MAT 137
Tutorial #5– Continuity and limit computations
October 18/19 , 2022
Due on Thursday, Oct 20 by 11:59pm via GradeScope

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

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- I understand that the collaboration of this worksheet is only among this group. I have not violated this rule while writing this worksheet.
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By signing this document, I agree that the statements above are true.

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1. Show that f is continuous at x_0 if and only if $\lim_{h\to 0}[f(x_0+h)-f(x_0)]=0$.

Hint: write down the formal $\varepsilon-\delta$ definitions of "f is continuous at x_0" and $\lim_{h\to 0}[f(x_0+h)-f(x_0)]=0$.

Pf. ① $\Rightarrow$ ②:

Let $L \in \mathbb{R}$.

Given that,

$\forall \varepsilon>0, \exists \delta>0$. s.t. $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$.

Take $h=x-x_0$, $x\neq x_0$

Since $x\to x_0$, $h\to 0$.

Substituting $x=x_0+h$. gives.

$\forall \varepsilon>0, \exists \delta>0$. s.t. $0<|h-0|<\delta \Rightarrow |f(x_0+h)-f(x_0)|<\varepsilon$.

Equivalent to $\lim_{h\to 0}[f(x_0+h)-f(x_0)]=0$. ∎

② $\Rightarrow$ ①

Given that.

$\lim_{h\to 0}[f(x_0+h)-f(x_0)]=0$, which is,

$\forall \varepsilon>0, \exists \delta>0$. s.t. $0<|h-0|<\delta \Rightarrow |f(x_0+h)-f(x_0)|<\varepsilon$.

Take $x=x_0+h$. gives. $h=x-x_0$.

$\forall \varepsilon>0$. $\exists \delta>0$, s.t. $0<|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$.

when $x=x_0$. i.e. $h=0$. gives.

$f(x_0+h)-f(x_0)=0. \Rightarrow f(x_0+h)=f(x_0) \Rightarrow f(x)=f(x_0)$.

Hence. $\forall \varepsilon>0, \exists \delta>0$, s.t. $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$.

i.e. f is continuous at x_0. ∎

2. Calculate the following limits:
Note: L'Hôpitals' Rule is not allowed to use for any limits in this worksheet.

(a) $\lim_{x \to 3} \frac{2-x}{x-3}$

① $\lim_{x\to 3^+} \frac{2-x}{x-3} = \lim_{x\to 3^+} (-1 - \frac{1}{x-3}) = -\infty$

② $\lim_{x\to 3^-} \frac{2-x}{x-3} = \lim_{x\to 3^-} (-1 - \frac{1}{x-3}) = +\infty$.

$\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$.

limit DNE.

(b) $\lim_{x \to 3} \frac{2-x}{(x-3)^2}$

Since $x \to 3$, $x \neq 3$.

$(x-3)^2 \to 0$, $2-x \to -1$

$\therefore \lim_{x\to 3} \frac{2-x}{(x-3)^2} = -\infty$

limit DNE.

(c) $\lim_{x \to 1} \frac{\sin x}{x}$

$\Rightarrow \frac{\lim_{x\to 1} \sin x}{\lim_{x\to 1} x} = \lim_{x\to 1} \sin x = \sin 1 = 0.841$

(d) $\lim_{x \to \infty} \frac{\sin x}{x}$

Since $\sin x \in [1, -1]$.

$\lim_{x\to\infty} \frac{\sin x}{x} \approx \lim_{x\to\infty} \frac{1}{x} = 0$

(e) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

$$= \lim_{x\to 0} \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$$

(f) $\lim_{x\to -\infty} \frac{\sqrt{x^2+1}+2x}{5x}$

$$\Rightarrow \lim_{x\to -\infty} \frac{\sqrt{x^2(1+\frac{1}{x^2})}+2x}{5x}$$

$$\Rightarrow \lim_{x\to -\infty} \frac{-x\sqrt{1+\frac{1}{x^2}}+2x}{5x} \approx \lim_{x\to -\infty} \frac{x}{5x} = \frac{1}{5}$$

More questions for practice (you are not required to return your work):

3. Calculate the following limits:

(a) $\lim_{x\to 1} \left(x^2 + 2^x\right)$

(b) $\lim_{h\to 2} \dfrac{h^3 - 5h^2 + 3h + 6}{h^3 - h^2 - 3h + 2}$

(c) $\lim_{t\to 0} \dfrac{t}{\sin(2t)}$

(d) $\lim_{x\to 0} \dfrac{\sin(2x)}{\sin(3x)}$

(e) $\lim_{z\to 0} \dfrac{\sin(2z^2)}{\cos(3z)\ \sin^2(5z)}$

(f) $\lim_{x\to 3} \dfrac{\tan(x-3)}{2x-6}$

(g) $\lim_{x\to 0} \dfrac{2e^x}{\sin(2e^x)}$

(h) $\lim_{t\to 0} \dfrac{1-\cos(3t)}{t^2}$

(i) $\lim_{y\to 1} \dfrac{\sqrt{y+4} - \sqrt{4y+1}}{\sqrt{y} - 1}$

(j) $\lim_{x\to 0} \dfrac{\sin\left(1-\cos x\right)}{x\tan(\pi x)}$

(k) $\lim_{u\to 2} \dfrac{1}{2-u}\left(\sqrt{\dfrac{u+2}{u-1}} - 2\right)$

(l) $\lim_{x\to\infty} \left[x + \sqrt{x^2 - x}\right]$

(m) $\lim_{x\to -\infty} \left[x + \sqrt{x^2 - x}\right]$

Hint: The answers to Questions 3l and 3m are different.
Answer: (a) 3, (b)-1, (c) $\frac{1}{2}$,(d) $\frac{2}{3}$,(e) $\frac{2}{25}$, (f)$\frac{1}{2}$, (g)$\frac{2}{\sin 2}$, (h) $\frac{9}{2}$, (i)$\frac{-3}{\sqrt{5}}$, (j)$\frac{1}{2\pi}$, (k)$\frac{3}{4}$,
(l)∞(There is no indeterminate form to begin with. It is $\infty + \infty$.),
(m) $\frac{1}{2}$ (Multiply and divide by the conjugate first. Make sure you get $1/2$ and not $-1/2$.)

4. Calculate the following limits (Hint: you may need to use Squeeze Theorem):

(a) $\lim_{x\to 0} \sin(x^2)\sin(\frac{1}{x})$

(b) $\lim_{x\to 1^-} \sqrt{1-x^2}\cos(\frac{1}{(x-1)^2})$

(c) $\lim_{x\to\infty} \dfrac{x^2(2+\sin^2 x)}{x+10}$

(d) $\lim_{x\to -\infty} \dfrac{5x^2 - \sin(3x)}{x^2+2}$

Answer: (a)0, (b) 0, (c)∞, (d)5.