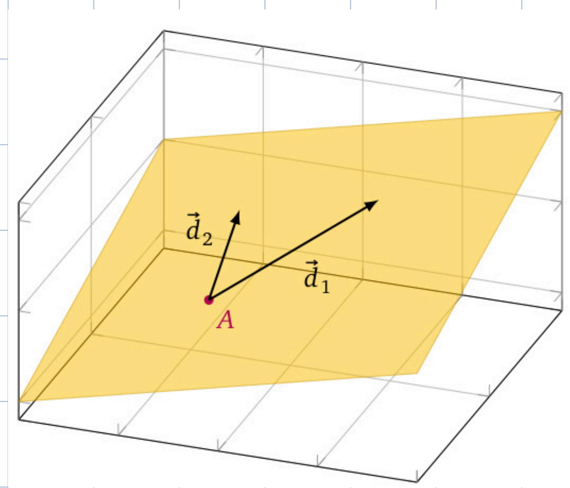
Planes

1. Vector form of a plane

A plane P is written in vector form if it is expressed as:

$$\vec{x} = t\vec{d_1} + s\vec{d_2} + \vec{p}$$

for some vectors $\vec{d_1}$ and $\vec{d_2}$ and point $\vec{p}$. That is, $P = \{\vec{x} \mid \vec{x} = t\vec{d_1} + s\vec{d_2} + \vec{p} \text{ for some } t, s \in \mathbb{R}\}$. $\vec{d_1}, \vec{d_2}$ are direction vectors.



2. Transform the line into vector form.

eg. Describe the plane $P \subseteq \mathbb{R}^3$ with equation $z = 2x + y + 3$ in vector form.

M1 1) Find 3 different & not on same line points.

$$A = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$$

2) Getting the direction vectors.

$$\vec{AC} = C - A = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix}$$

$$\vec{AB} = B - A = \begin{bmatrix}0\\1\\4\end{bmatrix} - \begin{bmatrix}0\\0\\3\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\vec{x} = t\vec{d_1} + s\vec{d_2} + A \longleftarrow$$ can actually be any point on the plane

$$= t\begin{bmatrix}1\\0\\2\end{bmatrix} + s\begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\3\end{bmatrix}$$

M2: 1) Use the other two variable to express one.

$$z = 2x + y + 3.$$

2) put into the form.

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x\\y\\2x+y+3\end{bmatrix} = x\begin{bmatrix}1\\0\\2\end{bmatrix} + y\begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\3\end{bmatrix}$$

3) Change the variable.

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = t\begin{bmatrix}1\\0\\2\end{bmatrix} + s\begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\3\end{bmatrix}. \quad s.t \in \mathbb{R}.$$

3. Intersection between two planes.

e.g. In vector form: $P_1 : \vec{x} = t\begin{bmatrix}1\\1\\0\end{bmatrix} + s\begin{bmatrix}-1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\2\\3\end{bmatrix}$

$P_2 : \vec{x} = t\begin{bmatrix}-1\\0\\2\end{bmatrix} + s\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}0\\0\\3\end{bmatrix}$

1) Looking for solution when $P_1 = P_2$.

$$a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\2\\3\end{bmatrix} = c\begin{bmatrix}-1\\0\\2\end{bmatrix} + d\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}0\\0\\3\end{bmatrix}$$

$$\Rightarrow a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\1\end{bmatrix} + c\begin{bmatrix}1\\0\\-2\end{bmatrix} + d\begin{bmatrix}-1\\-2\\-1\end{bmatrix} = \begin{bmatrix}-1\\-2\\0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -1 \\ 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right] \xrightarrow[r_3-r_2-r_1]{r_2-r_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_3\cdot(-1)} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{r_2+r_3} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{r_1+r_2-r_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right]$$

$$\therefore \begin{cases} a-2d=-2 \\ b-d=-2 \\ c=-1 \end{cases} \Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2d-2 \\ d-2 \\ -1 \\ d \end{bmatrix} = d\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

2) substitute back to equation.

substitute $a = 2t-2$; $b = t-2$. into P_1 gives.

$$\vec{x} = (2t-2)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (t-2)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2t \\ 2t \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

4. Hyperplane : ① $x = t_1 d_1 + t_2 d_2 + t_3 d_3 + \dots + t_n d_n + \vec{p}$

where $d_1, d_2, \dots, d_n$ are direction vectors. p is a point on the plane. The plane is in $\mathbb{R}^{n+1}$.

② There exists $\vec{n} \neq 0$ and $\vec{p}$ so that x is the set of solutions to the equation $\vec{n}(\vec{x}-\vec{p})=0$.

5. Normal Vector: a normal vector to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to all direction vectors for the line (or plane or hyperplane)

6. Normal Form: $(\vec{x}-\vec{p})\cdot\vec{n}=0$.

$$\vec{x}=t_1\vec{d_1}+t_2\vec{d_2}+\cdots+t_n\vec{d_n}+\vec{p}$$

$$\vec{x}-\vec{p}=t_1\vec{d_1}+t_2\vec{d_2}+\cdots+t_n\vec{d_n}$$ ← 平移回原点

$$\therefore (\vec{x}-\vec{p})\cdot\vec{n}=0.$$

e.g. $P: t\begin{bmatrix}1\\2\\3\end{bmatrix}+s\begin{bmatrix}4\\5\\6\end{bmatrix}+\begin{bmatrix}1\\2\\4\end{bmatrix}$, $\vec{n}=?$

$$\left(t\begin{bmatrix}1\\2\\3\end{bmatrix}+s\begin{bmatrix}4\\5\\6\end{bmatrix}\right)\cdot\vec{n}=0.$$

$$\begin{cases}\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=0\\ \begin{bmatrix}4\\5\\6\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=0\end{cases}\Rightarrow\begin{cases}x+2y+3z=0\\4x+5y+6z=0\end{cases}$$

$$\rightarrow\left[\begin{array}{ccc|c}1&2&3&0\\4&5&6&0\end{array}\right]\xrightarrow{r_2-4r_1}\left[\begin{array}{ccc|c}1&2&3&0\\0&-3&-6&0\end{array}\right]$$

$$\xrightarrow{r_2\cdot(-\frac{1}{3})}\left[\begin{array}{ccc|c}1&2&3&0\\0&1&2&0\end{array}\right]$$

$$\xrightarrow{r_1-2r_2}\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\end{array}\right]$$

$$\therefore\begin{cases}x=z\\y=-2z\end{cases}\rightarrow\vec{n}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}t\\-2t\\t\end{bmatrix}$$

$\therefore \begin{bmatrix}1\\-2\\1\end{bmatrix}$ is a normal vector.

normal form:
$\vec{n}\cdot(\vec{x}-\vec{p})=0$
$\Rightarrow\begin{bmatrix}1\\-2\\1\end{bmatrix}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}-\begin{bmatrix}1\\2\\4\end{bmatrix}\right)=0.$

e.g. $\vec{p}=\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\vec{n}=\begin{bmatrix}1\\1\\1\end{bmatrix}$. Plane?

$$\vec{n}\cdot(\vec{x}-\vec{p})=0.$$

$\vec{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}$.

$$\Rightarrow \vec{n}\cdot\vec{x}-\vec{n}\cdot\vec{p}=0\Rightarrow x+y+z-2=0.$$

$$\therefore x=-y-z+2.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y - z + 2 \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$