UNIVERSITY OF TORONTO
Faculty of Arts and Science

APRIL 2017 EXAMINATIONS

MAT301H1S

Duration: 3 Hours

No Aids Allowed

First Name: ______________________________

Last Name: ______________________________

Student Number: ______________________________

Notes: (1) This exam paper has 10 pages, including this front page. It has two parts: A and B. Part A consists of several short answer questions. You do not have to give any justification for your answers to the questions in Part A. Part B consists of eight questions to which you are required to give fully justified solutions.

(2) The numbers in [ ] indicate how many marks each (part of a) question is worth. The total mark for the exam is 100.

(3) You may use the back of the pages to continue your answer to a question, if you need more space. If you decide to do so, make sure you clearly indicate in the original space provided for the question where the solution continues. Otherwise your continuation may not be graded.

For grader's use only

Question/Part	A	B1	B2	B3	B4	B5	B6	B7	B8	Total
Mark										
/	16	10	10	12	10	10	12	10	10	100

Part A. [16, each question 2 marks] Answer each of the following questions in the space provided. You can simply give your final asnwer. No justification is necessary.

(A1) List all subgroups of μ_6.

(A2) How many elements of order 6 does μ_{90} have?

(A3) Which of the numbers 4, 7, 8, 10, 12 does not occur as the order of any element of S_7?

(A4) List all the elements of order 3 in the quotient group $\mathbb{R}/\mathbb{Z}$.

(A5) Give an example of an abelian group that has no proper subgroup of finite index.

(A6) What is the order of the element $((123), [2])$ of $S_3 \times \mathbb{Z}/4\mathbb{Z}$?

(A7) List all abelian groups of order 18 up to isomorphism.

(A8) Give an example of a non-cyclic abelian group of order 8.

Part B. Provide full answers to each of the following questions. Write your answers clearly and justify all your claims.

(B1) [10] Let G be a group. Define $\phi : G \to G$ by $\phi(g) = g^{-1}$. Show that ϕ is a homomorphism if and only if G is abelian.

(B2) [10] Let p be a prime number > 2. Let $d \mid p-1$. Let $\phi : U(p) \to U(p)$ be defined by $\phi([x]) = [x]^d$. Find (with justification) $|\mathrm{Im}(\phi)|$.

(B3) [12, 3 each part] In each part, give a brief argument for why there is no homomorphism as described.

(a) a surjective homomorphism $U(100) \to D_{10}$

(b) an injective homomorphism $U(8) \to \mathbb{Z}/100\mathbb{Z}$

(c) an injective homomorphism $D_8 \to S_5$

(d) a surjective homomorphism $\mathbb{C}^\times \to \mathbb{R}^\times$

(B4) [10] Let G be an abelian group. Let H be the subgroup of G consisting of all the elements of finite order, i.e.

$$H := \{g \in G : \ g \text{ has finite order}\}.$$

Show that the quotient group G/H has no nontrivial element of finite order. (In other words, show that the only element of finite order in G/H is the identity.)

(B5) [10] Find (with justification) all homomorphisms $\mathbb{Z}/10\mathbb{Z} \to A_4$.

(B6) [12, 6 each part] In each part show that the given groups are isomorphic.

(a) $\mathbb{C}^\times/\mathbb{R}_{>0}$, $\mathbb{R}/\mathbb{Z}$, and S, where S is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ under multiplication.

(b) $\mathbb{C}^\times$, $\mathbb{C}^\times/\mu_n$, and $GL_n(\mathbb{C})/SL_n(\mathbb{C})$

(B7) [10] Show that S_5 has no normal subgroup of index 3.

(B8) [10] Suppose G and K are finite groups such that $\gcd(|G|, |K|) = 1$. Show that the only homomorphism $G \to K$ is the trivial homomorphism. (In other words, show that if $\phi : G \to K$ is a homomorphism, then $\phi(g) = e$ for every $g \in G$.)

The end (Total Marks = 100).