Convergence of Random Variables

1). Convergence in Probability: A sequence $X_1, X_2, \dots$ of r.v. convergence in prob. to another r.v. Y if: $\forall \varepsilon > 0$, $\lim_{n\to\infty} P(|X_n - Y| \geq \varepsilon) = 0$.

$\iff \forall \varepsilon > 0$, $\lim_{n\to\infty} P(|X_n - Y| < \varepsilon) = 1$.

written as $\{X_n\} \xrightarrow{P} Y$

when its not identity func. try use the method as north does.

e.g. $Y \sim$ Uniform $[0,5]$ and $X_n = (1+\frac{1}{n})Y$. Does $X_n \xrightarrow{P} Y$?

Let $\varepsilon > 0$. WTS: $\lim_{n\to\infty} P(|X_n - Y| < \varepsilon) = 1$.

$|X_n - Y| = |(1+\frac{1}{n})Y - Y| = \frac{1}{n} Y \leq \frac{5}{n} < \varepsilon$. $\Rightarrow n > \frac{5}{\varepsilon}$, which

$\forall n > \frac{5}{\varepsilon}$, $P(|X_n - Y| < \varepsilon) = 1$, which $\lim_{n\to\infty} P(|X_n - Y| < \varepsilon) = 1$. i.e. $X_n \xrightarrow{P} Y$.

when its identical func. use the generalized property.

1). In general, $\forall \varepsilon > 0$, $P(|X_n - Y| \geq \varepsilon) \leq P(X_n \neq Y)$. which if $\lim_{n\to\infty} P(X_n \neq Y) = 0$. then $\{X_n\} \xrightarrow{P} Y$.

e.g. $U \sim$ Uniform$[0,1]$. $X_n = I_{U \leq \frac{1}{2} + \frac{1}{2^n}}$, $Y = I_{U \leq \frac{1}{2}}$. Does $X_n \xrightarrow{P} Y$?

Let $\varepsilon > 0$. $P(|X_n - Y| \geq \varepsilon) \leq P(X_n \neq Y)$

Since $\xrightarrow[\quad \frac{1}{2} \quad\quad \frac{1}{2}+\frac{1}{2^n} \quad]{U_1 \quad\quad U_2 \quad\quad U_3}$ only $U = U_2$ satisfies $X_n \neq Y$.

which is $P(\frac{1}{2} < U \leq \frac{1}{2} + \frac{1}{2^n}) = \frac{\frac{1}{2^n}}{1-0} = \frac{1}{2^n}$.

Since $\lim_{n\to\infty} P(X_n \neq Y) = \lim_{n\to\infty} \frac{1}{2^n} = 0$, gives $X_n \xrightarrow{P} Y$

2. Weak Law of Large Numbers: For any sequence of r.v. $X_1, X_2, \dots$ which are ① independent, each have the ② same μ, and each have ③ variance $< v$ for $v < \infty$, if $M_n = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$, then $M_n \to \mu$ in probability.

$Var(M_n) \leq \frac{v}{n}$

$E(M_n) = \mu$.

i.e. $\lim_{n\to\infty} P(|M_n - \mu| \geq \varepsilon) = 0$, $\forall \varepsilon > 0$.

proof: From Chebychev's Inequality $P(|M_n - \mu| \geq \varepsilon) \leq \frac{Var(M_n)}{\varepsilon^2} \leq \frac{v}{n\varepsilon^2}$

when $n \to \infty$. $\frac{v}{n\varepsilon^2} \to 0$ as needed. which $M_n \xrightarrow{P} \mu$.

1) $\{X_n\}$ are i.i.d., i.e., independent and identically distributed.

$\to$ All have same μ & σ^2. $\hookrightarrow$ X_n all have the same Prob.

If $\{X_n\}$ i.i.d., then no need cond. ③. $P(a \le X_n \le b)$ is same for all n.

e.g. $X_n = 1_{n\text{th coin Heads}}$. (Bernulli).

$\{X_n\}$ indp.. with $E(X_n) = \frac{1}{2} = \mu$, $Var(X_n) = \frac{1}{2} \cdot (1-\frac{1}{2}) = \frac{1}{4} < v < \infty$.

if $M_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$ is fraction of Head on first n. by WLLN, $M_n \xrightarrow{P} \mu$, i.e. $\lim_{n\to\infty} P(|M_n - \mu| \ge \varepsilon) = 0$. $\forall \varepsilon > 0$.

Can't directly apply to X_n, as for $0 < \varepsilon < \frac{1}{2}$, $P(|X_n - \frac{1}{2}| \ge \varepsilon) = 1$.

$X_n = 0$ or 1.

P_{51} EG.

3. Convergence Almost Surely: A sequence $X_1, X_2, \ldots$ of r.v. C.A.S. to another r.v. Y, if $P(X_n \to Y$ as a sequence$) = 1$, i.e. $P(\lim_{n\to\infty} X_n = Y) = 1$, written as $X_n \xrightarrow{a.s.} Y$.

def in math. $\forall \varepsilon > 0$. $\exists n \ge N$. s.t. $|X_n - Y| \le \varepsilon$.

1). If $X_n \xrightarrow{a.s.} Y$, then $X_n \xrightarrow{P} Y$.

e.g. $P(X_n = 7) = \frac{1}{9 \cdot 10^{k-1}}$, $P(X_n = 5) = 1 - [\frac{1}{9 \cdot 10^{k-1}}]$, i.e. one 7 in X_1 to X_9, one 7 in X_{10} to X_{99}, one 7 in X_{100} to X_{999}, ...

① $X_n \xrightarrow{P} 5$.

$P(|X_n - 5| \ge \varepsilon) \le P(X_n \ne 5) = P(X_n = 7)$; when $n \to \infty$. $P(X_n = 7) \to 0$

$\therefore \{X_n\} \xrightarrow{P} 5$.

② $X_n \not\xrightarrow{a.s.} 5$.

def in math.

$X_n \xrightarrow{?} 5$ as a sequence, i.e. $\forall \varepsilon > 0$. $\exists n \ge N$. s.t. $|X_n - 5| \le \varepsilon$.

However, we always have $X_n = 7$. which $|X_n - 5| = 2 > \varepsilon$.

① Intuition: $\forall \varepsilon > 0$. as $n \to \infty$.

$\longrightarrow X \xrightarrow{P} Y$: $P(|X_n - Y| \ge \varepsilon) \to 0$.

$\longrightarrow X \xrightarrow{a.s.} Y$: $P(\exists m \ge n$ with $|X_m - Y| \ge \varepsilon) \to 0$.

4. Strong law of large numbers (SLLN): For any sequence of r.v. $X_1, X_2, \ldots$ which are ① i.i.d., each with ② same μ, if $M_n = \frac{1}{n}(X_1 + \cdots + X_n)$.

then $M_n \xrightarrow{a.s.} \mu$

e.g. $X_n = 1_{n\text{th coin Heads}}$.

$\{X_n\}$ i.i.d. with $\mu = E(X) = \frac{1}{2}$.

① By WLLN. $\forall \varepsilon > 0$. $\lim_{n\to\infty} P(|M_n - \frac{1}{2}| \geq \varepsilon) = 0$. i.e. $P(M < 0.503) > 0.99$.

② By SLLN. $P(M_n \to \frac{1}{2}) = 1$. i.e. $P(\exists n : M_n < 0.503) = 1$.

$\Rightarrow P(\exists n : X_1 + \cdots + X_n < 0.503n) = 1$.