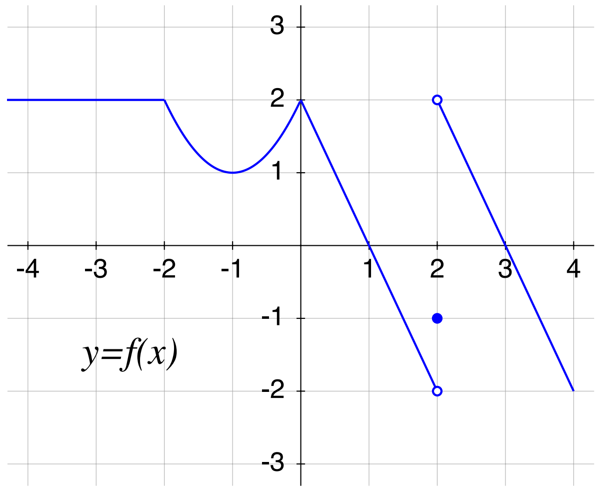
MAT 137Y – Practice problems
Unit 2 : Limits and continuity

1. Below is the graph of the function f:



Compute the following limits

(a) $\lim_{x \to 2} f(x)$ DNE

(b) $\lim_{x \to 0} f(f(x))$ -2

(c) $\lim_{x \to -3} f(f(x))$ -1

(d) $\lim_{x \to 0} f(2 \sec x)$ 2

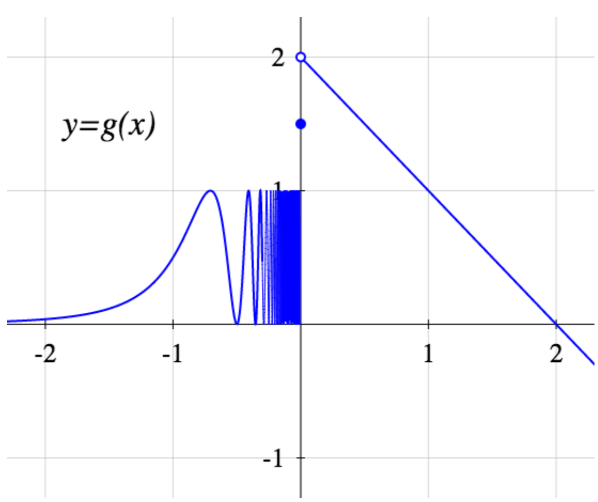
(e) $\lim_{x \to 2} (f(x))^2$ 4

2. Given a real number x, we defined the *floor of* x, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to x. For example, $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$, and $\lfloor -0.5 \rfloor = -1$.

(a) Sketch the graph of this function. At which points is the function $f(x) = \lfloor x \rfloor$ continuous? Which discontinuities are removable and which ones are non-removable?

(b) Consider the function $h(x) = \lfloor \sin x \rfloor$. Show that h has exactly one removable and one non-removable discontinuity inside the interval $(0, 2\pi)$.

3. Below is the graph of the function g:



For clarification, when $-1 < x < 0$, $g(x)$ "oscillates" between 0 and 1; as x approaches 0 from the left, these oscillations become faster and faster. The behaviour is similar to that of the function $f(x) = \sin(\pi/2x)$, which you can see on Video 2.2. Find the following limits:

(a) $\lim_{x\to 0^+} g(x)$

(b) $\lim_{x\to 0^+} \lfloor g(x) \rfloor$

(c) $\lim_{x\to 0^+} g(\lfloor x \rfloor)$

(d) $\lim_{x\to 0^-} g(x)$

(e) $\lim_{x\to 0^-} \lfloor g(x) \rfloor$

(f) $\lim_{x\to 0^-} \lfloor \frac{g(x)}{2} \rfloor$

(g) $\lim_{x\to 0^-} g(\lfloor x \rfloor)$

4. Compute the following limits

(a) $\lim_{x\to 1} \frac{x+1}{x+2}$

(b) $\lim_{x\to 2} \frac{x^2+3x-10}{x^2-4}$

(c) $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$

(d) $\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)}$

(e) $\lim_{x\to \infty} \frac{x^3+2x^2+1}{5x^3+6x-1}$

(f) $\lim_{x\to -\infty} \frac{x^5+2x^2+1}{5x^3+6x-1}$

(g) $\lim_{x\to \infty} \frac{\sqrt{x^4+2x+1}+3x^2+1}{x^2}$

(h) $\lim_{x\to 1} \frac{x^3-1}{x^2-2x+1}$

(i) $\lim_{x\to 0} \frac{\sin^{10}(2\sin^{10}(3x))}{x^{100}}$

5. Write the formal definition of the following concepts:

(a) $\lim_{x\to a} f(x) = L$

(b) $\lim_{x\to a} f(x)$ exists

(c) $\lim_{x\to a} f(x) \neq L$

(d) $\lim_{x\to a} f(x)$ doesn't exist

(e) $\lim_{x\to a^+} f(x) = L$

(f) $\lim_{x\to a} f(x) = \infty$

(g) $\lim_{x\to a^-} f(x) = -\infty$

(h) $\lim_{x\to \infty} f(x) = L$

(i) $\lim_{x\to -\infty} f(x) = \infty$

6. Prove the following claims directly from the formal definitions.

(a) $\lim_{x\to 2}(4x+1) = 9$

(b) $\lim_{x\to \infty} \frac{1}{x^2} = 0$

(c) $\lim_{x\to 1} x^3 = 1$

(d) $\lim_{x\to 1} \frac{1}{x^2+1} = \frac{1}{2}$

(e) $\lim_{x\to 0} \frac{x}{|x|}$ does not exist

(f) $\lim_{x\to 1^+} \frac{1}{1-x} = -\infty$

7. Let $a, L, M \in \mathbb{R}$. Let f be a function defined, at least, on an interval centered at a, except maybe at a. Prove that

$$\text{IF } \lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M \qquad \text{THEN } \lim_{x\to a} [f(x) - g(x)] = L - M.$$

Write a proof directly from the formal definitions, without using any of the limit laws.

8. Let $a \in \mathbb{R}$. Let f be a function defined at least on an interval centered at a, except possibly at a. Prove that

$$\text{IF } \lim_{x\to a} f(x) = \infty \qquad \text{THEN } \lim_{x\to a} \frac{1}{f(x)} = 0.$$

Write a proof directly from the formal definitions, without using any of the limit laws.

9. Construct a function f with domain $\mathbb{R}$ such that $\lim_{x\to 0} f(x) = 0$ but $\lim_{x\to 0} f(f(x)) \neq 0$.

10. Prove Theorem 3 on Video 2.16. More specifically:

    Let $a, L \in \mathbb{R}$. Let f be a function defined, at least, on an interval centered at a, except maybe at a. Let g be a function defined at least on an interval centered at L. Prove that

    $$\text{IF } \lim_{x\to a} f(x) = L \text{ and } g \text{ is continuous at } L \qquad \text{THEN } \lim_{x\to a} g(f(x)) = g(L).$$

    Write a proof directly from the formal definitions, without using any of the limit laws.

11. Use the Intermediate Value Theorem to prove that the equation

    $$\sin x = 2\cos^2 x + 0.5$$

    has at least one solution.

12. Use the Squeeze Theorem to explain why $\lim_{x\to 0} x\cos\frac{1}{x}$ exists, even though $\lim_{x\to 0} \cos\frac{1}{x}$ does not exist. Explain why the same argument does not work for $\lim_{x\to 0} xe^{1/x^2}$.

Bonus question:
Do you *really* understand the definition of limit?

13. Let f be a function. Let $a, L \in \mathbb{R}$. Assume that f is defined on some open interval around a, except maybe at a. Below is a list of nine statements.

    a. $\forall \varepsilon > 0, \exists \delta > 0$ such that $\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.

    b. $\forall \varepsilon > 0, \exists \delta > 0$ such that $\quad |x-a| < \delta \implies |f(x) - L| < \varepsilon$.

    c. $\forall \varepsilon > 0, \exists \delta > 0$ such that $\quad 0 < |x-a| < \delta \implies \mathbf{0} < |f(x) - L| < \varepsilon$.

    d. $\forall \varepsilon \geq \mathbf{0}, \exists \delta > 0$ such that $\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.

    e. $\forall \varepsilon > 0, \exists \delta \geq \mathbf{0}$ such that $\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.

    f. $\forall \varepsilon > 0, \exists \delta > 0$ such that $\quad 0 < |x-a| < \delta \implies |f(x) - L| \leq \varepsilon$.

    g. $\forall \delta > \mathbf{0}, \exists \varepsilon > \mathbf{0}$ such that $\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.

    h. $\forall \delta > \mathbf{0}, \exists \varepsilon > \mathbf{0}$ such that $\quad 0 < |x-a| < \varepsilon \implies |f(x) - L| < \delta$.

    i. $\exists \delta > \mathbf{0}$ such that $\forall \varepsilon > \mathbf{0}$, $\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.

    Match each of the statements above to one of the following (there may be repeats):

    A. Every function satisfies this statement.

    B. There isn't any function which satisfies this statement.

    C. This statement is (equivalent to) the definition of $\lim_{x \to a} f(x) = L$.

    D. This statement is (equivalent to) the definition of "f is continuous at a".

    E. This statement means that $\lim_{x \to a} f(x) = L$ and that, in addition, f does not take the value L anywhere on some interval centered at a, except maybe at a.

    F. This statement is equivalent to saying that f must be constantly equal to L on an interval centered at a, except maybe at a.

    G. This statement means that f is bounded on every interval centered at a.

Some answers and hints

1. (a) DNE (b) -2 (c) -1 (d) 2 (e) 4

2. (a) f is discontinuous at a when $a \in \mathbb{Z}$. f is continuous everywhere else. All the discontinuities are non-removable.

   (b) g has a removable discontinuity at $\frac{\pi}{2}$ and a non-removable discontinuity at π.

3. (a) 2 (b) 1 (c) 1.5 (d) DNE (e) DNE (f) 0 (g) 0.5

4. (a) 2/3 (d) 3/2 (g) 4
   (b) 7/4 (e) 1/5 (h) DNE
   (c) 1/4 (f) ∞ (i) $2^{10}3^{100}$

5. There are various equivalent ways to write each definition. The parts in blue (and only the parts in blue) are often omitted and are considered implicit.

   (a) $\forall \varepsilon > 0, \exists \delta > 0$ such that $(\forall x \in \mathbb{R},)$ $\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$
   (b) $\exists L \in \mathbb{R}$ such that $\forall \varepsilon > 0, \exists \delta > 0$ such that $(\forall x \in \mathbb{R},)$ $\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$
   (c) $\exists \varepsilon > 0$ such that $\forall \delta > 0, \exists x \in \mathbb{R}$ such that $\quad [0 < |x-a| < \delta$ and $|f(x) - L| \geq \varepsilon]$
   (d) $\forall L \in \mathbb{R}, \exists \varepsilon > 0$ such that $\forall \delta > 0, \exists x \in \mathbb{R}$ such that $\quad [0 < |x-a| < \delta$ and $|f(x) - L| \geq \varepsilon]$
   (e) $\forall \varepsilon > 0, \exists \delta > 0$ such that $(\forall x \in \mathbb{R},)$ $\quad a < x < a + \delta \implies |f(x) - L| < \varepsilon$
   (f) $\forall M \in \mathbb{R}, \exists \delta > 0$ such that $(\forall x \in \mathbb{R},)$ $\quad 0 < |x-a| < \delta \implies f(x) > M$
   (g) $\forall M \in \mathbb{R}, \exists \delta > 0$ such that $(\forall x \in \mathbb{R},)$ $\quad a - \delta < x < a \implies f(x) < M$
   (h) $\forall \varepsilon > 0, \exists K \in \mathbb{R}$ such that $(\forall x \in \mathbb{R},)$ $\quad x > K \implies |f(x) - L| < \varepsilon$
   (i) $\forall M \in \mathbb{R}, \exists K \in \mathbb{R}$ such that $(\forall x \in \mathbb{R},)$ $\quad x < K \implies f(x) > M$

6. (a) This is similar to the proof in Video 2.7.

   (b) WTS: $\forall \varepsilon > 0, \exists K \in \mathbb{R}$ such that $\forall x \in \mathbb{R}, \quad x > K \implies \left| \frac{1}{x^2} - 0 \right| < \varepsilon$

   - Fix $\varepsilon > 0$
   - Take $K = \frac{1}{\sqrt{\varepsilon}}$.
   - Fix $x \in \mathbb{R}$. Assume $x > K$. I need to verify that $\frac{1}{x^2} < \varepsilon$.

   $$\frac{1}{x^2} < \frac{1}{K^2} = \varepsilon.$$

   (c) This is similar to the proof in Video 2.8

   (d) WTS: $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in \mathbb{R}, \quad 0 < |x-1| < \delta \implies \left| \frac{1}{x^2+1} - \frac{1}{2} \right| < \varepsilon$

   - Fix $\varepsilon > 0$

- Take $\delta = \min\{1, 2\varepsilon/3\}$. Thus $\delta \leq 1$ and $\delta \leq 2\varepsilon/3$.
- Fix $x \in \mathbb{R}$. Assume $0 < |x-1| < \delta$. I need to verify that $\left|\frac{1}{x^2+1} - \frac{1}{2}\right| < \varepsilon$.
  By assumption, $0 \leq 1-\delta < x < 1+\delta \leq 2$. Thus $|1+x| < 3$.
  In addition $\frac{1}{x^2+1} \leq 1$.

$$\left|\frac{1}{x^2+1} - \frac{1}{2}\right| = \frac{|x+1||x-1|}{2(x^2+1)} < \frac{3\delta}{2\cdot 1} \leq \varepsilon.$$

(e) This is somewhat similar to the proof in Video 2.9.

(f) WTS $\forall M \in \mathbb{R}, \exists \delta > 0$ such that $\forall x \in \mathbb{R}, \quad 1 < x < 1+\delta \implies \frac{1}{1-x} < M$

- Fix $M \in \mathbb{R}$
- Next we need to choose δ. It is probably easiest to break this into two cases.
  - If $M \geq 0$, take $\delta = 1$ for example.
  - If $M < 0$ take $\delta = \frac{1}{|M|}$
- Fix $x \in \mathbb{R}$. Assume $1 < x < 1+\delta$. I need to verify that $\frac{1}{1-x} < M$.

$$\ldots$$

  (Pay careful attention to the signs. Sometimes you will be working with negative numbers.)

7. This proof is very similar to the one in Video 2.11.

8. WTS $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in \mathbb{R}, \quad 0 < |x-a| < \delta \implies \left|\frac{1}{f(x)}\right| < \varepsilon$

- Fix an arbitrary $\varepsilon > 0$.
- Using $\frac{1}{\varepsilon}$ as the bound in the definition of $\lim_{x\to a} f(x) = \infty$, we can conclude that

$$\exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, \quad 0 < |x-a| < \delta \implies f(x) > \frac{1}{\varepsilon}$$

  This is the value of δ I take.
- Let $x \in \mathbb{R}$. Assume $0 < |x-a| < \delta$. I need to verify that $\left|\frac{1}{f(x)}\right| < \varepsilon$.

  This follows immediately from knowing that $f(x) > \frac{1}{\varepsilon} > 0$.

9. This is definitely possible. You will need a function that is not continuous at 0, although being discontinuous at 0 is not enough.

10. I want to prove that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, \quad 0 < |x-a| < \delta \implies |g(f(x)) - g(L)| < \varepsilon.$$

- Fix an arbitrary $\varepsilon > 0$.
- First I use this value of ε in the definition of "g is continuous at L" to conclude that

  $$\exists \delta_0 > 0 \text{ such that } \forall y \in \mathbb{R}, \quad |y - L| < \delta_0 \implies |g(y) - g(L)| < \varepsilon.$$

  Second I use this value of δ_0 "as the epsilon" in the definition of "$\lim_{x\to a} f(x) = L$" to conclude that

  $$\exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, \quad 0 < |x - a| < \delta \implies |f(x) - L| < \delta_0.$$

  This is the value of δ I take.
- Fix $x \in \mathbb{R}$. Assume $0 < |x - a| < \delta$. I need to verify that $|g(f(x)) - g(L)| < \varepsilon$.
  - Since $0 < |x - a| < \delta$, we conclude that $|f(x) - L| < \delta_0$.
  - Since $|f(x) - L| < \delta_0$, we conclude that $|g(f(x)) - g(L)| < \varepsilon$.

11. Consider the function f defined by $f(x) = \sin x - 2\cos^2 x$. f has domain $\mathbb{R}$ and is continuous everywhere.

    $$f(0) = -2 < 0.5, \qquad f(\pi/2) = 1 > 0.5.$$

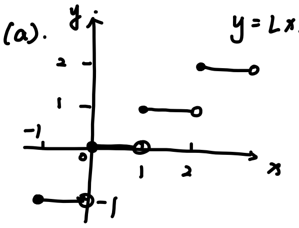
    Therefore, by the Intermediate Value Theorem, $\exists x \in (0, \pi/2)$ such that $f(x) = 0.5$.

12. This is similar to the argument in Video 2.12.

13. A. e B. d C. a, f, h D. b E. c F. i G. g

1. (a). DNE (b). -1 (c) -1 (d). -1 (e). DNE

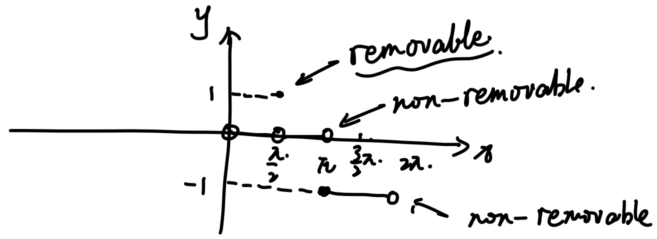
2. (a).



$\forall \varepsilon>0, \exists \delta>0$, s.t., $|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$.

non-removable

(b)



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两边的不可?

3. (a). 2 (b). 2 (c). DNE (d). DNE (e). DNE (f). DNE (g). DNE

4. (a). $\frac{2}{3}$

(b) $= \lim_{x\to 2} \frac{(x+5)(x-2)}{(x+2)(x-2)}$

$= \lim_{x\to 2} \frac{x+5}{x+2} = \frac{7}{4}$

(c) $= \lim_{x\to 1} \frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)}$

$= \lim_{x\to 1} \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)}$

$= \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{4}$

(d). $\lim_{x\to 0} \frac{3x \cdot \frac{\sin 3x}{3x}}{2x \cdot \frac{\sin 2x}{2x}}$

$= \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}$

(e). $\lim_{x\to\infty} \frac{x^3(1+2\cdot\frac{1}{x}+\frac{1}{x^2})}{x^3(5+6\cdot\frac{1}{x^2}-\frac{1}{x^3})}$

$= \frac{1}{5}$

(f). $\lim_{x\to -\infty} \frac{x^5(1+2\cdot\frac{1}{x^3}+\frac{1}{x^5})}{x^5(5\cdot\frac{1}{x^2}+6\cdot\frac{1}{x^4}-\frac{1}{x^5})}$.

$= \infty$

(g). $= \lim_{x\to\infty} \frac{\sqrt{x^4\cdot(1+2/x^3+1/x^4)}+3x^2+1}{x^2}$.

$= \lim_{x\to\infty} \frac{x^2(\sqrt{1+2/x^3+1/x^4}+3+\frac{1}{x^2})}{x^2}$.

$= 4$.

(h) $= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)^2}$

$= \lim_{x\to 1} \frac{x^2+x+1}{x-1}$.

DNE not continuous.

(i). DNE. not continuous.

5. (a). $\forall \varepsilon>0, \exists \delta>0$. s.t. $0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$

(b). $\forall \varepsilon>0, \exists \delta>0$, s.t. $0<|x-a|<\delta \Rightarrow |f(x)-A|<\varepsilon$. $\forall A \in \mathbb{R}$.

(c). $\forall \varepsilon>0$. $\exists \delta>0$. s.t. $0<|x-a|<\delta \Rightarrow f(x)-L>\varepsilon$ or $f(x)-L<\varepsilon$.

(d). $\forall \varepsilon > 0, \exists \delta > 0$. s.t. $0 < |x-a| < \delta \Rightarrow f(x) - A > \varepsilon$ or $f(x) - A < \varepsilon$. $\forall A \in \mathbb{R}$.

(e). $\forall \varepsilon > 0, \exists \delta > 0$. s.t. $0 < x-a < \delta \Rightarrow |f(x) - L| < \varepsilon$.

(f) $\forall M > 0, \exists \delta > 0$. s.t. $0 < |x-a| < \delta \Rightarrow f(x) > M$, $M \in \mathbb{R}$.

(g). $\forall M < 0, \exists \delta > 0$. s.t. $-\delta < x-a < 0 \Rightarrow f(x) < M$. $M \in \mathbb{R}$.

(h). $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $x > \delta \Rightarrow |f(x) - L| < \varepsilon$.

(i). $\forall M > 0. \exists \delta < 0$. s.t. $x < \delta \Rightarrow f(x) > M$. $M \in \mathbb{R}$.

6. (a) ① rough work:

$|f(x) - A| = |4x+1-9| = |4x-8| = 4|x-2| < \varepsilon$.

$\Rightarrow \delta = \frac{\varepsilon}{4}$

② pf. Let $\varepsilon > 0$.

Take. $\delta = \frac{\varepsilon}{4}$

Gives. $0 < |x-2| < \frac{\varepsilon}{4}$,

We have. $4|x-2| < \varepsilon$.

$\Rightarrow |4x-8| < \varepsilon$.

$\Rightarrow |(4x+1) - 9| < \varepsilon$.

$\Rightarrow |f(x) - L| < \varepsilon$. ■

(b). ① RW: $\forall \varepsilon > 0, \exists \delta > 0$. s.t. $x > \delta \Rightarrow |f(x) - A| < \varepsilon$.

$|\frac{1}{x^2} - 0| = \frac{1}{x^2} < \varepsilon$.

$\Rightarrow x > \frac{1}{\sqrt{\varepsilon}}$

pf. Let $\varepsilon > 0$.

Take $\delta = \frac{1}{\sqrt{\varepsilon}}$

Gives $x > \frac{1}{\sqrt{\varepsilon}}$,

We have. $\frac{1}{x} < \sqrt{\varepsilon}$.

$\Rightarrow \frac{1}{x^2} < \varepsilon$. since $\frac{1}{x^2} \geq 0$.

$\Rightarrow |\frac{1}{x^2} - 0| < \varepsilon$. ■

(c). ① RW. $|f(x) - A| = |x^3 - 1| = |(x-1)(x^2+x+1)| = |x-1||x^2+x+1| < |x^2+x+1|\delta < \varepsilon$.

$\delta < 1 \rightarrow |x-1| < 1 \rightarrow 0 < x < 2. \rightarrow 1 < x^2+x+1 < 7$

② pf. Let $\varepsilon > 0$.

Take. $\delta = \min\{1, \frac{\varepsilon}{7}\}$

We have. ① $\delta = 1$. $0 < |x-1| < 1$

$\Rightarrow 0 < x < 2. \rightarrow x^2+x+1 < 7$

② $\delta = \frac{\varepsilon}{7}$ $\quad 0 < |x-1| < \frac{\varepsilon}{7}$

Thus. $|x^3-1|$

$$= |x-1||x^2+x+1| < \frac{\varepsilon}{7} \cdot 7 = \varepsilon. \quad \blacksquare$$

(d) ① RW: $|f(x)-A| = \left|\frac{1}{x^2+1} - \frac{1}{2}\right| = \left|\frac{2}{2(x^2+1)} - \frac{x^2+1}{2(x^2+1)}\right|$

$$= \left|\frac{2-x^2-1}{2(x^2+1)}\right| = \left|\frac{-x^2+1}{2(x^2+1)}\right|$$

$$= \left|\frac{-(x+1)(x-1)}{2(x^2+1)}\right| < \varepsilon$$

$$= \left|\frac{-(x+1)}{2(x^2+1)}\right||x-1| < \left|\frac{-(x+1)}{2(x^2+1)}\right| \delta < \varepsilon.$$

$\underline{\delta < 1}$, $|x-1| < 1$

$\to 0 < x < 2$ $\quad \frac{3}{10} < \left|\frac{-(x+1)}{2(x^2+1)}\right| < \frac{1}{2}$ $\quad$, $\delta = 2\varepsilon$.

② Pf. Let $\varepsilon > 0$

Take $\delta = \min\{1, 2\varepsilon\}$.

We have. ① $\delta = 1$. $0 < |x-1| < 1$

$\to 0 < x < 2$. $\to \frac{3}{10} < \left|\frac{-(x+1)}{2(x^2+1)}\right| < \frac{1}{2}$.

② $\delta = 2\varepsilon$. $0 < |x-1| < 2\varepsilon$.

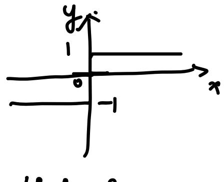
Thus. $|f(x) - L| = \left|\frac{1}{x^2+1} - \frac{1}{2}\right| = \left|\frac{-(x+1)}{2(x^2+1)}\right||x-1| < \frac{1}{2} \cdot 2\varepsilon = \varepsilon. \quad \blacksquare$

(e). $\lim_{x\to 0} \frac{x}{|x|}$ DNE.

① RW: E: $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $0 < |x| < \delta \Rightarrow \left|\frac{x}{|x|} - A\right| < \varepsilon$.

DNE: $\exists \varepsilon > 0$. s.t. $\forall \delta > 0$, $0 < |x| < \delta$ and. $\left|\frac{x}{|x|} - A\right| \geq \varepsilon$.

$\exists x \in \mathbb{R}$.



② Pf. Let $f(x) \in \mathbb{R}$.

Take $\varepsilon = 1$.

Let $\delta > 0$.

At least one of those will happen:

Case 1: $1 \notin (L-\varepsilon, L+\varepsilon)$

Then, $x \in \mathbb{R}$. $x > 0$. satisfying $0 < |x| < \delta$, $f(x) = 1$.

Case 2: $-1 \notin (L-\varepsilon, L+\varepsilon)$.

Then, $x \in \mathbb{R}$. $x < 0$, satisfying $0 < |x| < \delta$, $f(x) = -1$.

Therefore, it satisfies $0<|x|<\delta$ and $\left|\frac{x}{|x|}-A\right|\geq\varepsilon$.

(f). ① RW: $\forall A<0$, $\exists\delta>0$ s.t. $0<x-1<\delta$, $\frac{1}{1-x}<A$.

$$\underbrace{\frac{1}{1-x}<A}\ \rightarrow\ \underbrace{\frac{1}{-(x-1)}<A}.$$

$$\frac{1}{-A}>(x-1).\ \rightarrow\ (x-1)<-\frac{1}{A}=\delta.$$

② Pf. Let $A<0$.

Take $\delta=-\frac{1}{A}$

We have. $0<x-1<-\frac{1}{A}$.

$\Rightarrow\quad 1-x>\frac{1}{A}$

$\Rightarrow\quad \frac{1}{1-x}<A$. ∎

7. Let $a, L, M\in\mathbb{R}$.

Assuming that $\lim_{x\to a}f(x)=L\rightarrow\forall\varepsilon>0.\ \exists\delta_1>0$. s.t. $0<|x-a|<\delta_1\Rightarrow|f(x)-L|<\frac{\varepsilon}{2}$

$\lim_{x\to a}g(x)=M\longrightarrow\forall\varepsilon>0,\exists\delta_2>0$. s.t. $0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\varepsilon}{2}$.

Let $\varepsilon>0$.

Take. $\delta_3=\min\{\delta_1,\delta_2\}$.

We have, $0<|x-a|<\delta_3$. and,

$$|f(x)-g(x)-[L-M]|=|f(x)-L-g(x)+M|$$

$$=|(f(x)-L)-(g(x)-M)|\leq|f(x)-L|+|g(x)-M|<\varepsilon.$$ ∎

8. ① RW. Known: $\forall A>0$. $\exists\delta>0$. s.t. $0<|x-a|<\delta\Rightarrow f(x)>A$

WTS: $\forall\varepsilon>0$. $\exists\delta>0$. s.t. $0<|x-a|<\delta\Rightarrow\left|\frac{1}{f(x)}\right|<\varepsilon$.

$\Rightarrow\ \frac{1}{f(x)}<\varepsilon$.

$\Rightarrow\ \underbrace{f(x)>\frac{1}{\varepsilon}}$ $A=\frac{1}{\varepsilon}$.

②. pf. Assuming. $\lim_{x\to a}f(x)=\infty$. gives.

$\forall A>0,\exists\delta>0$ s.t. $0<|x-a|<\delta\Rightarrow f(x)>A$.

take $A=\frac{1}{\varepsilon}$. $f(x)>\frac{1}{\varepsilon}$.

Let $\varepsilon>0$.

Take $\delta>0$. $0<|x-a|<\delta$.

Since. $f(x)>\frac{1}{\varepsilon}$.

$\Rightarrow\ \frac{1}{f(x)}<\varepsilon$.

From $\frac{1}{f(x)}>0$, $\frac{1}{f(x)}=\left|\frac{1}{f(x)}\right|$.

Therefore. $\left|\frac{1}{f(x)}\right| < \varepsilon$. $\blacksquare$

10. ① RW: WTS: $\forall \varepsilon > 0$, $\exists\ \delta_1 > 0$. s.t. $0 < |x-a| < \delta_1 \Rightarrow |g(f(x)) - g(L)| < \varepsilon$.

Assu: $\forall \varepsilon > 0$, $\exists\ \delta_2 > 0$, s.t. $0 < |x-a| < \delta_2 \Rightarrow |f(x) - L| < \varepsilon$.

$\lim_{y \to L} g(y) = g(L)$.

$\hookrightarrow \forall \varepsilon > 0$. $\exists\ \delta_3 > 0$. s.t. $0 < |y-L| < \delta_3 \Rightarrow |g(y) - g(L)| < \varepsilon$.

② pf. Let $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = L$. $\lim_{y \to L} g(y) = g(L)$, gives.

$\forall \varepsilon > 0$. $\exists\ \delta_1 > 0$. s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$.

$\forall \varepsilon > 0$, $\exists\ \delta_2 > 0$. s.t. $0 < |y-L| < \delta_2 \Rightarrow |g(y) - g(L)| < \varepsilon$.

Taking. $\delta_2 = \varepsilon$, from $|f(x) - L| < \varepsilon$, $0 < |y-L| < \varepsilon$. gives.

$0 < |f(x) - L| < \delta_2$.

Therefore. $|g(f(x)) - g(L)| < \varepsilon$. $\blacksquare$

11. $\sin x = 2(1 - \sin^2 x) + 0.5$

$\Rightarrow \sin x = 2 - 2\sin^2 x + 0.5$

$\Rightarrow 2\sin^2 x + \sin x - 2.5 = 0$.

$\Rightarrow 4\sin^2 x + 2\sin x - 5 = 0$

Let $y = 4\sin^2 x + 2\sin x - 5$

when $x = 0$. gives. $y = -5$.

$x = \frac{\pi}{2}$ gives $y = 1$.

Therefore. from IVT. give. $x = 0$, $y < 0$; $x = \frac{\pi}{2}$. $y > 0$. y is continuous on $[0, \frac{\pi}{2}]$.

y has at least one solution.

12.