Problem Set 5

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Q8:

Let $T(n) = \begin{cases} 1 & n = 0, 1 \\ T(\lceil \frac{n}{2} \rceil)^2 + T(\lfloor \sqrt{n} \rfloor) & n \in \mathbb{N},\ n \geq 2. \end{cases}$

Let $P(n): \forall m, n \in \mathbb{N},\ T(m) \leq T(n)$.

I'll apply complete induction on n.

WTS: $\forall n \in \mathbb{N},\ P(n)$.

Base Case: $n=0$ or $n=1$.

When $n=0$, wts. $P(0)$, which $\forall m \leq 0,\ T(m) \leq T(0)$.

Since $m \in \mathbb{N}$, $m \leq 0$, gives $m=0$, which $T(m) = T(0) \leq T(0) = T(n)$.

When $n=1$, wts. $P(1)$, which $\forall m \leq 1$, $T(m) \leq T(1)$.

Since $m \in \mathbb{N}$, $m \leq 1$, gives $m=0$ or $m=1$.

① when $m=0$, $T(m) = T(0) = 1 = T(1) = T(n)$.

② when $m=1$, $T(m) = T(1) = 1 = T(1) = T(n)$.

When $n=2$, wts. $P(2)$, which $\forall m \leq 2$, $T(m) \leq T(n)$.

$T(n) = T(1)^2 + T(\lfloor \sqrt{2} \rfloor) = T(1)^2 + T(1) = 1+1 = 2$.

Since $m \in \mathbb{N}$, $m \leq 1$, gives $m=0$ or $m=1$, or $m=2$.

① when $m=0$, $T(m) = T(0) = 1 \leq 2 = T(2) = T(n)$.

② when $m=1$, $T(m) = T(1) = 1 \leq 2 = T(2) = T(n)$.

③ when $m=2$, $T(m) = T(2) = 2 = T(2) = T(n)$.

Thus, I've proved that the base case is true.

Induction Step: Let $n \in \mathbb{N}$, $n > 2$.

Induction Hypothesis: Let $k \in \mathbb{N}$, $0 \leq k < n$, $P(k)$ is True i.e. $\forall m \leq k$, $T(m) \leq T(k)$.

WTS: $P(n)$, i.e. $\forall m \leq n$, $T(m) \leq T(n)$.

Since $n\in\mathbb{N}$. $n>2$, gives $n\geq 3$. which $T(n) = T(\lceil \frac{n}{2}\rceil)^2 + T(\lfloor\sqrt{n}\rfloor)$

Take $k=n-1$. i.e. $n=k+1$, which. $T(k+1) = T(\lceil\frac{k+1}{2}\rceil)^2 + T(\lfloor\sqrt{k+1}\rfloor)$.

Since $k+1=n\geq 3$. gives. $0\leq \lceil \frac{n}{2}\rceil = \lceil \frac{k+1}{2}\rceil < n$, $0\leq \lfloor\sqrt{n}\rfloor = \lfloor\sqrt{k+1}\rfloor < n$.

Also, $\lceil \frac{k}{2}\rceil \leq \lceil \frac{k+1}{2}\rceil$, $\lfloor\sqrt{k}\rfloor \leq \lfloor\sqrt{k+1}\rfloor$, gives. $0\leq \lceil \frac{k}{2}\rceil \leq \lceil \frac{k+1}{2}\rceil < n$, $0\leq \lfloor\sqrt{k}\rfloor \leq \lfloor\sqrt{k+1}\rfloor < n$.

By I.H., gives. $T(\lceil \frac{k+1}{2}\rceil) \geq T(\lceil \frac{k}{2}\rceil)$, $T(\lfloor\sqrt{k+1}\rfloor) \geq T(\lfloor\sqrt{k}\rfloor)$. i.e.

$$T(n) = T(k+1) = T(\lceil \tfrac{k+1}{2}\rceil)^2 + T(\lfloor\sqrt{k+1}\rfloor) \geq T(\lceil \tfrac{k}{2}\rceil)^2 + T(\lfloor\sqrt{k}\rfloor) = T(k).$$ as. $k=n-1\geq 2$.

Let $m\leq n=k+1$.

① When $m=k+1$, gives.

$$T(m) = T(k+1) \leq T(k+1) = T(n)$$

② When $m<k+1$. i.e. $m\leq k$

Since. $0\leq m\leq k<n$, by I.H. gives. $T(m)\leq T(k)$.

Since $T(k)\leq T(n)$. gives. $T(m)\leq T(k)\leq T(n)$.

Therefore. the Inductive Step is True.

We've shown that $\forall n\in\mathbb{N}$, $P(n)$ is True. ■