Proof

1. A Bad Proof

"Thm": $\sqrt{xy} \le \frac{x+y}{2}$. $\xrightarrow{P(x)}$ wrong! start assuming what I want to prove.

"Pf":

$xy \le \left(\frac{x+y}{2}\right)^2$.

$xy \le \frac{x^2+y^2+2xy}{4}$.

$4xy \le x^2+2xy+y^2$.

$0 \le x^2-2xy+y^2 = (x-y)$. $Q(x)$

Rough Work. ↓ give an idea.

$P(x) \Rightarrow Q(x)$. doesn't mean $Q(x) \Rightarrow P(x)$ is true.

Should start with the things already known and ends up with $P(x)$.

Thm: Let $x, y \ge 0$. Then $\sqrt{xy} \le \frac{x+y}{2}$.

Pf. Since a square is always non-negative:

$0 \le (x-y)^2 = x^2-2xy+y^2$.

$\Leftrightarrow 4xy \le x^2+2xy+y^2$.

$\Leftrightarrow xy \le \frac{x^2+2xy+y^2}{4} = \left(\frac{x+y}{2}\right)^2$

Since both sides are non-negative:

$$\sqrt{xy} \le \sqrt{\left(\frac{x+y}{2}\right)^2} = \left|\frac{x+y}{2}\right| = \frac{x+y}{2} \quad \blacksquare$$

↑ x, y non-negative.

2. Proof from definition.

e.g. Prove $f(x) = 3x+7$ is increasing on $\mathbb{R}$ directly from the definition.

WTS $\forall x_1, x_2 \in \mathbb{R}$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

Pf. · Let $x_1, x_2 \in \mathbb{R}$. (fix & arbitrary)

· Assume $x_1 < x_2$.

$$3x_1 < 3x_2.$$

$$f(x_1) = 3x_1+7 < 3x_2+7 = f(x_2)$$

· I've shown $f(x_1) < f(x_2)$ ■

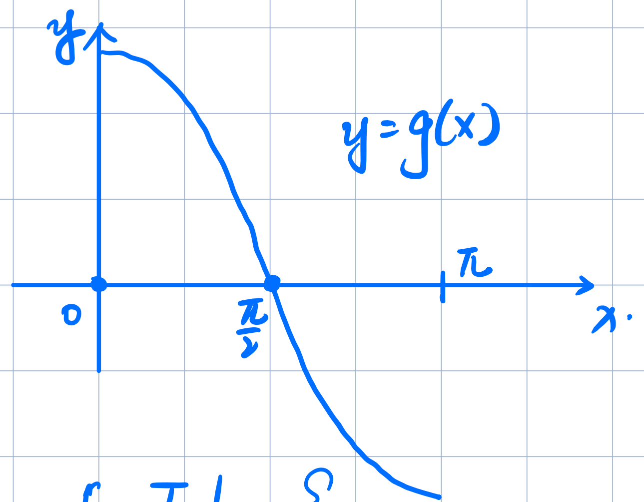
e.g. Prove $g(x) = \cos x$ is not increasing on $[0, \pi]$ directly from the definition.

$\text{No}[\forall x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow g(x_1) < g(x_2)]$

$\Leftrightarrow \exists x_1, x_2 \in I$ s.t. $x_1 < x_2$ and $g(x_1) \geq g(x_2)$

(g isn't increasing on I)

WTS $\exists x_1, x_2 \in [0, \pi]$ s.t. ($x_1 < x_2$ and $g(x_1) \geq g(x_2)$)



pf: Take $\begin{cases} x_1 = 0 \\ x_2 = \frac{\pi}{2}. \end{cases}$

· $0 < \frac{\pi}{2}$.

$\cdot\ g(0) = 1 \geq 0 = g(\frac{\pi}{2})$ ■

3. Proove a theorem.

e.g. Prove that the sum of increasing function is increasing.

1) Set-up. (draft).

Thm: Let f, g be functions on an interval I.

Let $h = f + g$.

2). Hyothesis (assume this is true)

If f, g increasing on I.

3) Conclusion (prove this).

Then h increasing on I.

pf. WTS $\forall x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow h(x_1) < h(x_2)$

Let $x_1, x_2 \in I$, (fix & arbitrary)

Assume $x_1 < x_2$.

Since f is an increasing function on I, $f(x_1) < f(x_2)$.

Since g is an increasing function on I, $g(x_1) < g(x_2)$.

Add both inequalities:

$$h(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = h(x_2).$$

Hence $h(x_1) < h(x_2)$. ■

4. Proof by induction

1) Procedure:

① Base case: Proof when n is smallest.

② Inductive step: Prove from $S_n \Rightarrow S_{n+1}$

not to prove S_{n+1}.

but to prove the implication from $S_n \Rightarrow S_{n+1}$.

e.g. Prove that $\forall n \geq 4$, $n! > 2^n$

Pf: (by induction on n).

Base case: ($n=4$) WTS: $4! > 2^4$.

$4! = 24$. $2^4 = 16$. ✓

Induction step. $\boxed{\text{Let } n \geq 4.}$ remember to write it.

Assume $n! > 2^n$. WTS $(n+1)! > 2^{n+1}$

$(n+1)! = (n+1) \cdot n! > (n+1) 2^n > 2 \cdot 2^n = 2^{n+1}$

by induction hypothesis.

because $n \geq 4$. so $n+1 > 2$.