Learning Objectives

> In this tutorial, you will practice doing proofs related to inner product spaces.

Before attending the tutorial, you should be able recall the definitions of the following terms:

- An inner product on a vector space V.
- An inner product space.
- An orthogonal set of vectors in V, and an orthogonal basis of V.
- An orthonormal set of vectors in V, and an orthonormal basis of V.

These definitions can be reviewed in the textbook, Damiano and Little Section 4.3.

Problems

1. Consider the following examples of vector spaces. Which of the following are inner product spaces under the given operation? Explain why or why not.

   (a) Let $V = \mathbb{R}^{n\times n}$, the space of all $n \times n$ matrices. Define $\langle A, B\rangle = \mathrm{Tr}(A + B)$. [1].

   (b) Let $V = \mathbb{R}^{n\times n}$. Define $\langle A, B\rangle = \mathrm{Tr}(A^T B)$.

   (c) Let V be the space of continuous functions from $[0, 1]$ to $\mathbb{R}$. Define $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$.

   (d) Let V be the space of polynomials in the variable t of degree less than or equal to 3. Define $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$.

2. Let $\vec{v}, \vec{w} \in \mathbb{R}^2$ and define $\langle \vec{v}, \vec{w}\rangle = \vec{v}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \vec{w}$.

   (a) Compute a formula for $\langle \vec{v}, \vec{w}\rangle$ in terms of the components of $\vec{v}$ and $\vec{w}$.

   (b) Show that $\langle \vec{v}, \vec{w}\rangle$ defines an inner product on $\mathbb{R}^2$, that is different from the dot product.

   (c) Can you come up with another matrix A such that $\langle \vec{v}, \vec{w}\rangle = \vec{v}^T A\vec{w}$ is also an inner product on $\mathbb{R}^2$?

3. Let V be an inner product space and let $\mathcal{U} = \{u_1, \ldots, u_n\}$ be an orthonormal basis of V.

   Theorem 1. *For every $v \in V$, $v = \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 + \cdots + \langle v, u_n\rangle u_n$.*

   Prove the theorem by using the following steps.

   (a) Let $v \in V$ and suppose $v = \sum_{i=1}^n c_i u_i$. What is $\langle v, u_1\rangle$? c_1.

   (b) Use your observation in the previous part to prove the theorem.

   (c) What would change about the theorem if $\mathcal{U}$ was only an **orthogonal** basis and not an orthonormal one?

[1] Recall that the trace is the sum of the diagonal entries of the matrix

(Q3.(b)). Let V be an inner product space.

Let $\mathcal{U} = \{u_1, \ldots, u_n\}$ be an orthonormal basis of V.

WTS: $\forall v \in V$, $v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$.

Since $\mathcal{U}$ is an orthonormal basis of V, according to theorem 1.6.3., (Let V be a vector space, and let S be a non-empty subset of V. Then S is a basis of V iff every vector $x \in V$ may be written uniquely as a lin. comb. of the vectors in S), we can assume $v = \sum_{i=1}^{n} c_i u_i$

Thus, if we can prove $\forall 1 \le i \le n$, where $i \in \mathbb{Z}$, $c_i = \langle v, u_i \rangle$ we can prove the theorem.

Let $1 \le i \le n$, where $i \in \mathbb{Z}$., by linearity of the inner product,

$$\langle v, u_i \rangle = \langle \sum_{i'=1}^{n} c_{i'} u_{i'}, u_i \rangle = \sum_{i'=1}^{n} c_{i'} \langle u_{i'}, u_i \rangle$$

$$= c_1 \langle u_1, u_i \rangle + c_2 \langle u_2, u_i \rangle + \cdots + c_i \langle u_i, u_i \rangle + \cdots + c_n \langle u_n, u_i \rangle.$$

Since $\mathcal{U}$ is an orthonormal basis of V, when $i' \neq i$, $\langle u_{i'}, u_i \rangle = 0$. and when $i' = i$, $\langle u_i, u_i \rangle = 1$.

Therefore. $\langle v, u_i \rangle = c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_i \cdot 1 + \cdots + c_n \cdot 0$.

$$= c_i$$

We've proved that $\forall 1 \le i \le n$, where $i \in \mathbb{Z}$, $c_i = \langle v, u_i \rangle$.

Since $v = \sum_{i=1}^{n} c_i u_i$, we've obtain $v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$.

Hence, we've proved the theorem. ∎.

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