MAT 137Y – Practice problems
Unit 4 : Transcendental functions

1. Your location is a function of time. What are the domain, the codomain, and the range of this function?

2. Let $f(x) = \dfrac{x+2}{x+1}$.

   (a) What are the domain and range of f?

   (b) Write an explicit equation for $f^{-1}(y)$.

   (c) What are the domain and range of f^{-1}?

   (d) Verify explicitly that $f^{-1}(f(x)) = x$ for every x in the domain of f.

   (e) Verify explicity that $f(f^{-1}(y)) = y$ for every y in the range of f.

3. Let $a \in \mathbb{R}$. Let f be a differentiable function at a. Assume $f'(a) \neq 0$. Let $b = f(a)$. Then you know that f^{-1} is differentiable at b and

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}. \tag{1}$$

   Now assume f has all its derivatives. It follows that $f^{(-1)}$ also has all its derivatives (you do not need to prove this).

   Find equations similar to (1) for $\left(f^{-1}\right)''(b)$ and $\left(f^{-1}\right)'''(b)$ in terms of the derivatives of f evaluated at a. You can do this in at least two ways. Either take derivatives (carefully!) from Equation (1) or differentiate the equation $f(f^{-1}(x)) = x$ multiple times.

4. Compute the derivatives of the following functions:

   (a) $f(x) = e^{3x+1}$

   (b) $f(x) = \ln(\cos x)$

   (c) $f(x) = e^{\tan e^x}$

   (d) $f(x) = x^{\sin x} + x^{\cos x}$

   (e) $f(x) = x^{x^x}$

   (f) $f(x) = \log_x 3$

   (g) $f(x) = \sqrt{1-x^2} + x \arcsin x$

5. In Video 4.14 we left it for you to complete the definition of arccos.

   (a) Give a full definition of arccos

   (b) What are the domain and the range of arccos?

   (c) Sketch its graph

   (d) Complete the statement at time 4:01 in Video 4.14.

6. Imitate the derivation in Video 4.13 to prove that

$$\frac{d}{dt}\left[\arctan t\right] = \frac{1}{1+t^2}$$

7. Compute

   (a) $\arcsin \sin 3$ (b) $\arccos \cos 3$ (c) $\arctan \tan 3$

8. Sketch the graphs of the following functions

   (a) $f(x) = \sin \arcsin x$
   (b) $f(x) = \tan \arctan x$
   (c) $f(x) = \arcsin \sin x$
   (d) $f(x) = \arctan \tan x$

9. Let f and g be functions. For simplicity, assume they both have domain $\mathbb{R}$. Two of the following statements are true, and one is false:

   (a) IF f and g are one-to-one, THEN $g \circ f$ is one-to-one.
   (b) IF $g \circ f$ is one-to-one, THEN g is one-to-one.
   (c) IF $g \circ f$ is one-to-one, THEN f is one-to-one.

   Which one is false? Show it with a counterexample. Which ones are true? Prove them.

10. For each $k \in \mathbb{Z}$, let I_k be the largest interval containing k such that the restriction of sin to I_k is one-to-one, and let α_k be the inverse of that restriction. For example, $\alpha_0 = \alpha_1 = \arcsin$, but α_2 is a different function.

    (a) Sketch the graphs of α_2 and α_6.
    (b) Calculate $\alpha_2(\sin 1)$ and $\alpha_6(\sin 1)$.
    (c) Obtain an equation for the derivatives of α_2 and α_6.
    Note: You should get two different answers.

Bonus question: hyperbolic functions

11. The "hyperbolic sine" (sinh) and the "hyperbolic cosine" (cosh) functions are defined by the equations:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

   (a) Compute $\cosh'(x)$ and $\sinh'(x)$.

   (b) Prove that for all $x \in \mathbb{R}$, $\cosh^2(x) - \sinh^2(x) = 1$.

   (c) The function sinh is one-to-one. (You may assume so). Its inverse function is called "hyperbolic arc sine" (arcsinh). Use a theorem from Video 4.4 to prove that arcsinh is differentiable without doing any calculations.

   (d) Find an explicit formula for $\operatorname{arcsinh}(y)$ by solving for x in the equation $\sinh(x) = y$.

   Note: If you are having trouble finding an expression for the inverse, consider the following easier questions first:

   - Solve for t: $t^2 - 6t + 4 = 0$.
   - Solve for t: $t^2 - 6at + 4 = 0$.
   - Solve for u: $(e^u)^2 - 6\,(e^u) + 4 = 0$.
   - Solve for u: $(e^u)^2 - 6a\,(e^u) + 4 = 0$.

   (e) Use your answer to Question 11d to obtain a formula for $\operatorname{arcsinh}'(y)$.

   (f) There is a faster way to obtain a formula for $\operatorname{arcsinh}'(y)$ without having to obtain an explicit formula for $\operatorname{arcsinh}(y)$ first! Start with the identity

$$\sinh(\operatorname{arcsinh}(y)) = y,$$

   take the derivative with respect to y on both sides, and use Questions 11b and 11a to obtain a formula for $\operatorname{arcsinh}'(y)$. This should agree with your result to Question 11e.

Some answers and hints

2. (a) The domain of f is $(-\infty,-1)\cup(-1,\infty)$. The range of f is $(-\infty,1)\cup(1,\infty)$.
   (b) $f^{-1}(y)=(2-y)/(y-1)$

3.
$$\left(f^{-1}\right)''(b)=\frac{-f''(a)}{\left(f'(a)\right)^3}\,, \qquad \left(f^{-1}\right)'''(b)=\frac{-f'''(a)f'(a)+3\left(f''(a)\right)^2}{\left(f'(a)\right)^5}\,.$$

4. (a) $f'(x)=3e^{3x+1}$
   (b) $f'(x)=-\tan x$
   (c) $f'(x)=e^{x+\tan(e^x)}\ \sec^2\left(e^x\right)$
   (d) $f'(x)=x^{\sin x}\left[\dfrac{\sin x}{x}+\cos x\ln x\right]+x^{\cos x}\left[\dfrac{\cos x}{x}-\sin x\ln x\right]$
   (e) $f'(x)=x^{x^x+x}\left[(\ln x)^2+\ln x+\dfrac{1}{x}\right]$
   (f) $f'(x)=\dfrac{-\ln 3}{x\left(\ln x\right)^2}=\dfrac{-\log_x 3}{x\ln x}$
   (g) $f'(x)=\arcsin x$

5. (a) arccos is the inverse function of the restriction of cos to $[0,\pi]$.
   (b) The domain of arccos is $[-1,1]$. The range is $[0,\pi]$.
   (c) Use desmos to verify your answer.
   (d) For all $0\le x\le\pi$ and for all $-1\le y\le 1$, $\quad x=\arccos y \iff y=\cos x$.

7. (a) $\pi-3$ (b) 3 (c) $3-\pi$

8. All four graphs should be different.
   (a) Analyze the function first when $-\pi/2\le x\le\pi/2$. Then analyze it for all other values.
   (c) Sketch the graph first when $-\pi/2\le x\le\pi/2$. Then when $\pi/2\le x\le 3\pi/2$. Then when $3\pi/2\le x\le 5\pi/2$. Then think of the full graph.

9. (b) is false.

10. (b) $\alpha_2(\sin 1)=\pi-1,\quad \alpha_6(\sin 1)=1+2\pi.$
    (c) $\alpha_2{}'(x)=\dfrac{-1}{\sqrt{1-x^2}},\quad \alpha_6{}'(x)=\dfrac{1}{\sqrt{1-x^2}}.$

11. (a) $\cosh'(x)=\sinh(x),\quad \sinh'(x)=\cosh(x).$
    (d) $\operatorname{arcsinh}(y)=\ln\left(y+\sqrt{1+y^2}\right)$
    (e) $\operatorname{arcsinh}'(y)=\dfrac{1}{\sqrt{1+y^2}}$