Definition of Integral

1. Under- & Over-Estimate

Let f be a bounded function on $[a, b]$

Let $P = \{x_0, x_1, \dots, x_N\}$ be a partition of $[a, b]$.

For each $i = 1, \dots, N$, let

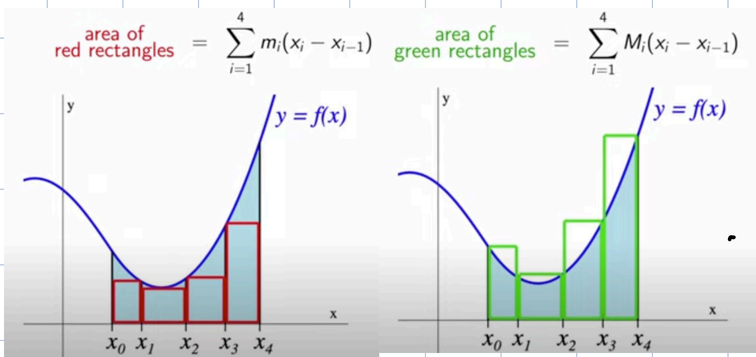
① m_i be the infimum of f on $[x_{i-1}, x_i]$.

② M_i be the supremum of f on $[x_{i-1}, x_i]$.

③ $\Delta x_i = x_i - x_{i-1}$.

The P-lower sum of f is the number $L_P(f) = \sum_{i=1}^{N} m_i \cdot \Delta x_i$

The P-upper sum of f is the number $U_P(f) = \sum_{i=1}^{N} M_i \cdot \Delta x_i$

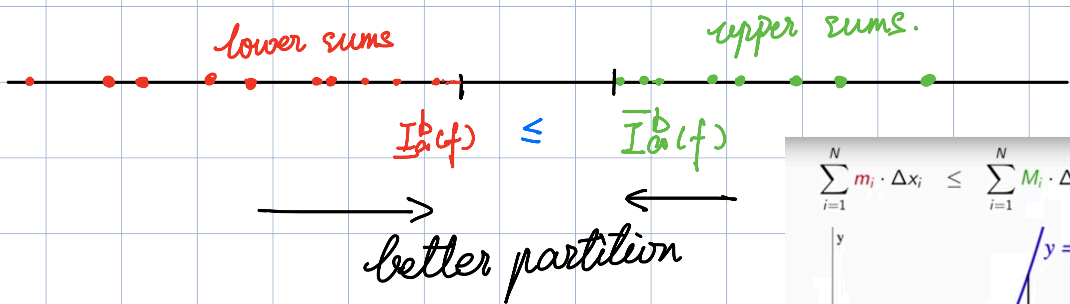


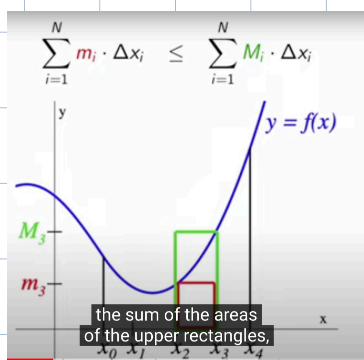
2. Lower & Upper Integral

Let f be a bounded function on $[a, b]$.

① lower integral (of f from a to b) $= \underline{I}_a^b(f) = \sup\{\text{lower sums of } f\}$

② upper integral (of f from a to b) $= \overline{I}_a^b(f) = \inf\{\text{upper sums of } f\}$





Q is finer than P.

3. Properties

1) For every partition P of $[a,b]$,

$$L_P(f) \le U_P(f).$$

2) Let P and Q be partitions of $[a,b]$.
If $P \subseteq Q$, then,

$$L_P(f) \le L_Q(f) \quad \text{and} \quad U_Q(f) \le U_P(f).$$

3) Let P and Q be any partitions of $[a,b]$.
Then $L_P(f) \le U_Q(f)$.

pf. Call $R = P \cup Q$. Then $P \subseteq R$, $Q \subseteq R$.

$$\underset{P_2}{L_P(f) \le L_R(f)} \underset{P_1}{\le} \underset{P_2}{U_R(f) \le U_Q(f)}$$