N^{th} Order non-Homo

$y_c(t)$：用Homo法求即可.

1. General sol^{n} : $y(t) = y_c(t) + y_p(t)$.

1) Undetermined Coefficient 求 $y_p(t)$.

① 找 $y_c(t) = C_1 y_1 + \cdots + C_n y_n$ as usual.

② 猜出 $y_p(t)$.

③ 求出 $y_p'(t), \cdots, y_p^{(n)}(t)$ substitute into ODE. (因为 $y_p(t)$ 也是解).

④ 求出 $y_p(t)$ 中 coeff. 的值.

⑤ $y(t) = y_c(t) + y_p(t)$.

先通过Homo找 $y_1 \cdots y_n$

2) Variation of Parameter (General)

① 求出 Wronskin.

$$\det(W[y_1 \cdots y_n]) = \det\left(\begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}_{n\times n} \right) = y_1^{(n-1)} \cdot W_1 + y_2^{(n-1)} \cdot W_2 + \cdots + y_n^{(n-1)} \cdot W_n.$$

(正负号放到 W 中).

② 求出 $W_1, \cdots, W_n$. 无y_1 无y_n $(+,-,+,-)$

$$W_1 = \begin{vmatrix} y_2 & \cdots & y_n \\ \vdots & & \vdots \\ y_2^{(n-2)} & \cdots & y_n^{(n-2)} \end{vmatrix}_{n-1\times n-1} \qquad W_n = \begin{vmatrix} y_1 & \cdots & y_{n-1} \\ \vdots & & \vdots \\ y_1^{(n-2)} & \cdots & y_{n-1}^{(n-2)} \end{vmatrix}_{n-1\times n-1}.$$

③ 求出 $u_1(t), \cdots, u_n(t)$. (记得加对应 C_i，在积分后).

$$u_i(t) = \int \frac{g(t)\cdot W_i(t)}{W(t)}\, dt.$$

④ $y(t) = u_1 y_1 + \cdots + u_n y_n$. $(= C_1 y_1 + \cdots + C_n y_n + y_p(t))$.

eg. Third Order:

① $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = y_1'' \cdot W_1 + y_2'' \cdot W_2 + y_3'' \cdot W_3.$

② $W_1 = \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix}$; $W_2 = -\begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix}$; $W_3 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

3). General Abel Theorem: $W(t) = c \cdot e^{-\int p_1(t)\, dt}$

p_1: n-1 阶导前系数.

(先写成 std. form.).

2.题

1) General sol^{n}.

e.g. $y''' - 2y'' + 4y' = e^{t} \cdot \sin(\sqrt{3}t)$.

Undetermined Coeff.

i). Solve Homo:

$r^3 - 2r^2 + 4r = 0. \Rightarrow r(r^2 - 2r + 4) = 0.$

$\Rightarrow r_1 = 0. \quad r_{2,3} = 1 \pm \sqrt{3}i$

$\therefore y(t) = C_1 + C_2 \cdot e^{t} \cdot \sin(\sqrt{3}t) + C_3 \cdot e^{t} \cdot \cos(\sqrt{3}t) + y_p(t).$

ii) Let $y_p(t) = t \cdot e^{t} \cdot (A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t))$. repeat once.

iii) $y_p'(t) = (A \cdot e^{t} \cdot \cos(\sqrt{3}t) + B \cdot e^{t} \cdot \sin(\sqrt{3}t)) + t \cdot e^{t} \cdot (A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)) +$

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Variation of Parameter:

i). Solve Homo and get y_1, y_2, y_3.

$r^3 - 2r^2 + 4r = 0. \Rightarrow r(r^2 - 2r + 4) = 0.$

$\Rightarrow r_1 = 0. \quad r_{2,3} = 1 \pm \sqrt{3}i$

$y_1(t) = 1$; $y_2(t) = e^{t} \cdot \cos(\sqrt{3}t)$; $y_3(t) = e^{t} \cdot \sin(\sqrt{3}t)$.

ii) Solve Wronskin:

$$W[y_1, \cdots y_n] = \det\left(\begin{vmatrix} 1 & e^{t}\cos(\sqrt{3}t) & e^{t} \cdot \sin(\sqrt{3}t) \\ 0 & e^{t}\cos(\sqrt{3}t) - \sqrt{3} \cdot e^{t} \cdot \sin(\sqrt{3}t) & e^{t} \cdot \sin(\sqrt{3}t) + \sqrt{3} \cdot e^{t} \cdot \cos(\sqrt{3}t) \\ 0 & \begin{matrix} e^{t} \cdot \cos(\sqrt{3}t) - \sqrt{3} \cdot e^{t} \cdot \sin(\sqrt{3}t) \\ -\sqrt{3}e^{t} \cdot \sin(\sqrt{3}t) - \sqrt{3} \cdot e^{t} \cdot \cos(\sqrt{3}t) \end{matrix} & \begin{matrix} e^{t} \cdot \sin(\sqrt{3}t) + \sqrt{3} \cdot e^{t} \cdot \cos(\sqrt{3}t) \\ +\sqrt{3} \cdot e^{t} \cdot \cos(\sqrt{3}t) - 3 \cdot e^{t} \cdot \sin(\sqrt{3}t) \end{matrix} \end{vmatrix}\right)$$

$$= \cdot W_1 + (e^{t} \cdot \cos(\sqrt{3}t) - \sqrt{3} \cdot e^{t} \cdot \sin(\sqrt{3}t) - \sqrt{3}e^{t} \cdot \sin(\sqrt{3}t) - \sqrt{3} \cdot e^{t}\cos(\sqrt{3}t))$$

$$\cdot W_2 + (e^{t}\sin(\sqrt{3}t) + \sqrt{3} \cdot e^{t} \cdot \cos(\sqrt{3}t) + \sqrt{3} \cdot e^{t} \cdot \cos(\sqrt{3}t) - 3 \cdot e^{t} \cdot \sin(\sqrt{3}t)) \cdot W_3.$$

$$W_1 = \det\left(\begin{vmatrix} e^{t}\cos(\sqrt{3}t) & e^{t} \cdot \sin(\sqrt{3}t) \\ e^{t}\cos(\sqrt{3}t) - \sqrt{3} \cdot e^{t} \cdot \sin(\sqrt{3}t) & e^{t} \cdot \sin(\sqrt{3}t) + \sqrt{3} \cdot e^{t} \cdot \cos(\sqrt{3}t) \end{vmatrix}\right).$$

$= e^{t} \cdot \cos(\sqrt{3}t) \cdot [e^{t} \cdot \sin(\sqrt{3}t) + \sqrt{3} \cdot e^{t} \cdot \cos(\sqrt{3}t)] - e^{t} \cdot \sin(\sqrt{3}t) \cdot [e^{t} \cdot \cos(\sqrt{3}t) - \sqrt{3} \cdot$

$e^{t} \cdot \sin(\sqrt{3}t)]$.

$= e^{2t} \cdot \sin(\sqrt{3}t) \cdot \cos(\sqrt{3}t) + \sqrt{3} \cdot e^{2t} \cdot \cos^2(\sqrt{3}t) - e^{2t} \sin(\sqrt{3}t) \cdot \cos(\sqrt{3}t) + \sqrt{3} \cdot e^{2t} \cdot \sin^2(\sqrt{3}t)$

$$= \sqrt{3}\cdot e^{2t}$$

$$\omega_2 = -\det\left(\begin{vmatrix} 1 & e^t\cdot\sin(\sqrt{3}t) \\ 0 & e^t\cdot\sin(\sqrt{3}t)+\sqrt{3}\cdot e^t\cdot\cos(\sqrt{3}t)\end{vmatrix}\right)$$

$$= -e^t\cdot\sin(\sqrt{3}t) - \sqrt{3}\cdot e^t\cdot\cos(\sqrt{3}t).$$

$$\omega_3 = \det\left(\begin{vmatrix} 1 & e^t\cdot\cos(\sqrt{3}t). \\ 0 & e^t\cdot\cos(\sqrt{3}t)-\sqrt{3}\cdot e^t\cdot\sin(\sqrt{3}t)\end{vmatrix}\right)$$

$$= e^t\cdot\cos(\sqrt{3}t) - \sqrt{3}\cdot e^t\cdot\sin(\sqrt{3}t).$$

$$\therefore \omega = \omega_1 + (e^t\cdot\cos(\sqrt{3}t)-\sqrt{3}\cdot e^t\cdot\sin(\sqrt{3}t)-\sqrt{3}e^t\cdot\sin(\sqrt{3}t)-\sqrt{3}\cdot e^t\cos(\sqrt{3}t))\cdot\omega_2 +$$

$$(e^t\sin(\sqrt{3}t)+\sqrt{3}\cdot e^t\cdot\cos(\sqrt{3}t)+\sqrt{3}\cdot e^t\cdot\cos(\sqrt{3}t)-3\cdot e^t\cdot\sin(\sqrt{3}t))\cdot\omega_3$$

$$= \sqrt{3}\cdot e^{2t}+[(1-\sqrt{3})\cdot e^t\cdot\cos(\sqrt{3}t)-2\sqrt{3}\cdot e^t\cdot\sin(\sqrt{3}t)]\cdot[-e^t\cdot\sin(\sqrt{3}t)-$$

$$\sqrt{3}\cdot e^t\cdot\cos(\sqrt{3}t)]+[-2\cdot e^t\cdot\sin(\sqrt{3}t)+2\sqrt{3}\cdot e^t\cdot\cos(\sqrt{3}t)]\cdot[e^t\cdot\cos(\sqrt{3}t)-$$

$$\sqrt{3}\cdot e^t\cdot\sin(\sqrt{3}t)]$$

$$= \cdots\cdots$$

iii) $u_1 = \int \frac{g(t)\cdot\omega_1}{\omega}\,dt$; $u_2 = \int \frac{g(t)\cdot\omega_2}{\omega}\,dt$; $u_3 = \int\frac{g(t)\cdot\omega_3}{\omega}\,dt$.

iv). $y(t) = u_1\cdot y_1 + u_2\cdot y_2 + u_3\cdot y_3$.

e.g. $y''' + y'' - y' - y = 8e^{-t}$.

Variation of Parameter:

i). $r^3+r^2-r-1=0$.

$\Rightarrow (r-1)\cdot(r^2+2r+1)=0$.

$\Rightarrow (r-1)\cdot(r+1)^2=0$.

$r_1 = 1$. $r_{2,3} = -1$.

$y_1 = e^t$; $y_2 = e^{-t}$; $y_3 = t\cdot e^{-t}$.

ii) $\omega = \det\left(\begin{vmatrix} e^t & e^{-t} & t\cdot e^{-t} \\ e^t & -e^{-t} & e^{-t}-t e^{-t} \\ e^t & e^{-t} & -2e^{-t}+t\cdot e^{-t}\end{vmatrix}\right)$

$$= e^t\cdot\omega_1 + e^{-t}\cdot\omega_2 + (-2e^{-t}+t\cdot e^{-t})\cdot\omega_3.$$

$\omega_1 = \det\left(\begin{vmatrix} e^{-t} & t\cdot e^{-t} \\ -e^{-t} & e^{-t}-te^{-t}\end{vmatrix}\right) = e^{-2t} - t\cdot e^{-2t} + t\cdot e^{-2t} = e^{-2t}$.

$\omega_2 = -\det\left(\begin{vmatrix} e^t & t\cdot e^{-t} \\ e^t & e^{-t}-te^{-t}\end{vmatrix}\right) = -(1-t-t) = 2t-1$

$$W_3 = \det\left(\begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix}\right) = -1-1 = -2.$$

$$\therefore W = e^t\cdot(e^{-2t}) + e^{-t}\cdot(2t-1) + (-2e^{-t}+t\cdot e^{-t})\cdot(-2)$$

$$= e^{-t} + 2t\cdot e^{-t} - e^{-t} + 4e^{-t} - 2t\cdot e^{-t}$$

$$= 4e^{-t}.$$

iii) $$u_1 = \int \frac{g(t)\cdot W_1}{W}\,dt = \int \frac{8e^{-t}\cdot e^{-2t}}{4e^{-t}}\,dt = \int 2\cdot e^{-2t}\,dt$$

$$= -e^{-2t} + C_1$$

$$u_2 = \int \frac{g(t)\cdot W_2}{W}\,dt = \int \frac{8e^{-t}\cdot(2t-1)}{4e^{-t}}\,dt = \int 2(2t-1)\,dt$$

$$= 2t^2 - 2t + C_2$$

$$u_3 = \int \frac{g(t)\cdot W_3}{W}\,dt = \int \frac{8e^{-t}\cdot(-2)}{4e^{-t}}\,dt = \int -4\,dt$$

$$= -4t + C_3.$$

iv) $$y(t) = (-e^{-2t}+C_1)\cdot e^t + (2t^2-2t+C_2)\cdot e^{-t} + (-4t+C_3)\cdot t\cdot e^{-t}$$

$$= C_1\cdot e^t + C_2\cdot e^{-t} + C_3\cdot t\cdot e^{-t} + (-e^{-t} + 2t^2\cdot e^{-t} - 2t\cdot e^{-t} - 4t^2\cdot e^{-t})$$

$$= C_1\cdot e^t + C_2\cdot e^{-t} + C_3\cdot t\cdot e^{-t} + e^{-t}\cdot(-2t^2 - 2t - 1).$$

$$= C_1^*\cdot e^t + C_2^*\cdot e^{-t} + C_3^*\cdot t\cdot e^{-t} \underbrace{-2t^2\cdot e^{-t}}_{y_p(t)}.$$