Cumulative Distribution Function

1. C.D.F.: Given a random variable X, it cumulative distribution func. is the func. $F_X: \mathbb{R}^1 \to [0,1]$, defined by $F_X(x) = P(X \le x)$, $\forall x \in \mathbb{R}$.

   1). X is discrete: $F_X(x) = \sum_{u \le x} P(X=u) = \sum_{u \le x} p_X(u)$

   ↘ probability func.

   2). X is a.cts.: $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$.

   ↘ density function

2. $P(X \in B)$.

   1). $B_1 = (a, b]$.

   $$P(X \in B_1) = P(a < X \le b) = P(X \le b) - P(X \le a) = F_X(b) - F_X(a).$$

   2). $B_2 = [a, b]$

   $$\begin{aligned} P(X \in B_2) = P(a \le X \le b) &= \lim_{n\to\infty} P(a - \tfrac{1}{n} < X \le b). \\ &= P(X \le b) - \lim_{n\to\infty} P(X \le a - \tfrac{1}{n}). \\ &= F_X(b) - \lim_{n\to\infty} F_X(a - \tfrac{1}{n}). \end{aligned}$$

   ① $P(X=a) = P(a \le X \le a) = F_X(a) - \lim_{n\to\infty} F_X(a - \frac{1}{n})$.

   ② When X is cts, $P(a \le X \le b) = F_X(b) - F_X(a)$.

   对cts.来说无须关注开闭.因为$P(X=x)=0$.

   3). $B_3 = [a, b)$.

   $$\begin{aligned} P(X \in B_3) = P(a \le X < b) &= \lim_{n\to\infty} (a - \tfrac{1}{n} < X \le b - \tfrac{1}{n}). \\ &= \lim_{n\to\infty} F_X(b - \tfrac{1}{n}) - \lim_{n\to\infty} F_X(a - \tfrac{1}{n}) \end{aligned}$$

   4). $B_4 = (a, b)$.

   $$\begin{aligned} P(X \in B_4) = P(a < X < b) &= \lim_{n\to\infty} (a < X \le b - \tfrac{1}{n}). \\ &= \lim_{n\to\infty} F_X(b - \tfrac{1}{n}) - F_X(a). \end{aligned}$$

   5) $B_5 = (a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_k, b_k]$. $a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$.

   $$\begin{aligned} P(X \in B_5) &= P(X \in (a_1, b_1]) + P(X \in (a_2, b_2]) + \cdots + P(X \in (a_k, b_k]). \\ &= F_X(b_1) - F_X(a_1) + F_X(b_2) - F_X(a_2) + \cdots + F_X(b_k) - F_X(a_k). \end{aligned}$$

3. Basic Properties of c.d.f.

1). Let F_X be the c.d.f of a random variable X. Then.

① $0 \le F_X(x) \le 1$. $\forall x$.

② $F_X(x) \le F_Y(y)$ whenever $x \le y$ (F_X is increasing).

③ $\lim_{x \to \infty} F_X(x) = 1$.

④ $\lim_{x \to -\infty} F_X(x) = 0$.

2). If a func. $F: \mathbb{R} \to \mathbb{R}$. satisfies ①-④ and is right cts., then there is a unique probability P on $\mathbb{R}$ s.t. F is c.d.f. of P.

3). If F_X is a c.d.f., then F_X is right cts.

4. Continuity of c.d.f.

1). Right cts. : $F_X(x+\frac{1}{n}) \to F_X(x)$.

$A = (-\infty, x]$, $A_n = (-\infty, x+\frac{1}{n}]$. $(-\infty, x] \subseteq (-\infty, x+\frac{1}{n}]$.

$\{A_n\} \downarrow A$, which $P(A_n) \to P(A)$.

2). Left cts : $F_X(x-\frac{1}{n}) \to F_X(x)$. $(-\infty, x] \nsubseteq (-\infty, x-\frac{1}{n}]$.

$A = (-\infty, x]$, $A_n = (-\infty, x-\frac{1}{n}]$.

$\{A_n\} \cancel{\nearrow} A$, which $P(A_n) \to P((-\infty, x)) = P(X < x)$. not $P(X \le x)$.

$= P(X \le x) - P(X = x)$.

when X is cts., $P(X=x) = 0$.

3). cts : left cts + right cts.

4). discts : $P(X=x) > 0$.

5. c.d.f. of discrete distribution.

$$F_X(x) = \sum_{u \le x} P(X=u) = \sum_{u \le x} p_X(u)$$

↘ probability func.

6. c.d.f of cts. dis. (X is an a.cts. variable).

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du.$$

↘ density function

(FTC).

1). From FTC1, $F_X'(x) = f_X(x)$.

2). $Z_1 \sim$ Uniform $[L, R]$: $F_{Z_1}(z) = \begin{cases} 0 & z < L. \\ \frac{z-L}{R-L} & L \le z < R. \\ 1 & z \ge R. \end{cases}$

3). $Z_2 \sim$ Exponential (λ): $F_{Z_2}(z) = \begin{cases} 0 & z < 0. \\ 1-e^{-\lambda z} & \text{otherwise}. \end{cases}$

e.g. $X \sim Ex(3)$. $P(X \ge 2.6) = ?$

$P(X \ge 2.6) = 1 - P(X < 2.6)$. Since X is cts.

$= 1 - P(X \le 2.6)$

$= 1 - F_X(2.6) = 1-(1-e^{-3\cdot(2.6)}) = e^{-7.8} = 0.00041$.

4). $Z_3 \sim N(\mu, \sigma^2)$.

① S.N.D.: $P(Z_3 \le x) = \Phi(x) = 1 - \Phi(-x)$.

e.g. $Z \sim N(0,1)$. $P(Z \le 1.43) = ?$

$P(Z \le 1.43) = \Phi(1.43) = 1 - \Phi(-1.43) \rightleftharpoons$ 察表.

$= 1 - (0.0764) = 0.9236$.

② $P(Z_3 \le x) = P(\mu + \sigma \overset{\text{S.N.D.}}{Z} \le x) = P(Z \le \frac{x-\mu}{\sigma})$.

$= \Phi(\frac{x-\mu}{\sigma}) = 1 - \Phi(-\frac{x-\mu}{\sigma})$.

e.g. $Z' \sim N(5, 16)$. $P(6 \le Z' \le 8)$.

$P(6 \le Z' \le 8) = P(6 \le 5+4Z \le 8)$

$= P(\frac{1}{4} \le Z \le \frac{3}{4})$. Since Z is cts.

$= F_Z(\frac{3}{4}) - F_Z(\frac{1}{4})$

$= \Phi(\frac{3}{4}) - \Phi(\frac{1}{4})$

$= \Phi(-\frac{1}{4}) - \Phi(-\frac{3}{4}) \doteq 0.1747$.

7. Mixture Distribution

Suppose now that $F_1, F_2, \dots, F_k$ are c.d.f.s. Let $p_1, p_2, \dots, p_k$ be $\mathbb{R}^+$ with $\sum_{i=1}^{k} p_i = 1$. define:

$$G(x) = p_1 F_1(x) + p_2 F_2(x) + \dots + p_k F_k(x).$$

G satisfies ①-④ and is right cts.; $G(x)$ is the mixture dis.

e.g. Y: the result of rolling one fair dice.

$Z \sim \text{Uniform}[2,5]$; $\omega \sim \text{Bernoulli}(\frac{1}{3})$.

Let $X = \begin{cases} Y & \omega = 1 \\ Z & \omega = 0. \end{cases}$; $F_X(4.4) = ?$ Is X discrete or cts.?

①.

$$F_X(4.4) = P(X \le 4.4). = P(X \le 4.4, \omega=1) + P(X \le 4.4, \omega=0).$$
$$= P(Y \le 4.4, \omega=1) + P(Z \le 4.4, \omega=0).$$
$$= P(Y \le 4.4) \cdot P(\omega=1) + P(Z \le 4.4) P(\omega=0).$$
$$= \frac{4}{6} \times \frac{1}{3} + \frac{4.4-2}{5-2} \cdot \frac{2}{3}$$
$$= \frac{4}{18} + \frac{8}{15}$$

$$F_X(x) = \frac{1}{3} F_Y(x) + \frac{2}{3} F_Z(x).$$

②.

$$P(X=2) = P(Y=2, \omega=1) + P(Z=2, \omega=0).$$
$$= \frac{1}{6} \times \frac{1}{3} + 0 \cdot \frac{2}{3} \quad (\text{As } Z \text{ is cts.})$$
$$= \frac{1}{18} \ne 0. \qquad \therefore X \text{ is not cts.}$$

③ $$\sum_{x=1}^{6} P(X=x) = \sum_{x=1}^{6} P(Y=x, \omega=1) + \sum_{x=2}^{6} P(Z=x, \omega=0).$$
$$= \sum_{x=1}^{6} P(Y=x) \cdot P(\omega=1) = \sum_{x=1}^{6} \frac{1}{6} \cdot \frac{1}{3} = 6 \times \frac{1}{6} \times \frac{1}{3} = \frac{1}{3} \ne 1.$$

$\therefore X$ is not discrete.