# 3 Question 3

(a)

```python
def q3_a_func(n: int) -> int:
    """Implement a Python function that takes a positive natural number n and returns a_n.

    Precondition: n is a positive natural number
    """
    if n == 1:
        # From the definition of a_n, when n is 1, return a_n equals to 1.
        return 1
    else:
        """ This is the recursion. Aim at returning the recursive value of a_n after reaching the base case when
        a_n equals to 1.
        """
        return q3_a_func(math.floor(math.sqrt(n))) * q3_a_func(math.floor(math.sqrt(n))) \
            + 2 * q3_a_func(math.floor(math.sqrt(n)))
```

Figure 1: Python function for Q3-a

Firstly, I will import sqrt, floor from math module.
Then, I write a comment to ensure it satisfies the pre-condition.
Besides, I handled the case when $n = 1$ in the 'if' statement.
Moreover, I handled $n \geq 2$ in the 'else' statement, which floor(sqrt(n)) is a positive natural number less than n since $\lfloor \sqrt{n} \rfloor \leq \sqrt{n} < n$ as $n > 1$

(b)

```python
def q3_b_func(n: int) -> int:
    """Implement a Python function that takes a positive natural number n and raises an exception if n is 1, otherwise
    it returns a_n.

    Precondition: n is a positive natural number
    """
    if n == 1:
        # By question requirement, when n is 1, raises an Exception.
        raise NotImplementedError
    elif n == 2 or n == 3:
        """Since when n equals to 2 or n equals to 3, the floor of square root of n is 1, and, in this function, we
        don't have the value of a_n when n equals 1. Thus, we need to manually add the value of a_n when n equals to 2
        and n equals to 3 to prevent the error when calling the recursive.
        """
        return 3
    else:
        """This is the recursion. Aim at returning the recursive value of a_n after reaching the case when a_n equals
        to 2 or a_n equals to 3.
        """
        return q3_b_func(math.floor(math.sqrt(n))) * q3_b_func(math.floor(math.sqrt(n))) \
            + 2 * q3_b_func(math.floor(math.sqrt(n)))
```

Figure 2: Python function for Q3-b

Firstly, I will import sqrt, floor from math module.
Then, I write a comment to ensure it satisfies the pre-condition.
Besides, I handled the case when $n = 1$ in the 'if' statement, and raise NotImplementedError.
Next, I handled the case when $n = 2$ *or* $n = 3$ in the 'elif' statement.
Moreover, I handled the case when $n \geq 4$, which similar to question a that floor(sqrt(n)) is a positive natural number less than n since $\lfloor \sqrt{n} \rfloor \leq \sqrt{n} < n$ as $n > 1$.

(c) When $n_0 = 2$, $n_0$ is the smallest natural $n_0$ so that $a_n$ is a multiple of 3 for each natural $n \geq n_0$.

If $n_0 = 1$, $a_n = a_1 = 1$ which is not a multiple of 3.

Given statement to prove: $\forall n \in \mathbb{N},\ n \geq n_0,\ P(n)$, which $P(n) : a_n$ is a multiple of 3.

Let $n \in \mathbb{N}$.

**Proof:** We prove this by complete induction on n.

**Base Case:** Let $2 \leq n < 4$.

$$\begin{aligned} P(2):\ a_2 &= (a_{\lfloor\sqrt{2}\rfloor})^2 + 2 \cdot a_{\lfloor\sqrt{2}\rfloor} \\ &= a_1^2 + 2 \cdot a_1 = 1^2 + 2 = 3 \text{is a multiple of 3.} \end{aligned}$$

$$\begin{aligned} P(3):\ a_3 &= (a_{\lfloor\sqrt{3}\rfloor})^2 + 2 \cdot a_{\lfloor\sqrt{3}\rfloor} \\ &= a_1^2 + 2 \cdot a_1 = 1^2 + 2 = 3 \text{is a multiple of 3.} \end{aligned}$$

Thus, I've proved the base case is true.

**Induction Step:** Let $n \in \mathbb{N}$. Let $n \geq 4$.

**Induction Hypothesis:** Assume $\forall k,\ 2 \leq k < n,\ P(k)$

Since $n \geq 4$, gives $\lfloor\sqrt{n}\rfloor < n$.

Since $\lfloor\sqrt{n}\rfloor < n$ and $4 \leq n$, gives $2 \leq \lfloor\sqrt{n}\rfloor$ as 2 is the smallest value of $\lfloor\sqrt{n}\rfloor$, which gives,

$$2 \leq \lfloor\sqrt{n}\rfloor < n$$

Since $\lfloor\sqrt{n}\rfloor$ is an integer which $\lfloor\sqrt{n}\rfloor \geq 2$, from induction hypothesis, we can always find $k' = \lfloor\sqrt{n}\rfloor$, which $P(k')$ is true and $a_{k'} = 3p,\ p \in \mathbb{N}$.

Thus gives,

$$\begin{aligned} a_n &= (\lfloor\sqrt{n}\rfloor)^2 + 2 \cdot a_{\lfloor\sqrt{n}\rfloor} \\ &= (a_{k'})^2 + 2 \cdot a_{k'} \\ &= (3p)^2 + 2 \cdot (3 \cdot p) \\ &= 9 \cdot p^2 + 6 \cdot p \\ &= 3 \cdot (3 \cdot p^2 + 2p) \end{aligned}$$

Let $q = 3 \cdot p^2 + 2 \cdot p$. Since $p \in \mathbb{N}$, gives $q \in \mathbb{N}$, which

$$a_n = 3q,\ q \in \mathbb{N} \text{ , where } a_n \text{ is a multiple of 3.}$$

I've proved that $P(n)$ is true.

To conclude, I've proved when $n_0 = 2$, $n_0$ is the smallest natural $n_0$ so that $a_n$ is a multiple of 3 for each natural $n \geq n_0$.

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