LAST (Family) NAME ______________________________

FIRST (Given) NAME ______________________________

Student Number: ______________________________

UNIVERSITY OF TORONTO
Faculty of Arts and Science
December 2018 Examinations
MAT301H1F
Duration – 3 hours
No aids are permitted.
Instructor: P. Eskandari

Exam Reminders:

- Fill out your name and student number at the top of this page.
- Do not begin writing the actual exam until the announcements have ended and the Exam Facilitator has started the exam.
- As a student, you help create a fair and inclusive writing environment. If you possess an unauthorized aid during an exam, you may be charged with an academic offence.
- Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. If any of these items is left in your pocket, it may be an academic offence.
- When you are done your exam, raise your hand for someone to come and collect your exam. Do not collect your bag and jacket before your exam is handed in.
- If you are feeling ill and unable to finish your exam, please bring it to the attention of an Exam Facilitator so it can be recorded before leaving the exam hall.
- In the event of a fire alarm, do not check your cell phone when escorted outside.

Exam Format and Grading Scheme:

- There are 10 questions on this exam, the last one of which is a bonus question. Including the bonus question (which is worth 6 marks), there are a maximum of 81 marks that can be collected on the exam. The exam will be graded out of 75.
- The numbers in [] indicate how many marks each question is worth. Unless otherwise indicated, multiple parts of a question have equal weights.
- Unless otherwise indicated, you are expected to provide full solutions to the questions, justifying all your claims.
- The exam consists of 18 pages, including the cover page. Once the exam begins, ensure that there are no missing pages. Pages 12-18 are intentionally left blank. You may use them for your rough work, or to continue your solution to a question, if you need more space. If you wish to do the latter, write a brief note in the original question space to indicate where your solution continues. Otherwise, your continuation will not be graded. Do not detach any of the pages.

Students must hand in all examination materials at the end.

1. [10] Answer the following questions. Briefly justify your answers in the space provided.

(a) Find the order of the element $\sigma = (123)(34)(12)$ of S_4.

(b) Find the number of elements of A_7 that have order 6.

(c) Let G be a cyclic group of order 10. How many isomorphisms $\mathbb{Z}/10\mathbb{Z} \to G$ are there?

(d) Find the order of the element $([2], (1234))$ of $\mathbb{Z}/6\mathbb{Z} \times S_4$.

(e) Give an example of an abelian group G and a subgroup $K \leq G$ such that G is not isomorphic to $K \times (G/K)$.

2. [9] Determine if each statement below is true or false. No explanation is necessary.

(a) Given any abelian group G of order 100, there are exactly 25 homomorphisms $\mathbb{Z}/25\mathbb{Z} \to G$.

(b) For any groups G and H, there exists a surjective homomorphism $G \twoheadrightarrow H$ if and only if H is isomorphic to a quotient of G.

(c) Every abelian group of order 21 is cyclic.

(d) If G is a group of order n, then for every divisor d of n, there is a subgroup of order d in G.

(e) If G is a cyclic group of order 20, then G has a quotient isomorphic to $\mathbb{Z}/10\mathbb{Z}$.

(f) If G and H are groups with $|G| = 120$ and $|H| = 20$, and $\phi : G \twoheadrightarrow H$ is a surjective homomorphism, then for every $h \in H$, the set

$$\{g \in G : \phi(g) = h\}$$

has 6 elements.

3. [10] In each part below, determine if there exists a homomorphism as described. Justify your answers.

(a) [3] a surjective homomorphism $U(100) \to D_{10}$

(b) [3] an injective homomorphism $U(15) \to \mathbb{Z}/24\mathbb{Z}$

(c) [4] a nontrivial homomorphism $\mathbb{Q} \to \mathbb{Z}$ (i.e. a homomorphism whose image is not zero)

4. [8] Given an integer a and a positive integer n, denote the residue class of a mod n by $[a]_n$. Let $\phi : U(27) \to U(9)$ be given by $\phi([a]_{27}) = [a]_9$.

(a) [3] Show that ϕ is a well-defined homomorphism.

(b) [2] Find $\ker(\phi)$.

(c) [3] Denote $\ker(\phi)$ by H. Find the order of the element $[4]_{27} \cdot H$ of the quotient group $U(27)/H$.

5. [6] Let G be a (possibly infinite) abelian group and p a prime number. Let

$$H := \{g \in G : \text{there is } \ell \geq 0 \text{ such that } g^{p^\ell} = e\}.$$

Note that H is a subgroup of G (you do not need to check this). Suppose that the index $[G : H]$ is finite. Show that $[G : H]$ is not divisible by p.

6. [8] (a) [6] Let G be a group and $H \leq G$ a normal subgroup of finite index n. Show that for every $g \in G$, the element g^n is in H.

(b) [2] Give an example that shows that the conclusion of part (a) may not hold if H is not normal. More explicitly, give an example of a group G, a subgroup $H \leq G$ of finite index, and an element $g \in G$ such that $g^{[G:H]} \notin H$.

7. [9] Show that the two groups given in each part are isomorphic.

(a) $\mathbb{C}^\times/\mu_6$ and $\mathbb{C}^\times$

(b) $\mathbb{R}/\mathbb{Z}$ and S, where S is the unit circle ($\{z \in \mathbb{C}^\times : |z| = 1\}$) under multiplication.

(c) $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ and $\mathbb{R}^\times$

8. [6] Let G and H be abelian groups and $\pi : G \to H$ a surjective homomorphism with kernel K. Suppose there exists a homomorphism $\psi : H \to G$ such that $\pi \circ \psi : H \to H$ is the identity map. Show that the map $\phi : K \times H \to G$ defined by

$$\phi((k, h)) = k \cdot \psi(h)$$

is an isomorphism.

9. [9]

(a) Find the order of the element [3] of $U(32)$.

(b) Give a complete list of abelian groups of order 16, up to isomorphism. Your list must contain exactly one group from each isomorphism class. No explanation is necessary.

(c) Show that $U(32)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

10. (Bonus) [6] Let G be a finite abelian group. Let $n = |G|$. Show that there are n homomorphisms $G \to \mathbb{C}^\times$.

Extra space. What you write here will not be graded unless you write next to the relevant question(s) "Continued on page 12".

Extra space. What you write here will not be graded unless you write next to the relevant question(s) "Continued on page 13".

Extra space. What you write here will not be graded unless you write next to the relevant question(s) "Continued on page 14".

Extra space. What you write here will not be graded unless you write next to the relevant question(s) "Continued on page 15".

Extra space. What you write here will not be graded unless you write next to the relevant question(s) "Continued on page 16".

Extra space. What you write here will not be graded unless you write next to the relevant question(s) "Continued on page 17".

Extra space. What you write here will not be graded unless you write next to the relevant question(s) "Continued on page 18".

The end (total marks to collect = 81, including 6 bonus marks).