2. The Well Ordering Principle (WOP): The Well Ordering Principle states that "any non empty subset of natural numbers **must have** a smallest element." Apply WOP to answer the following questions:

   a) Given a number $n > 1$, prove that the smallest divisor of n (divisor greater than 1) exists, and it is a prime number.

proof: Let $n>1$.

Let the set of divisor of n be A, which $A \subseteq \mathbb{N}$.

WTS: ① the smallest divisor of n exists ② it's a prime number.

Since every $n \in \mathbb{N}$, $n>1$, no matter it's prime or composite has at least 2 divisors which are 1 and itself n, which $A \neq \emptyset$.

Thus, by WOP, A must have a smallest element a_0, since A is non-empty and $A \subseteq \mathbb{N}$.

Assume the smallest element of A, a_0 is not a prime. which besides 1 and a_0 itself, a_0 has other divisors $p \in \mathbb{N}$, $p<a_0$, which we have $\exists k \in \mathbb{Z}$ s.t. $a_0 = k \cdot p$, $p \mid a_0$

However, according to transitivity, $p \mid a_0$ and $a_0 \mid n$ gives $p \mid n$.

Since $p<a_0$, means a_0 is not the smallest divisor of n, we've found the contradiction. ■

   b) Assume *'there cannot exist an infinitely long decreasing sequence of natural numbers'*. Use this assumption to prove the WOP. That is, prove that any **non-empty** set $S \subseteq \mathbb{N}$ must have a smallest element.

   Hint: start a proof by contradiction; assume S has no smallest element. Since $S \neq \emptyset$ choose $s_1 \in S$. Since S has no smallest elements then s_1 cannot be the smallest element of S ... Get a contradiction with the fact that there can't be an infinitely decreasing sequence of natural numbers.

proof: Let $S \subseteq \mathbb{N}$, $S \neq \emptyset$.

I'll prove this by contradiction.

Assume there cannot exist an infinitely long decreasing sequence of natural numbers.

Assume for contradiction: assume S has no smallest element.

Since $S \neq \emptyset$, choose $s_1 \in S$.

Since S has no smallest elements, then s_1 cannot be the smallest element of S, which $\exists s_2 \in S$, $s_2 < s_1$.

Since S has no smallest elements, then s_2 cannot be the smallest element of S, which $\exists s_2 \in S$, $s_3 < s_2$.

Since S has no smallest elements, then s_3 cannot be the smallest element of S, which $\exists s_4 \in S$, $s_4 < s_3$.

Continuing writing will give a sequence S' which $s_1, s_2, s_3, \ldots\ldots$ where the sequence S' is an infinitely long decreasing sequence of natural numbers. contradicts to the assumption.

Therefore, Any non-empty set $S \subseteq \mathbb{N}$ must have a smallest element.

■