Natural Number

$0 \notin \mathbb{N}$.

1. Learning Concepts.

1) A 'Theory' of numbers is a number of 'facts' about numbers.
2) 'Facts' are statements 'Logically derived' from 'generic properties'.

2. Properties of Natural Number.

1) Any natural number n is either 1 or it's $k+1$ for some other natural number k. ($\forall n \in \mathbb{N}, n=1 \vee (\exists k \in \mathbb{N}, k \neq n \wedge n=k+1)$).

2) For any natural number n, $n+1$ is also a natural number, and $n<n+1$ and there is no natural number m such that $n<m<n+1$.

3) Trichotomy Principle: For any two numbers m and n, exactly one of the three options holds: either $m=n$, or $m<n$, or $m>n$.
→ Corollary: if $m<n$, then $m \neq n$ or $m>n$.
→ Dichotomy: either $m \leq n$. or $m>n$.

$m \leq n \Leftrightarrow m<n$ or $m=n$.
for '$\leq$':
① $n \leq n$.
② $m \leq n \wedge n \leq m$. $\Rightarrow m=n$.
③ $k \leq n \wedge n \leq m$. $\Rightarrow k \leq m$.

4) Axioms of equivalence relation.

equivalence relation
① reflexive: $\forall n,\ n=n$.
② symmetry: $\forall m,n,\ m=n \Rightarrow n=m$.
③ transitivity: $\forall m,n,k,\ k=m \wedge m=n \Rightarrow k=n$.

5) Transitivity: if $n<m$ and $m<k$, then $n<k$.

6). If $m=n$ and $k \in \mathbb{N}$, then $m+k=n+k$ and $mk=nk$.

7). If $m<n$ and $k \in \mathbb{N}$. then $m+k<n+k$ and $mk<nk$.

8). Facts.

① $\forall k \in \mathbb{N}.\ 1 \le k$.

② $\forall k, m, n \in \mathbb{N}$. if $n < m$, then $n < m \cdot k$.

③ $\forall k, m, n \in \mathbb{N}$, if $n = mk$. then $m \le n$.

④ $\forall m, n, k, s \in \mathbb{N}$. if $n = mk$ and $m = n \cdot s$, then $m = n$.

Let $m, n, k, s \in \mathbb{N}$.

Assume $n = m \cdot k$, $m = n \cdot s$. gives. $n = n \cdot s \cdot k$, according to the cancellation, gives $s \cdot k = 1$. Since $s, k \in \mathbb{N}$, gives $s = k = 1$, which $n = m \cdot k = m \cdot 1 = m$.

3. Divisibility of natural number. For natural numbers n and m, we say n divides m ($n | m$), if there is a natural number q, s.t. $m = n \cdot q$.

1). For $n \in \mathbb{N}$, $1 | n$ and $n | n$. (Any $n \in \mathbb{N}$, $n < 1$, has at least two factors. (i.e. 1 and itself).)

$\exists q_1 \in \mathbb{N}.\ q = n \quad q \cdot 1 = n \cdot 1 = n$.

$\exists q_2 \in \mathbb{N}.\ q = 1.\ q \cdot n = 1 \cdot n = n$.

2). $\forall m, n \in \mathbb{N}$, If $m | n$, then $m \le n$. ← Can use with dichotomy.

Assume $m | n$, which $\exists q \in \mathbb{N}.\ m \cdot q = n$.

Assume for contradiction $m > n$.

Since $q \in \mathbb{N}$. from the fact, gives $m \cdot q > n$. which $n > n$. contradicts

a total order is a partial order which every every element of the set is comparable with every other element of the set.
(in this case. $\forall n, m \in \mathbb{N}.\ m \ne n. \Rightarrow$ $n | m$ or $m | n$).
e.g. $\ge$, $\le$......

3). Divisibility is an order relation (partial order).

partial order:

① reflexive: $\forall m \in \mathbb{N}.\ m | m$.

②. anti-symmetry: $\forall m, n \in \mathbb{N}$, $m | n$ and $n | m \Rightarrow m = n$.

Assume $m | n$ and $n | m$. gives. $\exists q_1 \in \mathbb{N}.\ m \cdot q_1 = n$. $\exists q_2 \in \mathbb{N}$, $n \cdot q_2 = m$.

Since $m, n, q_1, q_2 \in \mathbb{N}$, from the fact. since $m \cdot q_1 = n$. $n \cdot q_2 = m$. gives $m = n$.

③ transitivity: $\forall k, m, n \in \mathbb{N}$, $k | m$ and $m | n \Rightarrow k | n$

4). $\forall m, n, k \in \mathbb{N}.\ mk \mid n \Rightarrow m \mid n$ and $k \mid n$.

Let $m, n, k \in \mathbb{N}$.

Assume $mk \mid n$. gives. $\exists q \in \mathbb{N}.\ n = m \cdot k \cdot q$.

$\exists q_1 \in \mathbb{N}.\ q_1 = k \cdot q$, gives. $n = m \cdot q_1$. which. $m \mid n$.

$\exists q_2 \in \mathbb{N}.\ q_2 = m \cdot q$, gives $n = k \cdot q_2$, which. $k \mid n$.

5). $\forall a, b, k \in \mathbb{N},\ a \mid b \Rightarrow a \mid b \cdot k$.

6). linear combination: $a \mid b \wedge a \mid c \Rightarrow a \mid kb + c$.

$\longrightarrow a \mid b \wedge a \mid c \Rightarrow \forall k_1, k_2 \in \mathbb{N}.\ a \mid (bk_1 + ck_2)$.

$\longrightarrow a \mid b$ and $b \mid c \Rightarrow a \mid (kb + c)$.

e.g. A given natural number n cannot have more than n many factors.

Let $n \in \mathbb{N}$.

Since a factor of n divides n. let's say m. $m \mid n$.

Since $m \mid n$. gives $m \leq n$. Since $m \in \mathbb{N}$, $m \geq 1$. which $1 \leq m \leq n$, which are n ms.