MAT 137Y
Proof writing practice

These problems are for students who would like more practice writing proofs. None of these problems use any calculus, or linear algebra, or any advanced mathematics. They are just to practice plain proof writing.

1. Prove that if m and n are each divisible by 3, then so is $m + n$.

2. Prove that the product of an even integer and an odd integer is even.
   $2k$ $2t+1$

3. Prove that every integer that is divisible by 10 must be an even number.
   $10k$ $2 \mid (---)$.

4. Suppose that $a \in \mathbb{Z}$. Prove that if a is odd, then a^2 is odd.
   $a = 2k+1$ $a^2 = ----$

5. Suppose that $a \in \mathbb{Z}$. Prove that if a^2 is even, then a is even. negation X

6. Prove that there is no smallest, positive rational number. negation X

TUT.

7. Prove that for all $e > 0$, there exists $d > 0$ such that for all $x > 0$, if $x < d$, then $x^2 < e$.
   $\forall e>0.\ \exists d>0$ s.t. $\forall x>0,\ (x<d) \Rightarrow x^2<e$.
   $\exists e>0.\ \forall d>0.\ \exists x>0,\ x<d\ \&\ x^2 \geq e$.

8. Prove that the sum of any two rational numbers is a rational number. $\frac{p}{q} + \frac{s}{t}$ $p \in \mathbb{Z}$. $q \in \mathbb{Z}$ $q \neq 0$

TUT. 过程.

9. Prove that the sum of a rational number and an irrational number is irrational. $\Rightarrow$ 反证法.

10. Prove that the sum of two irrational numbers can be either rational or irrational. Why is it okay to prove "by example" here, whereas it is not okay to prove "by example" in general?

11. Prove that if n is an odd integer, then $n^2 - 1$ is a multiple of 8.
    $2k+1$

12. Prove that $\forall x \in [0, 1),\ \exists y \in [0, 1)$ such that $x < y$. $y = x + (1-x) \times 0.5$

13. Prove that $\exists y \in \mathbb{R}$ such that $\forall x \in [0, 2],\ x^2 + 1 < y$. $y = 5$

14. Prove that for every real number $x > 0$ and for every natural number $n \geq 2$,
$$(1 + x)^n > 1 + nx.$$
    TUT. $\forall x>0,\ x \in \mathbb{R}$. $\forall n \geq 2,\ n \in \mathbb{N}$

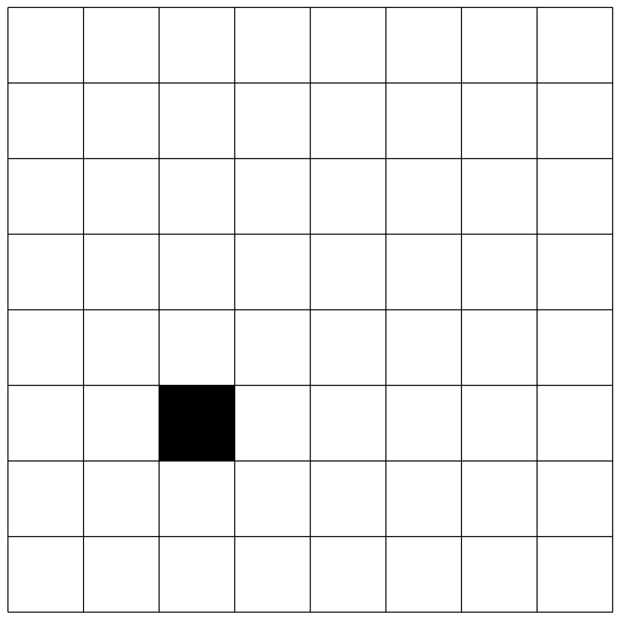
15. We want to calculate a formula for the number
$$A_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}.$$
where n is a natural number. Your formula, of course, will depend on n.

    (a) Calculate the first few values (A_1, A_2, A_3, A_4, ...). Then make a conjecture for the value of A_n.

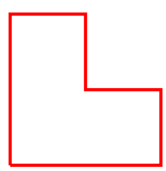
(b) State your conjecture as a theorem and prove it by induction.

16. We have a 2^N-by-2^N grid, where N is a positive integer, with exactly one square shaded black. For example:

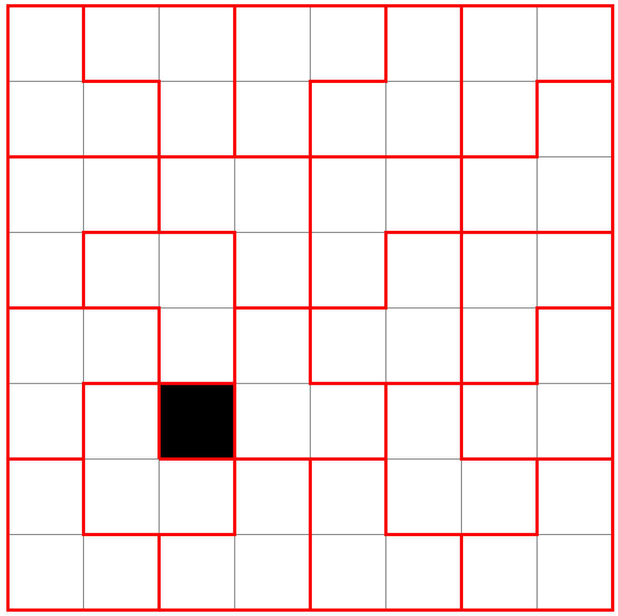
TUT



We want to tile the region (except the black square) with L-shaped pieces like this



For example, the grid above can be tiled like this:



Prove that it is always possible to do this, for every positive integer N and no matter where the initial black square is.

Hint: Write a proof by induction on N.

how to express it in math form.

is the core of mathematical induction

1. WTS: $m \bmod (3) \equiv 0$, $n \bmod (3) \equiv 0 \Rightarrow (m+n) \bmod (3) \equiv 0$

Pf. Let $m = 3k$. $k \in \mathbb{Z}$; $n = 3t$. $t \in \mathbb{Z}$

Then $m+n = 3k+3t = 3(k+t)$.

Since. $k \in \mathbb{Z}$, $t \in \mathbb{Z}$. $\frac{m+n}{3} = k+t \in \mathbb{Z}$.

To conclude. ......

7. Let $e > 0$.

Take $d > 0$., $d = \sqrt{e}$.

Let $x > 0$. $x < d = \sqrt{e}$.

Therefore. $x^2 < e$ ∎

14. WTS: $\forall x > 0. x \in \mathbb{R}$; $\forall n \geq 2. n \in \mathbb{N}$, $(1+x)^n > 1+nx$

Let $x > 0$. $x \in \mathbb{R}$.

Let $n \geq 2$. $n \in \mathbb{N}$

$(1+x)^n = \underbrace{(1+x)(1+x)(1+x) \cdots\cdots (1+x)}_{n}$.

Take 1 from all brackets. gives: $\underbrace{1 \times 1 \times 1 \times \cdots\cdots \times 1}_{n}$

Take 1 'x' and 1 from the other brackets gives. $\underbrace{x \times \underbrace{1 \times 1 \times \cdots\cdots \times 1}_{n-1} + x \times \underbrace{1 \times 1 \times \cdots\cdots \times 1}_{n-1} + \cdots\cdots + x \times \underbrace{1 \times 1 \times \cdots\cdots \times 1}_{n-1}}_{n} = nx$.

Take 2 'x's and 1 from ...... gives.

$x^2 \times \underbrace{1 \times 1 \times \cdots\cdots \times 1}_{n-2} + x^2 \times \underbrace{1 \times 1 \times \cdots\cdots \times 1}_{n-2} + x^2 \times \underbrace{1 \times 1 \times \cdots\cdots \times 1}_{n-2} = \frac{n(n-1)}{2} x^2$.

Therefore. $(1+x)^n$

$= 1 + nx + \frac{n(n-1)}{2} x^2 + \cdots\cdots$

Since $x > 0$, $n \geq 2$. $1+nx+\frac{n(n-1)}{2} x^2 > 1+nx$.

Since. $(1+x)^n > 1+nx+\frac{n(n-1)}{2} x^2$. gives. $(1+x)^n > 1+nx$. ∎

15. (a). $A_1 = \frac{1}{(1+1)!} = \frac{1}{2!} = \frac{1}{2}$

$A_2 = \frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{2 \times 3} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$A_3 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{1}{2} + \frac{2}{2 \times 3} + \frac{3}{2 \times 3 \times 4} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{23}{24}$.

$A_4 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{30} = \frac{119}{120}$

$\vdots$

$A_n = \frac{(n+1)! - 1}{(n+1)!}$

(b) basic. $A_1 = \frac{2!-1}{2!} = \frac{1}{2}$

Assume. $A_n = \frac{(n+1)!-1}{(n+1)!}$, $A_{n+1} = \frac{(n+2)!-1}{(n+2)!}$.

$$A_{n+1} = A_n + \frac{n+1}{(n+2)!}$$

$$= \frac{(n+1)!-1}{(n+1)!} + \frac{n+1}{(n+2)!}$$

$$= \frac{(n+2)[(n+1)!-1] + n+1}{(n+2)!}$$

$$= \frac{(n+2)! - n - 2 + n + 1}{(n+2)!}$$

$$= \frac{(n+2)!-1}{(n+2)!} \qquad \blacksquare$$

16. 2^N - by - 2^N.

↳ $2^{2n} = 4 + 3 \cdot 2k$