Derivative of Exponential and Logarithms.

what about 2^π? how to calculate?

1. Define Exponentials.

1) a^c for $c \in \mathbb{Q}$

① For $n \in \mathbb{Z}^+$, $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$.

② $a^{\frac{1}{n}} = \sqrt[n]{a}$ is defined by $(\sqrt[n]{a})^n = a$.

③ For $n, m \in \mathbb{Z}^+$, $a^{\frac{n}{m}} = [a^n]^{\frac{1}{m}}$

④ $a^0 = 1$, $a^{-c} = \frac{1}{a^c}$

2) The modern analysis solution.

① Define from scratch one of these two functions.

$E(x) = e^x$ or $L(x) = \ln(x)$

→ Option A: Differential equations.

Define $E(x) = e^x$ as the only function that satisfies $\begin{cases} E'(x) = E(x) \\ E(0) = 1 \end{cases}$ (Use Picard-Lindelöf Theorem)

→ Option B: Power Series.

Define: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

§ Ad: generalize easily if we want to use complex nums.

§ Disad: proving the properties of the exponential from its def harder.

→ Option C: Integrals

Define $\ln x = \int_1^x \frac{1}{t} dt$.

② Define other function as its inverse.

$$y = e^x \Leftrightarrow x = \ln y.$$

③ Define other exponentials as $a^c = e^{c\ln a}$.

e.g. $2^\pi = e^{\pi \ln 2}$

2. Derivative of Natural Exponentials.

noticing.
$y = \frac{a^{\Delta x}-1}{\Delta x}$
difference
$y = \frac{\ln(1+x)}{x}$
(special limit).

$f(x) = a^x$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^x(a^{\Delta x}-1)}{\Delta x}$$

$$= a^x \cdot \lim_{\Delta x \to 0} \frac{a^{\Delta x}-1}{\Delta x}$$

when $a = e$, $\lim_{\Delta x \to 0} \frac{e^{\Delta x}-1}{\Delta x} = 1$

$\therefore \frac{d}{dx} e^x = e^x$.

part of

1) Issue 1: Assuming the limit exists.

2) Issue 2: How to know there is one value of a which limit is 1?

3) Issue 3: How the exponential functions are well-defined always?

Use the trick when want to solve the derivative for inverse function (complex).

3. Derivative of Natural Logarithm.

$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x.$$

$$\Rightarrow e^{\ln x} \cdot \frac{d}{dx} \ln x = 1$$

$$\Rightarrow \frac{d}{dx} \ln x = \frac{1}{x}$$

4. Derivative of other exponentials.

A common trick

$$a^x = (e^{\ln a})^x = e^{x \cdot \ln a}.$$

in math: when want to solve the general problem for all value of a. and already know the answer for one value; then try to write the general case in terms of the case that can solve.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot\frac{d}{dx}[x\ln a]$$

$$= a^x\cdot\ln a$$

5. Derivative of other logarithm.

$$\log_a x = \frac{\ln x}{\ln a}$$

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{\frac{\ln a}{x} - 0}{(\ln a)^2} = \frac{1}{x\ln a}$$

6. Common Method

e.g. $f(x) = (\cos x)^{\sin x}$.

1) Method 1: $a^b = e^{b\ln a}$.

$$f(x) = (\cos x)^{\sin x} = e^{\sin x\cdot\ln\cos x}$$

$$f'(x) = e^{\sin x\cdot\ln\cos x}\cdot\left(\cos x\cdot\ln\cos x + \sin x\cdot\left(\frac{1}{\cos x}\cdot(-\sin x)\right)\right)$$

$$= (\cos x)^{\sin x}\cdot\left(\cos x\cdot\ln\cos x - \frac{\sin^2 x}{\cos x}\right).$$

2) Method 2: Logarithm Differentiation

$$\ln f(x) = \ln(\cos x)^{\sin x}$$

$$\Rightarrow \frac{d}{dx}\ln f(x) = \frac{d}{dx}\sin x\cdot\ln\cos x$$

$$\Rightarrow \frac{1}{f(x)}\cdot f'(x) = \cos x\cdot\ln\cos x - \frac{\sin^2 x}{\cos x}$$

$$\Rightarrow f'(x) = (\cos x)^{\sin x}\cdot\left(\cos x\cdot\ln\cos x - \frac{\sin^2 x}{\cos x}\right).$$

7. Proof of the power rule.

$$\frac{d}{dx}[x^c] = \frac{d}{dx}[e^{c\ln x}]$$

$$= e^{c\ln x}\cdot\frac{d}{dx}[c\ln x]$$

$$= x^c\cdot\frac{c}{x}$$

$$= cx^{c-1}$$