MAT 137Y: Calculus with proofs
Assignment 2
Due on Thursday, Oct 27 by 11:59pm via GradeScope

Instructions

This assignment has purposely been made shorter because of your upcoming midterm exam. However, working through this assignment will help you on the upcoming exam because many of the topics overlap! This problem set is based on Unit 2: Limits and Continuity. Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions. Remember you can submit in pairs or individually.

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

Full Name: Xuanqi Wei

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Full Name: ____________________

Student number: ____________________

I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Henry Wei

2) ____________________

1. (*Note:* Before you attempt this problem, solve Problem 1 and 3 from Practice problems for Unit 2 on Quercus or Problem 2.1-7 in the textbook. Otherwise you may find this question difficult.)

Sketch the graph of a function f that satisfies all 10 conditions below simultaneously. For this question, you do not need to prove or explain your answer, as long as the graph is correct and very clear.

(a) The domain of f is $\mathbb{R}$.

(b) $\lim_{x\to a} f(x)$ does not exists when $a \in \{-4, 2, 4\}$, and the limit exists for all other $a \in \mathbb{R}$.

(c) $\lim_{x\to 0} f(x) = 0$

(d) $\lim_{x\to 0} \lfloor f(x) \rfloor = 0$

(e) $\lim_{x\to -4} f(x) = \infty$

(f) $\lim_{x\to 4} f(x) = -\infty$

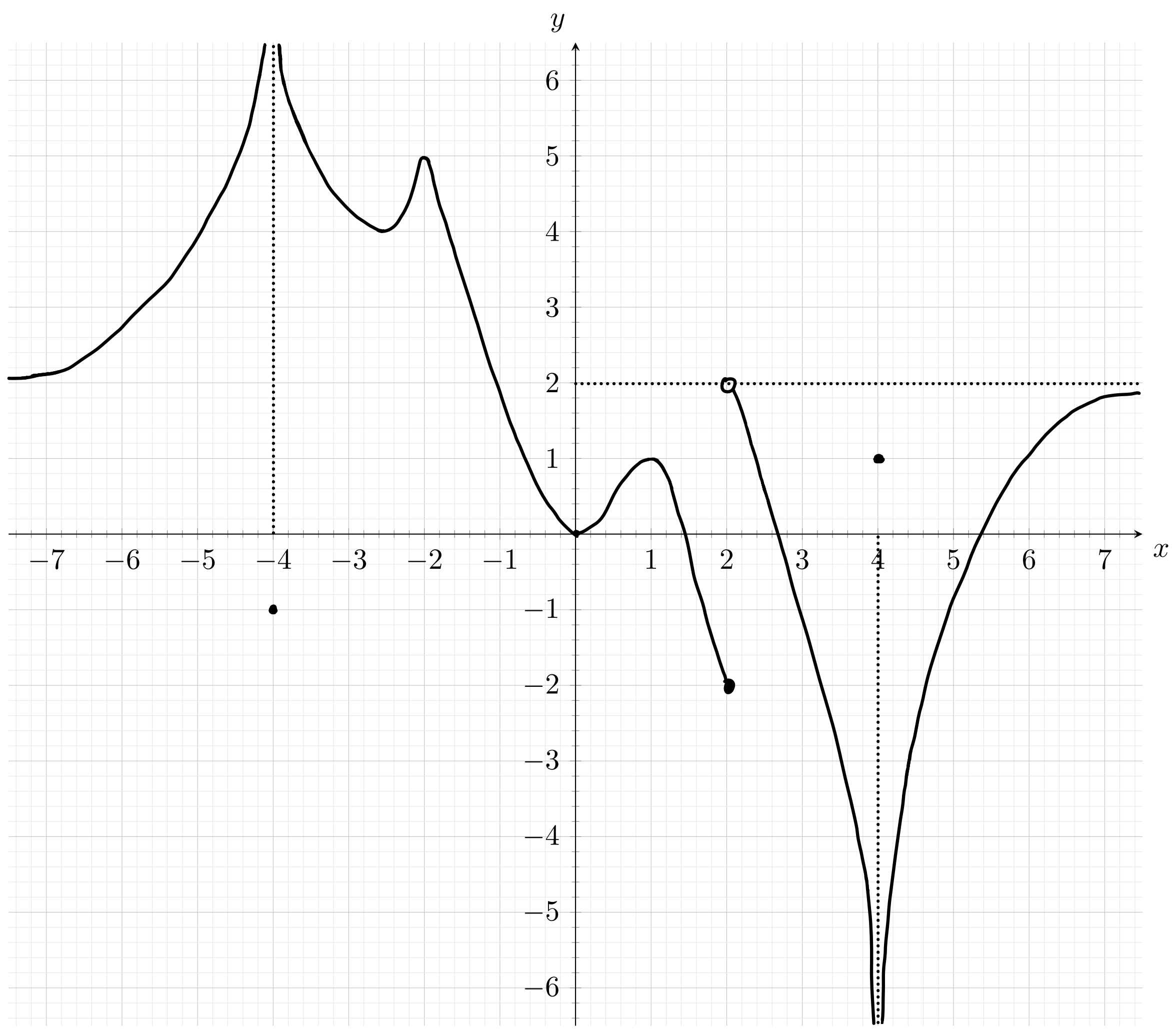
(g) $\lim_{x\to 2} [f(x)]^2 = 4$

(h) $\lim_{x\to 2^-} f(f(x)) = 5$

(i) $\lim_{x\to \infty} f(x) = 2$

(j) $\lim_{x\to 4} f(1 - e^{f(x)}) = 2$

To clarify, we want one single function f that satisfies all the conditions in all the parts, all at once. Make your graph tidy and unambiguous.



2. Let $f(x) = \frac{x-1}{\sqrt{x}-1}$

(a) What is the largest possible domain of f?

From $\sqrt{x}$ gives $x \geq 0$.

From $\sqrt{x} - 1 \neq 0$ gives $x \neq 1$.

Thus, $x \in [0, 1) \cup (1, +\infty)$

(b) Let $g(x) = \sqrt{x} + 1$. Find the largest subset $I \subset \mathbb{R}$ such that for all $x \in I$ we have $f(x) = g(x)$.

From (a), when $f(x)$ is defined, $x \in [0, 1) \cup (1, +\infty)$.

When $g(x) = f(x)$, gives

$$\sqrt{x}+1 = \frac{x-1}{\sqrt{x}-1} \quad \text{, when } x \neq 1.$$

$$\Rightarrow (\sqrt{x}+1)(\sqrt{x}-1) = x-1$$

$$\Rightarrow x-1 = x-1$$

$$\Rightarrow x \in \mathbb{R} \text{ and } x \neq 1.$$

Therefore. where $I = [0, 1) \cup (1, +\infty)$. $\forall x \in I$, $f(x) = g(x)$

(c) Prove, directly from the epsilon-delta definition of limit, that

$$\lim_{x\to 9} g(x) = 4$$

Don't use any of the limit laws or other theorems.

Pf. Let $\varepsilon > 0$.

Take $\delta = 3\varepsilon$, gives.

$|g(x) - 4| = |(\sqrt{x}+1) - 4|$

$\Rightarrow |\sqrt{x} - 3|$

$\Rightarrow \left|\dfrac{x-9}{\sqrt{x}+3}\right| = \dfrac{|x-9|}{|\sqrt{x}+3|}$, since $\sqrt{x}+3 \geq 0$, gives

$\Rightarrow |x-9| \cdot \dfrac{1}{\sqrt{x}+3}$

Since $\sqrt{x} \geq 0$, gives.

$|x-9| \cdot \dfrac{1}{\sqrt{x}+3} \leq |x-9| \cdot \dfrac{1}{3} < \delta \cdot \dfrac{1}{3} = 3\varepsilon \cdot \dfrac{1}{3} = \varepsilon.$

We've shown $|(\sqrt{x}+1)-4| < \varepsilon$, as needed.

Thus, we've proved that $\lim_{x\to 9} g(x) = 4$. ∎

(d) Prove that

$$\lim_{x\to 9} f(x) = 4$$

You may need to use some results from (b) and (c).

pf. from (b). gives:

when $x \in [0,1) \cup (1,+\infty)$. $f(x) = g(x)$.

from (c) gives:

$\forall \varepsilon > 0$, $\exists \delta_1 > 0$, s.t. $0 < |x-9| < \delta \Rightarrow |g(x) - 4| < \varepsilon$.

Take $\delta = \min(\delta_1, 1)$. gives.

when $\delta = \delta_1$. satisfying $\lim_{x\to 9} g(x) = 4$, which is.

$\forall \varepsilon > 0$, $\exists \delta_1 > 0$, s.t. $0 < |x-9| < \delta \Rightarrow |g(x) - 4| < \varepsilon$.

when $\delta = 1$,

$0 < |x-9| < 1 \Rightarrow 8 < x < 10$. which $x \in (8,10)$.

Since $(8,10) \subseteq (1,+\infty)$, $f(x) = g(x)$.

Therefore. we have,

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $0 < |x-9| < \delta \Rightarrow |f(x) - 4| < \varepsilon$.

i.e. $\lim_{x\to 9} f(x) = 4$.

∎

3. Let $a \in \mathbb{R}$. Let f and g be functions defined on $\mathbb{R}$. Is each of the following claims true or false? Prove your answer. Hint: often times, the easiest way to prove something is false is by providing a counter example and and proving that counter example satisfies the required conditions. It your answer is true, the proof should be a short, "one-line" proof using the properties of limits you already know (review section 2. 10 and section 2.12). You don't need to use the epsilon-delta definition in this question.

(a) IF $\lim_{x \to a} f(x)$ does not exist and $\lim_{x \to f(a)} g(x)$ does not exist,
THEN $\lim_{x \to a} g(f(x))$ does not exist.

○ True ✓ False

Take $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$; $g(x) = \begin{cases} \frac{1}{x^3} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

When $a=0$, which $x \to 0$. $\lim_{x \to 0^-} f(x) = -\infty$; $\lim_{x \to 0^+} f(x) = +\infty$

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$, gives $\lim_{x \to 0} f(x)$ DNE.

When $x \to f(a)$, which $f(a) = f(0) = 0$, $\lim_{x \to f(0)^-} g(x) = \lim_{x \to 0^-} g(x) = -\infty$,

$\lim_{x \to f(0)^+} g(x) = \lim_{x \to 0^+} g(x) = +\infty$. $\lim_{x \to f(a)^-} g(x) \neq \lim_{x \to f(a)^+} g(x)$. gives $\lim_{x \to f(a)} g(x)$ DNE.

Thus, when $x \neq 0$. $g(f(x)) = \frac{1}{(\frac{1}{x})^3} = x^3$

when $x = 0$, $g(f(x)) = 0$. gives.

$g(f(x)) = \begin{cases} x^3 & x \neq 0. \\ 0 & x = 0. \end{cases} \Rightarrow g(f(x)) = x^3$.

Apparently, $\lim_{x \to 0^-} g(f(x)) = \lim_{x \to 0^+} g(f(x)) = 0$.

Thus. $\lim_{x \to 0} g(f(x))$ exist and $\lim_{x \to 0} g(f(x)) = 0$. counters the statement in the question, which is false.

(b) IF $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, and $\lim_{x\to a} |f(x)-g(x)| = 0$,
THEN $\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$.

✓ True ○ False

pf. Assuming $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. gives.

Assuming $\lim_{x\to a} |f(x)-g(x)| = 0$.

Since $-|f(x)-g(x)| \leq f(x)-g(x) \leq |f(x)-g(x)|$. for $x \in \mathbb{R}$,

By limit law, $\lim_{x\to a} -|f(x)-g(x)| = \lim_{x\to a}(-1) \cdot \lim_{x\to a} |f(x)-g(x)|$

$= -1 \cdot 0 = 0.$

Thus, by squeeze theorem,

from $-|f(x)-g(x)| \leq f(x)-g(x) \leq |f(x)-g(x)|$,

$\lim_{x\to a} -|f(x)-g(x)| = 0 = \lim_{x\to a} |f(x)-g(x)|$, gives.

$\lim_{x\to a} [f(x)-g(x)] = 0$.

Since $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exists, gives,

$$\lim_{x\to a} [f(x)-g(x)] = 0$$

$$\Rightarrow \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = 0$$

$$\Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} g(x) \quad \blacksquare$$

4. Let $a \in \mathbb{R}$. Let f and g be functions defined on $\mathbb{R}$. Is each of the following claims true or false? Prove your answer. If it is true, prove it directly from the epsilon-delta definition of a limit. Hint: often times, the easiest way to prove something is false is by providing a counter example and and proving that counter example satisfies the required conditions.

   (a) IF $f(x)$ is continuous at a and $g(x)$ is not continuous at a,
   THEN $f(x)g(x)$ is not continuous at a.

   ○ True ✓ False

Providing that $f(x) = x$. and $g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, which are both defined on $\mathbb{R}$.

when $x \to 0$, apparently. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) = 0$.
which $f(x) = x$ is continuous when $x \to 0$.
when $x \to 0$, apparently, $\lim_{x\to 0^-} g(x) = -\infty$. $\lim_{x\to 0^+} g(x) = +\infty$, $g(0) = 0$.
$\lim_{x\to 0^+} g(x) \neq \lim_{x\to 0^-} g(x) \neq g(0)$, which $g(x) = \frac{1}{x}$ is not continuous when $x \to 0$.
Thus, $h(x) = f(x) \cdot g(x) = x \cdot \frac{1}{x} = 1$.
when $x \to 0$ for $h(x)$.
$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^-} h(x) = h(0) = 1$.
Therefore $h(x)$ is continuous when $x \to 0$.
Logically equivalent to $f(x) \cdot g(x)$ is continuous when $x \to 0$.
Counters the statement in the question.

(b) IF f is continuous on $\mathbb{R}$, $f(-1) = 10$ and $f(1) = 20$, THEN $\lim_{x\to\infty} f(\sin(x))$ does not exist.

☑ True ◯ False

WTS: $\forall L \in \mathbb{R}, \exists \varepsilon > 0, \forall M \in \mathbb{R}, \exists x \in \mathbb{R}$, s.t. $x > M$ and $|f(x) - L| \geq \varepsilon$.

pf. Let $L \in \mathbb{R}$. Take $\varepsilon = 1$. Let $M \in \mathbb{R}$.

Take $x = \max\{3M, \frac{\pi}{2} + k\pi \ (k \in \mathbb{Z})\}$. gives.

When $x = 3M$, $x = 3M > M$.

When $x = \frac{\pi}{2} + k\pi \ (k \in \mathbb{Z})$.

when k is an odd integer. $\sin(x) = 1$, which $f(\sin(x)) = 20$

when k is an even integer. $\sin(x) = -1$, which $f(\sin(x)) = 10$

At least one of below must be true.

① $20 \notin (L-1, L+1)$; i.e. $20 \notin (L-\varepsilon, L+\varepsilon)$. $\Rightarrow |20 - L| \geq \varepsilon$.
take $x = \frac{\pi}{2} + k\pi$, for some k is odd integer,
Then $f(\sin(x)) = 20$.

② $10 \notin (L-1, L+1)$; i.e. $10 \notin (L-\varepsilon, L+\varepsilon)$. $\Rightarrow |10 - L| \geq \varepsilon$.
Take $x = \frac{\pi}{2} + k\pi$. for some k is even integer.
Then $f(\sin(x)) = 10$.

Either way, it satisfies $x > M$ and $|f(\sin x) - L| \geq \varepsilon$. ■

(c) IF f is continuous on $\mathbb{R}$, $f(-2) = 10$ and $f(2) = 20$, THEN $\lim_{x\to\infty} f(\sin(x))$ does not exist.

○ True ✓ False

Providing that $f(x) = \begin{cases} 5x + 20 & x < -1 \\ 15 & -1 \le x \le 1 \\ 5x + 10 & x > 1. \end{cases}$, apparently, f is

continuous on $\mathbb{R}$. Due to $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1) = 15$. and.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 15$;

Assuming that $f(-2) = 10$ and $f(2) = 20$.

WTS: $\exists L \in \mathbb{R}$, s.t. $\forall \varepsilon > 0$, $\exists M \in \mathbb{R}$, s.t $x > M \Rightarrow |f(\sin x) - L| < \varepsilon$.

Take $L = 15$ Let $\varepsilon > 0$, Take $M = 0$, gives $x > 0$.

Since $x > 0$, $x \to \infty$ and $\sin x \in [-1, 1]$,

Take $t = \sin x$. $t \in [-1, 1]$

Therefore. $f(t) = 15$. gives that:

$|f(\sin x) - 15| = |15 - 15| = 0 < \varepsilon$

Thus, $\lim_{x\to\infty} f(\sin(x)) = 15$. which $\lim_{x\to\infty} f(\sin(x))$ exists,

counters the statement in the question.

5b) is quite challenging. You can try it however we will not require you to return your work. 5b) is not counted for credits.

5. We will try to prove the following theorem.

Theorem 1. *Let $a, b \in \mathbb{R}$ and $a < b$. Let f be a one-to-one continuous function on an interval (a, b). Then*
f is strictly increasing on (a, b), i.e. $\forall x_1, x_2 \in (a, b), x_1 < x_2 \implies f(x_1) < f(x_2)$
OR
f is strictly decreasing on (a, b), i.e. $\forall x_1, x_2 \in (a, b), x_1 < x_2 \implies f(x_1) > f(x_2)$.

(a) What happens when we drop continuity property? Find an example of one-to-one non-continuous function defined on $(-1, 1)$ which increases on $(-1, 0]$ and decreases on $(0, 1)$. No justification is necessary.

$$f(x) = \begin{cases} x & x \in [-1, 0] \\ -x+2 & x \in (0, 1) \end{cases}$$

when we drop the continuity property, which is provided as a counter example above, which is non-continuous but still strictly increasing when $x \in [-1, 0]$ and strictly decreasing when $x \in (0, 1)$.

(b) Prove this following theorem. Hint: you may need to use the Intermediate Value Theorem in your proof.