3 Question 3

(a)

```python
def q3_a_func(n: int) -> int:
    """Implement a Python function that takes a positive natural number n and returns a_n.

    Precondition: n is a positive natural number
    """
    if n == 1:
        # From the definition of a_n, when n is 1, return a_n equals to 1.
        return 1
    else:
        """ This is the recursion. Aim at returning the recursive value of a_n after reaching the base case when
        a_n equals to 1.
        """
        return q3_a_func(math.floor(math.sqrt(n))) * q3_a_func(math.floor(math.sqrt(n))) \
            + 2 * q3_a_func(math.floor(math.sqrt(n)))
```

Figure 1: Python function for Q3-a

(b)

```python
def q3_b_func(n: int) -> int:
    """Implement a Python function that takes a positive natural number n and raises an exception if n is 1, otherwise
    it returns a_n.

    Precondition: n is a positive natural number
    """
    if n == 1:
        # By question requirement, when n is 1, raises an Exception.
        raise Exception("Sorry, n must be greater than 1")
    elif n == 2 or n == 3:
        """Since when n equals to 2 or n equals to 3, the floor of square root of n is 1, and, in this function, we
        don't have the value of a_n when n equals 1. Thus, we need to manually add the value of a_n when n equals to 2
        and n equals to 3 to prevent the error when calling the recursive.
        """
        return 3
    else:
        """This is the recursion. Aim at returning the recursive value of a_n after reaching the case when a_n equals
        to 2 or a_n equals to 3.
        """
        return q3_b_func(math.floor(math.sqrt(n))) * q3_b_func(math.floor(math.sqrt(n))) \
            + 2 * q3_b_func(math.floor(math.sqrt(n)))
```

Figure 2: Python function for Q3-b

(c) When $n_0 = 2$, n_0 is the smallest natural n_0 so that a_n is a multiple of 3 for each natural $n \geq n_0$.

Given statement to prove: $\forall n \in \mathbb{N},\ n \geq n_0,\ P(n)$, which $P(n) : a_n$ is a multiple of 3.

Let $n \in \mathbb{N}$.

Proof: We prove this by complete induction on n.

Base Case: Let $2 \leq n < 4$.

$$\begin{aligned} P(2):\ \ a_2 &= (a_{\lfloor\sqrt{2}\rfloor})^2 + 2 \cdot a_{\lfloor\sqrt{2}\rfloor} \\ &= a_1^2 + 2 \cdot a_1 = 1^2 + 2 = 3 \text{is a multiple of } 3. \end{aligned}$$

$$
\begin{aligned}
P(3): \ a_3 &= (a_{\lfloor\sqrt{3}\rfloor})^2 + 2 \cdot a_{\lfloor\sqrt{3}\rfloor} \\
&= a_1^2 + 2 \cdot a_1 = 1^2 + 2 = 3 \text{is a multiple of 3.}
\end{aligned}
$$

Thus, I've proved the base case is true.

Induction Step: Let $n \geq 4$.

Induction Hypothesis: Assume $\forall k,\ 2 \leq k < n,\ P(k)$

Since $n \geq 4$, gives $\lfloor\sqrt{n}\rfloor < n$.

Since $\lfloor\sqrt{n}\rfloor < n$ and $4 \leq n$, gives $2 \leq \lfloor\sqrt{n}\rfloor$ as 2 is the smallest value of $\lfloor\sqrt{n}\rfloor$, which gives,

$$2 \leq \lfloor\sqrt{n}\rfloor < n$$

Since $\lfloor\sqrt{n}\rfloor$ is an integer which $\lfloor\sqrt{n}\rfloor \geq 2$, from induction hypothesis, we can always find $k' = \lfloor\sqrt{n}\rfloor$, which $P(k')$ is true and $a_{k'} = 3p,\ p \in \mathbb{N}$.

Thus gives,

$$
\begin{aligned}
a_n &= (\lfloor\sqrt{n}\rfloor)^2 + 2 \cdot a_{\lfloor\sqrt{n}\rfloor} \\
&= (a_{k'})^2 + 2 \cdot a_{k'} \\
&= (3p)^2 + 2 \cdot (3 \cdot p) \\
&= 9 \cdot p^2 + 6 \cdot p \\
&= 3 \cdot (3 \cdot p^2 + 2p)
\end{aligned}
$$

Let $q = 3 \cdot p^2 + 2 \cdot p$. Since $p \in \mathbb{N}$, gives $q \in \mathbb{N}$, which

$$a_n = 3q,\ q \in \mathbb{N} \text{ , where } a_n \text{ is a multiple of 3.}$$

I've proved that $P(n)$ is true.

To conclude, I've proved when $n_0 = 2$, n_0 is the smallest natural n_0 so that a_n is a multiple of 3 for each natural $n \geq n_0$.

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