The Limit Laws.

1. The Proving Plan.

1) Prove 'Basic Limits'.

① $\lim_{x\to a} x = a.$ ② $\lim_{x\to a} C = C.$

2) Prove the law.

Assume $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$

Then.

① $\lim_{x\to a} [f(x)+g(x)] = L+M.$

② $\lim_{x\to a} [f(x)g(x)] = LM.$

③ $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$ (assuming $M \neq 0$)

2. Proof.

1) Theorem

① Let $a, L, M \in \mathbb{R}$.

② Let f and g be functions defined at least on an interval centered at a, except maybe at a.

③ If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

then $\lim_{x\to a} [f(x)+g(x)] = L+M$. Call $h(x) = f(x)+g(x)$.

2) Structure of Proof.

① Let $\varepsilon > 0$.

② Take $\delta = \dots$.

③ Let $x \in \mathbb{R}$. Assume $0 < |x-a| < \delta$.

④ Conclude $|h(x) - (L+M)| < \varepsilon$.

3) Notice before proof.

① WTS: $\lim_{x\to a} h(x) = L+M$ $\quad \forall \varepsilon, \exists \delta \ldots\ldots$

$\longrightarrow$ fix an arbitrary ε.

$\longrightarrow$ need to find a value of δ that work for this ε.

② We know: $\lim_{x\to a} f(x) = L$

$\longrightarrow$ Can choose a value of ε.

$\longrightarrow$ $\exists$ a value of δ that works for that ε.

4) Rough Work.

$$|h(x) - (L+M)| = |(f(x)-L) + (g(x)-M)|$$
$$\le |f(x)-L| + |g(x)-M|$$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x)-L| < \frac{\varepsilon}{2}$.

$\exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow |g(x)-M| < \frac{\varepsilon}{2}$.

Take $\delta = \min\{\delta_1, \delta_2\}$ (can get both conclusions)

5) Proof.

Let $\varepsilon > 0$.

Take $\frac{\varepsilon}{2}$ in definition of $\lim_{x\to a} f(x) = L$. $\exists \delta_1 > 0$ s.t.

$0 < |x-a| < \delta_1 \Rightarrow |f(x)-L| < \frac{\varepsilon}{2}$.

Take $\frac{\varepsilon}{2}$ in definition of $\lim_{x\to a} g(x) = M$. $\exists \delta_2 > 0$ s.t.

$0 < |x-a| < \delta_2 \Rightarrow |g(x)-M| < \frac{\varepsilon}{2}$.

Take $\delta = \min\{\delta_1, \delta_2\}$

Let $x \in \mathbb{R}$. Assume $0 < |x-a| < \delta$, this implies:

$0 < |x-a| < \delta_1$. Thus $|f(x)-L| < \frac{\varepsilon}{2}$.

$0 < |x-a| < \delta_2$. Thus $|g(x)-M| < \frac{\varepsilon}{2}$

Then, $|h(x) - (L + M)| = |(f(x) - L) + (g(x) - M)|$

$$\leq |f(x) - L| + |g(x) - M|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

I have shown that $|h(x) - (L+M)| < \varepsilon$, as needed.