2. RRT and Tower of fields (from chapters 8 and 12)

a) Prove $\sqrt[4]{2} \notin \mathbb{Q}(\sqrt{2})$.

By definition: $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

Assume $\sqrt[4]{2} \in \mathbb{Q}\sqrt{2}$. which $\exists a_1, b_1 \in \mathbb{Q}$, s.t. $a_1 + b_1\sqrt{2} = \sqrt[4]{2}$.

$$\Rightarrow (a_1 + b_1\sqrt{2})^4 = 2. \text{ which}$$

$$\Rightarrow (a_1^2 + 2b_1^2 + 2a_1b_1\sqrt{2})^2 = 2$$

$$\Rightarrow a_1^4 + 4b_1^4 + 12a_1^2b_1^2 + 4a_1b_1a_1^2\sqrt{2} + 8a_1b_1b_1^2\sqrt{2} = 2.$$

$$\Rightarrow \sqrt{2}(4a_1b_1a_1^2 + 8a_1b_1b_1^2) = 2 - (a_1^4 + 4b_1^4 + 12a_1^2b_1^2).$$

$$\Rightarrow \sqrt{2} = \frac{2 - (a_1^4 + 4b_1^4 + 12a_1^2b_1^2)}{4a_1b_1a_1^2 + 8a_1b_1b_1^2}.$$

Since $a, b \in \mathbb{Q}$. from definition of a field, it's addition and multiplication is close. which $\frac{2 - (a_1^4 + 4b_1^4 + 12a_1^2b_1^2)}{4a_1b_1a_1^2 + 8a_1b_1b_1^2}$ is in $\mathbb{Q}$. However, from theorem 8.2.7. since 2 is a prime. $\sqrt{2} \notin \mathbb{Q}$. contradicts. Therefore. $\sqrt[4]{2} \notin \mathbb{Q}(\sqrt{2})$.

b) Looking for a field extension of $\mathbb{Q}$ that contains $\sqrt[4]{2}$, naturally we consider $\mathbb{Q}[\sqrt[4]{2}] = \{a + b\sqrt[4]{2} : a, b \in \mathbb{Q}\}$. Prove that $\mathbb{Q}[\sqrt[4]{2}]$ is not a field.

WTS: $\mathbb{Q}[\sqrt[4]{2}] = \{a + b\sqrt[4]{2} : a, b \in \mathbb{Q}\}$ is not a field.

Assume $\mathbb{Q}[\sqrt[4]{2}]$ is a field. I'll check its property from definition.

Let $x, y \in \mathbb{Q}[\sqrt[4]{2})$. which $x = a_1 + b_1\sqrt[4]{2}$.

$y = a_2 + b_2\sqrt[4]{2}$. where $a_1, a_2, b_1, b_2 \in \mathbb{Q}$.

$$x \cdot y = (a_1 + b_1\sqrt[4]{2})(a_2 + b_2\sqrt[4]{2}).$$
$$= a_1a_2 + a_1b_2\sqrt[4]{2} + a_2b_1\sqrt[4]{2} + b_1b_2(\sqrt[4]{2})^2$$
$$= a_1a_2 + (a_1b_2 + a_2b_1)\sqrt[4]{2} + b_1b_2\sqrt{2}.$$

Since $b_1b_2\sqrt{2} \notin \mathbb{Q}$ and can't be written as $b\sqrt[4]{2}$. where $b \in \mathbb{Q}$, gives $x \cdot y \notin \mathbb{Q}[\sqrt[4]{2}]$. doesn't satisfy the property of a field. which $\mathbb{Q}[\sqrt[4]{2}]$ is not a field.

c) Read Definition 12.2.16. If possible, find a shortest tower of fields such that the final field contains $\sqrt[4]{2}$, and if not, explain why it is impossible.

Let $F_0 = \mathbb{Q}$; $F_1 = \mathbb{Q}(\sqrt{2})$; $F_2 = F_1(\sqrt[4]{2})$.

From (a). $\mathbb{Q}(\sqrt{2})$ is a field. which F_1 is a field.

WTS: F_2 is a field, which $F_2 = F_1[\sqrt[4]{2}] = \{a + b\sqrt[4]{2} : a, b \in F_1\}$.

① WTS: $0, 1 \in F_1[\sqrt[4]{2}]$.

Since F_1 is a field. $0, 1 \in F_1$.

Take $a = 0$. $b = 0$; $a = 1$, $b = 0$, gives. $0, 1 \in F_1[\sqrt[4]{2}]$.

② WTS: Closed addition and multiplication.

Let $x, y \in F_1[\sqrt[4]{2}]$, $x = a_1 + b_1\sqrt[4]{2}$. $y = a_2 + b_2\sqrt[4]{2}$. where $a_1, a_2, b_1, b_2 \in F_1$.

$x + y = a_1 + b_1\sqrt[4]{2} + a_2 + b_2\sqrt[4]{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt[4]{2}$.

Since, F_1 is a field. it's closed addition. which $(a_1 + a_2), (b_1 + b_2) \in F_1$, gives $x + y \in F_1[\sqrt[4]{2}]$.

$$xy = (a_1 + b_1\sqrt[4]{2})(a_2 + b_2\sqrt[4]{2}).$$
$$= a_1a_2 + a_1b_2\sqrt[4]{2} + a_2b_1\sqrt[4]{2} + b_1b_2(\sqrt[4]{2})^2$$
$$= (a_1a_2 + b_1b_2\sqrt{2}) + (a_1b_2 + a_2b_1)\sqrt[4]{2}$$

Since F_1 is a field. it's closed addition and multiplication, which (a_1a_2), $(a_1b_2 + a_2b_1) \in F_1$.

Since $F_1 = \mathbb{Q}(\sqrt{2}) = \{p + q\sqrt{2} : p, q \in \mathbb{Q}\}$. gives $\sqrt{2} \in F_1$ when $p = 0$, $q = 1$. gives $(b_1b_2\sqrt{2}) \in F_1$. which $xy \in F_1[\sqrt[4]{2}]$.

③ WTS: $-x \in F_1[\sqrt[4]{2}]$.

Let $x \in F_1[\sqrt[4]{2}]$, i.e. $x = a + b\sqrt[4]{2}$, where $a, b \in F_1$.

Thus. $-x = -a - b\sqrt[4]{2}$, where $-a, -b \in F_1$.

$\therefore -x \in F_1[\sqrt[4]{2}]$.

④ WTS: $\frac{1}{x} \in F_1[\sqrt[4]{2}]$.

Let $x \in F_1[\sqrt[4]{2}]$. i.e. $x = a + b\sqrt[4]{2}$. where $a, b \in F_1$.

$$\frac{1}{x} = \frac{1}{a + b\sqrt[4]{2}} = \frac{a - b\sqrt[4]{2}}{(a + b\sqrt[4]{2})(a - b\sqrt[4]{2})} = \frac{a - b\sqrt[4]{2}}{a^2 - b^2\sqrt{2}} = \frac{a}{a^2 - b^2\sqrt{2}} - \frac{b}{a^2 - b^2\sqrt{2}}\sqrt[4]{2}.$$

Since $F_1 = \mathbb{Q}(\sqrt{2}) = \{p + q\sqrt{2} : p, q \in \mathbb{Q}\}$. gives $\sqrt{2} \in F_1$

Since $a, b, \sqrt{2} \in F_1$. $\left(\frac{a}{a^2 - b^2\sqrt{2}}\right)$, $\left(\frac{-b}{a^2 - b^2\sqrt{2}}\right) \in F_1$. gives

$$\frac{a}{a^2 - b^2\sqrt{2}} - \frac{b}{a^2 - b^2\sqrt{2}}\sqrt[4]{2} \in F_1[\sqrt[4]{2}], \text{ which } \frac{1}{x} \in F_1[\sqrt[4]{2}].$$

Therefore. $F_1[\sqrt[4]{2}]$ is a field. $F_1(\sqrt[4]{2})$. i.e. F_2.

Take $a = 0$, $b = 1$. gives $\sqrt[4]{2} \in F_2$.