L'Hôpelal's Rule

1. Theorem.

Let f, g be functions. Let $a \in \mathbb{R}$. I want to compute $\lim_{x\to a} \frac{f(x)}{g(x)}$. (also works when $a \to \pm\infty$ & side limits).

If 1) f and g are differentiable as $x \to a$.

2) g and g' are over 0 as $x \to a$.

3) The limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

4) The limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists or is ∞ or $-\infty$.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

① $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

② $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \pm\infty$, the $\lim_{x\to a} \frac{f(x)}{g(x)} = \pm\infty$.

③ $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ DNE, no info about $\lim_{x\to a} \frac{f(x)}{g(x)}$

e.g. $\lim_{x\to\infty} \frac{x}{\ln x}$ $\left(\frac{\infty}{\infty}\right)$

$= \frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty} \frac{1}{x}} = \lim_{x\to\infty} x = \infty$.

$\lim_{x\to 1} \frac{x^3-2x+1}{x^2+3x+2} = \frac{0}{6}$

$= 0$

$\lim_{x\to 0} \frac{\cos x - \cos(2x)}{xe^x - x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{-\sin x + 2\sin(2x)}{e^x + xe^x - 1}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{-\cos x + 4\cos(2x)}{1 + e^x + xe^x}$

$= \frac{3}{2}$.

2. When it goes wrong.

e.g. $\lim_{x\to\infty} \frac{x+\sin x}{2x+\cos x}$ $\left(\frac{\infty}{\infty}\right)$.

$= \lim_{x\to\infty} \frac{1+\cos x}{2-\sin x}$. DNE (oscillating).

$\lim_{x\to\infty} \frac{x+\sin x}{2x+\cos x}$

$= \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{2+\frac{\cos x}{x}} = \frac{1}{2}$

Core: change into $\left(\frac{0}{0}\right)$.

3. Different Forms.

1). $0\cdot\infty$

e.g. $\lim_{x\to\infty} x\cdot\left[1-e^{\frac{2}{x}}\right]$ $(\infty\cdot 0)$.

$= \lim_{x\to\infty} \frac{\left[1-e^{\frac{2}{x}}\right]}{\frac{1}{x}}$ $\left(\frac{0}{0}\right)$ change into $\left(\frac{0}{0}\right)$.

$= \lim_{x\to\infty} \frac{-e^{\frac{2}{x}}\cdot\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\left(-2e^{\frac{2}{x}}\right) = -2.$

2) $\infty-\infty$

e.g. $\lim_{x\to\infty}\left[\sqrt{x^2-x}-x\right]$ $(\infty-\infty)$

$= \lim_{x\to\infty}\left[\sqrt{x^2}\sqrt{1-\frac{1}{x}}-x\right]$

$= \lim_{x\to\infty} x\cdot\left[\sqrt{1-\frac{1}{x}}-1\right]$ $(\infty\cdot 0)$. change to $0\cdot\infty$.

$= \lim_{x\to\infty} \frac{\sqrt{1-\frac{1}{x}}-1}{\frac{1}{x}}$ $\left(\frac{0}{0}\right)$ change to $\left(\frac{0}{0}\right)$.

$= \lim_{x\to\infty} \frac{\left(\frac{1}{2}\cdot\left(1-\frac{1}{x}\right)^{-\frac{1}{2}}\right)\cdot\frac{1}{x^2}}{-\frac{1}{x^2}}$

$= \lim_{x\to\infty} \frac{-1}{2\sqrt{1-\frac{1}{x}}} = -\frac{1}{2}$

3) 1^{∞}

① Why it's indeterminate

e.g. Let $c\in\mathbb{R}$.

$\lim_{x\to0^+}\underset{\to 1}{(e^x)}^{\frac{c}{x}} \leftarrow \pm\infty$

$= \lim_{x\to0^+}\left[e^{x\cdot\frac{c}{x}}\right] = \lim_{x\to0^+} e^c = e^c$ ← can obtain different values when c changes.

Any tiny number close to 1 after ∞ to it can change a lot.

$\lim_{x\to\infty} 1.01^x = \infty \longleftrightarrow \lim_{x\to\infty} 0.99^x = 0.$

e.g. $\lim_{x\to0^+}(1-x)^{\frac{1}{x}}$ (1^{∞})

Call $f(x) = (1-x)^{\frac{1}{x}}$

$\ln f(x) = \ln(1-x)^{\frac{1}{x}} = \frac{1}{x}\ln(1-x)$

1) $\lim_{x\to 0^+} [\ln f(x)]$

$= \lim_{x\to 0^+} \frac{\ln(1-x)}{x}$ $\quad (\frac{0}{0})$.

$= \lim_{x\to 0^+} \frac{\frac{1}{1-x}\cdot(-1)}{1}$

$= \lim_{x\to 0^+} \frac{-1}{1-x} = -1$

2) $\lim_{x\to 0^+} f(x)$

$= \lim_{x\to 0^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$