# Problem Set 5

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Q2: Let $f_1 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f_1(n) = n^{\frac{n}{2}+1}$

Let $f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f_2(n) = n^{\lfloor \frac{n+3}{2} \rfloor}$.

WTS: ① $f_1$ is non-decreasing. i.e. $\forall n_1, n_2 \in \mathbb{N}.\ (n_1 \leq n_2) \Rightarrow f_1(n_1) \leq f_1(n_2)$.

② $f_2$ is non-decreasing. i.e. $\forall n_1, n_2 \in \mathbb{N}.\ (n_1 \leq n_2) \Rightarrow f_2(n_1) \leq f_2(n_2)$.

Since $n \in \mathbb{N}$, $n \geq 0$.

① When $n=0$. $f_1, f_2$ are constant function, which is non-decreasing.

②. When $n \geq 1$. $f_1, f_2$ are power-exponential function. which

$f_1(n) = n^{\frac{n+2}{2}} = e^{\frac{n+2}{2} \cdot \ln n}$. $\quad f_2(n) = n^{\lfloor \frac{n+3}{2} \rfloor} = e^{\lfloor \frac{n+3}{2} \rfloor \cdot \ln n}$.

Since $n \geq 1$, gives $\frac{n+2}{2} \geq 1$ and $\ln n \geq 0$. which $e^{\frac{n+2}{2} \cdot \ln n}$ is increasing. i.e. $f_1(n)$ is increasing.

Since $n \geq 1$, gives $\lfloor \frac{n+3}{2} \rfloor \geq 1$ and $\ln n \geq 0$. which $e^{\lfloor \frac{n+3}{2} \rfloor \cdot \ln n}$ is increasing. i.e. $f_2(n)$ is increasing.

Thus. $f_1(n)$ and $f_2(n)$ are non-decreasing function.

WTS: $\forall n \in \mathbb{N}.\ n$ is even $\Rightarrow f_1(n) = f_2(n)$.

Let $n \in \mathbb{N}$. Assume $n$ is even. i.e. $\exists k \in \mathbb{N}.$ s.t. $2k = n$.

$f_1(n) = f_1(2k) = n^{\frac{2k+2}{2}} = n^{k+1}$

$f_2(n) = f_2(2k) = n^{\lfloor \frac{2k+3}{2} \rfloor} = n^{\lfloor k + \frac{3}{2} \rfloor} = n^{k+1}$

Thus $f_1(n) = f_2(n)$.

WTS: $f_1 \notin \Theta(f_2)$.

I'll prove by contradiction.

Assume $f_1 \in \Theta(f_2)$. i.e. $\exists c_1, c_2, B \in \mathbb{R}^+$, s.t. $\forall n \geq B$, $c_1 f_2(n) \leq f_1(n) \leq c_2 f_2(n)$.

When $n$ is an odd number, i.e. $\exists k \in \mathbb{N}$. s.t. $n = 2k+1$, gives.

$f_1(n) = n^{\frac{2k+1+2}{2}} = n^{\frac{2k+3}{2}} = n^{k+\frac{3}{2}}$; $\quad f_2(n) = n^{\lfloor \frac{2k+1+3}{2} \rfloor} = n^{\lfloor \frac{2k+4}{2} \rfloor} = n^{k+2}$; which

$\frac{f_1}{f_2} = \frac{n^{k+\frac{3}{2}}}{n^{k+2}} = n^{\frac{3}{4}}$.

Since $n$ is an variable and $c_1, c_2$ is taken before $n$. we can't get such $c_1$, $c_2 = n^{\frac{3}{4}}$, $\forall n \geq B$, contradicts to the assumption.

Therefore. $f_1 \notin O(f_2)$. ∎