MAT 137
Tutorial #4– Proofs of limit
October 11/12 , 2022
Due on Thursday, Oct 13 by 11:59pm via GradeScope

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

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I confirm that:

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- I understand that the collaboration of this worksheet is only among this group. I have not violated this rule while writing this worksheet.
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By signing this document, I agree that the statements above are true.

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1. Let $x, y \in \mathbb{R}$. Prove that if $\forall \varepsilon > 0, |x - y| < \varepsilon$, then $x = y$. Hint: what is the equivalent statement of "If p, then q."?

pf. The equivalent statement of $\forall \varepsilon > 0$, $|x-y| < \varepsilon$. then $x = y$ is:

If $x \neq y$, then $\exists \varepsilon > 0$ s.t. $|x-y| \geq \varepsilon$.

Since $x \neq y$, gives, $x - y \neq 0$. therefore $|x-y| > 0$.

Take $\varepsilon > 0$. There exist $0 < \varepsilon \leq |x-y|$

To conclude. the statement: If $x \neq y$, then $\exists \varepsilon > 0$ s.t. $|x-y| \geq \varepsilon$. is correct.

Hence. if $\forall \varepsilon > 0$. $|x-y| < \varepsilon$. then $x = y$ is correct.

$\blacksquare$

2. Let f be a function. Let $a, M \in \mathbb{R}$. Assume that f is defined on some open interval around a, except maybe at a. Prove the following two statement are equivalent. We have discussed this in our lecture. Now let's write the formal proof. $\lim_{x \to a} f(x) = \infty$

(1) For every $M \in \mathbb{R}$, there exists $\delta > 0$ such that $0 < |x - a| < \delta \implies f(x) > M$.

(2) For every $M \in \mathbb{Z}$, there exists $\delta > 0$ such that $0 < |x - a| < \delta \implies f(x) > M$.

pf. (1) $\Rightarrow$ (2)

Given that $\forall M \in \mathbb{R}$. $\exists \delta > 0$. s.t. $0 < |x-a| < \delta \Rightarrow f(x) > M$.

Since $\mathbb{Z} \subseteq \mathbb{R}$. Take $M \in \mathbb{Z}$, take δ_1 satisfying $0 < |x-a| < \delta$. $\Rightarrow f(x) > M$.

Take $\delta_2 = \delta_1$. gives.

$\forall M \in \mathbb{Z}$, $\exists \delta_2 > 0$. s.t. $0 < |x-a| < \delta_2 \Rightarrow f(x) > M$

(2) $\Rightarrow$ (1).

Given that $\forall M \in \mathbb{Z}$. $\exists \delta > 0$, s.t. $0 < |x-a| < \delta \Rightarrow f(x) > M$.

Let $M_2 \in \mathbb{R}$.

There always exist $M_1 \in \mathbb{Z}$ s.t. $M_1 = \lfloor M_2 \rfloor + 1$.

Take δ_3 satisfying $0 < |x-a| < \delta_3 \Rightarrow f(x) > M_1$.

Therefore. $0 < |x-a| < \delta_3 \Rightarrow f(x) > M_2$.

Take $\delta_4 = \delta_3$. gives.

$\forall M \in \mathbb{R}$, $\exists \delta > 0$. s.t. $0 < |x-a| < \delta \Rightarrow f(x) > M$.

$\blacksquare$

3. Use the formal ε-δ definition of the limit to prove

$$\lim_{x\to 1} \frac{3x+2}{2x+3} = 1$$

Do not use any limit laws or any other theorems.

WTS: $\forall \varepsilon>0, \exists \delta>0$ s.t. $0<|x-1|<\delta \Rightarrow |f(x)-1|<\varepsilon$.

pf. Let $\varepsilon>0$.

Take $\delta = \min\{1, 3\varepsilon\}$

For $x \in \mathbb{R}$. Assume $0<|x-1|<\delta$.

if $\delta=1$. $0<|x-1|<1 \Rightarrow 0<x<2$

$\Rightarrow \left|\frac{1}{2x+3}\right| < \frac{1}{3}$.

if $\delta = 3\varepsilon$. $0<|x-1|<3\varepsilon$.

Thus. $\left|\frac{3x+2}{2x+3} - 1\right| = \left|\frac{x-1}{2x+3}\right|$

$= \left|\frac{1}{2x+3}\right| |x-1| < \frac{1}{3}\delta = \frac{1}{3}\cdot 3\varepsilon = \varepsilon$. ∎

This question is for your practice and you don't need to return your work.

4. Construct a function f with domain $\mathbb{R}$ such that $\lim_{x\to 0} f(x) = 0$ but $\lim_{x\to 0} f((f(x)) \neq 0$.
Hint: you can use geometry to help with your construction.