System of Linear Equations

1. Linear System: a collection of one or more linear equations involving the same variables

m个方程

$$\begin{cases} \boxed{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1} \Rightarrow \text{linear equation} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

n variables.

a_{ij}'s are called coefficients

e.g. $\frac{5x^2}{2x} = 1$ (x)

$$\Leftrightarrow \begin{cases} \frac{5x}{2} = 1 \\ x \neq 0 \end{cases}$$ 不等式组.

$5xy + 3y^2 + 3z = 2y - 6t$ (x) not linear.

1) Remark.

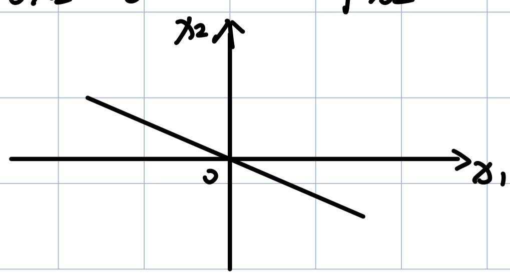
0, 1, 无数
解的数量.

A linear system is either:

① consistent $\begin{cases} \text{exactly one solution} \\ \text{infinite many solutions} \end{cases}$

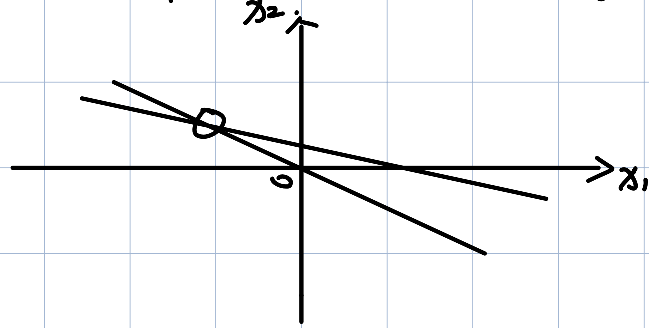
② inconsistent: no solutions.

e.g. ① $\begin{cases} 2x_1 + 3x_2 = 0 \\ 4x_1 + 6x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = -\frac{2}{3}x_1 \\ x_2 = -\frac{4}{6}x_1 = -\frac{2}{3}x_1 \end{cases}$



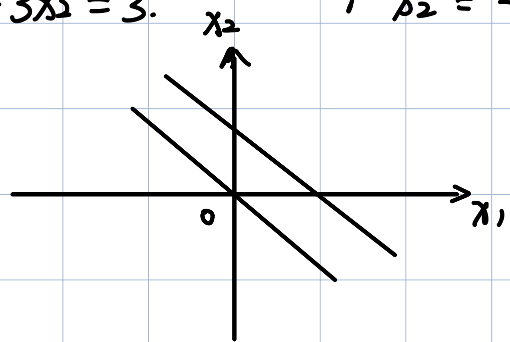
infinite solutions

② $\begin{cases} 2x_1 + 3x_2 = 0 \\ x_1 + 3x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_2 = -\frac{2}{3}x_1 \\ x_2 = -\frac{1}{3}x_1 + \frac{1}{3} \end{cases}$



1 solution

③ $\begin{cases} 2x_1 + 3x_2 = 0 \\ 2x_1 + 3x_2 = 3 \end{cases} \Rightarrow \begin{cases} x_2 = -\frac{2}{3}x_1 \\ x_2 = -\frac{2}{3}x_1 + 1 \end{cases}$



no solution.

2. Solving linear system by Matrix using row reduction.

1) Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

row → equation

column → variables.

coefficient matrix → 把方程组中系数拿出

加上一列b即是argumented matrix

① size: $m \times n$

row number, column number.

e.g. $\begin{cases} 3x_1 + x_3 = 9. \\ 2x_2 + 3x_3 = 4 \\ x_1 + 4x_2 = 7 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & 1 & 9 \\ 0 & 2 & 3 & 4 \\ 1 & 4 & 0 & 7 \end{array}\right]$

2) Row Operations

① Interchange : between 2 rows.

$\begin{cases} 2x+y=1 \\ 3x-y=2 \end{cases} \longrightarrow \begin{cases} 3x-y=2 \\ 2x+y=1 \end{cases}$

$\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

② Scaling : Multiply all entries in a row by a nonzero constant.

$\begin{cases} 3x-y=2 \\ 2x+y=1 \end{cases} \longrightarrow \begin{cases} 6x-2y=4 \\ 2x+y=1 \end{cases}$

$\left[\begin{array}{cc|c} 3 & -1 & 2 \\ 2 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 6 & -2 & 4 \\ 2 & 1 & 1 \end{array}\right]$

③ Replacement : replace one row by the sum of itself and a multiple of another row. ($r_1 \to r_1 + 2r_2$) 其它行可以是倍数.

$\begin{cases} 6x-2y=4 \\ 2x+y=1 \end{cases} \longrightarrow \begin{cases} 6x-2y=4 \\ 8x-y=5. \end{cases}$

$\left[\begin{array}{cc|c} 6 & -2 & 4 \\ 2 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 6 & -2 & 4 \\ 8 & -1 & 5 \end{array}\right]$

Row equivalent : a sequence of elementary row operations that transforms one matrix into the other:

④ Row operations are reversible.

⑤ Corollary: If the argumented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

⑥ Two linear systems are equivalent if they have the same solution set.

3. Some Definitions:

1) Nonzero Row: contains a least one nonzero entry.

Zero Row: contains only zero.

2) Leading Entry: the leftmost nonzero entry.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ leading entry.

3) Row Echelon Form (REF) (more than one matrix)

① All nonzero rows are above any rows of all zeros.

② Each LE of a row is in a column to the right of the leading entry of the row above it.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ✓

③ All entries in a column below a LE are zeros.

4) Row Reduced Echelon Form (RREF) (besides the above three) (unique for any nonzero matrix)

④ The LE in each nonzero row is 1.

⑤ Each leading 1 is the only nonzero entry in its column.

5) Pivot Position: a location in matrix corresponds to a LE 1 in it's RREF form.

e.g. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[\text{reduction}]{\text{row}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

PP1 PP2

pivot column.

有LE.

6) Basic Variable & Free Variable: The variables corresponding to the pivot columns are called basic variables and other are called free variables.

4. Solve a linear system.

1) Write the argumented matrix for the linear system.

e.g. $\begin{cases} x_1 + 2x_2 + x_3 + 2x_4 + x_5 = 2 \\ 2x_1 + x_2 + 3x_3 + 5x_4 + 5x_5 = 7 \end{cases}$

$$\begin{cases} 3x_1+6x_2+4x_3+9x_4+10x_5=11 \\ x_1+2x_2+4x_3+3x_4+6x_5=9. \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 5 & 5 & 7 \\ 3 & 6 & 4 & 9 & 10 & 11 \\ 1 & 2 & 4 & 3 & 6 & 9 \end{array}\right]$$

2) 按列，从左至右，从上至下 先变成REF，用 replacement，已处理好的便不用管（从左上至右下）

$$\xrightarrow[r_4-r_1]{\substack{r_2-2r_1 \\ r_3-3r_1}} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & -3 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 7 & 5 \\ 0 & 0 & 3 & 1 & 5 & 7 \end{array}\right]$$

$$\xrightarrow{r_4-3r_3} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & -3 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 7 & 5 \\ 0 & 0 & 0 & -8 & -16 & -8 \end{array}\right]$$

3) REF ⟶ RREF.

① Using Scaling to make LE being '1.'

$$\xrightarrow[r_4\times(-\frac{1}{8})]{r_2\times(-\frac{1}{3})} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & -1 & -1 \\ 0 & 0 & 1 & 3 & 7 & 5 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

② Using Replacement to make the column be '0'. (从下至上)

$$\xrightarrow{r_3-3r_4} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & -1 & -1 \end{array}\right.$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & 1 & 2 & | & 1 \end{bmatrix}$$

$$\xrightarrow[r_2+\frac{1}{3}r_4]{r_2+\frac{1}{3}r_3} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & | & 2 \\ 0 & 1 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & 1 & 2 & | & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{r_1-r_3 \\ r_1-2r_4}]{r_1-2r_2} \begin{bmatrix} 1 & 0 & 0 & 0 & -4 & | & -2 \\ 0 & 1 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & 1 & 2 & | & 1 \end{bmatrix}$$

4) 用 free variable 表示出 basic variable. (BV = FV)

$$\begin{cases} x_1 - 4x_5 = -2 \\ x_2 = 0. \\ x_3 + x_5 = 2. \\ x_4 + 2x_5 = 1. \end{cases} \Rightarrow \begin{cases} x_1 = 4x_5 - 2. \\ x_2 = 0 \\ x_3 = -x_5 + 2. \\ x_4 = -2x_5 + 1 \end{cases}$$

5) Write it in vector form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4x_5 - 2 \\ 0 \\ -x_5 + 2 \\ -2x_5 + 1 \\ x_5 \end{bmatrix} = x_5 \begin{bmatrix} 4 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$