MAT 137Y: Calculus with proofs
Assignment 1
Due on Thursday, Sept 29 by 11:59pm via GradeScope

Instructions

This problem set is based on Unit 1: Logic, sets, and notations. Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions. Remember you can submit in pairs or individually.

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

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I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) HenryWei

2) ____

1. In this problem, we will deal with subsets $A \subseteq \mathbb{R}$. Let's define two new concepts.

   (i) We say that A is *blow-up* if

   $$\exists x \in A \ s.t. \ \forall y \in A, x - y > 0 \Longrightarrow x - y \text{ is an odd integer.}$$

   (ii) We say that A is *step-down* if

   $$\forall x \in A, \exists y \in A \ s.t. \ x - y \geq 1 \ AND \ x - y \text{ is an even integer.}$$

   (Part 1) To help you understand the definitions, here are two sets

   $$A = \{-2, 0, 3, 3.5, 4\} \text{ and } B = \{2k : k \in \mathbb{Z}\}.$$

   (A) Prove that A is a blow-up set by using the definition.

Pf. Take $x=3$. , $x \in A$.

Let $y \in A$. when $y=-2$, $x-y=5>0$, $x-y$ is an odd integer., satisfying A is blow-up.

when $y=0$, $x-y=3>0$, $x-y$ is an odd integer. satisfying A is blow-up.

when $y=3$. $x-y=0$ not satisfying $x-y>0$.

when $y=3.5$. $x-y=-0.5$ not satisfying $x-y>0$.

when $y=4$. $x-y=-1$ not satisfying $x-y>0$.

Since. $P(x)$ is $x-y>0$ and for $x=0$, $y \in A$ when $y=3$ or $y=3.5$ or $y=4$ not satisfying $P(x)$ is true in this condition.

Thus. the whole statement is True and A is a blow-up set. ∎

   (B) Prove that B is a step-down set by using the definition.

Pf. Let $x \in B$

Take $y \in B$. $y = x-2$.

Since $B = \{2k \mid k \in \mathbb{Z}\}$. gives,

$x = 2k$, $k \in \mathbb{Z}$.

Therefore. $y = 2k-2 = 2(k-1) \in B$

Hence $x-y = 2k-(2k-2) = 2 \geq 1$

Moreover. $2 = 2 \cdot 1$ is an even number.

To conclude. we've shown $\forall x \in B$, $\exists y \in B$. s.t. $x-y \geq 1$ and $x-y$ is an even integer ∎

(Part 2) Below are six claims. Which ones are true and which ones are false? If a claim is true, prove it. If a claim is false, provide a counterexample and a justification of how the counterexample shows the claim is false.

(a) If A is blow-up, then A is step-down.

This statement is ◯ True ✓ False

Take $A=\{1\}$.

Take $x=1$, $x\in A$,

Let $y\in A$, apparently, $x=y=1$.

$x-y=0$, not satisfying $x-y>0$, which reveals $P(x)$ part of A is false.

Therefore, A is a blow-up set.

Let $x\in A$,

Take $y=1$, $y\in A$, Clearly, $x=y=1$.

Thus, $x-y=1-1=0$. not satisfying $x-y\geq 1$.

Therefore A is not a step-down set.

To conclude this statement is false.

(b) If A is step-down, then A is not blow-up.

This statement is ✓ True ◯ False

pf. Since A is step-down set. giving that:

$\forall x\in A$, $\exists y\in A$ s.t. $x-y\geq 1$ and $x-y$ is an even integer.

$x-y\geq 1>0$, which satisfies:

$\forall x\in A$, $\exists y\in A$ s.t. $x-y>0$ and $x-y$ is an even integer.

Thus, $\forall x\in A$, $\exists y\in A$ s.t. $x-y>0$ and $x-y$ is not an odd integer.

This statement is exactly the negation of blow-up set,

which is not blow-up.

To conclude, if A is step-down, then A is not blow-up. ■

(c) If A is not blow-up, then A is step-down.

This statement is ◯ True ✓ False

Take $A = \{\sqrt{3}k \mid k \in \mathbb{Z}\}$

Take the negation of blow-up, gives:

$\forall x \in A, \exists y \in A$ s.t. $x-y > 0$ and $x-y$ isn't an odd integer.

Let $x \in A$, $x = k\sqrt{3}$. take $y = (k-1)\sqrt{3}$, gives.

$x-y = k\sqrt{3} - (k-1)\sqrt{3} = \sqrt{3} > 0$, satisfying $x-y$ isn't an odd integer.

Therefore. A is not a blow-up.

Moremover. Let $x \in A$, $x = s\sqrt{3}$, $s \in \mathbb{Z}$;

Take $y = t\sqrt{3}$, $t \in \mathbb{Z}$

① when $t > s$. $\Rightarrow s-t < 0$.

$x-y = s\sqrt{3} - t\sqrt{3} = (s-t)\sqrt{3} < 0$, not satisfying $x-y \geq 1$.

① when $t = s$. $\Rightarrow s-t = 0$.

$x-y = s\sqrt{3} - t\sqrt{3} = (s-t)\sqrt{3} = 0$, not satisfying $x \geq 1$.

③ when $t < s$ $\Rightarrow s-t > 0$.

$x-y = s\sqrt{3} - t\sqrt{3} = (s-t)\sqrt{3} > 0$. may satisfy $x \geq 1$.

however, $s \in \mathbb{Z}$, $t \in \mathbb{Z}$. $s-t \in \mathbb{Z}$.

Therefore. $x-y$ is a multiple of $\sqrt{3}$. not satisfying $x-y$ is an even integer.

To conclude. 'if A is not blow-up. then A is step-down' is a false statement.

(d) If A is blow-up and $A \subset \mathbb{Z}$, then A is not step-down.

This statement is ✓ True ◯ False

pf. Since A is blow-up and $A \subset \mathbb{Z}$. gives.

$\exists x \in A$, s.t. $\forall y \in A$, $x-y>0 \Rightarrow x-y$ is an odd integer.

Take $x \in A$. Let $y \in A$.

① '$x-y>0$' and '$x-y$ is an odd integer.' are both true.

Since $A \subset \mathbb{Z}$, $\min\{x-y\} = 1$. $x-y \geq 1$.

Satisfying $\exists x \in A$ s.t. $\forall y \in A$, $x-y \geq 1 \Rightarrow x-y$ is not an even integer. is True.

②. '$x-y>0$' is false.

Therefore. '$x-y \geq 1$' is false.

Satisfying $\exists x \in A$ s.t. $\forall y \in A$, $x-y \geq 1 \Rightarrow x-y$ is not an even integer is True.

Since not step-down is:

$\exists x \in A$ s.t. $\forall y \in A$, $x-y \geq 1 \Rightarrow x-y$ is not an even integer

We've prove if A is blow-up. then A is not step-down. ∎

(e) If we have two sets $A, B \subset \mathbb{R}$ and $A \neq B$, A and B are both step-down, then $A \cup B$ is also step-down.

This statement is ✓ True ◯ False

pf. Given that $A, B \subset \mathbb{R}$. $A \neq B$. A and B are both step-down.

$\forall x \in A$, $\exists y \in A$. s.t. $x-y \geq 1$ and $x-y$ is an even integer.

$\forall x \in B$. $\exists y \in B$. s.t. $x-y \geq 1$ and $x-y$ is an even integer.

Let $A \cup B = C$. Let $x \in C$.

When $x \in (C \cap A)$. take $y_1 \in A$.

there exist $x-y_1 \geq 1$ and $x-y_1$ is an even integer.

When $x \in (C \cap B)$. take $y_2 \in B$.

there exist $x-y_2 \geq 1$ and $x-y_2$ is an even integer.

Since $C = (C \cap A) \cup (C \cap B). = A \cup B$.

To conclude. $\forall x \in C = A \cup B$, $\exists y \in C$. s.t. $x-y \geq 1$ and $x-y$ is an even integer. ∎

(f) If we have two sets $A, B \subset \mathbb{R}$ and $A \neq B$, A and B are both blow-up, then $A \cup B$ is also blow-up.

This statement is ◯ True ✓ False

Take $A = \{-1\} \cup \{2k \mid k \leq -1, k \in \mathbb{Z}\}$

$B = \{0\} \cup \{2k-1 \mid k \leq -1, k \in \mathbb{Z}\}$.

Take $x = -1$. $x \in A$.

Let $y \in A$.

① $x = -1$, $y = -1$

$x - y = 0$ not satisfying $x - y > 0$.

Thus. A is blow-up.

② $x = -1$, $y \in \{2k \mid k \leq -1, k \in \mathbb{Z}\}$.

$x - y = -1 - 2k \geq 1 > 0$.

Moreover, $-2k+1$ is an odd integer.

Thus A is blow-up.

Take $x = 0$. $x \in B$.

Let $y \in B$.

① $x = 0$, $y = 0$

$x - y = 0$. not satisfying $x - y > 0$.

Thus. B is blow-up.

② $x = 0$, $y \in \{2k-1 \mid k \leq -1, k \in \mathbb{Z}\}$.

$x - y = 0 - (2k-1) = 1 - 2k \geq 3 > 0$.

Moreover, $-2k+1$ is an odd integer.

Thus. B is blow-up.

Therefore. both A and B are blow-up, $A, B \subseteq \mathbb{R}$ and $A \neq B$.

$A \cup B = \{k \mid k \leq 0, k \in \mathbb{Z}\}$

Let $x \in A$.

Take $y = x - 2$, $y \in A$.

Gives. $x - y = x - (x-2) = 2 > 0$.

2 is not an odd integer.

Satisfying:

$\forall x \in A \cup B, \exists y \in A \cup B$ s.t. $x - y > 0$ and $x - y$ is not an odd integer.

Which is exactly the same as the negation of $A \cup B$ is blow-up.

Thus. $A \cup B$ is not blow-up.

To conclude. 'If we have two sets $A, B \subseteq \mathbb{R}$ and $A \neq B$, A and B are both blow-up, the $A \cup B$ is also blow-up' is a false statement.

2. Define $f(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)}$, where $n \in \mathbb{Z}^+$.
Here, $\mathbb{Z}^+$ is the set of all positive integers. Find a rational polynomial that is equal to $f(n)$.

$$f(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{i}{i+1}$$

Justify this equality by induction. Hint: compute $f(1), f(2), f(3), f(4)$ by hand first and then make a conjecture.

pf. ① Basic Case.

$$f(1) = \frac{1}{1\cdot(1+1)} = \frac{1}{2}$$

$$f(2) = \frac{1}{1\cdot(1+1)} + \frac{1}{2\cdot(2+1)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}.$$

$$f(3) = \frac{1}{1\cdot(1+1)} + \frac{1}{2\cdot(2+1)} + \frac{1}{3\cdot(3+1)} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$$

$$f(4) = \frac{1}{1\cdot(1+1)} + \frac{1}{2\cdot(2+1)} + \frac{1}{3\cdot(3+1)} + \frac{1}{4\cdot(4+1)} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}$$

② Induction procedure

Assuming $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{i}{i+1}$,

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(i+1)(i+2)}$$

$$= \frac{i}{i+1} + \frac{1}{(i+1)(i+2)}$$

$$= \frac{i(i+2)+1}{(i+1)(i+2)} = \frac{(i+1)^2}{(i+1)(i+2)}.$$

Since $i \geq 1$, $i+1 \neq 0$.

Therefore $\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{i+1}{i+2}$.

We have proved from. $i=n \rightarrow i=n+1$ ■