MAT 137
Problem Set A

This problem set is intended to help you prepare for Test 1. It is not comprehensive: it only contains some problems that were not included in past problem sets or in past tutorials. You do not need to turn in any of these problems.

For further preparation review pre-class quizzes, lecture notes, Problem Set 1, Tutorials 1 through 5, and Unit 1 and 2 practice problems.

Problems

1. Let f be some function for which you know only that

$$\text{if } 0 < |x-3| < 1, \quad \text{then} \quad |f(x)-5| < 0.1.$$

Which of the following statements are necessarily true?

(a) If $|x-3| < 1$ and $x \neq 3$, then $|f(x)-5| < 0.1$.
(b) If $|x-3| < 1$, then $|f(x)-5| < 0.1$.
(c) If $|x-2.5| < 0.3$, then $|f(x)-5| < 0.1$.
(d) $\lim_{x\to 3} f(x) = 5$.
(e) If $0 < |x-3| < 2$, then $|f(x)-5| < 0.1$.
(f) If $0 < |x-3| < 0.5$, then $|f(x)-5| < 0.1$.
(g) If $0 < |x-3| < \frac{1}{4}$, then $|f(x)-5| < \frac{1}{4}(0.1)$.
(h) If $0 < |x-3| < 1$, then $|f(x)-5| < 0.2$.
(i) If $0 < |x-3| < 2$, then $|f(x)-4.95| < 0.05$.
(j) If $\lim_{x\to 3} f(x) = L$, then $4.9 \leq L \leq 5.1$.

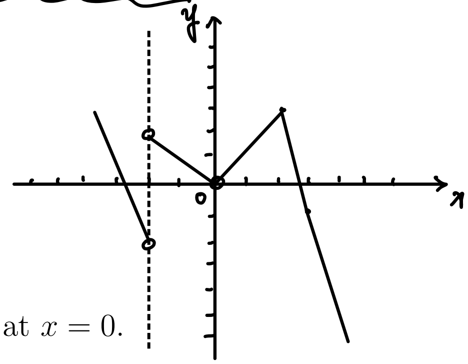
2. Prove the following version of the Squeeze Theorem:

Theorem. *Let f and g be functions with domain $\mathbb{R}$. IF*

- *for all x large enough, $f(x) \leq g(x)$*
- $\lim_{x\to\infty} f(x) = \infty$

THEN $\lim_{x\to\infty} g(x) = \infty$

3. Sketch the graph of a function g defined on $(-5, 5)$which satisfies the following properties at once:



- $\lim_{x \to 2} g(x) = 3$
- $\lim_{x \to 2} g(g(x)) = -1$
- $\lim_{x \to -2} g(x)$ does not exist
- $\lim_{x \to -2} [g(x)]^2$ exists
- g has a removable discontinuity at $x = 0$.

4. Let f be a function defined on an interval centered at 4.

   (a) Write the formal ϵ-δ definition of " f is continuous at 4".

   (b) Assume f is continuous at 4 and $f(4) > 0$. Prove that there exists an open interval I, centered at 4, such that

   $$\forall x \in I, f(x) > \frac{f(4)}{2}$$

   Write a proof directly from the above formal ϵ-δ definition of continuity. Do not use any of the limit laws or any other theorems.

5. Prove the following theorem:

   Theorem. *Let $a \in \mathbb{R}$ and let f be a function defined on an interval centered at a. IF f is continuous at a and $f(a) > 0$ THEN f is strictly positive on some interval centered at a.*

6. Prove the $\lim_{x \to 0^+} \frac{1}{x}$ does not exist from the formal $\varepsilon - \delta$ definition.
   Note that the formal definition of $\lim_{x \to a} f(x)$ does not exist and the definition of $\lim_{x \to a} f(x) = \infty$ are different.

7. Let $a \in \mathbb{R}$ and let f be a function with domain $\mathbb{R}$. Using the formal definition of the limit, prove that:

   IF $\lim_{x \to a} f(x) = \infty$ THEN $\lim_{x \to a} f(x)$ does not exist.

8. Define a sequence $\{x_n\}$ where $n \in \mathbb{Z}$ and $n \geq 0$ by the following rule:

   $$a_0 = 1, a_1 = 3, a_2 = 9, a_n = a_{n-1} + 3a_{n-2} + 9a_{n-3} \textit{ for } n \geq 3.$$

   Prove that $a_n = 3^n$ for all $n \in \mathbb{Z}$ and $n \geq 0$. Hint: use the induction method.

9. Here you will prove that $\lim\limits_{x\to\infty} \dfrac{x}{\ln x} = \infty$ without any use of L'Hopital's Rule.

   (a) For all positive integers $n \geq 1$, prove that $2^n \geq 2n$.

   (b) For all real numbers $x \geq 1$, prove that $2^x \geq x$.

   (c) Recall $e = 2.71828...$ is Euler's constant and e^x is the exponential function. Since $e > 2$, you may assume that $\lim\limits_{x\to\infty} \left(\frac{e}{2}\right)^x = \infty$. Show that

   $$\lim_{x\to\infty} \frac{e^x}{x} = \infty.$$

   (d) Using the formal definition of the limit, prove that

   $$\lim_{x\to\infty} \frac{x}{\ln x} = \infty.$$

10. A prime number is a positive integer $p \geq 2$ such that p can be divided, without remainder, only by itself and 1. The first few primes are $2, 3, 5, 7, 11, 13, 17$, and 19.

    For all real numbers $x \geq 1$, define the *prime counting function* $\pi(x)$ to be the number of primes $p \in \mathbb{Z}$ satisfying $p \leq x$. (Yes, the letter π here is used for a function instead of the special constant that we all know and love.)

    (a) Sketch the graph of $\pi(x)$ for $1 \leq x \leq 10$.

    (b) For which $a \in (1, \infty)$ is π discontinuous at a? No justification required.

    (c) Proven in 1896, the celebrated Prime Number Theorem states that

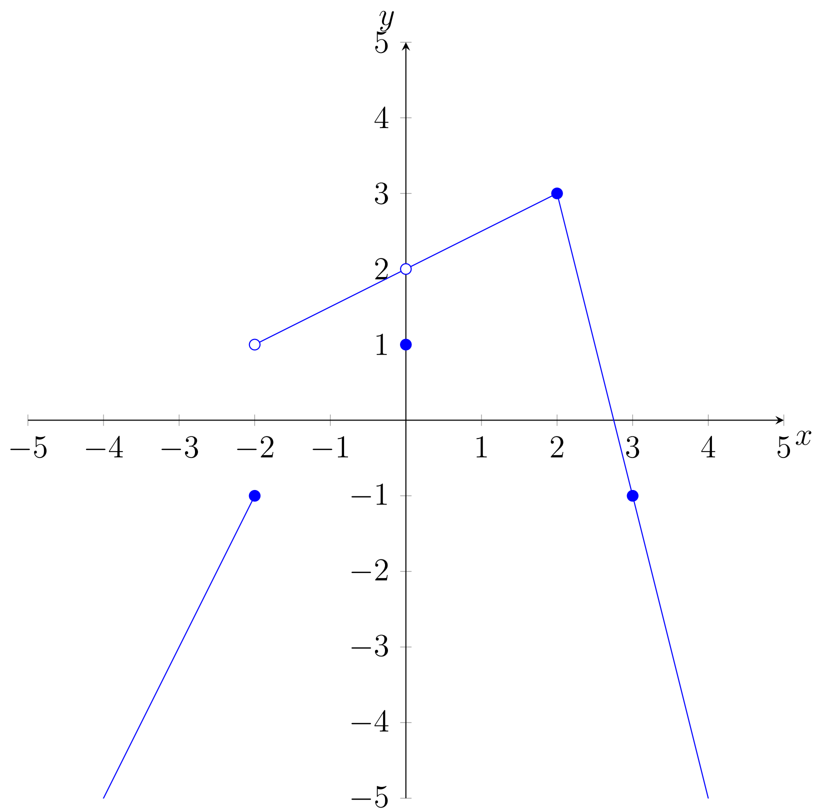
    $$\lim_{x\to\infty} \frac{\pi(x)}{\left(\frac{x}{\ln x}\right)} = 1.$$

    Assuming the Prime Number Theorem, show that there exists a real number $N \geq 1$ such that

    $$x > N \implies \frac{x}{2\ln x} < \pi(x) < \frac{2x}{\ln x}$$

    (d) Conclude that there are infinitely many primes.

Some answers and hints

1. The answer is "true" for exactly five of the statements.
2. Start by rewriting the first hypothesis as a formal mathematical statement. Your proof should follow a structure similar to the usual Squeeze Theorem (Video 2.13).
3. 

4. You can do (a). This is the solutions for (b):

Proof.

i. Take $\epsilon = \frac{f(4)}{2}$. By hypothesis, $\epsilon > 0$. I use this value of "ϵ" in the definition of " f is continuous at 4", and I get that:

$$\exists \delta > 0 \text{ such that } |x-4| < \delta \implies |f(x) - f(4)| < \epsilon$$

I will keep this value of δ.

ii. Then I define the interval I to be $(4-\delta, 4+\delta)$.
I will show that this interval satisfies

$$\forall x \in I, f(x) > \frac{f(4)}{2}$$

iii. Let $x \in I$. This is equivalent to $|x-4| < \delta$.
Therefore, it follows that $|f(x) - f(4)| < \epsilon = \frac{f(4)}{2}$. In particular,

$$-\frac{f(4)}{2} < f(x) - f(4) < \frac{f(4)}{2}$$

$$\frac{f(4)}{2} < f(x) < \frac{3f(4)}{2}$$

Thus, I have showed $\forall x \in I, f(x) > \frac{f(4)}{2}$.

$\square$

5. Use the same idea in Q4.

6. WTS:$\forall L \in \mathbb{R}, \exists \varepsilon > 0$ such that $\forall \delta > 0, \exists x \in R \ s.t.\ 0 < x < \delta \ and \ |f(x) - L| \geq \varepsilon.$

Proof. i) Let $L \in \mathbb{R}$ be fixed.

ii) Now we have two cases: $L \geq 0$ and $L < 0$.

iii) We first prove the case $L < 0$.
   - Take $\varepsilon = 1$.
   - Given any $\delta > 0$, let $x = \min\{\delta/2, 1\}$. We have $0 < x \leq \frac{\delta}{2}$ and $0 < x \leq 1$. Thus, we have $0 < x < \delta$ and $|f(x) - L| = |\frac{1}{x} - L| = \frac{1}{x} - L \geq 1 - L \geq 1 = \varepsilon$. We have proved the $L < 0$ case.

iv) We now prove the case $L \geq 0$.
   - Take $\varepsilon = 1$.
   - Given any $\delta > 0$, let $x = \min\left\{\delta/2, \frac{1}{L+1}\right\}$.
     We have $0 < x \leq \frac{\delta}{2}$ and $0 < x \leq \frac{1}{L+1}$. Thus, we have $0 < x < \delta$ and

$$f(x) = \frac{1}{x} \geq L + 1 \geq L$$

This shows that $|f(x) - L| = f(x) - L$, so

$$|f(x) - L| = \frac{1}{x} - L \geq (L+1) - L = 1 = \varepsilon.$$

We have proved the $L \geq 0$ case.

v) Therefore, we have proved $\lim_{x \to 0^+} \frac{1}{x}$ does not exist.

$\square$

7. The formal definitions for "$\lim_{x\to a} f(x) = \infty$" and "$\lim_{x\to a} f(x)$ does not exist" are absolutely necessary here. If you do not use the correct formal definitions, your proof cannot possibly be correct.

8. Let $S(n)$ be the statement "$a_n = 3^n$ for all $n \in \mathbb{Z}$ and $n \geq 0$".
Base case ($n = 0, 1, 2, 3$): $S(0)$ is true since $a_0 = 1 = 3^0$.
$S(1)$ is true since $a_1 = 3 = 3^1$.
$S(2)$ is true since $a_2 = 9 = 3^2$.
$S(3)$ is true since $a_3 = a_2 + 3a_1 + 9a_0 = 9 + 3 \cdot 3 + 9 \cdot 1 = 27 = 3^3$.
Inductive step: Fix some $k \geq 3$, and assume that for every $t \in Z$ satisfying $3 \leq t \leq k$, the statement $S(t)$ is true. We want to show that
$$S(k+1) : a_{k+1} = 3^{k+1}$$
follows.Now you can complete the rest of the proof.

9. (b) Define $n = \lfloor x \rfloor$ and use Question 9a. There should be no induction here.
(c) Use the results of Questions 2 and 9b. No formal limit definition is needed here.
(d) The formal limit definition is needed here. Use the result of Question 9c. At some point, your proof will need to explicitly use that the logarithm is an increasing function.

10. (a) Note $\pi(3.4) = \pi(3) = 2$ and $\pi(2.9) = 1$. Your graph should look vaguely similar to the floor function.
(b) π is discontinuous at $a \in (1, \infty)$ if and only if a is a prime number.
(d) Write the statement "there are infinitely many primes" in terms of limits. Then use Questions 2 and 9.