4. PMI.

   a) Prove using PMI that for any natural number n, there must exist a natural number m such that $n \leq m^2 \leq 2n$.

proof. WTP: $\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.$ s.t. $n \leq m^2 \leq 2n$.

Let $n \in \mathbb{N}$.

Base Case: $n = 1$.

when $n=1$, $1 \leq m^2 \leq 2$. take $m = 1$. which $m \in \mathbb{N}$, and. $1 \leq 1^2 \leq 2$ holds.

I've proved the base case is true.

Induction Step. Let $n \in \mathbb{N}$.

Induction Hypothesis. $\exists m \in \mathbb{N}$. s.t. $n \leq m^2 \leq 2n$.

WTP: $\exists m' \in \mathbb{N}$. s.t. $n+1 \leq (m')^2 \leq 2(n+1)$ holds.

① By I.H. when $m^2 = n$, take $m' = m+1$, which. $(m+1)^2 = m^2+2m+1$.

Since $m \in \mathbb{N}$. and $m^2 = n$. gives $m^2+2m+1 = n+2m+1 > n+1$, which $n+1 \leq (m')^2$

Since $n \in \mathbb{N}$. and $m^2 = n$. $4m^2 \leq n^2+4+4m^2 \Rightarrow 4m^2 \leq n^2+4+4n \Rightarrow 4m^2 \leq (n+2)^2 \Rightarrow 2m \leq n+2$

Since. $(m+1)^2 = m^2+2m+1$ and $m^2 = n$, gives. $(m+1)^2 = n+2m+1 \leq n+(n+2)+1 = 2n+3 < 2n+4 = 2(n+1)$.

Thus. when $m^2 = n$, $m' = m+1$. which $m' \in \mathbb{N}$, $n+1 \leq (m')^2 \leq 2(n+1)$ holds.

② By I.H. when $n < m^2 \leq 2n$, take $m' = m$

Since $n \in \mathbb{N}$, and according to the property of natural number, there is no natural number k. s.t. $n < k < n+1$,

gives. $n+1 \leq m^2 = (m')^2$.

Since $n < m^2 \leq 2n$, gives $m^2 \leq 2n+2$. which $m^2 = (m')^2 \leq 2n+2 = 2(n+1)$. as $m' = m$.

Thus. when $n < m^2 \leq 2n$, $m' = m$. which $m' \in \mathbb{N}$. $n+1 \leq (m')^2 \leq 2(n+1)$. holds.

Therefore, I've proved $\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.$ s.t. $n \leq m^2 \leq 2n$. $\blacksquare$

   b) Read Definition 10.1.1, 10.1.2, and 10.3.27 (the last one is only relevant for understanding the notation in 10.3.28). Carefully read the proof of Theorem 10.3.28, and use the idea of that proof to prove that for any natural number $n > 1$, if a set S has cardinality n, then S has exactly $\frac{n(n-1)}{2}$ many subsets of size 2.

WTP: $\forall n \in \mathbb{N}$, $n > 1$, if $|S| = n$, then S has exactly $\frac{n(n-1)}{2}$ subsets of size 2.

Let $n \in \mathbb{N}$, $n > 1$.

Base Case: $n = 2$.

Since $|S| = 2$, the only subset of size 2 is itself S, which we only have 1 subset satisfies the condition.

$\frac{n(n-1)}{2} = \frac{2 \cdot (2-1)}{2} = 1$. which says when $n=2$. the claim holds.

I've proved the base case is true.

Induction Step. Let $n \in \mathbb{N}$. $n \geq 2$.

Induction Hypothesis. Assume all set of size n has exactly $\frac{n(n-1)}{2}$ subsets of size 2.

Let $|S| = n+1$. Let $s_0 \in S$, gives. $S \setminus \{s_0\}$ has cardinality of n, which it satisfies the I.H., having exactly $\frac{n(n-1)}{2}$ subsets of size 2.

Let $T \subseteq S$. and assume $|T| = 2$.

① $s_0 \in T$.

Since we want the subset of size 2 and we've gotten $s_0 \in T$. we just need one more element in $S \setminus \{s_0\}$ to be a subset of size 2, which there are n choices from set $S \setminus \{s_0\}$

② $s_0 \notin T$.

When need to find two elements from set $S \setminus \{s_0\}$ to get a subset of size 2, T. Since $|S \setminus \{s_0\}| = n$. by I.H. there are $\frac{n(n-1)}{2}$ subsets of size 2.

Thus, in total. there will be $n + \frac{n(n-1)}{2} = \frac{2n}{2} + \frac{n(n-1)}{2} = \frac{2n+n^2-n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$

I've proved the induction step is true.

Therefore. $\forall n \in \mathbb{N}$, $n > 1$, if $|S| = n$, then S has exactly $\frac{n(n-1)}{2}$ subsets of size 2. $\blacksquare$