Existence & Uniqueness Thm. for 1^{st}-order ODE.

1. E & U for FOLODE: If func. p & q are cts. on open interval $I: \alpha < t < \beta$ containing the point $t = t_0$ (I.V.), then $\exists$ a unique func. $y = \phi(t)$ that satisfies the differential eq. $y' + p(t)y = q(t)$ for $\forall t \in I$, and also satisfies the I.V. $y(t_0) = y_0$ where y_0 is an arbitrary prescribed I.V..

找p,q cts 且包括 I.V. 的区间.

e.g. Find an Interval $\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2 \end{cases}$ has a unique sol^{n}.

1) 判断是否 Linear: Linear.

2) Transform to std. form: $y' + \frac{2}{t} y = 4t$.

3). 列出 $p(t)$. $q(t)$ 及其 cts 条件.

$p(t) = \frac{2}{t}$ cts. on $t > 0$ or $t < 0$.

$q(t) = 4t$ cts. on $\mathbb{R}$.

Since $t > 0$ containing $t_0 = 1$. $I = (0, \infty)$.

2. E & U for FONLODE: If the func. f and $\frac{\partial f}{\partial y}$ be cts. in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing point (t_0, y_0), then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique sol^{n} $y = \phi(t)$ of the IVP. $y' = f(t, y)$, $y(t_0) = y_0$.

e.g. $y' = -\frac{2t}{y}$, $y(0) = y_0$.

1) 判断是否 Linear.

It's non-linear ODE.

2) Transform into: $y' = f(t, y)$.

3). 列出 $f(t,y)$, 算 $\frac{\partial f}{\partial y}$, 及其 cts 条件.

$f(t,y) = -\frac{2t}{y}$. where cts when $y \neq 0$.

$\frac{\partial f}{\partial y} = \frac{2t}{y^2}$ where cts when $y \neq 0$.

4). 解 ODE. 看 $y = \phi(t)$ 是否有额外限制 (i.e. 在分母上).

解到最后(代入 I.V.).

因为 Non-L 所以用 separation of variables 来解.

$y \cdot y' = -2t$

$\Rightarrow \int y \cdot dy = -2\int t\, dt$

$\Rightarrow \frac{1}{2}y^2 = -t^2 + C$

$\Rightarrow y^2 = -2t^2 + C$

Since $y(0) = y_0$, gives $y_0^2 = C$, $y^2 = -2t^2 + y_0^2$ 无额外限制.

$\therefore y \neq 0$

e.g. $y' = 2ty^2$, $y(0) = y_0$.

It's non-linear.

$f(t,y) = 2ty^2$ where cts on $\mathbb{R}$.

$\frac{\partial f}{\partial y} = 4ty$ where cts on $\mathbb{R}$.

$\frac{1}{y^2}\, dy = 2t\, dt$

$\Rightarrow \int \frac{1}{y^2}\, dy = \int 2t\, dt$

$\Rightarrow -\frac{1}{y} = t^2 + C$

$\Rightarrow y = -\frac{1}{t^2+C}$

$y(0) = y_0 \Rightarrow -\frac{1}{C} = y_0 \Rightarrow C = -\frac{1}{y_0}$

$\Rightarrow y = -\frac{1}{t^2 - \frac{1}{y_0}} \Rightarrow y = -\frac{y_0}{y_0 t^2 - 1} = \frac{y_0}{1 - y_0 t^2}$ which $1 - y_0 t^2 \neq 0$.

$\Rightarrow t^2 \neq \frac{1}{y_0}$

$\Rightarrow t \neq \pm\frac{1}{\sqrt{y_0}}$

when $y_0 > 0$, $-\frac{1}{\sqrt{y_0}} < t < \frac{1}{\sqrt{y_0}}$

$y_0 \leq 0$, $(-\infty, \infty)$.