The Main Continuity Theorem.

1. The Theorem.

Any function we can construct with sum, product, quotient, and composition of polynomials, roots, trigonometric functions, exponentials, logarithms, and absolute values is continuous (on its domin).

2. How to prove

1) Step 1: Prove that the 'basic' functions are continuous.

$f(x) = c.$ $\qquad f(x) = e^x$ $\qquad f(x) = \sin x.$

$f(x) = x$ $\qquad f(x) = \ln x$ $\qquad f(x) = |x|.$

e.g. $\cos x = \sin(\frac{\pi}{2} - x)$; $2^x = e^{x\ln 2}$

$\tan x = \frac{\sin x}{\cos x}$; $\sqrt{x} = e^{\frac{1}{2}\ln x}.$

2) Step 2: Prove that sum, product, quotient and composition of continuous functions is continuous.

→ no limit law for composition!

3. Theorems regarding composite functions.

1) If f continuous at a, and g continuous at $f(a)$. then $g \circ f$ continuous at a.

means: $x \to a. \Rightarrow f(x)$ close to $f(a)$.

$y \to f(a) \Rightarrow g(y)$ close to $g(f(a))$.

concatenate:

$x \to a \Rightarrow g(f(x))$ close to $g(f(a))$.

2) If $\lim_{x \to a} f(x) = L$, and $f(x) \neq L$ for x on an interval centered at a, except maybe at a; $\lim_{y \to L} g(y) = M$, then $\lim_{x \to a} g(f(x)) = M$.

mean: $x \to a \Rightarrow f(x) \to L$.

$y \to L \Rightarrow g(y) \to M$.

concatenate:

$x \to a \Rightarrow g(f(x)) \to M$.

3) If $\lim_{x \to a} f(x) = L$, g is continuous at L, then $\lim_{x \to a} g(f(x)) = g(L)$. $\lim_{y \to L} g(x) = g(L)$.

means: $x \to a \Rightarrow f(x) \to L$.

$\lim_{y \to L} g(y) = g(L)$

$= y \to L \Rightarrow g(y) \to g(L)$.

concatenate:

$x \to a \Rightarrow g(f(x)) \to g(L)$.

Do we exist a function f such that $\lim_{x \to 0} f(x) = 0$ but $\lim_{x \to 0} f(f(x)) = 1$.