Eigen

考对出现

1. Eigenvalue & Eigenvector

$n \times n$ matrix A (lin trans $T: \mathbb{R}^n \to \mathbb{R}^n$) has a non-trivial ($\vec{x} \neq \vec{0}$). solution to $A\vec{x} = \lambda\vec{x}$.

λ is eigenvalue and $\vec{x}$ is eigenvector. corresponding to λ.

e.g. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\lambda = 2$. $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

1) $\vec{x} \neq \vec{0}$.

2). λ can be 0; if $\lambda = 0$, then matrix M or lin trans T is not invertible.

3) One eigenvalue λ has infinite many different eigenvectors.

4) $\vec{x}$ can't be a eigenvector of different eigenvalue λ_1, λ_2 at once.

5). One eigenvalue λ may ~~be~~ have several lin ind eigenvectors.

6) Not every matrix (lin trans) has eigenvalue in $\mathbb{R}$.

2. Characteristic Polynomial.

For a matrix A, the characteristic polynomial of A is $\mathrm{char}(A) = \det(A - \lambda I)$. and the roots of $\mathrm{char}(A)$ are the eigenvalues of A.

e.g. $A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. λ of A?

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 & 4 \\ 0 & -\lambda & 4 & -1 \\ 0 & 0 & 1-\lambda & -2 \\ 0 & 0 & 0 & -1-\lambda \end{bmatrix}$$

$\therefore \mathrm{char}(A) = \det(A - \lambda I) = (1-\lambda)^2 \cdot (-\lambda) \cdot (-1-\lambda) = 0$.

$\Rightarrow \lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = -1$.

3. Eigenspace.

If λ is the eigenvalue of $n \times n$ matrix A, eigenspace correspond to the eigenvalue λ is $E_\lambda = null(A-\lambda I)$, a space spanned by all eigenvectors that have eigenvalue λ.

e.g. $A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $\lambda = 1$. $E_\lambda = ?$

$E_\lambda = null(A - I)$

$= null\left(\begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix}\right).$

$\begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \vec{x} = \vec{0}$

$\Rightarrow \left[\begin{array}{cccc|c} 0 & 0 & 0 & 4 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cccc|c} 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$

$\xrightarrow[\substack{r_4 + \frac{1}{3} r_2 \\ r_1 \times (-1)}]{\substack{r_2 \times (\frac{1}{4}) \\ r_3 + \frac{1}{2} r_2}} \left[\begin{array}{cccc|c} 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{r_1 - r_2} \left[\begin{array}{cccc|c} 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\longrightarrow \begin{cases} x_2 = 4x_3. \\ x_4 = 0. \end{cases}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}$

$\therefore E_\lambda = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} \right\}$

4. Geometric & Algebric Multiplicity (property of λ).

1) GM: $\dim(E_\lambda)$.

$GM = 2.$

2) AM:

e.g. same as above.

$$\text{char}(A) = (1-\lambda)^{2} \cdot (-\lambda)^{1} \cdot (-1-\lambda)^{1}$$

$\lambda_1 = 1$ AM = 2.

$\lambda_2 = 0$ AM = 1

$\lambda_3 = -1$ AM = 1

3) GM ≤ AM ($d_i \le m_i$, $1 \le i \le k$)

4) Sum of AM ≤ matrix size. ($\sum_{i=1}^{k} m_i \le n$).

5. Theorems.

1). A: n x n. with λ and corresponding $\vec{x}$.

A^n ($n>0$) has λ^n and $\vec{x}$ unchanged.

$$A\vec{x} = \lambda\vec{x}, \quad A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda^2\vec{x}.$$

If A is invertible, A^{-1} has $\frac{1}{\lambda}$ and $\vec{x}$ unchanged.

$$A\vec{x} = \lambda\vec{x}, \quad A^{-1}(A\vec{x}) = A^{-1}(\lambda\vec{x}) \Rightarrow A^{-1}(\vec{x}) = \frac{1}{\lambda}\vec{x}.$$

2) The eigenvalues of a triangular matrix are the entries on its main diagonal.

e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{bmatrix} \quad \det(A-\lambda I) = (1-\lambda)(2-\lambda)(3-\lambda)$$

$\lambda = 1, 2, 3.$

3) A: n x n is invertible iff 0 is not a eigenvalue of A.

4). A: n x n matrix, $\lambda_1, \lambda_2, \dots, \lambda_m$ be distinct eigenvalues of A, with corresponding eigenvectors $\vec{v_1}, \vec{v_2}, \dots, \vec{v_m}$. Then $\vec{v_1}, \vec{v_2}, \dots, \vec{v_m}$ are lin ind.

5) A : 2x2 matrix. Σ of eigenvalues of A = $tr(A)$.

$tr(A)$ (trace of A) = Σ of diagonal entry.

e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$tr(A) = 1+4 = 5$