Non-homo System

1. Non-homo System: $\vec{x}' = P(t)\cdot\vec{x} + g(t)$.

1) General sol^{n}: $\vec{x} = \vec{x}_c(t) + \vec{x}_p(t) = c_1 x^{(1)} + \cdots + c_n x^{(n)} + \vec{x}_p(t)$.

2) 找 $\vec{x}_p(t)$ 方法.

① Diagonalization: A diagonalizable $\longleftrightarrow$ distinct eigenvalue λ.

i). $T = (\vec{v_1} \cdots \vec{v_n})$ be matrix of eigenvector of A.

$D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ be matrix of eigenvalue of A.

ii) Let $\vec{x} = T\cdot\vec{y}$. $\Rightarrow \vec{y}' = D\cdot\vec{y} + T^{-1}\cdot g(t)$. $\rightarrow$ solve 1^{st} order using integrating factor.

iii) $\vec{x} = T\cdot\vec{y}$.

② Undetermined Coefficient. (出现重复的,然后设 $\vec{x}_p(t)$).

注意: 若 $g(t)$ 出现 $e^{\lambda t}$ where λ is eigenvalue, let $x_p = at\,e^{\lambda t} + be^{\lambda t}$.

以前一项,现在两项.

③ Variation of Parameter (通用).

i). find $\varphi(t)$.

ii). Let $x(t) = \varphi(t)\cdot\vec{u}(t)$. $\Rightarrow \vec{u}(t)' = \varphi^{-1}(t)\cdot g(t)$.

iv). $\vec{u}(t) = \int \varphi^{-1}(t)\cdot g(t)\,dt + \vec{c}$.

v). $\vec{x}(t) = \varphi(t)\cdot\vec{u}(t)$.

2. 练习题.

e.g. $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$

① $\vec{x}_c(t)$: $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = (-2-\lambda)^2 - 1 = 0$.

$\therefore \lambda_1 = -3,\ \lambda_2 = -1$. $\Rightarrow (2+\lambda)^2 = 1$.

when $\lambda_1 = -3$. $A - \lambda I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = -v_2 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

when $\lambda_2 = -1$, $A - \lambda I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = v_2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\therefore \vec{x}_c(t) = c_1\cdot e^{-3t}\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\cdot e^{-t}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

② $\varphi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$.

Let $\vec{x}(t) = \varphi(t)\cdot\vec{u}(t)$. $\Rightarrow \vec{u}(t)' = \varphi^{-1}(t)\cdot g(t)$.

$\det(\varphi(t)) = e^{-4t} + e^{-4t} = 2e^{-4t}$.

$$\varphi^{-1}(t) = \frac{e^{4t}}{2}\begin{pmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix} = \begin{pmatrix} \frac{e^{3t}}{2} & -\frac{e^{3t}}{2} \\ \frac{e^{t}}{2} & \frac{e^{t}}{2} \end{pmatrix}$$

$$\therefore \vec{u}(t)' = \varphi^{-1}(t)\cdot g(t)$$

$$= \begin{pmatrix} \frac{e^{3t}}{2} & -\frac{e^{3t}}{2} \\ \frac{e^{t}}{2} & \frac{e^{t}}{2} \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \begin{pmatrix} e^{3t}\cdot e^{-t} - \frac{3}{2}t\cdot e^{3t} \\ 1 + \frac{3}{2}t\cdot e^{t} \end{pmatrix}.$$

$$\therefore \vec{u}_1(t) = \int e^{2t} - \frac{3}{2}t\cdot e^{3t}\,dt = \frac{e^{2t}}{2} - \int \frac{3}{2}t\cdot e^{3t}\,dt$$

$$= \frac{e^{2t}}{2} - \left(\frac{3}{2}t\cdot\frac{e^{3t}}{3} - \int \frac{3}{2}\cdot\frac{e^{3t}}{3}\,dt\right).$$

$$= \frac{e^{2t}}{2} - \left(\frac{t\cdot e^{3t}}{2} - \int \frac{e^{3t}}{2}\,dt\right).$$

$$= \frac{e^{2t}}{2} - \frac{t\cdot e^{3t}}{2} + \frac{e^{3t}}{6} + C_1$$

$$\vec{u}_2(t) = \int 1 + \frac{3}{2}t\cdot e^{t}\,dt = t + \frac{3}{2}\int t\cdot e^{t}\,dt$$

$$= t + \frac{3}{2}\left(t\cdot e^{t} - \int e^{t}\,dt\right).$$

$$= t + \frac{3}{2}\left(t\cdot e^{t} - e^{t}\right) + C_2$$

$$= t + \frac{3}{2}t\cdot e^{t} - \frac{3}{2}e^{t} + C_2$$

$$\therefore \vec{x}(t) = \varphi(t)\cdot\vec{u}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}\cdot\begin{pmatrix} \frac{e^{2t}}{2} - \frac{t e^{3t}}{2} + \frac{e^{3t}}{6} + C_1 \\ t + \frac{3}{2}t\cdot e^{t} - \frac{3}{2}e^{t} + C_2 \end{pmatrix}.$$

$$= \begin{pmatrix} e^{-3t}\left(\frac{e^{2t}}{2} - \frac{t e^{3t}}{2} + \frac{e^{3t}}{6} + C_1\right) + e^{-t}\cdot\left(t + \frac{3}{2}t\cdot e^{t} - \frac{3}{2}e^{t} + C_2\right) \\ -e^{-3t}\left(\frac{e^{2t}}{2} - \frac{t e^{3t}}{2} + \frac{e^{3t}}{6} + C_1\right) + e^{-t}\cdot\left(t + \frac{3}{2}t\cdot e^{t} - \frac{3}{2}e^{t} + C_2\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{-t} - \frac{1}{2}t + \frac{1}{6} + C_1\cdot e^{-3t} + t\cdot e^{-t} + \frac{3}{2}t - \frac{3}{2} + C_2\cdot e^{-t} \\ -\frac{1}{2}e^{-t} + \frac{1}{2}t - \frac{1}{6} - C_1 e^{-3t} + t\cdot e^{-t} + \frac{3}{2}t - \frac{3}{2} + C_2\cdot e^{-t} \end{pmatrix}$$

$$= C_1\cdot e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\cdot e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + t\cdot e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + t\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{4}{3} \\ -\frac{5}{3} \end{pmatrix}$$

e.g. $\vec{x} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$