Continuity of Probability

1. Write $\{A_n\} \nearrow A$, if:

① $\bigcup_n A_n = A$;

② they are nested increasing. i.e. $A_1 \subseteq A_2 \subseteq \ldots\ldots$

1) Continuity of Probabilities Theorem: If $\{A_n\} \nearrow A$, then $\lim_{n\to\infty} A_n = A$.

proof: Let $B_1 = A_1$. $B_n = A_n \cap A_{n-1}^c$ for $n \geq 2$. (we don't need to say $B_n = A_n \cap (A_1 \cup A_2 \cup \cdots \cup A_{n-1})^c$ as its nested increasing $\downarrow A_{n-1}$).

Gives $\{B_n\}$ is disjoint. and $A = \bigcup_{i=1}^{\infty} B_i$.

$P(A) = P(\bigcup_{i=1}^{\infty} B_n) = \sum_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)$.

↳ additivity as $\{B_n\}$ are disjoint.

Also since $A_n = \bigcup_{i=1}^{n} B_i$. $P(A_n) = P(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} P(B_i)$, gives.

$P(A) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P(A_n)$. ■

also apply.

2. Write $\{A_n\} \searrow A$, if:

① $\bigcap_n A_n = A$.

② 'nested' decreasing: i.e. $A_1 \supseteq A_2 \supseteq A_3 \ldots\ldots$

1) $\{A_n\} \searrow A$ iff $\{A_n^c\} \nearrow A^c$.

3. Exercises:

1) Filp infinite number of coins. P(all the coins are heads).

A: all the coins are H; A_n: n coins are H.

Since $A_n \supseteq A_{n+1}$ and $\bigcap_n A_n = A$, gives. $\{A_n\} \searrow A$.

Also apply continuity of propability theorem, $\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} (\frac{1}{2})^n = 0$.
$= P(A)$.

$P([a,b]) = b-a$

2) Suppose $0 \leq a < b \leq 1$. Pick a number between 0 and 1.

① $P((a,b))$

Let $A = (a, b)$ and $A_n = [a+\frac{1}{n}, b-\frac{1}{n}]$, which $A_{n+1} \supseteq A_n$

Since $A_{n+1} \supseteq A_n$, $\bigcup_n A_n = A$, gives $\{A_n\} \nearrow A$.

$P(A_n) = b - \frac{1}{n} - (a+\frac{1}{n}) = (b-a) - \frac{2}{n}$.

By continuity of probability, $P(A) = \lim_{n\to\infty} P(A_n) = b-a$.

② $P(\{a\})$.

Let $A = \{a\}$. $A_n = [a-\frac{1}{n}, a+\frac{1}{n}]$, which $A_{n+1} \supseteq A_n$.

$P(\{a\}) = \lim_{n\to\infty} P([a-\frac{1}{n}, a+\frac{1}{n}]) = \lim_{n\to\infty} (\frac{2}{n}) = 0$.