# Unit 2 Assignment

Q1. For an arbitrary polynomial $p \in P_1(\mathbb{R})$. which $p = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, $a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \in \mathbb{R}$, we obtain.

$$J(p): \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + \frac{a_1}{2}x^2 + a_0x$$, which is the general formula for $J(p)$.

Q2. Define $P$ is a vector space on $\mathbb{R}$, where $P(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}\}$

If the transformation $J: P_1(\mathbb{R}) \to W$. where $W \subseteq P_2(\mathbb{R})$, satisfies:

1) $J(\vec{u} + \vec{v}) = J(\vec{u}) + J(\vec{v})$ for all $\vec{u}, \vec{v} \in P_1(\mathbb{R})$.

2) $J(k\vec{u}) = kJ(\vec{u})$ for all $\vec{u} \in P_1(\mathbb{R})$ and $k \in \mathbb{R}$.

then $J$ is a linear combination.

p.f. 1). $\forall \vec{u}, \vec{v} \in P_1(\mathbb{R})$. gives. $\vec{u} = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$

$\vec{v} = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$, $b_0, b_1, b_2, \ldots, b_n \in \mathbb{R}$.

$$\begin{aligned}
J(\vec{u}+\vec{v}) &= J(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + b_0 + b_1x + b_2x^2 + \cdots + b_nx^n). \\
&= J((a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_n+b_n)x^n). \\
&= \frac{a_n+b_n}{n+1} \cdot x^{n+1} + \frac{a_{n-1}+b_{n-1}}{n} \cdot x^n + \cdots + \frac{a_1+b_1}{2}x^2 + (a_0+b_0)x. \\
&= \frac{a_n}{n+1}x^{n+1} + \frac{b_n}{n+1} \cdot x^{n+1} + \frac{a_{n-1}}{n} \cdot x^n + \frac{b_{n-1}}{n} \cdot x^n + \cdots + \frac{a_1}{2}x^2 + \frac{b_1}{2} \cdot x^2 + a_0x + b_0x. \\
&= \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + \frac{a_1}{2}x^2 + a_0x + \frac{b_n}{n+1}x^{n+1} + \frac{b_{n-1}}{n}x^n + \cdots + \frac{b_1}{2}x^2 + b_0x. \\
&= J(\vec{u}) + J(\vec{v}).
\end{aligned}$$

satisfies the first condition.

2) $\forall \vec{u} \in P_1(\mathbb{R})$, $\forall k \in \mathbb{R}$, gives $\vec{u} = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$

$$\begin{aligned}
J(k\vec{u}) &= J(k(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)) \\
&= J(ka_0 + ka_1x + ka_2x^2 + \cdots + ka_nx^n). \\
&= \frac{ka_n}{n+1} \cdot x^{n+1} + \frac{ka_{n-1}}{n}x^n + \cdots + \frac{ka_1}{2}x^2 + ka_0x \\
&= k\left(\frac{a_n}{n+1} \cdot x^{n+1} + \frac{a_{n-1}}{n} \cdot x^n + \cdots + \frac{a_1}{2} \cdot x^2 + a_0x\right). \\
&= kJ(\vec{u}).
\end{aligned}$$

satisfies the second condition.

Hence, I've proved that $J$ is a linear transformation by satisfying the two conditions. ∎

$\int$ ng

Q3. (i). $J(\alpha_1) = J(x) = \frac{1}{2}x^2 = a\beta_1 + b\beta_2$.

$$\Rightarrow \frac{1}{2}x^2 = a(x^2+x) + b(2x).$$

$$\Rightarrow \frac{1}{2}x^2 = ax^2 + (a+2b)x.$$

where, $\begin{cases} a = \frac{1}{2} \\ a+2b = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2} \\ b = -\frac{1}{4} \end{cases}$

(According to Q1, we can directly substitute $\alpha_1$ in $J$ and create a lin comb of $\beta_1$ and $\beta_2$ to make them equal to get the coefficient).

$$\therefore J(\alpha_1) = \frac{1}{2}\beta_1 - \frac{1}{4}\beta_2.$$

(ii). $J(\alpha_2) = J(1+x) = \frac{1}{2}x^2 + x = a\beta_1 + b\beta_2$.

$$\Rightarrow \frac{1}{2}x^2 + x = ax^2 + (a+2b)x.$$

where, $\begin{cases} a = \frac{1}{2} \\ a+2b = 1 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2} \\ b = \frac{1}{4} \end{cases}$

(According to Q1, we can directly substitute $\alpha_2$ in $J$ and create a lin comb of $\beta_1$ and $\beta_2$ to make them equal to get the coefficient)

$$\therefore J(\alpha_2) = \frac{1}{2}\beta_1 + \frac{1}{4}\beta_2.$$

Q4. According to Q3. we obtain that:

$$J(\alpha_1) = \frac{1}{2}\beta_1 - \frac{1}{4}\beta_2; \quad J(\alpha_2) = \frac{1}{2}\beta_1 + \frac{1}{4}\beta_2.$$

Therefore, we can obtain that: $[J(\alpha_1)_\beta, J(\alpha_2)_\beta]$

$$\text{matrix } [J]_\alpha^\beta = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Q5. We firstly get how $\alpha_1, \alpha_2$ combines $3x+2$,

$$3x+2 = ax + b(1+x).$$

$$\Rightarrow 3x+2 = ax + b + bx$$

$$\Rightarrow 3x+2 = (a+b)x + b.$$

where. $\begin{cases} a+b = 3. \\ b = 2. \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2. \end{cases}$

Therefore we times $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with the matrix we obtained in Q4. above, which $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\times 1 + \frac{1}{2}\times 2 \\ -\frac{1}{4}\times 1 + \frac{1}{4}\times 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{4} \end{bmatrix}$

$$\therefore J(3x+2) = \frac{3}{2}\beta_1 + \frac{1}{4}\beta_2.$$