Prove Limit From Def.

1. Structure of Proof.

e.g. $\lim_{x\to 3}(2x+1) = 7$.

1) WTS: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < |x-3| < \delta \Rightarrow |(2x+1)-7| < \varepsilon.$$

2) Let $\varepsilon > 0$ (fix)

3) Take $\delta =$ ..... (Since first fix ε, now can take δ depending on ε).

4) Let $x \in \mathbb{R}$, Assume $0 < |x-3| < \delta$

5) Conclude $|(2x+1)-7| < \varepsilon$.

2. Process of Proof

1) Rough Work

① Reverse Reasoning: Change $|f(x)-A|$ into an equation including $|x-x_0|$.

$$|f(x)-A| = k|x-x_0| < k\delta$$

② Take $\delta = \frac{\varepsilon}{k}$ (Any $\delta \le \frac{\varepsilon}{k}$ will work)

2) Proof.

① Let $\varepsilon > 0$

② Take $\delta = \frac{\varepsilon}{2}$.

③ Let $x \in \mathbb{R}$. Assume that $0 < |x-3| < \delta$.

④ Then:

$$|(2x+1)-7| = |2x-6| = 2|x-3| < 2\delta = \varepsilon.$$

QED.

3. Examples.

1) $\lim_{x\to 4}(x^2+1)=17$

① Rough Work.

$|(x^2+1)-17| = |x^2-16| = |x+4||x-4| < |x+4|\,\delta$

→ if we can make $|x+4|<C$ for some number C. then any $\delta \le \frac{\varepsilon}{C}$ will work. (we can take $\delta = \frac{\varepsilon}{C}$).

how to get it?

can put constrains on δ.

choose $\delta \le 1$ (guarantee that x and 4 are 'close') (can choose other numbers but 1 for simplicity)

then $|x-4|<1$;

$3<x<5 \Rightarrow 7<x+4<9.$

$\Rightarrow |x+4|<9$

→ So we just need $\delta \le 1$ and $\delta \le \frac{\varepsilon}{9}$ at the same time. Take $\delta = \min\{1, \frac{\varepsilon}{9}\}$

② Proof.

Let $\varepsilon > 0$.

Take $\delta = \min\{1, \frac{\varepsilon}{9}\}$.

For $x \in \mathbb{R}$, assume $0<|x-4|<\delta$.

if $\delta = 1$, $0<|x-4|<1 \Rightarrow 3<x<5$.

$\Rightarrow |x+4|<9$.

if $\delta = \frac{\varepsilon}{9}$, $0<|x-4|<\frac{\varepsilon}{9}$.

Thus, $|(x^2+1)-17| = |x^2-16|$

$= |x+4||x-4|$

$< 9\delta = 9 \cdot \frac{\varepsilon}{9} = \varepsilon.$

We have proven $|(x^2+1)-17| < \varepsilon$, as needed