Unit 1(a) Lecture Notes for MAT224

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These notes are intended for study purposes. You should be able to fill in these blanks from the notes you take during lecture and/or the textbook. You are welcome to use them to work ahead. Your completed copy of these notes should be submitted to the Quercus assignment called “Unit 1(a) Lecture Notes” by April 7, 2023. You can scan your handwritten answers or you can type them out. See the Unit 1 Homework for details.

§1 Vector Spaces 1.1

What is a vector space?

A vector space is a collection of vectors that can be added together and multiplied ('scaled') by numbers, called scalars, in a consistent way. Formally, a vector space is defined as a set V of vectors over a field $\mathbb{F}$, where vector addition and scalar multiplication satisfy the following axioms:

1. Commutativity: $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$.
2. Associativity: $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ and $a(b\mathbf{u})=(ab)\mathbf{u}$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$ and $a,b\in\mathbb{F}$.
3. Identity: There exists a vector $\mathbf{0}\in V$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}\in V$.
4. Inverse: For each $\mathbf{u}\in V$, there exists a vector $-\mathbf{u}\in V$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
5. Distributivity: $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$ and $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$ and $a,b\in\mathbb{F}$.
6. Scalar multiplication identity: $1\mathbf{u}=\mathbf{u}$ for all $\mathbf{u}\in V$.

Give three examples of vector spaces that are not $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{R}^n$ for any $n\in\mathbb{N}$.

The space of $n\times n$ complex matrices, denoted by $M_{n\times n}(\mathbb{C})$. Here, the vectors are $n\times n$ complex matrices, and addition and scalar multiplication are defined in the usual way. The dimension of this vector space is n^2.

The space of all polynomials with real coefficients of degree at most n, denoted by $\mathbb{R}_n[x]$. Here, the vectors are polynomials $a_0+a_1x+\cdots+a_nx^n$, and addition and scalar multiplication are defined in the usual way. The dimension of this vector space is $n+1$.

The space of all continuous functions from $[0,1]$ to $\mathbb{R}$, denoted by $C([0,1])$. Here, the vectors are functions $f:[0,1]\to\mathbb{R}$ that are continuous, and addition and scalar multiplication are defined pointwise. The dimension of this vector space is infinite, and it is an example of an infinite-dimensional vector space.

§2 Subspaces 1.2

What is the subspace criterion? What does it say?

The subspace criterion is a test that determines whether a subset of a vector space is itself a vector space. In other words, it gives conditions under which a subset of a vector space inherits the vector space structure from the original space. The subspace criterion states that a subset U of a vector space V is a subspace of V if and only if the following three conditions hold:

1. $\mathbf{0}_V \in U$: the zero vector of V is also in U.
2. Closure under vector addition: if $\mathbf{u}, \mathbf{v} \in U$, then $\mathbf{u} + \mathbf{v} \in U$.
3. Closure under scalar multiplication: if $\mathbf{u} \in U$ and a is a scalar, then $a\mathbf{u} \in U$.

Here, $\mathbf{0}_V$ denotes the zero vector of the vector space V. Condition 1 ensures that U contains the zero vector, which is a necessary condition for it to be a vector space. Conditions 2 and 3 ensure that U is closed under vector addition and scalar multiplication, respectively, which are the defining operations of a vector space.

What is another definition of a subspace that is NOT the subspace criterion?

A subset U of a vector space V is a subspace of V if and only if U is non-empty and closed under linear combinations.

Here, a linear combination of vectors in U is any expression of the form $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k$, where $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are vectors in U and $a_1, a_2, \ldots, a_k$ are scalars.

The condition of being non-empty means that U must contain at least one vector. The condition of being closed under linear combinations means that any linear combination of vectors in U must also be in U. In other words, U must be closed under linear combinations.

Give an example of a subspace of $M_{2\times 2}(\mathbb{R})$. Prove it is a subspace.

Let W be the set of all 2×2 symmetric matrices over $\mathbb{R}$, that is,

$$W = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \right\}.$$

We claim that W is a subspace of $M_{2\times 2}(\mathbb{R})$.

To prove that W is a subspace, we need to verify the two conditions:

1. W is non-empty: This is clear, since the zero matrix is in W.

2. W is closed under linear combinations: Let A, B be two matrices in W, and let k_1, k_2 be scalars. We need to show that $k_1A + k_2B$ is also in W.

$$k_1 + k_2 = \begin{pmatrix} k_1a + k_2a & k_1b + k_2b \\ k_1b + k_2b & k_1c + k_2c \end{pmatrix} = \begin{pmatrix} (k_1 + k_2)a & (k_1 + k_2)b \\ (k_1 + k_2)b & (k_1 + k_2)c \end{pmatrix}$$

which is a symmetric matrix. Therefore, $k_1A + k_2B \in W$, and W is closed under linear combinations.

Since W satisfies the two conditions for being a subspace, we conclude that W is a subspace of $M_{2\times2}(\mathbb{R})$.

Give an example of a subset S of a vector space V that is not a subspace. Explain why it is not a subspace.

Consider the set $S = (x, y) \in \mathbb{R}^2 : x + y = 1$, which is a subset of the vector space $\mathbb{R}^2$.

We claim that S is not a subspace of $\mathbb{R}^2$.

To see why S is not a subspace, we need to check one of the conditions of the subspace criterion that fails. Here, we see that S fails to satisfy the closure under vector addition property, which is the second condition of the subspace criterion.

To see this, consider the vectors $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (0, 1)$, which are both in S since $1 + 0 = 0 + 1 = 1$. However, their sum $\mathbf{u} + \mathbf{v} = (1, 1)$ is not in S, since $1 + 1 = 2 \neq 1$. Therefore, S is not closed under vector addition and hence not a subspace of $\mathbb{R}^2$.

§3 Linear Independence 1.3

Definition of Linear Independence: The set of vectors $\{v_1, v_2, \ldots v_n\}$ is linearly independent if and only if the only solution to the equation $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$ is the trivial solution $c_1 = c_2 = \cdots = c_n = \mathbf{0}$, where $\mathbf{0}$ is the zero vector.

Definition of Linear Dependence: The set of vectors $\{v_1, v_2, \ldots v_n\}$ is linearly dependent if and only if there exist scalars $c_1, c_2, \ldots, c_n$, not all zero, such that the equation $c_1 = c_2 = \cdots = c_n = \mathbf{0}$ has a nontrivial solution, where $\mathbf{0}$ is the zero vector.

Can a set of only one vector be linearly dependent? Why or why not?

A set containing only one vector cannot be linearly dependent, as there is no other vector to compare it to. This is because linear dependence refers to the property of a set of vectors such that at least one vector in the set can be written as a linear combination of the other vectors in the set.

$\mathbf{v}$ is linearly independent if and only if $c_1\mathbf{v} = \mathbf{0}$ implies $c_1 = 0$, for all $c_1 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$.

Can a set of 5 vectors in $\mathbb{R}^4$ be linearly independent?

Let $B = \{v_1, v_2, v_3, v_4, v_5\},\ v_i \in \mathbb{R}^4,\ i = 1, 2, \ldots 5$. Then B is not linearly independent. This is the consequence of the Steinitz exchange lemma, which states that for a vector space V, if L is a linearly independent set and S is a spanning set, then we have $\|L\| \leq \|S\|$. Since we may take S to be a basis of V, this implies that we must have $\|L\| \leq dim(V)$.
Similarly, In this example, $\|B\| \geq dim(V)$, therefore, a set of 5 vectors can't be linearly independent in $\mathbb{R}^4$.

Give an example of a proof that a set is linearly independent.
Let $v_1 = \begin{pmatrix}1 & 0 & 2\end{pmatrix}, v_2 = \begin{pmatrix}0 & 1 & -1\end{pmatrix}, v_3 = \begin{pmatrix}2 & -1 & 3\end{pmatrix}$ be three vectors in $\mathbb{R}^3$. We want to show that the set $S = v_1, v_2, v_3$ is linearly independent.

Suppose that there exist scalars c_1, c_2, c_3 such that $c_1v_1 + c_2v_2 + c_3v_3 = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

Then we have the system of linear equations:

$$\begin{cases} c_1 + 2c_3 = 0 \\ c_2 - c_3 = 0 \\ 2c_1 - c_2 + 3c_3 = 0 \end{cases}$$

Since the system has only the trivial solution $c_1 = c_2 = c_3 = 0$, we can conclude that S is linearly independent.
Therefore, the set $S = v_1, v_2, v_3$ is linearly independent.

Give an example of a proof that a set is linearly dependent.
Let $v_1 = \begin{pmatrix}1 & 2 & 3\end{pmatrix}, v_2 = \begin{pmatrix}-1 & 0 & 1\end{pmatrix}, v_3 = \begin{pmatrix}2 & 4 & 6\end{pmatrix}$ be three vectors in $\mathbb{R}^3$. We want to show that the set $S = v_1, v_2, v_3$ is linearly dependent.
We notice that $v_3 = 2v_1 + v_2$. Indeed, we have

$$2v_1 + v_2 - v_3 = 2 \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\4\\7\end{pmatrix} = v_3$$

Therefore, the set S is linearly dependent, since we have found a nontrivial linear combination of v_1, v_2, and v_3 that equals the zero vector:

$$2v_1 + v_2 - v_3 = 2 \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}-1\\0\\1\end{pmatrix} - \begin{pmatrix}2\\4\\6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = v_3$$

Which of the above two examples is also an example of disproving that a set is linearly independent? Why?
The second example is also an example of disproving that a set is linearly independent, because it shows that the set $S = v_1, v_2, v_3$ is not linearly independent by providing a nontrivial linear combination of v_1, v_2, and v_3 that equals the zero vector. This means that there exist scalars c_1, c_2, and c_3, not all zero, such that

$$c_1v_1 + c_2v_2 + c_3v_3 = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

which is the definition of linearly dependent. Specifically, we have shown that $2v_1 + v_2 - v_3 = \mathbf{0}$, so S is linearly dependent.