CSC236 Problem Set 2

Question 2.

(a). Let $h \in \mathbb{N}$, let b_h be the number of elements of T of height h.

Since we count the root of the tree as $h=1$, when we want the tree has height h, we should have at least one subtree with height $h-1$, the other would be at height h'.

Case 1: both are at height $h-1$, resulting in having $b_{h-1} \times b_{h-1}$ combinations.

Case 2: the other one is not at height $h-1$, which means, $0 \le h' < h-1$, since $h \in \mathbb{N}$, the combinations can be written as: $\sum_{i=0}^{h-2} b_i$

Since the subtree with height $h-1$ should also be counted, the combination: $b_{h-1} \times \sum_{i=0}^{h-2} b_i$

Since according to the question, mirror images are considered to be different, gives.

$$2 \cdot b_{h-1} \times \sum_{i=0}^{h-2} b_i \text{ combinations.}$$

Adding Case 1 and Case 2 together gives, $b_h = b_{h-1}^2 + 2 \cdot b_{h-1} \times \sum_{i=0}^{h-2} b_i$, which.

$$b_h \begin{cases} 1 & h=0 \\ 1 & h=1 \\ b_{h-1}^2 + 2 \cdot b_{h-1} \times \sum_{i=0}^{h-2} b_i & \forall h \in \mathbb{N},\ h \geq 1. \end{cases}$$

(b) Let $h \in \mathbb{N}$.

WTP: $\forall h \in \mathbb{N},\ b_{h+1} = a_{h+1}^2 - a_h^2$.

Base Case: $h=0$. or $h=1$.

when $h=0$, From (a). $b_1 = 1$.

Since $a_0 = 1$, $a_1 = a_0^2 + 1 = 1$, which $a_1^2 - a_0^2 = 1^2 - 0^2 = 1$.

Thus. $b_1 = 1 = a_1^2 - a_0^2$.

when $h=1$, From (a). $b_2 = b_1^2 + 2 \cdot b_1 \cdot \sum_{i=0}^{0} b_i = 1^2 + 2 \cdot 1 \cdot 1 = 3$.

$$a_2^2 - a_1^2 = (a_1^2+1)^2 - a_1^2 = ((a_0^2+1)^2+1)^2 - (a_0^2+1)^2$$
$$= (1^2+1)^2 - 1 = 3.$$

Thus. $b_2 = 3 = a_2^2 - a_1^2$

I've proved the base case is true.

Induction Step: Let $h \in \mathbb{N}$. $h > 0$.

Induction Hypothesis. Let $l \in \mathbb{N}$, $0 \leq l < h$, $b_{l+1} = a_{l+1}^2 - a_l^2$

WTP: $b_{h+1} = a_{h+1}^2 - a_l^2$.

From (a), gives, $\qquad b_1 = a_1^2 - a_0^2$.

$b_{h+1} = b_h^2 + 2 \cdot b_h \times \sum_{i=0}^{h-1} b_i$ (By I.H., take $l_1 = h-1$, gives, $b_h = a_h^2 - a_{h-1}^2$.

$\Rightarrow b_{h+1} = (a_h^2 - a_{h-1}^2)^2 + 2 \cdot (a_h^2 - a_{h-1}^2) \cdot (b_0 + b_1 + \cdots + b_{h-1})$ (By I.H. $l \in [0, h)$, take $l_2 = h-2$.

$\Rightarrow b_{h+1} = (a_h^2 - a_{h-1}^2)^2 + 2 \cdot (a_h^2 - a_{h-1}^2) \cdot (1 + a_1^2 - a_0^2 + a_2^2 - a_1^2 + a_3^2 - a_2^2 + \cdots + a_{h-1}^2 - a_{h-2}^2)$ ($l_3 = h-3 \cdots\cdots l_{h-1} = 0$, gives

$\Rightarrow b_{h+1} = (a_h^2 - a_{h-1}^2)^2 + 2 \cdot (a_h^2 - a_{h-1}^2) \cdot (a_{h-1}^2 + 1)$. (According to def$^{\underline{n}}$ of $a_{n+1} = a_n^2 + 1$).

$\Rightarrow b_{h+1} = (a_{h+1} - 1 - (a_h - 1))^2 + 2 \cdot (a_{h+1} - 1 - (a_h - 1)) \cdot (a_h - 1 + 1)$.

$\Rightarrow b_{h+1} = (a_{h+1} - a_h)^2 + 2 \cdot (a_{h+1} - a_h) \cdot a_h$

$\Rightarrow b_{h+1} = a_{h+1}^2 + a_h^2 - 2a_{h+1}a_h + 2a_{h+1}a_h - 2a_h^2$.

$\Rightarrow b_{h+1} = a_{h+1}^2 - a_h^2$.

I've proved the induction step is true.

Therefore, $b_{h+1} = a_{h+1}^2 - a_h^2$ for $h \in \mathbb{N}$.