Joint Distribution

1. Joint Distribution

P33.

2. Joint c.d.f.

Given random variable X and Y, their joint c.d.f. is the function $F_{X,Y}: \mathbb{R}^2 \to [0,1]$. $F_{X,Y}(x,y) = P(X \le x, Y \le y) \equiv P(X \le x \text{ and } Y \le y)$.

e.g. $X = \mathbb{1}$ first coin H.; $Y_1 = X$, $Y_2 = 1 - X$, $Y_3 = \mathbb{1}$ second coin H.

$$F_{X,Y_1}(x,y) = \begin{cases} 1 & x \ge 1 \text{ and } y \ge 1. \\ \frac{1}{2} & 0 \le \min[x,y] < 1 \\ 0 & x < 0 \text{ or } y < 0 \text{ or both}. \end{cases}$$

$F_{X,Y_1}(x,y)$	$x<0$	$0 \le x < 1$.	$x \ge 1$.
$y<0$.	0	0	0
$0 \le y < 1$.	0	$\frac{1}{2} \to (0,0)$.	$\frac{1}{2} \to (0,0)$.
$y \ge 1$.	0	$\frac{1}{2} \to (1,1)$.	$1. \to (0,0); (1,1)$.

$F_{X,Y_2}(x,y)$.

$F_{X,Y_3}(x,y)$.	$x<0$.	$0 \le x < 1$.	$x \ge 1$.
$y<0$	0	0	0
$0 \le y < 1$	0	$\frac{1}{4}$	$\frac{1}{2}$
$y \ge 1$.	0	$\frac{1}{2}$	1.

因为有 and.

1). properties of $F_{X,Y}(x,y) = P(X \le x, Y \le y) := P(X \le x$ and $Y \le y)$.

① $\forall y.\ \lim_{x \to -\infty} F_{X,Y}(x,y) = 0$; $\forall x.\ \lim_{y \to -\infty} F_{X,Y}(x,y) = 0$.

② Marginal cdf : tells us the individual ones.

$\longrightarrow \lim_{x \to \infty} F_{X,Y}(x,y) = F_Y(y)$.

$\longrightarrow \lim_{y \to \infty} F_{X,Y}(x,y) = F_X(x)$.

2). $P(a < X \le b,\ c < Y \le d) = P(a < X \le b, Y \le d) - P(a < X \le b, Y \le c)$.

$= [P(X \le b, Y \le d) - P(X \le a, Y \le d)] - [P(X \le b, Y \le c) - P(X \le a, Y \le c)]$.

$= F_{X,Y}(b,d) - F_{X,Y}(a,d) - F_{X,Y}(b,c) + F_{X,Y}(a,c)$.

3. Joint Probability Function (for discrete variables).

1). $p_{X,Y}(x,y) := P(X = x, Y = y)$.

e.g. $p_{X,Y_1}(1,1) = \frac{1}{2}$; $p_{X,Y_1}(0,0) = \frac{1}{2}$ (otherwise is 0).

$p_{X,Y_2}(1,0) = \frac{1}{2}$; $p_{X,Y_2}(0,1) = \frac{1}{2}$.

$p_{X,Y_3}(1,1) = \frac{1}{4}$

相当于 marginal.

2). $p_X(x) = P(X = x) = \sum_y P(X = x, Y = y) = \sum_y p_{X,Y}(x,y)$.

$p_Y(x) = P(Y = y) = \sum_x P(X = x, Y = y) = \sum_x p_{X,Y}(x,y)$.

3). $P(a \le X \le b,\ c \le Y \le b) := \sum_{a \le x \le b} \sum_{c \le y \le d} p_{X,Y}(x,y)$.

4. Joint Density Function (for cts. variables).

1) J.D.F. is valid if $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$. → joint density function

1). $P(a \le X \le b,\ c \le Y \le d) = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy$. for $a \le b$, $c \le d$.

2). $P(a \le X \le b) = \int_a^b \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$.

Since $P(a \le X \le b) = \int_a^b f_X(x)\,dx$, we have.

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$; $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$.

e.g. $f_{X,Y}(x,y) = \frac{15}{32} xy^2$. for $0 \le y \le x \le 2$. otherwise 0.

① $P(0 \le X \le \frac{1}{2},\ 0 \le Y \le \frac{1}{4})$.

Since $y \le x$. we have $0 \le Y \le \frac{1}{4}$ and $Y \le X \le \frac{1}{2}$.

$$\int_0^{\frac{1}{4}}\int_y^{\frac{1}{2}} \frac{15}{32} x y^2 \, dx\, dy = \int_0^{\frac{1}{4}} \left[\frac{15}{64}\left(\left(\frac{1}{2}\right)^2 - y^2\right) y^2\right] dy = \frac{17}{65536}.$$

② $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^x \frac{15}{32} x y^2 \, dy = \frac{5}{32} x^4.$$

Use $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$ to check, i.e. $\int_0^2 \frac{5}{32} x^4 \, dx = 1$

③ $f_Y(y)$.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_y^2 \frac{15}{32} x y^2 \, dx = \frac{15}{64}(4y^2 - y^4).$$