Learning Objectives

> In this tutorial, you will *practice using the definition of a vector space and practice identifying when a set is a subspace of a given vector space.*

In MAT 223, you studied the vector space $\mathbb{R}^n$. In MAT 224, you will see further examples of vector spaces.

Before beginning the tutorial, you should be able to define the following terms using precise mathematical language: a field, a vector space over a field, a subspace of a vector space. For more information, you may read the textbook Damiano and Little Sections 1.1 and 1.2.

Problems

1. Let n be a non-negative integer. Let

$$P_n = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n | a_0, \ldots, a_n \in \mathbb{R}\}$$

be the set of polynomials of degree less than or equal to n with real coefficients.

   (a) Describe how there is an addition and scalar multiplication on P_n.

   (b) Show that addition defined in part (a) on P_n satisfies the axioms of a vector space.

   (c) Show that the scalar multiplication defined in part (a) on P_n satisfies the axioms of a vector space.

2. In each part decide whether or not the given subset of P_n is a vector space together with the addition and scalar multiplication of P_n. If yes, provide a proof. If not, show which axiom of a vector space is violated.

   (a) The subset $S \subset P_n$ of polynomials with non-negative coefficients.

   (b) The subset $R \subset P_n$ of polynomials with rational coefficients.

   (c) Let $P := \bigcup_{n=1}^{\infty} P_n$. Explain what P is in words. Is P a vector space over $\mathbb{R}$?

3. Let $\mathcal{C}$ be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. We may give $\mathcal{C}$ the structure of a vector space, by defining by $(f + g)(x) = f(x) + g(x)$ and $(kf)(x) = kf(x)$ for all $f, g \in \mathcal{C}, k \in \mathbb{R}$ and $x \in \mathbb{R}$.

   Consider each of the following subsets of $\mathcal{C}$. In each part decide whether or not the given subset of $\mathcal{C}$ is a vector space together with the addition and scalar multiplication of $\mathcal{C}$. In other words, decide whether the given subset is a subspace of $\mathcal{C}$. If yes, provide a proof. If not, show which axiom of a vector space is violated.

   (a) The subset $\mathcal{C}^1 \subset \mathcal{C}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ that have a first derivative.

   (b) The set $\mathcal{C}^r \subset \mathcal{C}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ that have an r^{th} derivative, for any integer $r \geq 2$.

   (c) The set $V \subset \mathcal{C}^2$ of functions satisfying the equation $f'' = f$. [1]

   (d) The set $V \subset \mathcal{C}^2$ of functions satisfying the equation $f'' = f + 1$.

[1] You do not need to solve the differential equation to make a conclusion.

Q1. (a) To define the addition and scalar multiplication, we have:

$\forall p, q \in P_n$, where $p = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, $\forall k \in \mathbb{R}$.

$q = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$.

Gives,

$(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n)$.

$= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \cdots + (a_n + b_n) x^n$. (Addition).

$k(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$

$= ka_0 + ka_1 x + ka_2 x^2 + \cdots + ka_n x^n$. (scalar multiplication).

(b). Axioms for addition:

① $\forall p, q \in P_n$.

$p + q = (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n)$

$= (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n) + (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$

$= q + p$. (Commutative)

② $\forall p, q, r \in P_n$.

$(p + q) + r = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \cdots + (a_n + b_n) x^n +$
$(c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n)$.

$= [(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n)] +$
$(c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n)$.

$= (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (b_0 + c_0) + (b_1 + c_1) x + (b_2 + c_2)$
$x^2 + \cdots + (b_n + c_n) x^n$.

$= p + (q + r)$ (Associativity).

③ $\forall p \in P_n$. Take $\vec{0} = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$.

$\vec{0} + p = p$.

$\Rightarrow (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \cdots + (a_n + b_n) x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

Thus, $b_0 = b_1 = b_2 = \cdots = b_n = 0$.

$\therefore \vec{0} = 0 + 0x + 0x^2 + \cdots + 0x^n$ (Additive Identity).

④ $\forall p \in P_n$. Take $(-p) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$.

$p + (-p) = \vec{0}$

$\Rightarrow (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_n+b_n)x^n = 0 + 0x + 0x^2 + \cdots + 0x^n$.

$\Rightarrow b_0 = -a_0,\ b_1 = -a_1,\ b_2 = -a_2, \ldots, b_n = -a_n$.

$\therefore -p = -a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n$ (Additive Inverse).

(c) ① $\forall p, q \in P_n$. $c \in \mathbb{R}$.

$c(p+q) = c[(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n)]$.

$= c(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + c(b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n)$.

$= cp + cq$ (distributive).

② $\forall p \in P_n. \forall c, d \in \mathbb{R}$.

$(c+d)p = (c+d)(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$.

$= c(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + d(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$

$= cp + dp$ (distributive).

③ $\forall p \in P_n, \forall c, d \in \mathbb{R}$.

$(cd)p = (cd)(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$.

$= c(da_0 + da_1 x + da_2 x^2 + \cdots + da_n x^n)$.

$= c(dp)$

④ $\forall p \in P_n$.

$1 \cdot p = 1 \cdot (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$.

$= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$= p$.

Q2. (a) No. contradict additive inverse

(b). No. contradict scalar multiplication.

(c). Yes. P is a polynomial that can be at any degrees. When it goes sufficiently large, it's the same as the vector space we proved is Q1.

Q3 (a). Yes. C^1 is non-empty and according to derivative properties. Since those functions are continuous, we can obtain that.

$\forall f, g \in C^1, (f+g)' = f' + g'$

$\forall f \in C^1, \forall k \in \mathbb{R}. (kf)' = kf'$ where the addition and multiplication. Therefore, it's closed as a result of which is a vector space.

(b). Yes. for any $r \geq 2$. C^r is non-empty. Similar to (a), since those functions are continuous, we can obtain that.

$\forall f, g \in C^r$, where $r \geq 2$, $(f+g)' = f' + g'$

$\forall f \in C^r$, where $r \geq 2$, $\forall k \in \mathbb{R}$. $(kf)' = k \cdot f'$ where the addition and multiplication are closed, which C^r is a vector space.

(c) Yes. for $V \subset C^2$, V is non-empty since when $f(x) = 0$, $f'' = f$. Since those functions are continuous, we can obtain that.

$\forall f, g \in V, (f+g)'' = f'' + g'' = f+g$.

$\forall f \in V, \forall k \in \mathbb{R}. (kf)' = k \cdot f'$ where the addition and multiplication are closed provided by the question, which V is a vector space.

(d). No. the 0 function is not in V, violating additive identity.