Q1: WTS: if $a_{12}a_{21} \geq 0$. then A has two distinct real eigenvalues.

Suppose $a_{12}a_{21} \geq 0$.

To calculate the eigenvalues λ of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. we have.

$\det(A - \lambda I) = 0$, where $\lambda \in \mathbb{R}$ and I is the identity M of A.

$\Rightarrow \det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} = 0$.

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$.

$\Rightarrow a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21} = 0$.

$\Rightarrow \lambda^2 + (-a_{11}-a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$.

$$\begin{aligned} \Delta &= \sqrt{b^2 - 4ac} \\ &= \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})} \\ &= \sqrt{a_{11}^2 + a_{22}^2 + 2a_{11}a_{22} - 4a_{11}a_{22} + 4a_{12}a_{21}} \\ &= \sqrt{(a_{11}-a_{22})^2 + 4a_{12}a_{21}} \end{aligned}$$

Since $a_{11}, a_{22} \in \mathbb{R}$. $(a_{11}-a_{22})^2 \geq 0$.

Since $a_{12} \cdot a_{21} > 0$. $4a_{12} \cdot a_{21} \geq 0$.

Thus. $(a_{11}-a_{22})^2 + 4a_{12} \cdot a_{21} \geq 0$. which $\sqrt{(a_{11}-a_{22})^2 + 4a_{12} \cdot a_{21}} \geq 0$.

Therefore. A has two distinct real eigenvalues. ∎

Q2: Since T has only one eigenvalue λ, its characteristic polynomial is $(x-\lambda)^n \cdot q$, which q is a polynomial of degree $k-n$. where n is the geometric multiplicity of λ.

① $k < n$. then the degree of the characteristic polynomial is greater than k. so T can't be diagonalizable. and thus $[T]_\alpha^\alpha$ can't be similar to a diagonal matrix for any basis α.

In 4.1.14, $m_\lambda \leq k$ and in 4.2.6, $\dim(E_\lambda) \leq m_\lambda$. where m_λ is the alegebric multiplicity. gives.

$$n = \dim(E_\lambda) \leq m_\lambda \leq k.$$

$$\Rightarrow \quad n \leq k. \quad \text{contradicts to } n > k.$$

② $k = n$.

Since, by prop. 4.1.7, E_λ is a subspace of V
Since $k=n$, $\dim(E_\lambda)=k$. by corollary, $E_\lambda = V$.
Thus, there exist a basis $B=\{\vec{b_1}, \vec{b_2}, \dots, \vec{b_n}\}$ of eigenvalues s.t. $T(\vec{b_i})=\lambda\vec{b_i}$, $i\in\{1,\dots,k\}$.

Therefore. $$[T]_B^B = [[T(\vec{b_1})]_B \mid [T(\vec{b_2})]_B \mid \cdots \mid [T(\vec{b_k})]_B].$$
$$= \lambda[[\vec{b_1}]_B \mid [\vec{b_2}]_B \mid \cdots \mid [\vec{b_k}]_B].$$
$$= \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix}$$

T is a scalar multiplication of the identity transformation.

By theorem 2.7.5, $[T]_\alpha^\alpha = ([I\vec{v}]_\alpha^\beta)^{-1}[T]_\beta^\beta[I\vec{v}]_\alpha^\beta$
which $[T]_\alpha^\alpha$ are similar to $[T]_\beta^\beta$.

Hence. we've shown that $n=k$ results in $[T]_\alpha^\alpha$ being similar to a diagonal matrix for all bases α of v.

② $k<n$. then the degree of the characteristic polynomial is less than k, so the remaining $k-n$ roots of the polynomial must be distinct and different from λ. which correspond to a lin. independent set of eigenvalues of T. Since T has only n lin. independent eigenvectors. it's not diagonalizable.
Therefore, $[T]_\alpha^\alpha$ can't be similar to a diagonal matrix for any basis α in this case as well.

Q3. Since T is a linear transformation from V to itself, it can be represented by a matrix with respect to any basis of V.
Since $\dim(V)=3$, we can choose a basis for V consisting of three linearly independent vectors. denoted $\{v_1, v_2, v_3\}$.
The matrix representation of T with respect to this basis is a 3×3 matrix A with entry in $\mathbb{R}$. The characteristic polynomial of A is a

3 degree polynomial with coefficient in $\mathbb{R}$

By fundamental theorem of algebra: every non-constant polynomial with complex coefficients has at least one complex root.

Therefore, it has at least one real root.

This root corresponds to a real eigenvalue of T.

Q4. Suppose S is an invertible $n \times n$ matrix. We have:

$$\begin{aligned}
\det(ST - \lambda I) &= \det(S^{-1})\det(ST - \lambda I)\det(S) \\
&= \det(S^{-1}(ST - \lambda I)S) \\
&= \det(S^{-1}(ST)S - S^{-1}(\lambda I)S). \\
&= \det(ITS - \lambda(S^{-1}S)) \\
&= \det(TS - \lambda I)
\end{aligned}$$

Similarly, suppose T is an invertible $n \times n$ matrix.

$$\begin{aligned}
\det(TS - \lambda I) &= \det(T^{-1})\det(TS - \lambda I)\det(T). \\
&= \det(T^{-1}(TS - \lambda I)T). \\
&= \det(T^{-1}(TS)T - T^{-1}(\lambda I)T) \\
&= \det(ST - \lambda(T^{-1}T)) \\
&= \det(ST - \lambda I).
\end{aligned}$$

Therefore, no matter which is an invertible $n \times n$ matrix, $\det(TS - \lambda I) = \det(ST - \lambda I)$, we've shown TS and ST have the same set of real eigenvalues.

Q5. (a) $\lambda = 1$ is an eigenvalue of $S \circ T$.

(b) $S \circ T$ doesn't have any other real eigenvalues.

(c) $\lambda = 1$ is an eigenvalue of $T \circ S$.

Q6. Using Gram-Schmidt process: $\vec{u_k} = \vec{v_k} - \sum_{j=1}^{k-1} proj_{\vec{u_j}}(\vec{v_k})$, where

$proj_{\vec{u_j}}(\vec{v_k}) = \frac{\vec{u_j} \cdot \vec{v_k}}{|\vec{u_j}|^2} \cdot \vec{u_j}$, the normalized vector is $\vec{e_k} = \frac{\vec{u_k}}{|\vec{u_k}|}$.

① $\vec{u_1} = \vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$|\vec{u_1}| = \sqrt{1^2+1^2+0} = \sqrt{2}$.

$\vec{e_1} = \frac{\vec{u_1}}{|\vec{u_1}|} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$

② $\vec{u_2} = \vec{v_2} - proj_{\vec{u_1}}(\vec{v_2})$

$proj_{\vec{u_1}}(\vec{v_2}) = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|^2} \cdot \vec{u} = \frac{1}{(\sqrt{2})^2} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$

$\vec{u_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

$|\vec{u_2}| = \sqrt{(\frac{1}{2})^2 + (-\frac{1}{2})^2 + 1} = \frac{\sqrt{6}}{2}$.

$\vec{e_2} = \frac{\vec{u_2}}{|\vec{u_2}|} = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix}$.

③ $\vec{u_3} = \vec{v_3} - proj_{u_1}(\vec{v_3}) - proj_{u_2}(\vec{v_3})$

$= \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

$\vec{e_3} = \frac{\vec{u_3}}{|\vec{u_3}|} = \frac{\vec{u_3}}{\sqrt{(\frac{2}{3})^2 + (-\frac{2}{3})^2 + (\frac{2}{3})^2}}$

$= \frac{\vec{u_3}}{\frac{2\sqrt{3}}{3}} = \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$

Thus, the set of the orthonormal vector is $\left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix} \right\}$