Inverse

only $n\times n$ matrix has inverse.

1. Find Inverse Matrix. $\longrightarrow A\cdot B = I$

1) row reduction.

e.g. $A=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$. A^{-1} ?

计左边不能化到 Identity matrix 则 not invertible

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

I_3 $\qquad$ A^{-1}

2) use determinant

① 2×2: $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $A^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det(A)=ad-bc$. (invertible iff $\det(A)\neq 0$).

解方程:
① row reduction
② 两边同乘 inverse matrix.

e.g. Find inverse of $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$, use it to solve the system $\begin{cases} 8x_1+6x_2=2. \\ 5x_1+4x_2=-1 \end{cases}$

$$A^{-1}=\frac{1}{32-30}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}=\begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow A^{-1}\cdot A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=A^{-1}\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=A^{-1}\begin{bmatrix} 2 \\ -1 \end{bmatrix}=\begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

② 3×3: 见 AS. FM 1.4

2. Identity Matrix & Elementary Matrix.

1) Identity Matrix: I_n: all entries are 0 except diagonal.

e.g. $I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ...

① $AI_n = I_n A$ $\quad A: n\times n$.

② $A\cdot B = I \rightarrow A\cdot B = B\cdot A$.

2). Elementary Matrix: performing a single elementary row operation on an identity matrix.

左乘相当于作一个相同的 row operation

① $E: m\times m$. $A: m\times n$: EA. make A operate the same row operation that from $I_m \rightarrow E$.

e.g. $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$I_3 \xrightarrow{r_3 - 4r_1} E_1$ $\quad \therefore E_1 A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}$

$I_3 \xrightarrow{r_2 \leftrightarrow r_3} E_2$. $\quad \therefore E_2\cdot A = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$

$I_3 \xrightarrow{r_3 \cdot 5} E_3$ $\quad \therefore E_3\cdot A = \begin{bmatrix} a & b & c \\ g & h & i \\ 5d & 5e & 5f \end{bmatrix}$

② each elementary matrix E is invertible. (The inverse of E use the same type of row operation to back to I).

$(A\cdot B)^{-1} = B^{-1}\cdot A^{-1}$

$\longrightarrow A = E_1^{-1}\cdot E_2^{-1}\cdot \cdots \cdot E_k^{-1}$

$\longrightarrow A^{-1} = E_k\cdot E_{k-1}\cdot \cdots \cdot E_2\cdot E_1$.

$(E_k\cdot E_{k-1}\cdot \cdots \cdot E_2\cdot E_1)\cdot A = I$.

$\Rightarrow$ $A = I\cdot (E_k\cdot E_{k-1}\cdot \cdots \cdot E_2\cdot E_1)^{-1}$

$\Rightarrow A = E_1^{-1}\cdot E_2^{-1}\cdot \cdots \cdot E_{k-1}^{-1}\cdot E_k^{-1}$.

e.g. $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Find Es and E^{-1}s of A.

$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \xrightarrow[E_1]{r_2 - r_1} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \xrightarrow[E_2]{r_2\cdot(-1)} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow[E_3]{r_1 - 2r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $\quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. $\quad E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

By $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\therefore E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $E_2^{-1} = (-1)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$E_3^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

e.g. A is 3×3, $A^T + 5I$ is nonsingular

$(A^T+5I)^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$ find A.

$A^T + 5I = \left(\begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix}\right)^{-1}$

$\Rightarrow \quad A^T = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ -2 & -4 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$\Rightarrow \quad A^T = \begin{bmatrix} -4 & 3 & 0 \\ 0 & -6 & 0 \\ -2 & -4 & -4 \end{bmatrix}$

$\Rightarrow \quad A = \begin{bmatrix} -4 & 0 & -2 \\ 3 & -6 & -4 \\ 0 & 0 & -4 \end{bmatrix}$

3. Invertible Linear Transformation.

For a lin trans. $T: \mathbb{R}^n \to \mathbb{R}^m$ $(n=m)$. is invertible iff T is both one-to-one and onto. ($null(T) = \{\vec 0\}$; $range(T) = \mathbb{R}^m$).
($null(M) = \{\vec 0\}$; $col(M) = \mathbb{R}^m$).

$T(\vec v) = M\cdot\vec v$

1) $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible if the standard matrix of T is invertible.

2) $n = m$.

3) If T is invertable and $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a lin ind subset in $\mathbb{R}^n$, then $\{T(\vec{v_1}), T(\vec{v_2}), \ldots, T(\vec{v_k})\}$ is a lin ind subset in $\mathbb{R}^n$.

4). Properties when A is $n\times n$ invertable matrix

① A is row equivalent to $n\times n$ identity matrix.
→ after row oper.

② A has n pivot positions.

③ The equation $A\vec{x}=\vec{0}$ has only trivial solution.

$$A\vec{x}=0 \rightarrow A^{-1}\cdot A\vec{x}=A^{-1}\cdot\vec{0} \rightarrow \vec{x}=\vec{0}.$$

④ $\forall \vec{b}\in\mathbb{R}^n$. $A\vec{x}=b$ has at least one solution.

$$A\vec{x}=\vec{b} \rightarrow \vec{x}=A^{-1}\cdot\vec{b}.$$

⑤ The columns of A from a lin ind set.

⑥ The columns of A from a basis of $\mathbb{R}^n$.

⑦ The lin trans $T(\vec{x})=A\vec{x}$ is one-to-one.

⑧ $null(A)=\{\vec{0}\}$

⑨ $\dim(null(A))=0$

⑩ The lin trans $T(\vec{x})=A\vec{x}$ is onto.

⑪ The columns of A spans $\mathbb{R}^n$.

⑫ $col(A)=\mathbb{R}^n$.

⑬ $\dim(col(A))=n$

⑭ $rank(A)=n$

⑮ There is an $n\times n$ matrix C s.t. $C\cdot A=I$.

⑯ There is an $n\times n$ matrix D s.t. $A\cdot D=I$.

⑰ A^T is an invertible matrix.