Continuous but not Differentiable

1. Meaning

1) f is continuous at a means $\lim_{x \to a} f(x) = f(a)$

2) f is not differentiable at a means

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \text{ DNE.}$$

2. Conditions:

1) f has a corner at $x = a$.

① f is continuous at a.

② The limit in

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

DNE because the side limits are different.

e.g. $f(x) = |x|$. Is f differentiable at 0?

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$

- $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$
- $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$

the side limit are different, not differentiable

different from vertical asymptotes.

2) g has a vertical tangent line at $x = a$.

① g is continuous at a.

② $g'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \pm\infty$

e.g. $g(x) = x^{\frac{1}{3}}$. Is g differentiable at 0?

$\rightarrow g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0.} = \lim_{x \to 0} \frac{x^{\frac{1}{3}}}{x} = \lim_{x \to 0} \frac{1}{x^{\frac{2}{3}}} = \infty$

$\rightarrow$ if $x \neq 0$, $g'(x) = \frac{1}{3x^{\frac{2}{3}}}$ & $\lim_{x \to 0} g'(x) = \lim_{x \to 0} \frac{1}{3x^{\frac{2}{3}}} = \infty$

③ Difference: $\lim_{x \to a} \frac{g(x)-g(a)}{x-a} = \pm\infty \iff \underbrace{\lim_{x \to a} g'(x) = \pm\infty}_{\text{a bit stronger.}}$