Comparing Cardinality

1. Comparing:

所以当已知 $S \subseteq T$，那 T_0 就是 S 时. 可直接知 $|S| \le |T|$.

1). If S and T are sets, if there is a subset T_0 of T, s.t. $|S| = |T_0|$, then S has card. less than or equal to T, i.e. $|S| \le |T|$; $|S| < |T|$ if there is an injection $S \to T$ but not a surjection $S \to T$.

$|\mathbb{N}| < |\mathbb{R}|$.

2). Thm.: Any infinite set has card. greater than or equal to $\aleph_0$.

Let S be an infinite set.

Let $s_1, s_2, \dots \in S$.

Define $\underbrace{S^* = \{s_1, s_2, \dots\} \subset S}$ $\Rightarrow |S^*| \le |S|$.

→ important!: every infinite set has a countable subset. (重要启发步骤).

Since $f: \mathbb{N} \to S^*$ as $f(n) = s_n$ $\Rightarrow$ f is bijective.

$\Rightarrow |S^*| = |\mathbb{N}|$. $\Rightarrow |\mathbb{N}| \le |S|$

3). Cantor-Bernstein Thm.: If S and T are sets s.t. $|S| \le |T|$ and $|S| \ge |T|$, then $|S| = |T|$.

① Co.: If $S \subseteq T$ and there exists a func. $f: T \to S$ that is 1-1, then $|S| = |T|$.

Since $S \subseteq T$, $|S| \le |T|$.

Since f is 1-1, $|T| \le |S|$.

② Thm.: If $a < b$, then $|[a,b]| = |(a,b)| = |(a,b]| = |[a,b)|$.

Since $(a,b) \subset [a,b]$, $|(a,b)| \le |[a,b]|$.

Take $[a+\frac{b-a}{3}, b-\frac{b-a}{3}] \subset (a,b)$, $|[a+\frac{b-a}{3}, b-\frac{b-a}{3}]| \le |(a,b)|$.

Since $|[a+\frac{b-a}{3}, b-\frac{b-a}{3}]| = |[a,b]|$, $|[a,b]| \le |(a,b)|$.

pf or use? $f: [0,1] \to \sim$.
$g: [0,1] \to [a,b]$.
$g(x) = a + (b-a)x$.

4). Enumeration Principle: Every set that can be labeled by a countable set is countable.

e.g. $S = \{3 + \sqrt{m} + n^m : m, n \in \mathbb{N}\}$.

Labelling: $(3, +, \sqrt{\ }, m, +, n, m) \to$ countable set.

$\Rightarrow S$ is countable.

5). Power Set: If S is any set, then the set of all subset of S is called the power set of S and is denoted $P(S)$.

1). If S is a finite set with n elements, then $|P(S)| = 2^n$

2). (Thm.): $|S| < |P(S)|$.

$\longrightarrow$ If S is any set, then $\exists$ set T, s.t. $|T| > |S|$. (T can be $P(S)$)

$P(\mathbb{N})$.

6). The card. of the set of all sets of natural # is the same as the card. of the set of real #. That is $|P(\mathbb{N})| = c$ or $2^{\aleph_0} = c$