MAT 137
Tutorial #7– Linear Approximation and Newton's method
Nov 1/2 , 2022
Due on Thursday, Nov 3 by 11:59pm via GradeScope

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

I confirm that:

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- I understand that the collaboration of this worksheet is only among this group. I have not violated this rule while writing this worksheet.
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By signing this document, I agree that the statements above are true.

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1. Let $a \in \mathbb{R}$. Let f be a function that is differentiable at a. Define the function g by $g(x) = x^2 f(x)$. Prove that g is differentiable at a and $g'(a) = 2af(a) + a^2 f'(a)$.
Write a proof directly from the definition of the derivative. Do not use any differentiation rules, e.g. quotient rule or chain rule.

(a) Write out the definition of $f'(a)$ as a limit. Does $f'(a)$ exist?

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon$.

Yes. because f is a function that is differentiable at a.

(b) Prove that $\lim_{x \to a} f(x) = f(a)$. The proof should be short like one or two sentences.

We know If f is differentiable at a then f is continuous at a

so $\lim_{x \to a} f(x) = f(a)$ exist

(c) Prove that $g(x)$ is defined on an interval centered at a. Hint: can you prove $f(x)$ is defined near a and at a? Write out the epsilon-delta definition of (b) and you will get some idea.

Let $x \in \mathbb{R}$, $a \in \mathbb{R}$. Let $\varepsilon \in \mathbb{R}$, $\delta \in \mathbb{R}$.
Call $h(x) = x^2$. $g(x) = h(x) f(x)$.
Since $h(x)$ and $f(x)$ are both continous at a:

$\lim_{x \to a} h(x) = h(a)$ and $\lim_{x \to a} f(x) = f(a)$

Therefore, $\lim_{x \to a} g(x) = \lim_{x \to a} [h(x) f(x)] = \lim_{x \to a} h(x) \cdot \lim_{x \to a} f(x) = f(a) h(a) = g(a)$

Equivalently, $g(x)$ is continous at a. Therefore:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - a| < \delta \Rightarrow |g(x) - g(a)| < \varepsilon$

Keeping this δ, we have that $\forall x \in (a - \delta, a + \delta)$, $g(x)$ exists, which means $g(x)$ is defined on an interval centered at a.

(d) Prove that $g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$ exists and satisfies the desired formula.
Hint: use the same trick in the proof of the product rule and quotient rule. Remember to check that all the limits exist before applying limit laws.

Pf: Let $x \in \mathbb{R}$, $a \in \mathbb{R}$.

Call $h(x) = x^2$. Verify that $h(x)$ is differentiable at a:

$$\lim_{x\to a} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a}(x+a) = 2a$$

Therefore, $h'(a)$ exists and $h'(a) = 2a$

Since $g(x) = h(x) f(x)$,

$$g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{h(x)f(x) - h(a)f(a)}{x-a} = \lim_{x\to a} \frac{h(x)f(x) + h(x)f(a) - h(x)f(a) - h(a)f(a)}{x-a}$$

$$= \lim_{x\to a}\left[h(x)\frac{f(x)-f(a)}{x-a} + f(a)\frac{h(x)-h(a)}{x-a}\right]$$

Since $h(x)$ and $f(x)$ is continous at a, $\lim_{x\to a} h(x) = h(a)$, $f(a)$ exists and $\lim_{x\to a} f(a) = f(a)$.

Since $f(x)$ ans $h(x)$ is differentiable at a, $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to a}\frac{h(x)-h(a)}{x-a}$ exists,

Therefore, by limit law, $g'(a) = \lim_{x\to a} h(x) \cdot \lim_{x\to a}\frac{f(x)-f(a)}{x-a} + \lim_{x\to a} f(a) \cdot \lim_{x\to a}\frac{h(x)-h(a)}{x-a}$

$= h(a)\cdot f'(a) + f(a)\cdot h'(a) = a^2 f'(a) + 2af(a)$

Since $f(x)$ is differentiable at a, $f(a)$ and $f'(a)$ are defined.

Therefore, $g'(a)$ exists and $g'(a) = 2af(a) + a^2 f'(a)$

$\blacksquare$

2. Find an expression for $\frac{dy}{dx}$ by differentiation implicitly:$e^x \sin(y) = x + \sin(xy) - e$.
Then find the tangent line to this curve at $(e, 0)$.

$$e^x \sin y + e^x \cos y \frac{dy}{dx} = 1 + \cos(xy) \cdot [xy]'$$

$$\Rightarrow e^x \sin y + e^x \cos y \frac{dy}{dx} = 1 + \cos(xy) \left(y + x \cdot \frac{dy}{dx} \right)$$

$$\Rightarrow e^x \sin y + e^x \cos y \frac{dy}{dx} = 1 + \cos(xy) \cdot y + \cos(xy) \cdot x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} (e^x \cos y - \cos(xy) \cdot x) = 1 + \cos(xy) \cdot y - e^x \sin y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + \cos(xy) \cdot y - e^x \sin y}{e^x \cos y - \cos(xy) \cdot x}$$

substitute $(e, 0)$ gives.

$$\frac{dy}{dx} = \frac{1 + \cos 0 \cdot 0 - e^e \cdot \sin 0}{e^e \cos 0 - \cos 0 \cdot e}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^e - e}$$

Thus. substitute $\frac{dy}{dx} = \frac{1}{e^e - e}$ and $(e, 0)$ gives.

$$y - 0 = \frac{dy}{dx}(x - e)$$

$$\Rightarrow y = \frac{1}{e^e - e} x - \frac{1}{e^e - e} \cdot e$$

$$\Rightarrow y = \frac{1}{e^e - e} x - \frac{1}{e^{e-1} - 1}$$