Vector Calculation

1. Addition & Scalar Multiplication.

1) A vector $u \in \mathbb{R}^n = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ can be seen as a matrix with only one column and n rows.

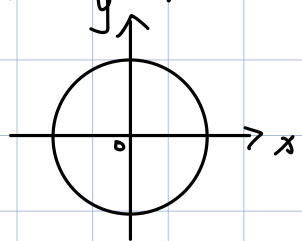
2) Vector Addition. $\vec{u}+\vec{v} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{pmatrix}$

3) Scalar Multiplication $\lambda\vec{u} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \vdots \\ \lambda u_n \end{pmatrix}$

2. Norm (Length of a vector).

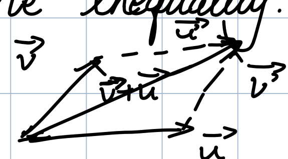
$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_1^2+u_2^2+\cdots+u_n^2}$$

e.g. $\{\vec{x} \in \mathbb{R}^2 \mid \|\vec{x}\| = 1\}$.



1) $\|\vec{u}\| \geq 0$, $\forall \vec{u} \in \mathbb{R}^n$; $\|\vec{u}\| = 0$ iff $\vec{u} = \vec{0}$

2) Triangle Inequality. $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$. $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$.



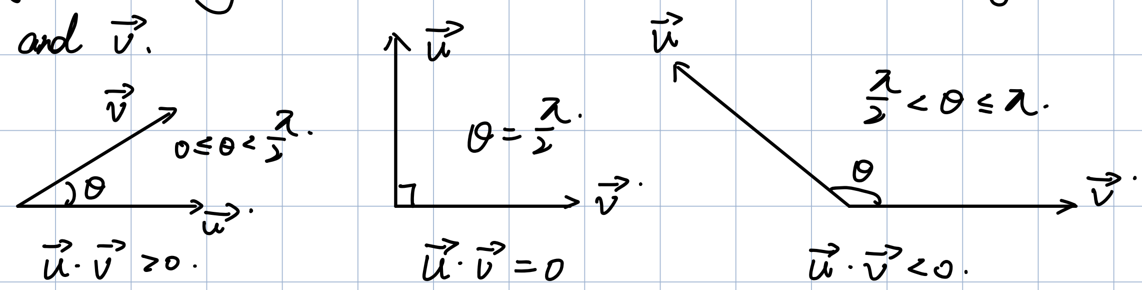
3) Unit Vector : $\hat{u} = \dfrac{\vec{u}}{\|\vec{u}\|}$

3. Dot Product.

1) Algebraically : $\vec{u}\cdot\vec{v} = \sum_{k=1}^{n} u_k v_k = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

正负性代表夹角大小

2) Geometrically: $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$, where θ is angle between $\vec{u}$ and $\vec{v}$.



3) Properties.

① $\vec{u}\cdot\vec{u}\ge 0$, $\forall\vec{u}\in\mathbb{R}^n$; $\vec{u}\cdot\vec{u}=0$ iff $\vec{u}=0$

② Bilinearity: $(\lambda\vec{u}+\vec{v})\cdot\vec{w} = \lambda\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$; $\forall\lambda\in\mathbb{R}$, $\forall u.v.w\in\mathbb{R}^n$.

③ Symmetry: $\forall\vec{u}.\vec{v}\in\mathbb{R}$, $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$

④ $\forall\vec{u}.\vec{v},\vec{w}\in\mathbb{R}^n$, $\vec{u}(\vec{v}-\vec{w})=0 \not\Rightarrow \vec{v}=\vec{w}$ (No Cancellation)

$\hookrightarrow$ $\vec{u}\cdot\vec{v}=0 \not\Rightarrow \vec{u}=\vec{0}$ or $\vec{v}=\vec{0}$.

e.g. $\vec{u}=\begin{bmatrix}1\\1\end{bmatrix}$. $\vec{v}=\begin{bmatrix}-1\\1\end{bmatrix}$.

$\vec{u}\cdot\vec{v} = 1+(-1) = 0$.

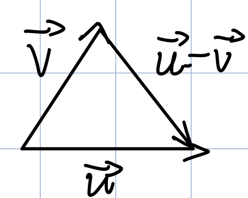
4) Angel Between two vectors.

$\theta(\vec{u},\vec{v}) = \arccos\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$ ($\arccos: [-1,1]\to[0,\pi]$)

5). Law of Cosines.

$c^2=a^2+b^2-2ab\cos\theta$.

$\Rightarrow \|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2+\|\vec{v}\|^2-2\vec{u}\cdot\vec{v}$.



4. Orthogonal & Orthonormal

1) Orthogonal (Perpendicular): $\vec{u}\cdot\vec{v}=0$ ($\vec{u}.\vec{v}\in\mathbb{R}^n$)

① $S=\{\vec{u}_1,\vec{u}_2,\dots,\vec{u}_k\}\subseteq\mathbb{R}^n$ is orthogonal if $\forall 1\le i,j\le k$, $i\ne j \Rightarrow \vec{u}_i\cdot\vec{u}_j=0$

e.g. $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

$\vec{0}$垂直于任意向量.

every vector's norm is 1.

2) Orthonormal: $\|u_i\| = 1$, $\forall\, 1 \le i \le k$.

3) Relationship with lin inde

① lin inde $\Rightarrow$ orthogonal (x)

② lin inde $\Rightarrow$ orthonormal (x)

e.g. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$.

③ orthogonal $\Rightarrow$ lin inde (x)

e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

④ orthogonal + non-zero $\Rightarrow$ lin inde (✓). $\quad V_i \neq \vec{0}$.

pf. Assume $S = \{v_1, v_2, \dots v_k\}$ for all $1 \le i, j \le k$. $\vec{v_i} \cdot \vec{v_j} = 0$, $i \neq j$.

WTS: $x_1 \vec{v_1} + x_2 \vec{v_2} + \dots + x_k \vec{v_k} = \vec{0}$, $x_1 = x_2 = \dots = x_k = 0$.

Since $\vec{v_1}(x_1 \vec{v_1} + x_2 \vec{v_2} + \dots + x_k \vec{v_k}) = \vec{0} \cdot \vec{v_k} = 0$

$\Rightarrow \underbrace{x_1 \cdot \vec{v_1} \cdot \vec{v_1}}_{x_1 \cdot \underbrace{\|\vec{v_1}\|}_{\neq 0}} + \underbrace{x_2 \vec{v_1} \cdot \vec{v_2}}_{0} + \dots + \underbrace{x_k \cdot \vec{v_1} \cdot \vec{v_k}}_{0} = 0$

$\therefore x_1 = 0$.

Similarly, times $\vec{v_2}, \vec{v_3}, \dots, \vec{v_k}$ everytime.

$\therefore x_1 = x_2 = x_3 = \dots = x_k = 0$.

⑤ orthonormal $\Rightarrow$ lin inde (✓).

5. Distance

The distance between 2 vectors $\vec{u}$ and $\vec{v}$ is given by $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$.

1) Properties.

① $d(\vec{u}, \vec{v}) \ge 0$, $\forall \vec{u}$; $d(\vec{u}, \vec{v}) = 0$ iff $\vec{u} = \vec{v}$

② $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$.

③ triangel inequality: $d(\vec{u}, \vec{w}) \le d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w})$

$\longrightarrow |x-y| \le |x-z| + |z-y|$