Critical Points & Stability

det(A) = 0
↓
singular.

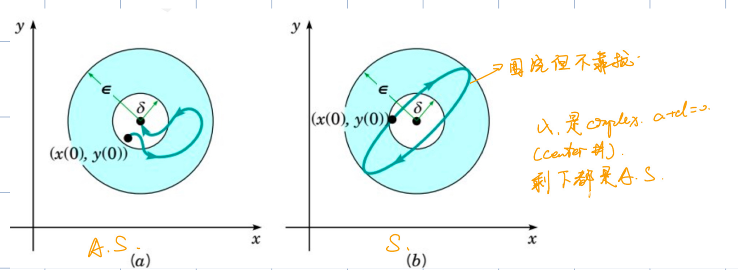
1. Critical Points (Equilibrium sol^{n} / cost. sol^{n}): $\vec{x}\,' = 0$.

   1) If A is non-sigular, then $\vec{x} = \vec{0}$ is the only C.P.
   (det(A) $\neq 0$)

2. Stability: For some initial value close to the C.P.

   1) stable: sol^{n} stay close to C.P. (在圈内).

   2) Asymptotically stable: sol^{n} converge to the C.P. (靠拢).



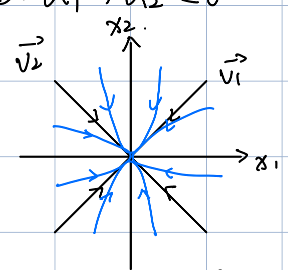
3. Trajectories & stability.

If det(A) ≠ 0. then (0,0) is the only critical point for the system.

   1) distinct $\lambda \in \mathbb{R}$: 负朝内，正朝外

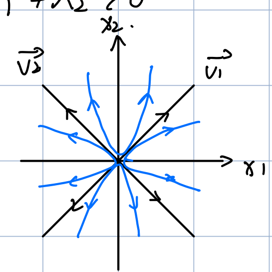
若 $\vec{v}_1, \vec{v}_2$ 非分布于4个象限，须用 col vector 以辅助(方法同 same λ).
↓
则来画 $\vec{v}_1\ \vec{v}_2$ 不在的2个象限(中间夹着的还是直线).

① $\lambda_1 \neq \lambda_2 < 0$.



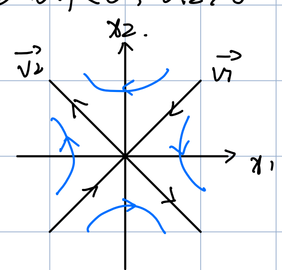
(0,0) is asy. stable (node sink)

② $\lambda_1 \neq \lambda_2 > 0$.

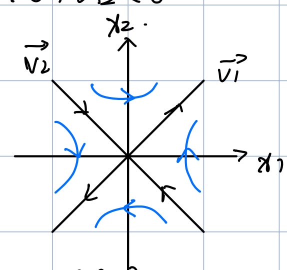


(0,0) is unstable (node source)

③ $\lambda_1 < 0,\ \lambda_2 > 0$.

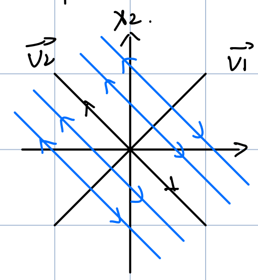


④ $\lambda_1 > 0,\ \lambda_2 < 0$.



saddle point (unstable).

⑤ $\lambda_1 = 0.\ \lambda_2 > 0.$



← 无方向. $\lambda_1 = 0$. 作为 $\vec{v}$ 的分界线

⑥ $\lambda_1 = 0.\ \lambda_2 < 0.$ 同理.

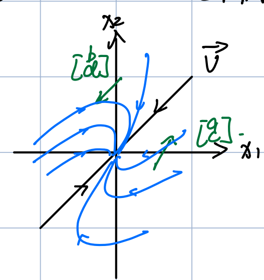
N/A.

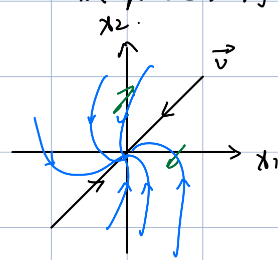
2). repeated $\lambda \in \mathbb{R}$.

点 $(1,0)$ 处 direction $\vec{v}$.　　点 $(0,1)$ 处 direction $\vec{v}$.

→ 共用一个 $\vec{v}$: 根据 $A \cdot \begin{bmatrix}1\\0\end{bmatrix}$, i.e. $\begin{bmatrix}a\\c\end{bmatrix}$; $A \cdot \begin{bmatrix}0\\1\end{bmatrix}$, i.e. $\begin{bmatrix}b\\d\end{bmatrix}$

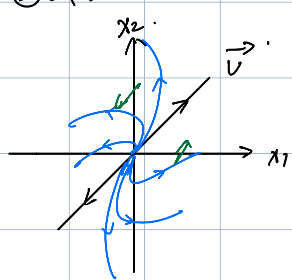
①. $\lambda < 0$. 画辅助 vector. 根据此方向作图.

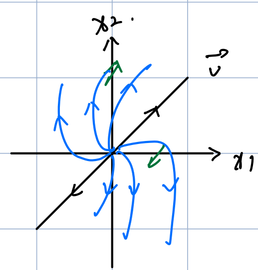




improper node sink (asy. sta.)

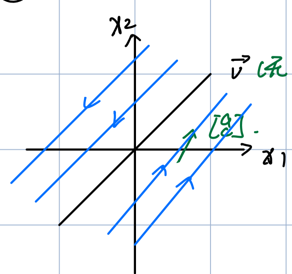
② $\lambda > 0$.





im. node source (unstable).

③ $\lambda = 0$.



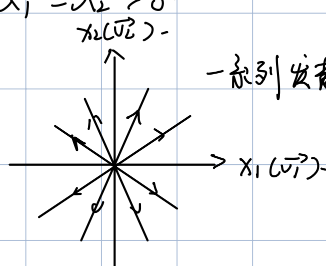
画 1 个 $\begin{bmatrix}a\\c\end{bmatrix}$ / $\begin{bmatrix}b\\d\end{bmatrix}$.

确定一边方向, 另一侧相反.

N/A.

→ 不同 $\vec{v}$

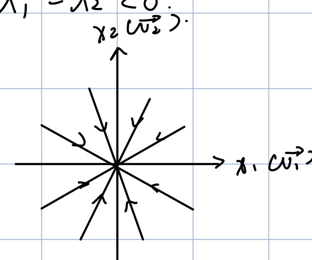
i). $\lambda_1 = \lambda_2 > 0$.



一系列发散的.

proper node source (unstable).

ii) $\lambda_1 = \lambda_2 < 0$.



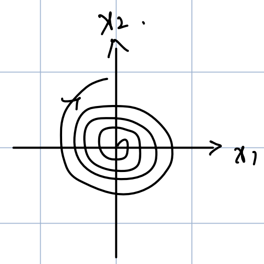
proper node sink (asy. sta.).

3) complex λ.

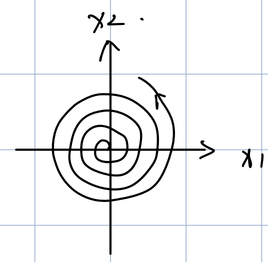
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

① $a+d$ (α) $\begin{cases} >0: \text{ spiral outward} \\ <0: \text{ spiral inward.} \\ =0: \text{ center.} \end{cases}$ ② b $\begin{cases} >0: \text{ 顺} \\ <0: \text{ 逆.} \end{cases}$

i) $a+d > 0$



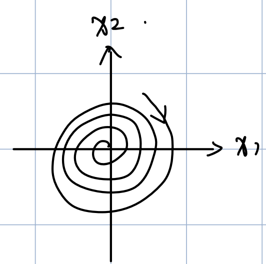
$b>0$ (C.W)



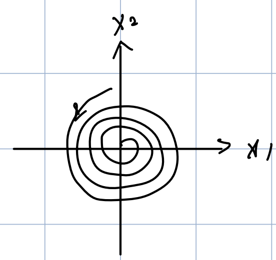
$b<0$ (C.C.W).

spiral source (un. s.)

ii) $a+d<0$.



$b>0$

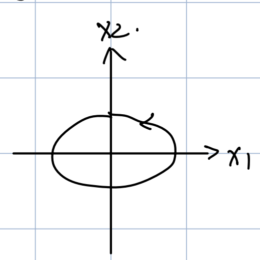


$b<0$

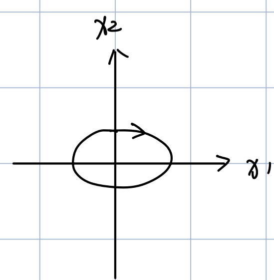
spiral sink (asy. s.).

iii) $a+d=0$.

只有 center point stable. 其它都为 asy. stable.



$b<0$.



$b>0$.

center point $\underbrace{\text{(stable)}}_{s}$.

4. Exercise.

1). Classify C.P. & Determine Stability $\longrightarrow$ calculate eigenvalue of A.

Sketch the trajectory $\longrightarrow$ calculate eigenvector correspondingly. (use $[\,]$; $[\,]$ if needed).

$\vec{v} = (b, -a)$. $\uparrow$ $A-\lambda I$.

e.g. $\vec{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x}$.

$\det(A-\lambda I) = \det \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda)+3 \overset{\text{set}}{=} 0$

$\Rightarrow 5-5\lambda-\lambda+\lambda^2+3=0$.

Since $\det(A) = 5+3 = 8 \neq 0$. $\Rightarrow$ $\lambda^2 - 6\lambda + 8 = 0$.

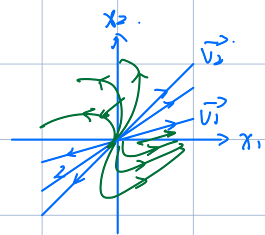
$(0,0)$ is the only C.P. $\Rightarrow (\lambda-2)(\lambda-4) = 0$.

Since $\lambda_1 \neq \lambda_2 > 0$. $(0,0)$ is. $\therefore \lambda_1 = 2;\ \lambda_2 = 4$.

node source, unstable.

For $\lambda_1 = 2$. $\vec{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$; $\lambda_2 = 4$. $\vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\therefore$ Trajectory:



Since $\vec{v_1}, \vec{v_2}$ not in all quadrants.

Direc. vec. at $(1,0)$: $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

at $(0,1)$: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

e.g. $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$.

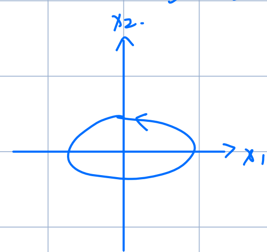
$\det(A - \lambda I) = \det\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - (-5) \overset{\text{set}}{=} 0$

Since $\det(A) = -4+5 = 1 \neq 0$. $\Rightarrow -(2-\lambda)(2+\lambda) + 5 = 0$

$(0,0)$ is the only c.p. $\Rightarrow -4 + \lambda^2 + 5 = 0$. $\Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$

Since $\lambda = \pm i$, $a+d = 0$, center point $(0,0)$ stable.

Since $b = -5 < 0$, C.C.W.



e.g. $\vec{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}$.

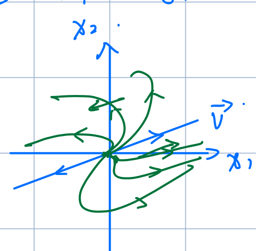
$\det(A - \lambda I) = \det\begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - (-4) \overset{\text{set}}{=} 0$

Since $\det(A) = -3+4 = 1 \neq 0$. $\Rightarrow -(3-\lambda)(1+\lambda) + 4 = 0$.

$(0,0)$ is only c.p. $\Rightarrow -(3 + 3\lambda - \lambda - \lambda^2) + 4 = 0$

$\lambda_1 = \lambda_2 = 1 > 0$ $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. $\Rightarrow -3 - 2\lambda + \lambda^2 + 4 = 0 \Rightarrow \lambda^2 - 2\lambda + 1 = 0$.

Tra. $\Rightarrow (\lambda - 1)^2 = 0$.



$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ at $(1,0)$; $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$ at $(0,1)$.