Basis Change in Quantum Mechanics

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1 Introduction

Suppose we have $|\psi\rangle$, and we have many representations for $|\psi\rangle$. However, suppose $|\psi\rangle$ has basis $\{|u_i\rangle\}$ and basis $\{|v_i\rangle\}$. We can write $|\psi\rangle$ as the following:

$$|\psi\rangle = \sum_i c_i |u_i\rangle$$
$$|\psi\rangle = \sum_j d_j |v_j\rangle$$

We can calculate the coefficient c_i, by using the following:

$$c_i = \langle u_i|\psi\rangle$$
$$d_j = \langle v_j|\psi\rangle$$

Both basis form an orthonormal basis, and based on the closure property, the following is true:

$$\begin{aligned} I &= \sum_i |u_i\rangle \langle u_i| \\ &= \sum_j |v_j\rangle \langle v_j| \end{aligned}$$

Now, let's focused on one coefficient:

$$\begin{aligned} c_i &= \langle u_i|\psi\rangle \\ &= \langle u_i| \sum_j |v_j\rangle \langle v_j|\psi\rangle \\ &= \sum_j \langle u_i|v_j\rangle \langle v_j|\psi\rangle \\ &= \sum_j d_j \langle u_i|v_j\rangle \end{aligned}$$

We are given two basis $|u_i\rangle$ and $|v_j\rangle$, and we want to transform one representation in u or v to another representation. This means we are trying to find the coefficients of the linear combination of u or v. Let's go back to the above equation,

$$\langle u_i|v_j\rangle = S_{ij}$$

is the overlap between basis $|u_i\rangle$ and $|v_j\rangle$(assign different index variable to different variable, less confusion). Each element S_{ij} is the inner product of basis $|u_i\rangle$ and $|u_j\rangle$. This creates the linear map Gram matrix S.

I see the matrix S as a function which takes an input in the represent of basis $|u_j\rangle$ with coefficients d_j, so:

$$S(\vec{d}) \to \vec{c}$$

If we are given two basis $|u_i\rangle$ and $|v_j\rangle$, then we can create the Gram matrix or the function that does the basis transformation.

1.1 Examples

Suppose we have Z-basis and X-basis, their basis is defined below:

$$\begin{aligned} |{+z}\rangle &= [1,0]^T, \quad |{-z}\rangle = [0,1]^T. \\ |{+x}\rangle &= [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T, \quad |{-x}\rangle = [\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]^T. \end{aligned}$$

then:

$$\begin{aligned} S &= \begin{pmatrix} \langle +z|+x\rangle & \langle +z|-x\rangle \\ \langle -z|+x\rangle & \langle -z|-x\rangle \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H \end{aligned}$$