MAT 137

Tutorial #3– The definition of limit

Oct 4-5, 2022

Due on Thursday, Oct 6 by 11:59pm via GradeScope

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- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

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Let f be a function. Let $a, L \in \mathbb{R}$. Assume that f is defined on some open interval around a, except maybe at a. As you know, the definition of the statement $\lim_{x \to a} f(x) = L$ is

(1) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$.

Below is a list of seven other statements. Write formal, rigorous proofs for these statements:

(a) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\quad 0 < |x - a| < \delta \implies |f(x) - L| \leq \varepsilon$.
What does this statement mean? Hint: what's the difference between (1) and (a)? can we say statement (1) implies this statement (a)? How about the $(a) \implies (1)$? are the epsilons in this statement and in the definition of $\lim_{x \to a} f(x) = L$ necessary to be the same?

The statement means exactly the same as the definition of the statement.

We can say $(1) \Leftrightarrow (a)$.

① $(1) \Rightarrow (a)$.

Since we already know:

$\forall \varepsilon > 0, \exists \delta_1 > 0$. s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$.

Let $\varepsilon > 0$. Take $\delta_2 = \delta_1$.

Since $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$,

Therefore. $0 < |x-a| < \delta_2 \Rightarrow |f(x) - L| < \varepsilon$.

$\leq \varepsilon$.

Hence. $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $0 < |x-a| < \delta$

$\Downarrow$

$|f(x) - L| \leq \varepsilon$

② $(a) \Rightarrow (1)$

Since we already know:

$\forall \varepsilon > 0, \exists \delta_1 > 0$, s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| \leq \varepsilon$.

Take δ_1 when $|f(x) - L| \leq \frac{\varepsilon}{2}$.

Let $\varepsilon > 0$. $\delta_2 = \delta_1$.

Since. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| \leq \frac{\varepsilon}{2}$.

Therefore. $0 < |x-a| < \delta_2 \Rightarrow |f(x) - L| \leq \frac{\varepsilon}{2} < \varepsilon$.

(b) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\quad 0 < |x - a| < \delta \implies 0 < |f(x) - L| < \varepsilon$. Compare this statement with the definition of $\lim_{x \to a} f(x) = L$. What does this statement mean? Hint: Can you find a function $f(x)$ that satisfies this statement? Compare $f(x) = x$ and $f(x) = x^2 \sin(\frac{1}{x})$. Use Desmos to graph these two functions. (https://www.desmos.com/calculator) Do these two functions satisfy this statement?

Comparing the following two statements:

(1): $\forall \varepsilon > 0, \exists \delta > 0$. s.t. $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

(b): $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow 0 < |f(x) - L| < \varepsilon$.

Clearly. the statement (b) restricts that $f(x)$ when x close to a is rigorously only close to L but not equal to L.

Therefore. for. $f(x) = x$. (1) works but (b) doesn't. because. every point on. the function $f(x) = x$ is well defined.

However. for. $f(x) = x^2 \sin(\frac{1}{x})$. (1) doesn't work but (b) does.

As when x is approaching 0, $f(a)$ can not be defined by an accurate number since $f(x)$ is oscillating around 0.

(c) For every $\varepsilon \geq 0$, there exists $\delta > 0$ such that $0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$. Can you find a function satisfy this statement? If yes, give one example. If no, can you explain why. Hint: can you write down the negation of this statement? The negation is true or false?

No.

Take the negation of the statement gives:

$\exists\, \varepsilon \geq 0$, s.t. $\forall \delta > 0$, $0 < |x-a| < \delta$ and $|f(x) - L| \geq \varepsilon$.

Since the function is defined on some open interval around a. $\forall \delta > 0$. we can ensure that $0 < |x-a| < \delta$ is true.

Moreover, $\min\{|f(x) - L|\} = 0$, which was satisfied by $\varepsilon \geq 0$.

Therefore the negation of the statement is correct. The (c) statement is false.

(d) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x-a| < \delta \implies |f(x) - L| < \varepsilon$.
Compare this statement with the definition of $\lim_{x \to a} f(x) = L$. What's the difference? What does this statement mean? Hint: sketch the graphs of the functions $f(x) = x$ and $f(x) = \begin{cases} x, & x \neq 1 \\ 2, & x = 1 \end{cases}$. Let $a = 1$ and $L = 1$. Check if they are satisfies (1) and (d).

The definition of $\lim_{x\to a} f(x) = L$ is $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

In this case, the difference is between $0 < |x-a| < \delta$ and $|x-a| < \delta$.

As for the statement in (d), solving the δ-inequality, gives,

$|x-a|$ maybe at 0.

Therefore x has the potential $x = a$, revealing $f(x)$ is continuous at a.

x close to a $\Rightarrow$ $f(x)$ close to L.

However, for $\lim_{x\to a} f(x) = L$, $\begin{cases} x \text{ close to } a \\ x \neq a \end{cases} \Rightarrow f(x)$ close to L.

$f(x) = x$ satisfy both (1) and (d).

$f(x) = \begin{cases} x, & x \neq 1 \\ 2, & x = 1 \end{cases}$ satisfy (1) but not (d)

(e) For every $\varepsilon > 0$, there exists $\delta \geq 0$ such that $\quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$. Can you find a function satisfies this statement? If yes, give one example. If no, can you explain why. Hint: what happens if I take $\delta = 0$?

Take $f(x) = x$. $a = 0$, $L = 0$.

WTS: $\forall \varepsilon > 0$, $\exists \delta \geq 0$. s.t. $0 < |x-a| < \delta \Rightarrow |x-0| < \varepsilon$.

Let $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{2}$. $\delta \geq 0$.

gives. $|x-a| < \frac{\varepsilon}{2}$.

$\Rightarrow |x-0| < \frac{\varepsilon}{2}$.

$\Rightarrow |x| < \frac{\varepsilon}{2} < \varepsilon$. ■

The statement: $\forall \varepsilon > 0$, $\exists \delta \geq 0$, s.t. $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
Since there exist $\delta > 0$ statisfying the statement. which is the rigorous definition of the limit, there exist $\delta \geq 0$ statisfying as well.
The only difference occurs when take $\delta = 0$. in this case, $0 < |x-a| < 0$, is always false. the whole statement is then True, revealing when taking $\delta = 0$. any functions that is defined on the open interval a, satisfies the statement in (e).

These two questions are for your practice and you don't need to return your work.

(f) For every $\delta > 0$, there exists $\varepsilon > 0$ such that $\quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$.
What does this statement mean? Hint: let $a = 0, L = 1$ and $f(x) = x$. For every $\delta > 0$, can you find the corresponding ε?

(g) There exists $\delta > 0$ such that for every $\varepsilon > 0$, $\quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$. What does this statement mean?