An Easy Way to Understand: How to Prove a Function has Limit - From Definition

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1 Introduction

Due to the difficulty in understanding how to prove a function has a limit from the definition, the author would like to summarize the proof procedure, letting readers easily solve the questions and have clear comprehension.

2 The Formal Definition of Limit (Single Variable)

Before introducing the author's summary of the solution, we must briefly understand the book's definition and proof procedure.
We say that

$$\lim_{x \to a} f(x) = L$$

when

$$\forall\ \epsilon > 0,\ \exists\ \delta > 0\ s.t.\ 0 < |x\ -\ a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

It's quite easy to understand the formal definition of a limit after apprehending the meaning of those two inequalities, so we will directly talk about the easy way to understand how to prove it.

3 Understanding the Procedure

The procedure could be mainly separated into two parts: the rough work for reverse reasoning and the proof. Let's focus on the reverse reasoning part first.
The author will raise a question first to make the steps much more specific and easier to understand.
Example: Prove that $\lim\limits_{x \to 1} (2x - 1) = 1$.

3.1 Reverse Reasoning

Rough Work: Since

$$|f(x) - L| = |(2x - 1) - 1| = 2|x - 1|,$$

To satisfy $|f(x) - L| < \epsilon$, we want

$$|x - 1| < \frac{\epsilon}{2}.$$

These two lines are the reverse-reasoning part of the proof procedure. Here are three probable questions that may be raised:

1. Why do we need to do the reverse reasoning?

   (a) Following the definition, given $\epsilon > 0$, we need to provide the δ in the $0 < |x - a| < \delta$. Because we couldn't directly find the value of δ, it's better to use reverse reasoning to find the δ, which is the reason for reverse-reasoning.

2. What can we get from reverse reasoning?

   (a) The result from reverse-reasoning is the value of δ, which could be an abstract parameter or an exact number.

3. How do we do the reverse reasoning?

   (a) This is the most essential part of the proof procedure, and we need to rephrase $|f(x) - L|$ to find the relationship between it and $|x - a|$.

3.2 The Proof

Let $\epsilon > 0$ be given and choose $\delta = \frac{\epsilon}{2}$.
Assume $0 < |x - 1| < \delta$, then we have:

$$|f(x) - 1| = |(2x - 1) - 1| < \epsilon$$

Hence, we have precisely shown that $\lim_{x \to 1}(2x - 1) = 1$.
Quad Erat Demonstrandum.

4 Conclusion

There are only two simple steps before obtaining the proof procedure of a function's limit from definition.

1. Idea: $\forall\ \epsilon > 0$, we only need to provide the δ's value in the $0 < |x - a| < \delta$.
2. Step 1 (Reverse Reasoning): Change $|f(x) - L|$ into an equation that includes $|x - a|$.
3. Step 2 (Finding the Relationship): Set δ as $\frac{\epsilon}{k}$, where k is the coefficient from the reverse reasoning.

Thus, we only need to set δ as the value we found during the reverse reasoning. Then we could simply say the proof is done.