Subspace & Basis

判断时先看 $\vec{0}$ 在不在.

1. Subspace (set in $\mathbb{R}^n$).

A subspace of $\mathbb{R}^n$ is any set W in $\mathbb{R}^n$ that satisfy

① zero vector in W

② each $\vec{u}, \vec{v}$ in W, $\vec{u}+\vec{v}$ is in W. (close under addition).

③ each $\vec{u}$ in W, $\forall c \in \mathbb{R}$, $c\vec{u}$ is in W. (close under scalar multiplication).

e.g. $W = \{A\vec{x} = \vec{0} \mid x \in \mathbb{R}^3\}$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

§ $A\vec{0} = \vec{0}$

§ $\forall \vec{u}, \vec{v} \in W$. $A\vec{u} = A\vec{v} = \vec{0}$

$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0}$.

$\therefore (\vec{u}+\vec{v}) \in W$.

§ $\forall \vec{u} \in W$, $A\vec{u} = \vec{0}$.

$A(c\vec{u}) = c(A\vec{u}) = c\vec{0} = \vec{0}$.

$\therefore c\vec{u} \in \mathbb{R}$.

e.g. $V \subseteq \mathbb{R}^2$ be the solution to $x+2y=0$. V a sub?

$\forall \vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$; $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in V$.

gives. $\begin{cases} x_1 + 2y_1 = 0 \\ x_2 + 2y_2 = 0 \end{cases}$

§ $0 + 2 \cdot 0 = 0$

§ $\vec{u} + \vec{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$

$= x_1 + x_2 + 2(y_1 + y_2) = x_1 + 2y_1 + x_2 + 2y_2 = 0$

$\therefore (\vec{u} + \vec{v}) \in V$.

§ $\lambda \vec{u} = \begin{bmatrix} \lambda x_1 \\ \lambda y_1 \end{bmatrix}$

$= \lambda x_1 + 2\lambda y_1 = \lambda(x_1 + 2y_1) = \lambda \cdot 0 = 0$

$\therefore \lambda \vec{u} \in V$.

$\therefore V$ is a subspace

1) Every subspace is a span and every span is a subspace.

2) Trival Subspace: the subset $\{\vec{0}\} \in \mathbb{R}^n$

2. Basis. (抽出精华描述 subspace)

A basis for a subspace W of $\mathbb{R}^n$ is ① a linear independent set is W that ② spans W.

e.g. find a basis for span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$.

① row reduction

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 1 & 3 & 2 & 3 & 0 \\ 3 & 0 & 3 & 4 & 0 \end{bmatrix} \xrightarrow[r_4 - 3r_1]{r_3 - r_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 3 & 2 & 2 & -1 \\ 0 & 0 & 3 & 1 & -3 \end{bmatrix} \xrightarrow[r_3 - 3r_2]{r_2 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 1 & -3 \end{bmatrix}$$

$$\xrightarrow[r_4 - 3r_3]{r_3 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & -2 \\ 0 & 0 & 0 & 5/2 & 3 \end{bmatrix}$$

$$\xrightarrow{r_4 \cdot \frac{2}{5}} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & -2 \\ 0 & 0 & 0 & 1 & 6/5 \end{bmatrix}$$

$$\xrightarrow{xxx} \begin{bmatrix} 1 & 0 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & 0 & -1/5 \\ 0 & 0 & 1 & 0 & -7/5 \\ 0 & 0 & 0 & 1 & 6/5 \end{bmatrix}$$

② determine the dim (pivot point numbers)

$\dim(\text{span}\{v_1, v_2, v_3, v_4, v_5\}) = 4$

③ Take basis

$\therefore$ basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$

1) $\dim(H)$: dimension of a non-zero subspace H. The number of vectors is any basis for H.

e.g. $\dim(\mathbb{R}^n) = n$.

$\dim(\text{span}\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}\}) = 2.$

① $\dim(\{\vec{0}\}) = 0.$

② the dimension of a given subspace is unique.

③ if $V = \mathbb{R}^n$, then any set with less than n vectors can't span $\mathbb{R}^n$, any set with more than n vector can't be lin inde.

顺序 matters.
$\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ 和 $\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\}$ 不是同一 basis.

2) Standard basis for $\mathbb{R}^n$: $\{\vec{e_1} \dots \vec{e_n}\}$