N^{th} Order ODE (const. coeff.)

最高次前系数为1

1. Std. Form: $\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$.

   1) Ex. & Uni.: p_i, g 都连续，则对应 Initial Value Prob. 有唯一解.

   2) 若所有 p_i 都连续，并且 $y_1,\dots,y_n$ 是解且 Wronskin 不永远为 0，则组成 fundamental set of sol^{n}. 且 $y_1,\dots,y_n$ lin. indep.

2. 题型.

   1) sol^{n} are sure to exist.

   e.g. $(x-1)y^{(4)} + (x+1)y'' + \tan x \cdot y = 0$.

   ① 化成 Std. Form.

   $$y^{(4)} + \frac{x+1}{x-1}y'' + \frac{\tan x}{x-1}\cdot y = 0.$$

   ② 分析：p_i cts 条件.

   $x \neq 1$. $x \neq \frac{\pi}{2} + k\pi$. $k \in \mathbb{Z}$. $x \in \mathbb{R}$.

   2) Determine whether funcs are lin. indep.

   e.g. $f_1(t) = 2t-3$; $f_2(t) = 2t^2+1$; $f_3(t) = 3t^2+t$.

   ① indep. if no non-trivial sol^{n} exists for $c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$.

   $c_1f_1 + c_2f_2 + c_3f_3 = 0$

   $\Rightarrow c_1\cdot(2t-3) + c_2\cdot(2t^2+1) + c_3(3t^2+t) = 0$.

   $\Rightarrow 2c_1 t - 3c_1 + 2c_2t^2 + c_2 + 3c_3t^2 + c_3t = 0$.

   $\Rightarrow (2c_2+3c_3)t^2 + (2c_1+c_3)t + (-3c_1+c_2) = 0$.

   $$\begin{cases} 2c_2+3c_3=0 \\ 2c_1+c_3=0 \\ -3c_1+c_2=0 \end{cases} \Rightarrow c_2 = 3c_1. \Rightarrow \begin{cases} 6c_1+3c_3=0 \\ 2c_1+c_3=0 \end{cases} \Rightarrow c_3 = -2c_1.$$

   $\therefore$ there's non-trivial sol^{n} $\longrightarrow$ dependent.

   3) Verify the sol^{n}: 求导代入.

   e.g. $y^{(4)} + y'' = 0$; 1, t, $\sin t$, $\cos t$.

For 1, $y''=0$, $y^{(4)}=0$. ✓

For t, $y''=0$, $y^{(4)}=0$ ✓

For $\sin t$. $y'=\cos t$, $y''=-\sin t$, $y'''=-\cos t$. $y^{(4)}=\sin t$. ✓

For $\cos t$. $y'=-\sin t$, $y''=-\cos t$, $y'''=\sin t$. $y^{(4)}=\cos t$ ✓

4) Determine the Wronskin: $W[y_1 \cdots y_n] = \begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & & y_n^{(n-1)} \end{vmatrix}$

e.g. $y^{(4)}+y''=0$; $1, t, \sin t, \cos t$

$$W[y_1 \dots y_4] = \begin{vmatrix} 1 & t & \sin t & \cos t \\ 0 & 1 & \cos t & -\sin t \\ 0 & 0 & -\sin t & -\cos t \\ 0 & 0 & -\cos t & \sin t \end{vmatrix} \Rightarrow \det = \det\left(\begin{vmatrix} 1 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \\ 0 & -\cos t & \sin t \end{vmatrix}\right) - t\det\left(\begin{vmatrix} 0 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \\ 0 & -\cos t & \sin t \end{vmatrix}\right)$$

$$+\sin t \cdot \det\left(\begin{vmatrix} 0 & 1 & -\sin t \\ 0 & 0 & -\cos t \\ 0 & 0 & \sin t \end{vmatrix}\right) - \cos t \cdot \det\left(\begin{vmatrix} 0 & 1 & \cos t \\ 0 & 0 & -\sin t \\ 0 & 0 & -\cos t \end{vmatrix}\right).$$

$$= \det\left(\begin{vmatrix} 1 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \\ 0 & -\cos t & \sin t \end{vmatrix}\right) = \det\left(\begin{vmatrix} -\sin t & -\cos t \\ -\cos t & \sin t \end{vmatrix}\right)$$

$$= -\sin^2 t - \cos^2 t = -1$$

5) General sol^{n} for n^{th}-order homo with const. coeff.

① 写 characteristic equation.

② 找到 C.E. 所有解.

③ 根据解的情况找对应 y_i.

i) n个不同实数解.

$$y_1 = e^{r_1 t};\ y_2 = e^{r_2 t};\ \cdots;\ y_n = e^{r_n t}$$

ii) s个相同实数解 (e.g. $r=r_0$ 重复了 s 次).

$$y_1 = e^{r_0 t};\ y_2 = t \cdot e^{r_0 t};\ \cdots;\ y_s = t^{s-1} \cdot e^{r_0 t}$$ 剩下照常看情况

iii) 复数解: $r_{1,2} = \alpha \pm \beta i$

$$y_{1,2} = e^{r_{1,2}t} = e^{(\alpha \pm \beta i)t} \Rightarrow y_1 = e^{\alpha t} \cdot \cos \beta t$$

$$y_2 = e^{\alpha t} \cdot \sin \beta t.$$

④ $y(t) = c_1 y_1 + \cdots + c_n y_n$.

e.g. $2y''' - 4y'' - 2y' + 4y = 0$.

i). $2r^3-4r^2-2r+4=0$.

ii) $(r-1)\cdot(r+1)(r-2)=0$, gives $r_1=1$. $r_2=-1$, $r_3=2$.

iii). $y_1(t)=e^{t}$; $y_2(t)=e^{-t}$; $y_3(t)=e^{2t}$

iv) $y(t)=C_1y_1+C_2y_2+C_3y_3$
$=C_1\cdot e^{t}+C_2\cdot e^{-t}+C_3\cdot e^{2t}$.

e.g. $y'''-3y''+3y'-y=0$.

i) $r^3-3r^2+3r-1=0$.

ii) $(r-1)^3=0$, gives $r_{1,2,3}=1$.

iii) $y_1(t)=e^{t}$; $y_2(t)=t\cdot e^{t}$; $y_3(t)=t^2\cdot e^{t}$

iv) $y(t)=c_1e^{t}+c_2 t\cdot e^{t}+c_3\cdot t^2\cdot e^{t}$.

e.g. $y^{(4)}+2y''+y=0$.

i) $r^4+2r^2+1=0$.

ii) $(r^2+1)^2=0 \Rightarrow r_{1,2}^2=-1$. $\Rightarrow r_{1,2}=\pm i$; $r_{3,4}=\pm i$; $\alpha=0$, $\beta=1$.

iii) $y_1(t)=\cos t$; $y_2(t)=t\cdot\cos t$; $y_3(t)=\sin t$; $y_4(t)=t\cdot\sin t$.

iv) $y(t)=C_1\cos t+C_2\sin t+C_3\cdot t\cdot\cos t+C_4\cdot t\cdot\sin t$.

e.g. $y'''-2y''+4y'=0$.

i) $r^3-2r^2+4r=0$.

ii). $r(r^2-2r+4)=0 \Rightarrow r_1=0$; $r_{2,3}=\dfrac{2\pm2\sqrt{3}i}{2}=1\pm\sqrt{3}i$ $\alpha=1$. $\beta=\sqrt{3}$.

iii). $y_1(t)=1$; $y_2(t)=e^{t}\cdot\cos(\sqrt{3}t)$; $y_3(t)=e^{t}\cdot\sin(\sqrt{3}t)$.

iv). $y(t)=C_1+C_2\cdot e^{t}\cdot\cos(\sqrt{3}t)+C_3\cdot e^{t}\cdot\sin(\sqrt{3}t)$.

⑥ 解 C.E. (complex root).

i) 设 $\lambda=\alpha+\beta i$

ii) 转换成 polar form 代入 C.E., $\lambda=r(\cos\theta+i\sin\theta)$, $r\geq0$, $\theta\in[0,2\pi]$.

iii) 用 De Moivre 去除外面的次方.

iv) 分离两边，把常数项化成 polar form.

v). 计算对应部分即可，注意 r 和 θ 范围.

e.g. $y^{(4)}+y=0$

i) $\lambda^4+1=0$.

ii). Let $\lambda=\alpha+\beta i=r\cdot(\cos\theta+i\sin\theta)$. $r\geq 0$. $\theta\in[0,2\pi]$.

Substitute, gives. $[r\cdot(\cos\theta+i\sin\theta)]^4+1=0$.

$\Rightarrow r^4\cdot(\cos4\theta+i\sin4\theta)+1=0$. by DeMoivre.

$\Rightarrow r^4\cdot(\cos4\theta+i\sin4\theta)=1\cdot(\cos\pi+i\sin\pi)$.

$\begin{cases} r^4=1 \\ \cos4\theta+i\sin4\theta=\cos\pi+i\sin\pi \end{cases} \Rightarrow \begin{matrix} r=1 \\ 4\theta=\pi,3\pi,5\pi,7\pi \end{matrix} \Rightarrow \begin{matrix} r=1 \\ \theta=\frac{\pi}{4},\frac{3}{4}\pi,\frac{5}{4}\pi,\frac{7}{4}\pi \end{matrix}$

$\therefore \lambda_1=\frac{\sqrt2}{2}+\frac{\sqrt2}{2}i$; $\lambda_2=-\frac{\sqrt2}{2}+\frac{\sqrt2}{2}i$; $\lambda_3=-\frac{\sqrt2}{2}-\frac{\sqrt2}{2}i$; $\lambda_4=\frac{\sqrt2}{2}-\frac{\sqrt2}{2}i$

iii) $y(t)=C_1\cdot e^{\frac{\sqrt2}{2}t}\cdot\cos(\frac{\sqrt2}{2}t)+C_2\cdot e^{\frac{\sqrt2}{2}t}\cdot\sin(\frac{\sqrt2}{2}t)+C_3\cdot e^{-\frac{\sqrt2}{2}t}\cdot\cos(\frac{\sqrt2}{2}t)+C_4\cdot e^{-\frac{\sqrt2}{2}t}\cdot\sin(\frac{\sqrt2}{2}t)$.