3. 'Subtraction' formalized. For this problem, m and n are natural numbers with $m < n$.

   a) Assume there is some natural number k such that $m + k = n$. Prove that k must be unique. (In this case, we formally denote such a unique k as the *difference* of n and m, and we denote it by $n - m$.)

proof. Assume k is not unique. which $m+k_1=n$. $m+k_2=n$, $k_1 \neq k_2$
Subtracting $(m+k_1)-(m+k_2)=n-n$. gives. $k_1-k_2=0$. which means k_1 and k_2 have no difference, meaning they are the same.

In the following two parts, we want to prove such a k must exist. (Remember, this question is an existential question, so we can apply WOP to it.)

   b) Define the set $S = \{w \in \mathbb{N} : n \leq m + w\}$, and show that S in non-empty.

Take $w=(n-m+1)$. gives.
$n \leq m+(n-m+1)$. since $m \in \mathbb{N}$ and $n \in \mathbb{N}$, subtracting m and n.
$\Rightarrow$ $0 \leq 1$.
Thus, $w \in S$ and S is non-empty.

   c) By the WOP the set S has a least element, call it k. Prove that $n = m + k$. (Hint: divide into two cases, $k = 1$, and $k = s + 1$ for some natural number s.)

proof. From b). we have $S=\{w \in \mathbb{N}; n \leq m+w\}$.
Let $k=S_{min}$, $S_{min} \in S$, which is the smallest element of S.
Since $k \in S$, $n \leq m+k$. and we want to show $n=m+k$.
Thus. we need to show $n<m+k$ is impossible.
Assume for contradiction, $n<m+k$ is possible.
① $k=1$.
Since $m \in \mathbb{N}$. $n \in \mathbb{N}$. $m<n$. the least difference is 1. which.
$m+1 \leq n$.
From assumption for contradiction, $n<m+k$, which gives.
$n<m+1$.
Thus. $m+1 \leq n < m+1$, contradicts. which $m+1<m+1$ is impossible.

② $k=s+1$, where $s \in \mathbb{N}$.
Since $s \in \mathbb{N}$. $s \geq 0$, gives. $s+1 \geq 1$. gives.
$m+1 \leq m+(s+1) \leq n$.
From assumption for contradiction, $n<m+k$, which gives.
$n< m+(s+1)$.
Thus. $m+(s+1) \leq n < m+(s+1)$. contradicts which $m+(s+1) < m+(s+1)$ is impossible.
Therefore. we've prove $n<m+k$ is impossible. and $k \in S$, which $n \leq m+k$. gives.
$n=m+k$. ∎