Problem Set 5

XUANQI WEI

1009353209

Q7:

(a). $T(3^k) = 3^2 \cdot T(\lfloor \frac{3^k}{3} \rfloor) + (3^k)^5$ by definition of T.

$= 3^2 \cdot T(3^{k-1}) + 3^{5k}$.

$= 3^2 \cdot (3^2 \cdot T(\lfloor \frac{3^{k-1}}{3} \rfloor + (3^{k-1})^5) + 3^{5k}$.

$= 3^{2 \cdot 2} \cdot T(3^{k-2}) + 3^2 \cdot 3^{5k-5} + 3^{5k}$.

$= 3^4 \cdot (3^2 \cdot T(\lfloor \frac{3^{k-2}}{3} \rfloor) + (3^{k-2})^5) + 3^2 \cdot 3^{5k-5} + 3^{5k}$.

$= 3^6 \cdot T(3^{k-3}) + 3^4 \cdot 3^{5k-10} + 3^2 \cdot 3^{5k-5} + 3^{5k}$.

$= 3^6 \cdot T(3^{k-3}) + 3^{5k-6} + 3^{5k-3} + 3^{5k}$.

$\vdots$

$= 3^{2m} \cdot T(3^{k-m}) + \sum_{i=0}^{m-1} 3^{5k-3i}$

(b). Let $Q(m)$: $\forall k, m \in \mathbb{N}, m \leq k, T(3^k) = 3^{2m} T(3^{k-m}) + \sum_{i=0}^{m-1} 3^{5k-3i}$.

Let $k \in \mathbb{N}$.

WTS: $\forall j \in \mathbb{N}, j \leq k \Rightarrow Q(j)$.

Let $j \in \mathbb{N}$.

Base Case: $j=0$ or $j=1$.

① When $j=0$. Assume $j \leq k$.

WTS: $Q(0)$.

Since $T(3^k) = 1 \cdot T(3^{k-0}) + 0$.

$= 3^{2 \cdot 0} \cdot T(3^{k-0}) + \sum_{i=0}^{-1} 3^{5k-3i}$

$= 3^{2j} \cdot T(3^{k-j}) + \sum_{i=0}^{j-1} 3^{5k-3i}$ as needed.

② When $j=1$. Assume. $j \leq k$.

WTS: $Q(1)$.

Since $k \geq j = 1$, gives $3^k > 1$. gives.

$$T(3^k) = 3^2 \cdot T(\lfloor \frac{3^k}{3} \rfloor) + (3^k)^5$$
$$= 3^2 \cdot T(3^{k-1}) + 3^{5k}.$$
$$= 3^{2\cdot 1} \cdot T(3^{k-1}) + \sum_{i=0}^{0} 3^{5k-3i}$$
$$= 3^{2j} \cdot T(3^{k-j}) + \sum_{i=0}^{j-1} 3^{5k-3i} \quad \text{as needed.}$$

Induction Step: Let $j \in \mathbb{N}$. $j > 1$.

Induction Hypothesis: Assume. $j \le k \Rightarrow Q(j)$.

WTS: $j+1 \le k \Rightarrow Q(j+1)$.

Assume. $j+1 \le k$.

Since $j+1 \le k$, gives. $k-j \ge 1$. which

$$T(3^{k-j}) = 3^2 \cdot T(\lfloor \frac{3^{k-j}}{3} \rfloor) + (3^{k-j})^5$$
$$= 3^2 \cdot T(3^{k-j-1}) + 3^{5k-5j} \quad (\star).$$

Since $j+1 \le k$, gives. $j < j+1 \le k$. by I.H., we have $Q(j)$. which

$$T(3^k) = 3^{2j} \cdot T(3^{k-j}) + \sum_{i=0}^{j-1} 3^{5k-3i} \quad \text{by } (\star).$$
$$= 3^{2j} \cdot (3^2 \cdot T(3^{k-j-1}) + 3^{5k-5j}) + \sum_{i=0}^{j-1} 3^{5k-3i}$$
$$= 3^{2(j+1)} \cdot T(3^{k-(j+1)}) + 3^{5k-3j} + \sum_{i=0}^{j-1} 3^{5k-3i}$$
$$= 3^{2(j+1)} \cdot T(3^{k-(j+1)}) + \sum_{i=0}^{(j+1)-1} 3^{5k-3i}$$, which is $Q(j+1)$ as needed. ∎

(c). Let $n = 3^k$.

Take $m = k$, gives.

$$T(n) = T(3^k) = 3^{2m} \cdot T(3^{k-m}) + \sum_{i=0}^{m-1} 3^{5k-3i} \quad \text{substitute gives.}$$
$$= 3^{2k} \cdot T(3^{k-k}) + \sum_{i=0}^{k-1} 3^{5k-3i}$$
$$= (3^k)^2 \cdot T(1) + \sum_{i=0}^{k-1} \frac{(3^k)^5}{(3^3)^i}$$
$$= n^2 + \sum_{i=0}^{k-1} \frac{n^5}{(3^3)^i}$$
$$= n^2 + n^5 \sum_{i=0}^{k-1} (\frac{1}{3^3})^i \quad (\star).$$

Since. $r = \frac{1}{3^3}$, $a = (\frac{1}{3^3})^0 = 1$, $1-r = 1-\frac{1}{3^3}$, $1-r^n = 1-(\frac{1}{3^3})^k$, where

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{1-(\frac{1}{3^3})^k}{1-\frac{1}{3^3}} = \frac{1-(\frac{1}{3^k})^3}{\frac{3^3-1}{3^3}} = \frac{1-(\frac{1}{n})^3}{\frac{3^3-1}{3^3}} = \frac{n^3-1}{n^3} \cdot \frac{3^3}{3^3-1}$$

Thus, $(\star)$, gives. $n^2 + n^5 \cdot (\frac{n^3-1}{n^3}) \cdot \frac{3^3}{3^3-1}$

$$= n^2 + \frac{(n^5 - n^2)\cdot 3^3}{3^3 - 1}$$

$$= \frac{n^2 3^3 - n^2 + n^5\cdot 3^3 - n^2\cdot 3^3}{3^3 - 1} = \frac{3^3\cdot n^5 - n^2}{3^3 - 1}$$, which $T(n) = \frac{3^3\cdot n^5 - n^2}{3^3-1}$.

(d). WTS: $\forall k \in \mathbb{N},\ n = 3^k \Rightarrow T(n) = \frac{3^3\cdot n^5 - n^2}{3^3 - 1}$

Let $k \in \mathbb{N}$.

Base Case: $k=0$.

Assume $n = 3^k$, gives $n = 3^0 = 1$.

Since $T(1) = 1$, substitute $n=1$ into $\frac{3^3\cdot n^5 - n^2}{3^3-1}$ gives, $\frac{3^3\cdot n^5 - n^2}{3^3-1} = \frac{3^3\cdot 1 - 1}{3^3 - 1} = 1$. where $T(n) = T(1) = 1 = \frac{3^3\cdot n^5 - n^2}{3^3 - 1}$.

Induction Step: Let $k \in \mathbb{N}$.

Induction Hypothesis: Assume $n = 3^k \Rightarrow T(n) = \frac{3^3\cdot n^5 - n^2}{3^3-1}$

WTS: $n' = 3^{k+1} \Rightarrow T(n') = \frac{3^3\cdot n'^5 - n'^2}{3^3 - 1}$

Assume $n' = 3^{k+1}$. gives $n' = 3^k\cdot 3$. which $n = 3^k = \frac{n'}{3}$.

By I.H., gives $T(n) = T(\frac{n'}{3}) = \frac{3^3\cdot(\frac{n'}{3})^5 - (\frac{n'}{3})^2}{3^3 - 1}$.

Since $k \in \mathbb{N}$, $k+1 \geq 1$, gives $n' = 3^{k+1} \geq 3^1 = 3 > 2$. gives.

$$T(n') = 3^2\cdot T(\lfloor \tfrac{n'}{3} \rfloor) + (n')^5$$ Since $\frac{n'}{3} = 3^k = n$ is divisible by 3.

$$= 3^2\cdot T(\tfrac{n'}{3}) + (n')^5$$

$$= 3^2\cdot \frac{3^3\cdot(\frac{n'}{3})^5 - (\frac{n'}{3})^2}{3^3-1} + (n')^5$$

$$= \frac{(n')^5 - (n')^2}{3^3 - 1} + (n')^5$$

$$= \frac{(n')^5 - (n')^2 + 3^3\cdot(n')^5 - (n')^5}{3^3-1}$$

$$= \frac{3(n')^5 - (n')^2}{3^3 - 1}$$ as needed. ■