CSC236 Problem Set 2

Question 3.

(a) For each $e \in E$, define T:

① $T(two) = 1$

② If $e_1, e_2 \in E$, then $T((e_1 \oplus e_2)) = T(e_1) + T(e_2)$ and $T((e_1 \otimes e_2)) = T(e_1) + T(e_2)$.

(b). Let $e \in E$.

WTP: $V(e) \geq 2 \cdot T(e)$

I'll prove a property first.

Property I: $\forall e \in E$, $T(e) \geq 1$

Let $e \in E$.

WTP: $T(e) \geq 1$

Base Case: $e = two$.

$T(two) = 1 \geq 1$ by def^{n} of T.

Induction Step: Let $e_1, e_2 \in E$.

Induction Hypothesis: Assume $T(e_1) \geq 1$, $T(e_2) \geq 1$

WTP: $T((e_1 \oplus e_2)) \geq 1$ and $T((e_1 \otimes e_2)) \geq 1$.

Since, by def^{n} of T. $T((e_1 \oplus e_2)) = T((e_1 \otimes e_2)) = T(e_1) + T(e_2)$

(By I.H.). $\geq 1 + 1 > 1$.

Therefore. I've proved $\forall e \in E$, $T(e) \geq 1$.

I'll start the prove then.

Base Case: $e = two$.

when $e = two$. according to the definition of V, $V(e) = 2$.

when $e = two$, according to the definition of T. $T(e) = 1$.

which $V(two) = 2 \times 1 = 2 \cdot T(two)$, gives $V(e) \geq 2 \cdot T(e)$ when $e = two$.

I've proved the base case is true.

Induction Step. Let $e_1, e_2 \in E$, s.t. $V(e_1) \geq 2 \cdot T(e_1)$. $V(e_2) \geq 2 \cdot T(e_2)$.

WTP: $V(e) \geq 2 \cdot T(e)$. where $e = e_1 \oplus e_2$ or. $e = e_1 \otimes e_2$

Case 1: $e = e_1 \oplus e_2$.

$$
\begin{aligned}
V(e) &= V(e_1) + V(e_2) \\
&\geq 2 \cdot T(e_1) + 2 \cdot T(e_2) \quad \text{According to I.H.} \\
&= 2 \cdot (T(e_1) + T(e_2)) \\
&= 2 \cdot T(e). \quad \text{According to def}^{\underline{n}} \text{ of } T.
\end{aligned}
$$

which in case 1. $V(e) \geq 2 \cdot T(e)$.

Before Case 2, I'll prove another property, which, $2T(e_1)T(e_2) \geq T(e_1) + T(e_2)$

Since $e_1, e_2 \in E$, by property I, $T(e_1) \geq 1$, $T(e_2) \geq 1$, which,

$$2T(e_2) = T(e_2) + T(e_2) \geq T(e_2) + 1.$$

$$\Rightarrow \quad 2T(e_2) - 1 \geq T(e_2), \text{ gives.}$$

$$T(e_1) \cdot (2T(e_2) - 1) \geq 2T(e_2) - 1 \quad (\text{since } T(e_1) \geq 1).$$

$$\Rightarrow \quad \geq T(e_2), \text{ gives.}$$

$$T(e_1) \cdot (2T(e_2) - 1) = 2 \cdot T(e_1) \cdot T(e_2) - T(e_1) \geq T(e_2).$$

$$\Rightarrow \quad 2T(e_1) \cdot T(e_2) \geq T(e_1) + T(e_2).$$

I'll call this property II.

Case 2: $e = e_1 \otimes e_2$.

$$
\begin{aligned}
V(e) &= V(e_1) \cdot V(e_2) \\
&\geq 2T(e_1) \cdot 2 \cdot T(e_2) \\
&\geq 2 \cdot (2 \cdot T(e_1) \cdot T(e_2)). \quad (\text{According to Property II}). \\
&\geq 2 \cdot (T(e_1) + T(e_2)) \\
&= 2 \cdot T((e_1 \otimes e_2)). \\
&= 2 \cdot T(e). \quad \text{According to def}^{\underline{n}} \text{ of } T.
\end{aligned}
$$

which in case 2. $V(e) \geq 2 \cdot T(e)$.

Therefore. I've proved. $\forall e \in E$. $V(e) \geq 2 \cdot T(e)$. ∎