The Squeeze Theorem.

1. The Theorem

Let $a, L \in \mathbb{R}$

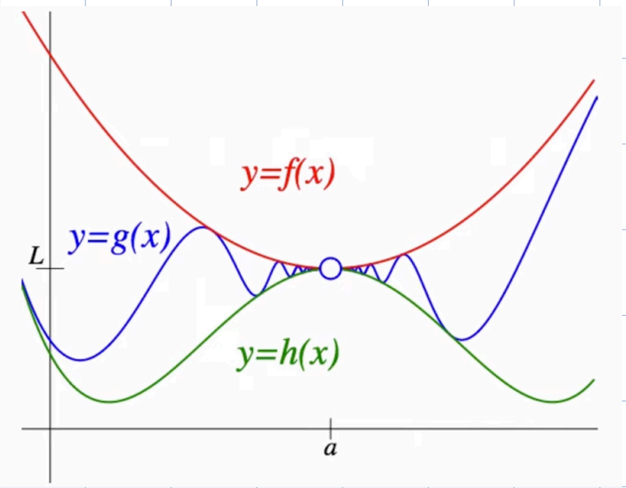
Let f, g, and h be functions defined near a, except possibly at a.

If,

$\exists p > 0$, s.t. $0 < |x-a| < p \Longrightarrow h(x) \le g(x) \le f(x)$.

$\lim_{x \to a} f(x) = L$; $\lim_{x \to a} h(x) = L$.

Then, $\lim_{x \to a} g(x) = L$.



2. Proof.

Same to notice as 2.6.

1) Rough Work.

① We need $0 < |x-a| < \delta \Rightarrow |g(x) - L| < \varepsilon$.

② $|g(x) - L| < \varepsilon$ is equivalent to $L - \varepsilon < g(x) < L + \varepsilon$.

③ We know $0 < |x-a| < p \Rightarrow h(x) < g(x) < f(x)$.

④ We know $\lim_{x \to a} f(x) = L$. $\exists \delta_1 > 0$ s.t.

$0 < |x-a| < \delta_1 \Rightarrow L - \varepsilon < f(x) < L + \varepsilon$.

⑤ We know $\lim_{x \to a} g(x) = L$. $\exists \delta_2 > 0$ s.t.

$0<|x-a|<\delta_2 \Rightarrow L-\varepsilon<g(x)<L+\varepsilon.$

③ Take $\delta=\min\{\delta_1, \delta_2, p\}$

2) Proof.

Let $\varepsilon>0$

Use same ε in the definition of $\lim_{x\to a} f(x)=L$.

$\exists\, \delta_1>0$, s.t. $0<|x-a|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon$

$\Rightarrow f(x)<L+\varepsilon.$

Use same ε in the definition of $\lim_{x\to a} h(x)=L$,

$\exists\, \delta_2>0$, s.t. $0<|x-a|<\delta_1 \Rightarrow |h(x)-L|<\varepsilon.$

$\Rightarrow L-\varepsilon<h(x)$

Take $\delta=\min\{\delta_1, \delta_2, p\}$

Let $x\in\mathbb{R}$. Assume $0<|x-a|<\delta$. This implies:

$\longrightarrow 0<|x-a|<\delta_2$ Thus $L-\varepsilon<h(x)$

$\longrightarrow 0<|x-a|<p$ Thus $h(x)\le g(x)\le f(x).$

$\longrightarrow 0<|x-a|<\delta_1$ Thus $f(x)<L+\varepsilon.$

Therefore, $L-\varepsilon<g(x)<L+\varepsilon.$

Equivalently, we have proven that $|g(x)-L|<\varepsilon$, as needed.