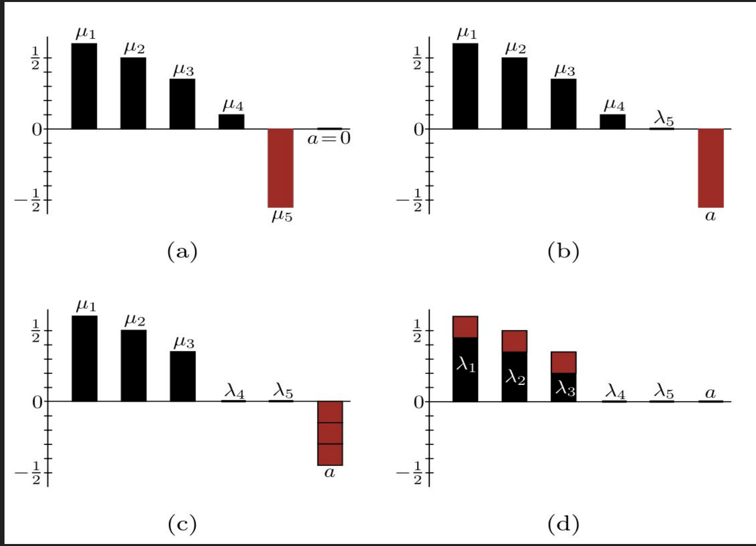
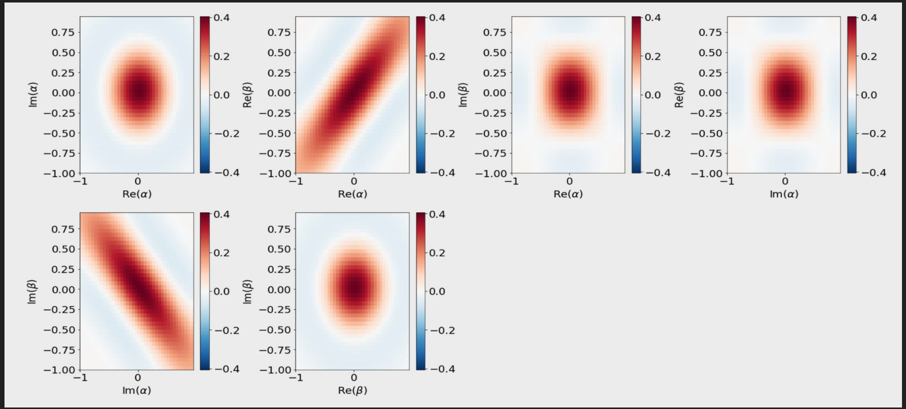
# Physicality constraint of the density matrix

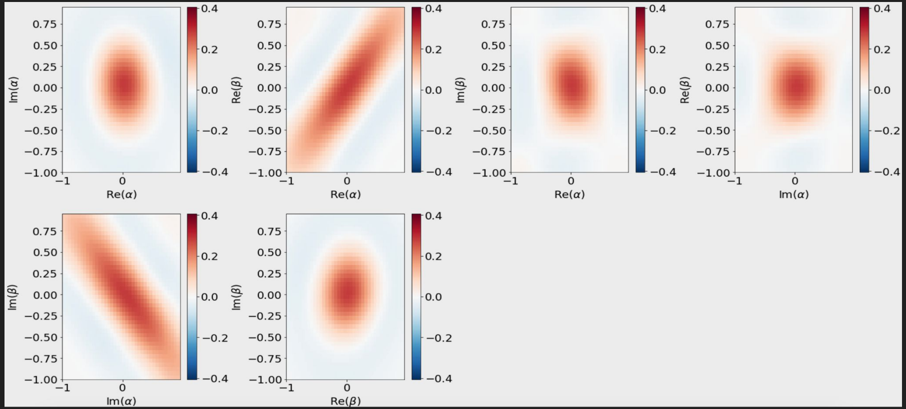
1. Trace 1 sum of the eigenvalues
2. Positive Semi-definite (eigenvalues of the density matrix must be positive and at least one eigenvalue is 0)




Efficient Method for Computing the Maximum- Likelihood Quantum State from Measurements with Additive Gaussian Noise

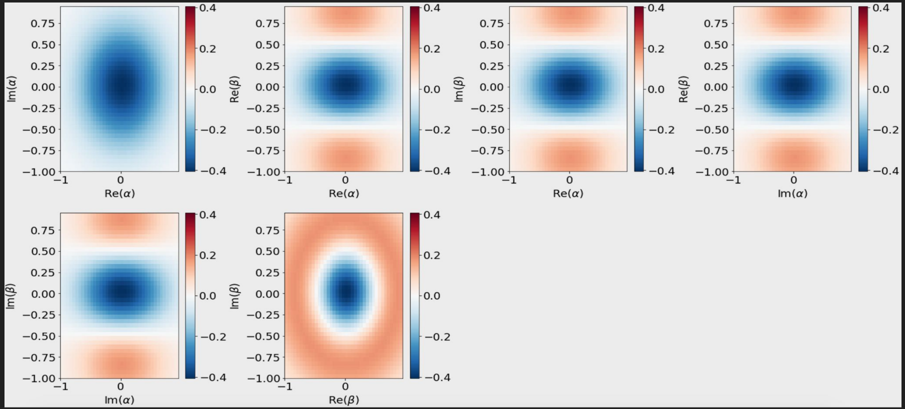
# Examples

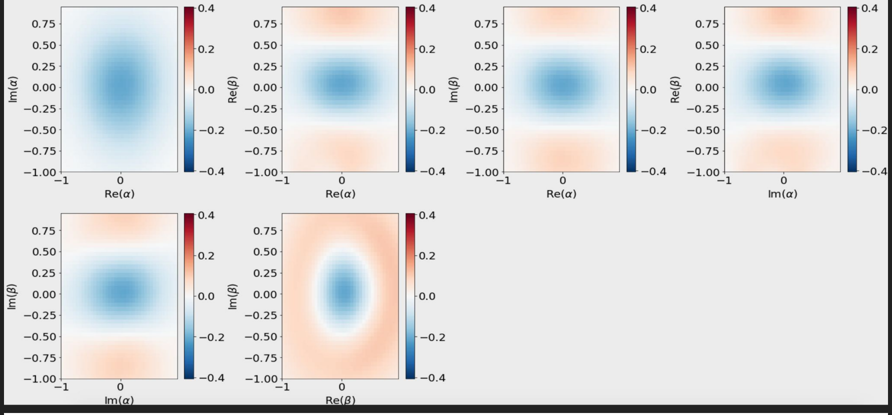



Fidelity = 0.8419223755187133

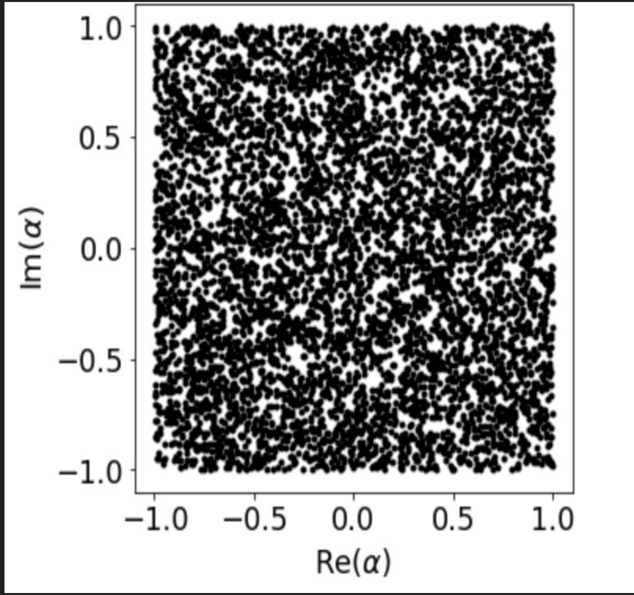
## Subject to 1% of Gaussian noise

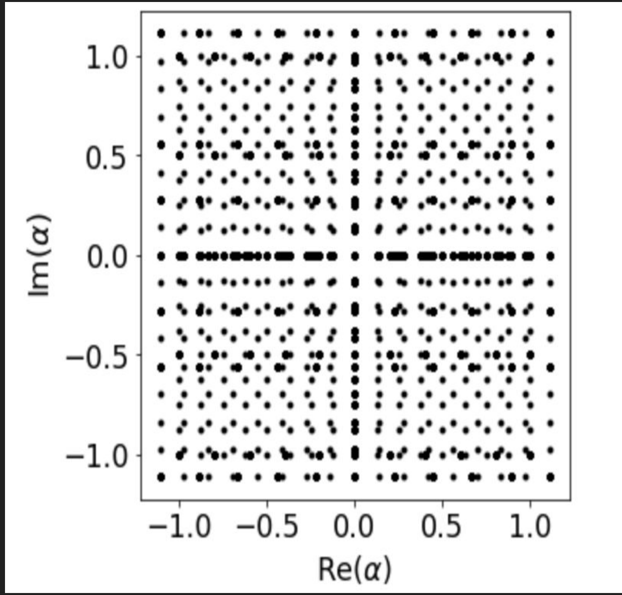



Fidelity = 0.724805167012505

# Problems in displacement array




Fidelity = 0.8633221671128446

Fidelity = 0.9665107157909298

Subject to 1% of Gaussian noise

# How to find the optimal displacement points?

- Monte Carlo Simulation
- Direct Optimization

# Monte Carlo Simulation

Randomly select points, and change the value. Subject to 1% of the Gaussian noise.

```
Best fidelity =  0.9457585646212795
Best displacement =  [((0.3967231795726567+0.2279057386279264j), (-0.6766283171068332+0.40153092509615496j)), ((0.6966779
[<matplotlib.lines.Line2D at
0x7fe6a4f18090>]WARNING:matplotlib.font_manager:findfont: Font family ['normal'] not found. Falling back to DejaVu Sans.
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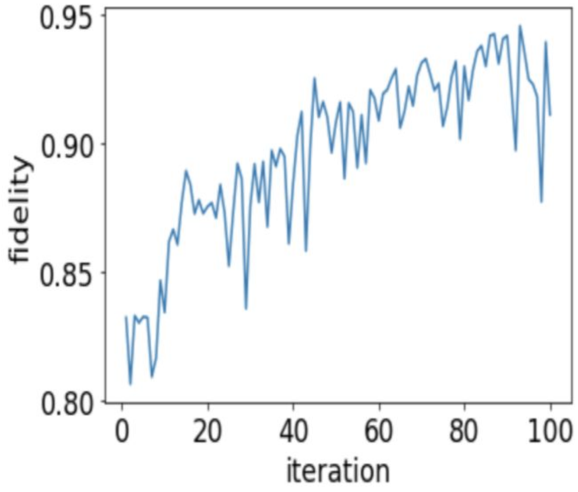

# Direct Optimization

Condition number measures the invertibility of a matrix. The lower the condition number, the better the matrix inversion will be. Therefore, we want to minimize the condition number. We can differentiate condition number with respect to the displacement.

Objective Function:

$$\kappa(\mathcal{M}) = \frac{|\lambda_{\text{max}}(\mathcal{M})|}{|\lambda_{\text{min}}(\mathcal{M})|}$$

# Gradient Descent and chain rule

Single Mode:

$$e^{-2|\alpha|^2}\sqrt{\frac{m!}{n!}}(-1)^{m-1}(2\alpha)^{n-m}L_m^{n-m}(4|\alpha|^2),$$

$$M = U\Sigma V^T$$

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MultiMode:

$$e^{-2|\alpha|^2}\sqrt{\frac{m!}{n!}}(-1)^{m-1}(2\alpha)^{n-m}L_m^{n-m}(4|\alpha|^2),$$

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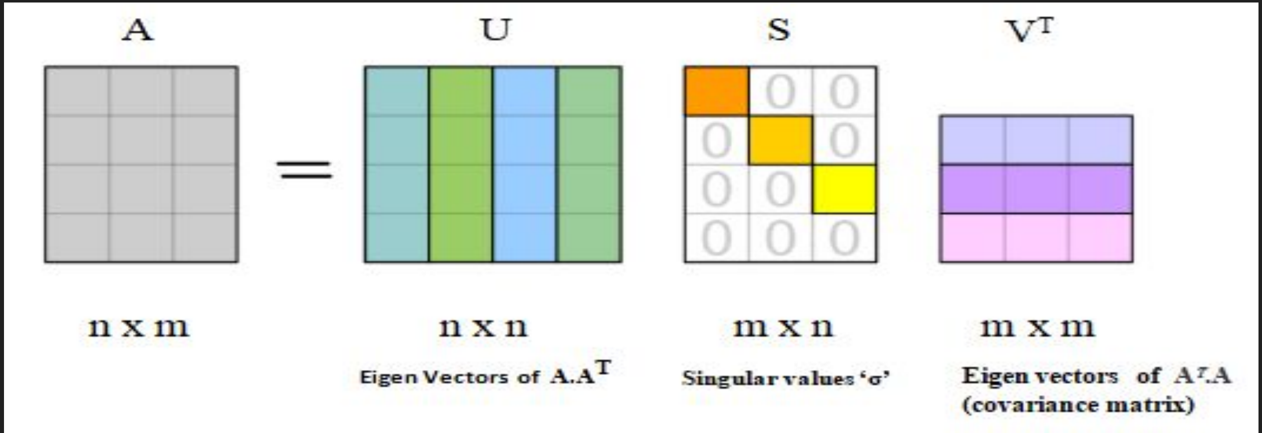
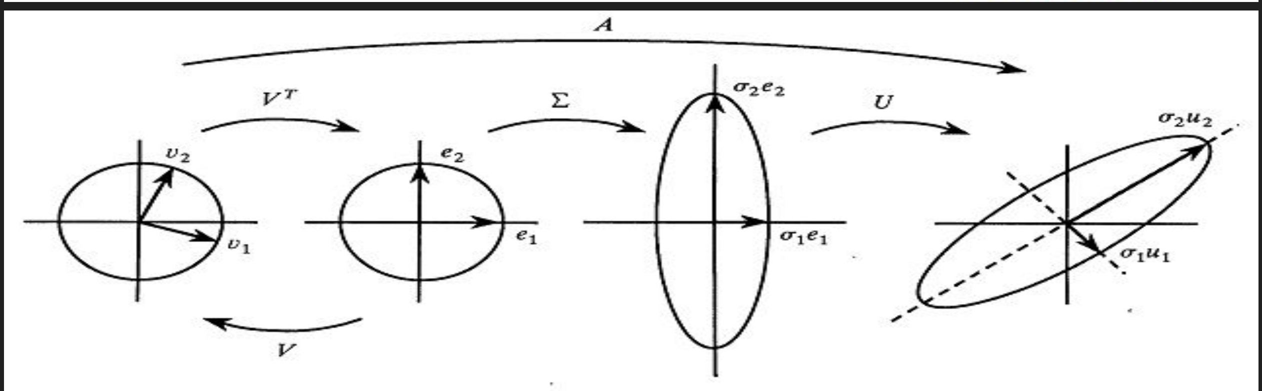
# SVD

$$M = U\Sigma V^T$$



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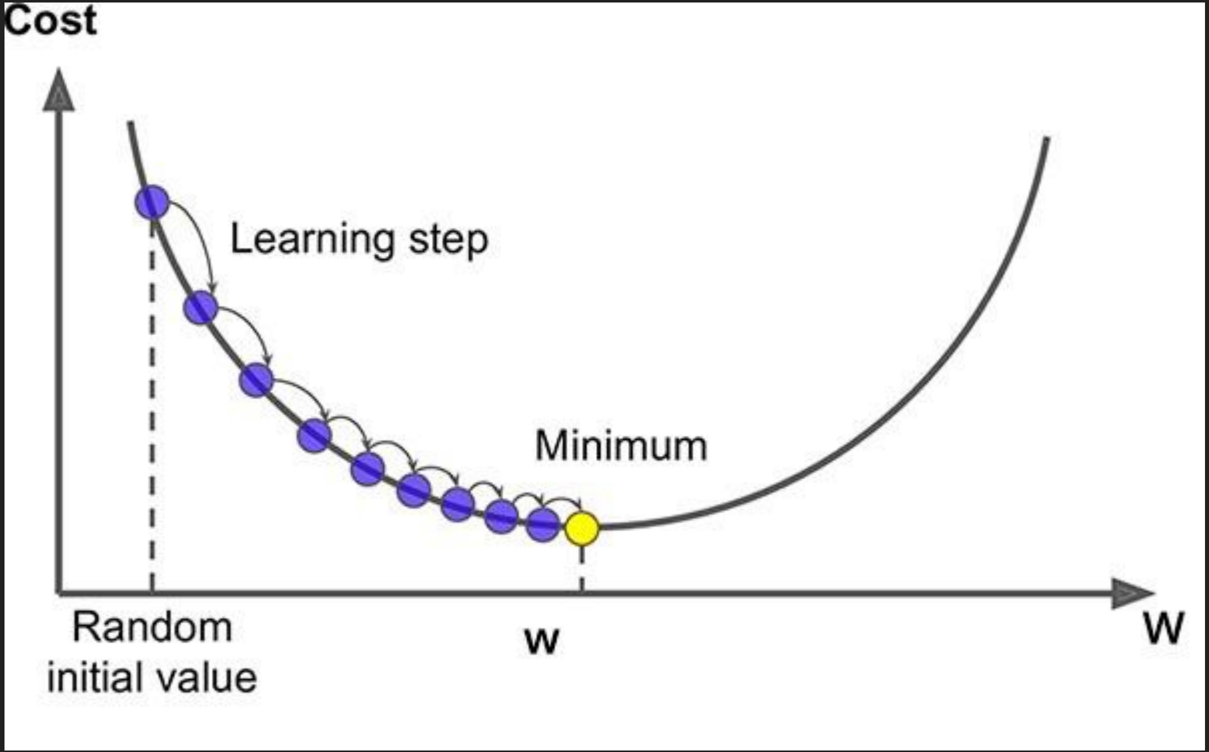
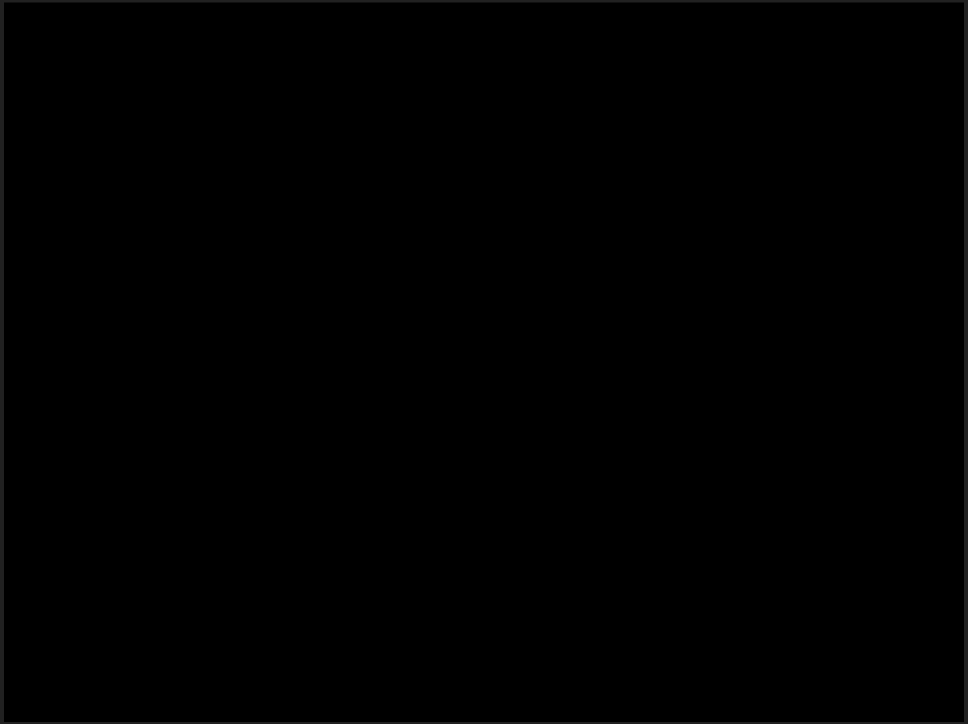
# Gradient Descent



# Demo



# Future Work

- Extend single mode to two modes
- Compare two result of the two methods: Monte Carlo and Direct Optimization

# Citations


1. Reinhold, Philip. *Controlling Error-Correctable Bosonic Qubits*. Diss. Yale University, 2019.
2. A. Smolin, J. M. Gambetta, and G. Smith. Efficient Method for Computing the Maximum- Likelihood Quantum State from Measurements with Additive Gaussian Noise. Phys. Rev. Lett., 108(7):070502 (2012). doi:10.1103/PhysRevLett.108.070502. (Cited on pages 202 and 206.)
3. K. Cahill and R. Glauber. Density Operators and Quasiprobability Distributions. Phys. Rev., 177(5):1882–1902 (1969). doi:10.1103/PhysRev.177.1882.

# Special Thanks

- Thanks to
    - Sean for being my amazing mentor
    - Chen for giving me this precious opportunity to work on this project
    - People who have helped me!