Proof Limit DNE from Def

1. Formal Def of 'limit DNE'.

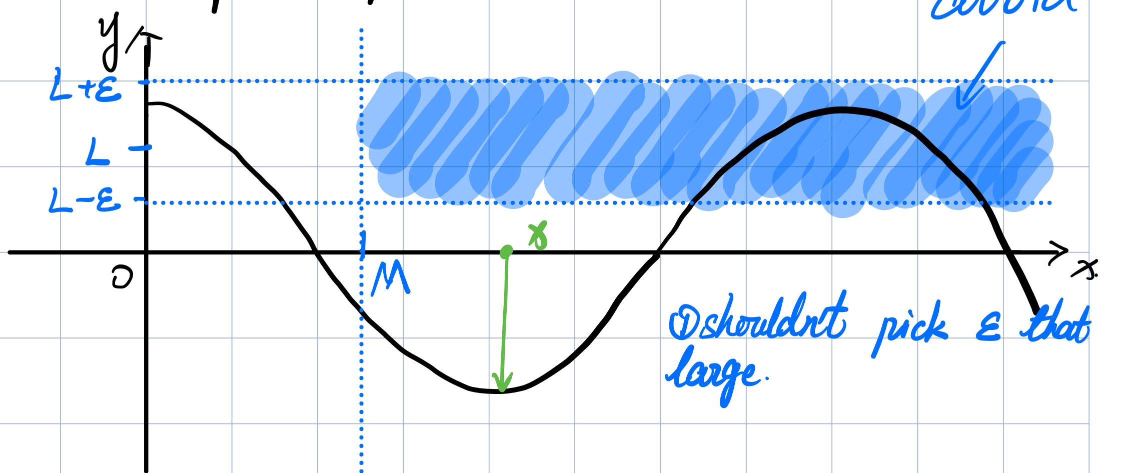
1) $x \to \infty$

① $\lim_{x\to\infty} f(x) = L$ means: $\forall \varepsilon > 0, \exists M \in \mathbb{R}$ s.t. $(\forall x \in \mathbb{R}).\ x > M \Rightarrow |f(x) - L| < \varepsilon$.

② $\lim_{x\to\infty} f(x) \neq L$ means: $\exists \varepsilon > 0$ s.t. $\forall M \in \mathbb{R}$. $(\exists x \in \mathbb{R})$. $x > M$ and $|f(x) - L| \geq \varepsilon$

③ $\lim_{x\to\infty} f(x)$ DNE means:

$\forall L \in \mathbb{R}$, $\exists \varepsilon > 0$ s.t. $\forall M \in \mathbb{R}$. $(\exists x \in \mathbb{R})$. $x > M$ and $|f(x) - L| \geq \varepsilon$.



2. Structure of proving limit DNE.

1) Let $L \in \mathbb{R}$.

2) Take $\varepsilon = \dots$ (with L in it)

3) Let $M \in \mathbb{R}$.

4) Take $x = \dots$ (with L or M in it).

5) Verify that $x > M$, and $|h(x) - L| \geq \varepsilon$.

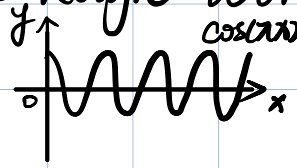
no matter whether it approches ∞ or not.

3. Example

oscillating.

1) Let $h(x) = \cos(\pi x)$. Proof from def. $\lim_{x\to\infty} h(x)$ doesn't exist.

1) Rough Work.



if we take $\varepsilon \leq 1$. Then 1 or -1 $\notin (L-\varepsilon, L+\varepsilon)$.

if we take $x \in \mathbb{Z}$, Then $h(x) = 1$ or -1

2) Proof.

Let $L \in \mathbb{R}$. Take $\varepsilon = 0.5$. Let $M \in \mathbb{R}$.

At least one of below must be true.

one of them goes out of the shaded region.

Case A: $1 \notin (L-\varepsilon, L+\varepsilon)$.

Take $x \in \mathbb{Z}$, even, satisfying $x > M$. Then $h(x) = 1$.

Case B. $-1 \notin (L-\varepsilon, L+\varepsilon)$.

Take $x \in \mathbb{Z}$. odd, satisfying $x > M$. Then $h(x) = -1$.

best to verify an think.

Either way, it satisfies $x > M$ and $|h(x) - L| \geq \varepsilon$.

4. Conclusion.

1) Proof DNE by using δ & ε, write the full form of negation first:

$\forall L \in \mathbb{R}, \exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R}$ s.t. $0 < |x - x_0| < \delta$ and $|f(x) - L| \geq \varepsilon$

No matter $\lim_{x\to x_0} f(x) = \pm\infty$ or not, we should write L here, because we regard it as DNE.; the only place might be changed is $0 < |x - x_0| < \delta$ (when its $x \to x_0^+$, then $0 < x - x_0 < \delta$; $x \to \infty$, $x > \delta$ ……)

2) During the KW:

① Let $L \in \mathbb{R}$.

② Take the value of ε depends on the reason why the function's limit DNE (not greater than range).

$\longrightarrow$ ∞: take any ε you like.

$\longrightarrow$ oscillating: smaller than range.

$\longrightarrow$ 分段函数 ($\lim_{x\to x_0^+} f(x) \neq \lim_{x\to x_0^-} f(x)$) 值域内 (涛风下方)

③ Let $\delta > 0$.

$0 < x < \frac{\delta}{n} < \delta$.

④ Take the value of x. (take $x = \min\{\frac{\delta}{n}, \cdots\}$

$\longrightarrow$ ∞: consider x satisfying $|f(x) - L| = f(x) - L \geq \varepsilon$.

$\longrightarrow$ oscillating: consider x satisfying: when one of the two boundary is not included in $O(L, \varepsilon)$.

$\longrightarrow$ 分段函数: x 在断裂点左右分别取值. 有

$f(x_1) = A$, $f(x_2) = B$. $|f(x_1) - L| + |f(x_2) - L|$.

$$\Rightarrow |f(x_1) - L| + |L - f(x_2)|$$

$$\geq |f(x_1) - L + L - f(x_2)|$$

$$= |f(x_1) - f(x_2)| = |A - B|$$

$$|A - B| = \boxed{2\varepsilon}$$