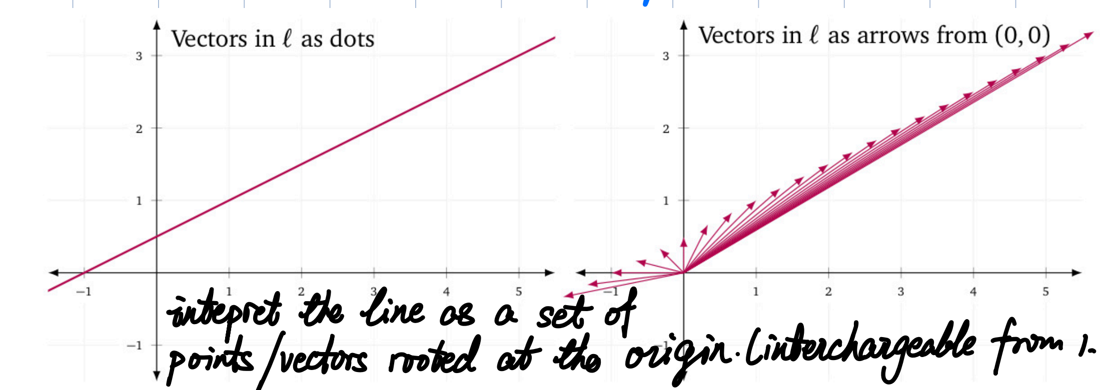
Sets of Vectors & Lines

☆ vector form is a specific shorthand notation. Can't be augmented any thing.

1. Vector Form of a Line: Let ℓ be a line and let $\vec{d}$ and $\vec{p}$ be vectors. If $\ell = \{\vec{x} \mid \vec{x} = t\vec{d} + \vec{p}$ for some $t \in \mathbb{R}\}$, we say the vector equation: can't be 'some' or 'all'

$$\vec{x} = t\vec{d} + \vec{p} \quad (\text{where } t \in \mathbb{R})$$

is ℓ expressed in vector form. The vector $\vec{d}$ is called a direction vector for ℓ.



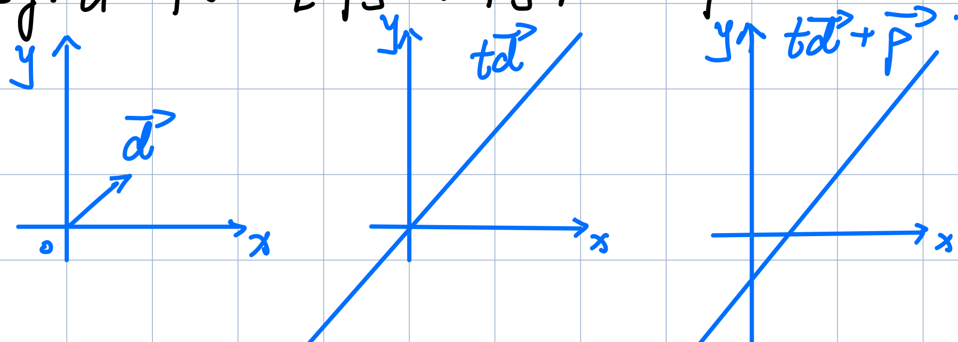
interpret the line as a set of points/vectors rooted at the origin. (interchangeable from 1.1)

1) Using coordinates when writing a line in vector form

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

the line passing through $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ (any point on the line) with $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ as a direction vector.

e.g. $\ell_1 = \{x = t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mid t \in \mathbb{R}\}$



2) Find vector form of the line $l \subseteq \mathbb{R}^2$ with equation.

e.g. $y = 2x + 3$.

method 1: ① find 2 points on equation (direction vector).

$P = (0, 3)$ $Q = (1, 5)$.

$$\vec{d} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

② let one of the point be $\vec{p}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

method 2: ① Use a variable to express other.

i.e. $y = 2x + 3$.

② List it out.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x+3 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

③ Change to t.

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

2. Determine the relationship between lines.

1) $l_1 = t \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} c_1 \\ d_1 \end{bmatrix}$; $l_2 = t \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} + \begin{bmatrix} c_2 \\ d_2 \end{bmatrix}$

① Parallel or identical (过同一点).

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \lambda \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \text{ for some } \lambda \in \mathbb{R}.$$

if two vectors are orthognal then use the same method.

② Orthogonal

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = 0 \quad (a_1 a_2 + b_1 b_2 = 0).$$

③ Intersect: 联立后看有没有解.

(consistent or inconsistent)

e.g. Determine if the lines l_1 & l_2, given in vector form as:

$$\vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix} \text{ and } \vec{x} = t\begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}4\\3\end{bmatrix}$$

are the same line.

① Give different parametric variables different names.

If $\vec{x} \in l_1$, $\vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix}$, where $t \in \mathbb{R}$.

If $\vec{x} \in l_2$, $\vec{x} = s\begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}4\\3\end{bmatrix}$, where $s \in \mathbb{R}$.

② Set their equations equal and solve.

$$t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix} = \vec{x} = s\begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}4\\3\end{bmatrix}.$$

$$\Leftrightarrow \begin{bmatrix}t+2\\t+1\end{bmatrix} = \begin{bmatrix}2s+4\\2s+3\end{bmatrix}.$$

$$\begin{cases} t+2 = 2s+4 \\ t+1 = 2s+3. \end{cases} \Leftrightarrow 0 = 2s - t + 2$$

This equation has a solution whenever $0 = 2s - t + 2$ has a solution. $l_1 = l_2$.

2) $\mathbb{R}^3$ and higher. $\begin{cases} \text{parallel (direction vector is a multiple of other).} \\ \text{intersect} \\ \text{skew (not parallel or intersect).} \end{cases}$

e.g. $\vec{x} = t(1, 3, -2) + (1, 2, 1)$; $\vec{x} = t(0, 2, 3) + (0, 3, 9)$.

to get point of intersection. we need to put $t=-1$ in $\vec{x_1}$ or $s=-2$ in $\vec{x_2}$.

if $\vec{x} \in l_1$, $\vec{x} = t(1,3,-2)+(1,2,1)$ where $t \in \mathbb{R}$.

if $\vec{x} \in l_2$, $\vec{x} = s(0,2,3)+(0,3,9)$ where $s \in \mathbb{R}$.

$\vec{x} = t(1,3,-2)+(1,2,1) = s(0,2,3)+(0,3,9)$.

$\Leftrightarrow (t+1, 3t+2, -2t+1) = (0, 2s+3, 3s+9)$

$$\begin{cases} t+1 = 0 \\ 3t+2 = 2s+3 \\ -2t+1 = 3s+9 \end{cases} \Leftrightarrow \begin{cases} t = -1 \\ s = -2 \end{cases}$$

$(-2)(-1)+1 = 3 = 3\times(-2)+9$

hence, the equations are consistent, and the lines intersect

(solution 2). $\begin{cases} t = -1 \\ 3t-2s = 1 \\ -2t-3s = 8. \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 3 & -2 & 1 \\ -2 & -3 & 8 \end{array}\right]$

$$\xrightarrow[r_3+2r_1]{r_2-3r_1} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -2 & 4 \\ 0 & -3 & 6 \end{array}\right]$$

$$\xrightarrow[r_3\times(-\frac{1}{3})]{r_2\times(-\frac{1}{2})} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{array}\right]$$

$\therefore \begin{cases} t = -1 \\ s = -2. \end{cases}$