MAT 137Y – Practice problems
Unit 5 : The Mean Value Theory and applications

1. For each of the following functions, find the intervals where they are increasing or decreasing, and find the local maxima and local minima.

   (a) $f(x) = x^3(x-1)^2$
   (b) $f(x) = x^{2/3}(x-1)$

2. Find the maximum and the minimum of the function $f(x) = \dfrac{\sin x}{3 + \cos x}$

3. How many solutions does the equation $x^5 + x^3 + 1 = 4\sin x$ have?

4. Prove there is a constant C such that

$$\arcsin \frac{1-x}{1+x} + 2 \arctan \sqrt{x} = C$$

   for all x in a certain domain. What is the largest domain for which this identity is true? What is the value of the constant C?

5. There are two versions of the theorem that says "positive derivative implies increasing function".

   (a) **Theorem 1.** Let $a < b$. Let f be a function defined on (a, b). IF
      - f is differentiable on (a, b)
      - $\forall x \in (a, b),\ f'(x) > 0$

      THEN f is increasing on (a, b).

   (b) **Theorem 2.** Let $a < b$. Let f be a function defined on $[a, b]$. IF
      - f is continuous on $[a, b]$
      - f is differentiable on (a, b)
      - $\forall x \in (a, b),\ f'(x) > 0$

      THEN f is increasing on $[a, b]$.

   Prove them.

6. Cauchy's Mean Value Theorem[1] is a generalization of the regular MVT. The theorem states:

   Let $a < b$. Let f and g be functions defined on $[a, b]$.
   IF (some hypotheses) THEN $\exists c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

   (a) The following proof is WRONG. Explain why:

[1] The main reason we care about this theorem is that it helps prove L'Hôpital's Rule.

Using MVT for f we get $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
Using MVT for g we get $g'(c) = \dfrac{g(b) - g(a)}{b - a}$.
Divide one equation by the other and we get the result we want.

(b) The idea of the proof is as follows. We want to apply Rolle's Theorem to the function $H(x) = f(x) - Mg(x)$ for a certain constant M. Which value of M satisfies $H(a) = H(b)$? What other assumptions do you need to apply Rolle's Theorem? What can you conclude?

(c) Fill in the missing hypotheses in the theorem and prove it.

(d) Show that when we choose g to be a very specific function, this becomes the regular MVT. (That is why we said the theorem is a generalization).

7. Let f be a twice-differentiable function defined on an open interval I. Assume there are three points in the graph of f that are on the same line. Prove that there exists $c \in I$ such that $f''(c) = 0$.

8. In this problem you will construct a generalization of Rolle's Theorem.[2] We will refer to the version you already know as the 1st Rolle Theorem.

(a) The 2nd Rolle's Theorem states:

Let $a < b$. Let f be a function defined on $[a, b]$. IF
- (some conditions about continuity and derivatives)
- $f(a) = f'(a) = 0$
- $f(b) = 0$

THEN $\exists c \in (a, b)$ such that $f''(c) = 0$.

Complete the statement of the theorem and prove it.

Hint: Use 1st Rolle Theorem on f on $[a, b]$. Then use 1st Rolle Theorem again (where?)

(b) Let N be a positive integer. The N-th Rolle's Theorem states:

Let $a < b$. Let f be a function defined on $[a, b]$. IF
- (some conditions about continuity and derivatives)
- (some conditions at a)
- $f(b) = 0$

THEN $\exists c \in (a, b)$ such that $f^{(N)}(c) = 0$.

Complete the statement of the theorem and prove it.

[2] This generalization will later help us prove Lagrange's Remainder Theorem for a Taylor polynomial.

Some answers and hints

1. (a) f is increasing on $(-\infty, 3/5]$, decreasing on $[3/5, 1]$,and increasing on $[1, \infty)$. There is a local maximum at $x = 3/5$ and a local minimum at $x = 1$.

   (b) f is increasing on $(-\infty, 0]$, decreasing on $[0, 2/5]$,and increasing on $[2/5, \infty)$. There is a local maximum at $x = 0$ and a local minimum at $x = 2/5$.

2. The maximum is $\dfrac{1}{\sqrt{8}}$ and the minimum is $\dfrac{-1}{\sqrt{8}}$.

3. The equation has 3 solutions. Imitate the proof in Video 5.6.

4. $C = \dfrac{\pi}{2}$. The identity is true for all $x \in [0, \infty)$.

   Did you take care of the case $x = 0$ properly? Notice that the function is not differentiable at 0 so we cannot get away simply with taking the derivative.

   For the proof, we need to notice that the function $f(x) = \arcsin \dfrac{1-x}{1+x} + 2 \arctan \sqrt{x}$ has domain $[0, \infty)$, is continuous on $[0, \infty)$, is differentiable on $(0, \infty)$, and $f'(x) = 0$ for all $x > 0$. It is necessary to verify each of these claims.

5. Imitate the proof in Video 5.9.

6. (b) $M = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

   (c) Once you have the function H and the correct hypotheses, the proof is very similar to the proof of the standard MVT (Video 5.8), but without the geometric interpretation. Make sure you include hypotheses that prevent you from dividing by 0.

7. Use the MVT twice on f (on different intervals). Then use the MVT once on f'. Each time you use the MVT, make sure you specify the function and the interval, and you verify the hypotheses. Drawing a picture will help.

8. (a) For the statement of the theorem, we need the hypotheses:

   - f is continuous on $[a, b]$
   - f' exists and is continuous on $[a, b)$
   - f'' exists on (a, b)

   If you write something like "f' exists/is continuous on $[a, c]$" then your statement does not make sense. The hypotheses should only use to a and b, not c or c' or other variables. For the proof:

   - Use 1st Rolle Theorem on f on $[a, b]$ (verify the hypotheses first!). Conclude there is $c_0 \in (a, b)$ such that $f'(c_0) = 0$.
   - Use 1st Rolle Theorem on f' on $[a, c_0]$ (verify the hypotheses first!). Conclude there is $c \in (a, c_0)$ such that $f''(c) = 0$.