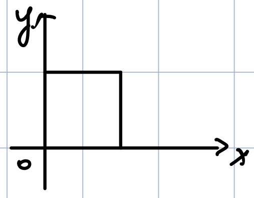
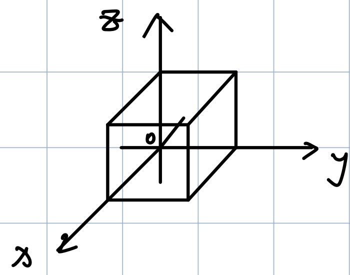
Determinant

1. Geometry

1) Unit-cube.: n-dim cube with sides given by standard basis vectors and lower-left corner located at the origin

e.g.





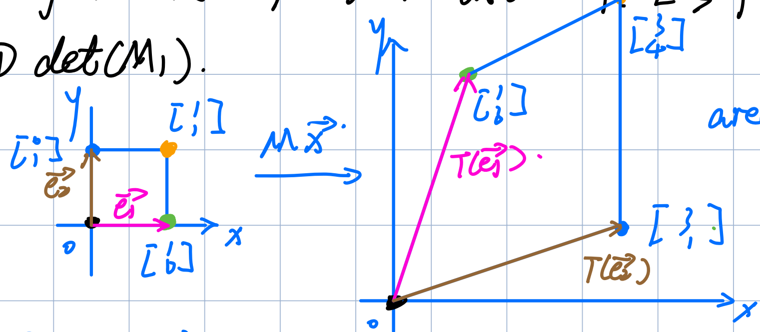
$$C_n = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e_i} \text{ for some } \alpha_1, \dots, \alpha_n \in [0,1]\} = [0,1]^n.$$

2) Determinant of a lin trans $X: \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit cube. ← -1 if changed.

det(X) = (orientation change or not)·(area after change).

e.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(\vec{x}) = M(\vec{x})$. $M_1 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

① $\det(M_1)$.



area = 5

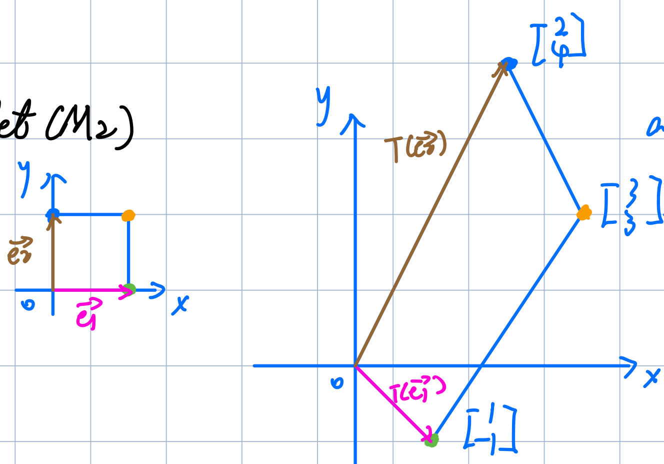
$\{\vec{e_1}, \vec{e_2}\}$ ⟶ $\{T(\vec{e_1}), T(\vec{e_2})\}$.

pos-ori　　　neg-ori

$\therefore \det(M_1) = (-1)\cdot 5 = -5$.

$|\det(X)| = \dfrac{\text{vol of } T(S)}{\text{vol of } S}$

② $\det(M_2)$



area = 6.

$\det(M_2) = 1 \times 6$
$= 6.$

$\det(M) = \det(M^T)$

$\det(T \circ M) = \det(T)\cdot\det(M)$

2. Algebra (compute).

1) Compute:

① 2x2 : $ad-bc$.

② 3x3.

e.g. $A=\begin{bmatrix}1&2&-3\\2&1&3\\0&4&5\end{bmatrix}$

①选0多的行/列

②标正负：左上开始为+，相邻相反

③展开

$\det(A)=1\times\det\begin{bmatrix}1&3\\4&5\end{bmatrix}-2\times\det\left(\begin{bmatrix}2&-3\\4&5\end{bmatrix}\right)+0\times\det\left(\begin{bmatrix}2&-3\\1&3\end{bmatrix}\right).$

$=5-12-2\times(10+12).$

$=-51$

③ 4x4: 化简到方便时同上拆 matrix 次数算

→ follow the law below.

upper tri: $\begin{bmatrix}a&b&c\\0&d&e\\0&0&f\end{bmatrix}$

lower tri: $\begin{bmatrix}a&0&0\\b&c&0\\d&e&f\end{bmatrix}$

2) If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.
(左上到右下).

e.g. $A=\begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}$, $\det(A)=?$

$\det(A)=1\times4\times6=24.$

3) row additive

$$\det\left(\begin{bmatrix}u+v\\A_2\\\vdots\\A_n\end{bmatrix}\right)=\det\left(\begin{bmatrix}u\\A_2\\\vdots\\A_n\end{bmatrix}\right)+\det\left(\begin{bmatrix}v\\A_2\\\vdots\\A_n\end{bmatrix}\right).$$

4) Elementary Matrix: $\det(E)=\begin{cases}-1 : & I\sim E \text{ interchanging.}\\ r : & I\sim E \text{ scalaring by } r.\\ +1 : & I\sim E \text{ row replacement.}\end{cases}$

5) $\det(A) = \det(A^T)$

$\det(A \cdot B) = \det A \cdot \det B.$

$\det(kA) = k\det(A).$

6) row operation: $B = EA$. $\det(B) = \det(EA)$ i.e.

① Interchanging: $\det B = -\det A$.

② Scaling: $\det B = r \cdot \det A$. 66.

③ Row replacement: $\det B = \det A$.

e.g. $A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 2 & 1 & 3 & -1 \\ 0 & 4 & 5 & -2 \\ 5 & 3 & 4 & 1 \end{bmatrix}$. $\det(A) = ?$

$$A \xrightarrow[r_4 - 5r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -3 & 9 & -9 \\ 0 & 4 & 5 & -2 \\ 0 & -7 & 17 & -19 \end{bmatrix} \xrightarrow{-\frac{1}{3}r_2} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 4 & 5 & -2 \\ 0 & -7 & 17 & -19 \end{bmatrix} \quad B_2$$

B_1

$$\xrightarrow[r_4 + 7r_2]{r_3 - 4r_2} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 17 & -14 \\ 0 & 0 & -4 & 2 \end{bmatrix} \quad B_3.$$

$\det(B_3) = \det\begin{bmatrix} 17 & -14 \\ -4 & 2 \end{bmatrix}$

$\det(A) = \det(B_1).$

$\det(B_2) = -\frac{1}{3}\det(B_1).$

$\det(B_3) = \det(B_2).$ $\therefore \det(B_3) = -\frac{1}{3}\det(A).$

$\Rightarrow \det(A) = (-3)\det(B_3).$

$\therefore \det(A) = (-3) \times (-22) = 66.$

7). A is invertible iff $\det(A) \neq 0$. i.e. A is nonsingular.