MAT 137
Tutorial #6– Linear Approximation and Newton's method
October 25/26 , 2022
Due on Thursday, Oct 27 by 11:59pm via GradeScope

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

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By signing this document, I agree that the statements above are true.

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1. In this problem we will estimate $\sqrt[3]{1.1}$. **For this question, you are not allowed to use a calculator.**

   (a) Consider the function $h(x) = \sqrt[3]{x}$. Notice that our goal is to compute $h(1.1)$. Write the equation of the line tangent to $y = h(x)$ at the point with x-coordinate 1. We will call this line L.

   given that $h(x) = x^{\frac{1}{3}} \Rightarrow h'(x) = \frac{1}{3} \cdot x^{-\frac{2}{3}}$

   when $x = 1$. gives the point: $(1, h(1))$.

   Thus the tangent line:

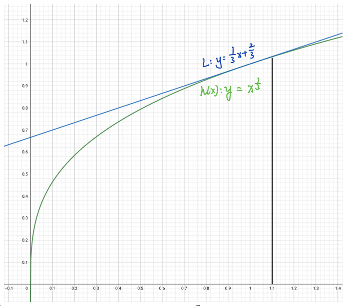
   $$y - h(1) = h'(1) \cdot (x-1).$$

   $$\Rightarrow \quad y = \frac{1}{3} \cdot (x-1) + 1$$

   $$\Rightarrow \quad y = \frac{1}{3}x + \frac{2}{3}$$

   Hence, $L: y = \frac{1}{3}x + \frac{2}{3}$.

   (b) Consider the point on L with x-coordinate 1.1 and the point on the graph of h with x-coordinate 1.1. Are their y-coordinates close to each other? You can use Desmos or Geogebra to graph h and L. Use this to obtain an approximate value for $\sqrt[3]{1.1}$.



   From the graph, it's clear that the y-coordinates of $y = x^{\frac{1}{3}}$, which is $h(x)$ and $y = \frac{1}{3}x + \frac{1}{3}$, which is L are close to each other.

   The approximate value for $\sqrt[3]{1.1}$ might be 1.03.

   (c) Let's say we use the same method (with the same line L) to approximate $\sqrt[3]{1.05}$ and $\sqrt[3]{1.2}$. In which case would we get a smaller error?

   Smaller error when approximating $\sqrt[3]{1.05}$

   (d) Without using the above method, what do you think the approximate value of $\sqrt[3]{28}$ is? Now, using the same line L as above, what value do you get? You will see that this is a very bad approximation. Why didn't it work?

   Without using the method above. the approximate value is around 3.05.

   Using the line L: $y = \frac{1}{3} \times 28 + \frac{2}{3} = 10$

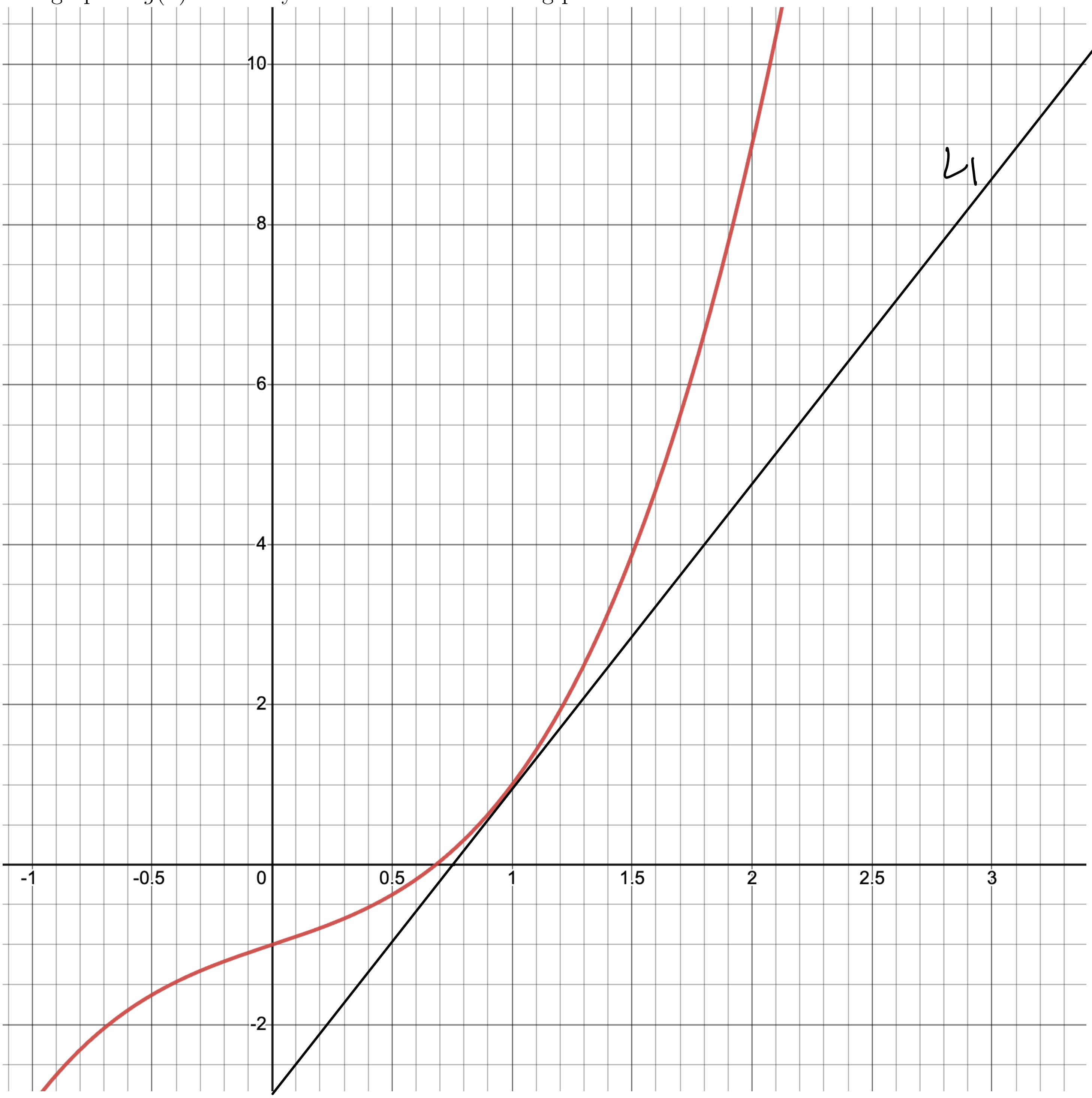
   Because this tangent line is used to approximate when $x = 1.1$.

   the tangent is different of point $(1.1, h(1.1))$ and $(28, h(28))$, making it not working.

2. Now you will try the whole thing by yourself. Use a similar method to the previous problem to obtain an approximate value for $\sqrt{3.9}$. **This question is for your practice and you don't need to return your work.**

3. For this problem, we define the function $g(x) = x^3 + x - 1$. We want to find a number x such that $g(x) = 0$. In other words, we want to solve the equation $x^3 + x - 1 = 0$. You may use a calculator for this question.

The graph of $g(x)$ is already included in the following picture.



(a) Calculate $g(0)$ and $g(1)$. This guarantees there has to be some number $0 < x < 1$ such that $g(x) = 0$. Why? Hint: which theorem we can imply here.

given that, $g(0) = 0+0-1 = -1 < 0$, $g(1) = 1+1-1 = 1 > 0$.

from intermediate value theorem: $f(a) < 0$; $f(b) > 0$. f is continuous on $[a, b]$.

Then $\exists c \in (a,b)$ s.t. $f(c) = 0$.

Therefore, this guarantees there has to be some number $0<x<1$ s.t. $g(x)=0$.

(b) We are going to make a bunch of successive guesses for the solution to the equation. None of them will be exact, but each one will be better than the previous one. Our first guess is going to be $x_1 = 1$. Write the equation of the line tangent to $y = g(x)$ at the point with x-coordinate x_1. We will call it L_1. Draw this line on the picture in the previous page.

when $x = 1$. $g(x) = 1$. $(1, g(1))$; $g'(x) = 3x^2+1$.

$L_1 : y - g(1) = g'(1)(x-1)$.

$\Rightarrow$ $y = 4(x-1)+1$

$\Rightarrow$ $y = 4x-3$

(c) We are looking for the point of the graph $y = g(x)$ that intersects the x-axis. Since this point is not too far from $(x_1, g(x_1))$, we can look for the point where the line L_1 intersects the x-axis instead. (Convince yourself that this makes sense!) Calculate this point. Call its x–coordinate x_2. This is our second guess.

when $y=0$ in L_1:

$0 = 4x-3$.

$\Rightarrow$ $x = \frac{3}{4}$.

$x_2 = \frac{3}{4}$.

(d) Calculate $g(x_2)$. Notice that $g(x_2)$ is not zero yet but it is closer to zero than $g(x_1)$. We are improving!

$g(x_2) = (\frac{3}{4})^3 + \frac{3}{4} - 1 = \frac{11}{64} < 1$.

(e) Now repeat the process you did in the last three steps, but starting with x_2 instead of with x_1. Write the equation of the line tangent to $y = g(x)$ at the point with x-coordinate x_2. We will call it L_2. Draw L_2 on the picture. Call the new value you obtain x_3. Calculate $g(x_3)$. Is it close enough to zero? Then obtain x_4. Is $g(x_4)$ close enough to 0?

given that: $f(a) < 0$, $f(b) > 0$, $f'(x) > 0$, $f''(x) \geq 0$.

Since $f(b)$ and $f''(x)$ both bigger than 0.

take $x_0 = b$. and make a tangent at $(x_0, f(x_0))$

the tangent line: $y - f(x_0) = f'(x)(x - x_0)$.

when $y = 0$. gives.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly. take tanget line of $(x_1, f(x_1))$ can get x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Concluding:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Therefore. $x_3 = x_2 - \frac{g(x_2)}{g'(x_2)}$

$$= \frac{3}{4} - \frac{\frac{11}{64}}{\frac{43}{18}} = \frac{59}{86}.$$

$$x_4 = x_3 - \frac{g(x_3)}{g'(x_3)}$$

$$= \frac{59}{86} - \frac{0.00894}{2.412} = 0.682$$

Keep track of your data:

n	x_n	$g(x_n)$
1	1	1
2	$\frac{3}{4}$	$\frac{11}{64}$
3	$\frac{59}{86}$	0.00894
4	0.682.	−0.000785
5	0.682	−0.000785

4. Now you try the whole thing by yourself. Use a similar method to the previous problem to find a solution to $x^3 - x - 1 = 0$.This question is for your practice and you don't need to return your work.