1. Structure.

1). Let $g(x) = \sum_{i=0}^{\infty} a_n x^n$.

2). 两边同乘 $x^n \longrightarrow$ 应当为最大 n，若有 a_{n+2}，则应乘 x^{n+2}.

3). 直接转成 Inf. Sum.

4). 把 Inf. Sum. 转成 $g(x)$；把 non-homo 部分转成 Inf. Sum 对应 func

5). 解出 $g(x)$.

6). 把 $g(x)$ 展开回 Inf. Sum. 找到 a_n.

2. Exercise.

e.g. $a_n - 3a_{n-1} = n$. for $n \geq 1$. and $a_0 = 1$.

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$.

$a_n \cdot x^n - 3a_{n-1} \cdot x^n = n \cdot x^n$.

$\Rightarrow \sum_{n=1}^{\infty} a_n \cdot x^n - 3\sum_{n=1}^{\infty} a_{n-1} \cdot x^n = \sum_{n=1}^{\infty} n \cdot x^n$.

从 n-1 开始避免有负数

$\Rightarrow (g(x) - 1) - 3 \cdot x \cdot (\sum_{n=1}^{\infty} a_{n-1} \cdot x^{n-1}) = x \cdot \sum_{n=1}^{\infty} n \cdot x^{n-1}$.

$\Rightarrow g(x) - 1 - 3x \cdot g(x) = \frac{x}{(1-x)^2}$

$\Rightarrow (1-3x) \cdot g(x) = \frac{x}{(1-x)^2} + 1$

$\Rightarrow g(x) = \frac{x}{(1-x)^2(1-3x)} + \frac{1}{1-3x} = \frac{x^2 - x + 1}{(1-x)^2(1-3x)}$

$= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-3x}$

$= \frac{A(1-x)(1-3x) + B(1-3x) + C(1-x)^2}{(1-x)^2(1-3x)}$

$= \frac{A(1-4x+3x^2) + B(1-3x) + C(1+x^2-2x)}{(1-x)^2(1-3x)}$

$\begin{cases} A + B + C = 1 \\ -4A - 3B - 2C = -1 \\ 3A + C = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{4} \\ B = -\frac{1}{2} \\ C = \frac{7}{4} \end{cases}$

$\therefore g(x) = \frac{1}{1-x} \cdot (-\frac{1}{4}) + (-\frac{1}{2}) \cdot \frac{1}{(1-x)^2} + \frac{7}{4} \cdot \frac{1}{1-3x}$

$= (-\frac{1}{4}) \cdot \sum_{n=0}^{\infty} x^n - \frac{1}{2} \cdot \sum_{n=0}^{\infty} (n+1) x^n + \frac{7}{4} \cdot \sum_{n=0}^{\infty} 3^n \cdot x^n$.

$= \sum_{n=0}^{\infty} \left[(-\frac{1}{4}) - \frac{1}{2}(n+1) + \frac{7}{4} \cdot 3^n\right] \cdot x^n$.

$\therefore a_n = -\frac{1}{4} - \frac{1}{2}n - \frac{1}{2} + \frac{7}{4} \cdot 3^n = \frac{7}{4} \cdot 3^n - \frac{1}{2}n - \frac{3}{4}$.