Cardinality

1. bijective: $f: A \to B$.

   1). Injective: $\forall x_1, x_2 \in A$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

   $(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$.

   2). Surjective: $\forall y \in B$, $\exists x \in A$. s.t. $f(x) = y$.

   3). Inverse function: If f is 1-1 func. mapping T onto S, then inverse of f, often denoted f^{-1}, is the func. mapping T onto S. defined by. $f^{-1}(t) = s$ when $f(s) = t$.

2. Cardinality: a measure of a set's size, meaning the # of elements in the set. Denoted as $|S|$.

   1). Same Card: the sets S and T have the same cardinality if there exists a bijective func. $f: S \to T$, and written as $|S| = |T|$; If no such bijection exists, then the sets have unequal card. and written as $|S| \neq |T|$.

   ① transitivity is reserved for equalcard. (Thm.).

   $f(x) = f_1(f_2(x))$.

   $|\mathbb{N}| = |\mathbb{Z}|$.
   (push down).
   $\mathbb{N}$: Even ∧ Odd
   ↓ ↓
   $\mathbb{Z}$: $\{0\} \cup \mathbb{Z}^+$ ∧ $\mathbb{Z}^-$

   ② $|\mathbb{N}| = |\mathbb{Q}|$ (P5). (Thm.).

   ③ $|\mathbb{N}| \neq |\mathbb{R}|$ (Thm.) (Cantor-Diagonal Thm.: $|[0,1]| \neq |\mathbb{N}|$).
   ↳ uncountable → infinite & $|\mathbb{R}| \neq |\mathbb{N}|$. $\quad |[0,1]| = |\mathbb{R}|$

   Count.: finite or $|S| = |\mathbb{N}|$.
   Uncount.: infinite & $|S| \neq |\mathbb{N}|$

   2). Card. under basic set operation.

   Assume that $|A_1| = |B_1|$ and $|A_2| = |B_2|$.

   ① $|A_1 \times A_2| = |B_1 \times B_2|$.

   ② If $A_1 \cap A_2 = \phi$. $B_1 \cap B_2 = \phi$. then $|A_1 \cup A_2| = |B_1 \cup B_2|$.

   3). The interval $[0,1]$ and $(0,1]$ have same card.

   Define. $f: [0,1] \to (0,1]$. as $f(x) = \begin{cases} \dfrac{1}{\frac{1}{x}+1} & \text{if } \exists n \in \mathbb{N}, \text{s.t. } x = \frac{1}{n}. \end{cases}$

$$\begin{cases} 1 & \text{if } x=0. \\ x & \text{otherwise}. \end{cases}$$

① $|[0,1)| = |(0,1]|$.

Define: $f: [0,1) \to (0,1]$. as $f(x) = \begin{cases} x. & x \in (0,1). \\ 1 & x=0. \end{cases}$

②. $|[0,1]| = |(0,1)|$.

Define $f: [0,1] \to (0,1)$. as $f(x) = \begin{cases} \frac{1}{\frac{1}{x}+2} & \text{if } \exists x \in \mathbb{N}.\ x = \frac{1}{n}. \\ \frac{1}{2} & \text{if } x = 0. \\ x & \text{if otherwise}. \end{cases}$

4). The card of natural number: $|\mathbb{N}| = \aleph_0$

The card of real #: $|\mathbb{R}| = c$. (also call card. of continuum).

① (Thm). $\forall a, b \in \mathbb{R}$. $a<b$., then $|[a,b]| = |[0,1]| = c$.

$\Rightarrow |[a,b]| = |[a,b)| = |(a,b]| = |(a,b)| = c$.

②. (Thm). $\forall a,b,c,d \in \mathbb{R}$, $a<b \wedge c<d$, then $|(a,b]| = |(c,d]|$.

$f: (0,1] \to (a,b]$, $f(x) = a+(b-a)x$.

$g: (0,1] \to (c,d]$. $g(x) = c+(d-c)x$.

3. Countable sets: A set is countable if it either finite or has the same cardinality as the set of natural number; Otherwise, it's uncountable

1) The union of a countable # of countable sets is countable (Thm).

2). A subset of a countable set is countable.

Assume S is countable, i.e. S is finite or $|\mathbb{N}| = |S|$.

Let $S^* \subseteq S$.

WTS: S^* is countable, i.e. S^* is finite or $|\mathbb{N}| = |S^*|$.

①. S is finite $\longrightarrow$ S^* is finite.

② If S is infinite and $|S| = |\mathbb{N}|$, $\exists$ bijection $f: \mathbb{N} \to S$. s.t

$S = \{f(1), f(2), \dots\}$.

Let $S^* = \{s_1, s_2, \dots\} \subseteq S$.

Since S, S^* are infinite and countable, by push-down method, $\exists g:$ $S \to S^*$ s.t. $g(f(i)) = s_i$, where g is bijective.

$\therefore |S| = |S^*|$. by transitivity, $|S^*| = |\mathbb{N}|$. i.e. ......

3). The set of all $\mathbb{R}$ between 0 and 1 is uncountable.

4). Let A and B be countable sets, then $A \times B$ is countable.