Q1. Let $a.b \in \mathbb{R}^+$. Let $M = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$

WTS: The procedure of turning $ax^2+2bxy-ay^2=1$ into $rx^2-ty^2=1$. by diagonalizing matrix .

We notice that $ax^2+2bxy-ay^2 = [x\ y]\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Since we obtain that $M^T = M$, which M is symmetric

According to spetral theorem, we start to diagonalize it

We want to find the eigenvalue.

$$\det(M-\lambda I) = 0$$

$$\Rightarrow \quad \det\begin{pmatrix} a-\lambda & b \\ b & -a-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \quad (-\lambda+a)(-\lambda-a) - b^2 = 0 \text{ , since } (m+n)(m-n) = m^2-n^2.$$

$$\Rightarrow \quad \lambda^2 - a^2 - b^2 = 0$$

$$\Rightarrow \quad \lambda^2 = a^2+b^2.$$

$$\Rightarrow \quad \lambda = \pm\sqrt{a^2+b^2}. \quad \lambda_1 = \sqrt{a^2+b^2} \ ; \ \lambda_2 = -\sqrt{a^2+b^2}$$

To find the matrix Q.

Since $a.b \in \mathbb{R}^+$, $a^2+b^2 > 0$, where the eigenvalues are distinct, the corresponding eigenvectors will be orthogonal. (theorem 4.5.7)

Thus, we can obtain the orthogonal basis of $\mathbb{R}^2$, which $\alpha = \{\vec{v_1}, \vec{v_2}\}$ where $\vec{v_1} \in E_{\lambda_1}$. $\vec{v_2} \in E_{\lambda_2}$.

According to collary 1.6.14, since $\dim(\alpha) = 2$, α is a basis for $\mathbb{R}^2$

Also, we can obtain that $Q = [\vec{v_1} \,|\, \vec{v_2}]$. which is the change of basis matrix from α to the standard basis, $Q = [I]_\alpha^\varepsilon$, where ε is the standard basis $\{\vec{e_1}, \vec{e_2}\}$.

To get the equation of hyperbola in the new basis, we need to use the Q we previously get by calculating: $M = Q\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}Q^{-1}$

$$\Rightarrow M = Q\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}Q^T$$

For the conic section, we substitute $M = Q\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}Q^T$, gives,

$$ax^2+2bxy-ay^2 = [x\,y]\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

$$= [x\ y]\, Q\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}Q^T\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \left(\begin{bmatrix} x & y \end{bmatrix} Q\right) \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \left(Q^T \begin{bmatrix} x \\ y \end{bmatrix}\right).$$

$$= \left(\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)^T (Q^T)^T\right) \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) \quad \text{where} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = Q^T \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= [I]_{\varepsilon}^{\alpha} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$= \left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right)^T \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

$$= r x^2 - t y^2$$