Learning Objectives

> In this tutorial, you will *practice working with linear transformations and bases of vector spaces.*

Before attending the tutorial, you should be able to write a complete mathematical definition of the following key words and concepts:

- The composition of two linear transformations.
- An invertible linear transformation
- An isomorphism between two vectorspaces
- The $\mathcal{B}$-coordinates of a vector $v \in V$, given a finite-dimensional vector space V and an ordered basis $\mathcal{B}$ of V.

The definitions can be found in the textbook Damiano and Little, 2.3-2.5.

Problems

1. Recall that P_n is the vector space of polynomials with real coefficients and degree at most n. Define a linear transformation $T_1 : P_3 \to P_2$ by $T_1(1) = 0, T_1(1+x) = 1, T_1(1+x+x^2) = 1+2x, T_1(1+x+x^2+x^3) = 2+2x$. Define a linear transformation $T_2 : P_2 \to P_3$ by $T_2(x^2) = x^3, T_2(x^2+x) = x^3+x^2, T_2(x^2+x+1) = x^3+x^2$.
   (a) Compute T_1 for an arbitrary element of P_3. [1] Compute T_2 for an arbitrary element of P_2.
   (b) Consider $T_2 \circ T_1 : P_3 \to P_3$. Compute $T_2 \circ T_1$ for an arbitrary element of P_3.
2. Let $M_{n\times n}$ be the vector space of $n \times n$ matrices.
   (a) Let $P \in M_{n\times n}$. Define the function $T_P : M_{n\times n} \to M_{n\times n}$ by $T_P(A) = PA$ for all $A \in M_{n\times n}$. Is T_P always linear? If so, is T_P ever an isomorphism?
   (b) Let P be an invertible $n \times n$ matrix. Prove that the function $A \mapsto PAP^{-1}$ from $M_{n\times n}$ to $M_{n\times n}$ is an isomorphism. (This transformation is called *conjugation by* P).
3. Consider the following two bases of $M_{2\times 2}(\mathbb{R})$, the vector space of 2×2 matrices:

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

   (a) Let $A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$. Find the coordinates $[A]_{\mathcal{E}}$ and $[A]_{\mathcal{B}}$.
   (b) Find a basis $\mathcal{A}$ of A such that $[A]_{\mathcal{A}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

[1] In other words, given $a, b, c, d \in \mathbb{R}$, what is $T_1(a + bx + cx^2 + dx^3)$?

Q1. (a). From the question, $T_1(1+0x+0x^2+0x^3)=0$ ...... (1)

$T_1(1+x+0x^2+0x^3)=1$ ...... (2)

$T_1(1+x+x^2+0x^3)=1+2x$ ...... (3)

$T_1(1+x+x^2+x^3)=2+2x.$ ...... (4).

Gives, (2) − (1): $T_1(0+x+0x^2+0x^3)=1$ ...... (5).

(3) − (2): $T_1(0+0x+x^2+0x^3)=2x$ ...... (6).

(4) − (3): $T_1(0+0x+0x^2+x^3)=1$ ...... (7).

Therefore. for arbitrary $a, b, c, d \in \mathbb{R}$.

$$T_1(a+bx+cx^2+dx^3) = a\cdot T_1(1+0x+0x^2+0x^3) + b\cdot T_1(0+x+0x^2+0x^3) + c\cdot T_1(0+0x+x^2+0x^3) + d\cdot T_1(0+0x+0x^2+x^3)$$

$$= a\cdot 0 + b\cdot 1 + c\cdot 2x + d\cdot 1$$

$$= b+2cx+d.$$

From the question, $T_2(0+0x+x^2) = x^3$ ...... (8)

$T_2(0+x+x^2) = x^3+x^2.$ ...... (9)

$T_2(1+x+x^2) = x^3+x^2.$ ...... (10)

Gives. (9) − (8): $T_2(0+x+0x^2)=x^2$ ...... (11)

(10) − (9): $T_2(1+0x+0x^2)=0$ ...... (12).

Therefore, for arbitrary $e. f. g \in \mathbb{R}$.

$$T_2(e+fx+gx^2) = e\cdot T_2(1+0x+0x^2) + f\cdot T_2(0+x+0x^2) + g\cdot T_2(0+0x+x^2)$$

$$= e\cdot 0 + f\cdot x^2 + g\cdot x^3.$$

$$= fx^2+gx^3.$$

(b) Since $T_2\circ T_1$ is also a linear transformation,

$$T_2\circ T_1 = (b+d+2cx)(fx^2+gx^3)$$

$$= bfx^2+bgx^3+dfx^2+dgx^3+2cfx^3+2cgx^4$$

$$= 2cgx^4+(2cf+bg+dg)x^3+(bf+df)x^2.$$

Q2. (a). ① It's always linear.

$\forall \alpha, \beta \in \mathbb{R}$. $\forall M_A, M_B \in M_{n\times n}$ gives.

$TP(\alpha M_A + \beta M_B) = P(\alpha M_A + \beta M_B) = \alpha PM_A + \beta PM_B = \alpha TP(M_A) + \beta TP(M_B)$.

Thus it satisfies the definition and theorem of lin. trans.

② Not always isomorphism.

when P is zero matrix, $TP(A) = PA = 0$ for all $M_A \in M_{n\times n}$.

which TP is the zero transformation.

Since the zero transformation is not one-to-one, TP is not isomorphism.

(b). To show it's both one-to-one and onto.

$\forall M_A, M_B \in M_{n\times n}$ s.t. $PM_AP^{-1} = PM_BP^{-1}$.

Gives, $M_APM_A^{-1} = M_BPM_B^{-1}$.

Since P is invertible, $M_A = P^{-1}M_BP$; which $M_A \to PM_AP^{-1}$ is one-to-one

$\forall M_C \in M_{n\times n}$. Since P is invertible,

$M_AP^{-1} = PM_C^{-1}$. Thus, $\exists M_A$ s.t. $M_AP^{-1} = PM_C^{-1}$.

$\Rightarrow PM_AP^{-1} = PM_C^{-1}P = M_C$

Take $M_A = PM_C^{-1}P^{-1} \Rightarrow PM_AP^{-1} = P(PM_C^{-1}P^{-1})P^{-1}$.

$= (PP^{-1})(PM_C^{-1})(P^{-1}P)$.

$= PM_C^{-1}$.

Hence. $M_A \to PM_AP^{-1}$ is onto.

Therefore, it's an isomorphism.

Q3. (a). For $[A]_\varepsilon$, $A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

$= 3 \cdot \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4$

$\therefore [A]_\varepsilon = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

For $[A]_B$, take $a, b, c, d \in \mathbb{R}$.

$A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} a+b & c-d \\ c+d & a-b \end{bmatrix}$, which,

$\begin{cases} a+b = 3 \\ a-b = -1 \\ c+d = -1 \\ c-d = 1 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \\ c = 0 \\ d = -1 \end{cases}$ gives $[A]_\beta = \beta_1 + 2\beta_2 - \beta_4$.

$\therefore [A]_B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$

(b). Consider $\mathcal{A} = \left\{ \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \right\}$. where $e, f, g, h \in \mathbb{R}$. gives.

$A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix}$

$\Rightarrow e = 3, f = 1, g = -1, h = -1.$

gives the basis $\mathcal{A} = \left\{ \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}$.

U3 Homework Exercise.

A: T must be the identity transformation. i.e. $T(v) = v$, for all v in V.

If $[T]_\alpha^\beta = [I]_\alpha^\beta$, $T(v) = v$ for all v is V, since any vector v is V can be written as a lin. comb. of the basis vectors a_i, for $i = 1, 2, \ldots, n$, and T is a linear transformation. Thus T is the identity transformation and $[T]_\alpha^\beta = [I]_\alpha^\beta$ iff. T is the identity transformation.