Vector Geometry

$proj_X \vec{v} \in X$.
(X 内离 $\vec{v}$ 最近的 vector).

1. Projection (a vector).

Let X be a set. The projection of the vector $\vec{v}$ onto X is the closest point in X to $\vec{v}$, written $proj_X \vec{v}$.

e.g. ① $X = \{\begin{bmatrix}1\\3\end{bmatrix}\}$. $\vec{v} = \begin{bmatrix}2\\1\end{bmatrix}$.

$proj_X \vec{v} = \begin{bmatrix}1\\3\end{bmatrix}$

② $X = \{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}\}$. $\vec{v} = \begin{bmatrix}2\\1\end{bmatrix}$

$\|\begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\| = \|\begin{bmatrix}-1\\2\end{bmatrix}\| = \sqrt{5}$

$\|\begin{bmatrix}2\\-1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\| = \|\begin{bmatrix}0\\-2\end{bmatrix}\| = 2$

$\therefore proj_X \vec{v} = \begin{bmatrix}2\\-1\end{bmatrix}$.

③ $X = span\{\begin{bmatrix}1\\3\end{bmatrix}\}$. $\vec{v} = \begin{bmatrix}2\\1\end{bmatrix}$

$\vec{x} = t\begin{bmatrix}1\\3\end{bmatrix} \in X$

$$\|\vec{x} - \vec{v}\| = \left\|\begin{bmatrix}t-2\\3t-1\end{bmatrix}\right\| = \sqrt{(t-2)^2 + (3t-1)^2}$$
$$= \sqrt{t^2+4-4t+9t^2+1-6t}$$
$$= \sqrt{10t^2-10t+5}$$
$$= \sqrt{10(t-\tfrac{1}{2})^2+\tfrac{5}{2}}$$

$\therefore$ when $t = \frac{1}{2}$, distance minimized.

$\therefore proj_X \vec{v} = \frac{1}{2}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{3}{2}\end{bmatrix}$.

④ $X = span\{\begin{bmatrix}1\\3\end{bmatrix}\} + \{\begin{bmatrix}0\\1\end{bmatrix}\}$ $\vec{v} = \begin{bmatrix}2\\1\end{bmatrix}$.

$\vec{x} = t\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} \in X$

$$\|\vec{x} - \vec{v}\| = \left\|\begin{bmatrix}t-2\\3t\end{bmatrix}\right\| = \sqrt{(t-2)^2+(3t)^2}$$
$$= \sqrt{t^2+4-4t+9t^2}$$
$$= \sqrt{10t^2-4t+4}$$
$$= \sqrt{10(t-\tfrac{1}{5})^2+\tfrac{90}{25}}$$

$\therefore$ when $t = \frac{1}{5}$, distance minimized.

$\therefore \text{proj}_X \vec{v} = \frac{1}{5}\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{5}\\ \frac{8}{5}\end{bmatrix}$

⑤ $X = \text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\3\\2\end{bmatrix}\right\} + \left\{\begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$, $\vec{v} = \begin{bmatrix}2\\3\\1\end{bmatrix}$. $\text{proj}_X \vec{v} = ?$

$\longrightarrow$ $\vec{v} - \text{proj}_X \vec{v}$ is the normal vector. let $\text{proj}_X \vec{v} = \begin{bmatrix}x\\y\\z\end{bmatrix}$

$\vec{d_1} = \begin{bmatrix}1\\2\\3\end{bmatrix}$; $\vec{d_2} = \begin{bmatrix}4\\3\\2\end{bmatrix}$

$$\begin{cases}(\vec{v} - \text{proj}_X \vec{v})\,\vec{d_1} = 0\\ (\vec{v} - \text{proj}_X \vec{v})\,\vec{d_2} = 0\end{cases} \Rightarrow \begin{cases}\begin{bmatrix}2-x\\3-y\\1-z\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} = 0\\ \begin{bmatrix}2-x\\3-y\\1-z\end{bmatrix}\begin{bmatrix}4\\3\\2\end{bmatrix} = 0\end{cases} \Rightarrow \begin{cases}11 - x - 2y - 3z = 0\\ 19 - 4x - 3y - 2z = 0\end{cases}$$

$\longrightarrow$ $\text{proj}_X \vec{v}$ is on X.

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = s\begin{bmatrix}1\\2\\3\end{bmatrix} + t\begin{bmatrix}4\\3\\2\end{bmatrix} + \begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$\begin{cases}x = s + 4t\\ y = 2s + 3t + 1\\ z = 3s + 2t + 2\end{cases}$$

$\longrightarrow$ solve it

$$\begin{cases}x + 2y + 3z = 11\\ 4x + 3y + 2z = 19\\ x - s - 4t = 0\\ y - 2s - 3t = 1.\\ z - 3s - 2t = 2.\end{cases} \longrightarrow \left[\begin{array}{ccccc|c}1 & 2 & 3 & 0 & 0 & 11\\ 4 & 3 & 2 & 0 & 0 & 19\\ 1 & 0 & 0 & -1 & -4 & 0\\ 0 & 1 & 0 & -2 & -3 & 1\\ 0 & 0 & 1 & -3 & -2 & 2\end{array}\right]$$

$$\xrightarrow[r_3 - r_1]{r_2 - 4r_1} \left[\begin{array}{ccccc|c}1 & 2 & 3 & 0 & 0 & 11\\ 0 & -5 & -10 & 0 & 0 & 25\\ 0 & -2 & -3 & -1 & -4 & -11\\ 0 & 1 & 0 & -2 & -3 & 1\\ 0 & 0 & 1 & -3 & -2 & 2\end{array}\right]$$

$$\xrightarrow[r_4 + \frac{1}{5}r_2]{r_3 + 2r_4} \left[\begin{array}{ccccc|c}1 & 2 & 3 & 0 & 0 & 11\\ 0 & -5 & -10 & 0 & 0 & 25\\ 0 & 0 & -3 & -5 & -10 & -9.\\ 0 & 0 & -2 & 0 & 0 & 6\\ 0 & 0 & 1 & -3 & -2 & 2\end{array}\right]$$

$$\xrightarrow[2_5 + \frac{1}{2}r_4]{r_4 + 2r_5} \left[\begin{array}{ccccc|c}1 & 2 & 3 & 0 & 0 & 11\\ 0 & -5 & -10 & 0 & 0 & 25\\ 0 & 0 & -3 & -5 & -10 & -9\\ 0 & 0 & 0 & -6 & -4 & 10\\ 0 & 0 & 0 & -3 & -2 & 5\end{array}\right]$$

$$\xrightarrow{r_5 - \frac{1}{2}r_4} \left[\begin{array}{ccccc|c}1 & 2 & 3 & 0 & 0 & 11\\ 0 & -5 & -10 & 0 & 0 & 25\\ 0 & 0 & -3 & -5 & -10 & -9\\ 0 & 0 & 0 & -6 & -4 & 10\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right]$$

$$\xrightarrow[r_4 \times (-\frac{1}{6})]{r_2 \times (-\frac{1}{5}),\ r_3 \times (-\frac{1}{3}).} \left[\begin{array}{ccccc|c}1 & 2 & 3 & 0 & 0 & 11\\ 0 & 1 & 2 & 0 & 0 & -5\\ 0 & 0 & 1 & \frac{5}{3} & \frac{10}{3} & 3\\ 0 & 0 & 0 & 1 & \frac{2}{3} & -\frac{5}{3}\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right]$$

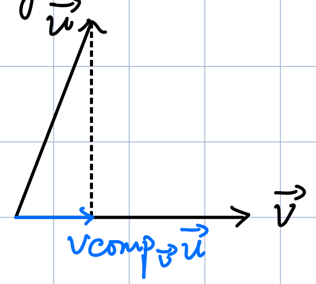
$$\xrightarrow{r_3 - \frac{5}{3}r_4} \left[\begin{array}{ccccc|c}1 & 2 & 3 & 0 & 0 & 11\\ 0 & 1 & 2 & 0 & 0 & -5\end{array}\right.$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 0 & 20/9 & 2/9 \\ 0 & 0 & 0 & 1 & 2/3 & -5/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[r_1-2r_2-3r_3]{r_2-2r_3} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 20/9 & 191/9 \\ 0 & 1 & 0 & 0 & -40/9 & -49/9 \\ 0 & 0 & 1 & 0 & 20/9 & 2/9 \\ 0 & 0 & 0 & 1 & 2/3 & -5/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \begin{cases} x = 191/9 - 20/9\,t \\ y = -49/9 + 40/9\,t \\ z = 2/9 - 20/9\,t \\ s = -5/3 - 2/3\,t. \end{cases} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -20/9 \\ 40/9 \\ -20/9 \end{bmatrix} + \begin{bmatrix} 191/9 \\ -49/9 \\ 2/9 \end{bmatrix}$$

2. Component

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be non-zero, then the component $\vec{u}$ in $\vec{v}$ direction is the vector in the direction of $\vec{v}$ so that $\vec{u} - \mathrm{vcomp}_{\vec v}\vec{u}$ is orthogonal to $\vec{v}$, written $\mathrm{comp}_{\vec v}\vec{u}$ and $\mathrm{vcomp}_{\vec v}\vec{u} = \underbrace{\frac{\vec u \cdot \vec v}{\|\vec v\|^2}}_{\text{系数}} \cdot \underbrace{\vec{v}}_{\text{跟}\vec v\text{同向}}$.



1) Properties.

① $\mathrm{vcomp}_{\vec u}\vec{v}$ is parallel to $\vec{u}$.

② $(\vec{u} - \mathrm{vcomp}_{\vec v}\vec{u}) \perp \vec{v}$

③ $\|\vec{u} - \mathrm{vcomp}_{\vec v}\vec{u}\| \le \|\vec{u} - \lambda\vec{v}\| \quad \forall \lambda \in \mathbb{R}.$

← $\vec v$上离$\vec u$最近的点.

2) Showing. $\mathrm{vcomp}_{\vec v}\vec{u} = \dfrac{\vec u \cdot \vec v}{\|\vec v\|^2} \cdot \vec{v}$

Since $\mathrm{vcomp}_{\vec v}\vec{u} = k\vec{v}$;

$$\vec{v} \cdot (\vec{u} - \mathrm{vcomp}_{\vec v}\vec{u}) = \vec{v}(\vec{u} - k\vec{v}) = 0$$

$$\Rightarrow \vec v \cdot \vec u - k\vec v \cdot \vec v = 0$$

$$\Rightarrow k = \frac{\vec v \cdot \vec u}{\|\vec v\|^2}$$

$$\therefore \mathrm{vcomp}_{\vec v}\vec{u} = k \cdot \vec{v} = \frac{\vec v \cdot \vec u}{\|\vec v\|^2} \cdot \vec{v}$$

only applies when its projecting onto the span of a single vector.

3. Projection and Component

$$\text{proj}_{\text{span}\{\vec{v}\}}\vec{u} = \text{vcomp}_{\vec{v}}\vec{u} \quad (\vec{v} \neq \vec{0}).$$

e.g. projection of $\vec{a} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ onto $L = \text{span}\left\{\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right\}$.

$$\text{proj}_{\text{span}\left\{\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right\}}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \text{vcomp}_{\begin{bmatrix} 1 \\ -4 \end{bmatrix}}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -25/17 \\ 100/17 \end{bmatrix}$$