2. Defining "negative integers". Let n be any natural number. Consider now the polynomial equation $\underline{x + n = z}$ and suppose this polynomial equation has a unique solution $x = \sim n$.

Let $\mathbb{Z}$ denote set of natural numbers adjoined with z and $\{\sim n : n \in \mathbb{N}\}$. Suppose we can extend the definitions of $<$ to $\mathbb{Z}$ (that is, $\sim n < \sim n+1$ for any natural number n, and $<$ is transitive), and $\mathbb{Z}$ satisfies all rules of $<$, addition, and multiplication originally defined on $\mathbb{N}$.

a) Prove that for any natural number n, $\sim n < z$. Continue to prove that for natural numbers $m < n$, we have $\sim n < \sim m$, and $n - m = n + (\sim m)$. (where $n - m$ is defined in PS1).

WTP: ① $\sim n < z$ ② $\forall m, n \in \mathbb{N},\ m<n \Rightarrow \sim n < \sim m$. ③ $n - m = n + (\sim m)$.

Let $n \in \mathbb{N}$. $m \in \mathbb{N}$.

①. WTP: $\sim n < z$.

I'll prove by contradiction.

Assume $\sim n \geq z$.

Since $n \in \mathbb{N}$, gives, $\sim n + n \geq n + z$.

Since according to the question, the solution of $x+n=z$ is $x=\sim n$. which $\sim n + n = z$, gives $z \geq n+z$. From Q1, gives $n+z = n$, which $z \geq n$.

Since $n \in \mathbb{N}$, $n \geq 1$ (from def^{n} of natural number).

However, in Q1, we have $z<1$, gives $z<n$. contradicts

Therefore, I've proved $\sim n < z$.

②. Assume $m<n$. WTP. $\sim n < \sim m$.

I'll prove by using contradiction.

Assume $\sim n \geq \sim m$.

Since $n \in \mathbb{N}$. add n on both side. gives $\sim n + n \geq \sim m + n$.

Since $n>m$, from assumption, gives. $\sim n + n \geq \sim m + n > \sim m + m$

However, from definition, $\sim n + n = z$, $\sim m + m = z$, as $n, m \in \mathbb{N}$.

Gives, $z > z$. contradicts.

Therefore, I've proved $\forall m, n \in \mathbb{N}$, $m<n \Rightarrow \sim n < \sim m$

③ WTP. $n - m = n + (\sim m)$.

From PS1, we have $n = m+k \Rightarrow n-m = k$.

Let $n = m+k$. gives $n - m = k$, $k \in \mathbb{N}$.

From the question. gives. $\sim m + m = z$, as $m \in \mathbb{N}$.

Adding $k \in \mathbb{N}$. gives. $\sim m + m + k = z + k$.

According to the definition of z. gives. $z + k = k$, which, $\sim m + m + k = k$.

Since $n = m+k$. gives, $\sim m + n = n - m$, which $n - m = n + (\sim m)$.

I've proved $n - m = n + (\sim m)$.

$\blacksquare$

b) Prove that $(\sim m)n = \sim(mn)$.

Let $m, n \in \mathbb{N}$.

Since $m, n \in \mathbb{N}$, gives $m \cdot n \in \mathbb{N}$. from the multiplication of natural number

By definition of z, gives. $\sim(m \cdot n) + m \cdot n = z$., as $m \cdot n \in \mathbb{N}$.

Also, by definition of z. gives. $\sim m + m = z$. as $m \in \mathbb{N}$.

Multiply n on both sides, gives, $[m + (\sim m)] \cdot n = z$.

By Q1(c). gives. $z \cdot n = z$. which. $[m + (\sim m)] \cdot n = z \cdot n = z$.

According to multiplication over addition, $[m + (\sim m)] \cdot n = m \cdot n + (\sim m) \cdot n = z$.

Since $\sim(m \cdot n) + m \cdot n = z$. gives. $\sim(m \cdot n) + m \cdot n = m \cdot n + (\sim m) \cdot n$.

Since $m \cdot n \in \mathbb{N}$. according to the cancellation rule, gives $(\sim m) \cdot n = \sim(m \cdot n)$

$\blacksquare$

We call the numbers $\sim n$ the "negative integers", denoted $-n$ henceforth. The set $\mathbb{Z}$ of all natural numbers, zero, and the negative integers is called the "integers".