3. Roots of unity (chapter 9)

a) Show, as a consequence of De Moivre's Theorem, that all roots of the equation $z^n = 1$ are located on the unit circle, and the roots come in conjugate pairs (except for real roots which are their own conjugates). Finally conclude that the roots form the vertices of a regular polygon with $n$ sides (or $n$ vertices) with one of the vertices located at the complex number 1.

Let $z = r(\cos\theta + i\sin\theta)$ be the polar form of $z$.

Since $z^n = 1$, gives $r = 1$.

By De Moivre's Theorem, $z^n = r^n(\cos(n\theta) + i\sin(n\theta)) = 1$

$$\Rightarrow \cos(n\theta) + i\sin(n\theta) = 1.$$

$$\Rightarrow \begin{cases} \cos(n\theta) = 1 \\ \sin(n\theta) = 0. \end{cases} \Rightarrow \begin{cases} n\theta = 2k\pi. \\ n\theta = k\pi. \end{cases} \Rightarrow n\theta = 2k\pi.$$

Overall. $\theta = k \cdot \frac{2\pi}{n}$, $k = 0, 1, \ldots, n-1$.

Since $z_k = \cos(n \cdot \frac{k \cdot 2\pi}{n}) + i\sin(n \cdot \frac{2k\pi}{n}) = \cos(2k\pi) + i\sin(2k\pi) = 1$, all roots are located on the unit circle. which they located on the unit circle with the same distance to the nearby points.

Since all of the roots are 1, which is real number, they are their own conjugates.

For the regular polygon, each vertices is the point of $z_k$ on the unit circle, which for complex number 1 having the last $z$, which when $k = n-1$.

Throughout this question, let $z_0$ be the first root in counterclockwise direction after 1.

b) Suppose you have $\mathrm{Re}(z_0)$. Explain, hypothetically, how you use value of $\mathrm{Re}(z_0)$ to construct all the roots of unity using compass and straight edge and using techniques of Greek Constructions as in 12.1. (You will return to this method in the following items.)

Having $\mathrm{Re}(z_0)$. Let $O$ be the center of unit circle and from $|OA|$ be the length of $\mathrm{Re}(z_0)$ which $A$ is the point on $x$-axis (Real-valued axis).

Thoughts: get a perpendicular line $l_1 \perp OM$ ($x$-axis). $\longrightarrow$ get the point of $l_1$ intersect unit circle $B$ $\longrightarrow$ get the line length of $B$ to unit circle intersect with $x$-axis. $C$. $\longrightarrow$ use $B$ as centre $|BC|$ as length. intersect with circle $D$ $\longrightarrow$ use $D$ as center and $|BC|$ as length intersect with $Z$ $\longrightarrow \ldots$ until the point overlap with $C$.

Steps:

1. get length $|OA|$ using compass.
2. use $A$ as centre, draw a circle with width $|OA|$. i.e. $\mathrm{Re}(z_0)$. intersect $x$-axis with $O$ and point $X$. (Axiom 4).
3. get a random width bigger than $|OA|$. and locate at $O$ and draw a circle $O_1$
4. use the same width as 3 and locate at point $X$ and draw a circle $O_2$.
5. $O_1$ and $O_2$ intersect at point $Y$, $Z$ (Axiom 4).
6. Get line $l_1$ from $Y$, $Z$ (Axiom 1). perpendicular to $OM$.
7. $l_1$ intersect with unit circle at $B$ (Axiom 5).
8. unit circle intersect with $x$-axis at $C$. (Axiom 5).
9. Use compass get the length $|BC|$.
10. Use width $|BC|$ locate at $B$ and draw a circle $C_3$.
11. $C_3$ intersect with unit circle at $D$ (Axiom 5). where $D$ is another root.
12. Use $D$ as centre and width $|BC|$ and draw a circle. $C_4$
13. $C_4$ intersect with unit circle at $Z$ (Axiom 5). where $Z$ is another root.
14. repeat until the new intersect point is $C$.