Tutorial # 8

The Spectral Theorem

Learning Objectives

> In this tutorial, you will practice doing proofs and calculations related to spectral theorems.

There are many different theorems in mathematics called the spectral theorem. The version most relevant to this tutorial is stated below.

Theorem 1. *Let V be a complex vector space. A self-adjoint linear transformation $T : V \to V$ admits an orthonormal eigenbasis. Moreover, all eignevalues of T are real.*

Theorem 2. *Let V be a real vector space. A linear transformation $T : V \to V$ admits an orthonormal eigenbasis if and only if T is self adjoint.*

You will also need the following definitions and proposition.

Definition. A square matrix O with real entry is called orthogonal if $O^T O = I$. In other words, Q is an orthogonal matrix if Q is invertible and $Q^{-1} = Q^T$.

Definition. A square matrix O with complex entry is called unitary if $O^* O = I$. In other words, Q is an orthogonal matrix if Q is invertible and $Q^{-1} = Q^*$.

Proposition 1. *A square matrix Q is orthogonal (or unitary) if and only if columns of Q are orthonormal.*

For more information about the spectral theorem, you may review Damiano and Little, Sections 4.6 and 5.3.

Problems

1. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by left multiplication by A.

   (a) Without doing any calculations, how do you know that the matrix A has an orthonormal eigenbasis?

   (b) Describe the transformation T_A **geometrically**. Find an orthonormal eigenbasis of A using geometric reasoning.

   (c) Write down an orthogonal matrix S and diagonal matrix D such that $S^T AS = D$. We say A is *orthogonally diagonalizable.*

2. Let $A = \begin{bmatrix} 1 & 0 & 2+2i \\ 0 & -3 & 0 \\ 2-2i & 0 & -1 \end{bmatrix}$.

   (a) Without doing any calculations, how do you know that the matrix A has an orthonormal eigenbasis in $\mathbb{C}^3$?

   (b) Given that $\begin{bmatrix} 1+i \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1+i \\ 0 \\ 1 \end{bmatrix}$ are two linearly independent eigenvectors of A, use the spectral theorem to find a third eigenvector that is linearly independent.

   (c) Find a unitary matrix U such that U^*AU is diagonal. We say A us *unitarily diagonalizable.*

3. (a) In class we proved that if A is Hermitian, then T_A admits an orthonormal eigenbasis with real eigenvalues. In other words, if A is hermitian, then A is unitarily diagonalizable with real eigenvalues.

   Prove the converse: if a matrix A with complex entries is unitarily diagonalizable with real eigenvalues, then A is a Hermitian matrix.

   (b) Prove Proposition 1 on the top of this worksheet.

4. Consider each of the following statements. If the statement is true, give a proof. If not, provide a counterexample.

   (a) Every complex symmetric matrix is diagonalizable.

   (b) If P is any 5×9 complex matrix, then the matrix PP^* has an orthonormal basis.

   (c) Every orthogonal matrix is orthogonally diagonalizable.

   (d) If a matrix A is diagonalizable, then A is unitarily diagonalizable.

   (e) If a matrix A has an orthonormal eigenbasis, then every eigenbasis for A is orthonormal.

Q4. (a). This statement is false. To provide a counter example: $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$

Since $A^T = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = A$ and A's entries include complex number, A is a complex symmetric matrix.

From the definition of the eigenvector v, with corresponding eigenvalue λ, we have $Av = \lambda v$.

We calculate by using $\det(A - \lambda I) = 0$, which,

$$\det \begin{pmatrix} 1-\lambda & i \\ i & -1-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-1-\lambda) - i^2 = 0$$

$$\Rightarrow (-\lambda+1)(-\lambda-1) - i^2 = 0$$

$$\Rightarrow \lambda^2 - 1 - (-1) = 0 \text{, since. } a^2 - b^2 = (a+b)(a-b)$$

$$\Rightarrow \lambda^2 = 0$$

$$\Rightarrow \lambda = 0.$$

To find the eigen vector. $(A - \lambda I)\vec{v} = 0$.

$$\left(\begin{array}{cc|c} 1 & i & 0 \\ i & -1 & 0 \end{array}\right) \xrightarrow{r_2 - i \cdot r_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right) \text{, which.}$$

$x_1 + i \cdot x_2 = 0.$

$\Rightarrow$ $x_1 = -ix_2$. we have. the set of solution of $\vec{v}$. to be $\left\{ x_2 \cdot \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$.

Since it doesn't have two linearly independent eigenvectors, the matrix A is not diagonalizable.

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