Monotonicity

1. Monotonicity.

Let f be a function defined on an inteval I.

① f is increasing on I when

$$\forall x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

② f is decreasing on I when

$$\forall x_1, x_2 \in I.\ x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

③ f is non-decreasing on I when

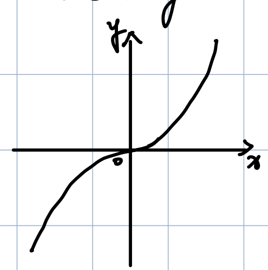
$$\forall x_1. x_2 \in I.\ x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$$

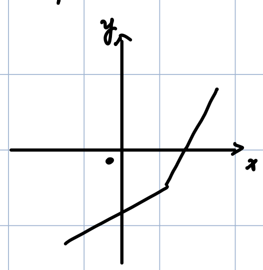
④ f is non-increasing on I when

$$\forall x_1. x_2 \in I.\ x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2).$$

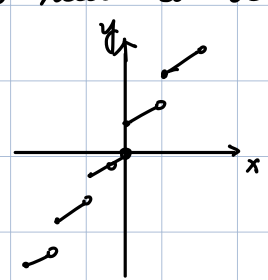
1) Note.

① increasing doesn't mean Positive derivative





① no need to be continuous.



2. Theorems.

1) Let $a<b$. Let f be a function defined on (a,b)

If $\forall x \in (a,b). f'(x) > 0$, then f is increasing on (a,b).

① proof.

Assume $\forall x \in (a, b)$. $f'(x) > 0$.

Let $x_1 . x_2 \in (a, b)$, $x_1 < x_2$.

Since $\lim_{x_1 \to x_2} \frac{f(x_1) - f(x_2)}{x_1 - x_2} > 0$, $x_1 - x_2 < 0$.

$f(x_1) - f(x_2) < 0 \Rightarrow f(x_1) < f(x_2)$.

$\therefore \forall x_1 . x_2 \in (a, b)$. $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ ■

2) Let $a < b$. Let f be a function defined on $[a, b]$.

If ① $\forall x \in (a, b)$. $f'(x) > 0$, and ② f is continuous on $[a, b]$.

Then f is increasing on $[a, b]$.

① Proof.

...... $\therefore f$ is increasing on (a, b).

Since f is continuous on $[a, b]$.

Take $x_3 \to a^+$, $x_3 < x_1 < x_2$, $f(x_3) < f(x_1) < f(x_2)$.

gives. $\lim_{x_3 \to a^+} f(x_3) < f(x_1)$. $\to f(a) \le f(x_3) < f(x_1)$.

Take $x_4 \to b^-$. $x_1 < x_2 < x_4$. $f(x_1) < f(x_2) < f(x_4)$.

gives. $\lim_{x_4 \to b^-} f(x_4) > f(x_2) \to f(b) \ge f(x_4) > f(x_2)$.

$\therefore f(a) < f(x) < f(b)$.

$\therefore f$ is increasing on $[a, b]$.

3. Summary

1). On open interval.

$f' = 0 \Rightarrow f$ constant.

$f' > 0 \Rightarrow f$ increasing.

$f' < 0 \Rightarrow f$ decreasing.

2). At a point: If $f'(x) = 0$ or DNE or ...... it can be anything.

4. Examples.

1). Find intervals where $f(x) = 8x^5 + 5x^4 - 20x^3$ is increasing / decreasing.

$$f'(x) = 40x^4 + 20x^3 - 60x^2.$$

$$= 20x^2(2x+3)(x-1).$$

		$x = -\frac{3}{2}$		$x=0.$		$x=1.$	
$f'(x).$	$+$	0	$-$	0	$-$	$0.$	$+$
f	$\nearrow$	LM	$\searrow$		$\searrow$	LM.	$\nearrow.$

$\therefore$ $f \uparrow$ on $(-\infty, -\frac{3}{2}]$. $f \searrow$ on $[-\frac{3}{2}, 1]$

$f \uparrow$ on $[1, +\infty)$