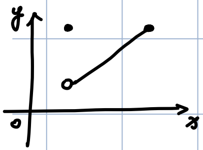
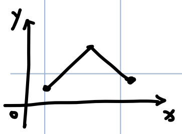
Rolle's Theorem

H①:



H②:



1. Rolle's Theorem.

Let $a < b$. Let f be a function defined on $[a, b]$.

If ① f is continuous on $[a, b]$.

② f is differentiable on (a, b).

③ $f(a) = f(b)$.

Then $\exists c \in (a, b)$. s.t. $f'(c) = 0$

← $[a, b]$ also but more rigorous. we want hypothesis as weak as possible.

1) Proof:

WTS: $\exists c \in (a, b)$. s.t. $f'(c) = 0$.

Assume: H①, H②, H③

Since f is continuous on $[a, b]$

By EVT. f must have a max & min on $[a, b]$.

① max and min at some $c \in (a, b)$.

Then c is also a local extremum.

By Local EVT, $f'(c) = 0$ or DNE.

Since f is differentiable. $f'(c) = 0$.

② max and min at end-points.

Since $f(a) = f(b)$. f must be constant.

$\forall x \in (a, b)$, $f'(x) = 0$.

∎

3. Rolle's Theorem Application. (how many zeros?).

1). A number $c \in \mathbb{R}$ is a zero of a function f when $f(c) = 0$

zero of f = solution of $f(x) = 0$.

2) How many zeros?

①. Use IVT to prove it has at least n.

② Use Rolle's theorem to prove it has at most n.

Assume f is continuous and differentiable everywhere.

Assume $f(x_1) = f(x_2) = 0$.

Use Rolle's Theorem, $\exists c \in (x_1, x_2)$. $f'(c) = 0$.

$\therefore$ Between any two zeros of f, there must be at least one zero of f'.

$\to$ # of zeros of $f' \geq$ # of zeros of $f - 1$.

$\Rightarrow$ # of zeros of $f \leq$ # of zeros of $f' + 1$.

e.g. How many zeros does $g(x) = x^6 + x^2 + x - 2$ have?

① at least:

$x = -2$ $\quad g(x) = 64 > 0$. $\qquad x = 0$. $g(0) = -2 < 0$.

$x = 2$. $\quad g(x) = 68 > 0$.

By IVT, at least 2 zeros.

② at most.

$g'(x) = 6x^5 + 2x + 1$. $\qquad g''(x) = 30x^4 + 2$. always positive.

# of zeros of $g'(x) \leq 0 + 1$.

# of zeros of $g(x) \leq 1 + 1 = 2$.

$\therefore$ 2 zeros for $f(x)$