Rational Numbers.

1. Definition: A rational number is a number of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$.

   1). The rational number $\frac{m_1}{n_1}$ is equal to $\frac{m_2}{n_2}$ when $m_1 n_2 = m_2 n_1$.

   2). The set is denoted as $\mathbb{Q}$.

   3). For rational number $\frac{m_1}{n_1}$, $\frac{m_2}{n_2}$.

   ① product: $\frac{m_1 \cdot m_2}{n_1 n_2}$.

   ② addition: $\frac{m_1 n_2 + m_2 n_1}{n_1 n_2}$.

   4). Multiplicative Inverse: $\frac{m}{n} \neq 0$ i.e. $m \neq 0$. $\frac{n}{m}$ is the M.I. of $\frac{m}{n}$.

   5). Lowest Term: m & n are relatively prime, i.e., $\gcd(m, n) = 1$.

   6). Definition of irrational number: $\mathbb{R} \setminus \mathbb{Q}$.

2. Thm. 8.2.7.: If p is a prime number, then $\sqrt{p}$ is irrational.

证 irrational 常用 contra.

proof. Assume $\sqrt{p}$ is rational, i.e. $\sqrt{p} = \frac{m}{n} = r$, where $m, n \in \mathbb{Z}$. $n \neq 0$. and $\frac{m}{n}$ is the lowest term.

$(\sqrt{p})^2 = (\frac{m}{n})^2 \Rightarrow p = \frac{m^2}{n^2}$. $\Rightarrow m^2 = n^2 \cdot p$.

$\Rightarrow p \mid m^2$.

Since p is prime, by Euclid Lemma, $p \mid m$. where $\exists k \in \mathbb{Z}$. s.t. $p \cdot k = m$. gives $(p \cdot k)^2 = n^2 \cdot p$.

$\Rightarrow p^2 \cdot k^2 = n^2 \cdot p$.

$\Rightarrow p \cdot k^2 = n^2$. $\Rightarrow p \mid n^2$.

Also, by Euclid Lemma, $p \mid n$.

Since $p \mid n \wedge p \mid m$, $\gcd(m, n) \geq p$. contradicts to $\frac{m}{n}$ is the lowest term i.e. $\gcd(m, n) = 1$. ∎

3. Thm. 8.2.8.: If the square root of a natural number is rational.

$\sqrt{N} \in \mathbb{Q}$
$\Rightarrow \sqrt{N} \in \mathbb{N}$.

then the square root is a natural number.

proof: Assume $N \in \mathbb{N}$, $\sqrt{N} \in \mathbb{Q}$. i.e. $\sqrt{N} = \frac{m}{n} = r$. where $m, n \in \mathbb{Z}$. $n \neq 0$, $\frac{m}{n}$. is the lowest term.

WTS: $\sqrt{N} \in \mathbb{N}$.

Since $\sqrt{N} = \frac{m}{n} \Rightarrow N = \frac{m^2}{n^2} \Rightarrow m^2 = n^2 \cdot N$.

Let p be a prime.

WTS: $p \nmid n$.

Assume $p \mid n$, gives $p \mid n^2 \cdot N$. i.e. $p \mid m^2$.

By Euclid Lemma. $p \mid m$. gives $\gcd(m,n) \geq p \neq 1$.

Since n is not divisible by any prime. $n = 1$.

Thus $\sqrt{N} = m$ i.e. $\sqrt{N} \in \mathbb{N}$ (as required).

4. Rational Root Theorem:

If $\frac{m}{n}$ is a rational root of the polynomial $a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$, where a_j are int. and m and n are relatively prime. then $m \mid a_0$ and $n \mid a_k$.

WTS: $m \mid a_0$ and $n \mid a_k$.

proof: Assume $\frac{m}{n}$ is a r.r with lowest term of:

$p(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$. gives $p(\frac{m}{n}) = 0$. and $\gcd(m,n) = 1$.

$\Rightarrow p(\frac{m}{n}) = a_k (\frac{m}{n})^k + a_{k-1} \cdot (\frac{m}{n})^{k-1} + \cdots + a_1 (\frac{m}{n}) + a_0 = 0$. $(\ast)$.

Times n^k on both sides

$a_k m^k + a_{k-1} m^{k-1} n + \cdots + a_1 m \cdot n^{k-1} + a_0 n^k = 0$

① $\Rightarrow m(a_k m^{k-1} + a_{k-1} m^{k-2} n + \cdots + a_1 \cdot n^{k-1}) = -a_0 n^k$.

$\Rightarrow m \mid (a_0 \cdot n^k)$.

Since $\gcd(m,n) = 1$, gives $m \nmid n$. which $m \nmid n^k$.

Thus $m \mid a_0$.

② $\Rightarrow n(a_0 n^{k-1} + a_1 m \cdot n^{k-2} + \cdots + a_{k-1} m^{k-1}) = -a_k m^k$.

$\Rightarrow n \mid (a_k \cdot m^k)$.

Since $\gcd(m,n)=1$, gives. $n \nmid m$, which $n \nmid m^k$.

Thus. $n \mid a_k$.

5. Field Properties:

Let F be a field: $F[\sqrt{p}] = \{a+b\sqrt{p} : a, b \in F\}$.

1). $0, 1 \in F$.

2). Close addition & multiplication

if $x, y \in F \Rightarrow x+y \in F \wedge x \cdot y \in F$.

3). $x \in F \rightarrow -x \in F$.

4). $x \in F \rightarrow \frac{1}{x} \in F$.

6. Check whether a num is rational.

1). R.R.T.:

① One '$\sqrt{\ }$': directly power it to remove '$\sqrt{\ }$'.

Show irrational: e.g. Show $\sqrt[7]{\frac{8}{9}}$ is irrational.

Assume rational: Assume $\sqrt[7]{\frac{8}{9}}$ is rational. which $\sqrt[7]{\frac{8}{9}} = r = \frac{m}{n}$. where $m, n \in \mathbb{Z}$. $n \neq 0$.

$\sqrt[7]{\frac{8}{9}} = r \Rightarrow r^7 = \frac{8}{9}. \Rightarrow 9r^7 - 8 = 0$.

By R.R.T. $m \mid -8$, $n \mid 9$, gives.

$m = \pm 1, \pm 8, \pm 2, \pm 4$. $n = \pm 1, \pm 9, \pm 3$. gives.

possible $\frac{m}{n}$: $\pm 1, \pm 9, \pm 3, \pm \frac{1}{9}, \pm \frac{1}{3}, \pm 8, \pm \frac{8}{9}, \pm \frac{8}{3}, \pm \frac{1}{8}, \pm \frac{9}{8}$.

$\pm \frac{3}{8}, \pm \frac{1}{2}, \pm \frac{9}{2}, \pm \frac{3}{2}, \pm 2, \pm \frac{2}{9}, \pm \frac{2}{3}, \pm 4, \pm \frac{4}{9}, \pm \frac{4}{3}$.

$\pm \frac{1}{4}, \pm \frac{9}{4}, \pm \frac{3}{4}$.

Substitute. gives. not equals to 0. $\rightarrow$ no rational root.

$\rightarrow$ r is irrational.

② More than one '$\sqrt{\ }$': Leave on $\sqrt{\ }$ and make the rest into $\mathbb{Q}$. to prove irrational.

eg. Show $\sqrt{3}+\sqrt{5}$ is irrational.

$\vdots$