AN ANALYSIS OF THE RESTRICTED EULER PROBLEM USING SYMPLECTIC INTEGRATORS

ABSTRACT. The Three-Body Problem is far from fully solved despite centuries of effort. The restricted Euler Problem is a special case in which two bodies are fixed in place, resulting in two poisson-commuting conserved quantities, allowing the system to be fully integrable by the Liouville-Arnold theorem.

We analysed the restricted Euler problem using an order-4 symplectic integrator, which conserves the Hamiltonian. We used this integrator to simulate the restricted Euler problem and recovered known orbits from the literature.

1. INTRODUCTION TO THE RESTRICTED EULER PROBLEM

Consider a two-dimensional system and consider the z-axis plotted against the x-axis. The two masses m_+ and m_- are located at $(0, b)$ and $(0, -b)$ respectively whereas the third mass can move freely around (Ó'Mathúna, 2008). We then try to find the trajectory of the third particle based on Newton's gravitational law. As two of the masses are stationary, this problem is also called the problem of two fixed centers.

The problem is of interest because it is one of the restricted cases of the three-body problem in which the system is fully integrable, *i.e.* the system is fully analytic, allowing for prediction of orbits.

We first express the system in terms of planar prolate spheroidal coordinates:

$$x = \pm\sqrt{R^2 - b^2}\sin\sigma$$
$$z = R\cos\sigma$$

where $R \geq 0$, $\sigma \in [0, \pi]$. It is similar to the polar coordinate system but with two foci. Without loss of generality, we consider $b = 1$.

1.1. Expressing prolate spheroidal coordinates using cartesian coordinates.
In this section, we will express prolate spheroidal coordinates in terms of cartesian coordinates. Knowing the relation between both systems is useful as both are commonly used. We will focus on $x > 0$. When $x < 0$, the values of R and σ correspond to those for $|x|$.

As $\cos\sigma$ is only negative in the range $\frac{\pi}{2} < \sigma < \pi$, we have $z > 0$ for $0 < \sigma < \frac{\pi}{2}$ and $z < 0$ for $\frac{\pi}{2} < \sigma < \pi$. Due to the nature of the solution, which will be apparent below, We will consider these two cases separately, and conclude by discussing $z = 0$.

1.1.1. *Case for $z > 0$.* Using $R = \frac{z}{\cos\sigma}$, we can substitute one of the equations into the other:

$$x = \sin\sigma\sqrt{\left(\frac{z}{\cos\sigma}\right)^2 - 1}$$

$$= \sqrt{\frac{z^2 - \cos^2\sigma}{\cos^2\sigma}}\sin\sigma$$

$$x^2 = \frac{(z^2 - 1 + \sin^2\sigma)(\sin^2\sigma)}{1 - \sin^2\sigma}$$

$$x^2 = \sin^2\sigma(x^2 + z^2 - 1 + \sin^2\sigma),$$

where we have used $\cos\sigma = \sqrt{1 - \sin^2\sigma}$. This is only valid as $0 \leq \sigma \leq \frac{\pi}{2}$, which corresponds to $z > 0$. This is why we considered z separately.

By letting $u = \sin^2\sigma$, we have:

$$u^2 + (x^2 + z^2 - 1)u - x^2 = 0.$$

Using the quadratic formula, and noting that $\sqrt{u} = \sin\sigma$, we arrive at

$$\arcsin\sqrt{u} = \sigma_{z>0},$$

where

$$u = \frac{-(x^2 + z^2 - 1) + \sqrt{(x^2 + z^2 - 1)^2 + 4x^2}}{2}.$$

1.1.2. *Case for $z < 0$.* We have assumed $x > 0$ and found the solution for $z > 0$. We want to find the solution for $z < 0$ while keeping $x > 0$.

Note that $\sin(\pi - \sigma) = \sin\sigma$ and $\cos(\pi - \sigma) = -\cos\sigma$. As $x \propto \sin\sigma$ and $z \propto \cos\sigma$, by letting $\sigma_{z<0} = \pi - \sigma_{z>0}$, we can find solutions for $z < 0$.

1.1.3. *Case for $z = 0$.* We have:

$$\sigma = \begin{cases} \arcsin\sqrt{u} & \text{for } z > 0, \\ \pi - \arcsin\sqrt{u} & \text{for } z < 0, \end{cases}$$

where

$$u = \frac{-(x^2 + z^2 - 1) + \sqrt{(x^2 + z^2 - 1)^2 + 4x^2}}{2}.$$

By continuity, we have $\sigma = \frac{\pi}{2}$. Another way to think of it is that this coordinate system is the polar coordinate system being stretched in the oblate direction and mirrored at the z-axis. Therefore similar to the polar coordinate system, when $z = 0$, the angle is $\frac{\pi}{2}$.

1.2. **Listing the conserved quantities.** The conserved quantities are important to both verifying theoretical models and computational results and showing the system is integrable.

It is important to note that both linear and angular momentum are not conserved as two of the bodies are fixed in place, implying the existence of external forces. This can be illustrated by releasing the third body a distance away from the two fixed bodies. The initial linear and angular momentum are zero but the third body is attracted by the two fixed bodies and accelerates towards them.

We define two constants, g and h below (Dullin and Montgomery, 2016):

$$g = \frac{1}{2}(zp_x - xp_z)^2 + \frac{1}{2}b^2p_z^2 + bz\left(\frac{m_1}{\sqrt{(z+b)^2 + x^2}} - \frac{m_2}{\sqrt{(z-b)^2 + x^2}}\right)$$

$$h = \frac{1}{2}(p_z^2 + p_x^2) - \frac{m_1}{\sqrt{(z+b)^2 + x^2}} - \frac{m_2}{\sqrt{(z-b)^2 + x^2}},$$

where b as defined above is the distance of the two centers from the origin, and p_x and p_z are momentum in the x and z directions respectively. We take $b = 1$, and therefore:

$$g = \frac{1}{2}(zp_x - xp_z)^2 + \frac{1}{2}p_z^2 + bz\left(\frac{m_1}{\sqrt{(z+1)^2 + x^2}} - \frac{m_2}{\sqrt{(z-1)^2 + x^2}}\right)$$

$$h = \frac{1}{2}(p_z^2 + p_x^2) - \frac{m_1}{\sqrt{(z+1)^2 + x^2}} - \frac{m_2}{\sqrt{(z-1)^2 + x^2}},$$

g is a symmetry in the phase space, and h is the energy of the third particle. It can also be shown that g is a first integral of motion and the two constants commute under the Poisson bracket, and hence the system is integrable by the Liouville-Arnold theorem.

2. Choosing an integration method for numerical simulation

Below we aim to choose an integration method that can be used for a general three-body problem, so we assume all three bodies can move freely. As we are integrating a Liouville integrable system, we choose symplectic integrators. which conserves the hamiltonian. This is discussed extensively in *Computer Implementation of Symplectic Integrators and Their Applications to the N-Body Problem* (Casey, 2020), which we will use for the whole section.

Starting from $i = 1$, denote each particle by m, *i.e.* m ranges from 1 to 3. Using x and v to represent positions and velocities respectively, we first run:

$$v_m^{(i)} = v_m^{(i-1)} + c_i a(x_m^{(i-1)})dt$$

looping for each m, in order of $m = 1$, $m = 2$, then $m = 3$. c_i is an integer, which is further explained below. dt is the stepsize in the numerical simulation.

After this step, we run:

$$x_m^{(i)} = x_m^{(i-1)} + d_i v_m^{(i)}dt.$$

looping for each m, as above. We have d_i as an integer. After these two steps are run, we increase i by 1. This process repeats until i reaches n, where n is defined as the order of the symplectic integrator. The whole process is illustrated in the example below, corresponding to $n = 2$:

$$v_1^{(1)} = v^{(0)} + c_1 a(x^{(0)})dt$$

$$v_2^{(1)} = v^{(0)} + c_1 a(x^{(0)})dt$$

$$v_3^{(1)} = v^{(0)} + c_1 a(x^{(0)})dt$$

$$x_1^{(1)} = x^{(0)} + d_1 v^{(1)}dt$$

$$x_2^{(1)} = x^{(0)} + d_1 v^{(1)}dt$$

$$x_3^{(1)} = x^{(0)} + d_1 v^{(1)}dt$$

$$v_1^{(2)} = v^{(1)} + c_2 a(x^{(1)})dt$$

$$v_2^{(2)} = v^{(1)} + c_2 a(x^{(1)})dt$$

$$v_3^{(2)} = v^{(1)} + c_2 a(x^{(1)})dt$$

$$x_1^{(2)} = x^{(1)} + d_2 v^{(2)}dt$$

$$x_2^{(2)} = x^{(1)} + d_2 v^{(2)}dt$$

$$x_3^{(2)} = x^{(1)} + d_2 v^{(2)}dt$$

65   For compactness, we can put all the c_i and d_i into a vector:

$$\mathbf{c} = [c_1, c_2, ..., c_{n-1}, c_n]$$
$$\mathbf{d} = [d_1, d_2, ..., d_{n-1}, d_n].$$

66   Furthermore, we can put $\mathbf{c}$ and $\mathbf{d}$ into a matrix :

$$\mathbf{A} = \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix}.$$

67   2.1. **Types of symplectic integrators.** It has also been shown that (Casey, 2020):

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

68   leads to a First-Order Symplectic Integration Method. This is the Euler Method.
69   Meanwhile

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

70   leads to a Second-Order Symplectic Integration Method (Casey, 2020). This is the
71   Verlet Method. Also,

$$\mathbf{A} = \begin{bmatrix} \frac{7}{24} & \frac{3}{4} & \frac{-1}{24} \\ \frac{2}{3} & \frac{-2}{3} & 1 \end{bmatrix}$$

72   leads to a Third-Order Symplectic Integration Method (Casey, 2020). This will be
73   referred to as the Ruth Method. At last,

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2(2-2^{1/3})} & \frac{1-2^{1/3}}{2(2-2^{1/3})} & \frac{1-2^{1/3}}{2(2-2^{1/3})} & \frac{1}{2(2-2^{1/3})} \\ \frac{1}{2-2^{1/3}} & \frac{-2^{1/3}}{2-2^{1/3}} & \frac{1}{2-2^{1/3}} & 0 \end{bmatrix}$$

74   leads to a Fourth-Order Symplectic Integration Method (Casey, 2020). This will be
75   referred to as the Neri method.
76   In theory, a nth order symplectic method is supposed to have an error of $O(h^n)$,
77   where h is the stepsize of the system. Therefore, the error should drop as the order of
78   the method increases.

79   2.2. **Metrics for choosing an integration method.** We used several metrics to eval-
80   uate the precision of the numerical methods specified above. As we built this code with
81   the purpose of simulating the general three-body problem, these metrics were compared
82   across three different orbits, namely the Figure-8 orbit, the Bumblebee Orbit, and the
83   Moth Orbit. Each orbit was simulated with a total of $100,000$ steps, using a stepsize of
84   0.0001.

The first metric compares the error of energy to the initial energy, which can be named
as energy deviation:

$$\frac{\Delta E}{E_0} = \left| \frac{E_{\max} - E_{\min}}{E_0} \right|$$

85   where E_0 is the energy at the start of the simulation.
86   The second metric is named the momentum difference. For many systems, the initial
87   momentum is zero. Therefore, a momentum deviation cannot be well defined. Instead,
88   we will only check the momentum difference, which can be defined as:

$$\text{Max } \Delta p_x = |p_{\text{max},x} - p_{\text{min},x}|$$
$$\text{Max } \Delta p_y = |p_{\text{max},y} - p_{\text{min},y}|.$$

The third metric is the run time. In our scenario, most simulations are relatively short so it is not as important as the metrics above.

2.2.1. *A Table for comparison.* Below all the metrics are compared with each orbit. First, the Euler method has the highest energy inaccuracy by far, while having a higher

Method	Orbit	Energy Deviation	Max Δp_x	Max Δp_y	Run Time (s)
Neri (Order 4)	Figure-8	6.124×10^{-14}	1.040×10^{-13}	4.852×10^{-14}	29.60
Ruth (Order 3)	Figure-8	1.280×10^{-13}	5.307×10^{-14}	3.048×10^{-14}	25.91
Verlet (Order 2)	Figure-8	5.893×10^{-9}	3.841×10^{-14}	2.232×10^{-14}	12.31
Euler (Order 1)	Figure-8	3.601×10^{-5}	3.686×10^{-14}	3.009×10^{-14}	15.53
Neri (Order 4)	Bumblebee	8.058×10^{-3}	4.591×10^{-14}	9.246×10^{-14}	27.79
Ruth (Order 3)	Bumblebee	8.058×10^{-3}	7.394×10^{-14}	4.249×10^{-14}	23.84
Verlet (Order 2)	Bumblebee	9.676×10^{-2}	2.287×10^{-14}	1.882×10^{-14}	11.56
Euler (Order 1)	Bumblebee	2.4329	3.625×10^{-14}	2.312×10^{-14}	15.71
Neri (Order 4)	Moth	1.459×10^{-9}	4.874×10^{-14}	8.693×10^{-14}	28.06
Ruth (Order 3)	Moth	2.143×10^{-8}	3.486×10^{-14}	3.185×10^{-14}	22.39
Verlet (Order 2)	Moth	4.775×10^{-5}	2.387×10^{-14}	3.835×10^{-14}	11.55
Euler (Order 1)	Moth	0.02070	2.031×10^{-14}	2.183×10^{-14}	13.76

TABLE 1. Comparison of Ruth, Neri, Verlet, and Euler Methods Across Different Orbits

running time than the Verlet method. The latter result is surprising and is possibly due to the 0 entry in the d_2 corresponding to the Verlet Method. Besides, it can be seen that the Neri Method conserves energy much better than other methods, especially in the Moth Orbit simulation. More interestingly, the Verlet Method conserves Momentum slightly better than methods with higher orders. However, the difference is too small to be significant (all of order 10^{-14}) and may not be accurate they are close to the machine error for double-precision floating-point numbers in Python, which is $O(10^{-16})$.

Even though the Neri method causes a longer run time, the difference is negligible. Therefore, the Neri method is chosen.

3. RESULTS

3.1. **Types of Orbits.** We have reproduced three types of orbits using as described in *Syzygies in the two center problem*(Dullin and Montgomery, 2016).

First, we reproduced the satellite orbit, in which the z-coordinate of the third body never changes sign. In other words, the third body never seems to pass through the mid-point between the two fixed centers. Next, we reproduced the planetary orbit, in which the third body almost forms a complete ellipse around the two centers. Last, we reproduced the lemniscate orbit, in which the third body passes through the line connecting the two fixed centers, unlike both satellite and planetary orbits. The trajectories of the orbits are included in my GitHub repository and my blog.

112 3.2. **Predicting type of orbit from integrals of motion.** Referring to g and h
113 defined above, the value of h must be negative as we are solving a bounded system, but
114 the values for g greatly vary. It has been shown in literature (Dullin and Montgomery,
115 2016) that g and h, the integrals of motion shown above, can predict the type of orbit.
116 In general, negative g corresponds to a satellite orbit, a slightly positive g (say, between
117 0 and 1) corresponds to a lemniscate orbit, whereas a larger g corresponds to a planetary
118 orbit. This is different for different combinations of m_+ and m_-, but we only considered
119 the symmetric case. We have verified that the Satellite Orbit corresponds to $g < 0$
120 and $h < 0$, the Lemniscate Orbit corresponds to $0 < g < 1$ and $-1 < h < 0$, and the
121 Planetary Orbit corresponds to $g > 1$ and $-0.5 < h < 0$, which agrees with literature.
122 (Dullin and Montgomery, 2016).

123 3.3. **Textbook verification by comparing eccentricities and semi-major axes.**
124 To conclude, we can compare with the constants defined in *Integrable Systems in Celestial*
125 *Bodies* (Ó'Mathúna, 2008), p.58-59. It is defined that M and N are constants, where:

$$\frac{1}{2} \frac{\left(R^2 - b^2 \cos^2 \sigma\right)^2}{R^2 - b^2} \dot{R}^2 = ER^2 + \mu R + M$$
$$\frac{1}{2} \left(R^2 - b^2 \cos^2 \sigma\right)^2 \dot{\sigma}^2 = -Eb^2 \cos^2 \sigma + \frac{(m_+ - m_-)\mu b \cos \sigma}{m_+ + m_-} + N.$$

126 We have $\mu = G(m_1 + m_2)$ and G is the gravitational constant between particles, which
127 is usually taken as 1 for simplicity.
128 It is stated in the textbook that when $\eta^2 < 1$, it corresponds to a closed elliptic orbit
129 of eccentricity η and semimajor axis p. The two constants are defined as:

$$p = \frac{O^2}{2}, \text{where } O = \sqrt{2N} \quad (O \text{ is a constant with units of angular momentum}).$$
$$\eta = \frac{1}{p}.$$

130 In our code, only the planetary orbit satisfies $\eta^2 < 1$. It agrees with theoretical results
131 as the planetary orbit is the only type of closed orbit.
132 To verify our estimate of the semi-major axis p is accurate, we have also measured
133 the semi-major axis of a planetary orbit directly from its trajectories in the x-z plane.
134 If the code is accurate, using $\eta = \frac{1}{p}$, we should obtain $p = 2.35$ if we use an orbit with
135 $\eta = 0.425$. Instead, we obtained p is 2.53 with a standard deviation of 0.133, showing a
136 small discrepancy. This inaccuracy should be investigated over a range of η values.

137 ## 4. Conclusion

138 We selected an integration method and used it to analyze Euler's Three Body Prob-
139 lem. It is demonstrated that the three types of orbits in the Euler Problem can be
140 recreated. Further work could involve simulating and exploring different cases discussed
141 in *Integrable Systems in Celestial Bodies* (Ó'Mathúna, 2008), as well as comparing re-
142 sults with other integrator types, such as regularized integrators and adaptive step-size
143 integrators.

144 ## References

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