Learning Objectives

> In this tutorial, you will show how the eigenvectors and eigenvalues of linear transformations are related to eigenvectors and eigenvalues of their matrix representations.

Before attending the tutorial, you should be able recall the definitions of eigenvectors, eigenvalues, characteristic polynomial, and diagonalization for matrices. These definitions can be reviewed in the textbook, Damiano and Little Sections 4.1 and 4.2. To extend these definitions to linear transformations, we make the following **new** definitions:

Definition: Let V be a vector space of dimension n, and let $T : V \to V$ be a linear transformation. The *determinant of* T, denoted by $\det(T)$, is defined to be $\det([T]_{\mathcal{B}})$, where $\mathcal{B}$ is any basis for V.

Definition: Let V be a vector space of dimension n. The *characteristic polynomial of the linear transformation* $T : V \to V$ is the polynomial in the variable λ given by $\det([T - \lambda \mathrm{id}]_{\mathcal{B}})$, where $\mathcal{B}$ is any basis of V.

Definition: Let V be an n dimensional vector space and let $T : V \to V$ be a linear transformation. A basis $\mathcal{B} = \{b_1, \cdots, b_n\}$ of V is called an *eigenbasis* if all b_i's are eigenvectors of T.

Problems

1. For the following transformations, find an eigenvector using any methods you can think of, including basic geometry, if this is possible. What are the corresponding eigenvalues?

   (a) $V = \mathbb{R}^2$, $T =$ left multiplication by $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

   (b) $V = \mathbb{P}_3$ the space of polynomials of degree less than or equal 3 in the variable t, $T(f) = f'$.

2. (a) Is the definition of $\det(T)$ well defined? What may go wrong? Prove that this definition is well-defined. [1]

   (b) Let λ be an eigenvalue of $T : V \to V$ and let $\vec{v}$ be an eigenvector with eigenvalue λ. Then

   $$(T - \lambda \mathrm{id})\vec{v} = \vec{0}.$$

   This means that the linear transformation $T - \lambda \mathrm{id}$ has a nontrivial kernel. Suppose $\mathcal{B}$ is a basis for V. Discuss with your group why we can deduce

   $$\det([T - \lambda \mathrm{id}]_{\mathcal{B}}) = 0.$$ [2]

   (c) Conversely, show that if $\det([T - \lambda \mathrm{id}]_{\mathcal{B}}) = 0$, then λ is an eigenvalue of T.

   (d) Show that $\det([T - \lambda \mathrm{id}]_{\mathcal{B}})$ does not depend on our choice of basis $\mathcal{B}$. That is choose a different basis $\mathcal{A}$ and show that $\det([T - \lambda \mathrm{id}]_{\mathcal{B}}) = \det([T - \lambda \mathrm{id}]_{\mathcal{A}})$.

   (e) Discuss with your group why part (d) shows that the characteristic polynomial on the top of this page is well defined.

[1] You may need to use the fact that $\det(AB) = \det(A)\det(B)$ for any matrices A, B and $\det(A^{-1}) = \det(A)^{-1}$ if A is invertible.

[2] Recall from MAT223 that a matrix have non-zero determinant if and only if it is invertible.

(f) Let A and B be similar matrices and let $r \in F$. Prove that $A - rI$ and $B - rI$ are similar.

(g) Let $\mathcal{B}$ be a basis of V. Show that

$$([T - \lambda \mathrm{id}]_{\mathcal{B}}) = [T]_{\mathcal{B}} - \lambda I$$

Conclude the characteristic polynomial of T is equal to $\det([T]_{\mathcal{B}} - \lambda I)$.

(h) To find all the eigenvalues λ of T, we find roots of $\det([T]_{\mathcal{B}} - \lambda I) = 0$. Use the previous parts to justify why this method works. Discuss with your groups what happens if we choose a different basis.

(i) Find the characteristic polynomial and eigenvalues of the transformations in problem 1 using this method.

3. (a) Suppose $\mathcal{B} = \{b_1, \cdots, b_n\}$ is an eigenbasis for T with the corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$. Write down $[T]_{\mathcal{B}}$.

(b) Suppose the matrix if T with respect to some basis $\mathcal{A}$ is diagonal. What can we say about the vectors in $\mathcal{A}$?

(c) Prove: T is diagonalizable if and only if for any basis $\mathcal{A}$ of V, $[T]_{\mathcal{A}}$ is similar to a diagonal matrix.

(d) Are maps given in problem 1 diagonalizable?

Q(2) (c). Suppose $\det([T-\lambda id]_B)=0$.

$(T-\lambda id)(\vec{v}) = T(\vec{v}) - \lambda id(\vec{v}) = \lambda\vec{v}-\lambda\vec{v}=0$.

Thus, $\vec{v}$ is the kernel of $T-\lambda id$.

Since, $\vec{v}$ is non-trivial (according to definition of eigenvalue).

the kernel of $T-\lambda id$ is nontrivial.

Since B is a basis for V, the matrix representation of $T-\lambda id$ with respect to B is $[T-\lambda id]_B$

Since the determinant $=0$, it means the transformation is not invertible, which implies $T-\lambda id$ has a non-trivial kernel

Thus, λ is an eigenvalue of T.