Unit 4(a) Lecture Notes for MAT224

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What is an eigenvalue?
What is an eigenvector?
How are they related?
An eigenvalue of a square matrix A is a scalar λ for which there exists a non-zero vector $\mathbf{v}$ such that the following equation holds:

$$A\vec{v} = \lambda\vec{v}$$

The vector $\mathbf{v}$ is called an eigenvector of A corresponding to the eigenvalue λ.

Give an example of a linear transformation T, one of its eigenvalues λ and a corresponding eigenvector v. Confirm that $T(v) = \lambda v$. No other calculations are required.
Suppose we have the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix:

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$$

We claim that $\lambda = -1$ is an eigenvalue of A, with corresponding eigenvector $\mathbf{v} = \begin{pmatrix} 1 & 2 \end{pmatrix}$. To confirm this, we can compute:

$$Av = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda v$$

Thus, $\mathbf{v}$ is indeed an eigenvector of A corresponding to the eigenvalue $\lambda = -1$. Note that $T(\mathbf{v}) = A\mathbf{v} = \lambda\mathbf{v} = -\mathbf{v}$, which confirms that $\mathbf{v}$ is an eigenvector of T with eigenvalue λ.

Give an example of a matrix A and explain how you find its characteristic polynomial and eigenvalues. Include your calculations.
Suppose we have the matrix:

$$A = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}$$

To find the characteristic polynomial of A, we need to compute the determinant of the matrix $A - \lambda I$, where λ is a variable and I is the 2×2 identity matrix. That is,

$$A - \lambda I = \begin{pmatrix} 3 - \lambda & 2 \\ 4 & -1 - \lambda \end{pmatrix}$$

So the characteristic polynomial of A is $p(\lambda) = \lambda^2 - 2\lambda - 5$. To find the eigenvalues of A, we need to solve the equation $p(\lambda) = 0$. Using the quadratic formula, we have the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$.

Give an example of a matrix A, one of its eigenvalues λ, and explain how you can find a basis of its corresponding eigenspace E_λ. Is it possible to find more than one basis for E_λ? If so, how?

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$$

We found in a previous answer that one of the eigenvalues of A is $\lambda = -1$. To find a basis for the eigenspace E_{-1}, we need to find all solutions to the equation

$$(A - \lambda I)v = 0$$

where $\mathbf{0}$ is the 2×1 zero vector. That is, we need to find all vectors $\mathbf{v} = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ such that

$$\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3v_1 - v_2 = 0$$

$$4v_1 - 2v_2 = 0$$

Simplifying the second equation gives $2v_1 - v_2 = 0$, which we can substitute into the first equation to obtain $v_1 = \frac{1}{3}v_2$. So the eigenvectors of A corresponding to $\lambda = -1$ are of the form $\begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}v_2 & v_2 \end{pmatrix}$. To find a basis for E_{-1}, we can choose one nonzero vector that satisfies this condition. A convenient choice is $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

It is possible to find more than one basis for E_λ. In fact, any nonzero scalar multiple of an eigenvector corresponding to λ is also an eigenvector corresponding to λ, and thus it is a basis for E_λ. In our example, $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ is also an eigenvector of A corresponding to $\lambda = -1$, and it is a scalar multiple of $\mathbf{v}$. So $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ is also a basis for E_{-1}.

An **involution** is a function f such that $f^2 = I$. Another way to say this is $(f \circ f) = I$ or

$$f(f(x)) = I(x) = x$$

An example of an involution is the transpose operator $f(A) = A^T$, where A^T is the transpose of matrix A. From Questions 15 and 16 at the end of textbook section 4.1, what do we know about the eigenvalues and eigenspaces of involutions?

From Questions 15 and 16 at the end of textbook section 4.1, we know that the only possible eigenvalues of an involution are 1 and -1.

If $\lambda = 1$, then the eigenspace E_λ consists of all vectors v such that $f(v) = v$, i.e., $v = f(v) = f(f(v)) = f(\lambda v)$. So the eigenspace E_1 is the set of all fixed points of f.
If $\lambda = -1$, then the eigenspace E_λ consists of all vectors v such that $f(v) = -v$, i.e., $v = f(v) = f(f(v)) = f(-\lambda v)$. So the eigenspace E_{-1} is the set of all vectors that are mapped to their negation by f.

Let W be a subspace of vector space V. In Section 4.4 we will be talking about **orthogonal projections**. These are the function $p_W(x) = w$ whenever $x = x_1 + w$, for $w \in W$ and $x_1 \in W^\perp$.

An example of an orthogonal projection is p_1 which projects a vector in $\mathbb{R}^3$ onto the $x - y$ plane:

$$p\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Such a projection will have the property $p^2(x) = p(x)$. What do we know about the eigenvalues of p? (see question 17 at the end of textbook section 4.1)
From question 17 at the end of textbook section 4.1, we know that the only possible eigenvalues of an orthogonal projection p are 0 and 1.
If $\lambda = 1$, then the eigenspace E_λ consists of all vectors v such that $p(v) = v$. In other words, v is already in the subspace W and is not affected by the projection.
If $\lambda = 0$, then the eigenspace E_λ consists of all vectors v such that $p(v) = 0$. In other words, v lies entirely in the orthogonal complement $W^\perp$ and is completely projected onto the zero vector.

Give an example of a matrix A and its characteristic polynomial p. Confirm that $p(A) = 0$ by calculating $p(A)$.
Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

To find the characteristic polynomial of A, we compute $|A - \lambda I|$:

$$(1 - \lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$$

Therefore, the characteristic polynomial of A is $p(\lambda) = \lambda^2 - 2\lambda - 3$.
To confirm that $p(A) = 0$, we substitute A into $p(\lambda)$:

$$p(A) = A^2 - 2A - 3I = \begin{pmatrix} -6 & -6 \\ -6 & -6 \end{pmatrix} = 0$$

Therefore, $p(A) = 0$, as expected.

Let A and B be **similar** matrices. What does that mean? In what way are the eigenvalues of similar matrices related? Find the answer in the textbook or your notes from class and summarize the proof.

Two matrices A and B are called similar if there exists an invertible matrix P such that $B = P^{-1}AP$. The matrices A and B are essentially the same, just represented in different coordinate systems.

The eigenvalues of similar matrices are the same. To prove this, suppose A and B are similar matrices, and let λ be an eigenvalue of A with corresponding eigenvector v. Then we have $Av = \lambda v$. Multiplying both sides by P^{-1} on the left and P on the right, we get $P^{-1}AP(P^{-1}v) = \lambda(P^{-1}v)$, which can be written as $B(P^{-1}v) = \lambda(P^{-1}v)$. This shows that $P^{-1}v$ is an eigenvector of B with eigenvalue λ. Therefore, any eigenvalue of A is also an eigenvalue of B and vice versa, and they have the same eigenvectors.

How can you tell the number of DISTINCT eigenvalues that a matrix A will have? How can you tell the multiplicity of $(x - \lambda)$ for each each eigenvalue λ as it appears in the characteristic polynomial?

To determine the number of distinct eigenvalues of a matrix A, we can compute the characteristic polynomial $p(\lambda)$ of A and count the number of distinct roots of $p(\lambda)$. Each distinct root corresponds to a distinct eigenvalue.

To determine the multiplicity of $(x - \lambda)$ for each eigenvalue λ, we need to look at the powers of $(x - \lambda)$ in the factorization of $p(x)$. Specifically, the multiplicity of $(x - \lambda)$ is equal to the power of $(x - \lambda)$ in the factorization of $p(x)$ as a product of linear factors. For example, if the characteristic polynomial is $p(x) = (x - 2)^3(x - 5)^2$, then 2 has algebraic multiplicity 3 and 5 has algebraic multiplicity 2.

Consider a matrix M with characteristic polynomial q where $q(x) = x^3 - x$. How can we relate each of the following matrices to M with a lower exponent?

(i) M^3: If $A = M^3$, then the characteristic polynomial of A is $q(x) = x^3 - x$. Therefore, A has the same eigenvalues as M, which are $\lambda_1 = -1, \lambda_2 = 0$, and $\lambda_3 = 1$. The eigenvectors of A corresponding to λ_1, λ_2, and λ_3 are $(M(v_1))^3, (M(v_2))^3$, and $(M(v_3))^3$ respectively.

(ii) M^4: To relate M^4 to M, we can use the fact that $M^3 = M - I$. Multiplying both sides of this equation by M gives $M^4 = M^2 - M$, and we can express M^2

in terms of M using the above calculations: $M^2v = \lambda^2 v$ for any eigenvector v of M. Therefore, M^4 is the zero matrix.

(iii) M^{2k} for $k \in \mathbb{N}$: Since M has characteristic polynomial $q(x) = x^3 - x$, we know that $q(M) = M^3 - M = 0$. Therefore, we can write $M^3 = M$. Now, for any even integer k, we have $M^{2k} = (M^3)^k = M^{3k}$. Since $3k$ is also an even integer, we can repeat this process and get $M^{3k} = (M^3)^k = M^k$. Therefore, $M^{2k} = M^k$ for any even integer k

(iv) M^{2k+1} for $k \in \mathbb{N}$: Now, notice that $q(x) = x^3 - x = x(x^2 - 1) = x(x+1)(x-1)$, so the eigenvalues of M are 0, 1, and -1. We know from the theory of diagonalization that if a matrix A is diagonalizable with diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$ for any positive integer k. In other words, we can compute M^{2k+1} by diagonalizing M and raising the diagonal matrix to the power of $2k+1$. Therefore, we can conclude that M^{2k+1} is similar to D^{2k+2}, which is a diagonal matrix whose entries are 0, 1^{2k+2}, and $(-1)^{2k+2}$.

§1 4.2 Diagonalizability

What does it mean for a matrix A to be **diagonalizable**?
What does it mean for a transformation T to be **diagonalizable**?
How are these two concepts related?

A matrix A is diagonalizable if it is similar to a diagonal matrix, i.e., if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.
A transformation T is diagonalizable if there exists a basis of V consisting of eigenvectors of T. In other words, T is diagonalizable if there exists an invertible matrix P such that $P^{-1}TP$ is a diagonal matrix.
These two concepts are related in that a matrix A is diagonalizable if and only if the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is diagonalizable. More specifically, if A is diagonalizable, then T is diagonalizable with eigenvectors corresponding to the diagonal entries of the diagonalized form of A. Conversely, if T is diagonalizable, then the matrix representation of T with respect to the basis of eigenvectors is a diagonal matrix, and the matrix A whose columns are the coordinates of the eigenvectors with respect to the standard basis is diagonalizable and similar to this diagonal matrix.

At the bottom of page 176 in the textbook, its says "In order for a linear mapping or a matrix to be diagonalizable, it must have enough linearly independent eigenvectors to form a basis of V." Consider a diagonalizable matrix A. What does the previous statement say about the eigenvectors of A?
The statement means that in order for a diagonalizable matrix A to exist, there must be enough linearly independent eigenvectors of A to form a basis for the vector space V on which A is acting. In other words, the eigenvectors of A must span the entire vector space V. This is because, as mentioned earlier, a diagonalizable matrix is one that can be written in the form $A = PDP^{-1}$, where D is a diagonal matrix and P is a matrix whose columns are the eigenvectors of A. If there are not enough linearly independent eigenvectors of A, then the matrix P will not be invertible, and A will not be diagonalizable.

What do Propositions 4.2.4, 4.2.6, Corollary 4.2.5, 4.2.8, 4.2.9, and Theorem 4.2.7 tell us about how to find a basis of V made up of eigenvectors of T for a diagonalizable linear transformation $T : V \to V$?

Taken together, these results tell us that in order to find a basis of V made up of eigenvectors of T for a diagonalizable linear transformation $T : V \to V$, we need to find a set of linearly independent eigenvectors of T that spans V. If such a set exists, then it forms a basis of V, and we can use it to diagonalize T. Conversely, if we know that T is diagonalizable, then we know that there exists a basis of V made up of eigenvectors of T. We can find this basis by finding the eigenvectors of T and checking that they are linearly independent.

Give an example of a matrix $[T]_\alpha^\alpha$ corresponding to a linear transformation $T : V \to V$ and a basis β of V (not necessarily equal to α), such that the elements of β are eigenvectors of the matrix.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x, y) = (3x + 2y, 2x + 3y)$, and let $\beta = (1, 1), (-1, 1)$ be a basis of $\mathbb{R}^2$. To find the matrix $[T]_\beta^\beta$, we need to compute the action of T on each basis vector in β and write the result as a linear combination of the basis vectors in β. That is:

$$T(1, 1) = (3(1) + 2(1), 2(1) + 3(1)) = (5, 5) = 5(1, 1) + 0(-1, 1)$$
$$T(-1, 1) = (3(-1) + 2(1), 2(-1) + 3(1)) = (-1, 1) = 0(1, 1) + 1(-1, 1)$$

Therefore, the matrix $[T]_\beta^\beta$ is:

$$[T]_\beta^\beta = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice that the elements of β are eigenvectors of $[T]_\beta^\beta$, with corresponding eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 1$.