U3 HW

1. Two $n\times n$ matrices A and B are said to be similar if there exists. a $n\times n$ matrix P s.t. $B=P^{-1}AP$. where P is invertible

Since, the determinant of a matrix is a scalar value that can be computed using the cofactor expansion along any row or column,

$$\det(B)=\det(P^{-1}AP).$$

$$\Rightarrow \det(B)=\det(P^{-1})\det(A).\det(P).$$

$$\Rightarrow \frac{\det(B)}{\det(P)}=\det(P^{-1})\cdot\det(A).$$

$$\Rightarrow \det(B)=\det(A)\cdot\det(P).$$

Thus. $\det(P^{-1}AP)=\det(A)\cdot\det(P)$.

$$\Rightarrow \det(A)\cdot\det(P)=\det(P^{-1})\cdot\det(A)\cdot\det(P)$$

$$\Rightarrow \det(A)=\det(P^{-1})\cdot\det(A).$$

$$\Rightarrow \det(P^{-1})=1.$$

Hence. $\det(A)=\det(B)$.

2. Suppose α and β are two bases for V. Let P be the change of basis matrix from $\alpha\to\beta$. Thus. the columns of P are the coordinates of the basis vectors of α with respect to β. gives.

$[T]_\alpha^\alpha=P^{-1}[T]_\beta^\beta P$. since $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ are the same linear transformation based on different basis.

Therefore, $\det([T]_\alpha^\alpha)=\det(P^{-1}[T]_\beta^\beta P)$.

$$\Rightarrow \det([T]_\alpha^\alpha)=\det(P^{-1})\cdot\det([T]_\beta^\beta)\cdot\det(P).$$

$\Rightarrow \det([T]_\alpha^\alpha)=\det([T]_\beta^\beta)$. means that the det of the matrix representation is independent of the choice of basis.

3. Given $A \in M_{n \times n}(\mathbb{R})$ where A is a block diagonal matrix with 2×2 blocks $B(1)$ & $B(2)$

Let $B(1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $B(2) = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, where $a, b, c, d, e, f, g, h \in \mathbb{R}$.

Gives, $A = \begin{bmatrix} B(1) & \\ & B(2) \end{bmatrix} = \begin{bmatrix} a & b & & \\ c & d & & \\ & & e & f \\ & & g & h \end{bmatrix}$ ↳ making $B(1)$ & $B(2)$ arbitrarily.

Therefore $\det(A) = (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + (-1)^{1+3} a_{13} \det(A_{13}) + (-1)^{1+4} a_{14} \det(A_{14})$.

$= a \det\left(\begin{bmatrix} d & & \\ & e & f \\ & g & h \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} c & & \\ & e & f \\ & g & h \end{bmatrix}\right) + 0 + 0$.

$= a \cdot (d \det(B_{11})) - b(c \det(C_{11}))$

$= a \cdot d \cdot (eh - gf) - b \cdot (c \cdot (eh - gf))$.

$= (eh - gf)(ad - bc)$

$= \det(B(1)) \cdot \det(B(2))$

$\det(B(1)) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\det(B(2)) = \begin{vmatrix} e & f \\ g & h \end{vmatrix} = eh - gf$.

Hence, $\det A = \det(B(1)) \cdot \det(B(2))$ ∎

4. Let $B(1) = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. $B(2) = [j]$, where $a, b, c, d, e, f, g, h, i, j \in \mathbb{R}$.

① $A = \begin{bmatrix} a & b & c & \\ d & e & f & \\ g & h & i & \\ & & & j \end{bmatrix} \Rightarrow \det(A) = a \cdot \det\begin{vmatrix} e & f & \\ h & i & \\ & & j \end{vmatrix} - b \det\begin{vmatrix} d & f & \\ g & i & \\ & & j \end{vmatrix} + c \cdot \det\begin{vmatrix} d & e & \\ g & h & \\ & & j \end{vmatrix}$.

$= a \cdot (e\begin{vmatrix} i & \\ & j \end{vmatrix} - f\begin{vmatrix} h & \\ & j \end{vmatrix}) - b(d\begin{vmatrix} i & \\ & j \end{vmatrix} - f\begin{vmatrix} g & \\ & j \end{vmatrix}) + c \cdot (d \cdot \begin{vmatrix} h & \\ & j \end{vmatrix} - e \cdot \begin{vmatrix} g & \\ & j \end{vmatrix})$.

$= a \cdot (e \cdot i \cdot j - f \cdot h \cdot j) - b(d \cdot i \cdot j - f \cdot g \cdot j) + c \cdot (d \cdot h \cdot j - e \cdot g \cdot j)$.

$= aeij - efhj - bdij + bf \cdot gj + cdhj - cegj$.

$= j \cdot (aei - efh - bdi + bfg + cdh - ceg)$.

$= \det(B(1)) \cdot \det(B(2))$

$\det(B(1)) = \det\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= a \cdot \det\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \det\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$= a(ei - hf) - b(di - gf) + c \cdot (dh - eg)$

$= aei - ahf - bdi - bgf + cdh - ceg$.

$\det(B(2)) = j$.

② $A = \begin{bmatrix} j & & & \\ & a & b & c \\ & d & e & f \\ & g & h & i \end{bmatrix}$

$\det(A) = j \cdot (\det\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix})$.

$= \det(B(2)) \cdot \det(B(1))$

Hence, $\det A = (\det(B(1)) \cdot (\det(B(2)))$ is still true if one of the block is 3×3 and the other is 1×1. ∎

5. $\forall A \in M_{N\times N}(\mathbb{R})$, $\forall k_i \in \mathbb{Z}^+$, where $i \in \mathbb{Z}^+$.

$\det A = (\det B_{k_1}) \cdot (\det B_{k_2}) \cdots (\det B_{k_n})$, where $\sum_{i=1}^{n} k_i = N$

6. Let V be an n-dimensional vector space with basis $\alpha = \{a_1, a_2, \dots, a_n\}$. Gives the matrix representation of the linear transformation T with respect to α is $[T]_\alpha^\alpha = I_n$. the $n \times n$ identity matrix.

To define T on basis elements in α more specifically.

$$T(a_1) = [T]_\alpha^\alpha a_1 = e_1$$
$$T(a_2) = [T]_\alpha^\alpha a_2 = e_2.$$
$$\vdots$$
$$T(a_n) = [T]_\alpha^\alpha a_n = e_n,$$ where $e_1, e_2, \dots, e_n$ are the standard basis vectors in $\mathbb{R}^n$; revealing T maps each basis vector of V to the corresponding standard basis vector in $\mathbb{R}^n$. Besides, since a basis for V is being mapped to a basis for $\mathbb{R}^n$, T is an isomorphism between V and $\mathbb{R}^n$.

7. $[I]_\alpha^\beta \cdot [I]_\beta^\alpha = I_{n\times n}$, which states that the product of the change of basis matrix from α to β and from β to α is equal to the $n \times n$ identity matrix. This means the fact that a change of basis matrix is an invertible linear transformation, and the inverse of $[I]_\alpha^\beta$ is $[I]_\beta^\alpha$.

Moreover, geometrically, these matrices represent the transformations that change the basis of a vector space:

① $[I]_\alpha^\beta$: lin. trans. maps each vector in the α-basis to its coordinate representation in β-basis.

② $[I]_\beta^\alpha$: lin. trans. maps each vector in the β-basis to its coordinate representation in α-basis.

③ $I_{n\times n}$: the identity matrix $I_{n\times n}$ represents the linear transformation that leaves each vector unchanged.