Theory of Angular Momentum

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1 Introduction

$$J = L + S$$

L is in the position space, and S is in the spin space.Now, let's just only focus on S and consider only 2 spin 1/2 particles. The total spin operator is defined below:

$$S = S_1 + S_2$$

and it can be represented as the following:

$$S = S_1 \otimes I + I \otimes S_2$$

The commutator of spin operator on different particle always commutes:

$$[S_{1x}, S_{2x}] = 0$$

In general the following will be true:

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$$

and

$$S_i \in \{\sigma_x, \sigma_y, \sigma_z, I\}$$

These are the Pauli operators. The eigenvalues of the various spin operators are denoted as follows:

$$\begin{aligned} S^2 = (S_1 + S_2)^2 &: s(s+1)\hbar^2 \\ S_z = S_{1z} + S_{2z} &: m\hbar \\ S_{1z} &: m_1\hbar \\ S_{2z} &: m_2\hbar \end{aligned}$$

1.1 Total Spin Operator

1.2 single spin $1/2$ particle

$$
\begin{aligned}
S^2 &= \sum_i S_i^2 \\
&= S_x^2 + S_y^2 + S_z^2
\end{aligned}
$$

$$
S_x = \frac{\hbar}{2}\sigma_x, \quad S_y = \frac{\hbar}{2}\sigma_y, \quad S_z = \frac{\hbar}{2}\sigma_z
$$

After substitute the definitionns back to the total spin operator, we will have the following:

$$
S^2 = \frac{3}{4}\hbar^2 I
$$

1.3 two spin $1/2$ particles

$$
\begin{aligned}
S^2 &= S_1^2 + S_2^2 + 2S_1S_2 \\
&= \frac{3}{4}\hbar^2 I + \frac{3}{4}\hbar^2 I + 2S_1S_2 \\
&= \frac{3}{2}\hbar^2 I + 2S_1S_2
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= [S_{1x}, S_{1y}, S_{1z}] \\
S_2 &= [S_{2x}, S_{2y}, S_{2z}] \\
S_1 \cdot S_2 &= S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}
\end{aligned}
$$

$$
(S_{1x}) = S_x \otimes I = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (S_{2x}) = I \otimes S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

$$
(S_{1y}) = S_y \otimes I = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad (S_{2y}) = I \otimes S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}
$$

$$
(S_{1z}) = S_z \otimes I = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (S_{2z}) = I \otimes S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
$$

$$(S_1 \cdot S_2) = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

S^2 has eigenvalues of $2\hbar^2$ with degeneracy of 3, 3-fold degenerate, $s = 1$. 0 with degeneracy of 1-fold degenerate, $s = 1$.
Eigenvalue $2\hbar^2$ has eigenvectors of

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \frac{\sqrt{2}}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Eigenvalue 0 has eigenvector of

$$\frac{\sqrt{2}}{2}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

This means $s = 1$ has 3-fold degeneracy and this implies $m = \{0, \pm 1\}$. $s = 0$ has 1-fold degeneracy and this implies $m = 0$.

1.4 More Examples

$$S^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle, \quad S_z |s, m\rangle = \hbar m |s, m\rangle, \quad \text{with } s = \frac{1}{2}, \text{and } m = \pm\frac{1}{2}.$$

The aboves are the spin operators and its eigenket and eigenvalues and in the case of one spin 1/2 particle. This is 2-dimensional Hilbert space. The basis vectors are:

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle, \quad \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

These states are labeled as $|+\rangle$ and $|-\rangle$, respectively.
We now consider the case in which our system features two spin one-half particles. For the first particle we have the triplet of spin operators $S^{(1)}$ acting on the vector space V_1 spanned by:

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle_1, \quad \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_1$$

For the second particle we have the triplet spin operators $S^{(2)}$ acting on the vector space V_2 spanned by:

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_2, \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle_2$$

We have two particles and each lives in a 2-dimensional Hilbert space. If we want to represent them jointly, then the joint Hilbert space will be:

$$V = V_1 \otimes V_2$$

The basis vectors are:

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle_2, \quad \left|\frac{1}{2},\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle_2, \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle_2, \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle_2.$$

or

$$\left|++\right\rangle, \left|+-\right\rangle, \left|-+\right\rangle, \left|--\right\rangle.$$

Since this is two spin 1/2 particle, the Hilbert space is 4 and we have 4 basis states, then the possibilities for multiplets of total spin s are:

1. Four singlets ($s = 0$), this means for each s, the values of is $m = 0$.
2. Two doublets ($s = \frac{1}{2}$), this means for each s, the values of is $m = \pm\frac{1}{2}$.
3. One doublet ($s = \frac{1}{2}, m = \pm\frac{1}{2}$) and two singlets ($s = 0, m = 0$).
4. One triplet ($s = 1, m = \pm 1, 0$) and one singlet ($s = 0, m = 0$).
5. One $s = \frac{3}{2}, m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ multiplet.

Inspecting the two spin 1/2 particles equation above, $m = \pm 1, 0$ by adding the second of ket number of both particles. This implies that we have one triplet and one singlet, $s = 1, m = \pm 1, 0$ and $s = 0, m = 0$. We can write this in the representation of direct sum as below:

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0.$$

Basically, in the tensor notation we can tell the spin of each particle and in the direct sum notation we can tell the degeneracy or algebraic multiplicity of each eigenvalue s. Geometric multiplicity of an eigenvalue is number of unique of eigenvectors associated with that eigenvalue.

$$\begin{aligned}
m = -1 &: \left|\frac{1}{2},-\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle_2 \\
m = 0 &: \left|\frac{1}{2},-\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle_2, \left|\frac{1}{2},\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle_2 \\
m = 1 &: \left|\frac{1}{2},\frac{1}{2}\right\rangle_1 \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle_2
\end{aligned}$$

For a simpler notation, we can write:

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle \to |\uparrow\rangle\,, \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle \to |\downarrow\rangle\,.$$

We then have:

$$\begin{aligned}
|1,1\rangle &= |\uparrow\uparrow\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
|1,-1\rangle &= |\downarrow\downarrow\rangle \\
|0,0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
\end{aligned}$$

In the representation of $|s,m\rangle$, s is the total angular momentum and m is the total z-projection of angular momentum.

2 Formal Theory of Angular Momenta

The following notes are mainly taken and summarized from Sakurai.

We have two representations here: one is in terms of simultaneous eigenkets of $J_1^2, J_2^2, J_{1z}, J_{2z}$ with eigenket of $|j_1 j_2; m_1 m_2\rangle$, and another one is in terms of simultaneous eigenkets of J^2, J_1^2, J_2^2, J_z with eigenket of $|j_1 j_2; jm\rangle$. Now the question is how do we go from $|j_1 j_2; m_1 m_2\rangle$ to $|j_1 j_2; jm\rangle$ or vice versa.This is very similar with the basis transformation note! We know that:

$$\sum_{m_1 m_2} |j_1 j_2; m_1 m_2\rangle \langle j_1 j_2; m_1 m_2| = I$$

because it forms the complete orthnormal bases.

$$\begin{aligned}
|j_1 j_2; jm\rangle &= \sum_{m_1 m_2} |j_1 j_2; m_1 m_2\rangle \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \\
&= \sum_{m_1 m_2} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle |j_1 j_2; m_1 m_2\rangle \\
&= \sum_{m_1 m_2} c_{m_1 m_2} |j_1 j_2; m_1 m_2\rangle
\end{aligned}$$

$c_{m_1 m_2}$ is the Clebsch–Gordan(cg) coefficient, m_1, m_2 are the eigenvalues of J_{1z}, J_{2z},respectively. j_1, j_2 are the eigenvalues of J_1, J_2,respectively. m is the eigenvalue of J_z. Since all the cg coefficients are real, this formed an orthogonal matrix ($Q^T = Q^{-1}$) which linearly transformed one basis to another basis.
Here are some important properties about cg coefficient:

1. The coefficients vanish unless, $m = m_1 + m_2$.
2. The coefficients vanish unless, $|j_1 - j_2| \leq j \leq j_1 + j_2$.

Ok, let's prove the first property.

$$(J_z - J_{1z} - J_{2z}) |j_1 j_2; jm\rangle = 0$$

because $J = J_{1z} + J_{2z}$, total spin operator. Now, let's multiply $\langle j_1 j_2; m_1 m_2|$ on the left.

$$\langle j_1 j_2; m_1 m_2| (J_z - J_{1z} - J_{2z}) |j_1 j_2; jm\rangle = (m - m_1 - m_2) \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle = 0$$

This implies $m = m_1 + m_2$.

The proof for the second property is purely counting. Why angular momenta j is lower bounded by $|j_1 - j_2|$ and upper bounded by $j_1 + j_2$? The maximum number that I can get is by adding them together, thus $j_1 + j_2$. For the lower bound, it is $|j_1 - j_2|$ because we are thinking j is like the length of two vectors adding together. A length cannot be negative. For example, $j_1 = 3$ and $j_2 = 1$, the smallest number that I can make is $3 - 2 = 1$. Basically, taking the largest j_i,sd and subtract the smallest j_i.

$$N = \sum_{j_1 - j_2}^{j_1 + j_2} (2j + 1) = (2j_1 + 1)(2j_2 + 1)$$

sadasd We are enumerating though all the possible values of j, start from the lower bound and all the way to the upper bound. This is in the representation of $|j_1 j_2; jm\rangle$.

3 Important Notes

Suppose I have two spin 1/2 particles, in the uncoupled tensor product basis and we want to write them in the coupled basis, it will be written as the following:

$$(j_1, m_1) \otimes (j_2, m_2) \to (J, M)$$

Two spin 1/2 particles, then $j_1 = 1/2$ and $j_2 = 1/2$. $m_1 = \pm 1/2$ and $m_2 = \pm 1/2$. J is the total angular momentum, $|j_1 - j_2| \leq J \leq j_1 + j_2$.
(j, m) is the uncoupled basis:

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle \approx |\uparrow\uparrow\rangle$$
$$\left|\frac{1}{2}\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{-1}{2}\right\rangle \approx |\uparrow\downarrow\rangle$$
$$\left|\frac{1}{2}\frac{-1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle \approx |\downarrow\uparrow\rangle$$
$$\left|\frac{1}{2}\frac{-1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{-1}{2}\right\rangle \approx |\downarrow\downarrow\rangle$$

and (J, M) is the coupled basis:

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle.$$

They are related by an unitary transformation.
Suppose we want to go from coupled to uncoupled basis, then the transformation is the following:

$$|J, M\rangle = \sum_{m_1, m_2} |j_1 m_1, j_2 m_2\rangle \langle j_1 m_1, j_2 m_2 | J, M\rangle$$

$\langle j_1 m_1, j_2 m_2 | J, M\rangle$ is the Clebsch-Gordan coefficients. For example,

$$|10\rangle = c_1 |\uparrow\uparrow\rangle + c_2 |\uparrow\downarrow\rangle + c_3 |\downarrow\uparrow\rangle + c_4 |\downarrow\downarrow\rangle$$

c_1, c_2, c_3, c_4 are given by the Clebsch-Gordan coefficients.

Suppose we want to find $\langle j_1 m_1 j_2 m_2 | JM\rangle$, steps to use c.g. coefficients:

1. j_1, j_2 determines which table to use.
2. m_1, m_2 determines the rows of the table.
3. J, M determines the column.
4. $x \to \sqrt{x}$.

3.1 Recursion Relations for the Clebsch–Gordan Coefficients

4 Some Notations