Homo System

1. Homo System: $\vec{x}' = P(t)\cdot\vec{x}$.

$$\Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}' = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

1) Wronskin: Let $x^{(1)}, \dots, x^{(n)}$ be n sol^{n} to the system.

$$W = \det(X(t)) = \det[x_1^{(1)} \cdots x_n^{(n)}] = \det\left(\begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix}\right).$$

① sol^{n} are lin. indep. iff Wronskian $\neq 0$.

② if sol^{n} are lin. indep, then each sol^{n} $\vec{x} = \vec{x}(t) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ of the system can be expressed in exactly one way.

$\{x^{(1)}, \dots, x^{(n)}\}$ is a fundamental set of sol^{n}.

2) Abel's Theorem: If $x^{(1)} \cdots x^{(n)}$ are sol^{n} on interval $\alpha < t < \beta$, $W[x^{(1)}, \dots, x^{(n)}]$ identically zero / never vanish.

① Abel's Formula: $W = c \cdot e^{\int (p_{11} + p_{22} + \cdots + p_{nn})\, dt}$.

2. 题

1) General sol^{n}.

① Coefficient Matrix A 求 eigenvalues & eigenvectors.
(快速写出见下题)

② 由 eigenvalue 写 general sol^{n}:

i) distinct $\lambda \in \mathbb{R}$: $\vec{x}^{(i)} = e^{\lambda_i t}\cdot \underbrace{\vec{v_i}}_{\text{eigenvector}}$.

$$\vec{x} = c_1 e^{\lambda_1 t}\cdot\vec{v_1} + \cdots + c_n\cdot e^{\lambda_n t}\cdot\vec{v_n}.$$

ii) repeated $\lambda \in \mathbb{R}$:

(a) one eigen-vec: 找 $\vec{w} = k\vec{v} + \vec{r}$.

$\rightarrow (A - \lambda I)\cdot\vec{w} = \vec{v}$ 求 $\vec{w}$.

$\rightarrow \vec{w} = k\vec{v} + \vec{r}$ 求出 $\vec{r}$.

$\rightarrow \vec{x} = c_1\cdot e^{\lambda t}\cdot\vec{v} + c_2\cdot e^{\lambda t}\cdot(t\vec{v} + \vec{r})$.

(b) two eigen-vec: $\vec{x} = c_1 e^{\lambda t} \cdot \vec{v_1} + c_2 \cdot e^{\lambda t} \cdot \vec{v_2}$.

iii). $\lambda \in \mathbb{C}$: 同一个λ，找对应 $\vec{v}$ 共用.

(a) 求 $e^{\lambda t} \cdot \vec{v} = e^{\alpha t} \cdot (\cos\beta t + i\sin\beta t) \cdot \vec{v}$.

(b) $\vec{x} = c_1 \cdot \underbrace{Re(e^{\lambda t} \cdot \vec{v})}_{e^{\alpha t} \cdot \cos(\beta t) \cdot \vec{v}} + c_2 \cdot \underbrace{Im(e^{\lambda t} \cdot \vec{v})}_{e^{\alpha t} \cdot \sin(\beta t) \cdot \vec{v}}$.

③ $\vec{x} = c_1 \cdot x^{(1)} + \cdots + c_n \cdot x^{(n)}$.

e.g. $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}$.

或使用 $(A-\lambda I)\vec{v}=0$ 也可求出.

i). $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$. $\Rightarrow A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda) + 4 = 0$.

$\therefore \lambda_1 = 2$. $\lambda_2 = -1$. $\Rightarrow -6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$.

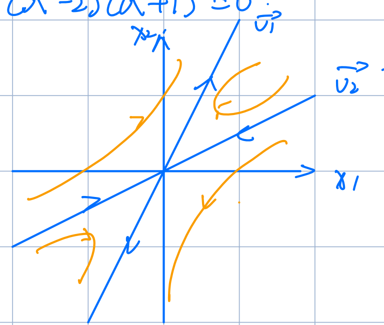
when $\lambda_1 = 2$, $A - \lambda_1 I = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$. gives. $\Rightarrow \lambda^2 - \lambda - 2 = 0$.

$\vec{v_1} = \underbrace{\begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}}$ $= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. $\Rightarrow (\lambda - 2)(\lambda + 1) = 0$.

快速求eigen-vector.

when $\lambda_2 = -1$. $A - \lambda_2 I = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$. gives.

$\underbrace{\vec{v_2} = \begin{pmatrix} b \\ -a \end{pmatrix}} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.



ii) $\vec{x}(t) = c_1 \cdot e^{2t} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \cdot e^{-t} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

e.g. $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$.

i). $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} = -(4-\lambda^2) + 5 = 0$.

$\therefore \lambda_1 = i$, $\lambda_2 = -i$ $\Rightarrow \lambda^2 = -1$.

when $\lambda_1 = i$. $A - \lambda I = \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}$, gives. $\vec{v_1} = \begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} -5 \\ -(2-i) \end{pmatrix} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$.

ii) $e^{\lambda t} \cdot \vec{v} = e^{\alpha t} \cdot (\cos\beta t + i\sin\beta t) \cdot \vec{v}$.

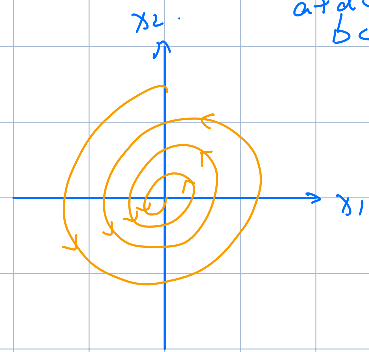
$= (\cos t + i\sin t) \cdot \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$

$= \begin{pmatrix} 5\cos t + 5\sin t \cdot i \\ 2\cos t - i\cos t + 2i\sin t + \sin t \end{pmatrix}$.

$= \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + \begin{pmatrix} 5 \cdot \sin t \\ 2\sin t - \cos t \end{pmatrix} \cdot i$.

iii) $\vec{x} = c_1 \cdot \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \cdot \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$

$a + d < 0$
$b < 0$



e.g. $\vec{x} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \vec{x}$.

i). $A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. $A - \lambda I = \begin{pmatrix} -1-\lambda & 0 \\ 1 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2 = 0$.

$\therefore \lambda_{1,2} = -1$.

when $\lambda = -1$, $A - \lambda I = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ [can't use $\begin{pmatrix} b \\ -a \end{pmatrix}$]

$(A-\lambda I)\cdot\vec{v} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = 0,\ v_2$ free. $\therefore$ let $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

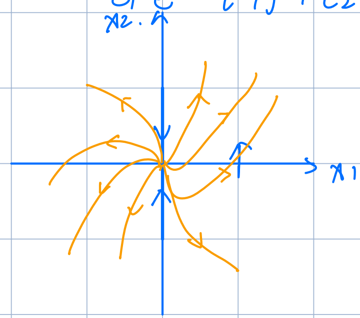
ii). $(A-\lambda I)\,\vec{\omega} = \vec{v}$

$\Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \omega_1 = 1.\ \omega_2$: free.

$\Rightarrow \vec{\omega} = \begin{pmatrix} 1 \\ \omega_2 \end{pmatrix} = k\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = k\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\therefore \vec{r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

iv). $\vec{x}(t) = c_1\cdot e^{-t}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2\cdot e^{-t}\cdot\left(t\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$.

$= c_1\cdot e^{-t}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2\cdot e^{-t}\cdot t\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2\cdot e^{-t}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.



$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

3) Fundamental Matrix: $\varphi(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_n^{(1)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}$ where $x_1 \cdots x_n$ are fundamental set of col$^{\underline{n}}$ for $x' = P(t)x$.

① $\vec{x}(t) = c_1 \cdot \vec{x}^{(1)} + \cdots + c_n \cdot x^{(n)} = \varphi(t) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \varphi(t) \cdot \vec{c}$.

② Sepecific Fundamental Matrix: $\Phi(t)$.

i) $\vec{x}(t) = \Phi(t) \cdot \vec{x}(0)$. ← Initial cond. 有时间. e.g. $\vec{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$; $\vec{x}(t) = \Phi(t) \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

ii) $\Phi(0) = I$ ← Identity.

iii) $\Phi'(t) = A \cdot \Phi(t)$. ← coeff.

4) 求 $\Phi(t)$:

法一: $\Phi(t) = \varphi(t) \cdot \varphi^{-1}(0)$.

① 解 homo, 写 $\varphi(t)$.

② 求 $\varphi(0) \longrightarrow \varphi^{-1}(0)$.

法二: Initial Condition Method.

① 解 homo, $\vec{x} = c_1 x^{(1)} + c_2 x^{(2)}$.

② Let $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 求对应 c_1, c_2, then $\begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix} = c_1 x^{(1)} + c_2 \cdot x^{(2)}$.

③ Let $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. 求对应 c_1, c_2, then $\begin{pmatrix} \phi_{21} \\ \phi_{22} \end{pmatrix} = c_1 \cdot x^{(1)} + c_2 \cdot x^{(2)}$.

④ $\Phi(t) = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$.

e.g. $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}$.

$\vec{x}(t) = c_1 \cdot e^{2t} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \cdot e^{-t} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

法一: $\varphi(t) = \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2 \cdot e^{-t} \end{pmatrix} \Rightarrow \varphi(0) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\det(\varphi(0)) = 4 - 1 = 3$.

$\therefore \varphi(0)^{-1} = \frac{1}{3} \cdot \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{\text{ad交换, bc}\cdot(-1)} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

$\therefore \Phi(t) = \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2 \cdot e^{-t} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

$= \begin{pmatrix} \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} & -\frac{2}{3}e^{2t} + \frac{2}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t} \end{pmatrix}$

法二: Let $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix} = -\frac{1}{3} e^{2t} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{2}{3} e^{-t} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}e^{2t} + \frac{2}{3}e^{-t} \\ -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t} \end{pmatrix}$

Let $\vec{x}(0) = \begin{pmatrix}0\\1\end{pmatrix}$, $c_1\begin{pmatrix}2\\1\end{pmatrix} + c_2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}\phi_{12}\\\phi_{22}\end{pmatrix} = \frac{2}{3}e^{2t}\begin{pmatrix}2\\1\end{pmatrix} - \frac{1}{3}e^{-t}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}\frac{4}{3}e^{2t} - \frac{1}{3}e^{-t}\\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t}\end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{pmatrix} = \cdots$$

e.g. $\vec{x}' = \begin{pmatrix}2 & -5\\1 & -2\end{pmatrix}\vec{x}$.

$$\vec{x} = c_1\cdot\begin{pmatrix}5\cos t\\2\cos t+\sin t\end{pmatrix} + c_2\cdot\begin{pmatrix}5\sin t\\2\sin t-\cos t\end{pmatrix}$$

$$\psi(t) = \begin{pmatrix}5\cos t & 5\sin t\\ 2\cos t+\sin t & 2\sin t-\cos t\end{pmatrix}.$$

$$\psi(0) = \begin{pmatrix}5 & 0\\2 & -1\end{pmatrix}. \Rightarrow \det(\psi(0)) = -5$$

$$\Rightarrow \psi(0)^{-1} = -\frac{1}{5}\cdot\begin{pmatrix}-1 & 0\\-2 & 5\end{pmatrix} = \begin{pmatrix}\frac{1}{5} & 0\\ \frac{2}{5} & -1\end{pmatrix}$$

$$\therefore \Phi(t) = \psi(t)\cdot\psi(0)^{-1}$$

$$= \begin{pmatrix}5\cos t & 5\sin t\\ 2\cos t+\sin t & 2\sin t-\cos t\end{pmatrix}\begin{pmatrix}\frac{1}{5} & 0\\ \frac{2}{5} & -1\end{pmatrix}$$

$$= \begin{pmatrix}\cos t+2\sin t & -5\sin t\\ \sin t & \cos t-2\sin t\end{pmatrix}$$

e.g. $\vec{x} = \begin{pmatrix}-1 & 0\\1 & -1\end{pmatrix}\vec{x}$.

$$\vec{x}(t) = c_1\cdot e^{-t}\cdot\begin{pmatrix}0\\1\end{pmatrix} + c_2\cdot e^{-t}\cdot t\cdot\begin{pmatrix}0\\1\end{pmatrix} + c_2\cdot e^{-t}\cdot\begin{pmatrix}1\\0\end{pmatrix}.$$

$$\psi(t) = \begin{pmatrix}0 & e^{-t}\\ e^{-t} & t\cdot e^{-t}\end{pmatrix} \Rightarrow \psi(0) = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}.$$

$$\Rightarrow \det(\psi(0)) = -1 \Rightarrow \psi(0)^{-1} = \frac{1}{-1}\cdot\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}.$$

$$\Phi(t) = \psi(t)\cdot\psi(0)^{-1}$$

$$= \begin{pmatrix}0 & e^{-t}\\ e^{-t} & t\cdot e^{-t}\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}e^{-t} & 0\\ t\cdot e^{-t} & e^{-t}\end{pmatrix}$$