Chapter 1

Linear Algebra

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Introduction

Linear algebra is a crucial subject in mathematics and it is one of the first things a mathematics undergrad expands their knowledge on after a brief introduction in M2. This chapter shall revisit some of the basics and expands further on it, giving it a more rigorous mathematical background. If you are familiar with the dot product, feel free to read Chapter 4, Vector Calculus.

The $\mathbb{R}^2$ plane is the Cartesian coordinate system that we are all familiar with. The $\mathbb{R}$ denotes that the coordinates are composed of real numbers, and the ^{2} denotes that the plane is two dimensional. There are many operations defined in the $\mathbb{R}^2$ plane that are defined similarly in $\mathbb{R}$.

Example 1.0.1. Consider two coordinates (u_1, u_2) and (v_1, v_2) in the $\mathbb{R}^2$ plane, where u_1, u_2, v_1, v_2 are real numbers. Let us name the coordinates as $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively, then

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2)$$

Let k be a real number, then

$$k\boldsymbol{u} = k(u_1, u_2) = (ku_1, ku_2)$$

1.1 Dot Product

You may ask, then, is it possible to multiply two coordinates in $\mathbb{R}^3$ together? For example,

$$(u_1, u_2, u_3)(v_1, v_2, v_3) = ???$$

There may be two answers in your head: maybe it is equivalent to a number, maybe it is equivalent to another coordinate, and both are correct! We shall discuss the former here, and the later in chapter 4.4 If you are interested in it, take a stab at Serge Lang's *Linear Algebra* or his *Introduction to Linear Algebra*. It's crucial to realise that these "multiplication" methods are only invented because it serves a purpose and proves itself to be useful. Mathematicians didn't make these up so that everything can be "multiplied". [1]

The former answer, is in fact, the main topic of this chapter, the dot product. For $\mathbb{R}^2$, it's defined as follows:

Definition 1.1.1. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be 2 coordinates in $\mathbb{R}^2$. Their dot product: $\boldsymbol{u} \cdot \boldsymbol{v}$ or $< \boldsymbol{u}, \boldsymbol{v} >$ is defined to be:

$$u_1v_1 + u_2v_2$$

[1] As an additional example, for complex numbers, it is important to realise that all square roots of complex numbers **are** complex numbers as well. Hence, there would not be any more problems about square roots that stems from the complex plane. Mathematicians, similarly, didn't invent complex numbers only because they wanted to take square roots of negative numbers

For $\mathbb{R}^3$, it is defined similarly by adding one more term, u_3v_3. Please note that if a, b are real numbers, $a \cdot b$ is just plain multiplication. i.e. $3 \cdot 4 = 12$

Example 1.1.1. Find the dot product between (5, -5) and (3, 4)

$$5 \cdot 3 + -5 \cdot 4 = -5$$

Of course, for the dot product to be a convincing extension of multiplication. It has to satisfy some basic properties and prove that it is worthy.

Theorem 1.1.1. The dot product in $\mathbb{R}^n$ satisfy the below properties: Let u, v and w be three coordinates in $\mathbb{R}^n$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$$
$$\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$$

If x is a real number, then

$$\langle x\boldsymbol{u}, \boldsymbol{v} \rangle = x\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{u}, x\boldsymbol{v} \rangle$$

Try to prove these properties on your own!

1.2 Orthogonality

Let us show one more property of the dot product. Let u and v be two coordinates in $\mathbb{R}^2$. They are defined to be **orthogonal** or **perpendicular** if $u \cdot v = 0$. To understand the reasoning behind it will require some basic knowledge in coordinate geometry. We shall prove it for $\mathbb{R}^2$, note that it is true for $\mathbb{R}^n$ in general as well.

Proof. Let us consider the line that is drawn by connecting $\boldsymbol{u}$ and the origin. Hence, let m_u be the slope of the line

$$m_u = \frac{u_2 - 0}{u_1 - 0} = \frac{u_2}{u_1}$$

As the line passes through the origin, $c = 0$. Hence the equation is $y = u_2/u_1 \cdot x$

Similarly, for the line connecting v and the origin, the equation is $y = v_2/v_1 \cdot x$ From coordinate geometry, we know for these two lines to be perpendicular, the product of their slope $= -1$. Hence,

$$\begin{aligned} \frac{u_2 v_2}{u_1 v_1} &= -1 \\ u_2 v_2 &= -u_1 v_1 \\ \boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 &= 0 \end{aligned} \tag{1.1}$$

□

Example 1.2.1. *Consider the line $2x + 3y + 4z = 0$ in $\mathbb{R}^3$, then (x, y, z) is orthogonal to $(2, 3, 4)$. Hence the solution will be a 2d plane with coordinates that are perpendicular to $(2, 3, 4)$. Note that this plane still passes through the origin as $(0, 0, 0)$ is a solution to the equation.

Example 1.2.2. *Prove that there does **not** exist a non-trivial coordinate in $\mathbb{R}^2$ that is orthogonal to two non-trivial coordinates v and w unless $\boldsymbol{v} = c\boldsymbol{w}$ for some real number c.[2]

Proof. Let us assume that the statement is true and that there exist $\boldsymbol{u} = (u_1, u_2)$ such that

$$\begin{cases} u_1 v_1 + u_2 v_2 = 0 & (1.2) \\ u_1 w_1 + u_2 w_2 = 0 & (1.3) \end{cases}$$

Let c be a real number such that $cw_1 = v_1$. If c doesn't exist, then $w_1 = 0$, hence

$$\begin{aligned} 0 \cdot u_1 + u_2 w_2 &= 0 \\ u_1 w_1 &= 0 \end{aligned} \tag{1.4}$$

[2]Non-trivial in this context means that the coordinates are not equal to the origin.

$w_2 \neq 0$ as otherwise $\boldsymbol{w}$ will be trivial. Hence u_2 must be 0. If c exists, then consider the equation $(1,2) - c \cdot (1,3)$

$$\begin{gathered} u_1(v_1 - cw_1) + u_2(v_2 - cw_2) = 0 - c \cdot 0 \\ u_2(v_2 - cw_2) = 0 \end{gathered} \tag{1.5}$$

As it was assumed that $v_1 = cw_1$, if $v_2 - cw_2 = 0$ There would exist some c such that $\boldsymbol{v} = c\boldsymbol{w}$, which contradicts with the assumption that the statement is true. Hence u_2 must be 0.

Similarly, let d be a real number such that $v_2 = dw_2$ ..., using the same method above, we would find that u_1 also must be 0. Hence u is the origin, which contradicts the assumption. Hence, we have proven this theorem by proof of contradiction. [3]

□

1.3 Norm

The dot product also provides a generalization or a way to define common concepts in mathematics. For example, when we are talking about the length of a "coordinate" $\boldsymbol{u}$ in $\mathbb{R}^2$. What we mean is

$$\sqrt{u_1^2 + u_2^2}$$

which is in fact

$$||\boldsymbol{u}|| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = \sqrt{u_1^2 + u_2^2}$$

We use $||\boldsymbol{u}||$ to define the length of a coordinate in the $\mathbb{R}^n$ plane. We could also define the **unit vector**, denoted by a small hat, $\hat{\boldsymbol{u}}$. We could find the unit vector of any other vector by

[3]Fun fact, the idea of this example came from Mr. Ching, the principal of this school and previously the head of the mathematics department. He proved that there could not exist n+1 linear independent vectors in $\mathbb{R}^n$, which is a more general restatement of the example

$$\hat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{||\boldsymbol{u}||}$$

It could be seen that all unit vectors have a norm of 1.

1.4 Pythagoras theorem

Theorem 1.4.1. Pythagoras theorem If v, w are perpendicular, i.e. $v \cdot w = 0$, then

$$||\boldsymbol{v} + w||^2 = ||\boldsymbol{v}||^2 + ||w||^2$$

Proof.

$$\begin{aligned} ||\boldsymbol{v} + \boldsymbol{w}||^2 &= (\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} + \boldsymbol{w}) \\ &= \boldsymbol{v} \cdot \boldsymbol{v} + 2(\boldsymbol{v} \cdot \boldsymbol{w}) + \boldsymbol{w} \cdot \boldsymbol{w} \\ &= \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} \end{aligned} \tag{1.6}$$

□

Note that the theorem is true even if we replace $\boldsymbol{v}+\boldsymbol{w}$ with $\boldsymbol{v}-\boldsymbol{w}$

1.5 Projection

Let $\boldsymbol{w}$ be a coordinate where $||\boldsymbol{w}|| \neq 0$. Then for any $\boldsymbol{v}$, we could find a number c such that $(\boldsymbol{v}-c\boldsymbol{w})\cdot\boldsymbol{w} = 0$. We call c the component of $\boldsymbol{v}$ along $\boldsymbol{w}$. What we are doing is basically expressing $\boldsymbol{v}$ as a sum of $c\boldsymbol{w}$ and $\boldsymbol{v} - c\boldsymbol{w}$, which are orthogonal to each other. For the calculation of c:

$$\begin{aligned} (\boldsymbol{v} - c\boldsymbol{w}) \cdot \boldsymbol{w} &= 0 \\ \boldsymbol{v} \cdot \boldsymbol{w} - c\boldsymbol{w} \cdot \boldsymbol{w} &= 0 \\ \boldsymbol{v} \cdot \boldsymbol{w} &= c(\boldsymbol{w} \cdot \boldsymbol{w}) \\ c &= \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\boldsymbol{w} \cdot \boldsymbol{w}} \end{aligned} \tag{1.7}$$

Example 1.5.1. Find the component of (4, 5) along (2, 0):
Let $\boldsymbol{v} = (4,5)$ and $\boldsymbol{w} = (2,0)$

$$\begin{aligned} c &= \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\boldsymbol{w}\cdot\boldsymbol{w}} \\ &= \frac{8-0}{4} \\ &= 2 \end{aligned} \tag{1.8}$$

As expected, the answer is 2.
We could observe that

$$\begin{aligned} (\boldsymbol{v} - c\boldsymbol{w})\cdot\boldsymbol{w} &= (4-2\cdot 2, 5-0)\cdot(2,0) \\ &= (0,5)\cdot(2,0) \\ &= 0 \end{aligned} \tag{1.9}$$

Example 1.5.2. *Find the area of the triangle drawn between the origin, $\boldsymbol{u}$, and $\boldsymbol{v}$.

To solve this, first imagine the triangle drawn. The area of the triangle equals to the length of the base times the height divided by 2. In this case, if we pick the length of the base to be $||\boldsymbol{v}||$. The height will be $||\boldsymbol{v} - c\boldsymbol{w}||$ where c is the component of v along w. Hence the area is equal to $||\boldsymbol{v}|| \cdot ||\boldsymbol{v} - c\boldsymbol{w}||/2$.

For example, to find the area of the triangle between the origin, $(4,5)$ and $(2,0)$. Let $\boldsymbol{v} = (4,5)$ and $\boldsymbol{w} = (2,0)$, then $\boldsymbol{v} - c\boldsymbol{w} = (0,5)$. Hence the area is

$$\begin{aligned} ||(0,5)||\cdot||(2,0)||/2 &= 5\cdot 2/2 \\ &= 5 \end{aligned} \tag{1.10}$$

1.6 Basis

In daily life and in $\mathbb{R}^2$, the basis we used the most often is $(1,0)$ and $(0,1)$. For example, we refer to a point in space by coordinates.

Such as $(3, 4)$, which is equal to $3(1, 0) + 4(0, 1) = (3, 4)$. However, what if we use a different basis?

Example 1.6.1. Given a coordinate $\boldsymbol{u} = (u_1, u_2)$, which is expressed using the basis of $(1, 0)$ and $(0, 1)$. What will its coordinates be if instead that basis of $(1, 0)$ and $(1, 1)$ is used?

Note that if we stick with (u_1, u_2) in the new basis, the coordinate expressed back in the original basis will become $u_1(1, 0) + u_2(1, 1) = (u_1 + u_2, u_2) \neq (u_1, u_2)$.

To solve this, let $\boldsymbol{k} = (k_1, k_2)$ be the new coordinates that uses the new basis yet denotes the same point as $\boldsymbol{u}$. If we express $\boldsymbol{k}$ using the original basis, the point derived is $(k_1 + k_2, k_2)$. As the point derived is equivalent to (u_1, u_2). We could make a simultaneous equation to solve this problem.

$$\begin{cases} k_1 + k_2 = u_1 & (1.11) \\ \qquad k_2 = u_2 & (1.12) \end{cases}$$

Hence, $k_1 = u_1 - k_2 = u_1 - u_2$. As a result,

$$k = (u_1 - u_2, u_2)$$

1.7 Matrix multiplication

You may notice that the simultaneous equations of example 1.6.1 could be expressed as two dot products, i.e.

$$\begin{cases} (1, 1) \cdot k = u_1 & (1.13) \\ (0, 1) \cdot k = u_2 & (1.14) \end{cases}$$

This interesting 2 by 2 grid of numbers could be called a matrix, and would help simplify the notations used when we are changing basis (which is something we do a lot in linear algebra and has significant importance) In fact, now will be a swell time to introduce the concept of matrices and matrix multiplication. Let A be a 2 by 2 matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then we can define the multiplication between a 2 by 2 matrix, A, and a 2d coordinate, u, which outputs a 2d coordinate, v. Then,

$$A\boldsymbol{u} = \boldsymbol{v}$$

where

$$\boldsymbol{v} = (\boldsymbol{A_1} \cdot \boldsymbol{u}, \boldsymbol{A_2} \cdot \boldsymbol{u})$$

Where $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ denote the first and second row column vectors of A. Using our example, $\boldsymbol{A_1} = (1, 1)$ and $\boldsymbol{A_2} = (0, 1)$

Example 1.7.1. Let $\boldsymbol{u}$ be $(3, 4)$ and a 2 by 2 matrix, A be

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Find $A\boldsymbol{u}$

Firstly, we find out that $\boldsymbol{A_1} = (0, -1)$ and $\boldsymbol{A_2} = (1, 0)$. Hence $A\boldsymbol{u}$ is equal to,

$$\begin{aligned} A\boldsymbol{u} &= ((0, -1) \cdot (3, 4), (1, 0) \cdot (3, 4)) \\ &= (-1 \cdot 4, 1 \cdot 3) \\ &= (-4, 3) \end{aligned} \tag{1.15}$$

You could also express this equivalence by saying, if we express the coordinate (3, 4) in the basis of (0, 1) and (-1, 0). Then the coordinate is (-4, 3) in the basis of (1, 0) and (0, 1).

Also, note how this matrix multiplication seemed to have "rotated" $\boldsymbol{u}$ anti-clockwise by 90 degrees. Try to give an explanation on why this occurred.

From this observation, we could define matrix multiplication for other dimensions. Let M be a matrix with m rows (Horizontal) and n columns (Vertical), let u be a vector of dimension n and v be a vector of dimension m. Then

$$M\boldsymbol{u} = (\boldsymbol{A_1} \cdot \boldsymbol{u}, \boldsymbol{A_2} \cdot \boldsymbol{u} ... \boldsymbol{A_m} \cdot \boldsymbol{u}) = \boldsymbol{v}$$

u must be of dimension n, as the column coordinates of M have a dimension of n. v must be of dimension m, as there are a total of m terms.

Example 1.7.2. Let $\boldsymbol{u}$ be $(3, 4, 5)$ and a 2 by 3 matrix, M be

$$M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Find $M\boldsymbol{u}$

$$\begin{aligned} M\boldsymbol{u} &= (2 \cdot 3 + 1 \cdot 4, 1 \cdot 3 + 5 \cdot 5) \\ &= (10, 28) \end{aligned} \tag{1.16}$$

Conclusion

Sadly, we will have to end the chapter here due to spacing concerns, albeit quite abruptly. For those seeking to learn more about linear algebra, I wholeheartedly recommend 3blue1brown's YouTube series, Essence of Linear Algebra or read Serge Lang's book on linear algebra.

While learning through YouTube or other online resources has been a norm recently. I still think that the value of a book has not diminished. Many well-written books have been acclaimed and used for decades. The learning experience they provide are usually much more coherent and structured. While mathematical textbooks are quite expensive. The information is significantly more condensed than novels / other genres. They are worth the price and getting it printed will make it easier for you to pick it up. You could literally spend a year of your free time on a mathematics textbook. Really try to at least get a kindle if you cannot obtain a physical copy.

Chapter 4

Applied Mathematics

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Introduction

In this chapter, we shall give some properties that are found in DSE physics - "Force and Motion" more mathematical background using linear algebra and calculus. To ease the readability of this chapter, **no** further concepts from physics will be introduced, and examples involved are to be kept as simple and fundamental as possible. Please briefly revisit Chapter 1 before proceeding on-wards. Note that the apostrophe will be used to denote differentiation.

4.1 Vector functions

Before we begin, I would like to introduce the concept of putting functions into vectors. Let us assume that we would like to plot $y = f(x)$ in $\mathbb{R}^2$ plane. Usually we will write $y = f(x)$. However, we could also express it as a vector $(x, f(x))$. There are numerous benefits brought by such change. First of all, the x-coordinate doesn't necessarily have to be x. For example, given an implicit function

$$x^2 + y^2 = 1$$

It is impossible to express it fully as $y = f(x)$. However if we instead use

$$(\cos\theta, \sin\theta),\ 0 \leq \theta < 2\pi$$

We would be able to express the unit circle in vector form with relative ease. As you can see, this is much more flexible than sticking with $y = f(x)$. This is called a **curve parameterization** in mathematics. [1]. Differentiation of such vector functions are also intuitive. For example, given a a function (x, x^2). Taking its derivative means to take derivative of all of the coordinates inside, i.e. $(1, 2x)$. Try entering (t, t^2) or $(\cos t, \sin t)$ in Desmos or other graphing calculators and play around!

Let us now prove a widely used theorem about the dot products between vectors, akin to the chain rule.

Theorem 4.1.1. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors that have functions with respect to t as their coordinates (i.e. $\boldsymbol{u} = (t, t^2)$ etc). Then

$$\frac{d}{dt}(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{u}' \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{v}'$$

Proof. It could be seen that $\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2$. Remember that the functions consist in the coordinates. Hence the chain rule must be used.

[1] As you may notice, one curve may have different parameterizations. Some parametrizations also have special properties, such as the norm of its derivative being constant, which is called parametrization by arc length. In physics terms, it would mean that the velocity of the curve is constant

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{u}\cdot\boldsymbol{v}) &= \frac{d}{dt}(u_1v_1+u_2v_2)\\ &= u_1v_1'+u_1'v_1+u_2v_2'+u_2'v_2\\ &= u_1v_1'+u_2v_2'+u_1'v_1+u_2'v_2\\ &= \boldsymbol{u}'\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{v}'\end{aligned} \tag{4.1}$$

For dimensions higher than 2, similar methods could be used to prove the statement by adding u_3v_3, u_4v_4 etc. □

4.2 Motion equations

Note that although the equations are familiar, the equation below applies to $\mathbb{R}^2$ and $\mathbb{R}^3$ as they are expressed in vector notation. As $v^2 = u^2+2as$ could be derived from conservation of energy, it should come naturally after reading section 4.5

Theorem 4.2.1. If the acceleration vector, $\boldsymbol{a}$, is constant, then the displacement is equal to

$$s = \frac{t^2}{2}\boldsymbol{a} + t\boldsymbol{u_v} + \boldsymbol{u_s}$$

Where $\boldsymbol{u_v}$ and $\boldsymbol{u_s}$ are the velocity and the displacement vectors when $t = 0$ respectively.

Proof. We know that,

$$\frac{d}{dt}\boldsymbol{s} = \boldsymbol{v},\ \ \frac{d}{dt}\boldsymbol{v} = \boldsymbol{a} \tag{4.2}$$

Hence,

$$\begin{aligned}\boldsymbol{v} &= \int \boldsymbol{a}dt + \boldsymbol{u_v}\\ &= t\boldsymbol{a} + \boldsymbol{u_v}\end{aligned} \tag{4.3}$$

where $\boldsymbol{u_v}$ is the velocity vector at $t = 0$.
Similarly,

$$\begin{aligned} \boldsymbol{s} &= \int \boldsymbol{v} dt + \boldsymbol{u_s} \\ &= \int t\boldsymbol{a} + \boldsymbol{u_v} dt + \boldsymbol{u_s} \\ &= \frac{t^2}{2}\boldsymbol{a} + t\boldsymbol{u_v} + \boldsymbol{u_s} \end{aligned} \tag{4.4}$$

where $\boldsymbol{u_s}$ is the displacement vector at $t = 0$

□

4.3 Uniform Circular Motion

Horizontal uniform circular motion in $\mathbb{R}^2$ is all about two assumptions, that is the norm of the displacement vector and the velocity vector are both constant. [2] Only using this information, a lot could already be deduced about the motion of the object. Let us start by proving a useful theorem.

Theorem 4.3.1. Let u and u' be two vectors in the 2 dimensional plane. Given that the norm of u is constant, then $u \cdot u' = 0$

Proof. Consider the derivative of $u \cdot u$. According to the theorem 4.1.1

$$\frac{d}{dt}\boldsymbol{u} \cdot \boldsymbol{u} = 2\boldsymbol{u}' \cdot \boldsymbol{u} \tag{4.5}$$

As it was given that the norm of $\boldsymbol{u}$ is a constant, there exist a number c such that

[2] if we translate the plane such that the origin is the centre of rotation

$$\begin{aligned} ||\boldsymbol{u} \cdot \boldsymbol{u}|| &= c \\ \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} &= c \\ \boldsymbol{u} \cdot \boldsymbol{u} &= c^2 \end{aligned} \tag{4.6}$$

Hence,

$$\begin{aligned} \frac{d}{dt} c^2 &= 2\boldsymbol{u}' \cdot \boldsymbol{u} \\ 0 &= \boldsymbol{u}' \cdot \boldsymbol{u} \end{aligned} \tag{4.7}$$

$\square$

In uniform circular motion, the norm / length of the displacement vector and the velocity vector is constant. Hence $||\boldsymbol{s}||$ and $||\boldsymbol{v}|| = ||\boldsymbol{s}'||$ are both constants. I shall use the term radius and displacement interchangeably.

Hence, we could use the theorem 4.3.1 to show that $\boldsymbol{s} \cdot \boldsymbol{s}'$ and $\boldsymbol{s}' \cdot \boldsymbol{s}''$ are both 0 ($\boldsymbol{s} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{a} = 0$) Hence the velocity is orthogonal to the displacement, and the acceleration is orthogonal to the velocity / parallel to the displacement. [3]

In fact, one of the ways you could parameterize uniform circular motion is by the following equation

$$\boldsymbol{s} = (r\cos\theta, r\sin\theta) = (r\cos wt, r\sin wt)$$

where w is the angular velocity and r is the radius, both being constants. Note that $wt = \theta$. The object moves from (r, 0) at t = 0 to (0, r), (-r, 0) then back to (0, r). You could observe that $||\boldsymbol{s}|| = \sqrt{r^2(cos^2wt + sin^2wt)} = r$. By taking derivatives

[3] If 2 vectors are parallel, they are scalar multiples of each other.

$$\boldsymbol{s}' = \boldsymbol{v} = (-rw\sin wt, rw\cos wt)$$

$$\boldsymbol{s}'' = \boldsymbol{a} = (-rw^2\cos wt, -rw^2\sin wt)$$

We could observe that $||\boldsymbol{a}|| = rw^2$, which is also what we learn in physics. We could also verify that $\boldsymbol{s}\cdot\boldsymbol{s'} = 0$

$$\begin{aligned}\boldsymbol{s}\cdot\boldsymbol{s'} &= -rw\sin wt\cdot r\cos wt + rw\cos wt\cdot r\sin wt \\ &= -rw^2\sin wt\cos wt + rw^2\sin wt\cos wt = 0\end{aligned} \tag{4.8}$$

4.4 Work Done

The full definition of work done is,

$$\int_a^b F(\boldsymbol{s})d\boldsymbol{s}$$

Where $\boldsymbol{s}$ is the displacement, $F(\boldsymbol{s})$ is the function of the force at position s, b and a are the displacements at some time t_2 and t_1 respectively. As $d\boldsymbol{s}/dt = \boldsymbol{v}$, $d\boldsymbol{s} = \boldsymbol{v}\,dt$. Work done could also be expressed as[4]

$$\int_{t_1}^{t_2} \boldsymbol{F}\cdot\boldsymbol{v}dt$$

$\boldsymbol{F}$ and $\boldsymbol{v}$ denotes respectively the function of the force and the velocity with respect to time. t_1 and t_2 are the time the object reaches position a and b. We could immediately apply this to a few examples.

Example 4.4.1. Show why work done is 0 in uniform circular motion

[4] Think of the dot product as a logical extension of multiplication

$$\int_{t_1}^{t_2} \boldsymbol{F} \cdot \boldsymbol{v} dt = m \int_{t_1}^{t_2} \boldsymbol{a} \cdot \boldsymbol{v} dt$$

As the acceleration and the velocity is perpendicular at all times. $\boldsymbol{a} \cdot \boldsymbol{v} = 0$ The indefinite integral is always 0.

Example 4.4.2. Show why Work Done is taught to be $Fs \sin \theta$

In DSE-esque questions, the force is normally constant and hence $\boldsymbol{F}(\boldsymbol{s})$ outputs a constant vector (i.e. The force doesn't depend on the displacement) As such,

$$\int_a^b \boldsymbol{F}(\boldsymbol{s}) d\boldsymbol{s} = \boldsymbol{F}(\boldsymbol{s}) \int_a^b d\boldsymbol{s} = \boldsymbol{F}(\boldsymbol{s}) \cdot (\boldsymbol{b} - \boldsymbol{a})$$

From linear algebra, we know that $\boldsymbol{F} \cdot \boldsymbol{s} = ||\boldsymbol{F}|| ||\boldsymbol{s}|| \sin \theta$. [5]

4.5 Energy

Example 4.5.1. Show that $P.E. = mgh$

In this example, using the original definition is actually better. As potential energy means the work done by the gravitational force on an object, we only have to consider the gravitational force. Hence, $\boldsymbol{F}(\boldsymbol{s}) = m\boldsymbol{g}$.

Let c be a positive constant. Remember that the norm of a real coordinate is always positive. Realize that $\boldsymbol{g} \cdot \boldsymbol{s} = (0, -c) \cdot (s_x, h) = -ch = -\sqrt{g \cdot g} h = -||g||h$. Hence,

[5] It is very important to know that the angle between vectors are imaginary constructs that we **define**. The dot product between u and v is not defined to be $||\boldsymbol{u}|| ||\boldsymbol{v}|| \sin \theta$, it is $\sin \theta$ that is defined to be $||\boldsymbol{u}|| ||\boldsymbol{v}|| / \boldsymbol{u} \cdot \boldsymbol{v}$

$$\begin{aligned}\int_a^b m\boldsymbol{g}\,ds &= m(\boldsymbol{g}\cdot\boldsymbol{b}-\boldsymbol{g}\cdot\boldsymbol{a})\\ &= -m||\boldsymbol{g}||h_2+m||\boldsymbol{g}||h_1\\ &= m||\boldsymbol{g}||(h_1-h_2)\end{aligned} \tag{4.9}$$

Note that we ignored the horizontal components of the displacements b and a because they were multiplied by 0.

Example 4.5.2. *Show that $K.E. = 1/2\cdot mv^2$

Firstly, in this equation, v^2 actually means $||\boldsymbol{v}||^2=\boldsymbol{v}\cdot\boldsymbol{v}$. Consider

$$\int_{t_1}^{t_2}\boldsymbol{F}\cdot\boldsymbol{v}dt=\frac{m}{2}\int_{t_1}^{t_2}2\boldsymbol{a}\cdot\boldsymbol{v}dt \tag{4.10}$$

Recall Theorem 4.1.1, which showed that

$$\frac{d}{dt}(\boldsymbol{v}\cdot\boldsymbol{v})=\boldsymbol{v}\cdot\boldsymbol{a}+\boldsymbol{a}\cdot\boldsymbol{v}=2\boldsymbol{a}\cdot\boldsymbol{v}$$

Hence,

$$\begin{aligned}\frac{m}{2}\int_{t_1}^{t_2}2\boldsymbol{a}\cdot\boldsymbol{v}dt &= \frac{m}{2}\int_{t_1}^{t_2}\frac{d}{dt}(\boldsymbol{v}\cdot\boldsymbol{v})dt\\ &= \frac{m}{2}v_2\cdot v_2-\frac{m}{2}v_1\cdot v_1\end{aligned} \tag{4.11}$$

Example 4.5.3. **Show that K.E. + P.E. is kept constant.

This statement only holds true when there does not exist any other force other than the gravitational force during $t_1<t<t_2$. Hence, we could assume that $m\boldsymbol{g}$ is the only force that is applied on the object from $t_1<t<t_2$. [6]

[6] It is very important to realize that the two equations are equivalent simply because both equations have the same force in mind.

$$\frac{m}{2}\boldsymbol{v_2}\cdot\boldsymbol{v_2} - \frac{m}{2}\boldsymbol{v_1}\cdot\boldsymbol{v_1} = m\boldsymbol{g}(h_1 - h_2)$$

$$\frac{m}{2}\boldsymbol{v_2}\cdot\boldsymbol{v_2} + m||\boldsymbol{g}||h_2 = \frac{m}{2}\boldsymbol{v_1}\cdot\boldsymbol{v_1} + m||\boldsymbol{g}||h_1$$

If there are more forces involved, then the initial R.H.S. of the equation will become more complicated. Also note that the L.H.S. is always the same.

To advance further, the knowledge of cross products is required, something that I have avoided due to their tedium. While its calculations are tedious, its properties are not. As such, let us ignore the calculations for now.

Definition 4.5.1. The cross product, $\boldsymbol{u}\times\boldsymbol{v}$, between two vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$ in the $\mathbb{R}^3$ plane, is a $\mathbb{R}^3$ vector that have these properties:[7]

$$(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{u} = 0$$
$$(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{v} = 0$$
$$||\boldsymbol{u}\times\boldsymbol{v}|| = |det(\boldsymbol{u},\boldsymbol{v})| = ||\boldsymbol{u}||||\boldsymbol{v}||\sin\theta$$

As the cross product is perpendicular to two vectors at once, you could imagine that u and v forms a 2d plane in 3d space, and the cross product lies on a line that is perpendicular to that plane and passes through the origin.

For those unfamiliar with the determinant of a matrix, $||u\times v||$ is equal to the area of a parallelogram that is drawn by connecting the origin, u, v, and u + v, which is twice the area of the triangle drawn by connecting the origin, u and v. Hence, it is equal to $||u||||v||sin\theta$ For example, $||(1,0,0)\times(0,1,0)|| = 1$ and $||(2,0,0)\times(4,5,0)|| = 5\cdot 2 = 10$. (Refer to example 1.5.2)

[7] It's still very important to remember that any mentioning of angles in linear algebra are arbitrary constructs that **we define**. It's $\sin\theta$ that is defined to be $||\boldsymbol{u}\times\boldsymbol{v}||/||\boldsymbol{u}||/||\boldsymbol{v}||$

You may realize that there are two possible orientation for the cross product. It could be in theory either be placed perpendicularly upwards or downwards from the plane drawn by u and v. This ambiguity is solved by the famous *right hand rule*. In essence, by using your right hand, face your index finger towards u and your middle finger towards v. The orientation of your thumb is the orientation of the cross product (i.e. $(1,0,0) \times (0,1,0)$ points "upwards" while $(0,1,0) \times (1,0,0)$ points "downwards".)

4.6 Torque

In 3d, the torque could be generalized as

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

$$\tau = ||\boldsymbol{\tau}|| = ||\boldsymbol{r}||||\boldsymbol{F}|| \sin\theta$$

From this definition, you could see that anticlockwise rotations will point upwards the plane of the rotation and clockwise rotations will point downwards.

Example 4.6.1. Let there be a sphere that's centered at the origin with a radius of 5. A force pointing towards the x axis $(1,0,0)$ is acted on $(3,4,0)$ with a magnitude of 5. $(F = (5,0,0))$ Find the torque of the force.

From imagining the 2d diagram. We know that the rotation is a clockwise one. This could be determined from the right hand rule. As there's an anticlockwise "movement" from $\boldsymbol{F}$ to $\boldsymbol{r}$. $\boldsymbol{F} \times \boldsymbol{r}$ points upwards, hence $\boldsymbol{r} \times \boldsymbol{F}$ points downwards. Hence the rotation is clockwise.

From some coordinate geometry, we know that the area of the parallelogram spanned by $\boldsymbol{F}$ and $\boldsymbol{r}$ is equal to $\tau = 5 \cdot 4 = 20$. Hence $\boldsymbol{\tau} = (0,0,-20)$.

4.7 Advanced uniform circular motion

The parameterisation that was featured earlier at 4.3 only applies in the 2 dimensional case and was used to prove the ratios between the norm of the displacement, the velocity and the acceleration, and that they were perpendicular. However, they could be further generalized so that they could be dealt with in the case of 3 dimensions, just like how the moment could be further generalized in 4.6. Note that as usual, we will only use the xy plane.

We will start from the angular velocity. As we know from physics,

$$||\boldsymbol{w}|| = \frac{||\boldsymbol{s}||}{||\boldsymbol{v}||}$$

Hence, we only have to figure out an unit vector that points to the direction we intended, and multiply it with the norm. Similar to the moment, we will adopt the same convention that an anticlockwise velocity will have an angular velocity that points upwards. Hence, we only have to find that unit vector and multiply the norm to find out the angular velocity. As $\boldsymbol{s}$ and $\boldsymbol{v}$ are perpendicular in uniform circular motion (Refer to example 4.3), Hence, $||\boldsymbol{s} \times \boldsymbol{v}|| = ||\boldsymbol{s}||||\boldsymbol{v}||$ as $\sin\theta = 1$

$$\boldsymbol{u} = \frac{\boldsymbol{s} \times \boldsymbol{v}}{||\boldsymbol{s} \times \boldsymbol{v}||} = \frac{\boldsymbol{s} \times \boldsymbol{v}}{||\boldsymbol{s}||||\boldsymbol{v}||}$$

Hence,

$$\boldsymbol{w} = ||\boldsymbol{w}||\boldsymbol{u} = \frac{||\boldsymbol{v}||(\boldsymbol{s} \times \boldsymbol{v})}{||\boldsymbol{s}||||\boldsymbol{s}||||\boldsymbol{v}||} = \frac{\boldsymbol{s} \times \boldsymbol{v}}{\boldsymbol{s} \cdot \boldsymbol{s}}$$

Conclusion

I think at this moment, it is obvious that mathematics plays a large role in physics. Understanding the mathematics behind will enrich the foundation in physics and pave way for more difficult concepts. Sadly, its role on DSE physics' "fourth" chapter, Electricity and

Magnetism, will be left undiscussed as it's beyond the scope of this journal. For those who love mathematics, it's important to recognize its applications from other fields and seek inspirations from there.