MVT Application

1. Zero derivative implies constant.

1). Let $a<b$. Let f be a function defined on $[a, b]$.

If ① $\forall x \in (a, b)$. $f'(x)=0$.

② f is continuous on $[a, b]$.

Then f is constant on $[a, b]$

→ any arbitrary two values are the same.

2). Proof.

WTS: $\forall x_1, x_2 \in [a, b]$. $f(x_1) = f(x_2)$.

Let $x_1, x_2 \in [a, b]$, $x_1 < x_2$.; Assume ① & ②.

H①: f is continuous on $[a, b]$. $\rightarrow$ f is continuous on $[x_1, x_2] \subseteq [a, b]$.

H②: f is differentiable on (a, b). $\rightarrow$ f is differentiable on $(x_1, x_2) \subseteq (a, b)$.

By MVT. $\exists c \in (x_1, x_2)$. s.t. $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$.

Since $f'(c)=0$, $f(x_1)=f(x_2)$.

3). Let $a<b$. Let f be a function defined on (a, b).

If $\forall x \in (a, b)$. $f'(x)=0$.

Then f is constant on (a, b).

4). Proof.

WTS. $\forall x_1, x_2 \in (a, b)$, $x_1 < x_2$. $f(x_1) = f(x_2)$.

Since f is differentiable on (a, b). $\forall c \in (a, b)$.

$\lim_{x\to c} f(x)-f(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot (x-c) = f'(c) \cdot 0 = 0$.

Gives. $\lim_{x\to c} f(x) = f(c)$. which $f(x)$ is continuous on (a, b)

H① : $f(x)$ is continuous on (a, b) $\rightarrow$ $f(x)$ is continuous on $[x_1, x_2] \subseteq (a, b)$

H②: $f(x)$ is differentiable on (a, b) $\rightarrow$ $f(x)$ is differentiable on (x_1, x_2)

By MVT. $\exists c \in (x_1, x_2)$. s.t. $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

Since $f'(c)=0$, $f(x_2)=f(x_1)$.

Goal: use f' to obtain info of $f(x)$.

5). Application. 1.

e.g. Prove that there exist a constant C s.t.

$$\arctan\sqrt{\frac{1-x}{1+x}} = C - \frac{1}{2}\arcsin x.$$

Equivalently showing $\arctan\sqrt{\frac{1-x}{1+x}} + \frac{1}{2}\arcsin x = C$.

$F(x) = \arctan\sqrt{\frac{1-x}{1+x}} + \frac{1}{2}\arcsin x$. defined on $[-1,1]$

$\forall x \in (-1,1)$. $F'(x) = -\frac{1}{2(1+x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} + \frac{1}{2}\cdot\frac{1}{(1-x^2)^{\frac{1}{2}}} = 0$

Therefore, $\exists C \in \mathbb{R}$. s.t. $\forall x \in (-1,1)$, $F(x) = C$.

Since F is continuous on $[-1,1]$. $\forall x \in [-1,1]$, $F(x) = C$.

Take any $x \in [-1,1]$.

$C = F(0) = \arctan 1 + \frac{1}{2}\arcsin 0 = \frac{\pi}{4}$.

6) Application 2: Integral.

e.g. If f satisfies $\forall x \in \mathbb{R}$, $f'(x) = x^2$, then $\exists C \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}$, $f(x) = \frac{1}{3}x^3 + C$.

If h has zero derivative on an open interval I. then h is constant on I.

If f and g have same derivative on an open interval I, then $f-g$ is constant on I.

Since $\forall x \in \mathbb{R}$, $f'(x) = x^2$. gives $\frac{d}{dx}[f(x)] = \frac{d}{dx}[\frac{1}{3}x^3]$.

$f(x) - \frac{1}{3}x^3 = C$.

$\therefore f(x) = \frac{1}{3}x^3 + C$ are all solutions to $f'(x) = x^2$.