Generating Function

1. Ordinary Generating Function. ($P(x) = \sum_{n=0}^{\infty} a_n x^n$)

1) $P_A(x) + P_B(x) = \sum_{n=0}^{\infty} (a_n + b_n) \cdot x^n$.

若两个不 $n=0$ 开始，则按照大的开始加，多出来的单独加.

e.g. $\sum_{n=1}^{\infty} a_n x^n + \sum_{n=3}^{\infty} b_n x^n = a_1 x + a_2 x^2 + \sum_{n=3}^{\infty} (a_n + b_n) x^n$.

2) $P_A(x) \cdot P_B(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \cdot b_{n-k} \right) \cdot x^n$.

3). Common G.F.

① $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$; $\frac{1}{1-x^k} = \sum_{n=0}^{\infty} (x^k)^n$.

② $\frac{1}{1-ax} = \sum_{n=0}^{\infty} (ax)^n$.

$\frac{1}{1+ax} = \sum_{n=0}^{\infty} (-ax)^n$.

④ $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) \cdot x^n$.

$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot x^n$

⑤ $\frac{1}{(1-ax)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot (ax)^n$.

$\frac{1}{(1+ax)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot (-ax)^n$.

$S_n = \frac{a(1-r^n)}{1-r}$.

⑤. Finite: $1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$.

⑥ 若不是从第一项开始加.

→ 能提出来，则提出来 e.g. $x^2 + x^3 + \cdots = x^2 \cdot \sum_{n=0}^{\infty} x^n$.

→ 求补全再减.

2. Exponential Generating Function ($P(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$).

1). $P_A(x) \cdot P_B(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} \cdot a_k \cdot b_{n-k} \right) \cdot x^n$

2) Common G.F.

① $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n$.

② $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot x^n$.

③. $e^x + e^{-x} = \sum_{n=0}^{\infty} \frac{1}{2n!} \cdot x^{2n}$. ⟹ $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{e^x + e^{-x}}{2}$

④ $e^x - e^{-x} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \cdot x^{2n+1}$. $\Rightarrow x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{e^x - e^{-x}}{2}$.

← order matters.

3. Exercises. (先判断 OGF / EGF).

1). O.G.F.

① 转换成 $A_1 + \cdots + A_k = p$; $\forall i$, $0 \le A_i \le q$ 要求解 (A_i 代表一个种类).

② 根据 $0 \le A_i \le q$ 列出每个 type 的 G.F. 相乘再写成 Summation 形式.

G.F. 每一项 ax^b: a: # of ways to contribute. b: contribute length.

③ 求对应 a_p.

e.g. 10 types of objects.

① one for each type; ways in total. take 8.

② unlimited; ways in total take 8.

① $\begin{cases} A_1 + \cdots + A_{10} = 8. \\ 0 \le A_i \le 1. \end{cases}$

$\therefore (1+x)^{10} = \sum_{k=0}^{10} \binom{10}{k} \cdot x^k$. $\therefore x^8 \longrightarrow a_8 = \binom{10}{8}$.

② $\begin{cases} A_1 + \cdots + A_{10} = 8. \\ 0 \le A_i. \end{cases}$

$(1+x^2+x^3+\cdots\cdots)^{10} = (\sum_{n=0}^{\infty} x^n)^{10} = (\frac{1}{1-x})^{10} = \sum_{n=0}^{\infty} \binom{n+9}{n} \cdot x^n$.

$\therefore x^8 \longrightarrow a_8 = \binom{17}{8}$.

e.g. Let a_n be # of int. sol^{n} to $9x_1 + x_2 + 3x_3 + x_4 = n$. where $x_1 \ge 0$. $0 \le x_2 \le 5$. $0 \le x_3 \le 2$. $x_4 \ge 0$; Find G.F. for $\{a_n\}$; determine a_n.

$\begin{cases} 9x_1 + x_2 + 3x_3 + x_4 = n. \\ 0 \le x_2 \le 5.\ 0 \le x_3 \le 2.\ x_1, x_4 \ge 0. \end{cases}$

$\underbrace{(1+x^9+x^{18}+\cdots)}_{\frac{1}{1-x^9}} \cdot \underbrace{(1+x+x^2+x^3+x^4+x^5)}_{\frac{1-x^6}{1-x}} \cdot \underbrace{(1+x^3+x^6)}_{\frac{1-(x^3)^3}{1-x^3}} \cdot \underbrace{(1+x+x^2+\cdots)}_{\frac{1}{1-x}}$.

$= \frac{1}{1-x^9} \cdot \frac{1-x^6}{1-x} \cdot \frac{1-x^9}{1-x^3} \cdot \frac{1}{1-x} = \frac{(1+x^3)(1-x^3)}{(1-x)^2(1-x^3)}$.

$= \frac{1+x^3}{(1-x)^2} = (1+x^3) \cdot \frac{1}{(1-x)^2} = (1+x^3) \cdot \sum_{n=0}^{\infty} (n+1) \cdot x^n$.

$$= \sum_{n=0}^{\infty} (n+1)\cdot x^n + \sum_{n=0}^{\infty} (n+1)\cdot x^{n+3}.$$
$$= \sum_{n=0}^{\infty} (n+1)\cdot x^n + \sum_{n=3}^{\infty} (n-2)\cdot x^n.$$
$$= \sum_{n=3}^{\infty} (n+1)\cdot x^n + 1 + 2\cdot x + 3\cdot x^2 + \sum_{n=3}^{\infty} (n-2)\cdot x^n.$$
$$= \sum_{n=3}^{\infty} (2n-1)\cdot x^n + 1 + 2x + 3x^2.$$

$\therefore a_n = 2n-1.$

e.g. Use G.F.: how many four-elements subsets of $S=\{1,2,\dots,15\}$. contain no consecutive integers.

$\underline{1\ \ 2}$ 3 $\underline{4}$ 5 $\underline{6\ \ 7}$ 8 $\underline{9}$ 10 $\underline{11\ \ 12\ \ 13\ \ 14\ \ 15}$.

$$\begin{cases} A_1+A_2+A_3+A_4+A_5 = 11. \\ A_1 . A_5 \geq 0;\ A_2, A_3 . A_4 \geq 1. \end{cases}$$ 没明确说上限可不管.

$(1+x+x^2+\cdots)^2\cdot(x+x^2+\cdots)^3.$

$$\Rightarrow \left(\frac{1}{1-x}\right)^2\cdot\left(\frac{x}{1-x}\right)^3 = \frac{x^3}{(1-x)^5} = x^3\cdot\sum_{n=0}^{\infty}\binom{n+4}{n}x^n = \sum_{n=0}^{\infty}\binom{n+4}{n}x^{n+3}.$$

$\therefore x^{11} \longrightarrow a_n = \binom{12}{8}.$

e.g. Find # of integral sol$^{\text{n}}$ $x_1+x_2+x_3=n$. $-2\leq x_1\leq 2$, $x_2\geq 0$, $x_3\geq 0$. x_2, x_3 are even.

$$\begin{cases} x_1+x_2+x_3 = n. \\ x_2, x_3 \geq 0.\ x_2 . x_3 . \text{ even}.\ -2\leq x_1\leq 2. \end{cases} \Rightarrow \begin{cases} x_1+x_2+x_3 = n+2. \\ 0\leq x_1\leq 4.\ x_2, x_3\geq 0.\ x_2, x_3 \text{ even}. \end{cases}$$

$$\underbrace{(1+x+x^2+x^3+x^4)}_{\frac{1-x^5}{1-x}}\cdot\underbrace{(1+x^2+x^4+\cdots)^2}_{\left(\frac{1}{1-x^2}\right)^2}.$$

$$\Rightarrow \frac{1-x^5}{1-x}\cdot\frac{1}{(1-x^2)^2} \Rightarrow \frac{1-x^5}{(1-x)^3(1+x)^2} =$$

e.g. b_n: # of ternery str. len n. starting with at least one 2 or 3 then followed by zero or more 12 (as a whole). Find G.F. of b_n.

$(2/3)^{\geq 1}(12)^{\geq 0}.$

$(2x+(2x)^2+(2x)^3+\cdots)\cdot(1+x^2+x^4+\cdots).$

$$\overset{\text{II}}{\frac{2x}{1-2x}} \qquad \overset{\text{II}}{\frac{1}{1-x^2}}$$

$$\Rightarrow \frac{2x}{1-2x}\cdot\frac{1}{1-x^2} = \frac{A}{1-2x} + \frac{B}{1+x} + \frac{C}{1-x}$$

$$\Rightarrow \frac{2x}{(1-2x)(1-x^2)} = \frac{A(1-x^2)+B(1-3x+2x^2)+C(1-x-2x^2)}{(1-2x)(1+x)(1-x)}$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ -A+2B-2C=0 \\ -3B-C=2 \end{cases} \Rightarrow \begin{cases} A=\frac{4}{3} \\ B=-\frac{1}{3} \\ C=-1 \end{cases}$$

$$\Rightarrow \frac{\frac{4}{3}}{1-2x} + \left(-\frac{1}{3}\right)\cdot\frac{1}{1+x} + (-1)\cdot\frac{1}{1-x} = \frac{4}{3}\cdot\sum_{n=0}^{\infty} 2^n\cdot x^n - \frac{1}{3}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot x^n - \sum_{n=0}^{\infty} x^n$$

e.g. Let c_n # of ternary str. len. n without 122 substr. Find G.F. c_n.
$\{0,1,2\}$-str.

larger or equal to 0 '2' or '3' + substr. start with '1'.

+ 0 or more '1' '2' '3'.

Considering all substr. start w/ '1'.

① $'1'^{\geq 1} + '3' (+ '2/3')^{\geq 0}$

$$(x+x^2+\cdots)\cdot x\cdot(1+(2x)+(2x)^2+\cdots)$$

$$= \frac{x}{1-x}\cdot x\cdot\frac{1}{1-2x} = \frac{x^2}{(1-x)(1-2x)}$$

② $'1'^{\geq 1} + '2'$

$$(x+x^2+\cdots)\cdot x = \frac{x^2}{1-x}$$

③ $'1'^{\geq 1} + '2' + '3' (+ '2/3')^{\geq 0}$

$$(x+x^2+\cdots)\cdot x^2\cdot(1+(2x)+(2x)^2+\cdots) = \frac{x^3}{(1-x)(1-2x)}$$

Also $(2/3)^{\geq 0}$: $\frac{1}{1-2x}$

$$((1)/(2)/(3))^{\geq 0} : \frac{1}{1-((1)/(2)/(3))} = \frac{1}{1-\frac{2x^2+x^3}{(1-x)(1-2x)}}$$

$$1^{\geq 0} : \frac{1}{1-x}$$

2) E.G.F. (order matters). (去除内部 order).

1) 转换成 str. 问题 : $\{A,B,C,D\}$-str. len n. 每个出现几次/奇偶次.

2). 放球问题. n 个球 ~ k 个盒子. $\longleftrightarrow$ $[k]$ 构造 n len. str.

3). func: $f:[n]\to[k]$. $\longleftrightarrow$ 对 $f([n])$ 贴 k 个标签.

$\longleftrightarrow$ [k] 构造 n len. str.

e.g. # of sequence len 10, 5 diff. sym.

$$\left(1+x+\frac{x^2}{2!}+\cdots\right)^5=(e^x)^5=\sum_{n=0}^{\infty}\frac{1}{n!}\cdot(5x)^n=\sum_{n=0}^{\infty}\frac{5^n}{n!}\cdot x^n.$$

$\therefore x_{10}\longrightarrow a_{10}=5^{10}.$

e.g. # of seq. len. 8.; 1/2/3 'a's; 2/3/4 'b's; 0/2/4 'c's.

$$\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\cdot\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)\cdot\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}\right).=\sum_{n=0}^{\infty}\frac{C_n}{n!}x^n.$$

$8=1+3+4=2+2+4=2+4+2=3+3+2$

$$\therefore\frac{C_8}{8!}=\left(1\cdot\frac{1}{3!}\cdot\frac{1}{4!}\right)+\left(\frac{1}{2!}+\frac{1}{2!}+\frac{1}{4!}\right)+\left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{2!}\right)+\left(\frac{1}{3!}+\frac{1}{3!}+\frac{1}{2!}\right).$$

$\Rightarrow C_8=8!\cdot(\cdots).$

e.g. # of ways to distribute. n obj. into 5 boxes.; ① even # of objects are in box 5.; ② positive even # of obj. into box 5. $\rightarrow(-1)$.

$$\underbrace{\left(1+x+\frac{x^2}{2!}+\cdots\right)^4}_{=(e^x)^4}\cdot\underbrace{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)}_{=\frac{e^x+e^{-x}}{2}}.$$

$$\Rightarrow\frac{1}{2}e^{4x}\cdot e^x+\frac{1}{2}e^{4x}\cdot e^{-x}=\frac{1}{2}e^{5x}+\frac{1}{2}e^{3x}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\cdot(5x)^n+\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\cdot(3x)^n$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}(5^n+3^n)\cdot x^n.$$

$$\therefore C_n=\frac{3^n+5^n}{2}$$

e.g. Find # of len. n. seq. where each digit is from [k]. Also, each # in [p]. is used at least once. where $0<p<k$.

$f:[k]\to[n].\longrightarrow$ 用 $\{1,\dots,k\}$ 构造 n len. str. $0\sim p$ used ≥ 1.

$$\underbrace{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)}_{=e^x-1}{}^{p}\cdot\underbrace{\left(1+x+\frac{x^2}{2!}+\cdots\right)}_{=e^x}{}^{k-p}.$$

$$\Rightarrow(e^x-1)^p\cdot(e^x)^{k-p}=e^{(k-p)x}\cdot\sum_{i=0}^{p}\binom{p}{i}(-1)^i(e^x)^{p-i}$$
$$=\sum_{i=0}^{p}\binom{p}{i}\cdot(-1)^i\cdot e^{(k-i)x}\qquad(\star).$$

Find $\frac{x^n}{n!}$ coeff.:

Since $e^{(k-i)x}=\sum_{n=0}^{\infty}\frac{(k-i)^n}{n!}\cdot x^n$, gives.

$$(\star) = \sum_{i=0}^{p} \binom{p}{i} (-1)^i \cdot \sum_{n=0}^{\infty} \frac{(k-i)^n}{n!} \cdot x^n.$$

$$\therefore \; c_n = \sum_{i=0}^{p} \binom{p}{i} (-1)^i (k-i)^n.$$