Nonlinear Autonomous System.

1. non-linear Autonomous System.

$$\begin{cases} x' = F(x,y) \\ y' = G(x,y). \end{cases} \Rightarrow \vec{x}' = \underbrace{A\vec{x}}_{\text{lin.}} + \underbrace{\vec{g}(\vec{x})}_{\text{non-lin.}}.$$

2. Exercise.

求C.P. $\begin{cases} \vec{x}' = 0 \\ \vec{y}' = 0 \end{cases}$ 考虑所有情况

1). 判断 non-lin. auto. sys. is locally lin.

① Write in $A\vec{x} + \vec{g}(\vec{x})$ form.

② Near C.P. $(0,0)$: i.e. $Dg = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix}$.

i). $\vec{g}$ has cts. partials. $\to$ Jaccob of g every entry are. cts.

ii). $\lim_{\vec{x} \to \vec{0}} \frac{\|\vec{g}\|}{\|\vec{x}\|} = 0$.

③ Near other C.P.: F & G has. cts. partial here. ($J(x_0, y_0)$ entry).

2). Locally Linear system 在 C.P. 的 Stability 判定

① 求 Jaccob. at C.P. i.e. $J(x_0, y_0) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$.

$F \cdot G$ 代入 (x_0, y_0) 后，求 Jaccob.

② 求 Jaccob. 的 eigenvalues ($\det(J - \lambda I)$).

eigenvectors.

③ 看表.

e.g. $\begin{cases} x' = x(x-y+1). \\ y' = y(x-2). \end{cases}$ ① classify C.P. ② Sketch Trajectory.

除非问，不然跳过验证 local lin.

① Find

$x' = x^2 - xy + x$; $y' = xy - 2y$. $\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ -2y \end{pmatrix} + \begin{pmatrix} x^2 - xy \\ xy \end{pmatrix}$.

$x' = 0 \Rightarrow x(x-y+1) = 0$.　　　　$= \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x^2 - xy \\ xy \end{pmatrix}$.

$\Rightarrow$ ① $x = 0$　　② $x - y = -1$.

$y' = 0 \Rightarrow y(x-2) = 0$.

$\Rightarrow$ ① $y = 0$　　② $x = 2$.

when $y=0$, gives. ① $x = 0$.　　　　C.P.1 $= (0,0)$.

② $x = y - 1 = -1$　　　　C.P.2 $= (-1,0)$.

when $x = 2$, gives. ① $x = 0$ ✗.

② $y = x + 1 = 3$.　　　　C.P.3 $= (2,3)$.

② Classify.

For $(0,0)$, $J_1[F,G](0,0) = \begin{pmatrix} 2x-y+1 & -x \\ y & x-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$.

$\det(J_1 - \lambda I) = \det\begin{vmatrix} 1-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) \overset{\text{set}}{=} 0$.

$\Rightarrow \lambda_1 = 1$. $\lambda_2 = -2$.

$\therefore$ saddle point, unstable.

For $(-1,0)$. $J_2[F,G](-1,0) = \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}$

$\det(J_2 - \lambda I) = \det\begin{vmatrix} -1-\lambda & 1 \\ 0 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda) \overset{\text{set}}{=} 0$

$\Rightarrow \lambda_1 = -1$. $\lambda_2 = -3$.

$\therefore$ node sink.　asymptotic stable.

For $(2,3)$. $J_3[F,G](2,3) = \begin{pmatrix} 2 & -2 \\ 3 & 0 \end{pmatrix}$.

$\det(J_3 - \lambda I) = \det\begin{vmatrix} 2-\lambda & -2 \\ 3 & 0-\lambda \end{vmatrix} = (2-\lambda)(-\lambda) + 6 \overset{\text{set}}{=} 0$

$\therefore \lambda = \frac{2 \pm \sqrt{-20}}{2} = \frac{2 \pm 2\sqrt{5}i}{2}$　　$\Rightarrow -2\lambda + \lambda^2 + 6 = 0$.

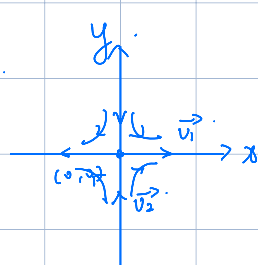
$= 1 \pm \sqrt{5}i$.　　$\Rightarrow \lambda^2 - 2\lambda + 6 = 0$.

$\therefore a + d = 2 > 0$. $b = -2 < 0$.

$\therefore$ spiral node source.　unstable.

③ Sketch

For $(0,0)$.



$(J_1 - \lambda_1 I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

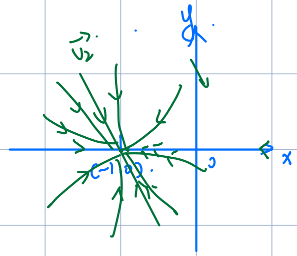
$\Rightarrow v_2 = 0$; v_1 free.

$\therefore \vec{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$(J_1 - \lambda_2 I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

注意. ab中任一为0，用. $(A - \lambda I)\cdot\vec{v} = 0$. 来算 $\vec{v}$.

For $(-1,0)$.



∴ $\vec{v_2} = \begin{pmatrix}0\\1\end{pmatrix}$. ⇒ $v_1 = 0$. v_2 free.

$(J_2 - \lambda_1 I)\vec{v} = \vec{0}$ ⇒ $\begin{pmatrix}0 & 1\\0 & -2\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

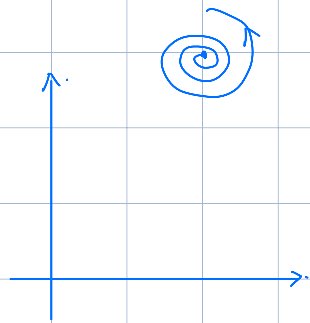
∴ $\vec{v_1} = \begin{pmatrix}1\\0\end{pmatrix}$. ⇒ $v_2 = 0$; v_1 free.

$(J_2 - \lambda_2 I)\cdot\vec{v} = \vec{0}$ ⇒ $\begin{pmatrix}2 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

∴ $\vec{v_2} = \begin{pmatrix}1\\-2\end{pmatrix}$. ⇒ $v_2 = -2v_1$.

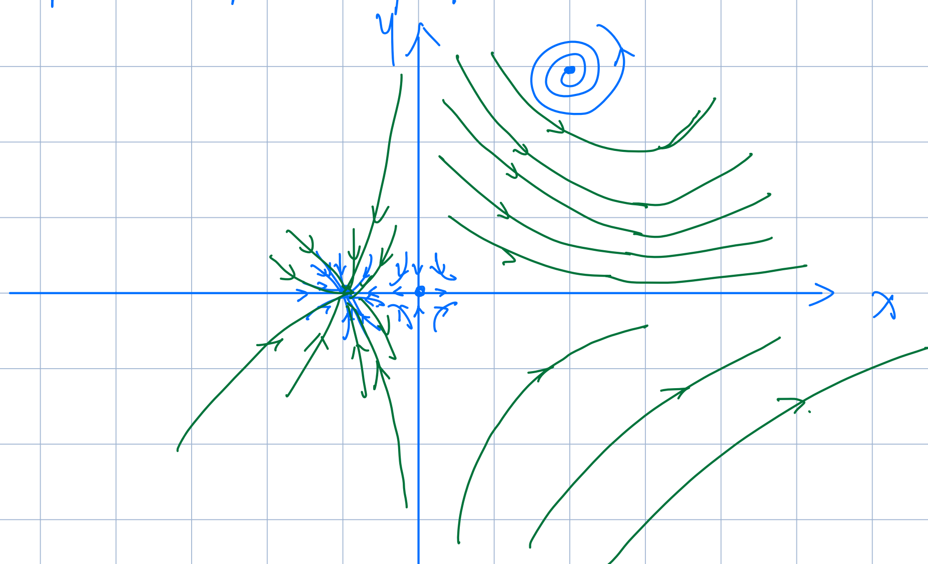
$\begin{bmatrix}a\\c\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$. $\begin{bmatrix}b\\d\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$.

For $(2,3)$.



$a+d>0$. $b<0$.

For all. C.P. (先挪到同一坐标下再连起来).



e.g. $\begin{cases} x' = -2xy \\ y' = x^2+y^2-1 \end{cases}$.

① Find C.P. $\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-2xy\\x^2+y^2-1\end{pmatrix}$.

$x' = 0 = -2xy$. ⇒ ① $x=0$ ② $y=0$.

$y'=0 = x^2+y^2-1$ ⇒ $x^2+y^2=1$.

when $x=0$. $y^2=1$ ⇒ $y=\pm1$. C.P.1 $= (0,1)$.

C.P.2 $= (0,-1)$.

when $y=0$ $x^2=1$ ⇒ $x=\pm1$. C.P.3 $= (1,0)$.

C.P.4 $= (-1,0)$.

② . $J[F,G](x,y) = \begin{pmatrix} -2y & -2x \\ 2x & 2y \end{pmatrix}$.

For $(0,1)$. $J(0,1) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \det(J_1 - \lambda I) = (-2-\lambda)(2-\lambda) = 0$

$\Rightarrow \lambda_1 = -2$. $\lambda_2 = 2$. saddle point unstable.

For $(0,-1)$. $J(0,-1) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \det(J_2 - \lambda I) = (2-\lambda)(-2-\lambda) = 0$.

$\Rightarrow \lambda_1 = 2$. $\lambda_2 = -2$. saddle point. unstable.

For $(1,0)$. $J(1,0) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \Rightarrow \det(J_3 - \lambda I) = \lambda^2 + 4 = 0$.

$a+d = 0$. center point stable. $\Rightarrow \lambda = \pm 2i$.

For $(-1,0)$. $J(-1,0) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \Rightarrow \det(J_4 - \lambda I) = \lambda^2 + 4 = 0$.

$a+d = 0$. center point stable. $\Rightarrow \lambda = \pm 2i$.

③ . For $(0, 1)$ $\lambda_1 = -2$. $J_1 - \lambda_1 I = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

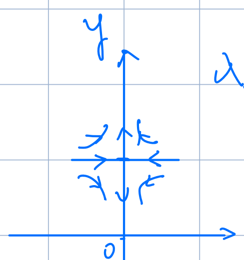
$\Rightarrow v_2 = 0$. v_1 free.

$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\lambda_2 = 2$. $J_1 - \lambda_2 I = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow v_1 = 0$ v_2 free.

$\vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



For $(0,-1)$. $\lambda_1 = 2$. $J_2 - \lambda_1 I = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

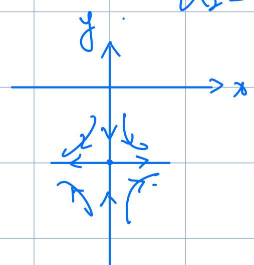
$\Rightarrow v_2 = 0$. v_1 free.

$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

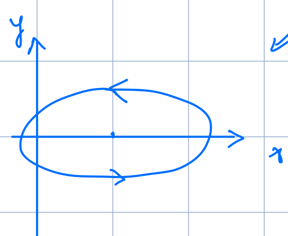
$\lambda_2 = -2$. $J_2 - \lambda_2 I = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

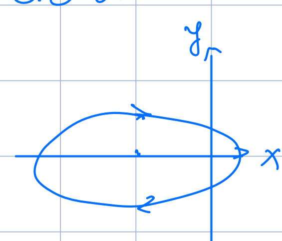
$\Rightarrow v_1 = 0$. v_2 free.

$\vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



For $(1,0)$. $a+d = 0$. $b < 0$. C.C.W.





For $(-1,0)$. $a+d = 0$. $b > 0$. C.W.

Overall.