2^{nd}-order Homogenous Differential Equation with const. coefficient.

$r^2 \to y''$
$r \to y'$
$1 \to y^{(0)}$

1. Std. Form of SOLODE: $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$.

$g(t) = 0$: homo.

2. Solving Constant Coefficient: $ay'' + by' + cy = 0$.

1). Characteristic Equation. $ar^2 + br + c = 0$.

C.E. sol^n	$y_1(t)$.	$y_2(t)$.
$r_1, r_2 \in \mathbb{R}, r_1 \neq r_2$.	$e^{r_1 t}$	$e^{r_2 t}$
$r_1, r_2 \in \mathbb{R}. r_1 = r_2$.	e^{rt}	$t \cdot e^{rt}$.
$r = \pm\alpha + \beta i$	$e^{\alpha t} \cdot \cos(\beta t)$.	$e^{\alpha t} \cdot \sin(\beta t)$

General sol^n

$y(t) = c_1 y_1(t) + c_2 y_2(t)$.

$\Updownarrow$

Wronskian is non-zero.

e.g. $2y'' - 3y' + y = 0$.

①. Write C.E. and solve.

$2r^2 - 3r + 1 = 0$.

$\Rightarrow (2r-1)(r-1) = 0. \quad r_1 = \frac{1}{2} \quad r_2 = 1$.

$\therefore y_1(t) = e^{\frac{t}{2}} \quad y_2(t) = e^{t}$.

$\therefore y(t) = C_1 \cdot e^{\frac{t}{2}} + C_2 \cdot e^{t}$

e.g. $y'' - y' + \frac{y}{4} = 0$.

$r^2 - r + \frac{1}{4} = 0$.

$\Rightarrow (r - \frac{1}{2})^2 = 0. \quad r = \frac{1}{2}$.

$y(t) = C_1 \cdot e^{\frac{t}{2}} + C_2 \cdot t \cdot e^{\frac{t}{2}}$.

e.g. $16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1$.

$16r^2 - 8r + 145 = 0$.

$\Rightarrow r = \frac{8 \pm \sqrt{8^2 - 4 \times 16 \times 145}}{2 \times 16} = \frac{8 \pm 96i}{32} = \frac{1}{4} \pm 3i. \quad \alpha = \frac{1}{4}. \quad \beta = 3$.

$y_1(t) = e^{\alpha t} \cdot \cos(\beta t) = e^{\frac{t}{4}} \cdot \cos(3t)$

$y_2(t) = e^{\alpha t} \cdot \sin(\beta t) = e^{\frac{t}{4}} \cdot \sin(3t)$.

$y(t) = C_1 e^{\frac{t}{4}} \cdot \cos(3t) + C_2 \cdot e^{\frac{t}{4}} \cdot \sin(3t)$.

$y(0) = C_1 = -2 \quad ; \quad y'(t) = \frac{C_1}{4} \cdot e^{\frac{t}{4}} \cdot \cos(3t) - 3C_1 e^{\frac{t}{4}} \cdot \sin(3t) + \frac{C_2}{4} \cdot e^{\frac{t}{4}} \cdot \sin(3t)$

$+ 3C_2 e^{\frac{t}{4}} \cdot \cos(3t) \cdot = 1$.

$\Rightarrow \frac{C_1}{4} + 3C_2 = 1 \quad \Rightarrow C_2 = \frac{1}{2}$

$\therefore y(t) = -2e^{\frac{t}{4}}\cdot\cos(3t) + \frac{1}{2}e^{\frac{t}{4}}\cdot\sin(3t).$

3. Euler's Equation: $ax^2y'' + bxy' + cy = 0$. plug $y = x^r$.

与上面相同.同样用于这些情况.

1). C.E.: $ar(r-1) + br + c = 0$.

C.E. sol$^{\underline{s}}$.	$y_1(t)$	$y_2(t)$.	General sol$^{\underline{n}}$.
$r_1, r_2 \in \mathbb{R}, r_1 \neq r_2$.	x^{r_1}	x^{r_2}.	$y = c_1y_1 + c_2y_2$.
$r_1, r_2 \in \mathbb{R}. r_1 = r_2$.	x^r	$x^r\cdot\ln x$.	
$r = \alpha \pm \beta i$	$x^\alpha\cdot\cos(\beta\cdot\ln x)$	$x^\alpha\cdot\sin(\beta\cdot\ln x)$.	

e.g. $2x^2y'' + 3xy' - 15y = 0$. $y(1) = 0$, $y'(1) = 1$.

$2r(r-1) + 3r - 15 = 0$.

$\Rightarrow 2r^2 - 2r + 3r - 15 = 0$.

$\Rightarrow 2r^2 + r - 15 = 0$.

$\Rightarrow (2r-5)(r+3) = 0$. $r_1 = \frac{5}{2}$, $r_2 = -3$.

$y_1(x) = x^{\frac{5}{2}}$, $y_2(x) = x^{-3}$.

$\therefore y(x) = c_1x^{\frac{5}{2}} + c_2x^{-3}$.

$y(1) = c_1 + c_2 = 0$. $y'(x) = \frac{5}{2}c_1\cdot x^{\frac{3}{2}} - 3c_2\cdot x^{-4}$

$y'(1) = \frac{5}{2}c_2 - 3c_2 = 1$.

$\therefore \begin{cases} c_1 + c_2 = 0. \\ \frac{5}{2}c_1 - 3c_2 = 1 \end{cases} \Rightarrow 3c_1 = -3c_2$.

$\Downarrow$.

$\frac{5}{2}c_1 + 3c_1 = 1 \Rightarrow \frac{11}{2}c_1 = 1 \Rightarrow c_1 = \frac{2}{11}$.

$\Downarrow$

$c_2 = -\frac{2}{11}$.

$\therefore y(x) = \frac{2}{11}x^{\frac{5}{2}} - \frac{2}{11}x^{-3}$.

e.g. $x^2y'' - 7xy' + 16y = 0$.

e.g. $x^2\cdot y'' + 3xy + 4y = 0$.

3. Wronskin Determinant: $W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0)$

$= W[y_1, y_2](t_0)$.

e.g. The Wron. of f and g is $3e^{4t}$; if $g(t) = e^{2t}$, then find $f(t)$.

已知的放第一列

$W = \begin{vmatrix} g(t) & f(t) \\ g'(t) & f'(t) \end{vmatrix} = 3e^{4t}$.

$\Rightarrow g(t)\cdot f'(t) - f(t)\cdot g'(t) = 3e^{4t}$

$\Rightarrow e^{2t}\cdot f'(t) - 2e^{2t}\cdot f(t) = 3e^{4t}$. FOLODE. (Integral factor).

$\Rightarrow f'(t) - 2\cdot f(t) = 3$.

$\mu(t) = e^{\int -2\,dt} = e^{-2t}$.

$(e^{-2t}\cdot f(t))' = 3\cdot e^{-2t}$.

$\Rightarrow e^{-2t}\cdot f(t) = 3\int e^{-2t}\,dt$.

$\Rightarrow e^{-2t}\cdot f(t) = -\frac{3}{2}e^{-2t} + C$.

$\Rightarrow f(t) = -\frac{3}{2} + C\cdot e^{2t}$.

1). For $y'' + p(t)y' + q(t)y = g(t)$. $y(t) = y_0$; $y'(t) = y_0'$

$\rightarrow W[y_1, y_2](t_0) \neq 0$: always has unique sol$^{\underline{n}}$ regardless of y_0, y_0'

$\rightarrow W[y_1, y_2](t_0) = 0$: many I.V. cannot be satisfied no matter how C_1 & C_2 are chosen.

4. Abel's Theorem: If y_1, y_2 are sol$^{\underline{n}}$ to SOLODE

常联立W def$^{\underline{n}}$ 得到等式 ↓ 知 y_1, 求 y_2.

$L[y] = y'' + p(t)y' + q(t)y = 0$. then the Wronskin is given by $W[y_1, y_2](t) = C\cdot e^{-\int p(t)\,dt}$

e.g. $\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + q(t)y = 0$., $q(t)$ is cts for all real t.

① Find $W[y_1, y_2](t)$ if $W[y_1, y_2](0) = -2$; does sol$^{\underline{n}}$ form a fundamental set?

② if $y_1(t) = e^t$, find $y_2(t)$.

①. $W[y_1, y_2](t) = C\cdot e^{-\int 8\,dt} = C\cdot e^{-8t}$.

$W[y_1, y_2](0) = C\cdot e^0 = -2 \Rightarrow C = -2$.

$\therefore W[y_1, y_2](t) = -2e^{-8t}$. since $W \neq 0$, y_1, y_2 are lin. independent. $\therefore$ Yes.

②. $W = \begin{vmatrix} e^{t} & y_2(t) \\ e^{t} & y_2'(t) \end{vmatrix} = e^{t}\cdot y_2'(t) - e^{t}\cdot y_2(t) = -2e^{-8t}$.

$\Rightarrow \quad y_2'(t) - y_2(t) = -2e^{-9t}$. FOLODE.

$\mu(x) = e^{\int p(t)dt} = e^{\int -1 dt} = e^{-t}$, multiply on both sides,

$(e^{-t}\cdot y_2(t))' = -2e^{-10t}$

$\Rightarrow \quad e^{-t}\cdot y_2(t) = -2\int e^{-10t}\, dt$.

$\Rightarrow \quad e^{-t}\cdot y_2(t) = -2\cdot(-\frac{1}{10})e^{-10t} + C$.

$\Rightarrow \quad y_2(t) = \frac{1}{5}e^{-9t} + c\cdot e^{t}$.

$\therefore y(t) = C_1 y_1(t) + C_2 y_2(t)$

$= C_1 e^{t} + C_2\cdot(\frac{1}{5}e^{-9t} + c\cdot e^{t})$.

$= \underline{C_1 e^{t}} + \frac{1}{5}C_2\cdot e^{-9t} + \underline{C_2\cdot c\cdot e^{t}}$.

$= C_1 e^{t} + \frac{1}{5}C_2 e^{-9t}$.

e.g. $(x\sin x + \cos x)y'' - (x\cos x)y' + (\cos x)y = 0$.

① Find $W[y_1, y_2](t)$

②. $y_1 = x$. $y_2 = ?$

③. $y(0) = 1$. $y'(0) = 1$

①. $y'' - \frac{x\cos x}{x\sin x + \cos x}y' + \frac{\cos x}{x\sin x + \cos x}y = 0$.

$W = C\cdot e^{-\int p(x)dx} = C e^{-\int -\frac{x\cos x}{x\sin x + \cos x}dx}$

$u = x\sin x + \cos x \Rightarrow du = x\cos x\, dx$

$\therefore Ce^{\int \frac{1}{u}du} = Ce^{\ln|u|} = C\cdot(x\sin x + \cos x)$.

②. $W[y_1, y_2](x) = \begin{vmatrix} x & y_2(x) \\ 1 & y_2'(x) \end{vmatrix} = x\cdot y_2'(x) - y_2(x) = C\cdot x\sin x + C\cdot\cos x$.

$\Rightarrow \quad y_2'(x) - \frac{1}{x}y_2(x) = C\cdot\sin x + C\cdot\frac{\cos x}{x}$.

$\mu(x) = e^{\int p(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$

$\therefore \quad (\frac{1}{x}\cdot y_2(x))' = (C\cdot\sin x + C\cdot\frac{\cos x}{x})\cdot\frac{1}{x}$.

$\Rightarrow \quad \frac{1}{x}\cdot y_2(x) = C\int \frac{1}{x}\sin x + \frac{1}{x^2}\cos x\, dx$.

$\Rightarrow \quad \frac{1}{x}\cdot y_2(x) = C\cdot\int \sin x\cdot \underline{x^{-1}} + \underline{\cos x}\cdot x^{-2}dx$.

$\Rightarrow \quad \frac{1}{x}y_2(x) = C\cdot(-\frac{\cos x}{x}) + C$.

$$\Rightarrow \quad y_2(x) = -C\cdot\cos x + C\cdot x.$$

$$\therefore y(x) = C_1\cdot y_1(x) + C_2\cdot y_2(x)$$
$$= C_1\cdot x + C_2(-C\cdot\cos x + C\cdot x)$$
$$= C_1\cdot x + C_2\cdot\cos x.$$

③. $y(0) = C_2 = 1.$ $\quad y'(x) = C_1 - C_2\sin x.$

$$\Rightarrow y'(0) = C_1 = 1.$$

$$\therefore y(x) = x + \cos x.$$

Generalize.

5. Reduction of order: For $y'' + p(t)y' + q(t) = 0$, $y_2(t) = y_1(t)\cdot\int\frac{e^{-\int p(t)dt}}{(y_1(t))^2}dt.$

e.g. $2t^2y'' + 3ty' - y = 0$, $t > 0$; $y_1(t) = t^{-1}.$

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0.$$

$$y_2(t) = y_1(t)\cdot\int\frac{e^{-\int p(t)dt}}{(y_1(t))^2}dt = \frac{1}{t}\cdot\int t^2\cdot e^{-\int\frac{3}{2}\cdot t^{-1}dt}dt$$
$$= \frac{1}{t}\cdot\int t^2\cdot t^{-\frac{3}{2}}dt = \frac{1}{t}\cdot\left(\frac{2}{3}t^{\frac{3}{2}} + C\right).$$
$$= \frac{2}{3}t^{\frac{1}{2}} + Ct^{-1}.$$

$$y(t) = C_1\cdot t^{-1} + C_2\left(\frac{2}{3}t^{\frac{1}{2}} + C\cdot t^{-1}\right).$$
$$= C_1 t^{-1} + C_2\cdot\frac{2}{3}t^{\frac{1}{2}}.$$