d) Determine a tower of fields in which the polynomial $x^2 - \sqrt{3}x - 1$ has a root.

① Calculate the root of $x^2 - \sqrt{3}x - 1$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{3} \pm \sqrt{3+4}}{2} = \frac{\sqrt{3} \pm \sqrt{7}}{2}$$

② Define. $F_0 = \mathbb{Q}$. $F_1 = \mathbb{Q}(\sqrt{3})$. $F_2 = F_1(\sqrt{7})$.

WTS: $\frac{\sqrt{3} \pm \sqrt{7}}{2}$ is in the tower of field.

$x_1 = \frac{\sqrt{3}+\sqrt{7}}{2} = \frac{\sqrt{3}}{2} + \frac{\sqrt{7}}{2}$; $x_2 = \frac{\sqrt{3}-\sqrt{7}}{2} = \frac{\sqrt{3}}{2} - \frac{\sqrt{7}}{2}$.

Since $F_2 = F_1(\sqrt{7}) = \{a_1 + b_1\sqrt{7} : a_1, b_1 \in F_1\}$.

$F_1 = \mathbb{Q}(\sqrt{3}) = \{a_2 + b_2\sqrt{3} : a_2, b_2 \in \mathbb{Q}\}$.

For x_1, take $a_1 \in F_1$, where $a_2 = 0$, $b_2 = \frac{1}{2}$. where $a_2, b_2 \in \mathbb{Q}$ gives $a_1 = \frac{\sqrt{3}}{2}$.

take $b_1 \in F_1$, where $a_2 = \frac{1}{2}$, $b_2 = 0$. where $a_2, b_2 \in \mathbb{Q}$. gives. $b_1 = \frac{1}{2}$. which.

$x_1 = \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \sqrt{7} = \frac{\sqrt{3}+\sqrt{7}}{2} \in F_2$.

For x_2. take $a_1 \in F_1$. where $a_2 = 0$, $b_2 = \frac{1}{2}$. where $a_2, b_2 \in \mathbb{Q}$ gives $a_1 = \frac{\sqrt{3}}{2}$.

take $b_1 \in F_1$, where $a_2 = -\frac{1}{2}$, $b_2 = 0$. where $a_2, b_2 \in \mathbb{Q}$. gives. $b_1 = -\frac{1}{2}$. which.

$x_2 = \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \sqrt{7} = \frac{\sqrt{3}-\sqrt{7}}{2} \in F_2$.

e) Prove using RRT that the polynomial $x^3 + \sqrt{3}x + 1$ has no rational roots. (Hint: RRT does not apply if the coefficients of the polynomial are not rational. You need to transform the polynomial somehow.)

I'll transform it at first, by multiplying $(x^3+1-\sqrt{3}x)$. gives

$$p(x) = (x^3+1+\sqrt{3}x)(x^3+1-\sqrt{3}x) = (x^3+1)^2 - (\sqrt{3}x)^2$$
$$= (x^3+1)^2 - 3x^2 = x^6 + 2x^3 - 3x^2 + 1$$

Assume $r_0 = \frac{m}{n}$ is the rational root with lowest term of the polynomial $p(x)$

From RRT, gives. $m \mid 1$ and $n \mid 1$. given. the possible value for m: ± 1.

the possible value for n: ± 1.

Hence, the possible value for $\frac{m}{n}$: ± 1.

Subsitute into $p(x)$, gives $p(\frac{m}{n})_1 = 1 + 2 - 3 + 1 = 1 \neq 0$, which $\frac{m}{n} = 1$ is not the solution.

$$p(\tfrac{m}{n})_2 = 1 + 2(-1)^3 - 3 \cdot (-1)^2 + 1$$
$$= 1 - 2 - 3 + 1 = -3 \neq 0$$. which $\frac{m}{n} = -1$ is not the solution.

Thus. $p(x) = x^6 + 2x^3 - 3x^2 + 1$ has not rational roots, which $(x^3+1+\sqrt{3}x)(x^3+1-\sqrt{3}x)$ has no rational solution which $x^3 + \sqrt{3}x + 1$ has no rational solution.

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