Unit 2(a) Lecture Notes for MAT224

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24-26 January 2023

§1 1.6 Bases and Dimension Continued...

Write two definitions for a set S to be a basis for a vector space V.

1. A set S is a basis for a vector space V if and only if every element of V can be expressed as a linear combination of the vectors in S, and S is linearly independent.

2. A set S is a basis for a vector space V if and only if S spans V and S is linearly independent.

Consider the statements of theorems: 1.6.3, 1.6.6, and 1.6.10, 1.6.14, 1.6.18, Lemma 1.6.8, and Corollary 1.6.11.

How much can they tell you about the answers to the following questions? Let $V = span\{s_1, s_2, \ldots, s_n\}$ for some $n \in \mathbb{N}$

(a) Does V have at least one basis?

Yes, V has at least one basis. This is because any spanning set can be reduced to a basis by removing any linearly dependent vectors. The resulting set will still span V and will be linearly independent, making it a basis for V. If the original set is already linearly independent, then it is already a basis for V. Therefore, every vector space has at least one basis.

(b) Do all bases of a given vector space have the same number of elements?

Yes, all bases of a given vector space have the same number of elements, which is called the dimension of the vector space. This is a fundamental result in linear algebra known as the dimension theorem.

(c) If a subspace W of vector space V has a basis, can that basis be extended (have vector(s) added to it) to a basis for all of V?

Yes, it is always possible to extend a basis of a subspace W to a basis of the larger vector space V. This is known as the "basis extension theorem" or 'Steinitz exchange lemma'. The basic idea behind the theorem is that we can take any linearly independent set of vectors in V that is not already in the span of the basis of W, and then replace some of the vectors in the basis of W with the new linearly independent vectors to get a new basis for V. The theorem provides a systematic way to make this replacement while still maintaining linear independence and ensuring that the resulting set spans all of V.

(d) Does it make more sense to talk about **a** dimension of vector space V or **the** dimension of vector space V? In other words, is there more than one candidate for the value of $dim(V)$?

It makes more sense to talk about 'the' dimension of vector space V. The dimension of a vector space is a well-defined concept, and it is unique for a given vector space. While different bases of V may have different numbers of vectors, they will all have the same number of vectors, which is the dimension of V. Therefore, we can talk about 'the' dimension of V rather than 'a' dimension.

§2 2.1 Linear Transformations

Give two definitions for “linear transformation”.

1. A linear transformation is a function $T : V \to W$ between two vector spaces V and W that preserves the operations of addition and scalar multiplication. In other words, for any vectors $\mathbf{u}, \mathbf{v} \in V$ and any scalar $c \in \mathbb{R}$, the following properties hold:
   a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
   b) $T(c\mathbf{u}) = cT(\mathbf{u})$
2. A linear transformation is a function $T : V \to W$ that satisfies the following two properties for any vectors $\mathbf{u}, \mathbf{v} \in V$ and any scalars $c, d \in \mathbb{R}$:
   a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
   b) $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$

Is it possible to determine the pre-image of a vector $\vec{w} \in W$, if you know its image? In other words, let $T : V \to W$ be a linear transformation and let $\vec{w} \in T(W)$. So $T(v) = w$ for some $v \in V$. Can we tell which $v \in V$ is sent to W by V.
Not necessarily. If $\vec{w}$ is in the range of T, then there exists at least one vector $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. However, there may be other vectors in V that are also mapped to $\vec{w}$ by T. In fact, if T is not injective (i.e., if there exist distinct vectors $\vec{v}_1, \vec{v}_2 \in V$ such that $T(\vec{v}_1) = T(\vec{v}_2) = \vec{w}$), then there are infinitely many vectors in V that are mapped to $\vec{w}$ by T. Therefore, without additional information about T or $\vec{w}$, we cannot determine the pre-image of $\vec{w}$ with certainty.

Consider Proposition 2.1.14. It suggests that, if we know the value of $T(b_i)$ for every $b_1, b_2, \ldots, b_n$ in a basis B of V and linear transformation $T : V \to W$, then we can find the value of $T(v)$ for any $v \in V$. Explain how we would do this.

For example, if $T(1) = 5$, $T(x) = 1 + 2x$, and $T(x^2) = 1 - 2x + 3x^2$, what is the value of $T(2x + 4)$?
To find the value of $T(v)$ for any $v \in V$, we first express v as a linear combination of the basis vectors $b_1, b_2, \ldots, b_n$ of V. That is, we write $v = c_1b_1 + c_2b_2 + \cdots + c_nb_n$ for some scalars $c_1, c_2, \ldots, c_n$.
Then, we can use the linearity of T to compute $T(v)$. Specifically, we have:

$$T(v) = T(c_1b_1 + c_2b_2 + \ldots + c_nb_n) = c_1T(b_1) + c_2T(b_2) + \ldots + c_nT(b_n)$$

So, to find the value of $T(v)$ for any $v \in V$, we simply need to know the values of $T(b_i)$ for every b_i in a basis B of V.

In the given example, let $B = 1, x, x^2$ be a basis of V. We are given that $T(1) = 5$, $T(x) = 1 + 2x$, and $T(x^2) = 1 - 2x + 3x^2$. To find $T(2x + 4)$, we express $2x + 4$ as a linear combination of the basis vectors in B:

$$2X + 4 = 2X + 0 \cdot 1 + 0 \cdot x^2 = 2x \cdot 1 + 0 \cdot (1 + 2x) + 0 \cdot (1 - 2x + 3x^2)$$

Then, we use the linearity of T to compute $T(2x + 4)$:

$$\begin{aligned} T(2x + 4) &= T(2x \cdot 1 + 0 \cdot (1 + 2x) + 0 \cdot (1 - 2x + 3x^2)) \\ &= 2xT(1) + 0 \cdot T(x) + 0 \cdot T(x^2) \\ &= 2x \cdot 5 + 0 \cdot (1 + 2x) + 0 \cdot (1 - 2x + 3x^2) \\ &= 10 \end{aligned}$$

Therefore, we have $T(2x + 4) = 10$.

§3 2.2 Linear Transformations Between Finite Dimensional Vector Spaces

Give three examples of linear transformations $T : V_1 \to V_2$.

1. Let V_1 and V_2 be Euclidean spaces of dimension n and m, respectively, and let A be an $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. This is an example of a linear transformation because it satisfies the two properties of additivity and homogeneity: $T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$ and $T(k\mathbf{x}) = A(k\mathbf{x}) = kA\mathbf{x} = kT(\mathbf{x})$ for all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and scalars k.

2. Let V_1 and V_2 be the spaces of polynomials of degree at most 2 and 3, respectively. Define $T : V_1 \to V_2$ by $T(p(x)) = xp(x)$, where $p(x) = a_0 + a_1x + a_2x^2$ is a polynomial in V_1. This is a linear transformation because it satisfies additivity and homogeneity: $T(p(x) + q(x)) = x(p(x) + q(x)) = xp(x) + xq(x) = T(p(x)) + T(q(x))$ and $T(kp(x)) = kxp(x) = kT(p(x))$ for all polynomials $p(x), q(x) \in V_1$ and scalars k.

3. Let V_1 and V_2 be the spaces of continuous functions on $[0, 1]$ and $[1, 2]$, respectively. Define $T : V_1 \to V_2$ by $T(f(x)) = f(x + 1)$, where $f(x)$ is a function in V_1. This is a linear transformation because it satisfies additivity and homogeneity: $T(f(x) + g(x)) = (f(x) + g(x) + 1) = f(x + 1) + g(x + 1) = T(f(x)) + T(g(x))$ and $T(kf(x)) = kf(x + 1) = kT(f(x))$ for all functions $f(x), g(x) \in V_1$ and scalars k.

Give an example of a vector space V, two bases $\alpha = \{a_1, a_2, \dots, a_n\}$ and $\beta = \{b_1, b_2, \dots, b_n\}$ of V, a transformation $T : V \to V$, and its matrix forms $[T]_\alpha^\beta$ and $[T]_\beta^\alpha$.

Let V be the vector space $\mathbb{R}^2$ over the field $\mathbb{R}$, and let $\alpha = a_1, a_2$ and $\beta = b_1, b_2$ be two bases of V defined as follows:

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; a_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; b_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; b_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Let $T : V \to V$ be a linear transformation defined by $T(x, y) = (x + y, y - x)$ for all $(x, y) \in V$. We want to find the matrix representations $[T]\alpha^\beta$ and $[T]\beta^\alpha$ of T with respect to the bases α and β.

To find $[T]\alpha^\beta$, we first find the images of the basis vectors under T:

$$T(a_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 1b_1 + 0b_2$$

$$T(a_2) = T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 1b_1 - 1b_2$$

Therefore, $[T]_\alpha^\beta = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

To find $[T]_\beta^\alpha$, we first find the pre-images of the basis vectors under T. Solving the equations $T(x, y) = b_1$ and $T(x, y) = b_2$, we get:

$$T^{-1}(b_1) = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix} \textit{ and } T^{-1}(b_2) = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$$

respectively. Therefore,

$$[T]_\beta^\alpha = (T^{-1}(b_1) \;\; T^{-1}(b_2)) = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

What is the process of getting from a linear transformation T to its matrix form $[T]_\alpha^\beta$? Is it possible to go backwards and determine T from its matrix?

To obtain the matrix form $[T]_\alpha^\beta$ of a linear transformation $T : V \to W$ with respect to bases α and β of V and W respectively, we follow these steps:

1. For each basis vector a_i in α, we apply T to it and express the result as a linear combination of the vectors in β. This gives us a column vector $[![T(a_i)]!]\beta$ in the matrix $[T]\alpha^\beta$.

2. We arrange the column vectors obtained in step 1 side by side to form the matrix $[T]_\alpha^\beta$.

To go backwards and determine T from its matrix $[T]_\alpha^\beta$, we use the following steps:

1. We choose a basis α for the domain V and a basis β for the codomain W.

2. We use the matrix $[T]\,\alpha^\beta$ to define a function $T\alpha^\beta : \mathbb{R}^n \to \mathbb{R}^m$ where n is the dimension of V and m is the dimension of W. This function takes a column vector in $\mathbb{R}^n$ as input and outputs a column vector in $\mathbb{R}^m$ obtained by multiplying the matrix $[T]_\alpha^\beta$ by the input column vector.

3. We define the function $T : V \to W$ by setting $T(a_i) = \sum_{j=1}^m [T]_{\alpha_j}^{\beta_i} w_j$, where α_j is the jth basis vector in α and w_j is the jth basis vector in β.