CSC165 Problem Set 2

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1 Q1: Number Theory

(a) Fact from the worksheet:

$\forall a, b, \in \mathbb{Z}, a \neq 0 \vee b \neq 0 \Rightarrow (\exists p_1, q_1 \in \mathbb{Z}, gcd(a,b) = p_1 a + q_1 b) \wedge (\forall d \in \mathbb{Z}^+, (\exists p_2, q_2 \in \mathbb{Z}, d = p_2 a + q_2 b) \Rightarrow d \geq gcd(a,b))$

Proof:

Let $n \in \mathbb{N}$

Since $n \in \mathbb{N},\ n \in \mathbb{Z}\ and\ n \geq 0$, gives,

$$9n + 1 \geq 1\ and\ 10n + 1 \geq 1; 9n + 1 \in \mathbb{Z}\ and\ 10n + 1 \in \mathbb{Z}$$

Thus, $9n + 1 \neq 0$ and $10n + 1 \neq 0$.

By the fact form the worksheet(as listed above), we have,

$$\forall d \in \mathbb{Z}^+, (\exists p_2, q_2 \in \mathbb{Z}, d = p_2(9n+1) + q_2(10n+1)) \Rightarrow d \geq gcd(9n+1, 10n+1))$$

Let $d_1 = 1$, which $d \in \mathbb{Z}^+$, gives

$$d_1 = 10(9n+1) + 9(10n+1) \text{ and } gcd(9n+1, 10n+1) \leq d_1 = 1$$

According to fact 4 in worksheet 4, gives,

$$\forall n, m \in \mathbb{N}, n \neq 0 \vee m \neq 0 \Rightarrow gcd(n,m) \geq 1$$

Since $9n + 1 \neq 0$ and $10n + 1 \neq 0$, $gcd(9n+1, 10n+1) \geq 1$.

Since $gcd(9n+1, 10n+1) \geq 1$ and $gcd(9n+1, 10n+1) \leq 1$, gives,

$$gcd(9n+1, 10n+1) = 1$$

Using the worksheet fact, and using a = 9n + 1, b = 10n + 1 and d = 1, we can conclude that gcd(9n + 1, 10n + 1) = 1. ■

(b) Proof:

Let $m, n \in \mathbb{Z}$, assume $n \mid m\ \wedge Prime(n)$

Given $\exists k_1 \in \mathbb{Z}\ s.t.\ m = k_1 n$ according to $n \mid m$

Suppose $n \mid (m+1)$, then $\exists k_2 \in \mathbb{Z}\ s.t.\ m + 1 = k_2 n$

Subtract two equations, gives,

$$k_2 n - k_1 n = (m+1) - m$$
$$n(k_2 - k_1) = 1$$

Since n is a prime, $n > 1$; Since $k_1, k_2 \in \mathbb{Z}, (k_2 - k_1) \in \mathbb{Z}$

Therefore, $n \cdot (k_2 - k_1) \neq 1$ (for $n > 1$ but $0 < \frac{1}{k_2 - k_1} \leq 1$), gives,

It contradicts to $n(k_2 - k_1) = 1$

Thus, $n \nmid (m{+}1)$ ■

2 Q2: Floors and Ceilings

(a) Proof

Let $x \in \mathbb{N}$. We'll separate the proof into two cases: Either x is even or x is odd.

Case 1: Let x be even

By definition: $\exists k \in \mathbb{Z}, x = 2k$ (Since floor functions have no effect on integers, and $\frac{x}{2}$ is an integer)

$$\left\lceil \frac{x-1}{2} \right\rceil = \left\lceil \frac{2k-1}{2} \right\rceil = \left\lceil k - \frac{1}{2} \right\rceil = k = \frac{x}{2} = \left\lfloor \frac{x}{2} \right\rfloor$$

Case 2: Let x be odd

By definition: $\exists k \in \mathbb{Z}, x = 2k - 1$

$$\left\lceil \frac{x-1}{2} \right\rceil = \left\lceil \frac{2k-1-1}{2} \right\rceil = \lceil k-1 \rceil = k-1$$
$$= \left\lfloor k - \frac{1}{2} \right\rfloor \text{ works since k is an integer}$$
$$= \left\lfloor \frac{2k-1}{2} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor$$

■

(b) (i) We want to prove $\forall x \in \mathbb{R}, \lceil x-1 \rceil = \lceil x \rceil - 1$

Proof

Let $x \in \mathbb{R}$, gives,

$$x \leq \lceil x \rceil < x + 1$$
$$x - 1 \leq \lceil x \rceil - 1 < x$$

According to the fact that,

$$0 \leq \lceil x \rceil - x < 1$$
$$0 \leq \lceil x-1 \rceil - (x-1) < 1$$
$$x - 1 \leq \lceil x-1 \rceil < x$$

We want to prove that $\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, x - 1 \leq n < x$.

Let $x \in \mathbb{R}$. Assume there're two integers between x - 1 and x, which,

$$\exists n_1, n_2 \in \mathbb{Z}, x - 1 \leq n_1 < n_2 < x, \text{which } n_1 \neq n_2$$

Since $n_1 \neq n_2$, $n_1 < n_2$ and $n_1, n_2 \in \mathbb{Z}$, gives, $min(n_2 - n_1) = 1$

Since $n_2 < x$, gives, $x - n_2 > 0$

Thus, $min(x - n_1) > 1$, gives, $min(x - (x-1)) > 1$

However, x - (x - 1) = 1, which contradicts to $min(x - (x-1)) > 1$.

Therefore, we've proved that $\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, x - 1 \leq n < x$

Since $\lceil x \rceil - 1 \in \mathbb{Z}$ and $x - 1 \leq \lceil x \rceil - 1 < x$, gives, $\lceil x \rceil - 1$ is the only integer between x and $x - 1$.

Since $\lceil x - 1 \rceil \in \mathbb{Z}$ and $x - 1 \leq \lceil x - 1 \rceil < x$, gives, $\lceil x - 1 \rceil$ is the only integer between x and $x - 1$

Therefore, we have $\forall x \in \mathbb{R}, \lceil x - 1 \rceil = \lceil x \rceil - 1$ as needed. ■

(ii) We want to disprove the statement and prove: $\exists x, y \in \mathbb{R}, \lceil xy \rceil \neq \lceil x \rceil \lfloor y \rfloor$

Proof

Let x = 0.5, y = 2, gives:

$$\lceil xy \rceil = \lceil 0.5 \cdot 2 \rceil = \lceil 1 \rceil = 1 \neq 2 = 1 \cdot 2 = \lceil 0.5 \rceil \lfloor 2 \rfloor = \lceil x \rceil \lfloor y \rfloor$$

Therefore, we have $\exists x, y \in \mathbb{R}, \lceil xy \rceil \neq \lceil x \rceil \lfloor y \rfloor$ as needed. ■

3 Q3: Induction

a) Proof: We want to prove $\forall n \in \mathbb{N}, 9 \mid 11^n - 2^n$.

Base Case: Let n = 0. We want to prove $9 \mid 11^0 - 2^0$, meaning $\exists k_0 \in \mathbb{Z}, 11^0 - 2^0 = 9k_0$:

$$11^0 - 2^0 = 1 - 1 = 0 = 9 \cdot 0$$

when $k_0 = 0,\ 11^0 - 2^0 = 9k_0$, saying that:

$$\text{Therefore, } \exists k_0 \in \mathbb{Z},\ \text{when n} = 0,\ 9 \mid 11^0 - 2^0$$

Hence, we've proven the base case.

Induction Hypothesis: Let $n \in \mathbb{N}$. Assume $9 \mid 11^n - 2^n$ which, gives,

$$\exists k \in \mathbb{Z}, 11^n - 2^n = 9k$$

Induction Step: Want to prove $9 \mid 11^{n+1} - 2^{n+1}$, meaning $\exists k_1 \in \mathbb{Z}, 11^{n+1} - 2^{n+1} = 9k_1$

Let $w = 11^n + 2k$

$$\begin{aligned}
&11^{n+1} - 2^{n+1} \\
&= 11(11^n) - 2(2^n) \\
&= 9(11^n) + 2(11^n) - 2(2^n) \\
&= 9(11^n) + 2(11^n - 2^n) \\
&= 9(11^n) + 2(9k) \text{ By Induction Hypothesis} \\
&= 9 \cdot (11^n + 2k) \\
&= 9 \cdot w \ (9 \mid 9 \cdot w \ is\ True)
\end{aligned}$$

when $k_1 = w,\ 11^{n+1} - 2^{n+1} = 9k_1$.

Thus, $\exists k_1 \in \mathbb{Z},\ 9 \mid 11^{n+1} - 2^{n+1}$ ■

b) Proof: We want to prove $\forall n \in \mathbb{N}, P_n = \prod_{i=0}^{n-1} P_i + 2$

Base Case: Let n = 0. We want to prove $P_0 = \prod_{i=0}^{0-1} P_i + 2$ where P_n is a Pierre Number.
By Pierre Number Definition,

$$P_0 = 2^{2^0} + 1 = 2 + 1 = 3$$

Since, when $n < j$, then $\prod_{i=j}^{n} f(i) = 1$, gives,

$$\prod_{i=0}^{0-1} P_i + 2 = \prod_{i=0}^{-1} P_i + 2 = 1 + 2 = 3$$

Therefore, when n = 0, $P_0 = \prod_{i=0}^{0-1} P_i + 2$.
Hence, we've proven the base case.

Induction Hypothesis: Let $n \in \mathbb{N}$. Assume $P_n = \prod_{i=0}^{n-1} P_i + 2$

Induction Step: Want to prove $P_{n+1} = \prod_{i=0}^{(n+1)-1} P_i + 2$

$$\begin{aligned}
&\prod_{i=0}^{(n+1)-1} P_i + 2 \\
=&\prod_{i=0}^{n} P_i + 2 \\
=&\prod_{i=0}^{n-1} P_i \cdot P_n + 2 \\
=&(P_n - 2) \cdot P_n + 2 \\
=&{P_n}^2 - 2P_n + 2 \text{ (By the I.H.)} \\
=&(P_n - 1)^2 + 1 \\
=&(2^{2^n} + 1 - 1)(2^{2^n} + 1 - 1) + 1 \text{ (By definition of } P_n = 2^{2^n} + 1) \\
=&2^{2(2^n)} + 1 \\
=&2^{2^{n+1}} + 1
\end{aligned}$$

Since, by Pierre Number Definition, $P_{n+1} = 2^{2^{n+1}} + 1$, gives,

$$P_{n+1} = \prod_{i=0}^{(n+1)-1} P_i + 2$$

Hence, we've proven the induction successfully.

$$Thus,\ \forall n \in \mathbb{N},\ P_n = \prod_{i=0}^{n-1} P_i + 2$$

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