MAT246 Online Quiz 2

Xuanqi Wei 1009353209

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1 Question 1

Proof: Let $k \in \mathbb{N}$. Let $n \in \mathbb{N}$
Assume k does not divide n^2, which n^2 has no factorization, which $n^2 = k \cdot p$, where $p \in \mathbb{N}$. Meaning k is not a divisor of n^2.(Assumption 1)
Assume for contradiction: k can divide n, which n has a factorization $n = k \cdot q$, where $q \in \mathbb{N}$.
Since $n^2 = n \cdot n$ (from the definition of n^2), gives

$$\begin{aligned} n^2 &= n \cdot n \text{ (According to the definition of } n^2) \\ &= (k \cdot q) \cdot (k \cdot q) \text{ (According to the Assumption for Contradiction)} \\ &= k \cdot (q \cdot k \cdot q) \text{ (According to the Commutative Law of Multiplication)} \end{aligned}$$

Since $q \in \mathbb{N}$, $k \in \mathbb{N}$, we have $(q \cdot k \cdot q) \in \mathbb{N}$. Thus, $\exists p \in \mathbb{N},\ s.t.\ p = (q \cdot k \cdot q)$ where $n^2 = k \cdot p$, which is a factorization of n^2, contrdicts to the Assumption 1 that n^2 has no factorization.
Therefore, to conclude, for natural numbers k and n, if k does not divide n^2, then k cannot divide n either.

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2 Question 2

Proof: Let $k \in \mathbb{N}$.
Assume there are exactly k many natural numbers r, such that $1 \leq r \leq k$.
To prove there is exactly (k+1) many natural numbers r such that $1 \leq r \leq (k+1)$, there contains mainly parts, there are no less than $(k+1)$ many natural numbers r and there are no more than $(k+1)$ many natural numbers r.
Part 1: There are no less than $(k+1)$ many natural numbers r such that $1 \leq r \leq (k+1)$.
Considering the set of natural numbers between 1 and k, which $1 \leq r \leq k$, {1, 2, 3, ..., k}, which contains exactly k natural numbers. Extending the set listed above to include $(k+1)$, the set becoms {1, 2, 3, ..., k, (k+1) }. This set contains k natural numbers from the original set {1, 2, 3, ..., k} and one more element, which is $(k+1)$. Since the original set contains k natural numbers, and I've added one more $(k+1)$ to form the new set {1, 2, 3, ..., k, (k+1) }, there are no fewer than $(k+1)$ natural numbers r, which $1 \leq r \leq (k+1)$.
Part 2: There are no more than $(k+1)$ many natural numbers r such that $1 \leq r \leq (k+1)$.
Assume for contradiction: There are more than $(k+1)$ many natural numbers r such that $1 \leq r \leq (k+1)$.
Saying there are p natural numbers between 1 and $(k+1)$, according to the assuption for contradiction, $m > (k+1)$. Representing the the set of natural numbers between 1 and $(k+1)$ by using the index, gives: $\{x_1,\ x_2,\ x_3,\ ...,\ x_m\}$, such that $1 \leq x_1 < x_2 < ... < x_m \leq (k+1)$.
However, since all the numbers x_i are distinct, there are m numbers and $m > (k+1)$, this means that at least one of the numbers must be greater than (k+1), which contradicts to the assumption for contradiction as we assume all the numbers in $\{x_1,\ x_2,\ x_3,\ ...,\ x_m\}$ are between 1 and $(k+1)$.
Combining both proofs, we have shown that there are there are no less than $(k+1)$ many natural numbers r and there are no more than $(k+1)$ many natural numbers r such that $1 \leq r \leq (k+1)$, which there is exactly (k+1) many natural numbers r such that $1 \leq r \leq (k+1)$.

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3 Question 3

Proof: Let S be the set of all natural numbers for which the theorem, $\forall n \in \mathbb{N}$, there are exactly n many natural numbers r such that $1 \leq r \leq n$, is true. We want to show that S contains all of the natural numbers. We do this by showing that S has properties A and B.
For property A, we need to check that there is exactly one natural number r such that $1 \leq r \leq 1$. It's apparent that, in this case, $r = 1$, which there is exactly one natural number satisfying.
To verify property B, let k be in S. We must show that $(k+1)$ is in S. To state the procedure more clearly, I'm expanding the sentence. For a natural number k, assume there are exactly k many natural numbers r, such that $1 \leq r \leq k$. Show that there is exactly $(k+1)$ many natural numbers r such that $1 \leq r \leq (k+1)$.
We observed that for property B, we've already proved it in Question 2, which it's true.
Therefore, S is the set of natural numbers by the Principle of Mathematical Induction. To conclude, for each natural number n, there are exactly n many natural numbers r such that $1 \leq r \leq n$.

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