Conditioning & Independence

1. Conditioning

1). Discrete: Suppose X and Y are discrete with joint probability $p_{X,Y}$.

① $p_{X|Y}(x|y) = \frac{P(X=x, Y=y)}{P(Y=y)}$; $p_{Y|X}(y|x) = \frac{P(X=x, Y=y)}{P(X=x)}$.

② $P(a \le Y \le b \mid X=x) = \sum_{a \le y \le b} P(Y=y \mid X=x)$

$$= \sum_{a \le y \le b} p_{Y|X}(y|x)$$
$$= \sum_{a \le y \le b} \frac{P(X=x, Y=y)}{P(X=x)}.$$
$$= \frac{P(a \le Y \le b, X=x)}{P(X=x)}.$$

2). Continuous: Suppose X and Y have density function $f_{X,Y}(x,y)$.

① $P(a \le Y \le b \mid X=x) = \int_a^b f_{Y|X}(y|x)\, dy$.

$$= \int_a^b \frac{f_{X,Y}(x,y)}{f_X(x)}\, dy.$$

e.g. $f_{X,Y}(x,y) = \frac{15}{32} x y^2$. $0 \le y \le x \le 2$; $P(0 \le Y \le 1 \mid X = \frac{3}{2}) = ?$

Step 1: Get $f_X(x)$, using marginal.

$$f_X(x) = \int_0^x \frac{15}{32} x y^2\, dy = \frac{5}{32} x^4.$$

Step 2. get the function.

$$\frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{15}{32} x y^2}{\frac{5}{32} x^4} = 3x^{-3} y^2.$$

Step 3: Use definition. substitute x.

$$\int_0^1 3x^{-3} y^2\, dy = \int_0^1 3 \cdot \left(\frac{3}{2}\right)^{-3} y^2\, dy = \frac{8}{27}.$$

2. Independence

1) General.

① $P(X \in B, Y \in C) = P(X \in B) \cdot P(Y \in C)$.

② $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$.

2) Discrete:

e.g. B, C are interval. e.g. $B = [-\infty, x]$. $C = [-\infty, y]$.

$P(X \le x, Y \le y) = P(X \le x) \cdot P(Y \le y)$. i.e.

$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$.

① $P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$. i.e.

$p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)$

② $p_{X|Y}(x|y) = P(X = x)$. ; $p_{Y|X}(y|x) = P(Y = y)$.

3). Continuous.

① $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$.

② $f_{Y|X}(y|x) = f_Y(y)$.