Approaches to Prove an Important Limit

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1 Introduction

The limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ is a fundamental result in calculus, often used in various applications and proofs. In this article, we will explore three approaches to proving this limit:

- Taylor series expansion
- Squeeze Theorem
- L'Hôpital's Rule

2 Approach 1: Taylor Series Expansion

The Taylor series expansion of $\sin x$ around $x = 0$ is given by:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Dividing both sides by x, we get:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

Taking the limit as $x \to 0$, all the higher-order terms involving $x^2, x^4, \ldots$ tend to 0, so:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

This approach leverages the fact that the Taylor series provides an accurate approximation of $\sin x$ near 0, showing that the ratio $\frac{\sin x}{x}$ approaches 1 as x approaches 0.

3 Approach 2: Squeeze Theorem

We start by noting that for all $x \in R$, the following inequality holds:

$$\cos x \leq \frac{\sin x}{x} \leq 1 \quad \text{for } 0 < x < \frac{\pi}{2}.$$

As $x \to 0$, $\cos x \to 1$. Hence, by the Squeeze Theorem:

$$\lim_{x\to 0} \cos x = 1 \quad \text{and} \quad \lim_{x\to 0} 1 = 1,$$

which implies:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

This approach uses the fact that $\frac{\sin x}{x}$ is squeezed between two functions that both tend to 1 as $x \to 0$.

4 Approach 3: L'Hôpital's Rule

The limit $\lim_{x \to 0} \frac{\sin x}{x}$ is an indeterminate form of type $\frac{0}{0}$. L'Hôpital's Rule applies in this case, allowing us to differentiate the numerator and the denominator:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1}.$$

Since $\cos 0 = 1$, we have:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

This approach applies a fundamental tool in calculus, L'Hôpital's Rule, to directly evaluate the limit.

5 Conclusion

In this article, we have explored three different, yet rigorous, approaches to proving that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. Each method provides a different perspective on the problem and is a valuable tool in calculus.