Discrete Random Variables.

1. Discrete random variable: $\sum_{x\in\mathbb{R}} P(X=x) = 1$. (all of its P is on individual values).

1). Probability Function: $p_X(x) = P(X=x)$

① If $\sum_{x\in\mathbb{R}} P(X=x) = 1$, there's a distinct sequence $x_1, x_2, \dots \in \mathbb{R}$, and correspond P, $p_1, p_2, \dots \geq 0$, with $\sum_i p_i = 1$ s.t. $P(X=x_i) = p_i$.

② $p_X(x_i) = p_i$ for all i, with $p_X(x) = 0$ for all $x \notin \{x_1, x_2, \dots\}$.

e.g. Flip 2 coins. Let $X = $ # of Heads.

$P(X=0) = \frac{\binom{2}{0}}{2^2} = \frac{1}{4}$; $P(X=1) = \frac{\binom{2}{1}}{2^2} = \frac{1}{2}$; $P(X=2) = \frac{\binom{2}{2}}{2^2} = \frac{1}{4}$.

$\therefore x_1 = 0, x_2 = 1, x_3 = 2$; $p_1 = \frac{1}{4}, p_2 = \frac{1}{2}, p_3 = \frac{1}{4}$.

$\therefore p_X(0) = \frac{1}{4}, p_X(1) = \frac{1}{2}, p_X(2) = \frac{1}{4}$.

2. Binomial Distribution: $X \sim \text{Binomial}(n, \theta)$.

θ: successful rate.

1). Bernulli Distribution: $X \sim \text{Bernulli}(\theta)$.

e.g. throw a basket ball; $P(\text{score}) = \theta$.

$P(X=0) = p_X(0) = 1-\theta$; $P(X=1) = p_X(1) = \theta$.

$n=1$.

2) Binomial Distribution: $X \sim \text{Binomial}(n, \theta)$.

Binomial Theorem:

$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + \binom{n}{n}a^0 b^n$.

① $p_X(k) = P(X=k) = \binom{n}{k}\theta^k (1-\theta)^{n-k}$; for $k \in \{0, 1, 2, \dots, n\}$.

↳ starts from 0.

② $\sum_{k=0}^{n} P(X=k) = \sum_{k=0}^{n} \binom{n}{k}\theta^k (1-\theta)^{n-k} = [\theta + (1-\theta)]^n = 1$.

3). Poisson Distribution: $X \sim \text{Poisson}(\lambda)$, where $\lambda = n\cdot\theta$.

→ poisson approx. when n is large for Bin.

apply when n is large and θ is small.

① $P(Z=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

proof: Since $\lambda = n\cdot\theta \Rightarrow \theta = \frac{\lambda}{n}$, $Z \sim \text{Binomial}(n, \frac{\lambda}{n})$

$P(Z=k) = \binom{n}{k}\theta^k (1-\theta)^{n-k}$.

$= \frac{n!}{(n-k)!\cdot k!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$.

$$= \frac{n\cdot(n-1)\cdot(n-2)\cdots\cdots(n-k+1)}{k!}\cdot\left(\frac{\lambda}{n}\right)^k\cdot\left(1-\frac{\lambda}{n}\right)^{n-k}.$$

$$= \frac{n\cdot(n-1)\cdot(n-2)\cdots\cdots(n-k+1)\cdot\lambda^k}{k!\cdot n^k}\cdot\left(1-\frac{\lambda}{n}\right)^{n-k}.$$

When $n\to\infty$, for a fixed value k. we have $k \ll n$.

Thus, $\frac{n}{n}\to 1$, $\frac{n-1}{n}\to 1$, $\frac{n-2}{n}\to 1$, $\frac{n-3}{n}\to 1$, ..., $\frac{n-k+1}{n}\to 1$, gives.

$$\frac{n\cdot(n-1)\cdot(n-2)\cdots\cdots(n-k+1)}{n^k}\to 1.$$

Also. $\left(1-\frac{\lambda}{n}\right)^{n-k}\approx\left(1-\frac{\lambda}{n}\right)^n$

Considering $e^x = 1+x+\frac{x^2}{2!}+\cdots\cdots$, when $x\in\mathbb{R}$. x is small. $e^x = 1+x$.

Take $x = -\frac{\lambda}{n}$, gives. $\left(1-\frac{\lambda}{n}\right)^n\approx\left(e^{-\frac{\lambda}{n}}\right)^n = e^{-\lambda}$

Therefore. we have, when $n\to\infty$, gives.

$$P(z=k) = \frac{n\cdot(n-1)\cdot(n-2)\cdots\cdots(n-k+1)}{n^k}\cdot\frac{\lambda^k}{k!}\cdot e^{-\lambda}$$

$$= e^{-\lambda}\cdot\frac{\lambda^k}{k!}. \quad \blacksquare$$

② $\sum_{k=0}^{\infty} P(Z=k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\cdot\left(1+\lambda+\frac{\lambda^2}{2!}+\cdots\right) = e^{-\lambda}\cdot e^{\lambda} = 1.$

3. Geometric Distribution: $Z\sim \text{Geometric}(\theta)$.

θ: probability for Z.

1) $P(Z=k) = \theta\cdot(1-\theta)^k$

↳ k starts from 0.

2) $\sum_{k=0}^{n} P(X=k) = \sum_{k=0}^{n}\theta\cdot(1-\theta)^k = \theta\cdot[1+(1-\theta)+(1-\theta)^2+\cdots]$ $(0<\theta<1)$.

$$= \theta\cdot\frac{1}{1-(1-\theta)} = \theta\cdot\frac{1}{\theta} = 1.$$

3). Always, $\{Z > M\}\searrow\{Z=\infty\}$.

4. Law of Total Probability – Discrete Random Variable Version: If X is a discrete random variable, with possible values $x_1, x_2, \dots$, and correspond probabilities $p_1, p_2, \dots$ and B be any events, then.

$$P(B) = \sum_i P(X=x_i)\cdot P(B\mid X=x_i) = \sum_i p_i\, P(B\mid X=x_i).$$

e.g. roll a die. flip a number of coins equal to the number showing on the die. Let $X = \#$ of Heads. $P(X=3)$.

B: $X=3$; Y: num on die, $Y=\{1,2,3,4,5,6\}$. use values of Y as partition.

$$P(X=3) = \sum_{i=1}^{6} P(Y=y)\cdot P(X=3 \mid Y=y)$$

$$= \sum_{i=3}^{6} P(Y=y)\cdot P(X=3 \mid Y=y). \rightarrow \frac{\binom{y}{3}}{2^y}.$$

$$= P(Y=3)\cdot P(X=3 \mid Y=3) + P(Y=4)\cdot P(X=3 \mid Y=4) + P(Y=5)\cdot P(X=3 \mid Y=5) + P(Y=6)\cdot P(X=3 \mid Y=6)$$

$$= \frac{1}{6}\cdot\left(P(X=3 \mid Y=3) + P(X=3 \mid Y=4) + P(X=3 \mid Y=5) + P(X=3 \mid Y=6)\right).$$

$$= \frac{1}{6}\cdot\left(\frac{\binom{3}{3}}{2^3} + \frac{\binom{4}{3}}{2^4} + \frac{\binom{5}{3}}{2^5} + \frac{\binom{6}{3}}{2^6}\right). = \frac{1}{6}.$$