Linear Combination & Span

1. Linear Combination:

Let V be a vector space ($\mathbb{R}^n$), given vectors $v_1, v_2, \dots, v_k \in V$ and scalars $c_1, c_2, \dots, c_k \in \mathbb{R}$, the vector:

$y = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$ is called a linear combination of v_1, v_2 with coefficient $c_1, c_2, \dots c_k$.

e.g. $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ $v_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $v_3 = \begin{bmatrix}7\\8\\9\end{bmatrix}$ $v_4 = \begin{bmatrix}3\\5\\4\end{bmatrix}$, $v = \begin{bmatrix}3\\3\\3\end{bmatrix}$, v a lincomb of $v_{1\sim4}$?

① 每行都有 pivot.
② DNE zero row.

($v = x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4$ 是否有解).

$$\left[\begin{array}{cccc|c}1&4&7&3&3\\2&5&8&5&3\\3&6&9&4&3\end{array}\right] \xrightarrow[r_3-3r_1]{r_2-2r_1} \left[\begin{array}{cccc|c}1&4&7&3&3\\0&-3&-6&-1&-3\\0&-6&-12&-5&-6\end{array}\right]$$

$$\xrightarrow[r_2\cdot(-\frac{1}{3})]{r_3-2r_2} \left[\begin{array}{cccc|c}1&4&7&3&3\\0&1&2&\frac{1}{3}&1\\0&0&0&-3&0\end{array}\right]$$

$$\xrightarrow{r_3\times(-\frac{1}{3})} \left[\begin{array}{cccc|c}1&4&7&3&3\\0&1&2&\frac{1}{3}&1\\0&0&0&1&0\end{array}\right]$$

$$\xrightarrow{r_2-\frac{1}{3}r_3} \left[\begin{array}{cccc|c}1&4&7&3&3\\0&1&2&0&1\\0&0&0&1&0\end{array}\right]$$

$$\xrightarrow[r_1-3r_3]{r_1-4r_2} \left[\begin{array}{cccc|c}1&0&-1&0&-1\\0&1&2&0&1\\0&0&0&1&0\end{array}\right]$$

$$\therefore \begin{cases}x_1 = x_3 - 1\\ x_2 = -2x_3 + 1\\ x_4 = 0.\end{cases} \rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = t\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + \begin{bmatrix}-1\\1\\0\\0\end{bmatrix} \quad t \in \mathbb{R}.$$

$\therefore$ consistent and has infinite solutions.

e.g. $v_1 = \begin{bmatrix}3\\3\\5\end{bmatrix}$, $v_2 = \begin{bmatrix}2\\4\\6\end{bmatrix}$. $v_3 = \begin{bmatrix}4\\7\\8\end{bmatrix}$. for every vector $\vec{v}$ in $\mathbb{R}^3$. Is $\vec{v}$ a lin comb of $v_1 \sim v_3$?

$\forall \vec{x} \in \mathbb{R}^3$. where $\vec{x} = \begin{bmatrix}a\\b\\c\end{bmatrix}$. gives

$$\vec{x} = c_1\begin{bmatrix}3\\3\\5\end{bmatrix} + c_2\begin{bmatrix}2\\4\\6\end{bmatrix} + c_3\begin{bmatrix}4\\7\\8\end{bmatrix}.$$

$$\left[\begin{array}{ccc|c}3&2&4&a\\3&4&7&b\\5&6&8&c\end{array}\right] \xrightarrow[r_3-\frac{5}{3}r_1]{r_2-r_1} \left[\begin{array}{ccc|c}3&2&4&a\\0&2&3&b-a\\0&\frac{8}{3}&\frac{4}{3}&c-\frac{5}{3}a\end{array}\right] \xrightarrow{r_3-\frac{4}{3}r_2} \left[\begin{array}{ccc|c}3&2&4&a\\0&2&3&b-a\\0&0&\frac{8}{3}&c-\frac{4}{3}b-\frac{1}{3}a\end{array}\right]$$

$2\times\frac{4}{3}$

$$\xrightarrow[r_3\times\frac{3}{8}]{r_1\times\frac{1}{3}\ r_2\times\frac{1}{2}} \left[\begin{array}{ccc|c}1&\frac{2}{3}&\frac{4}{3}&\frac{a}{3}\\0&1&\frac{3}{2}&\frac{1}{2}(b-a)\\0&0&1&\frac{3}{8}c-\frac{1}{2}b-\frac{1}{8}a\end{array}\right]$$

$$\xrightarrow[r_1 - \frac{2}{3}r_2 - \frac{4}{3}r_3]{r_2 - \frac{3}{2}r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{11}{24}a + \frac{1}{6}b - \frac{1}{8}c \\ 0 & 1 & 0 & -\frac{9}{16}c + \frac{5}{4}b - \frac{5}{16}a \\ 0 & 0 & 1 & \frac{3}{8}c - \frac{1}{2}b - \frac{1}{8}a \end{array}\right]$$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{11}{24}a + \frac{1}{6}b - \frac{1}{8}c \\ -\frac{5}{16}a + \frac{5}{4}b - \frac{9}{16}c \\ -\frac{1}{8}a - \frac{1}{2}b + \frac{3}{8}c \end{bmatrix}$$

2. Non-negative Linear Combination:

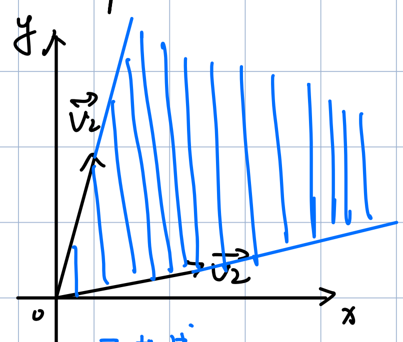
$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_k\vec{v_k} \qquad (\forall\, 1 \le i \le k,\ c_i \geq 0).$$

e.g. $\vec{v_1} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. is v a non-neg. lincomb of v_1, v_2

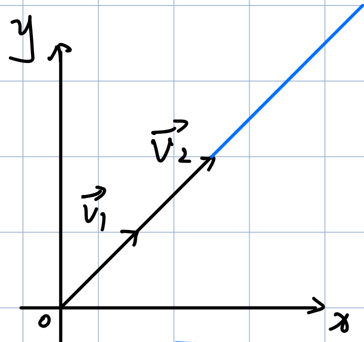
$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} 3 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

判断是否是只须有一组coefficient能满足即可.

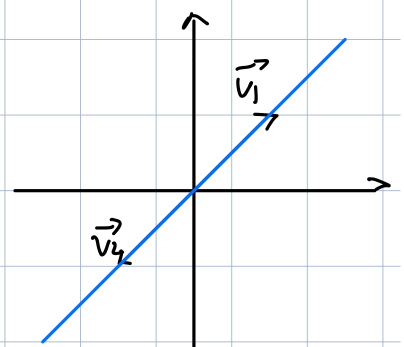
1) Graphs



不共线
(包括边界).



共线同向.
(射线上所有).



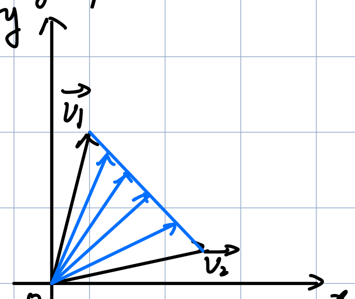
共向反向.

3. Convex Linear Combination.

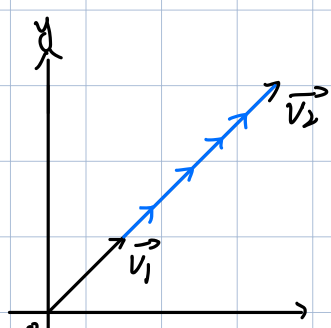
$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_k\vec{v_k} \qquad (\forall\, 1 \le i \le k,\ c_i \geq 0 \text{ and } \sum_{i=1}^{k} c_i = 1).$$

e.g. $\begin{bmatrix} \frac{5}{2} \\ 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

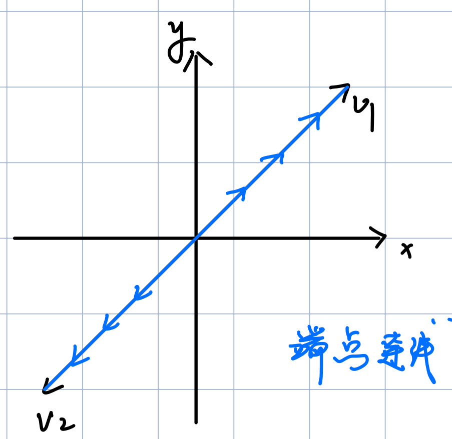
1) Graphs.



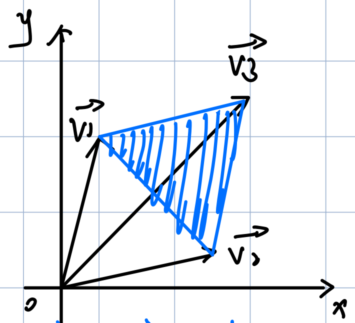
端点连线



端点连线



端点连线



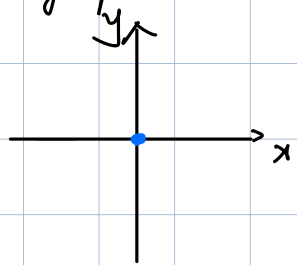
端点连线及其内部，

4. Trival Linear Combination

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_k\vec{v_k}. \quad (c_1 = c_2 = \dots = c_k = 0).$$

∴ $\vec{0}$ is a trivial lincomb of $v_1 \dots v_k$.

1) Graph.



5. Span:

The spanning set of vectors $V = \{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$. is the collection of all linear combinations of $v_1, v_2, \dots, v_k$.

$$\text{span } V: \left\{ \sum_{k=1}^{n} c_k v_k \,\middle|\, c_k \in \mathbb{R} \right\}.$$

e.g. $v_1 = \begin{bmatrix}1\\2\end{bmatrix}$. $v_2 = \begin{bmatrix}3\\4\end{bmatrix}$

$\text{span}\{v_1, v_2\} = \{a v_1 + b v_2 \mid a, b \in \mathbb{R}\}$.

e.g. $\text{span}\left\{\text{span}\left\{\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}1\\4\\9\end{bmatrix}\right\}\right\} = ?$

$= \text{span}\left\{a\begin{bmatrix}1\\3\\5\end{bmatrix} + b\begin{bmatrix}1\\4\\9\end{bmatrix}\right\}$

$= c_1(a_1 v_1 + b_1 v_2) + c_2(a_2 v_1 + b_2 v_2) + \dots + c_k(a_k v_1 + b_k v_2).$

$= \square\, v_1 + \triangle\, V_2.$

↳ lin comb of v_1. v_2.

∴ $\text{span}\left\{\begin{bmatrix}1\\3\\5\end{bmatrix}. \begin{bmatrix}1\\4\\9\end{bmatrix}\right\}$

e.g. $\text{span}\left\{\text{span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\} + \left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}\right\} = ?$

① $\mathrm{span}\{\underbrace{\mathrm{span}\{[\begin{smallmatrix}1\\3\end{smallmatrix}]\} + \{[\begin{smallmatrix}2\\1\end{smallmatrix}]\}}_{\Rightarrow\ \subseteq \mathbb{R}.}\} \subseteq \mathbb{R}.$ $\therefore \subseteq \mathbb{R}.$

② $\mathbb{R} \subseteq \mathrm{span}\{\mathrm{span}\{[\begin{smallmatrix}1\\3\end{smallmatrix}]\} + \{[\begin{smallmatrix}2\\1\end{smallmatrix}]\}\}.$

$\because [\begin{smallmatrix}3\\4\end{smallmatrix}] \in \mathrm{span}\{v_1\} + \{v_2\}$, $[\begin{smallmatrix}2\\1\end{smallmatrix}] \in \mathrm{span}\{v_1\} + \{v_2\}$.

$\therefore \underbrace{\mathrm{span}\{[\begin{smallmatrix}3\\4\end{smallmatrix}][\begin{smallmatrix}2\\1\end{smallmatrix}]\}}_{\mathbb{R}^2} \subseteq \mathrm{span}\{\mathrm{span}\{[\begin{smallmatrix}1\\3\end{smallmatrix}]\} + \{[\begin{smallmatrix}2\\1\end{smallmatrix}]\}\}$

$\therefore \mathbb{R}^2.$

e.g. $\mathrm{span}\{v_1, v_2\} + \mathrm{span}\{v_3, v_4\} = ?$

$= a_1v_1 + b_1v_2 + c_1v_3 + d_1v_4 + a_2v_1 + b_2v_2 + c_2v_3 + d_2v_4 + \cdots +$
$a_kv_1 + b_kv_2 + c_kv_3 + d_kv_4$

$= \square v_1 + \triangle v_2 + \bigcirc v_3 + \hexagon v_4$

$= \mathrm{span}\{v_1, v_2, v_3, v_4\}.$ (重要结论).

1) $\mathrm{span}\{v_1, v_2\} = \mathbb{R}^2 \Longrightarrow [\begin{smallmatrix}a\\b\end{smallmatrix}] = c_1v_1 + c_2v_2$ always 有解.
(用 matrix 解, 如第一页例 2).

2) Translated Matrix.

$\mathrm{span}\{v_1 \dots v_k\} + \{\vec{p}\}$ where $\vec{p} \notin \mathrm{span}\{v_1 \dots v_k\}$.

3) Redundancy Theorem.

$\vec{v}$ don't contribute anything

Let V be a vector space, $v_1, v_2, \dots, v_p \in V$, and $v \in \mathrm{span}\{v_1, v_2, \dots, v_p\}$
then $\mathrm{span}\{v_1, v_2, \dots, v_k, v\} = \mathrm{span}\{v_1, v_2, \dots, v_k\}$
e.g. $\mathbb{R}^2 = \mathrm{span}\{[\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}][\begin{smallmatrix}1\\1\end{smallmatrix}]\} = \mathrm{span}\{[\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}]\}.$

6. Relationship between span and lin comb.

1). $\vec{v}$ is a linear combination of $\{v_1, v_2, \dots, v_k\}$
$\Leftrightarrow \vec{v} \in \mathrm{span}\{v_1, v_2, \dots, v_k\}.$

$\Leftrightarrow \exists\, c_1, c_2, \ldots, c_k \in \mathbb{R}$ s.t. $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}$

7. Big Theorem.

Let A be an $m \times n$ matrix. Then following statements are equivalent.

1) For each $b \in \mathbb{R}^m$, the equation $A\vec{x} = b$ has a solution.

e.g. $$\underbrace{\begin{bmatrix} 3 & 2 \\ 2 & -1 \\ 4 & 0 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}_{\vec{b}}$$

$$\Rightarrow x\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \vec{b}$$

2) For each $b \in \mathbb{R}^m$ is a linear combination of columns of A.

3) For each $b \in \mathbb{R}^m$ is in the span of columns of A.

4) The columns of A spans $\mathbb{R}^m$

5) A has a pivot position in every row.