Spinors

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1 Motivation from physics

A light wave is consists of electro and magnetic waves and traveling in space with time, thanks to Clark Maxwell for discovering this. A vertical polarization of light wave means the E-field is oscillating up and down in the y-axis. A horizontal polarization of light wave means the E-field is oscillating in and out in the x-axis.

1.1 Jones Vectors

Jones vectors are used to describe electromagnetic wave in the physical space and it is written as below:

$$\begin{bmatrix} A^x e^{i\phi_x} \\ A^y e^{i\phi_y} \end{bmatrix}$$

The first and second components are the horizontal and vertical polarization of the light wave, respectively. A^x, A^y are the amplitudes, $e^{i\phi_x}, e^{i\phi_y}$ are the phases. $\vec{H}$ is defined to be the postive x-axis, $-\vec{H}$ is defined to be the negative x-axis, $\vec{V}$ is defined to be the positive y-axis, $-\vec{V}$ is defined to be the negative y-axis. $\vec{D}$ is defined to be 45 degrees in between $\vec{H}$ and $\vec{V}$. $\vec{A}$ is defined to be negative 45 degrees in between $\vec{H}$ and $-\vec{V}$. However, in the polarization space, $[H, V]$,$[A, D]$ are defined to be the antipodal points on the Poincare sphere. Physically, H and V are orthogonal because they're the x and y-axis. However, in polarization space, they are anti-parallel. Two cycle in space polarization space is equals to one cycle in the physical space which makes it a spinors.

2 Polarization of light wave, SU(2), S(3), O(3), SO(3), and U(2)

1. $U(3), O(3)$: all the **unitary/orthogonal matrices** in a $3D$ space. It contains both Rotations and Reflections. However, the determinant of rotational matrix is 1 and the determinant of reflection matrix is -1.

2. $SU(3), SO(3)$: all the **special unitary/orthogonal matrices** in a $3D$ space that also have determinant of 1. This restricts us to only rotational matrices, and not reflection matrices.

2.1 Polarization of light wave

There are H, V, L, R, A, D state, they are correspond to horizontal, vertical, left-circular, right-circular, anti-diagonal, diagonal polarized light. $[H, V], [L, R], [A, D]$ are antipodal points. They are the 6 antipodal points on the sphere in the polarization space.

In real life, we can use physical devices such as polarizers and waveplates to represent the transformation of Jones vectors. In the mathematical world, they are Jones matrices.

Physical device waveplates basically rotates one polarization into another through a phase delay. This is because when light travel through a crystal it slows down. After exiting the crystal, the phase will be different because the speed it experienced during the crystal compared with light travel in vacuum. Birefringent crystal: before entering the crystal both horizontal and vertical components of the lightwave are alighted, after exiting the crystal, the vertical component's phase will be different now compared with no crystal.

2.2 Why Jones vectors are spinors

We have physical and polarization spaces. Here we are considering only the eletrical field and not the magnetic field. Rotating θ in the physical space is the same as rotating 2θ in the polarization space. The definition of spinors is if we rotation it 1 cycle or 360 degrees, then it will have a phase of -1. If we rotate the polarization space by one cycle, it will have a phase of -1, thus, it is a spinors $\rightarrow$ angle-doubling relationship between the physical and polarization space.

2.3 Visualization Sphere

We can have visualization sphere to see all the possible states and evolutions of the states. In different context this sphere is named differently. In the context of polarizations, this sphere is called Poincare sphere. In the context of quantum states, this sphere is called Bloch sphere. In the context of complex projective line, this sphere is called Riemann sphere.

In summary:

(a) spinors is a member of complex projective line.

(b) pair of complex numbers $[\alpha, \beta]^T$

(c) overall scaling/phase ignored: $Ae^{i\phi}[\alpha, \beta]^T$

(d) only the ratio β/α matters.

3 Pauli vectors and Pauli matrices

Two reflection is equals to one rotation.

3.1 Equivalence of Quaternions, SU(2), and Sigma matrices

3.1.1 Quaternion

A Quaternion consists of 3 complex units and four real numbers.

$$q = a + bi + cj + dk$$

It follows the below rules:

$$i^2 = -1, \quad j^2 = -1, \quad k^2 = -1$$

$$\begin{aligned} ij &= k \\ jk &= i \\ ki &= j \end{aligned}$$

Quaternion units are anti-commutative.

$$\begin{aligned} ij &= -ji \\ jk &= -kj \\ ki &= -ki \end{aligned}$$

$$qq^{-1} = 1$$

3.1.2 Pauli matrices

$$\sigma_x^2 = I, \quad \sigma_y^2 = I, \quad \sigma_z^2 = I.$$

$$\begin{aligned} \sigma_x \sigma_y &= -\sigma_y \sigma_x \\ \sigma_y \sigma_z &= -\sigma_z \sigma_y \\ \sigma_z \sigma_x &= -\sigma_x \sigma_z \end{aligned}$$

Pauli matrices are also anti-commutative.

$$\sigma_i \sigma_j = \begin{cases} +1 & \textbf{if } i = j \\ -1 & \textbf{if } i \neq j \end{cases} \tag{1}$$

3.1.3 connections between quaternion and Pauli matrices

$$i \leftrightarrow -\sigma_y\sigma_z$$
$$j \leftrightarrow -\sigma_z\sigma_x$$
$$k \leftrightarrow -\sigma_x\sigma_y$$

See i as the x-axis and $\sigma_y\sigma_z$ is the yz-plane which is perpendicular to the x-axis. This holds for j and k as well. This means:

$$i^2 = j^2 = k^2 = ijk = -1$$

and holds for the equivalent Pauli matrices as well. They are isomorphic.

4 Pauli matrices and Spinors

we have Pauli matrices: $\sigma_x, \sigma_y, \sigma_z$ and a vector can be written as the following:

$$\vec{v} = xi + yj + zk$$

Now, replace the basis vectors with Pauli matrices:

$$\begin{aligned} v &= x\sigma_x + y\sigma_y + z\sigma_z \\ &= \begin{bmatrix} z & x - yi \\ x + yi & -z \end{bmatrix} \end{aligned}$$

This is the Pauli vector. Suppose we want to rotate Pauli vector by some angle θ, then:

$$SU(2)vSU(2)^\dagger$$

We can factor v into spinors, then:

$$SU(2)\begin{bmatrix} \zeta^1 \\ \zeta^2 \end{bmatrix}\begin{bmatrix} -\zeta^1 & \zeta^2 \end{bmatrix}SU(2)^\dagger$$

However, each $SU(2)$ only performs half rotation. Spinors only rotate only half as the vector does. If the Pauli vector has determinant of 0, then it is factorizable into spinors. If it has determinant of non-zero, then it is non-factorizable. Pauli vector $\rightarrow$ two Pauli spinors. summary:

(a) Pair of complex numbers ζ^1, ζ^2.
(b) Pauli spinors are obtained from factoring Pauli vector.
(c) multiple solutions from factoring.
(d) ratio $\frac{\zeta^1}{\zeta^2}$ is unique for a given Pauli vector.
(e) If det(V) = 0, then it can be factored into Pauli spinors. If $\det(V) \neq 0$, then we can split V into 4 components, then do factoring on these 4 matrices.
(f) column and row spinor only rotated with one SU(2).

5 Lie algebra and Lie group

Group is discrete transformation, and Lie group is continuous transformation. For example, SU(2) is a member of Lie group. Suppose we have s as the generator, and the Lie group will be obtained through exponential map, $e^{\theta s}$. This creates an rotational matrix. Interestingly, we can also take the exponential map of the derivative operator and times a function $f(x)$ and it is written as the following:

$$e^{a\frac{d}{dx}}(f(x)) = f(x+a)$$

Therefore, $e^{a\frac{d}{dx}}$ will be the translation operator. For example, the momentum operator in quantum mechanics:

$$p_x = -i\hbar\frac{d}{dx}$$

is the momentum is the generator of translation. In general, the following will not be true:

$$e^A e^B \neq e^{A+B}$$

The taylor series of both sides are written below:

$$\begin{aligned} e^A e^B &= I + A + B + \frac{1}{2}(A^2 + 2AB + B^2) + \dots \\ e^{A+B} &= I + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \dots \end{aligned}$$

If $[A, B] = 0$, then the above equation is true. In other words, if the generators A and B communtes, then the above equation is true. Google BCH formula for the detail of this. Now, let's focus on the Lie algebra $so(3)$:

$$g_{xy} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g_{yz} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad g_{zx} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

How to obtain all the generators? We can use Lie bracket to obtain all of them. Lie Bracket is basically the commutator of the two of the generators.

$$\begin{aligned} [g_{xy}, g_{yz}] &= g_{zx}, \\ [g_{zx}, g_{xy}] &= g_{yz}, \\ [g_{yz}, g_{zx}] &= g_{xy}. \end{aligned}$$

Keep taking nested commutator until we get repeated generator. Here is more abstract definition of Lie bracket:

(a) Alternating: $[x, x] = 0$

(b) Jacobi Identity: $[x,[y,z]]+[z,[x,y]]+[t,[z,x]]=0$

Note, in so(3) the above Lie brackets are true. However, in general it will not be true. Suppose we have a set of generators:

$$\{g_1, g_2, \ldots, g_n\}$$

The commutator:

$$[g_i, g_j] \neq g_k$$

instead:

$$[g_i, g_j] = \sum_k f_{ij}^k g_k$$

f_{ij}^k is the structure coefficients.

Lie group SO(3) is continuous which forms a continuous curved space and it is also called manifold. Each point on the space will be one specific SO(3) matrix and the trajectory it formed to create these points will be increasing the θ of its corresponding generator. The generator of that path will located at $\theta = 0$.

$$\frac{d}{dx} e^{\theta M}|_{\theta=0} = M$$

There are infinite number of curves through the identity with the same tangent vector. Tangent vector a generates the unique curve $A(\theta) = e^{\theta a}$. Sum of tangent vectors is still a tangent vector and scalar multiple of tangent vectors is still a tangent vector.

One important thing: the generators of the Lie algebra are like basis vectors.

6 Irreducible Representations of SU(2)

Group Representation = function ρ from group member in G to an invertible $n \times n$ matrix.

$$\rho : SU(2) \to GL(n)$$
$$SU(2) : \{U \in \mathcal{C}^{2\times 2} : U^{-1} = U^\dagger, det(U) = +1\}$$

$$\tilde{\rho} : SO(3) \to GL(n)$$
$$SO(3) : \{U \in \mathcal{R}^{3\times 3} : U^{-1} = U^T, det(R) = +1\}$$

Reducible: If an representation can be written as the direct sum, then it is reducible.
Irreducible: If an representation cannot be written as the direct sum, then it is irreducible.

Let's focus on spin 1/2 representation of su(2), we know the generators are:

$$
\begin{aligned}
g_{yz} &= -\frac{1}{2}\sigma_y\sigma_z = -\frac{1}{2}\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}\\
g_{zx} &= -\frac{1}{2}\sigma_z\sigma_x = -\frac{1}{2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\\
g_{xy} &= -\frac{1}{2}\sigma_x\sigma_y. = -\frac{1}{2}\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}
\end{aligned}
$$

They satisified the traceless property and anti-symmetric property. Now, we have ladder operators: lowering and raising operators.

$$g_+ = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \quad g_- = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

$$g_+ \left|\uparrow\right\rangle = 0, \quad g_+ \left|\downarrow\right\rangle = \left|\uparrow\right\rangle, \quad g_- \left|\downarrow\right\rangle = 0, \quad g_- \left|\uparrow\right\rangle = \left|\downarrow\right\rangle.$$

Ok, we can rewrite both ladder operators in terms of the generators:

$$
\begin{aligned}
g_+ = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} &= \frac{1}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\\
&= -i\frac{1}{2}\begin{bmatrix}0 & i\\ i & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\\
g_+ &= ig_{yz} - g_{zx}
\end{aligned}
$$

$$
\begin{aligned}
g_- = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} &= \frac{1}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\\
&= -i\frac{1}{2}\begin{bmatrix}0 & i\\ i & 0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\\
g_- &= ig_{yz} + g_{zx}
\end{aligned}
$$

$$g_z = \begin{bmatrix}1/2 & 0\\ 0 & -1/2\end{bmatrix}$$

We have these 3 operators:

$$g_+ = ig_{yz} - g_{zx}, \quad g_- = ig_{yz} + g_{zx}, \quad g_z = ig_{xy}.$$

bilinearity of commutator/Lie Bracket:

$$[Z, aX + bY] = a[Z, X] + b[Z, Y]$$
$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$

both directions work and this is why it is called bilinear.

$$[g_+, g_-] = 2i[g_{yz}, g_{zx}] = 2ig_{xy} = 2g_z$$
$$[g_z, g_+] = +g_+$$
$$[g_z, g_-] = -g_-$$

This creates new basis.

$$g_z |m\rangle = m |m\rangle$$
$$g_+ |m\rangle = \alpha |m+1\rangle$$
$$g_- |m\rangle = \beta |m-1\rangle$$

$\alpha = \sqrt{j(j+1) - m(m+1)}$ and $\beta = \sqrt{j(j+1) - m(m-1)}$. j is the spin or highest eigenvalue of the particle.

7 Tensor Product Representations of su(2) [Clebsch-Gordan coefficients]

Two operations that affect the dimensions: tensor product $\otimes$ and direct sum $\oplus$. $\otimes$: dim(i),dim(j) $\rightarrow$ dim(ij), whereas, $\oplus$: dim(i),dim(j) $\rightarrow$ dim$(i + j)$.
Suppose we have linear map L and K, we can construct the direct sum by putting L and K on the block diagonal. L on the left corner K on the right corner. Lie algebra only understand Lie bracket, it is the only operation that is defined on the Lie algebra group.

7.1 How to find irreducible representations of coupled basis?

Suppose we have 3 spin 1/2 particles written as $SU(2) \otimes SU(2) \otimes SU(2)$, and the dimension will be 8×8 and this implies it has 8 basis. We have the following member of Lie group:

$$A(t) = e^{at}, \quad B(t) = e^{bt}, \quad C(t) = e^{ct}.$$

We can find all the generators of the Lie algebra by taking the derivative and set the parameter $t = 0$.

$$\frac{d}{dt}(A(t) \otimes B(t) \otimes C(t))|_{t=0} = a \otimes I \otimes I + I \otimes b \otimes I + I \otimes I \otimes c$$

This is the generator of the coupled basis and it consists of 3 generators of individual particle. The above equation holds, only if a, b, c are the member of the Lie algebra or the generators. We know for spin-1/2 particle, we have the following basis:

$$\begin{aligned} g_z &\to g_z \otimes I \otimes I + I \otimes g_z \otimes I + I \otimes I \otimes g_z \\ g_+ &\to g_+ \otimes I \otimes I + I \otimes g_+ \otimes I + I \otimes I \otimes g_+ \\ g_- &\to g_- \otimes I \otimes I + I \otimes g_- \otimes I + I \otimes I \otimes g_- \end{aligned}$$

We can use g_z to retrieve the eigenvalue, and g_+ and g_z to retrieve eigenstates. In this case:

$$g_z \left|m_1 m_2 m_3\right\rangle = (m_1 + m_2 + m_3) \left|m_1 m_2 m_3\right\rangle$$

We know the highest weight vector will be $\left|\uparrow\uparrow\uparrow\right\rangle$:

$$\begin{aligned} g_z \left|\uparrow\uparrow\uparrow\right\rangle &= \frac{3}{2} \left|\uparrow\uparrow\uparrow\right\rangle \\ g_+ \left|\uparrow\uparrow\uparrow\right\rangle &= 0 \end{aligned}$$

Now, starts with the highest weigh vector and use lowering operator to build the rest of the eigenvectors:

$$\begin{aligned} g_- \left|\uparrow\uparrow\uparrow\right\rangle &= \alpha(\left|\downarrow\uparrow\uparrow\right\rangle + \left|\uparrow\downarrow\uparrow\right\rangle + \left|\uparrow\uparrow\downarrow\right\rangle) \\ g_-\alpha(\left|\downarrow\uparrow\uparrow\right\rangle + \left|\uparrow\downarrow\uparrow\right\rangle + \left|\uparrow\uparrow\downarrow\right\rangle) &= 2(\left|\downarrow\downarrow\uparrow\right\rangle + \left|\downarrow\uparrow\downarrow\right\rangle + \left|\uparrow\downarrow\downarrow\right\rangle) \\ g_- 2(\left|\downarrow\downarrow\uparrow\right\rangle + \left|\downarrow\uparrow\downarrow\right\rangle + \left|\uparrow\downarrow\downarrow\right\rangle) &= 6 \left|\downarrow\downarrow\downarrow\right\rangle \end{aligned}$$

Ok, now we have 4 basis vectors. This means there be more 4 more basis vectors and we have to find more highest weight vectors by using g_+. This turns out to be:

$$\begin{aligned} g_+(\left|\uparrow\uparrow\downarrow\right\rangle - \left|\uparrow\downarrow\uparrow\right\rangle) &= 0 \\ g_+(\left|\uparrow\downarrow\uparrow\right\rangle - \left|\downarrow\uparrow\uparrow\right\rangle) &= 0 \end{aligned}$$

,then we can use g_- to find the last 2 vectors:

$$\begin{aligned} g_-(\left|\uparrow\uparrow\downarrow\right\rangle - \left|\uparrow\downarrow\uparrow\right\rangle) &= \left|\downarrow\uparrow\downarrow\right\rangle - \left|\downarrow\downarrow\uparrow\right\rangle \\ g_-(\left|\uparrow\downarrow\uparrow\right\rangle - \left|\downarrow\uparrow\uparrow\right\rangle) &= \left|\uparrow\downarrow\downarrow\right\rangle - \left|\downarrow\uparrow\downarrow\right\rangle \end{aligned}$$

In sum:

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$$

We can use weight diagram to calculate the more spin 1/2 particles faster. Suppose we want to calculate $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$:

$$\begin{aligned}
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= (\frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}) \otimes \frac{1}{2} \\
&= \frac{3}{2} \otimes \frac{1}{2} \oplus \frac{1}{2} \otimes \frac{1}{2} \oplus \frac{1}{2} \otimes \frac{1}{2} \\
&= 2 \oplus 1 \oplus 1 \oplus 0 \oplus 1 \oplus 0
\end{aligned}$$

In direct sum representation, it allows states which aren't eigenstates of the operator of the number of particles, and the basis of the Fock space is composed of the basis vectors of the individual Hilbert spaces of given particle numbers.