Problem Set 5

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Q5:

Let $P(n)$: $T(n) = \begin{cases} 1 & n=0 \\ 2n-1 & n\geq 1 \end{cases}$ is the closed form

WTS: $\forall n \in \mathbb{N}.\ P(n)$.

Base Case: $n=0$ or $n=1$ or $n=2$.

When $n=0$, by Q3. $T(n)=1$.

When $n=1$. by Q3. $T(n) = 1 = 2\times 1 - 1 = 2n-1$.

When $n=2$. by Q3. $T(n) = T(1) + T(1) + 1 = 2+1 = 3 = 2\times 2 - 1 = 2n-1$.

Induction Step: Let $n \in \mathbb{N}$. $n>2$.

Induction Hypothesis: Assume. $\forall k \in \mathbb{N},\ 0 \leq k < n.\ P(k)$.

WTS: $P(n)$.

Since $n>2$. by Q3. gives. $T(n) = T(\lfloor \frac{n+2}{3} \rfloor) + T(\lceil \frac{2n-2}{3} \rceil) + 1$

Since $n \geq 3$. gives. $2n > 2$.

Since $n\geq 3$, $0 < \frac{n+2}{3} - 1 < \lfloor \frac{n+2}{3} \rfloor \leq \frac{n+2}{3}$, gives ① $\lfloor \frac{n+2}{3} \rfloor \leq \frac{n+2}{3} < \frac{n+2n}{3} = n$.
② $\lfloor \frac{n+2}{3} \rfloor \geq 1$.

Since $n \geq 3$, $0 < \frac{2n-2}{3} \leq \lceil \frac{2n-2}{3} \rceil < \frac{2n-2}{3} + 1$, gives ① $\lceil \frac{2n-2}{3} \rceil < \frac{2n-2}{3} + 1 = \frac{2n+1}{3} < \frac{2n+n}{3} = n$.
② $\lceil \frac{2n-2}{3} \rceil \geq 1$.

Thus. $1 \leq \lfloor \frac{n+2}{3} \rfloor < n$ and $1 \leq \lceil \frac{2n-2}{3} \rceil < n$. as shown.

Since $\lfloor \frac{n+2}{3} \rfloor, \lceil \frac{2n-2}{3} \rceil \in \mathbb{N}$, by I.H., $P(\lfloor \frac{n+2}{3} \rfloor)$ and $P(\lceil \frac{2n-2}{3} \rceil)$, which gives.

$$T(\lfloor \tfrac{n+2}{3} \rfloor) = 2\cdot \lfloor \tfrac{n+2}{3} \rfloor - 1 \text{ and } T(\lceil \tfrac{2n-2}{3} \rceil) = 2\cdot \lceil \tfrac{2n-2}{3} \rceil - 1.$$

Hence $T(n) = T(\lfloor \frac{n+2}{3} \rfloor) + T(\lceil \frac{2n-2}{3} \rceil) + 1 = 2\cdot \lfloor \frac{n+2}{3} \rfloor - 1 + 2\cdot \lceil \frac{2n-2}{3} \rceil - 1 + 1$.
$= 2(\lfloor \frac{n+2}{3} \rfloor + \lceil \frac{2n-2}{3} \rceil) - 1$.

Since for a given modulus m, each integer is congruent to exactly one of the numbers in the set $\{0, 1, 2, \dots, m-1\}$, take $m=3$.

① $n \equiv 0 \pmod 3$.

By definition, $3 \mid (n-0)$. i.e. $\exists k \in \mathbb{Z}$. s.t. $3k = n$. substitute into $T(n)$.

$$T(n) = 2 \cdot (\lfloor \frac{3k+2}{3} \rfloor + \lceil \frac{6k-2}{3} \rceil) - 1.$$

$$= 2 \cdot (k + 2k) - 1 = 6k - 1 = 2n - 1.$$

② $n \equiv 1 \pmod 3$.

By definition. $3 \mid (n-1)$. i.e. $\exists k \in \mathbb{Z}$. s.t. $3k = n-1$, gives $n = 3k+1$. sub. $T(n)$.

$$T(n) = 2 \cdot (\lfloor \frac{3k+1+2}{3} \rfloor) + \lceil \frac{6k+2-2}{3} \rceil) - 1$$

$$= 2 \cdot (k+1+2k) - 1$$

$$= 6k + 2 - 1 = 6k + 1 = 2(3k+1) - 1 = 2n - 1$$

③ $n \equiv 2 \pmod 3$.

By definition. $3 \mid (n-2)$. i.e. $\exists k \in \mathbb{Z}$. s.t. $3k = n-2$. gives. $n = 3k+2$. sub. $T(n)$.

$$T(n) = 2 \cdot (\lfloor \frac{3k+2+2}{3} \rfloor + \lceil \frac{6k+4-2}{3} \rceil) - 1$$

$$= 2 \cdot (\lfloor \frac{3k+4}{3} \rfloor + \lceil \frac{6k+2}{3} \rceil) - 1.$$

$$= 2 \cdot (k+1 + 2k+1) - 1 = 6k+3 = 2(3k+2) - 1 = 2n - 1.$$

Thus. I've shown. l.S. holds.

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