Exact Differential Eq.

Non-Linear;
Non-Separable.
↑
x, y 间有加减.

1. For func. $M(x,y)+N(x,y)y'=0$.

   1) Exact Eq.: $M_y(x,y)=N_x(x,y)$. i.e. $\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$.

   ⋮

2. Solving ODE. when there's '+', '−' between x. y.

   e.g. ① $(y\cos x+2xe^y)+(\sin x+x^2e^y-1)y'=0$.

   ② $(3xy+y^2)+(x^2+xy)y'=0$.

   1). Check Exact or not.

   ①: $M_y=\cos x+2xe^y$. $N_x=\cos x+2xe^y$. ∴ Exact.

   ②: $M_y=3x+2y$. $N_x=2x+y$. ∴ Non-Exact.

   2). For Exact:

   ① $\frac{\partial F}{\partial x}=M(x,y)$ 同时积分. 从而找到 potential $F(x,y)$ 表达式: $\int M(x,y)dx+B(y)$

   $F(x,y)=\int y\cos x+2xe^y dx=y\sin x+e^yx^2+B(y)$.

   ② 联立 $\frac{\partial F}{\partial y}=N(x,y)$ 与上面的 $\frac{\partial F}{\partial y}$ 求出 $B'(y)$: $\frac{\partial}{\partial y}\int M(x,y)dx+B'(y)=N(x,y)$

   $\sin x+x^2e^y+B'(y)=\sin x+x^2e^y-1$

   $\Rightarrow$ $B'(y)=-1$.

   ③ 积 $B'(y)$ 得到 $F(x,y)$ 最后表达式; Implicit sol^{n} 为 $F(x,y)=c$. $\forall c\in\mathbb{R}$.

   $B'(y)=-1 \Rightarrow B(y)=-y$

   ∴ $F(x,y)=y\sin x+e^yx^2-y$

   3) For non-exact (转成 exact).

   ① Compute M_y-N_x. 看 $\frac{M_y-N_x}{N}$ or $\frac{M_y-N_x}{M}$ 会消元只剩一个.

   $M_y-N_x=3x+2y-2x-y=x+y$. which $\frac{M_y-N_x}{N}=\frac{x+y}{x(x+y)}=\frac{1}{x}$.

   ② 根据①中选择. 列出 $\frac{d\mu}{dx}=\frac{M_y-N_x}{N}\mu$ or $\frac{d\mu}{dy}=-\frac{M_y-N_x}{M}\mu$

$$\frac{d\mu}{dx} = \frac{1}{x}\cdot\mu.$$

③. 解出 $\mu(x)$ or $\mu(y)$，有时会给 $\mu(x,y)$.

可不用加C.

$$\frac{1}{\mu}d\mu = \frac{1}{x}dx \Rightarrow \int\frac{1}{u}du = \int\frac{1}{x}dx \Rightarrow \ln|u| = \ln|x| \Rightarrow \mu(x) = x.$$

④. 在原式两边同乘 $\mu(x)$ 或 $\mu(y)$. (Integrating factor).

$$x(3xy+y^2) + x(x^2+xy)\cdot y' = 0.$$

← 可验证此时是否为exact.

⑤ Solve by using exact.

$$F(x,y) = \int M(x,y)\,dx + B(y) = \int 3x^2y + xy^2\,dx + B(y) = x^3y + \frac{1}{2}x^2y^2 + B(y).$$

$$\frac{\partial F}{\partial y} = x^3 + x^2y + B'(y) = N(x,y) = x^3 + x^2y. \Rightarrow B'(y) = 0.$$

↖此时为用乘后的 $N(x,y)$.

$$\therefore B(y) = C. \quad C \in \mathbb{R}.$$

$$\therefore F(x,y) = x^3y + \frac{1}{2}x^2y^2 + C, \quad C \in \mathbb{R}.$$

e.g. $x^2y^3 + x(1+y^2)y' = 0.$ with $\mu(x,y) = \frac{1}{xy^3}$.

$$\frac{1}{xy^3}\cdot x^2y^3 + \frac{1}{xy^3}\cdot x(1+y^2)y' = 0.$$

$$\Rightarrow \quad x + \left(\frac{1}{y^3} + \frac{1}{y}\right)y' = 0.$$

$$F(x,y) = \int M\,dx + B(y) = \int x\,dx + B(y) = \frac{1}{2}x^2 + B(y).$$

$$\frac{\partial F}{\partial y} = B'(y) = \frac{1}{y^3} + \frac{1}{y}. \Rightarrow B(y) = \int \frac{1}{y^3} + \frac{1}{y}\,dy = -\frac{1}{2y^2} + \ln|y|.$$

$$\therefore F(x,y) = \frac{1}{2}x^2 - \frac{1}{2y^2} + \ln|y|.$$