Problem Set 3.

Xuanqi Wei

1009353209

Q2.

(a). WTS: $2 < \sqrt{5} < 3$

Since $\forall x, y \in \mathbb{R}$, $0 \le x < y$ gives $\sqrt{x} < \sqrt{y}$, we have,

From the property of natural num., gives, $4 < 5 < 9$.

From the definition of square root. gives. $\sqrt{4} < \sqrt{5} < \sqrt{9}$.

$$\Rightarrow 2 < \sqrt{5} < 3 \quad \blacksquare$$

(b). Let $m, n > 0$.

Let $\frac{m}{n} = \sqrt{5}$.

From (a). gives. $2 < \sqrt{5} < 3$, since $\frac{m}{n} = \sqrt{5}$, gives.

$$2 < \sqrt{5} = \frac{m}{n} < 3. \text{ (since } m, n > 0\text{, from property of inequality).}$$

$$\Rightarrow 2 \cdot n < m < 3n.$$

Also gives: $\frac{2}{m} < \frac{1}{n} < \frac{3}{m} \Rightarrow \frac{m}{3} < n < \frac{m}{2}$

Thus, for m. L.B. is $2n$. U.B is $3n$; for n. L.B is $\frac{m}{3}$. U.B is $\frac{m}{2}$.

(c). From $(\sqrt{5}+2)\cdot(\sqrt{5}-2) = 1$. Since $(\sqrt{5}-2) \neq 0$; $(\sqrt{5}+2) \neq 0$, gives

① $\sqrt{5}+2 = \frac{1}{\sqrt{5}-2}$

$\Rightarrow \sqrt{5} = \frac{1-2(\sqrt{5}-2)}{\sqrt{5}-2}$

$\Rightarrow \sqrt{5} = \frac{5-2\sqrt{5}}{\sqrt{5}-2}$

② $\sqrt{5}-2 = \frac{1}{\sqrt{5}+2}$

$\Rightarrow \sqrt{5} = \frac{1+2(\sqrt{5}+2)}{\sqrt{5}+2}$

$\Rightarrow \sqrt{5} = \frac{5+2\sqrt{5}}{\sqrt{5}+2}$

(d). Let $m, n \in \mathbb{N}$. $m, n > 0$.

Let $\sqrt{5} = \frac{m}{n}$.

WTS: $\exists m', n' \in \mathbb{N}$, $m', n' > 0$. $n' < n$ s.t. $\sqrt{5} = \frac{m'}{n'}$.

I'll use ① from (c).

Since $\sqrt{5}=\frac{m}{n}$. gives. $\sqrt{5}=\frac{5-2\cdot\frac{m}{n}}{\frac{m}{n}-2}$.

$\Rightarrow \sqrt{5}=\frac{\frac{5n-2m}{n}}{\frac{m-2n}{n}}$ (Since $n>0$, time n for nominater and denominator).

$\Rightarrow \sqrt{5}=\frac{5n-2m}{m-2n}$.

Take $n'=m-2n$, since $m,n\in\mathbb{N}$, $n'\in\mathbb{Z}$.

WTS: $0<n'<n$.

Since from (b). gives. $2n<m<3n$.

$\Rightarrow 0<m-2n<n$. (Since $n'=m-2n$).

$\Rightarrow 0<n'<n$.

I've shown $0<n'<n$ ∴ $n'\in\mathbb{N}^+$

Take $m'=5n-2m$, since $m,n\in\mathbb{N}$, $m'\in\mathbb{Z}$.

WTS. $0<m'$.

Since $\sqrt{5}=\frac{m'}{n'}$, and $n'>0$, gives. $m'>0$. (accord to property of fraction).

I've shown $m'>0$, $m'\in\mathbb{N}^+$

Therefore. I've proved such n' and m' exist. ∎

(e). Let $m,n\in\mathbb{N}^+$

Assume for contradiction, $\sqrt{5}=\frac{m}{n}$.

Let $S=\{n\in\mathbb{N}^+ : \exists m\in\mathbb{N}^+, \sqrt{5}=\frac{m}{n}\}$.

Since by assumption, gives $S\neq\phi$. and $S\subseteq\mathbb{N}$, by WOP, gives.

$\exists n_0\in S, \forall n\in S, n_0\leq n$, which n_0 is the least element in S.

However, from (d), $\exists n'\in\mathbb{N}^+, \exists m'\in\mathbb{N}^+, \sqrt{5}=\frac{m'}{n'}$, which $n'\in S$ and $n'<n_0$, which contradicts to n_0 is the smallest element in S.

Therefore. S is an empty set and the assumption is false.

To provide a summary, since $\sqrt{5}$ is not a quotient of positive natural numbers, $\sqrt{5}$ is not a quotient with nominator 0, and $\sqrt{5}$ is not a quotient of negative natural numbers as we can multiply nominator and denominator by '-1', where $\sqrt{5}$ is not a quotient of integers. From definition of rational numbers, $\sqrt{5}$ is irrational. ∎