PL Equivalence and Scope.

1. Logical Equivalences.

1) Contrapositive

① $\phi \to \psi = \sim\psi \to \sim\phi$

② $\forall\alpha(\phi \to \psi) = \forall\alpha(\sim\psi \to \sim\phi)$

③ $\exists\alpha(\phi \to \psi) = \exists\alpha(\sim\psi \to \sim\phi)$

2) Biconditional

① $\phi \leftrightarrow \psi = (\phi \to \psi) \wedge (\psi \to \phi)$

② $\forall\alpha(\phi \leftrightarrow \psi) = \forall\alpha(\phi \to \psi) \wedge \forall\alpha(\psi \to \phi)$

③ $\exists\alpha(\phi \leftrightarrow \psi) = \exists\alpha(\phi \to \psi) \wedge \exists\alpha(\psi \to \phi)$

3) Exportation

① $\forall\alpha(\phi \wedge \psi \to \chi)$

$\equiv \forall\alpha(\phi \to (\psi \to \chi))$

$\equiv \forall\alpha(\psi \wedge \phi \to \chi)$

$\equiv \forall\alpha(\psi \to (\phi \to \chi))$

4) Quantifier Negation

① $\sim\forall\alpha\phi$ $\xrightarrow[\text{Practice}]{\text{In}}$ $\sim\forall\alpha(F\alpha \to G\alpha)$.

$\equiv \exists\alpha\sim\phi$ $\qquad \equiv \exists\alpha(F\alpha \wedge \sim G\alpha)$.

② $\sim\exists\alpha\phi$ $\xrightarrow[\text{Practice}]{\text{In}}$ $\sim\exists\alpha(F\alpha \wedge G\alpha)$.

$\equiv \forall\alpha\sim\phi$ $\qquad \equiv \forall\alpha(F\alpha \to \sim G\alpha)$

2. Quantifier Scope: subformula of which the quantifier is the main operator (the part that quantifier is the main operator)

e.g. $\forall\alpha\phi$, $\exists\alpha\psi$, $\forall\alpha(\ )$, $\exists\alpha(\ )$ $\quad$ $\forall, \exists, \rightarrow, \leftrightarrow, \vee, \wedge, \sim$

1) Bounded & Free Variables: All variables should be bounded or it's meaningless.

$\forall\alpha(\ldots\alpha)$ or $\exists\alpha(\ldots\alpha)$ ✓

$\forall\alpha(\ldots\alpha\ldots\beta)$ or $\exists\alpha(\ldots\alpha\ldots\beta)$ X

$\phi\alpha$ X

e.g. $\forall x(Fx \rightarrow \exists y(Gy \wedge Gb)) \rightarrow Ax$ bad scope.

'b' is not a variable, it's a name.

$\exists z \forall y((Fz \wedge \sim Hy) \leftrightarrow (Fy \vee P))$ good scope.

A symbolic sentence has no free variables.

2) The variable under the same scope means exactly the same.

3. Restrictive & Non-Restrictive Clause.

1) Restrictive Clause

e.g. Corporate lawyers that are rich work long hours.

F': a is a lawyer; G': a works for a corporation

H': a is rich; A': a works long hours.

$\forall x(Fx \wedge Gx \wedge Hx \rightarrow Ax)$

$\forall x(Fx \wedge Gx \rightarrow (Hx \rightarrow Ax))$

$\forall x(Fx \to (Gx \to (Hx \to Ax)))$

2) Non-Restrictive Clause: saying sth. that we can remove from the sentence and say it separately.

e.g. Corporate lawyers, who are rich, work long hours.

F': a is a lawyer; G': a works for a corporation

H': a is rich; A': a works long hours.

$\forall x(Fx \wedge Gx \to Hx) \wedge \forall x(Fx \wedge Gx \to Ax)$

$\forall x(Fx \to (Gx \to Hx)) \wedge \forall x(Fx \to (Gx \to Hx))$.

$\forall x(Fx \wedge Gx \to Hx \wedge Ax)$

4. Examples

1) Only dog are cute

D': {1} is a dog; C': {1} is cute.

$\forall x(Dx \to Cx)$ ✗ don't look at 'only'; do the rest then analyse 'only'

$\forall x(Cx \to Dx)$ ✓

$\forall x(\sim Dx \to \sim Cx)$ ✓

only = only if

P only if Q

$= P \to Q$

2) B': {1} is breaded; C': {1} is cute; A': {1} is a cat.

① Only <u>breaded cats</u> are cute

$\forall x(Cx \to Bx \wedge Ax)$ or $\forall x(\sim(Bx \wedge Ax) \to \sim Cx)$

② Only <u>breaded</u> cat are cute.☆

$\forall x(Ax \to (Cx \to Bx))$ or $\forall x(Ax \to (\sim Bx \to Cx))$

(amongst cat, only cute ones are breaded).

3) Every pet that is not a cat is cute unless it behaves like a cat.