Differentiation Rules

1. Differentiable and Continuous.

1) Theorem: Let $c \in \mathbb{R}$; Let f be a function defined at and near c.

If f is differentiable at c, then f is continuous at c.

2). f is differentiable at c means

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} \text{ exists.}$$

f is continuous at c means:

$$\lim_{x \to c} f(x) = f(c)$$

if $\lim_{x \to c} [f(x)] = f(c)$
then $\lim_{x \to c} [f(x) - f(c)]$
$= f(c) - f(c) = 0$

3) Proof.

Assume f is differentiable at c.

Then $\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \left[\frac{f(x)-f(c)}{x-c} \cdot (x-c) \right]$

$$= \left[\lim_{x \to c} \frac{f(x)-f(c)}{x-c} \right] \cdot \left[\lim_{x \to c} (x-c) \right]$$

$$= f'(c) \cdot 0 = 0.$$

2. Differentiation Rules

1) Over all:

f, g are functions. $c \in \mathbb{R}$ is a constant.

① $\frac{d}{dx}[c] = 0$

② $\frac{d}{dx}[x^c] = cx^{c-1}$

③ $(f+g)' = f' + g'$

④ $(cf)' = cf'$

⑤ $(f \cdot g)' = f'g + fg'$

⑥ $\left(\frac{f}{g}\right)' = \frac{f'g + fg'}{g^2}$

2) The product rule

① Theorem:

→ Let $a \in \mathbb{R}$; Let f and g be functions defined at and near a.

→ We define the function h by $h(x) = f(x)g(x)$

If f and g are differentiable at a, and

$$h'(a) = f'(a)g(a) + f(a)g'(a)$$

② Proof: $h(x) = f(x) \cdot g(x)$

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x-a} = \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x-a}$$

$$= \lim_{x \to a} \left[\frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a} \right]$$

$$= \lim_{x \to a} \left[\frac{f(x)g(x) - f(x)g(a)}{x-a} \right] + \lim_{x \to a} \left[\frac{f(x)g(a) - f(a)g(a)}{x-a} \right]$$

$$= \lim_{x \to a} [f(x)] \cdot \lim_{x \to a} \left[\frac{g(x) - g(a)}{x-a} \right] + g(a) \lim_{x \to a} \left[\frac{f(x) - f(a)}{x-a} \right]$$

$$= f(a) \cdot g'(a) + g(a) f'(a)$$

f is differentiable at a.

f is continuous at a.

$$h'(a) = f(a) \cdot g'(a) + g(a) \cdot f'(a)$$