Fundamental Subspaces

① $col(A)$: columns of A correspond to pivot columns of $rref(A)$ form a basis of $col(A)$
$row(A)$: non-zero rows of $rref(A)$ form a basis of $col(A)$

1. Subspaces of Matrix

For any matrix M.

1) Row Space: $row(M)$: the span of the rows of M.

2) Column Space: $col(M)$: the span of the columns of M.

e.g. $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$ (2×3) $(m \times n)$

$$row(A) = span\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \right\} \subseteq \mathbb{R}^3 \quad (\mathbb{R}^n)$$

$$col(A) = span\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\} \subseteq \mathbb{R}^2 \quad (\mathbb{R}^m).$$

3) Null Space (kernel): $null(M)$ ($ker(M)$): the set of solution to $A\vec{x} = \vec{0}$.

p.f. $null(A) = \{ x \in \mathbb{R}^n \mid A\vec{x} = 0 \}$ is a subspace in $\mathbb{R}^n$

① $\vec{0} \in null(A)$.

② let $\vec{u}, \vec{v} \in null(A) \longrightarrow A\vec{u} = A\vec{v} = \vec{0}$

$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0}$

$\therefore \vec{u} + \vec{v} \in null(A)$.

③ let $c \in \mathbb{R}$, $A(c\vec{u}) = c(A\vec{u}) = c \cdot \vec{0} = \vec{0}$

$\longrightarrow c\vec{u} \in null(A)$

$\therefore null(A)$ is subspace.

e.g. $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$, $col(A)$. $row(A)$. $null(A)$

① row reduction

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow[r_3 - 2r_2]{3r_2 + r_1} \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow{5r_3 - r_2} \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

row reduction 可能会变 col sp. 不变 row sp.

弄清楚 aimed pivot point

$$\xrightarrow[\frac{1}{5}\cdot r_2]{-\frac{1}{3}\cdot r_1} \begin{bmatrix} 1 & -2 & 1/3 & -1/3 & 7/3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_1 - \frac{1}{3}r_2} \begin{bmatrix} 1 & -2 & 0 & -1 & 3. \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

② row(A) & col(A).

col(A): pivot col in A: $\text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$

row(A): pivot row in A: $\text{span}\left\{ \begin{bmatrix} -3 \\ 6 \\ -1 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \\ -1 \end{bmatrix} \right\}$

(or in rref(A)) $\to \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -2 \end{bmatrix} \right\}$

③ solve $A\vec{x} = 0$ for null(A)

$$\begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_3 = -2x_4 + 2x_5 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + h \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \text{null}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$