CSC165H1 Problem Set 2

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Question 1: Number representation

For each $n \in \mathbb{N}$ and $k \in \mathbb{Z}^+$, define $C(n,k)$ to be:

$$\exists a_1, ..., a_k \in \mathbb{N}, (\forall i \in \mathbb{Z}^+, 1 \leq i \leq k \Rightarrow a_i \leq i) \wedge (n = \sum_{i=1}^{k} a_i \cdot i!)$$

Prove that: $\forall n \in \mathbb{N}, \forall k \in \mathbb{Z}^+, n < (k+1)! \Rightarrow C(n,k)$

Proof.

Let $n \in \mathbb{N}$. I will prove $\forall k \in \mathbb{Z}^+, n < (k+1)! \Rightarrow C(n,k)$ by induction on k.

Base Case:

Let $k = 1$. I need to prove $n < (k+1)! \Rightarrow C(n,k)$.

First, assume $n < (k+1)!$. Then, I will prove $C(n,k)$. That is, I will prove:

$\exists a_1 \in \mathbb{N}, (\forall i \in \mathbb{Z}^+, 1 \leq i \leq 1 \Rightarrow a_i \leq i) \wedge (n = \sum_{i=1}^{1} a_i \cdot i!)$, since $k = 1$.

Let $a_1 = n$. We know $n \in \mathbb{N}$, and we know $n < (k+1)! = (1+1)! = 2! = 1 \cdot 2 = 2$ by assumption. Therefore, $0 \leq n \leq 1$, which implies $0 \leq a_1 \leq 1$ because $a_1 = n$

I will first prove that $\forall i \in \mathbb{Z}^+, 1 \leq i \leq 1 \Rightarrow a_i \leq i$.

Let $i \in \mathbb{Z}^+$. Assume $1 \leq i \leq 1$. I need to prove $a_i \leq i$.

Since $1 \leq i \leq 1$, $i = 1$. Therefore, $a_i = a_1 \leq 1 = i$ because I have shown $0 \leq a_1 \leq 1$ above. I have proven that $\forall i \in \mathbb{Z}^+, 1 \leq i \leq 1 \Rightarrow a_i \leq i$ as needed.

I will then prove that $n = \sum_{i=1}^{1} a_i \cdot i!$.

$\sum_{i=1}^{1} a_i \cdot i! = a_1 \cdot 1! = a_1 \cdot 1 = a_1 = n$.

I have proven $n = \sum_{i=1}^{1} a_i \cdot i!$. Therefore, I have proven $C(n,k)$, which completes the proof for the base case.

Inductive Step:

Let $m \in \mathbb{Z}^+$. Assume $n < (m+1)! \Rightarrow C(n,m)$. *(This is the induction hypothesis)*

I need to prove that $n < ((m+1)+1)! \Rightarrow C(n, m+1)$.

Assume $n < ((m+1)+1)! = (m+2)!$. I need to prove $C(n, m+1)$.

Since $m > 0$, $m+1 > 1$. Since $m+1 > 1$, $(m+1)! = \prod_{j=1}^{m+1} j > 0$.

Since $n \geq 0$ and $(m+1)! > 0$, we know $\frac{n}{(m+1)!} \geq 0$. Therefore, $\frac{n}{(m+1)!} - 1 \geq -1$.

By the fact from a worksheet that $\forall x \in \mathbb{R}, x - 1 < \lfloor x \rfloor \leq x$, we can conclude:

$$-1 \leq \frac{n}{(m+1)!} - 1 < \left\lfloor \frac{n}{(m+1)!} \right\rfloor \leq \frac{n}{(m+1)!}$$

$$0 \leq \left\lfloor \frac{n}{(m+1)!} \right\rfloor \leq \frac{n}{(m+1)!} \qquad \left(\text{since } \left\lfloor \frac{n}{(m+1)!} \right\rfloor \in \mathbb{Z}\right)$$

Therefore, $\left\lfloor \frac{n}{(m+1)!} \right\rfloor \in \mathbb{N}$. Let $a_{m+1} = \left\lfloor \frac{n}{(m+1)!} \right\rfloor$. Then we have:

$$\frac{n}{(m+1)!} - 1 < \left\lfloor \frac{n}{(m+1)!} \right\rfloor = a_{m+1} \leq \frac{n}{(m+1)!}$$
$$(\frac{n}{(m+1)!} - 1) \cdot (m+1)! < a_{m+1}(m+1)! \leq \frac{n}{(m+1)!} \cdot (m+1)! \qquad \text{(because } (m+1)! > 0\text{)}$$
$$n - (m+1)! < a_{m+1}(m+1)! \leq n$$
$$n - a_{m+1}(m+1)! < (m+1)!$$

Also, since $a_{m+1}(m+1)! \leq n$ we have shown above, we have:

$$a_{m+1}(m+1)! \leq n$$
$$-n \leq -a_{m+1}(m+1)!$$
$$0 = n - n \leq n - a_{m+1}(m+1)!$$

$n - a_{m+1}(m+1)! \in \mathbb{Z}$ because n, a_{m+1} and $(m+1)!$ are all integers. Since $n - a_{m+1}(m+1)! \geq 0$ and $n - a_{m+1}(m+1)! \in \mathbb{Z}$, $n - a_{m+1}(m+1)! \in \mathbb{N}$. Then, by the induction hypothesis, since $n - a_{m+1}(m+1)! \in \mathbb{N}$ and since $n - a_{m+1}(m+1)! < (m+1)!$ holds as proven above, we can conclude that $C(n - a_{m+1}(m+1)!, m)$ is True. That is:

$$\exists a_1, ..., a_m \in \mathbb{N}, (\forall i \in \mathbb{Z}^+, 1 \leq i \leq m \Rightarrow a_i \leq i) \land (n - a_{m+1}(m+1)! = \textstyle\sum_{i=1}^{m} a_i \cdot i!)$$

I keep these $a_1, ..., a_m$. Now I will prove $C(n, m+1)$ by proving that for these $a_1, ..., a_m$ and the a_{m+1} I have, $(\forall i \in \mathbb{Z}^+, 1 \leq i \leq m+1 \Rightarrow a_i \leq i) \land (n = \sum_{i=1}^{m+1} a_i \cdot i!)$ is True.

First, I need to prove $\forall i \in \mathbb{Z}^+, 1 \leq i \leq m+1 \Rightarrow a_i \leq i$.

Let $i \in \mathbb{Z}^+$. Assume $1 \leq i \leq m+1$. I will prove $a_i \leq i$ by cases. Since $1 \leq m < m+1$ and $1 \leq i \leq m+1$, one of the following case must be true: $1 \leq i \leq m$ or $m < i \leq m+1$. Divide up the proof into these two cases:

Case 1: Assume $1 \leq i \leq m$.

I have proven that $\forall i \in \mathbb{Z}^+, 1 \leq i \leq m \Rightarrow a_i \leq i$ above from the fact that $C(n - a_{m+1}(m+1)!, m)$ is true using the induction hypothesis. Since $i \in \mathbb{Z}^+$ and $1 \leq i \leq m$ by the assumption of case 1, we can conclude that $a_i \leq i$

Case 2: Assume $m < i \leq m+1$.

Since $i \in \mathbb{Z}^+$, we have $i = m+1$. Then we have:

$$a_i = a_{m+1} = \left\lfloor \frac{n}{(m+1)!} \right\rfloor \leq \frac{n}{(m+1)!}$$
$$a_i \leq \frac{n}{(m+1)!} < \frac{(m+2)!}{(m+1)!} \qquad \text{(since } n < (m+2)! \text{ by assumption)}$$
$$a_i < \frac{(m+2)!}{(m+1)!} = \frac{(m+1)! \cdot (m+2)}{(m+1)!} = m+2$$
$$a_i \leq m+1 = i \qquad \text{(because } a_{m+1} \in \mathbb{N}\text{)}$$

Since for all possible cases, I have proven $a_i \leq i$, we can conclude $a_i \leq i$ is always true. Therefore, I have proven $\forall i \in \mathbb{Z}^+, 1 \leq i \leq m+1 \Rightarrow a_i \leq i$.

Then, I will prove $n = \sum_{i=1}^{m+1} a_i \cdot i!$.

I have proven above that $n - a_{m+1}(m+1)! = \sum_{i=1}^{m} a_i \cdot i!$ from the fact that $C(n - a_{m+1}(m+1)!, m)$ is true using the induction hypothesis.

Therefore, we have:

$$\begin{aligned} n - a_{m+1}(m+1)! &= \sum_{i=1}^{m} a_i \cdot i! \\ n &= a_{m+1}(m+1)! + \sum_{i=1}^{m} a_i \cdot i! \\ n &= \sum_{i=1}^{m+1} a_i \cdot i! \end{aligned}$$

I have proven $n = \sum_{i=1}^{m+1} a_i \cdot i!$ as needed.

I have proven $\forall i \in \mathbb{Z}^+, 1 \leq i \leq m+1 \Rightarrow a_i \leq i$ and $n = \sum_{i=1}^{m+1} a_i \cdot i!$. That is, I have proven $C(n, m+1)$. This completes the proof for the inductive step and thus completes the proof.

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Question 2: Induction

(a) (i) Prove: $\forall m \in \mathbb{N}, P(m, 0)$

Proof.

Let $m \in \mathbb{N}$. I need to prove $P(m, 0)$. That is, I need to prove $|F_{m,0}| = \frac{(m+n)!}{m! \cdot 0!} = \frac{m!}{m!} = 1$.
We know $A_m = \{a \mid a \in \mathbb{N} \wedge a \leq m\}$, and $B_0 = \{b \mid b \in \mathbb{N} \wedge b \leq 0\} = \{0\}$ because 0 is the only natural number that is less than or equal to 0, meaning that it is the only possible element that could satisfy the condition of B_0.

For the set $F_{m,0}$, we know all elements of it must be functions with domain A_m and codomain B_0, and must satisfy the following conditions:

$$[\forall k, l \in A_m, k \leq l \Rightarrow f(k) \leq f(l)] \wedge f(m) = 0.$$

All functions in $F_{m,0}$ must have domain A_m and codomain B_0, meaning that all values in A_m must be mapped to the only value in B_0, which is 0. Since there is only one possible way of mapping, there is only one function that has the desired domain and codomain: $f(a) = 0$, where $a \in A_m$. This function is the only potential element of $F_{m,0}$.

Now I will verify if this function f satisfies all conditions of $F_{m,0}$.

Let $k, l \in A_m$. Assume $k \leq l$. I want to show $f(k) \leq f(l)$. Since $\forall a \in A_m, f(a) = 0$, we have $f(k) = 0 = f(l)$, which satisfies $f(k) \leq f(l)$.

Then I will want to show $f(m) = 0$. Since $m \in \mathbb{N}$ and $m \leq m$, we know $m \in A_m$. Therefore, $f(m) = 0$ because $\forall a \in A_m, f(a) = 0$.

I have shown f satisfies the conditions that $[\forall k, l \in A_m, k \leq l \Rightarrow f(k) \leq f(l)] \wedge f(m) = 0$. Therefore, f is an element of $F_{m,0}$ and is its only element. Thus, $|F_{m,0}| = 1$ as needed.

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(ii) Prove: $\forall n \in \mathbb{N}, P(0, n)$

Proof.

Let $n \in \mathbb{N}$. I need to prove $P(0, n)$. That is, I need to prove $|F_{0,n}| = \frac{(0+n)!}{0! \cdot n!} = \frac{n!}{n!} = 1$.
We know $B_n = \{b \mid b \in \mathbb{N} \wedge b \leq n\}$, and $A_0 = \{a \mid a \in \mathbb{N} \wedge a \leq 0\} = \{0\}$ because 0 is the only natural number that is less than or equal to 0, meaning that it is the only possible element that could satisfy the condition of A_0.

For the set $F_{0,n}$, we know all elements of it must be functions with domain A_0 and codomain B_n, and must satisfy the following conditions:

$$[\forall k, l \in A_0, k \leq l \Rightarrow f(k) \leq f(l)] \wedge f(0) = n.$$

All functions in $F_{0,n}$ must have domain A_0, meaning that there is only one value in its domain, which is 0. Also, all functions in $F_{0,n}$ must satisfies the condition that $f(0) = n$. Since there is only one value in the domain, and this value must be mapped to n, there is only one possible way of mapping and therefore only one potential function: $f(a) = n$, where $a \in A_0$. This is the only potential element of $F_{0,n}$.

Since $n \in \mathbb{N}$ and $n \leq n$, we have $0 \in B_n$. Therefore, the function f has the desired domain and codomain.

Now I will verify if this function f satisfies all conditions of $F_{0,n}$.

Let $k, l \in A_0$. Assume $k \leq l$. I have $f(k) = f(l) = f(0) = n$, which satisfies $f(k) \leq f(l)$ and $f(0) = n$

I have shown f satisfies the conditions that $[\forall k, l \in A_0, k \leq l \Rightarrow f(k) \leq f(l)] \wedge f(0) = n$. Therefore, f is an element of $F_{0,n}$ and is its only element. Thus, $|F_{0,n}| = 1$ as needed.

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(iii) Prove: $\forall m, n \in \mathbb{N}, P(m, n+1) \wedge P(m+1, n) \Rightarrow P(m+1, n+1)$

Proof.

Let $m, n \in \mathbb{N}$. Assume $P(m, n+1) \wedge P(m+1, n)$. That is, assume $|F_{m,n+1}| = \frac{(m+n+1)!}{m! \cdot (n+1)!}$ and $|F_{m+1,n}| = \frac{(m+n+1)!}{(m+1)! \cdot n!}$. I need to prove $P(m+1, n+1)$. That is, I need to prove $|F_{m+1,n+1}| = \frac{(m+n+2)!}{(m+1)! \cdot (n+1)!}$.

We know all functions in $F_{m+1,n+1}$ must satisfies that $\forall k, l \in A_{m+1}, k \leq l \Rightarrow f(k) \leq f(l)$ and $f(m+1) = n+1$. Therefore, since $m, m+1 \in A_{m+1}$ and $m < m+1$, we can conclude that for all functions in $F_{m+1,n+1}$, $f(m) \leq f(m+1) = n+1$. Therefore, all functions in $F_{m+1,n+1}$ must satisfy one and only one of the following conditions: $f(m) = n+1$ or $f(m) \leq n$.

Therefore, based on these two conditions, we can break $F_{m+1,n+1}$ into two subsets: $C_{m+1,n+1} = \{f \mid f \in F_{m+1,n+1} \wedge f(m) = n+1\}$ and $D_{m+1,n+1} = \{f \mid f \in F_{m+1,n+1} \wedge f(m) \leq n\}$. Since every element of $F_{m+1,n+1}$ is also an element of one and only one of the two sets $C_{m+1,n+1}$ and $D_{m+1,n+1}$, we know $|F_{m+1,n+1}| = |C_{m+1,n+1}| + |D_{m+1,n+1}|$.

First, I will show $|C_{m+1,n+1}| = |F_{m,n+1}|$. We know, by conditions of set $F_{m+1,n+1}$, that for all functions in $C_{m+1,n+1}$, the value $m+1$ in its domain must be mapped to $n+1$. Therefore, for all functions in $C_{m+1,n+1}$, the way of mapping only depends on other values in its codomain, i.e., only depends on values in $A_{m+1} \setminus m+1 = A_m$ because $f(m+1) = n+1$ always holds. Therefore, to find $|C_{m+1,n+1}|$, we only need to find the number of ways of mapping from A_m to $B_n + 1$ that satisfies $f(m) = n+1$ and $\forall k, l \in A_m, k \leq l \Rightarrow f(k) \leq f(l)$, which is exactly $|F_{m,n+1}|$.

Then, I will show $|D_{m+1,n+1}| = |F_{m+1,n}|$. We know, by conditions of set $F_{m+1,n+1}$, that for all functions in $D_{m+1,n+1}$, the value $m+1$ in its domain must be mapped to $n+1$. Therefore, for all functions in $D_{m+1,n+1}$, the way of mapping only depends on other values in its codomain, i.e., only depends on values in $A_{m+1} \setminus m+1 = A_m$ because $f(m+1) = n+1$ always holds. Therefore, to find $|D_{m+1,n+1}|$, we only need to find the number of ways of mapping from A_m to $B_n + 1$ that satisfies $f(m) \leq n$ and $\forall k, l \in A_m, k \leq l \Rightarrow f(k) \leq f(l)$. Similarly, the number of ways of mapping in $F_{m+1,n}$ also only depends on A_m since $f(m+1)$ is always mapped for $f \in F_{m+1,n}$ so it won't affect the number of ways of mapping. For functions in $F_{m+1,n}$, we know $\forall k, l \in A_{m+1}, k \leq l \Rightarrow f(k) \leq f(l)$ and $f(m+1) = n$. Equivalently, they also satisfies $f(m) \leq f(m+1) = n$ since $m < m+1$. Therefore, $F_{m+1,n}$ contains all the ways of mapping that satisfy $\forall k, l \in A_m \subset A_{m+1}, k \leq l \Rightarrow f(k) \leq f(l)$ and $f(m) \leq n$, which is exactly what we are looking for. Thus, $|D_{m+1,n+1}| = |F_{m+1,n}|$.

I have shown $|C_{m+1,n+1}| = |F_{m,n+1}|$ and $|D_{m+1,n+1}| = |F_{m+1,n}|$. Therefore, we have:

$$|F_{m+1,n+1}| = |C_{m+1,n+1}| + |D_{m+1,n+1}|$$
$$|F_{m+1,n+1}| = |F_{m,n+1}| + |F_{m+1,n}|$$
$$|F_{m+1,n+1}| = \frac{(m+n+1)!}{m! \cdot (n+1)!} + \frac{(m+n+1)!}{(m+1)! \cdot n!} \qquad \text{(by assumption)}$$
$$|F_{m+1,n+1}| = \frac{(m+n+1)! \cdot (m+1)}{(m+1)! \cdot (n+1)!} + \frac{(m+n+1)! \cdot (n+1)}{(m+1)! \cdot (n+1)!}$$
$$|F_{m+1,n+1}| = \frac{(m+n+1)! \cdot (m+1) + (m+n+1)! \cdot (n+1)}{(m+1)! \cdot (n+1)!} = \frac{(m+n+1)! \cdot (m+n+2)}{(m+1)! \cdot (n+1)!}$$
$$|F_{m+1,n+1}| = \frac{(m+n+2)!}{(m+1)! \cdot (n+1)!} \text{ as needed.}$$

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(b) Using results from (a), prove: $P(1,1) \wedge P(2,2)$

Proof.

From (a)(i) and (ii), we know $\forall m \in \mathbb{N}, P(m,0)$ and $\forall n \in \mathbb{N}, P(0,n)$. Then since $1,2 \in \mathbb{N}$, we can conclude $P(1,0)$ and $P(0,1)$ and $P(2,0)$ and $P(0,2)$are true.

From (a)(iii), we know $\forall m,n \in \mathbb{N}, P(m,n+1) \wedge P(m+1,n) \Rightarrow P(m+1,n+1)$. Since $0 \in \mathbb{N}$, and we know $P(0,0+1) \wedge P(0+1,0)$ is true as proven above, we can conclude $P(0+1,0+1)$ is true, i.e., $P(1,1)$ is true. And since $1,0 \in \mathbb{N}$, and we know $P(1,0+1) \wedge P(1+1,0)$, we can conclude $P(1+1,0+1)$ is true, i.e., $P(2,1)$ is true. Also, since we know $P(0,1+1) \wedge P(0+1,1)$, we can conclude $P(0+1,1+1)$ is true, i.e., $P(1,2)$ is true. Finally, since we know $P(1,1+1) \wedge P(1+1,1)$ is true, we can conclude $P(1+1,1+1)$ is true, i.e., $P(2,2)$ is true.

I have proven $P(1,1) \wedge P(2,2)$ as needed.

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(c) Using induction and result from (a), prove: $\forall t \in \mathbb{N}, Q(t)$.

Proof.

I will prove this using induction on t.

Base Case:

Let t = 0. I will prove Q(t). That is, I will prove $\forall m, n \in \mathbb{N}, m + n = 0 \Rightarrow P(m, n)$.

Let $m, n \in \mathbb{N}$. Assume $m + n = 0$. Therefore, $m = 0$ and $n = 0$. I need to prove $P(m, n)$. That is, I need to prove $P(0, 0)$.

From (a)(i), we know $\forall m \in \mathbb{N}, P(m, 0)$. Since $m \in \mathbb{N}$, we can conclude $P(0, 0)$ is true as needed.

Inductive Step:

Let $s \in \mathbb{N}$. Assume $Q(s)$. That is, assume $\forall m_0, n_0 \in \mathbb{N}, m_0 + n_0 = s \Rightarrow P(m_0, n_0)$. I need to prove $Q(s + 1)$. That is, I need to prove $\forall m, n \in \mathbb{N}, m + n = s + 1 \Rightarrow P(m, n)$.

Let $m, n \in \mathbb{N}$. Assume $m + n = s + 1$. I will prove $P(m, n)$ by cases.

Since $m, n \in \mathbb{N}$ and $m + n = s + 1 \geq 1$, we know $m \geq 0$, $n \geq 0$ and $m + n \geq 1$, so it is impossible that $m = n = 0$. Therefore, one of the following 3 cases must be true: $m = 0$ and $n \geq 1$, $m \geq 1$ and $n = 0$, or $m \geq 1$ and $n \geq 1$. Divide up the proof into these three cases:

Case 1: Assume $m = 0$ and $n \geq 1$.

From (a)(ii), we know $\forall n \in \mathbb{N}, P(0, n)$. Since $n \in \mathbb{N}$, we can conclude $P(0, n)$ is true. That is, $P(m, n)$ is true since $m = 0$.

Case 2: Assume $m \geq 1$ and $n = 0$.

From (a)(i), we know $\forall m \in \mathbb{N}, P(m, 0)$. Since $m \in \mathbb{N}$, we can conclude $P(m, 0)$ is true. That is, $P(m, n)$ is true since $n = 0$.

Case 3: Assume $m \geq 1$ and $n \geq 1$.

Since $m + n = s + 1$, we have $m + (n - 1) = s$. Since $n \geq 1$ by assumption of case 3, we have $n - 1 \geq 0$ and therefore $n - 1 \in \mathbb{N}$. Since $n - 1 \in \mathbb{N}$, $m \in \mathbb{N}$, and $m + (n - 1) = s$, by the induction hypothesis, we can conclude that $P(m, n - 1)$ is true.

Since $m + n = s + 1$, we have $n + (m - 1) = s$. Since $m \geq 1$ by assumption of case 3, we have $m - 1 \geq 0$ and therefore $m - 1 \in \mathbb{N}$. Since $m - 1 \in \mathbb{N}$, $n \in \mathbb{N}$, and $(m - 1) + n = s$, by the induction hypothesis, we can conclude that $P(m - 1, n)$ is true.

From (a)(iii), we know $\forall m, n \in \mathbb{N}, P(m, n + 1) \wedge P(m + 1, n) \Rightarrow P(m + 1, n + 1)$.

Since $m - 1, n - 1 \in \mathbb{N}$, and since $P(m - 1, (n - 1) + 1) \wedge P((m - 1) + 1, n - 1)$ is true as prove above, we can conclude $P((m - 1) + 1, (n - 1) + 1)$ is true. That is, $P(m, n)$ is true.

I have proven that for all possible cases, $P(m, n)$ is true. Therefore, we can conclude $P(m, n)$ is always true. This completes the inductive step and thus the proof.

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(d) Using result from (c), prove: $\forall m, n \in \mathbb{N}, P(m, n)$

Proof.

Let $m, n \in \mathbb{N}$. I need to prove $P(m, n)$.

Let $s = m + n$. Since $m, n \in \mathbb{N}$, $s \in \mathbb{N}$.

From (c), we know $\forall t \in \mathbb{N}, Q(t)$. Since $s \in \mathbb{N}$, we can conclude $Q(s)$ is true. That is, $\forall m_0, n_0 \in \mathbb{N}, m_0 + n_0 = s \Rightarrow P(m_0, n_0)$.

Since $m, n \in \mathbb{N}$ and $m + n = s$, we can conclude $P(m, n)$ as needed.

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Question 3: Asymptotic notation

a) Disprove that $n^n \in \mathcal{O}(n!)$.

Proof.

I will prove $n^n \notin \mathcal{O}(n!)$ by contradiction.

First, assume for a contradiction that this statement is false, i.e., assume $n^n \in \mathcal{O}(n!)$ is True. That is, I assume for a contradiction that $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^n \leq c \cdot n!$ by the definition of Big-O. I will derive a contradiction from this.

I keep these c and n_0. Then for c and n_0, I know the statement $\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^n \leq c \cdot n!$ is true.

Let $n' = \lceil c + n_0 \rceil + 1$.

Since $c, n_0 \in \mathbb{R}^+$, $c + n_0 \in \mathbb{R}^+$. Then, by the fact from a worksheet that $\forall x \in \mathbb{R}, x \leq \lceil x \rceil < x + 1$, we can conclude:

$$\begin{aligned} 0 < c + n_0 &\leq \lceil c + n_0 \rceil \\ 1 &< \lceil c + n_0 \rceil + 1 = n' \\ 2 &\leq n' && \text{(because } n' \in \mathbb{Z}) \end{aligned}$$

We can also conclude:

$$\begin{aligned} c < c + n_0 \leq \lceil c + n_0 \rceil &< \lceil c + n_0 \rceil + 1 \\ c &< n' \quad \text{, and} \\ n_0 < c + n_0 \leq \lceil c + n_0 \rceil &< \lceil c + n_0 \rceil + 1 \\ n_0 &< n', \end{aligned}$$

Since $n' \in \mathbb{Z}$ and $n' \geq 2$, $n' \in \mathbb{N}$. Then, by the fact that $\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^n \leq c \cdot n!$, we can conclude $n'^{n'} \leq c \cdot n'!$ because we have proven $n_0 < n'$, which satisfies $n' \geq n_0$.

Then, we have:

$$\begin{aligned} 0 < n'^{n'} &\leq c \cdot n'! \\ \ln(n'^{n'}) &\leq \ln(c \cdot n'!) \\ n' \ln n' &\leq \ln c + \ln(n'!) = \ln c + \ln[1 \cdot, ..., \cdot (n'-1) \cdot n'] \\ \sum_{i=1}^{n'} \ln n' = n' \ln n' &\leq \ln c + [\ln 1 +, ..., + \ln(n'-1) + \ln n'] = \ln c + \sum_{i=1}^{n'} \ln i \\ \sum_{i=1}^{n'} (\ln n' - \ln i) = \sum_{i=1}^{n'} (\ln \frac{n'}{i}) &\leq \ln c \end{aligned}$$

But on the other hand, I know $\sum_{i=1}^{n'} (\ln \frac{n'}{i}) = \ln \frac{n'}{1} + \sum_{i=2}^{n'} (\ln \frac{n'}{i}) = \ln n' + \sum_{i=2}^{n'} (\ln \frac{n'}{i})$ since $n' \geq 2$. $\sum_{i=2}^{n'} (\ln \frac{n'}{i}) \geq 0$ because for every $2 \leq i \leq n'$, $\frac{n'}{i} \geq 1$ and therefore $\ln \frac{n'}{i} \geq 0$. We have proven above that $c < n'$, so we have:

$$c < n'$$

$$\ln c < \ln n' \leq \ln n' + \sum_{i=2}^{n'} (\ln \frac{n'}{i}) = \sum_{i=1}^{n'} (\ln \frac{n'}{i}) \qquad \text{(since } \sum_{i=2}^{n'} (\ln \frac{n'}{i}) \geq 0\text{)}$$

$$\ln c < \sum_{i=1}^{n'} (\ln \frac{n'}{i})$$

I have proven both $\sum_{i=1}^{n'} (\ln \frac{n'}{i}) > \ln c$ and $\sum_{i=1}^{n'} (\ln \frac{n'}{i}) \leq \ln c$ are true, which is a contradiction. Thus, the statement $n^n \in \mathcal{O}(n!)$ is false.

I have proven $n^n \notin \mathcal{O}(n!)$ as needed.

■

b) Prove that if $a, b \in \mathbb{R}$ and $b > 0$, then $(n+a)^b \in \Theta(n^b)$.

Translation. $\forall a, b \in \mathbb{R}, b > 0 \Rightarrow (n+a)^b \in \Theta(n^b)$

Proof.

Let $a, b \in \mathbb{R}$. Assume $b > 0$. I will prove $(n+a)^b \in \Theta(n^b)$ by cases. Since $a \in \mathbb{R}$, one of the following cases must be true: $a < 0$ or $a \geq 0$. Divide up the proof into these two cases:

Case 1: Assume $a < 0$.
I will prove $(n+a)^b \in \Theta(n^b)$. That is, by the definition of Θ, I will prove:

$\exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 n^b \leq (n+1)^b \leq c_2 n^b$

Let $c_1 = \frac{1}{2^b}$. $c_1 \in \mathbb{R}^+$ because $2 > 0$ and then $2^b > 0$. Let $c_2 = 1 \in \mathbb{R}^+$. Let $n_0 = -2a+1$. $n_0 \in \mathbb{R}^+$ because $a < 0$ by assumption and then $-2a > 0$ and then $n_0 = -2a+1 > 1$.
Let $n \in \mathbb{N}$. Assume $n \geq n_0$. I need to prove $c_1 n^b \leq (n+1)^b \leq c_2 n^b$.
Since $n \geq n_0$ by assumption, we have:

$$\begin{aligned}
n &\geq n_0 = -2a + 1 > -2a \\
-n &< 2a \\
n = -n + 2n &< 2a + 2n = 2(a+n) \\
n^b &< [2(a+n)]^b = 2^b(a+n)^b && \text{(because } b > 0 \text{ and } n > 0\text{)} \\
c_1 n^b &< 2^b(a+n)^b \cdot c_1 = (a+n)^b && \text{(because } c_1 > 0\text{)}
\end{aligned}$$

Also, we have: $n \geq n_0 = -2a + 1 > -2a$

$$\begin{aligned}
n > n + a &> -a > 0 && \text{(because } a < 0 \text{ by assumption)} \\
c_2 n^b = n^b &> (n+a)^b && \text{(because } b > 0\text{)}
\end{aligned}$$

I have proven $c_1 n^b < (a+n)^b < c_2 n^b$, which satisfies $c_1 n^b \leq (n+1)^b \leq c_2 n^b$. Therefore, I have proven $(n+a)^b \in \Theta(n^b)$.

Case 2: Assume $a \geq 0$.
I will prove $(n+a)^b \in \Theta(n^b)$. That is, by the definition of Θ, I will prove:

$\exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 n^b \leq (n+1)^b \leq c_2 n^b$

Let $c_1 = 1 \in \mathbb{R}^+$. Let $c_2 = 2^b \in \mathbb{R}^+$. Let $n_0 = a+1$. $n_0 = a+1 \in \mathbb{R}^+$ since $a \geq 0$ by assumption. Let $n \in \mathbb{N}$. Assume $n \geq n_0$. I need to prove $c_1 n^b \leq (n+1)^b \leq c_2 n^b$.
Since $a \geq 0$ by assumption, we have:

$$\begin{aligned}
n + a &\geq n > 0 \\
(n+a)^b &\geq n^b = c_1 b && \text{(because } b > 0 \text{ by assumption)}
\end{aligned}$$

Since $n \geq n_0$ by assumption, we have:

$$\begin{aligned}
n &\geq n_0 = a + 1 > a \geq 0 \\
n + n &> a + n \geq 0 \\
c_2 n^b = 2^b n^b = (2n)^b &> (a+n)^b && \text{(because } b > 0 \text{ by assumption)}
\end{aligned}$$

I have proven $c_1 n^b \leq (a+n)^b < c_2 n^b$, which satisfies $c_1 n^b \leq (n+1)^b \leq c_2 n^b$. Therefore, I have proven $(n+a)^b \in \Theta(n^b)$.

Since for all possible cases, I have proven $(n+a)^b \in \Theta(n^b)$, we can conclude this statement is always true. This completes the proof.

■

Question 4: More asymptotic notation

a) Prove: if $f : \mathbb{N} \to \mathbb{R}^{\geq 0}, k \in \mathbb{R}^+$, and $f(n) \in \mathcal{O}(n^k)$, then $\log_2(f(n)) \in \mathcal{O}(\log_2 n)$.

Proof.

Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Let $k \in \mathbb{R}^+$. Assume $f(n) \in \mathcal{O}(n^k)$. That is, assume:

$$\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow f(n) \leq c_1 \cdot n^k$$

I need to prove $\log_2(f(n)) \in \mathcal{O}(\log_2 n)$. That is, I need to prove:

$$\exists c_2, n_2 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_2 \Rightarrow \log_2(f(n)) \leq c_2 \cdot \log_2 n$$

Let $c_2 = k + \log_2(c_1 + 1)$. We know $c_2 \in \mathbb{R}^+$ because $c_1 + 1 > 1$ so $\log_2(c_1 + 1) > 0$ and therefore $c_2 > 0$. Let $n_2 = max(n_1, 2) \in \mathbb{R}^+$. Let $n \in \mathbb{N}$. Assume $n \geq n_2$. Then we know $n \geq n_1$ and $n \geq 2$. Therefore, $\log_2 n \geq \log_2 2 = 1$. I need to prove $\log_2(f(n)) \leq c_2 \cdot \log_2 n$.

Since we know $n \in \mathbb{N}$ and $n \geq n_1$, then by the assumption that $f(n) \in \mathcal{O}(n^k)$, we can conclude that $f(n) \leq c_1 \cdot n^k$. Then we have:

$$
\begin{aligned}
f(n) &\leq c_1 \cdot n^k \\
\log_2(f(n)) &\leq \log_2(c_1 \cdot n^k) && \text{(because } 2 > 1,\ f(n) > 0 \text{ and } c_1 \cdot n^k > 0) \\
\log_2(f(n)) &\leq \log_2((c_1 + 1)n^k) && \text{(because } (c_1 + 1)n^k > c_1 \cdot n^k) \\
\log_2(f(n)) &\leq \log_2(c_1 + 1) + k \log_2 n \leq \log_2(c_1 + 1) \cdot \log_2 n + k \log_2 n && \text{(since } \log_2 n \geq 1) \\
\log_2(f(n)) &\leq (\log_2(c_1 + 1) + k) \cdot \log_2 n = c_2 \log_2 n
\end{aligned}
$$

I have proven $\log_2(f(n)) \leq c_2 \log_2 n$ as needed.

■

b) Prove: if $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f_1 \in \mathcal{O}(g_1)$, and $f_2 \in \mathcal{O}(g_2)$, then $f_1 + f_2 \in \mathcal{O}(max(g1, g2))$.

Proof.

Let $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Assume $f_1 \in \mathcal{O}(g_1)$ and $f_2 \in \mathcal{O}(g_2)$. That is, assume:

$\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow f_1(n) \leq c_1 \cdot g_1(n)$

and

$\exists c_2, n_2 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_2 \Rightarrow f_2(n) \leq c_2 \cdot g_2(n)$

I want to prove $f_1 + f_2 \in \mathcal{O}(max(g1, g2))$. That is, I want to prove:

$\exists c_3, n_3 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_3 \Rightarrow (f_1 + f_2)(n) \leq c_3 \cdot max(g_1, g_2)(n)$

Let $c_3 = c_1 + c_2 \in \mathbb{R}^+$. Let $n_3 = max(n_1, n_2) \in \mathbb{R}^+$. Let $n \in \mathbb{N}$. Assume $n \geq n_3$. I need to prove $(f_1 + f_2)(n) \leq c_3 \cdot max(g_1, g_2)(n)$.

Since $n \geq n_3 = max(n_1, n_2)$, we have $n \geq n_1$ and $n \geq n_2$. Therefore, by the assumptions that $f_1 \in \mathcal{O}(g_1)$ and $f_2 \in \mathcal{O}(g_2)$, we can conclude $f_1(n) \leq c_1 \cdot g_1(n)$ and $f_2(n) \leq c_2 \cdot g_2(n)$.

Therefore, we have:

$$
\begin{aligned}
f_1(n) + f_2(n) &\leq c_1 \cdot g_1(n) + c_2 \cdot g_2(n) \\
(f_1 + f_2)(n) &\leq c_1 \cdot g_1(n) + c_2 \cdot g_2(n) \leq c_1 \cdot max(g_1(n), g_2(n)) + c_2 \cdot max(g_1(n), g_2(n)) \\
(f_1 + f_2)(n) &\leq (c_1 + c_2) \cdot max(g_1(n), g_2(n)) = c_3 \cdot max(g_1, g_2)(n)
\end{aligned}
$$

I have proven $(f_1 + f_2)(n) \leq c_3 \cdot max(g_1, g_2)(n)$ as needed.

■