Derivations

1. Rules

1) Types:

① Elimination Rule: remove the connective (automatic moves)

② Introduction Rule: introduce the connective.

2) Inference Rules for '$\to$'

① Modus Ponens (MP) (elimination)

$$\frac{(\phi \to \psi) \qquad \phi}{\psi}$$

Pr1 If p, then q.

Pr2 p.

C Therefore, q

ϕ	ψ	$\phi \to \psi$
T	T	T
T	F	
F	T	
F	F	

√ sound and truth perserving.

② Modules Tollens (MT) (elimination)

$$\frac{(\phi \to \psi) \qquad \sim\psi}{\sim\phi}$$

sound and truth perserving.

2) Inference Rules for '$\sim\sim$'.

① Double Negation (DN)

$$\frac{\sim\sim\phi}{\phi}$$ (elimination)

$$\frac{\phi}{\sim\sim\phi}$$ (introduction).

4) Inference Rules for 'V'. (disjunction)

① Modus Tollendo Ponens (MTP) (elimination)

$$\frac{\phi \vee \psi \qquad \sim\phi}{\psi} \qquad \frac{\phi \vee \psi \qquad \sim\psi}{\phi}$$

② Addition (ADD) (introduction)

$$\frac{\phi}{\phi \vee \psi} \qquad \frac{\phi}{\psi \vee \phi}$$

5) Inference Rule for '$\wedge$' (conjunction)

① Simplification (S) (elimination)

$$\frac{\phi \wedge \psi}{\phi} \qquad \frac{\phi \wedge \psi}{\psi}$$

② Adjunction (ADJ) (introduction)

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \qquad \frac{\phi \qquad \psi}{\psi \wedge \phi}$$

6) Inference Rule for '$\longleftrightarrow$' (biconditional)

① Biconditional – Conditional (BC) (elimination)

$$\frac{\phi \longleftrightarrow \psi}{} \qquad \frac{\phi \longleftrightarrow \psi}{}$$

$\phi \to \psi \qquad\qquad \psi \to \phi$

⑥ Conditional — Biconditional (CB) (introduction)

$$\frac{\begin{matrix}\phi \to \psi \\ \psi \to \phi\end{matrix}}{\phi \leftrightarrow \psi} \qquad\qquad \frac{\begin{matrix}\phi \to \psi \\ \psi \to \phi\end{matrix}}{\phi \leftrightarrow \psi}$$

7) Repetition (R)

$$\frac{\phi}{\phi}$$

bring the line out of show box in it.

e.g. ①

$(P \vee \sim R) \to (Q \wedge \sim(P \to S))$

$P \vee \sim R$

$\therefore Q \wedge \sim(P \to S)$ (√)

②

$Q \wedge (P \to S)$

P

$\therefore S$ (X)

③

$\sim(S \vee P) \to \sim R$

R

$\therefore S \vee P$ (X)

the negation of $\sim R$ is $\sim\sim R$ so we can't directly writes it.

④

$\sim P \to \sim(S \to Z)$

$\sim\sim(S \to Z)$

$\therefore \sim\sim P$ (√)

⑤

$\sim\sim(\sim P \vee \sim X)$

can only do DN when its at front i.e. start with $\sim\sim$

$\therefore \sim P \vee \sim X$ (✓)

⑥ $P \leftrightarrow \sim\sim Q$

$\therefore P \leftrightarrow Q.$ (X)

2. Derivation Structure

1) Classic.

(Line nums)	(Symb Sen.)	(Justifications)
1	~~Show~~: Δ	
2	Premise	PR1
3	Premise	PR2
4	Derived Sentence	line nums rule
⋮		
n	DS	line nums rule
n+1	Δ	line nums rule
n+2		n+1 DD

2) Components:

① Line Numbers: line numbered sequentially from 'show line'.

② Symbolic Sentences: a sentence, a 'Show' line or a 'Cancellation' line.

③ Boxes & Indentation:

→ 1 space from 'show' (left)

→ 1 space from cancellation line (right).

④ Justifications:

→ Premises: PR1, PR2, etc.

→ Inference Rules: Abb. form: 2 3 MP; 4 DN

→ Derivation Type: n+1 DD; n+1 CD; n+1 ID

→ Assumptions: A line containing an assumption must occur directly below a show line; ASS ID; ASS CD.

3) Available Lines: can be applied to inference rules.

① Premises ② unboxed lines ③ cancelled and unboxed 'show line'.

4) A Derivation is complete iff.

① Every 'show' line is crossed off

② All lines that are not 'show' lines are boxed off.

③ Every line except 'show' lines is properly justified

4) Direct Derivation Example.

e.g. Y. $X \to (Y \to Z)$. $\sim X \to \sim W$. W. $\therefore \sim\sim Z$.

1	~~show~~ $\sim\sim Z$	(show conditions)
2	$\sim\sim W$	Pr4 DN
3	$\sim X \to \sim W$	Pr3
4	$\sim\sim X$	2 3 MT

5	X	4 DN
6	$Y \to Z$	5 Pr2 MP
7	Y	Pr1
8	Z	6 7 MP
9	$\sim\sim Z$	8 DN
10		9 DD.

↖写一个

5) Abbreviations Example: drop unnecessary annotation; don't restate Prs; don't need a new line for DD; multiple moves at once.

e.g. $(P \to \sim Q) \to (\sim R \to S)$. $\sim S$. $\sim(P \to \sim Q) \to T$. $T \to S$.
$\therefore R$.

1	~~show~~ R	(show conditions)
2	$\sim T$	Pr2 Pr4 MT
3	$P \to \sim Q$	2 Pr3 MT DN
4	$\sim R \to S$	3 Pr1 MT
5	R	4 Pr2 MT DN DD.

6) Conditional Derivatives

CD到不能CI为止. 先别开始证.

① Structure

1	~~show~~ $\phi \to \psi$	
2	ϕ	ASS CD
3	~~show~~ ψ	
4	...	

subderivation for ψ (nested derivation)

5

6 ϕ Line num DD

7 3 CD 写个

e.g. $T \to S$. $\sim T \to \sim R$. $\therefore R \to S$.

1 ~~show~~ $R \to S$ (show conditions)

2 R ASS CD

3 ~~show~~ S (show conditions)

4 $\sim\sim R$ 2 DN

5 T 4 Pr2 MT DN

6 S Pr1 5 MP DD

7 3 CD

7) Indirect Derivations: assume $\sim\phi$ and derive a contradiction, then ass is false. for reductio

should be in the box that is currently working.

So sometime to use repeat rule

① Structure:

1 ~~show~~ ϕ

2 $\sim\phi$ ASS ID

3 ...

4 ...

5 ψ 不一定要用 one sentence 来说明 contradict!

6 $\sim\psi$

7 5 6 ID 写2个

e.g. $P \to \sim Q$. $R \to Q$. $\sim R \to \sim P$. $\therefore \sim P$.

1 ~~show~~ $\sim P$

2	P	ASS ID
3	$\sim Q$	2 Pr1 MP
4	$\sim R$	Pr2 3 MT
5	$\sim P$	4 Pr3 MP
6		2 5 ID.

8) Mixed Derivation:

① look at most recent 'show' line.

② if it's '$\phi \to \psi$', start a CD

③ if it's any other, start a ID

Can do some automatic break down after 'show' line.

e.g. $(P \wedge \sim R) \to T$. $(S \leftrightarrow T) \wedge \sim (P \wedge S)$. $\sim P \vee \sim R$

$\therefore Z \vee (P \to W)$

1	~~show~~ $Z \vee (P \to W)$		
2	$S \leftrightarrow T$	some auto moves.	Pr2 S
3	$\sim (P \wedge S)$		Pr2 S
4	~~show~~ $P \to W$	when finds Z doesn't exist in Pr.	
5	P		ASS CD
6	~~show~~ W		
7	$\sim W$		ASS ID
8	$\sim\sim P$	因为~P negation 是~~P，所以要写	5 DN
9	$\sim R$		8 Pr3 MTP
10	P	bring in box	5 R
11	$P \wedge \sim R$		9 10 ADJ
12	T		11 Pr1 MP

13	$S \leftrightarrow T$	2 R
14	$T \to S$	13 BC
15	S	12 14 MP
16	$P \wedge S$	10 15 ADJ
17	$\sim(P \wedge S)$	3 R
18		16 17 ID
19		6 CD
20	$Z \vee (P \to W)$	4 ADD DD

9) Tautology is zero premise proof.

e.g. $\therefore P \wedge Q \leftrightarrow \sim(\sim P \vee \sim Q)$

1	~~show~~ $P \wedge Q \leftrightarrow \sim(\sim P \vee \sim Q)$	
2	~~show~~ $P \wedge Q \to \sim(\sim P \vee \sim Q)$	1 BC
3	$P \wedge Q$	ASS CD
4	P	3 S
5	Q	3 S
6	~~show~~ $\sim(\sim P \vee \sim Q)$	
7	$\sim P \vee \sim Q$	ASS ID
8	P	4 R.
9	Q	5 R
10	$\sim Q$	7 8 MTP
11		9 10 ID
12		6 CD
13	~~show~~ $\sim(\sim P \vee \sim Q) \to P \wedge Q$	1 BC

14	$\sim(\sim P \vee \sim Q)$	13 ASS CD
15	~~show~~ $P \wedge Q$ 无 → 就取 ID.	
16	$\sim(P \wedge Q)$	15 ASS ID
17	~~show~~ P. ☆ show conjuncts seperately	
18	$\sim P$	17 ASS ID.
19	$\sim P \vee \sim Q$	18 ADD
20	$\sim(\sim P \vee \sim Q)$	14 R.
21		19 20 ID.
22	~~show~~ Q	
23	$\sim Q$	22 ASS ID.
24	$\sim P \vee \sim Q$	23 ADD.
25	$\sim(\sim P \vee \sim Q)$	14 R.
26.		24 25 ID.
27.	$P \wedge Q$	17 22 ADJ.
28.		16 27 ID
29.		15 CD.
30.	$P \wedge Q \leftrightarrow \sim(\sim P \vee \sim Q)$	2 13 CB
31.		30 DD

10). Contradiction Generator (skill in derivative).: the main connective is '$\sim$', and can do nothing in it.

e.g. $\sim(P \to Q \vee R)$; what we do is to 'show $P \to Q \vee R$' to benefit our derivation there is negation elimination in 4.2 which works more convenient.