Change of Variable

1. Change of Variable.

Suppose X is a variable, and $h: \mathbb{R} \to \mathbb{R}$ is some function. Define $Y = h(X)$, i.e. $\forall s \in S$, $Y(s) = h(X(s))$. Y is another variable.

2. Discrete: X is discrete, $P(X = x_i) = p_i$ where $p_i \geq 0$, $\sum_i p_i = 1$.

直接把 $h(x)$ 代进去解.

1). Y is discrete.

① $P(Y=y) = P(h(X)=y) = P(X \in \{x : h(x) = y\})$.

② $p_Y(y) = \sum_{x: h(x)=y} p_X(x)$.

e.g. X = roll a fair die.; $Y = (X-3)^2$; $P(Y=4) = ?$

$P(Y=4) = P(X \in \{x : (x-3)^2 = 4\}) = P(X \in \{1, 5\})$.

$= P(x=1) + P(x=5)$.

$= \frac{1}{3}$.

3. Cts.: X is a cts., X has density func. $f_X(x)$, $h(X) = Y$.

1). h is no strictly increasing: Y not neccessarily be a cts..

2). h is strictly increasing (decreasing): Y is a cts., with:

$$f_Y(y) = f_X(h^{-1}(y)) \cdot \frac{1}{|h'(h^{-1}(y))|} \quad ①$$

← for decreasing.

$$= f_X(h^{-1}(y)) \cdot \frac{1}{h'(x)}$$

$$= f_X(h^{-1}(y)) \cdot (h^{-1})'(y)..$$

Inverse func.

if $y = h(x)$.

$(h^{-1})'(y) = \frac{1}{h'(x)}$

$= \frac{1}{h'(h^{-1}(y))}$

proof: ①. $P(a \leq Y \leq b) = P(h^{-1}(a) \leq X \leq h^{-1}(b)) = \int_{h^{-1}(a)}^{h^{-1}(b)} f_X(x)\, dx$.

substitute $x = h^{-1}(y)$.

$\Rightarrow dx = \frac{1}{h'(h^{-1}(y))} dy$.

$\Rightarrow P(a \leq Y \leq b) = \int_a^b \left[f_X(h^{-1}(y)) \cdot \frac{1}{h'(h^{-1}(y))} \right] dy$.

$= \int_a^b f_Y(y)\, dy$. (from def$^{\text{n}}$).

②. $F_Y(y) = P(Y \leq y) = P(h(X) \leq y) = P(X \leq h^{-1}(y))$.

$= F_X(h^{-1}(y))$.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(h^{-1}(y)) = f_X(h^{-1}(y)) \cdot [h^{-1}(y)]'$$

$$= f_X(h^{-1}(y)) \cdot \frac{1}{h'(h^{-1}(y))}$$

e.g. $X \sim$ Exponential(5), $Y = X^2$, $f_Y(y) = ?$

$$f_Y(y) = f_X(h^{-1}(y)) \cdot \frac{1}{h'(h^{-1}(y))}$$

① $y = h(x) = x^2$, $h'(x) = 2x$, $h^{-1}(y) = \sqrt{y}$.

② $f_X(x) = \lambda e^{-\lambda x}$.

③ $f_Y(y) = 5 \cdot e^{-5\sqrt{y}} \cdot \frac{1}{2\sqrt{y}}$

3). $Z \sim N(0,1)$. $Y = \mu + \sigma Z$, then $Y \sim N(\mu, \sigma^2)$.

Examples. P.32.