MAT 137Y - Practice problems
Unit 11 - Sequences

1. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Write down the formal definition of the following concepts. You have already seen some of these in lecture.

   (a) The sequence is convergent.
   (b) The sequence is divergent.
   (c) The sequence is divergent to ∞.
   (d) The sequence is divergent to $-\infty$.
   (e) The sequence is increasing.
   (f) The sequence is eventually increasing.
   (g) The sequence is decreasing.
   (h) The sequence is non-decreasing.
   (i) The sequence isn't decreasing.
   (j) The sequence isn't non-decreasing.
   (k) The sequence isn't eventually decreasing.
   (l) The sequence is bounded above.
   (m) The sequence is not bounded above.
   (n) The sequence is bounded.

   Hints:
   Are all your variables introduced or properly quantified in Question 1a?
   All of the statements are *different*. Figuring out exactly how they are different is part of your job. Do not look at the solutions yet, or you will waste any possible learning opportunity.

2. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by the recurrence relation:

$$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= 1 - a_n \quad \text{for } n \geq 1 \end{aligned}$$

   I am going to calculate its limit with a nifty trick. Let us call L the limit of the sequence. Then:

$$\begin{aligned} \lim_{n\to\infty} [a_{n+1}] &= \lim_{n\to\infty} [1 - a_n] \\ \left[\lim_{n\to\infty} a_{n+1}\right] &= \left[\lim_{n\to\infty} 1\right] - \left[\lim_{n\to\infty} a_n\right] \end{aligned} \tag{1}$$

   Since $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$, Equation (1) becomes $L = 1 - L$. Therefore $L = 1/2$.
   The above argument is WRONG. Why?

3. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by the recurrence relation:

$$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= \sqrt{2 + a_n} \quad \text{for } n \geq 1 \end{aligned}$$

   (a) Compute a_2, a_3, and a_4.
   (b) Prove by induction that $0 < a_n < 2$ for all $n \geq 1$.
   (c) Prove that the sequence is increasing.

(d) Find the limit of the sequence.

Hint: Do something similar to Question 2, but correct.

(e) Why was the argument in Question 2 incorrect, but the argument in Question 3d was correct?

4. Prove that if a sequence is increasing and unbounded above, then it is divergent to ∞. Write a formal proof directly from the definitions. (This is basically Theorem 3 in Video 11.4.)

5. Compute the following limits

(a) $\displaystyle\lim_{n\to\infty} \frac{2n! + 3\ln n}{5n! + 9\ln n}$ (b) $\displaystyle\lim_{n\to\infty} \frac{n!}{e^n + n^{100}}$ (c) $\displaystyle\lim_{n\to\infty} \frac{(2n+1)^2 + 2^n}{(n+1)^2 + 2^{n+3}}$

6. In this problem we will only consider sequences that are **POSITIVE AND DIVERGENT TO** ∞.

For each of the following statements, decide whether they are true or false. If true, prove it. If false, give a counterexample.

(a) IF $\{x_n\}_n$, $\{y_n\}_n$, $\{z_n\}_n$ are sequences such that $x_n << y_n$ and $y_n << z_n$
THEN $x_n << z_n$.

(b) For every sequence $\{x_n\}_n$, there exists a sequence $\{y_n\}_n$ such that $y_n << x_n$

(c) IF $\{x_n\}_n$ and $\{y_n\}_n$ are sequences such that $x_n << y_n$
THEN there exists a sequence $\{z_n\}_n$ such that $x_n << z_n << y_n$.

(d) For every sequence $\{x_n\}_n$, there exists a sequence $\{y_n\}_n$ such that for every $a > 0$, $(x_n)^a << y_n$

7. The following is a well-known result known as Stirling's formula:

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$$

For this problem, you may assume we already know this identity to be true. Use it to calculate the limits of the four sequences below.

(a) $\displaystyle\lim_{n\to\infty} \frac{n!e^n}{n^{n+1/2}}$

(b) $\displaystyle\lim_{n\to\infty} \frac{(2n)!}{e^{-2n}(2n)^{2n}\sqrt{n}}$

(c) $\displaystyle\lim_{n\to\infty} \frac{(2n)!\sqrt{n}}{(n!)^2\, 4^n}$

(d) $\displaystyle\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$

Some answers and hints

1. (a) $\exists L \in \mathbb{R},\ \forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |a_n - L| < \varepsilon$
   (b) $\forall L \in \mathbb{R},\ \exists \varepsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N}, \quad (n \geq n_0 \text{ and } |a_n - L| \geq \varepsilon)$
   (c) $\forall M \in \mathbb{R},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies a_n > M$
   (d) $\forall M \in \mathbb{R},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies a_n < M$
   (e) $\forall n \in \mathbb{N}, \quad a_n < a_{n+1}$
   Equivalently, $\forall n, m \in \mathbb{N}, \quad n < m \implies a_n < a_m$
   (f) $\exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies a_n < a_{n+1}$
   (g) $\forall n \in \mathbb{N}, \quad a_n > a_{n+1}$
   (h) $\forall n \in \mathbb{N}, \quad a_n \leq a_{n+1}$
   (i) $\exists n \in \mathbb{N}, \quad a_n \leq a_{n+1}$
   (j) $\exists n \in \mathbb{N}, \quad a_n > a_{n+1}$
   (k) $\forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}, \quad (n \geq n_0 \text{ and } a_n \leq a_{n+1})$
   (l) $\exists A \in \mathbb{R},\ \forall n \in \mathbb{N}, \quad a_n \leq A$
   (m) $\forall A \in \mathbb{R},\ \exists n \in \mathbb{N}, \quad a_n > A$
   (n) $\exists A, B \in \mathbb{R},\ \forall n \in \mathbb{N}, \quad B \leq a_n \leq A$
   Equivalently, $\exists C \in \mathbb{R},\ \forall n \in \mathbb{N}, \quad |a_n| \leq C$

2. In the "proof" we are assuming the limit exists. We have only proven that either the limit does not exist (the sequence is divergent) or the limit is 1/2.

3. (c) There are various ways to do this. For example, we can prove by induction that for all $n \geq 1$, $a_{n+1} > a_n$ by noticing that
   $$a_{n+1} > a_n \quad \implies \quad \dots \quad \implies \quad \sqrt{2 + a_{n+1}} > \sqrt{2 + a_n}.$$
   Alternatively, it can also be proven without induction by analyzing in which domain the function $f(x) = \sqrt{2+x} - x$ is positive.
   (d) The limit is 2.
   If you imitate the proof in Question 2 without justifying that the sequence is convergent first, then your proof is incorrect.

4. Assume $\{a_n\}_n$ is increasing and unbounded above. We want to prove that
   $$\forall M \in \mathbb{R},\ \exists n_0 \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, \quad n \geq n_0 \implies a_n > M.$$
   - Fix $M \in \mathbb{R}$.
   - Since M is not an upper bound of $\{a_n\}_n$, there exists $n_0 \in \mathbb{N}$ such that $a_{n_0} > M$.
   - Now verify that this same value of n_0 works. Fix $n \in \mathbb{N}$ and assume $n \geq n_0$. Then...

5. (a) $2/5$ (b) ∞ (c) $1/8$

6. They are all TRUE.

   (a) Make sure you prove this using the definition of “<<”.

   (b) For your proof, fix $\{x_n\}_n$, then construct $\{y_n\}_n$ depending on $\{x_n\}_n$, and verify that it works. Remember that $\{y_n\}_n$ must be positive and divergent to ∞. A common error is to take something like $y_n = x_n/2$, which does not work.

   (c) For your proof, fix $\{x_n\}_n$ and $\{y_n\}_n$, then construct $\{z_n\}_n$ depending on $\{x_n\}_n$ and $\{y_n\}_n$, and verify that it works. A common error is to take something like $z_n = (x_n + y_n)/2$, which does not work.

   (d) For your proof, fix $\{x_n\}_n$, then construct $\{y_n\}_n$ depending on $\{x_n\}_n$, and verify that it works. You are not allowed to make $\{y_n\}_n$ depend on a; the same $\{y_n\}_n$ must work for all values of a (why?)

7. (a) $\sqrt{2\pi}$ (b) $2\sqrt{\pi}$ (c) $\dfrac{1}{\sqrt{\pi}}$ (d) $\dfrac{1}{e}$