Coordinates & Change of Basis

∴ 顺序 matters.
e.g. $[\vec{v}]_B = \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$
$\vec{v} = \alpha_2 \vec{v_1} + \alpha_1 \vec{v_2} + \cdots + \alpha_n \vec{v_n}$.

1. Coordinates

Let $B = \{\vec{b_1}, \vec{b_2}, \dots, \vec{b_n}\}$ be a basis for a subspace V and let $\vec{v} \in V$. The representation of $\vec{v}$ in the B basis (coordinate of $\vec{v}$ in B) is notated $[\vec{v}]_B$. is the column matrix:

$$[\vec{v}]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ unique satisfy $\vec{v} = \alpha_1 \vec{b_1} + \alpha_2 \vec{b_2} + \dots + \alpha_n \vec{b_n}$ (一一对应).

(Conversely, $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_B = \alpha_1 \vec{b_1} + \alpha_2 \vec{b_2} + \dots + \alpha_n \vec{b_n}$ is notation for the lin comb of $\vec{b_1}, \vec{b_2}, \dots, \vec{b_n}$ with coefficient $\alpha_1, \alpha_2, \dots, \alpha_n$.)

e.g. $\vec{v} = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}$

① β is the standard basis in $\mathbb{R}^n$. $[\vec{v}]_\beta$

$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. take $[\vec{v}]_\beta = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\therefore \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} z = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}$

$\therefore [\vec{v}]_\beta = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}$.

② $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix} \right\}$

take $[\vec{v}]_\beta = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\therefore \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} y + \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix} z = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 1 & 1 & 5 & 3 \\ 1 & 3 & -1 & -5 \end{array}\right] \xrightarrow{\times\times\times} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore [\vec{v}]_\beta = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

e.g. $[\vec{v}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix} \right\}$. $\vec{v} = ?$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 1 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot 2 + \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix} \cdot 3$.

$= \begin{bmatrix} 5 \\ 18 \\ 4 \end{bmatrix}$

1) $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in standard basis better write $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{E}}$.

2) $\vec{x} \neq [\vec{x}]_{\mathcal{E}}$.

vector. $\rightarrow$ list of numbers.

3) [true vector]$_{\mathcal{X}}$ = list of numbers ; [List of numbers]$_{\mathcal{X}}$ = true $\vec{v}$.

e.g. $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{E}}$; $[\vec{x}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

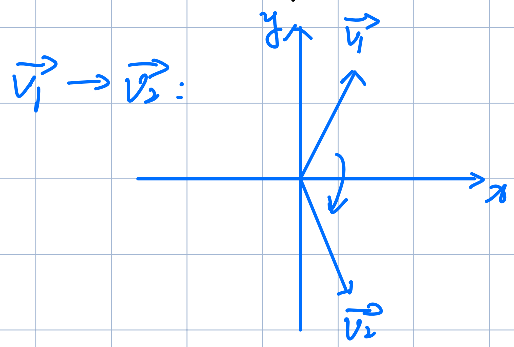
(顺序很重要)
$B = \{\vec{b_1}, \vec{b_2}\}$
$\vec{b_1} \rightarrow \vec{b_2}$:
顺时针 短: neg-oriented.
逆时针 短. po-oriented

2. Orientation of a basis.

The ordered basis $B = \{\vec{b_1}, \vec{b_2}, \dots, \vec{b_n}\}$

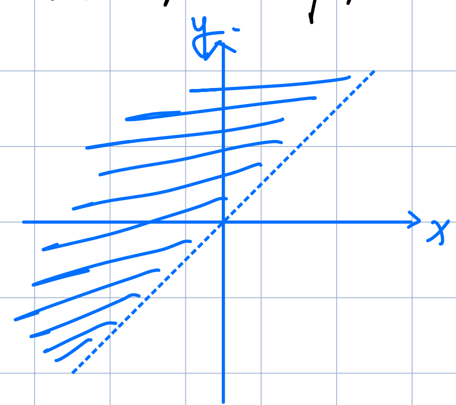
Righ-handed (positively oriented): can be continuously transformed the standard basis ($\vec{b_i} \rightarrow \vec{e_i}$) while remain lin indep throughout transformation ; otherwise, B is left-handed or neg-oriented.

e.g. $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$



顺短: neg-oriented.

e.g. $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\{\vec{v} \in \mathbb{R}^2 \mid \{\vec{u}, \vec{v}\}$ positive oriented in $\mathbb{R}^2\}$.

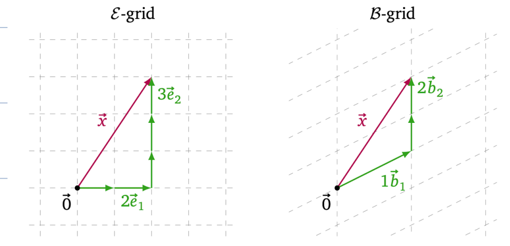


3. Change of Basis.

$\vec{x} \in \mathbb{R}^n$

represent in basis A $\quad$ represent in basis B.

$[\vec{x}]_A$ $\qquad\qquad$ $[\vec{x}]_B$



1). Change of Basis Matrix.

Let A and B be bases for $\mathbb{R}^n$. The matrix M is called a change of basis matrix which converts from A to B. if for all $\vec{x} \in \mathbb{R}^n$.

$$M[\vec{x}]_A = [\vec{x}]_B.$$

notato $M = [B \leftarrow A]$.

$$M = [I]_A^B = [\,[\vec{a}_1]_B, [\vec{a}_2]_B, \dots, [\vec{a}_n]_B\,]$$

e.g. $A = \left\{ \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \right\}$. $B = \left\{ \begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}-1\\3\end{bmatrix} \right\}$ $M[B \leftarrow A] = ?$

$M = \left[\begin{bmatrix}1\\0\end{bmatrix}_B \begin{bmatrix}0\\1\end{bmatrix}_B \right]$

$\left[\begin{bmatrix}1\\0\end{bmatrix}\right]_B$ where $\vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$. take $[\vec{v}_1]_B = \begin{bmatrix}x\\y\end{bmatrix}$

$\begin{bmatrix}1\\2\end{bmatrix} x + \begin{bmatrix}-1\\3\end{bmatrix} y = \begin{bmatrix}1\\0\end{bmatrix}$.

$\rightarrow \quad \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3/5\\-2/5\end{bmatrix}$

$\left[\begin{bmatrix}0\\1\end{bmatrix}\right]_B$ where $\vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$. take $[\vec{v}_2]_B = \begin{bmatrix}x\\y\end{bmatrix}$

$\begin{bmatrix}1\\2\end{bmatrix} x + \begin{bmatrix}-1\\3\end{bmatrix} y = \begin{bmatrix}0\\1\end{bmatrix}$

$\rightarrow \quad \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1/5\\1/5\end{bmatrix}$.

$\therefore M = \begin{bmatrix}3/5 & 1/5\\ -2/5 & 1/5\end{bmatrix}$

e.g. $\alpha = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$; $\beta = \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}-1\\3\\2\end{bmatrix} \begin{bmatrix}0\\2\\5\end{bmatrix} \right\}$; $M[\beta \leftarrow \alpha]$

$M[\beta \leftarrow \alpha] = \left[\begin{bmatrix}1\\0\\0\end{bmatrix}_\beta \begin{bmatrix}0\\1\\0\end{bmatrix}_\beta \begin{bmatrix}0\\0\\1\end{bmatrix}_\beta \right]$

① $\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $[\vec{v}_1]_\beta = \begin{bmatrix}x\\y\\z\end{bmatrix}$.

$\begin{bmatrix}1\\2\\3\end{bmatrix} x + \begin{bmatrix}-1\\3\\2\end{bmatrix} y + \begin{bmatrix}0\\2\\5\end{bmatrix} z = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 3 & 2 & 0 \\ 3 & 2 & 5 & 0 \end{array}\right] \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & -2 \\ 0 & 5 & 5 & -3 \end{array}\right]$

$$\xrightarrow[r_2 \cdot \frac{1}{5}]{r_3 - r_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 2/5 & -2/5 \\ 0 & 0 & 3 & -1 \end{array}\right]$$

$$\xrightarrow{r_3 \times \frac{1}{3}} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 2/5 & -2/5 \\ 0 & 0 & 1 & -1/3 \end{array}\right]$$

$$\xrightarrow{r_2 - \frac{2}{5} r_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -4/15 \\ 0 & 0 & 1 & -1/3 \end{array}\right]$$

$$\xrightarrow{r_1 + r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11/15 \\ 0 & 1 & 0 & -4/15 \\ 0 & 0 & 1 & -1/3 \end{array}\right]$$

$\therefore [\vec{v_1}]_\beta = \begin{bmatrix} 11/15 \\ -4/15 \\ -1/3 \end{bmatrix}$

② $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $[\vec{v_2}]_\beta = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} y + \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} z = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[\vec{v_2}]_\beta = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$.

③ $\vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $[\vec{v_3}]_\beta = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} y + \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$[\vec{v_3}]_\beta = \begin{bmatrix} -2/15 \\ -2/15 \\ 1/3 \end{bmatrix}$

$$\therefore M[\beta \leftarrow \alpha] = \begin{bmatrix} 11/15 & 1/2 & -2/15 \\ -4/15 & 1/2 & -2/15 \\ -1/3 & -1/2 & 1/3 \end{bmatrix}$$

2) Chain.

$$\underset{m \times n}{[C \leftarrow A]} = [C \leftarrow B] \cdot \underset{n \times k}{[B \leftarrow A]}.$$

$$\Downarrow$$

$$[\vec{x}]_C = [C \leftarrow A][\vec{x}]_A = [C \leftarrow B][B \leftarrow A] \cdot [\vec{x}]_A.$$

3) An $n \times n$ matrix is invertible iff it's a change of basis matrix.

$M = [B \leftarrow A]$, $M^{-1} = [A \leftarrow B]$

$MM^{-1} = [B \leftarrow A][A \leftarrow B] = [B \leftarrow B] = I$