Similarity & Diagonalization

invertible:
① $\det(P) \neq 0$.
② $\lambda \neq 0$.

1. Similarity

A, B: $n\times n$ matrices.

$A \sim B$ (A is similar to B): if there is an invertible $n\times n$ matrix P s.t. $P^{-1}AP = B$

1) $A \sim A$

2) if $A\sim B$, then $B \sim A$

$$P^{-1}AP = B \Rightarrow A = PBP^{-1}$$

3) If $A\sim B$, $B\sim C$, then $A \sim C$.

$$A = PBP^{-1}$$
$$B = HCH^{-1}$$
$$\Rightarrow A = P(HCH^{-1})P^{-1} = PHC(PH)^{-1}.$$

4) If $A \sim B$:

① $\det(A) = \det(B)$.

$$\det(A) = \det(PBP^{-1}) = \det P \cdot \det B \cdot \det P^{-1}$$
$$= \det B.$$

② A is invertible iff B is invertible.

③ $\operatorname{rank}(A) = \operatorname{rank}(B)$.

④ $\operatorname{char}(A) = \operatorname{char}(B)$.

$$\det(A-\lambda I) = \det(PBP^{-1} - \lambda I)$$
$$= \det(PBP^{-1} - \lambda PIP^{-1})$$
$$= \det(P(B-\lambda I)P^{-1}).$$
$$= \det(B-\lambda I).$$

A and B have same eigenvalues $\not\Rightarrow$ $A\sim B$

⑤ A and B have same eigenvalues.

5) $A \sim B \not\Rightarrow$ A and B are row equivalent (A can obtain from row operation of B).

除 diagonal 之外都为 0.

2. Diagonalization

$A: n\times n$ is diagonalizable if there is a diagonal matrix D s.t. $A \sim D$. i.e. $P^{-1}AP = D$. i.e. $A = PDP^{-1}$.

1) $A: n\times n$ is diagonalizable iff A has n lin ind eigenvectors.

$\lambda_1 \cdots\cdots \lambda_n$. $\vec{v_1} \cdots\cdots \vec{v_n}$.

$P = [\vec{v_1}, \cdots\cdots, \vec{v_n}]$. $D = \begin{bmatrix} \lambda_1 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

Converse is false.

2) If $A: n\times n$ has n distinct eigenvalues $\rightarrow$ A is diagonalizable

3) If $A: n\times n$ has eigenvalue $\lambda_1, \ldots, \lambda_k$, then A is diagonalizable iff $\sum_{i=1}^{k} \dim(E_{\lambda_i}) = n$.

i.e. sum of algebric multiplicity = sum of geometric multiplicity

is judge whether diagonalizable.

4) Calculating whether diagonalizable.

① character polynomial to solve eigenvalues.

② get eigenspace using $E_{\lambda_1} = null(A-\lambda_1 I) \ldots E_{\lambda_n} = null(A-\lambda_n I)$.

(③ $\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_n}) = n$?)

④ write PDP^{-1}.

e.g. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is diagonalizable? find P, D s.t. $A = P \cdot D P^{-1}$.

$\det(A-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right)$

$= (1-\lambda)\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 1 \\ 1-\lambda & 0 \end{bmatrix}\right)$.

$= (1-\lambda)((1-\lambda)^2-1) = (1-\lambda)(2-\lambda)(-\lambda) = 0.$

$\therefore \lambda_1 = 1. \quad \lambda_2 = 2. \quad \lambda_3 = 0.$

Since A has n distinct eigenvalues. it's diagonalizable.

$$\text{null}(A-\lambda_1 I) = \text{null}\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right)$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{cases} x_1 = 0 \\ x_2 = -x_3 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$$

$$\text{null}(A-\lambda_2 I) = \text{null}\left(\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\right)$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{r_3+r_1} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{r_3+r_2} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{r_2\times(-1) \\ r_1\times(-1)}]{r_1+r_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\text{null}(A-\lambda I) = \text{null}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{r_3-r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{r_3+r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{r_1 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore E_{\lambda_3} = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$\therefore P = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. Diagonalization & Change of coordinate.

e.g. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are eigen-vectors for A. T_A be transformation induced by A.

1) Is A diagonalizable?

$\vec{v_1}, \vec{v_2}, \vec{v_3}$ are lin ind.

$\therefore$ ✓.

2) Find λ of A.

$A\vec{v_1} = \lambda_1 \vec{v_1}$ $\qquad$ $A\vec{v_2} = \lambda_2 \vec{v_2}$

$\rightarrow \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\vec{v_1} \quad \therefore \lambda_1 = 2 \qquad \rightarrow \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = -\vec{v_2} \quad \therefore \lambda_2 = -1$

$A\vec{v_3} = \lambda_3 \vec{v_3}$

$\rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \vec{v_3} \qquad \therefore \lambda_3 = 1$

3). Compute $A\vec{w}$. where $\vec{w} = 2\vec{v_1} - \vec{v_2}$.

$A\vec{w} = A(2\vec{v_1} - \vec{v_2})$

$= 2A\vec{v_1} - A\vec{v_2}$

$= 4\vec{v_1} + \vec{v_2}$.

4). Compute. $T_A\vec{u}$ where $\vec{u} = a\vec{v_1} + b\vec{v_2} + c\vec{v_3}$

$T_A\vec{u} = A \cdot \vec{u} = A \cdot (a\vec{v_1} + b\vec{v_2} + c\vec{v_3})$

$= aA\vec{v_1} + bA\vec{v_2} + cA\vec{v_3}$

$= 2a\vec{v_1} - b\vec{v_2} + c\vec{v_3}$

5). By (1), $\alpha = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is lin ind, therefore its a basis of $\mathbb{R}^3$.
If $[\vec{x}]_\alpha = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, compute $T_A\vec{x}$ in α basis.

$[T_A\vec{x}]_\alpha = A\cdot\vec{x}$.

$\vec{x} = \vec{v_1} + 3\vec{v_2} + 4\vec{v_3}$. from (4) $\vec{u} = a\vec{v_1} + b\vec{v_2} + c\vec{v_3}$.

$\therefore T_A\vec{x} = 2\vec{v_1} - 3\vec{v_2} + 4\vec{v_3}$.

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6). Compute $A^{100}\vec{x}$ where $[\vec{x}]_\alpha = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$.

$A^{100}\vec{x} = (PDP^{-1})^{100} = PDP^{-1}PDP^{-1} \cdots PDP^{-1}$

$= PD^{100}\cdot P^{-1}$

$= P\cdot\begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot P^{-1}$.

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