Unit 2(b) Lecture Notes for MAT224

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31 January-2 February, 2023

§1 2.2 Linear Transformations Between Finite Dimensional Vector Spaces Continued...

We know that the product $[T]_\alpha^\beta e_i = [T(\alpha_i)]_\beta$ where α_i is the i^{th} basis element in α and the subscript β means "as a linear combination of elements in basis β".

Note that e_i is the vector with 1 in the i^{th} entry and zero everywhere else.

Now write $\vec{v}$ as a linear combination of standard basis elements $e_1, e_2, \ldots, e_n$.

Explain why the product $[T]_\alpha^\beta[\vec{v}]$ is equal to a linear combination of vectors in basis β. What do the e_i's represent here?

Suppose $\vec{v}$ can be written as a linear combination of the standard basis vectors $e_1, e_2, \ldots, e_n$ as follows:

$$v = c_1 e_1 + c_2 e_2 + \ldots + c_n e_n$$

where $c_1, c_2, \ldots, c_n$ are scalars.
Then, we can apply the formula $[T]_\alpha^\beta e_i = [T(\alpha_i)]\beta$ to each basis vector e_i to obtain:

$$[T]_\alpha^\beta e_1 = [T(\alpha_1)]\beta$$
$$[T]_\alpha^\beta e_2 = [T(\alpha_2)]\beta$$
$$\ldots$$
$$[T]_\alpha^\beta e_n = [T(\alpha_n)]\beta$$

Using the linearity of T, we can write:

$$\begin{aligned}
[T]_\alpha^\beta \vec{v} &= [T]_\alpha^\beta (c_1 e_1 + c_2 e_2 + \cdots + c_n e_n) \\
&= c_1 [T]_\alpha^\beta e_1 + c_2 [T]_\alpha^\beta e_2 + \cdots + c_n [T]_\alpha^\beta e_n \\
&= c_1 [T(\alpha_1)]_\beta + c_2 [T(\alpha_2)]_\beta + \cdots + c_n [T(\alpha_n)]_\beta
\end{aligned}$$

Thus, $[T]_\alpha^\beta \vec{v}$ is a linear combination of the vectors $[T(\alpha_1)]_\beta, [T(\alpha_2)]_\beta, \ldots, [T(\alpha_n)]_\beta$ in basis β.
The e_i's represent the standard basis vectors of the vector space, which can be used to write any vector in the space as a linear combination. In this context, e_i's are used to write $\vec{v}$ as a linear combination of standard basis vectors, which is then used to express $[T]_\alpha^\beta \vec{v}$ as a linear combination of vectors in basis β.

§2 2.3 Kernel and Image

Define the set Ker(T) for a linear transformation $T : V \to W$.
The kernel, or null space, of a linear transformation $T : V \to W$ is the set of all vectors in the domain V that map to the zero vector in the range W.
In other words, the kernel of T, denoted as $\text{Ker}(T)$, is the set of all vectors $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$, where $\vec{0}$ is the zero vector in W.
Formally, we can write: $Ker(T) = \{v \in V : T(V) = 0\}$

Prove that Ker(T) is a subspace of V. Notice that elements in Ker(T) are in V by definition.
To prove that the kernel of a linear transformation $T : V \to W$, denoted as $\text{Ker}(T)$, is a subspace of V, we need to show that it satisfies the three properties of a subspace:

1. The zero vector is in $\text{Ker}(T)$. By definition, the zero vector in V is mapped to the zero vector in W by any linear transformation. Therefore, $\vec{0} \in \text{Ker}(T)$.

2. $\text{Ker}(T)$ is closed under vector addition. Let $\vec{u}, \vec{v} \in \text{Ker}(T)$. This means that $T(\vec{u}) = \vec{0}$ and $T(\vec{v}) = \vec{0}$. We need to show that $\vec{u} + \vec{v} \in \text{Ker}(T)$. To do so, we evaluate $T(\vec{u} + \vec{v})$ as follows:

$$T(u + v) = T(u) + T(V) = 0 + 0 = 0$$

   Therefore, $\vec{u} + \vec{v} \in \text{Ker}(T)$, and $\text{Ker}(T)$ is closed under vector addition.

3. $\text{Ker}(T)$ is closed under scalar multiplication. Let $\vec{v} \in \text{Ker}(T)$ and let k be a scalar. This means that $T(\vec{v}) = \vec{0}$. We need to show that $k\vec{v} \in \text{Ker}(T)$. To do so, we evaluate $T(k\vec{v})$ as follows:

$$T(kv) = kT(v) = k0 = 0$$

Therefore, $k\vec{v} \in \text{Ker}(T)$, and $\text{Ker}(T)$ is closed under scalar multiplication.
Since $\text{Ker}(T)$ satisfies all three properties of a subspace, it is indeed a subspace of V.

Define the set Image(T) for a linear transformation $T : V \to W$.
The image, or range, of a linear transformation $T : V \to W$ is the set of all vectors in the range W that can be obtained as the output of T when the input is a vector in the domain V.
In other words, the image of T, denoted as $\text{Im}(T)$, is the set of all vectors $\vec{w} \in W$ that can be expressed as $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$.

Formally, we can write:

$$Im(T) = \{w \in W : w = T(V) \; for \; some \; v \in V\}$$

Prove that Image(T) is a subspace of W. Notice that elements in Image(T) are in W by definition.

1. The zero vector is in $\text{Im}(T)$. By definition, $T(\vec{0}) = \vec{0}$ for any linear transformation T, so $\vec{0} \in \text{Im}(T)$.

2. $\text{Im}(T)$ is closed under vector addition. Let $\vec{w}_1, \vec{w}_2 \in \text{Im}(T)$. This means that there exist vectors $\vec{v}_1, \vec{v}_2 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$ and $T(\vec{v}_2) = \vec{w}_2$. We need to show that $\vec{w}_1 + \vec{w}_2 \in \text{Im}(T)$. To do so, we evaluate $T(\vec{v}_1 + \vec{v}_2)$ as follows:

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$$

   Therefore, $\vec{w}_1 + \vec{w}_2 \in \text{Im}(T)$, and $\text{Im}(T)$ is closed under vector addition.

3. $\text{Im}(T)$ is closed under scalar multiplication. Let $\vec{w} \in \text{Im}(T)$ and let k be a scalar. This means that there exists a vector $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. We need to show that $k\vec{w} \in \text{Im}(T)$. To do so, we evaluate $T(k\vec{v})$ as follows:

$$T(kv) = kT(v) = kw$$

   Therefore, $k\vec{w} \in \text{Im}(T)$, and $\text{Im}(T)$ is closed under scalar multiplication.

Since $\text{Im}(T)$ satisfies all three properties of a subspace, it is indeed a subspace of W.

Is it possible to determine the pre-image of a vector $\vec{w} \in W$, if you know its image? In other words, let $T : V \to W$ be a linear transformation and let $\vec{w} \in T(W)$. So $T(v) = w$ for some $v \in V$. Can we tell which $v \in V$ is sent to W by V.

It is not always possible to determine the pre-image of a vector $\vec{w} \in W$ if you only know its image under a linear transformation $T : V \to W$.
Consider a simple example where $T : \mathbb{R}^2 \to \mathbb{R}$ is defined by $T(x, y) = x$. Let $\vec{w} = 1 \in \mathbb{R}$. We know that $T(1, 0) = 1$, so $(1, 0)$ is one possible pre-image of $\vec{w}$. However, $T(0, 1) = 0$ and $(0, 1)$ is also a valid pre-image of $\vec{w}$.
In this example, we can see that the pre-image of $\vec{w}$ is not unique. This is because the kernel of T (i.e., the set of all vectors that are mapped to the zero vector in W) is non-trivial, and so there are multiple vectors in V that get mapped to the same vector in W.
What are some of the things you can do with the matrix form of T (in other words $[T]_\alpha^\beta$) to learn about the properties of T?
The matrix form of a linear transformation T with respect to bases α and β (i.e., $[T]_\alpha^\beta$) contains useful information about the properties of T. Here are some things you can do with the matrix form of T:

1. Determine if T is injective: The linear transformation T is injective if and only if the kernel of T is trivial, i.e., $\operatorname{Ker}(T) = \vec{0}$. The matrix form of T can help us determine if $\operatorname{Ker}(T)$ is trivial by checking if the matrix is invertible.

2. Determine if T is surjective: The linear transformation T is surjective if and only if the image of T is equal to the entire range space W. The matrix form of T can help us determine if T is surjective by checking if every element of W can be written as a linear combination of the columns of $[T]_\alpha^\beta$.

3. Find the rank of T: The rank of T is the dimension of the image of T. The rank of T can be determined by finding the number of linearly independent columns in $[T]_\alpha^\beta$.

4. Find the nullity of T: The nullity of T is the dimension of the kernel of T. The nullity of T can be determined by finding the number of free variables in the row reduced form of $[T]_\alpha^\beta$.

5. Find the eigenvalues and eigenvectors of T: The matrix form of T can be used to find the eigenvalues and eigenvectors of T by solving the characteristic equation $\det([T]_\alpha^\beta - \lambda I) = 0$.

6. Determine if T is diagonalizable: The linear transformation T is diagonalizable if and only if there exists a basis of V consisting of eigenvectors of T. The matrix form of T can help us determine if T is diagonalizable by checking if there are enough linearly independent eigenvectors to form a basis of V.

For a linear transformation $T : V \to W$, how can you use two of the following quantities to determine the third:

$$dim(Ker(T)) + dim(Im(T)) = dim(V)$$

We can use the formula dim(Ker(T)) + dim(Im(T)) = dim(V) to determine any one of the three quantities if we know the other two. We just need to rearrange it.

§3 2.4 Application of the Dimension Theorem

Give examples of deciding if a linear transformation $T : V \to W$ is injective or not based on:

- its kernel: a linear transformation $T : V \to W$ is injective if and only if its kernel is trivial, that is, if $ker(T) = \vec{0}_V$. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T(x, y) = (x, y, 0)$. The kernel of T is the set of all vectors of the form (x, y) such that $T(x, y) = (0, 0, 0)$, which implies $x = y = 0$. Therefore, $ker(T) = (0, 0)$, which is the trivial subspace of $\mathbb{R}^2$. Since the kernel is trivial, T is injective.

- its image: a linear transformation $T : V \to W$ is injective if and only if its image is a subspace of W with the same dimension as the domain, that is, if T is one-to-one, then $dim(Im(T)) = dim(V)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(x, y) = (x, 0)$. The image of T is the x-axis in $\mathbb{R}^2$, which is a subspace of dimension 1. Since the domain of T is also $\mathbb{R}^2$, which has dimension 2, T cannot be injective.

- the relative dimensions of V and W: a linear transformation $T : V \to W$ is injective if and only if the kernel of T is trivial, that is, if $dim(Ker(T)) = 0$. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T(x, y) = (x, y, 0)$. Since the range of T is a subspace of $\mathbb{R}^3$ that lies in the xy-plane, its dimension is 2. Since the domain of T is $\mathbb{R}^2$, which has dimension 2, T is injective if and only if the kernel of T is trivial. But the kernel of T consists of all vectors of the form $(0, 0, z)$, which has dimension 1, so T is not injective.

- the relative dimensions of Im(T) and W: If Im(T) has the same dimension as W, then T must be surjective (onto), but it may or may not be injective. $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + y, z)$, i.e. the transformation that adds the first two coordinates and ignores the third. In this case, Im(T) is the plane $z = 0$ in $\mathbb{R}^2$, which has lower dimension than $\mathbb{R}^2$, so T cannot be injective.

Give the definition of "surjective" and explain how you can decide if a linear transformation T is surjective based on the relative dimensions of V and W.
A linear transformation $T : V \to W$ is said to be surjective (or onto) if every element of W is the image of at least one element in V under T, i.e., if $\mathrm{Im}(T) = W$.
To determine whether a linear transformation $T : V \to W$ is surjective based on the relative dimensions of V and W, we can use the following theorem:
T is surjective if and only if $\dim(\mathrm{Im}(T)) = \dim(W)$.

Let $T : V - - > W$ be a linear transformation where dim(V)=dim(W).

(i) If T is surjective, does it also have to be injective? Why or why not?

No, if $T : V \to W$ is a linear transformation where $\dim(V) = \dim(W)$ and T is surjective, it does not have to be injective.
Here is an example: Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(x, y) = (x, 0)$. This transformation maps every vector in V onto the x-axis in W, which covers all of W and makes T surjective. However, T is not injective, since every vector in the x-axis in W is the image of infinitely many vectors in V (all vectors of the form (x, y) where y is any real number).
Therefore, it is possible for a linear transformation $T : V \to W$ to be surjective without being injective, even when $\dim(V) = \dim(W)$.

(ii) If T is injective, does it also have to be surjective? Why or why not?
If $T : V \to W$ is a linear transformation where $\dim(V) = \dim(W)$ and T is injective, then T is also surjective.
This is because if T is injective, then every element in the image of T corresponds to exactly one element in V (its pre-image). Since $\dim(V) = \dim(W)$, this means that T is also surjective, since it maps all of the n-dimensional space V onto all of the n-dimensional space W.

Consider Proposition 2.4.11. Let $\vec{w}_0 \notin Ker(T)$ for some linear transformation $T : V \to W$. Then we know we can write every vector $\vec{v} \in V$ as $\vec{v} = \vec{w}_0 + \vec{k}$ for some $\vec{k} \in Ker(T)$. If we replace w_0 with w_1 then does k stay the same?

In other words, $\vec{v} = \vec{w}_1 + \vec{k}$ for the same $\vec{k}$ as before or $\vec{v} = \vec{w}_1 + \vec{k}'$ for a potentially different $\vec{k}' \in Ker(T)$?
No, the vector $\vec{k}$ may change. If we replace $\vec{w}_0$ with $\vec{w}_1$, then we have $\vec{v} = \vec{w}_1 + \vec{k}'$ for some $\vec{k}' \in Ker(T)$. It is possible that $\vec{k}' \neq \vec{k}$. To see why, consider a simple example where $V = W = \mathbb{R}^2$, and let T be the linear transformation that projects onto the x-axis. Then $Ker(T)$ consists of all vectors of the form $(0, y)$, and any vector of the form (x, y) can be written as $(x, y) = (x, 0) + (0, y)$, where $(x, 0)$ is a vector on the x-axis and $(0, y)$ is in $Ker(T)$. Now, if we replace $(1, 0)$ with $(0, 1)$, then any vector of the form (x, y) can be written as $(0, 1) + (x, y - 1)$, where $(0, 1)$ is not on the x-axis and $(x, y - 1)$ is in $Ker(T)$. Thus, the vector $\vec{k}$ has changed from $(0, y)$ to $(x, y - 1)$.