# CSC165 Problem Set 1

Jonah Kisluk, Nick Pestov, Xuanqi Wei

September 10, 2023

# Contents

# 1 Question 1: Propositional Formulas

1. $(\neg p \Leftrightarrow q) \Rightarrow q$

| p | q | $(\neg p \Leftrightarrow q) \Rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Table 1: (i)

(ii) Equivalent Formula: $(\neg p \wedge \neg q) \vee (q \wedge p) \vee q$

$$\begin{aligned}
&(\neg p \wedge \neg q) \vee (q \wedge p) \vee q \\
\equiv&\neg((\neg\neg p \vee q) \wedge \neg(\neg q \vee \neg p) \vee q) \ (De\ Morgan's)(Double\ Negation) \\
\equiv&\neg(\neg p \Rightarrow q) \vee \neg(q \Rightarrow \neg p) \vee q \ (Implication) \\
\equiv&\neg((\neg p \Rightarrow q) \wedge (q \Rightarrow \neg p)) \vee q \ (DeMorgan's) \\
\equiv&\neg(\neg p \Leftrightarrow q) \vee q \ (Bi - Implication) \\
\equiv&(\neg p \Leftrightarrow q) \Rightarrow q \ (Implication)
\end{aligned}$$

2. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow r)$

| p | q | r | $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow r)$ |
|---|---|---|---|
| T | T | T | T |
| T | T | F | T |
| T | F | T | T |
| T | F | F | T |
| F | T | T | T |
| F | T | F | F |
| F | F | T | T |
| F | F | F | F |

Table 2: (i)

(ii) Equivalent Formula: $(p \wedge q \wedge \neg r) \vee ((p \wedge \neg q) \vee r)$

$$\begin{aligned}
&(p \wedge q \wedge \neg r) \vee ((p \wedge \neg q) \vee r) \\
\equiv&(\neg\neg p \wedge \neg(\neg q \vee r)) \vee ((\neg\neg p \wedge \neg q) \vee r \ (De\ Morgan's)(DoubleNegation) \\
\equiv&\neg(\neg p \vee (\neg q \vee r)) \vee (\neg(\neg p \vee q) \vee r) \ (De\ Morgan's) \\
\equiv&\neg(\neg p \vee (q \Rightarrow r)) \vee (\neg(p \Rightarrow q) \vee r) \ (Implication) \\
\equiv&\neg(p \Rightarrow (q \Rightarrow r)) \vee ((p \Rightarrow q) \Rightarrow r) \ (Implication) \\
\equiv&(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow r) \ (Implication)
\end{aligned}$$

# 2 Question 2: Translating Statements

(a) $\forall c \in C,\ \forall s \in S,\ \neg CS(s) \vee \neg Fail(s,c)$

(b) $\exists c \in C,\ \forall s \in S,\ CS(s) \Rightarrow Study(s,c)$

(c) $\exists s \in S,\ \forall c \in C,\ CS(s) \wedge \neg Study(s,c)$

(d) $\forall c \in C,\ (\exists s_1 \in S,\ Study(s_1,c) \wedge (\forall s_2 \in S,\ Study(s_2,c) \Rightarrow s_1 = s_2))$

(e) $\forall s \in S,\ \exists c_1,\ c_2 \in C,\ Study(s,c_1) \wedge Study(s,c_2) \wedge c_1 \neq c_2 \wedge (\forall c_3 \in C,\ Study(s,c_3) \Rightarrow c_3 = c_1 \vee\ c_3 = c_2)$

# 3 Question 3: Choosing a Universe and Predicates

(a) $For\ each\ x, z \in \mathbb{N}\ define\ P(x,z)\ by\ x < z$

$First\ Statement: False.\ Since\ x = 166\ makes\ both\ P(x,165)\ and\ P(x,0) false.$
$Second\ Statement: True.\ First\ Part,\ \forall x \in \mathbb{N},\ P(x,165),\ is\ false\ since\ not\ all\ all\ natural\ number\ are\ less\ than\ 165.\ Hence,\ expression\ is\ vacuously\ true,\ as\ anticedent\ is\ not\ satisfied.$
$Hence,\ it\ satisfies\ the\ requirement\ of\ the\ question.$

(b) $For\ each\ x, z \in \mathbb{N}\ define\ P(x,z)\ by\ x \geq z$

$First\ Statement: True.\ Since\ not\ all\ natural\ numbers\ are\ greater\ or\ equal\ to\ 165,\ but\ all\ natural\ numbers\ by\ definition\ are\ greater\ than\ or\ equal\ to\ 0,\ so\ the\ statement\ is\ True.$
$Second\ Statement: True.\ Since\ not\ all\ natural\ numbers\ are\ bigger\ or\ equal\ to\ 165,\ for\ example\ \ 50\ is\ a\ natural\ number\ but\ it\ is\ smaller\ than\ 165,\ the\ antecedent\ is\ false.$
$Therefore,\ no\ matter\ what\ the\ consequent\ is,\ the\ statement\ is\ vacuously\ True.$
$Hence,\ it\ satisfies\ the\ requirement\ of\ the\ question.$

(c) $Let\ S: \{1,2,3\},\ T: \{2,3,4\},\ for\ each\ x \in S,\ y \in T\ define\ P(x,y)\ by\ y > x\ and\ Q(x)\ by\ x > 2$

$First\ Statement: False.\ Since\ for\ any\ arbitrary\ number\ x\ in\ S,\ there\ exists\ a\ number\ y = 4\ in\ T,\ which\ is\ bigger\ than\ any\ number\ in\ S,\ making\ the\ antecedent\ True.\ However,\ not\ all\ numbers\ in\ S\ are\ bigger\ than\ 2.\ Thus,\ Ture\ implies\ False\ is\ False.$
$Second\ Statement: True.\ For\ any\ arbitrary\ number\ x\ in\ S,\ x\ is\ not\ always\ bigger\ than\ 2,\ which\ the\ antecedent\ is\ False.\ Thus,\ False\ implies\ False\ is\ True.$
$Hence,\ it\ satisfies\ the\ requirement\ of\ the\ question.$

# 4 Question 4: Pierre Numbers

(a) $For\ each\ n \in \mathbb{N}\ define\ PierreNumber(n)\ by\ \exists k \in \mathbb{Z}\ s.t.\ n = 2^{2^k} + 1$

(b) $\forall n \in \mathbb{N},\ PierreNumber(n) \Rightarrow (\exists j \in \mathbb{Z},\ 2j + 1 = n)$

(c) $\forall n \in \mathbb{N},\ PierreNumber(n) \Rightarrow \exists x \in \mathbb{N},\ x > n \wedge PierreNumber(x)$

(d) $\forall k \in \mathbb{N}, Prime(2^{2^k} + 1) \Rightarrow k \leq 4$