SIMULATING THE THREE-BODY PROBLEM WITH THE CHOICE OF AN INTEGRATION METHOD

ABSTRACT. The Three-Body Problem involves three bodies, m_1, m_2, m_3, which attract each other according to Newton's inverse square law. Despite centuries of effort, it is far from fully solved.

We first pick an integration method for numerical simulation by defining metrics to evaluate their precision. We use the integration method to plot the trajectories of orbits corresponding to Euler's three-body problem. We attempt to convert the cartesian coordinates to planar prolate spheroidal coordinates and use these coordinates to measure and compare the values of constants described in the book Integrable Systems in Celestial Mechanics, thus classifying orbits.

1. CHOOSING AN INTEGRATION METHOD FOR NUMERICAL SIMULATION

Since the Three-Body Problem does not have a useful analytical form, much of the current research is based on numerical simulations. An integration method will be selected below, based on different criteria.

1.1. **Definition of order of an integration method.** Global error is the cumulative error over all steps compared to the exact solution of the differential equation (Higham, 2024). An integration method has order n if the global error is of order $O(h^n)$, where h is the stepsize. For example, Euler Method has the form $y_{n+1} = y_n + hf(t_n, y_n)$. The global error is of order $O(h)$ and hence is of order 1. This means the global error decreases linearly with the step size h. The approximation becomes more accurate as h becomes smaller, but more computational steps are required.

To determine if an integration method is suitable, it is important to check if the conserved quantities, namely the total energy, linear momentum and angular momentum are conserved. In this report, we mainly check the total energy and linear momentum.

1.2. **Definition of a symplectic integrator.** This subsection is based on the report *Computer Implementation of Symplectic Integrators and Their Applications to the N-Body Problem* (Casey, 2020). Conservation of energy can be achieved through symplectic integrators.

$$\begin{aligned} v^{(i)} &= v^{(i-1)} + c_i a(x^{(i-1)})dt \\ x^{(i)} &= x^{i-1} + d_i v^{(i)} dt. \end{aligned}$$

For compactness, we can express

$$\mathbf{A} = \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

where

$$\begin{aligned} \mathbf{c} &= [c_1, c_2, ..., c_{i-1}, c_i] \\ \mathbf{d} &= [d_1, d_2, ..., d_{i-1}, d_i]. \end{aligned}$$

It has been shown that:

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$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

leads to a First-Order Symplectic Integration Method. This is the Euler Method. Meanwhile

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

leads to a Second-Order Symplectic Integration Method. This is the Verlet Method. Also,

$$\mathbf{A} = \begin{bmatrix} \frac{7}{24} & \frac{3}{4} & \frac{-1}{24} \\ \frac{2}{3} & \frac{-2}{3} & 1 \end{bmatrix}$$

leads to a Third-Order Symplectic Integration Method. This will be referred to as the Ruth Method. At last,

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2\left(2-2^{1/3}\right)} & \frac{1-2^{1/3}}{2\left(2-2^{1/3}\right)} & \frac{1-2^{1/3}}{2\left(2-2^{1/3}\right)} & \frac{1}{2\left(2-2^{1/3}\right)} \\ \frac{1}{2-2^{1/3}} & \frac{-2^{1/3}}{2-2^{1/3}} & \frac{1}{2-2^{1/3}} & 0 \end{bmatrix}$$

leads to a Fourth-Order Symplectic Integration Method. This will be referred to as the Neri method.

Notice that the velocity is always updated before the position. This is illustrated using the Ruth Method:

$$\begin{aligned} v^{(1)} &= v^{(0)} + c_1 a(x^{(0)})dt \\ x^{(1)} &= x^{(0)} + d_1 v^{(1)} dt \\ v^{(2)} &= v^{(1)} + c_2 a(x^{(1)})dt \\ x^{(2)} &= x^{(1)} + d_2 v^{(2)} dt \\ v^{(3)} &= v^{(2)} + c_3 a(x^{(2)})dt \\ x^{(3)} &= x^{(2)} + d_3 v^{(3)} dt \end{aligned}$$

1.3. **Metrics for choosing an integration method.** To choose the most suitable integration method, we will define several metrics to evaluate the precision of the numerical methods specified above. These metrics will be compared across three different orbits, namely the Figure-8 orbit, the Bumblebee Orbit, and the Moth Orbit. Each orbit will be simulated with a total of $100,000$ steps, using a step size of 0.0001.

1.3.1. *First Metric - Energy Deviation.* The Energy Deviation is given by (Zuntz, 2023/2024):

$$\frac{\Delta E}{E_0} = \left| \frac{E_{\max} - E_{\min}}{E_0} \right|$$

where E_0 is the energy at the start of the simulation. A good model requires:

$$\frac{\Delta E}{E_0} < 1.$$

1.3.2. *Second Metric - Momentum Difference.* For many systems, the initial Momentum is zero. Therefore, Momentum Deviation cannot be well defined. Instead, we will only check the Momentum Difference, which can be defined as:

$$\text{Max } \Delta p_x = |p_{\text{max},x} - p_{\text{min},x}|$$
$$\text{Max } \Delta p_y = |p_{\text{max},y} - p_{\text{min},y}|.$$

1.3.3. *Third Metric - Run Time.* Run Time is important for particularly long simulations. In our case, most simulations are relatively short so it is not as important as the metrics above. Nonetheless, a faster simulation method is advantageous, provided that it does not compromise the quality of the output.

1.3.4. *A Table for comparision.* Below all the metrics are compared with each orbit. It

Method	**Orbit**	**Energy Deviation**	**Max Δp_x**	**Max Δp_y**	**Run Time (s)**
Neri (Order 4)	Figure-8	6.124×10^{-14}	1.040×10^{-13}	4.852×10^{-14}	29.60
Ruth (Order 3)	Figure-8	1.280×10^{-13}	5.307×10^{-14}	3.048×10^{-14}	25.91
Verlet (Order 2)	Figure-8	5.893×10^{-9}	3.841×10^{-14}	2.232×10^{-14}	12.31
Euler (Order 1)	Figure-8	3.601×10^{-5}	3.686×10^{-14}	3.009×10^{-14}	15.53
Neri (Order 4)	Bumblebee	8.058×10^{-3}	4.591×10^{-14}	9.246×10^{-14}	27.79
Ruth (Order 3)	Bumblebee	8.058×10^{-3}	7.394×10^{-14}	4.249×10^{-14}	23.84
Verlet (Order 2)	Bumblebee	9.676×10^{-2}	2.287×10^{-14}	1.882×10^{-14}	11.56
Euler (Order 1)	Bumblebee	2.4329	3.625×10^{-14}	2.312×10^{-14}	15.71
Neri (Order 4)	Moth	1.459×10^{-9}	4.874×10^{-14}	8.693×10^{-14}	28.06
Ruth (Order 3)	Moth	2.143×10^{-8}	3.486×10^{-14}	3.185×10^{-14}	22.39
Verlet (Order 2)	Moth	4.775×10^{-5}	2.387×10^{-14}	3.835×10^{-14}	11.55
Euler (Order 1)	Moth	0.02070	2.031×10^{-14}	2.183×10^{-14}	13.76

TABLE 1. Comparison of Ruth, Neri, Verlet, and Euler Methods Across Different Orbits

is somewhat surprising that the Verlet Method has the lowest running time, even lower than that of the Euler Method. More interestingly, the Verlet Method also conserves Momentum slightly better than methods with higher orders. However, the difference is too small to be significant (all of order 10^{-14}). It can be seen that the Neri Method conserves Energy much better than other methods, especially in the Moth Orbit simulation.

Even though the Neri method causes a significantly longer run time, it is not a significant limiting factor as most simulations are considerably short. Therefore, the Neri method is chosen. Using this method, we can analyze a special case in the three-body problem, named Euler's three-body problem.

2. An analysis of Euler's restricted three-body problem

2.1. **Setup of Euler's restricted three-body problem.** In this section, we mainly consider the setup in the book *Integrable Systems in Celestial Mechanics* (Ó'Mathúna, 2008), namely the Euler Problem. Consider the z-axis plotted against the x-axis. The two masses m_+ and m_- are located at $(0, 1)$ and $(0, -1)$ respectively whereas the third mass can move around freely. As two of the masses are stationary, this problem is also called the problem of two fixed centers.

It is a convention to express the system in terms of planar prolate spheroidal coordinates:

$$x = \pm\sqrt{R^2 - b^2}\sin\sigma$$
$$z = R\cos\sigma$$

where $R \in (0, \infty]$, $\sigma \in [0, \pi]$.

The advantage of this form is we can express:

$$r_+{}^2 = (R - b\cos\sigma)^2$$
$$r_-{}^2 = (R + b\cos\sigma)^2$$

where r_+ is the distance from m_+ to the third body, and r_- is the distance from m_- to the third body.

2.2. **Expressing prolate spheroidal coordinates using cartesian coordinates.** Given the $\pm$ sign for x, without loss of generality, we can consider $x > 0$.

As $\cos\sigma$ is only negative in the range $\frac{\pi}{2} < \sigma < \pi$, we have $z > 0$ for $0 < \sigma < \frac{\pi}{2}$ and $z < 0$ for $\frac{\pi}{2} < \sigma < \pi$. We will consider these two cases separately.

2.2.1. *Case for $z > 0$.* Using $R = \frac{z}{\cos\sigma}$, we can substitute one of the equations into the other:

$$\begin{aligned} x &= \sin\sigma\sqrt{\left(\frac{z}{\cos\sigma}\right)^2 - 1} \\ &= \sqrt{\frac{z^2 - \cos^2\sigma}{\cos^2\sigma}}\sin\sigma \\ x^2 &= \frac{(z^2 - 1 + \sin^2\sigma)(\sin^2\sigma)}{1 - \sin^2\sigma} \\ x^2 &= \sin^2\sigma(x^2 + z^2 - 1 + \sin^2\sigma) \end{aligned}$$

Where we have used $\cos\sigma = \sqrt{1 - \sin^2\sigma}$. This is valid as $0 < \sigma < \frac{\pi}{2}$. By letting $u = \sin^2\sigma$, we have:

$$u^2 + (x^2 + z^2 - 1)u - x^2 = 0$$

Using the Quadratic Formula, and noting that $\sqrt{u} = \sin\sigma$, :

$$\arcsin\sqrt{u} = \sigma_{z>0}$$

Where

$$u = \frac{-(x^2 + z^2 - 1) + \sqrt{(x^2 + z^2 - 1)^2 + 4x^2}}{2}.$$

2.2.2. *Case for $z < 0$.* We have assumed $x > 0$ and found the solution for $z > 0$. Now we want to find the solution for $z < 0$ while keeping $x > 0$.

Note that $\sin(\pi - \sigma) = \sin\sigma$ and $\cos(\pi - \sigma) = -\cos\sigma$. As $x \propto \sin\sigma$ and $z \propto \cos\sigma$, by letting $\sigma_{z<0} = \pi - \sigma_{z>0}$, we can find solutions for $z < 0$.

To summarize:

$$\sigma = \begin{cases} \arcsin\sqrt{u} & \text{for } z > 0, \\ \pi - \arcsin\sqrt{u} & \text{for } z < 0, \end{cases}$$

where

$$u = \frac{-(x^2 + z^2 - 1) + \sqrt{(x^2 + z^2 - 1)^2 + 4x^2}}{2}.$$

Finding R is very straightforward given σ and hence is omitted.

2.3. **Listing the conserved quantities.** In this subsection, we discuss the conserved quantities in the setup. This is important for classifying orbits in later parts of the report. First, we will state two quantities that are not conserved.

2.3.1. *Linear Momentum and Angular Momentum are not conserved.* Both linear momentum and angular momentum are not conserved as two of the bodies are fixed in place and do not react to the force exerted by the other point mass. This can be illustrated by releasing a point mass a distance away from the two fixed bodies. The initial momentum and linear momentum are zero but the point mass is attracted by the two fixed bodies and accelerates towards them. This leads to a non-zero linear momentum and angular momentum over time.

Below we describe quantities that are conserved.

2.3.2. *Energy is conserved.* The kinetic energy of the moving point mass is converted to the potential energy between itself and the two fixed masses. There is no energy lost.

2.3.3. *g and h - constants that define type of orbit, are conserved (Dullin and Montgomery, 2016).* This is slightly modified from the cited paper due to a different set-up:

$$g = \frac{1}{2}\left(zp_x - xp_z\right)^2 + \frac{1}{2}d^2p_z^2 + dz\left(\frac{m_1}{\sqrt{(z+d)^2+x^2}} - \frac{m_2}{\sqrt{(z-d)^2+x^2}}\right)$$
$$h = \frac{1}{2}\left(p_z^2 + p_x^2\right) - \frac{m_1}{\sqrt{(z+d)^2+x^2}} - \frac{m_2}{\sqrt{(z-d)^2+x^2}}.$$

Where d is the distances of the two centers from the origin. Both g and h will be used to determine the type of orbit. Note that h is simply the energy of the third particle.

2.3.4. *C_1 and C_2 - constants that define eccentricity and semi-major axis or some orbits, are conserved.* These two additional conserved quantities are denoted in the book mentioned in the early parts of this section (Ó'Mathúna, 2008):

$$\frac{1}{2}\frac{\left(R^2 - d^2\cos^2\sigma\right)^2}{R^2 - d^2}\dot{R}^2 = ER^2 + \mu R + C_1$$
$$\frac{1}{2}\left(R^2 - d^2\cos^2\sigma\right)^2\dot{\sigma}^2 = -Ed^2\cos^2\sigma + \mu\beta d\cos\sigma + C_2$$

where $\mu = G(m_+ + m_-)$ and $\beta = \frac{m_+ - m_-}{m_+ + m_-}$, with d described above. It can be verified that $C_1 + C_2 = 0$. The constants are used to define the eccentricities of orbits. More will be explained below.

2.4. **Describing the types of orbits.** Given these constants, we have rediscovered three types of orbits as described in *Syzygies in the two center problem*(Dullin and Montgomery, 2016). This is described below:

2.4.1. *Satellite Orbit.* The characteristic of this orbit is that the z-coordinate of the third body never changes sign. In other words, the third body never seems to pass through the mid-point between the two fixed centers.

2.4.2. *Planetary Orbit.* The characteristic of a planetary orbit is that the third body almost forms a complete ellipse around the two centers. This is explained further below.

2.4.3. *Lemniscate Orbit.* The characteristic of a lemniscate orbit is that the third body passes through the line connecting the two fixed centers, unlike both satellite and planetary orbits.

2.5. **Textbook verification by comparing eccentricities and semi-major axes.** Using constants defined above, we can check the robustness of our code with results in the literature. From the same textbook *Integrable Systems in Celestial Bodies* (Ó'Mathúna, 2008) we can define these constants:

$$
\begin{aligned}
C &= \sqrt{2C_2} \\
p &= \frac{C^2}{2} \\
a &= -\frac{1}{E} \quad (E \text{ is total energy of the system}) \\
e &= \sqrt{1-\frac{p}{a}} \\
\eta &= \frac{1}{p}
\end{aligned}
$$

It is stated in the textbook that when $\eta^2 < 1$, it corresponds to a closed elliptic orbit of eccentricity η and semimajor axis p. It has been found that the value of η is 0.425 for planetary orbit, with a p value of 2.35. For other orbits, the value of η exceeds 1. Indeed, it can be verified from our plots that the planetary orbit corresponds best to a closed elliptic orbit. To confirm the semimajor axis is indeed p, We have found the semi-major axis for the first 5 orbits for planetary orbit. As the two fixed centers are the foci of the ellipses, we can find the corresponding eccentricity also by:

$$\text{Eccentricity} = \frac{1}{\text{Semi-Major Axis}}$$

It has been found that the theoretical values roughly correspond to the measured values. The measured value for p is 2.53 with a standard deviation of 0.133, compared to a theoretical value of 2.35.

3. Errors in simulation

It has been found that the values of g and h are not constants in our simulation; they vary with time. Therefore, in the sections above, we merely take their averages. This may be due to inaccuracies in translating the formulae from the original paper, as the setup is slightly different. Another possible reason is an error in the Python code or a misunderstanding of the coordinate system used.

For the values of C_1 and C_2, there are huge spikes occasionally. Therefore we only take the average of the first few values for our simulation. We speculate this is because the value of R can be close to 1, so the denominator $R^2 - d^2$ can be close to 0, which causes huge numerical errors.

4. Conclusion

Throughout this report, we have chosen an integration method and used it to analyze Euler's Three Body Problem. It is shown that the three types of orbits can be recreated and the measured constants correspond well to the theoretical values. However, these are only very elementary analyses of the many results from the book. Furthermore, there are many more special cases of the three-body problem that are fully analytical and can be plotted for comparison.

References

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