Unit 3(a) Lecture Notes for MAT224

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§1 2.5 Composition of Linear Transformations

Let $S : U \to V$ and $T : V \to W$ be linear transformations. Prove that TS is also a linear transformation. What is its domain? What is its codomain?

To show that TS is a linear transformation, we need to show that it satisfies the two properties of linearity:

1. Additivity: $(TS)(\mathbf{u} + \mathbf{v}) = (TS)(\mathbf{u}) + (TS)(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v} \in U$.
2. Homogeneity: $(TS)(a\mathbf{u}) = a(TS)(\mathbf{u})$ for all scalars a and vectors $\mathbf{u} \in U$.

Let $\mathbf{u}, \mathbf{v} \in U$ and $a \in \mathbb{R}$. Then we have:

$$\begin{aligned}
(TS)(\mathbf{u} + \mathbf{v}) &= T(S(\mathbf{u} + \mathbf{v})) && \text{(definition of composition)} \\
&= T(S(\mathbf{u}) + S(\mathbf{v})) && \text{(linearity of } S\text{)} \\
&= T(S(\mathbf{u})) + T(S(\mathbf{v})) && \text{(linearity of } T\text{)} \\
&= (TS)(\mathbf{u}) + (TS)(\mathbf{v}) && \text{(definition of composition)}
\end{aligned}$$

Therefore, TS satisfies the additivity property. Similarly, we have:

$$\begin{aligned}
(TS)(a\mathbf{u}) &= T(S(a\mathbf{u})) && \text{(definition of composition)} \\
&= T(aS(\mathbf{u})) && \text{(linearity of } S\text{)} \\
&= aT(S(\mathbf{u})) && \text{(linearity of } T\text{)} \\
&= a(TS)(\mathbf{u}) && \text{(definition of composition)}
\end{aligned}$$

Therefore, TS satisfies the homogeneity property. Hence, TS is a linear transformation.
The domain of TS is U, since the domain of S is U and T takes as input the output of S, which is a vector in V. The codomain of TS is W, since T maps vectors in V to vectors in W.

Consider linear transformations $S : U \to V$ and $T : V \to W$.
We say that $A \subset B$ if $\forall a \in A$ we can show that $a \in B$ as well.

(Proposition 2.5.6) Prove that:

(i) $Ker(S) \subset Ker(TS)$

To prove that $Ker(S) \subset Ker(TS)$, we need to show that if a vector $\mathbf{u}$ is in $Ker(S)$, then it is also in $Ker(TS)$.

Suppose $\mathbf{u} \in Ker(S)$. Then by definition, we have $S(\mathbf{u}) = \mathbf{0}_V$, where $\mathbf{0}_V$ is the zero vector in V. Now consider the vector $(TS)(\mathbf{u})$. By the definition of composition, we have:

$$(TS)(u) = T(S(u)) = T(0_V) = 0_W$$

where the last equality follows from the fact that T is a linear transformation and maps the zero vector in V to the zero vector in W. Therefore, we have shown that if $\mathbf{u} \in Ker(S)$, then $(TS)(\mathbf{u}) = \mathbf{0}_W$, which means that $\mathbf{u} \in Ker(TS)$.

Hence, we have shown that $Ker(S) \subset Ker(TS)$.

(ii) $Im(TS) \subset Im(T)$

To prove that $Im(TS) \subset Im(T)$, we need to show that every vector in the image of TS is also in the image of T. That is, if $\mathbf{w} \in Im(TS)$, we need to show that $\mathbf{w} \in Im(T)$.

Since $\mathbf{w} \in Im(TS)$, there exists a vector $\mathbf{u} \in U$ such that $(TS)(\mathbf{u}) = \mathbf{w}$. Now consider the vector $S(\mathbf{u}) \in V$. Since T maps vectors in V to vectors in W, we have $T(S(\mathbf{u})) \in Im(T)$. But we also know that $TS(\mathbf{u}) = T(S(\mathbf{u}))$, so we have:

$$w = TS(u) = T(S(u)) \in Im(T)$$

Therefore, we have shown that if $\mathbf{w} \in Im(TS)$, then $\mathbf{w} \in Im(T)$, which implies that $Im(TS) \subset Im(T)$.

Why does it not make sense to compare $Ker(T)$ and $Ker(TS)$?

It does not make sense to compare $Ker(T)$ and $Ker(TS)$ because they are not necessarily related in any meaningful way.

Recall that the kernel (also known as the null space) of a linear transformation is the set of all vectors that the transformation maps to the zero vector. In other words, $Ker(T)$ is the set of all vectors $\mathbf{v}$ in the domain of T such that $T(\mathbf{v}) = \mathbf{0}$. On the other hand, $Ker(TS)$ is the set of all vectors $\mathbf{u}$ in the domain of S such that $(TS)(\mathbf{u}) = \mathbf{0}$. Note that $Ker(TS)$ involves both S and T, while $Ker(T)$ only involves T.

While it is true that $Ker(T) \subset Ker(TS)$ (as we showed in a previous answer), this does not necessarily mean that $Ker(T)$ and $Ker(TS)$ are comparable in any other way. In general, the dimensions and properties of $Ker(T)$ and $Ker(TS)$ can be very different, and it is not meaningful to compare them without additional information.

Similarly, why does it not make sense to compare $Im(S)$ and $Im(TS)$?

It does not make sense to compare $Im(S)$ and $Im(TS)$ in general because they are not necessarily related in any meaningful way.

Recall that the image (also known as the range) of a linear transformation is the set of all vectors in the codomain that can be obtained by applying the transformation to vectors in the domain. In other words, $Im(S)$ is the set of all vectors in V that can be written as $S(\mathbf{u})$ for some vector $\mathbf{u}$ in the domain of S. Similarly, $Im(TS)$ is the set of all vectors in W that can be written as $(TS)(\mathbf{u})$ for some vector $\mathbf{u}$ in the domain of S.

While it is true that $Im(S) \subset Im(TS)$, this does not necessarily mean that $Im(S)$ and $Im(TS)$ are comparable in any other way. In general, the dimensions and properties of $Im(S)$ and $Im(TS)$ can be very different, and it is not meaningful to compare them without additional information.

If W is a subspace of V, what do we know about the dimensions of W and V? How does this relate to Corollary 2.5.7?

Corollary 2.5.7 says that:

(i) $dim(Ker(S)) \leq dim(Ker(TS))$

(ii) $dim(Im(TS)) \leq dim(Im(T))$

If W is a subspace of V, then we know that the dimension of W is less than or equal to the dimension of V, since every basis of W is a linearly independent subset of V and hence can be extended to form a basis of V.
Corollary 2.5.7 says that if $S : U \to V$ and $T : V \to W$ are linear transformations, then $dim(Ker(S)) \leq dim(Ker(TS))$ and $dim(Im(TS)) \leq dim(Im(T))$. These inequalities relate to the rank-nullity theorem, which says that for any linear transformation $T : U \to V$, we have:

$$dim(Ker(T)) + dim(Ker(TS)) = dim(U)$$

Using this theorem, we can see that $dim(Im(TS)) = dim(V) - dim(Ker(TS))$ and $dim(Im(T)) = dim(W) - dim(Ker(T))$. Thus, the inequality $dim(Im(TS)) \leq dim(Im(T))$ can be rewritten as:

$$dim(V) - dim(Ker(TS)) \leq dim(W) - dim(Ker(T))$$
$$dim(Ker(T)) \leq dim(Ker(TS))$$

which is the first inequality in Corollary 2.5.7.
Therefore, if W is a subspace of V, then we can apply Corollary 2.5.7 to the linear transformations $S : 0 \to W$ and $T : W \to V$, which gives us:

$$dim(Ker(S)) \leq dim(Ker(TS)) \text{ and } dim(Im(TS)) \leq dim(Im(T))$$

Since S is the zero transformation from the trivial vector space 0 to W, we have $Ker(S) = 0$ and $dim(Ker(S)) = 0$. Therefore, the first inequality simplifies to:

$$0 \leq dim(Ker(T))$$

which is always true. The second inequality simplifies to:

$$dim(W) \leq dim(Im(T))$$

which is true since W is a subspace of V and hence $Im(T)$ contains W as a subspace. Therefore, we have shown that if W is a subspace of V, then Corollary 2.5.7 holds for the linear transformations $S : 0 \to W$ and $T : W \to V$.

Consider the matrices $[T]_\alpha^\beta$, $[S]_\gamma^\delta$, and $[TS]_\epsilon^\zeta$. Which of bases α, β, γ, δ, ϵ, and ζ have to be equal to each other in order for the following equation to be true:

$$[T]_\alpha^\beta [S]_\gamma^\delta = [TS]_\epsilon^\zeta$$

For the equation $[T]_\alpha^\beta [S]_\gamma^\delta = [TS]_\epsilon^\zeta$ to be true, the bases β and γ must be equal, since the number of columns of $[T]_\alpha^\beta$ must equal the number of rows of $[S]_\gamma^\delta$.
Moreover, the basis ϵ of TS must be the same as the basis α of T, since the left-hand side of the equation has basis α and the right-hand side has basis ϵ. Therefore, we must have $\epsilon = \alpha$.
Finally, the basis ζ of TS must be the same as the basis δ of S, since the number of columns of $[TS]_\epsilon^\zeta$ must equal the number of columns of $[S]_\gamma^\delta$. Therefore, we must have $\zeta = \delta$.
In summary, for the equation $[T]_\alpha^\beta [S]_\gamma^\delta = [TS]_\epsilon^\zeta$ to be true, we must have $\beta = \gamma$, $\epsilon = \alpha$, and $\zeta = \delta$.
In what ways are Propositions 2.5.4 and 2.5.14 similar? How are they different?
Proposition 2.5.4 and Proposition 2.5.14 are both about linear transformations, but they are different in their statements and conclusions.
Proposition 2.5.4 states that if $T : V \to W$ is a linear transformation, then T is injective if and only if $\mathrm{Ker}(T) = 0$. In other words, T maps distinct vectors in V to distinct vectors in W if and only if the only vector in V that is mapped to 0 is the zero vector.
On the other hand, Proposition 2.5.14 states that if $T : V \to W$ is a linear transformation, then T is surjective if and only if $\mathrm{Im}(T) = W$. In other words, for every vector $w \in W$, there exists at least one vector $v \in V$ such that $T(v) = w$.
So while both propositions are about linear transformations, they address different properties of such transformations. Proposition 2.5.4 is concerned with injectivity, which is a property related to the kernel of the transformation, while Proposition 2.5.14 is concerned with surjectivity, which is a property related to the image of the transformation.

§2 2.6 Inverses of a Linear Transformation and Isomorphisms

Consider linear transformation $T : V \to W$, where V and W are finite dimensional vector spaces.

What does it mean for T to be:

(i) injective: For a linear transformation $T : V \to W$ to be injective, it means that for any two distinct vectors v_1 and v_2 in V, their images under T are also distinct in W. Formally, T is injective if and only if for any $v_1, v_2 \in V$, if $T(v_1) = T(v_2)$, then $v_1 = v_2$.

(ii) surjective: For a linear transformation $T : V \to W$ to be surjective, it means that every vector in W is the image of at least one vector in V under T. Formally, T is surjective if and only if for any $w \in W$, there exists at least one $v \in V$ such that $T(v) = w$.

(iii) invertible: A linear transformation $T : V \to W$ is invertible if and only if there exists a linear transformation $T^{-1} : W \to V$ such that $T \circ T^{-1} = \mathrm{id}_W$ and $T^{-1} \circ T = \mathrm{id}_V$, where id_V and id_W are the identity transformations on V and W, respectively.

Also, how are these concepts related? The concepts of injectivity, surjectivity, and invertibility are all related to the idea of how a linear transformation $T : V \to W$ maps vectors in V to vectors in W.
Injectivity, or one-to-one-ness, means that each vector in V is mapped to a unique vector in W. In other words, no two distinct vectors in V are mapped to the same vector in W. Geometrically, this means that the linear transformation does not "squish" any vectors in V to the same point in W.
Surjectivity, or onto-ness, means that every vector in W is the image of at least one vector in V. Geometrically, this means that every point in W is "hit" by at least one vector in V, or equivalently, that the image of T spans all of W.
Invertibility means that there is a way to "undo" the effect of the linear transformation T on any vector in V by applying T^{-1}, and vice versa. This means that the linear transformation T is both injective and surjective, since invertibility implies that no two distinct vectors in V are mapped to the same vector in W and that every vector in W is the image of at least one vector in V. Geometrically, invertibility means that the linear transformation does not "squish" or "fold" any vectors in V, but instead preserves the full geometric structure of V in W.

Show that the inverse transformation of a bijection is also a linear transformation. In other words, if T is a bijective linear transformation, show that T^{-1} is also linear. To show that the inverse transformation of a bijection is also a linear transformation, we need to verify that T^{-1} satisfies the two properties of linearity:

1. $T^{-1}(u+v) = T^{-1}(u) + T^{-1}(v)$ for all vectors u, v in the codomain of T^{-1}
2. $T^{-1}(cu) = cT^{-1}(u)$ for all vectors u in the codomain of T^{-1} and all scalars c.

To prove property 1, let u and v be vectors in the codomain of T^{-1}, so that $T(T^{-1}(u)) = u$ and $T(T^{-1}(v)) = v$. Since T is bijective, it is invertible, so we can apply T^{-1} to both sides of the equation $T(T^{-1}(u+v)) = u+v$ to get $T^{-1}(u+v) = T^{-1}(u) + T^{-1}(v)$, as desired.
To prove property 2, let u be a vector in the codomain of T^{-1} and let c be a scalar. Again, since T is bijective and invertible, we can apply T^{-1} to both sides of the equation $T(T^{-1}(cu)) = cu$ to get $T^{-1}(cu) = cT^{-1}(u)$, as desired.
Therefore, T^{-1} satisfies the two properties of linearity, and is therefore a linear transformation.
Two vector spaces V and W are said to be isomophic if we can define a bijection between them. In this situation, we call the bijection and "isomorphism". Define an isomorphism between the vector space of polynomials of degree at most 4, $P_3(\mathbb{R}$ and the vector space of 2×2 matrices $M_{2\times 2}(\mathbb{R})$. To define an isomorphism between $P_3(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$, we need to find a bijective linear transformation $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$.
Let $1, x, x^2, x^3$ be a basis for $P_3(\mathbb{R})$ and let $A_{11}, A_{12}, A_{21}, A_{22}$ be the standard basis for $M_{2\times 2}(\mathbb{R})$. We can define a linear transformation $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ as follows:

$$T(a + bx + cx^2 + dx^3) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In other words, we map a polynomial of degree at most 3 to a 2×2 matrix by setting the top-left entry to the constant coefficient, the top-right entry to the coefficient of x, the bottom-left entry to the coefficient of x^2, and the bottom-right entry to the coefficient of x^3.
To show that T is a bijection, we need to show that it is both injective and surjective.
To show that T is injective, we need to show that if $T(p) = T(q)$ for two polynomials p and q in $P_3(\mathbb{R})$, then $p = q$. Suppose that $T(p) = T(q)$, i.e., that the matrices corresponding to p and q are equal. Then we have:

$$\begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} = \begin{pmatrix} a_q & b_q \\ c_q & d_q \end{pmatrix}$$

This implies that $a_p = a_q$, $b_p = b_q$, $c_p = c_q$, and $d_p = d_q$. But these coefficients uniquely determine the polynomials p and q, so we must have $p = q$, and hence T is injective.

To show that T is surjective, we need to show that for every 2×2 matrix A in $M_{2\times2}(\mathbb{R})$, there exists a polynomial p in $P_3(\mathbb{R})$ such that $T(p) = A$. This is straightforward: given a matrix A, we can construct a polynomial p of degree at most 3 whose coefficients are the entries of A in the appropriate order, and then we have $T(p) = A$. Hence T is surjective.

Therefore, T is a bijective linear transformation between $P_3(\mathbb{R})$ and $M_{2\times2}(\mathbb{R})$, and hence $P_3(\mathbb{R})$ and $M_{2\times2}(\mathbb{R})$ are isomorphic.

What must be true of the dimensions of two vectors space V and W if they are isomorphic to each other?

If two vector spaces V and W are isomorphic, then they must have the same dimension. This is because an isomorphism between V and W is a bijective linear transformation, which preserves the dimension of the vector space. Specifically, if $T : V \to W$ is an isomorphism, then for any basis β of V, the set $T(\beta)$ is a basis of W, and since bases have the same size, V and W must have the same dimension.

Proposition 2.6.11 States that if $T : V \to W$ is an isomorphism between finite-dimensional vector spaces, then for any choice of bases α for V and β for W,

$$[T^{-1}]_{\beta}^{\alpha} = \left([T]_{\alpha}^{\beta}\right)^{-1}$$

Confirm that this is true when T is the derivative, $V = \{ax^2 + bx \mid a, b \in \mathbb{R}\}$ with basis $\{x, x^2\}$, and $W = \{mx + b \mid m, b \in \mathbb{R}\}$ with basis $\{1, x\}$.
Let $T : V \to W$ be the derivative operator, where $V = ax^2 + bx \mid a, b \in \mathbb{R}$ with basis x, x^2, and $W = mx + b \mid m, b \in \mathbb{R}$ with basis $1, x$.
First, let's find the matrix representation of T with respect to these bases. For any $ax^2 + bx \in V$, we have:

$$T(ax^2 + bx) = \frac{d}{dx}(ax^2 + bx) = 2ax + b$$

So the matrix representation of T with respect to the basis x, x^2 for V and the basis $1, x$ for W is:

$$[T]_{\alpha}^{\beta} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

where $\alpha = x, x^2$ and $\beta = 1, x$.
To find the inverse of this matrix, we can use the formula:

$$(A^{-1})_{i,j} = \frac{(-1)^{i,j}}{det(A)} det(A_{j,i})$$

where $A_{j,i}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A. In this case, we have:

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}^{-1} = \frac{1}{(0)(0) - (1)(2)} \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & 0 \end{pmatrix}$$

Now, we can find $[T^{-1}]_{\beta}^{\alpha}$ using the formula from Proposition 2.6.11:

$$[T^{-1}]_{\beta}^{\alpha} = ([T]_{\alpha}^{\beta})^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & 0 \end{pmatrix}$$

Therefore, we have confirmed that Proposition 2.6.11 holds for this particular example.

Why is it okay to talk about T^{-1} here, even though the derivative in general does not have an inverse function?

In this case, we are not talking about the inverse function of the derivative as a function from $\mathbb{R}$ to $\mathbb{R}$. Instead, we are considering the inverse function of the linear transformation $T : V \to W$ that is defined as $T(p(x)) = p'(x)$, where V is the vector space of polynomials of degree at most 2, and W is the vector space of polynomials of degree at most 1.

Since T is a linear transformation and is bijective, it has an inverse transformation $T^{-1} : W \to V$, which is also a linear transformation. We can use this inverse transformation to calculate the matrix representation of T^{-1} in terms of the chosen bases for V and W, as given by Proposition 2.6.11.

§3 2.7 Change of Basis

What is a change of basis matrix? Give an example, including the two bases. A change of basis matrix is a matrix that allows us to represent the same vector with respect to different bases.
Suppose we have a vector space V with two bases $\alpha = v_1, v_2$ and $\beta = w_1, w_2$. Let P be the matrix whose ith column is the coordinate vector of w_i with respect to the basis α. Then P is the change of basis matrix from α to β.
To be more specific, let's consider the following example:
Let V be the vector space of polynomials of degree at most 2 with real coefficients. Let $\alpha = 1, x, x^2$ and $\beta = 1 + x, x - 1, 2 - x^2$ be two bases for V. We want to find the change of basis matrix from α to β, which we'll call P. To do this, we find the coordinate vectors of the vectors in β with respect to α and form them into a matrix.

$$[1 + x]\alpha = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad [x - 1]\alpha = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad [2 - x^2]_\alpha = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

Thus, the change of basis matrix from α to β is

$$P = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

This matrix tells us how to take the coordinate vector of a polynomial with respect to α and convert it to a coordinate vector with respect to β.

Consider $[I]_\alpha^\beta$ and vector $[v]_\alpha$. Show why the product $[I]_\alpha^\beta[v]_\alpha$ produces $[v]_\beta$, the vector whose elements are the coefficients when we write vector v as a linear combination of elements in basis β.

Let $\alpha = v_1, v_2, ..., v_n$ be a basis for the vector space V, and $\beta = w_1, w_2, ..., w_n$ be another basis for V. Let I be the identity transformation from V to V, and let $[I]_\alpha^\beta$ be the change of basis matrix from α to β.

Now, let v be a vector in V, and let $[v]\alpha$ be the coordinate vector of v with respect to α. That is, $[v]\alpha = \begin{bmatrix} a_1 & a_2 & \vdots & a_n \end{bmatrix}$, where $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$.

We want to show that $[I]\alpha^\beta[v]\alpha = [v]\beta$. To do this, we first need to find the coordinate vector $[v]\beta$ of v with respect to β.

Since α and β are both bases for V, we can write v as a linear combination of elements in β:

$$v = b_1w_1 + b_2w_2 + \ldots + b_nw_n$$

for some scalars $b_1, b_2, \ldots, b_n$. To find these scalars, we can use the fact that $[I]_\alpha^\beta$ is the change of basis matrix from α to β. In other words, for any vector u in V, we have:

$$[u]_\beta = [I]_\alpha^\beta[u]_\alpha$$

Applying this to v, we get:

$$[v]_\beta = [I]_\alpha^\beta[v]_\alpha$$

Therefore, $[I]\alpha^\beta[v]\alpha$ produces $[v]_\beta$, the vector whose elements are the coefficients when we write vector v as a linear combination of elements in basis β.

What has to be true of linear transformation T in order for $[T]_\alpha^\beta$ to equal the identity $n \times n$ matrix which looks like:

$$I_{n\times n}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

If $[T]\alpha^\beta$ is the identity matrix $In \times n$, then it means that T sends each element of the basis α to the corresponding element of the basis β. In other words, $T(\alpha_i) = \beta_i$ for $1 \le i \le n$, where α_i and β_i denote the i-th elements of the bases α and β, respectively. This implies that T is a bijection and $dim(V) = dim(W)$, since T maps a basis of V to a basis of W. Moreover, since T is the identity map on each element of the basis α, it follows that T is also the identity map on V, i.e., $T(v) = v$ for all $v \in V$. Therefore, T is both injective and surjective, and hence an isomorphism.

Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces V with bases α and α' and W with bases β and β'. Let $I_V : V \to V$ and $I_W : W \to W$ be the respective identity transformations.

Theorem 2.7.5 tells us that

$$[T]_{\alpha'}^{\beta'} = [I_W]_{\beta}^{\beta'} \cdot [T]_{\alpha}^{\beta} \cdot [I_V]_{\alpha'}^{\alpha}$$

Explain which transformation each of those matrices represents. Also, explain why the result of

$$[I_W]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[I_V]_{\alpha'}^{\alpha}[v]_{\alpha'}$$

is written in terms of basis elements of β'.
In the expression $[T]\alpha'^{\beta'} = [I_W]\beta^{\beta'} \cdot [T]\alpha^{\beta} \cdot [I_V]\alpha'^{\alpha}$, we have:

- $[I_W]_{\beta}^{\beta'}$ is the matrix representing the identity transformation I_W with respect to the bases β and β' of W.
- $[T]_{\alpha}^{\beta}$ is the matrix representing the linear transformation T with respect to the bases α and β of V and W respectively.
- $[I_V]_{\alpha'}^{\alpha}$ is the matrix representing the identity transformation I_V with respect to the bases α' and α of V.

The result of $[\mathrm{I}_W]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[I_V]_{\alpha'}^{\alpha}[v]_{\alpha'} isavectorin \mathrm{W} thatisobtainedbythefollowingsteps:$

- Starting with the vector $[v]\alpha'$ in V, we use the matrix $[I_V]\alpha'^{\alpha}$ to express it as a linear combination of the basis vectors of α.
- We then apply the linear transformation T to this linear combination of basis vectors to obtain a linear combination of the basis vectors of β.
- Finally, we use the matrix $[I_W]_{\beta}^{\beta'}$ to express this linear combination of basis vectors of β as a vector in W with respect to the basis β'.

Since the output of this expression is a vector in W with respect to the basis β', it makes sense that the coefficients of the linear combination are written in terms of the basis elements of β'.

Give an example two bases α and α' for $P_2(\mathbb{R})$ and a polynomial $v \in P_2(\mathbb{R})$ so that $[v]_\alpha$ and $[v]_{\alpha'}$ have different coordinates.

Also find the change of basis matrix $[I]_\alpha^{\alpha'}$ so that

$$[I]_\alpha^{\alpha'}[v]_\alpha = [v]_{\alpha'}$$

Let $\alpha = 1, x, x^2$ be the standard basis for $P_2(\mathbb{R})$ and let $\alpha' = 1, (x-1), (x-1)^2$ be another basis for $P_2(\mathbb{R})$. Consider the polynomial $v = 2x^2 + 3x + 1 \in P_2(\mathbb{R})$.
To find $[v]_\alpha$, we express v as a linear combination of the elements of α:

$$v = 1(1) + 3(x) + 2(x^2)$$

So we have $[v]_\alpha = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$.
To find $[v]_{\alpha'}$, we first express α in terms of α':

$$1 = (x-1) + 2((x-1)^2)$$

$$x = (x-1) + (x-1)^2$$

$$x^2 = (x-1)^2 + 2(x-1)$$

Thus, we have:

$$\begin{aligned} v &= 1(1) + 3(x) + 2(x^2) \\ &= 1[(x-1) + 2((x-1)^2)] + 3[(x-1) + (x-1)^2] + 2[(x-1)^2 + 2(x-1)] \\ &= 5(x-1)^2 + 7(x-1) + 2 \end{aligned}$$

So we have $[v]_{\alpha'} = \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}$.
To find the change of basis matrix $[I]_\alpha^{\alpha'}$, we need to express each element of α in terms of α'. Using the equations we found earlier, we have:

$$[1]_{\alpha'} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ [(x-1)]_{\alpha'} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \ [(x-1)^2]_{\alpha'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So the change of basis matrix is:

$$[I]_\alpha^{\alpha'} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$