Learning Objectives

> In this tutorial, you will *practice doing calculations and proofs related to linear transformations and change-of-basis.*

Before attending the tutorial, you should be able to write a complete mathematical definition of the following key words and concepts:

- The $\mathcal{B}$-coordinates of a vector $v \in V$, given a finite-dimensional vector space V and an ordered basis $\mathcal{B}$ of V, and the coordinate isomorphism $\gamma_{\mathcal{B}} : V \to F^{\dim V}$.
- The change of basis matrix $[I]_{\mathcal{A}}^{\mathcal{B}}$ given two bases $\mathcal{A}, \mathcal{B}$ of V.
- The matrix $[T]_{\mathcal{A}}^{\mathcal{B}}$ of a linear transformation $T : V \to W$, given a basis $\mathcal{A}$ of V and a basis $\mathcal{B}$ of W.

The relevant definitions can be found in the textbook Damiano and Little, Chapter 2.7.

Problems

1. Let $A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$, and consider the bases

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

   of the vector space $M_{2\times 2}$ of 2×2 matrices. [1]

   (a) Find $[I_2]_{\mathcal{E}}$ and $[A]_{\mathcal{E}}$. (Recall, for example, $[I_2]_{\mathcal{E}}$ is the coordinate vector of I_2 relative to the ordered basis $\mathcal{E}$ for $M_{2\times 2}$.)

   (b) Find $[I_2]_{\mathcal{B}}$ and $[A]_{\mathcal{B}}$.

   (c) Find a basis $\mathcal{C}$ of $M_{2\times 2}$ such that $[A]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

   (d) Find a matrix C such that $C[B]_{\mathcal{B}} = [B]_{\mathcal{C}}$ for all B in $M_{2\times 2}$.

   (e) Find a matrix D such that $D[B]_{\mathcal{C}} = [B]_{\mathcal{E}}$ for all B in $M_{2\times 2}$.

   (f) Find a matrix F such that $F[B]_{\mathcal{B}} = [B]_{\mathcal{E}}$ for all B in $M_{2\times 2}$.

   (g) Draw a diagram relating the linear transformations corresponding to the matrices F, C and D.

2. Consider the bases $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ and $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$ of $\mathbb{R}^2$. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation defined by $T[v]_{\mathcal{E}} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} [v]_{\mathcal{E}}$.

[1] Part of this problem appeared on Tutorial 4 already, you may use your results from the previous tutorial.

(a) Find the change of basis matrices $C_1 = [I]^{\mathcal{E}}_{\mathcal{B}}$ and $C_2 = [I]^{\mathcal{B}}_{\mathcal{E}}$. Explain how they are related.

(b) Use composition of maps to construct a linear transformation that takes in $[\vec{v}]_{\mathcal{B}}$ as input and gives $[T\vec{v}]_{\mathcal{B}}$ as output.

(c) Compute a matrix for $[T]^{\mathcal{B}}_{\mathcal{B}}$ using your answer in previous part.

(d) Compare your answer in the previous part to the matrix

$$\begin{bmatrix} [T(\begin{bmatrix} 1 \\ 1 \end{bmatrix})]_{\mathcal{B}}, [T(\begin{bmatrix} -1 \\ 1 \end{bmatrix})]_{\mathcal{B}} \end{bmatrix}.$$

3. Let V and W be n and m dimensional F-vector spaces and let $\mathcal{B}$ and $\mathcal{A}$ be bases for V and W respectively. Let $\gamma_{\mathcal{B}} : V \to F^n$ and $\gamma_{\mathcal{A}} : W \to F^m$ denote the coordinate isomorphisms. Let $S : V \to W$ be a linear transformation.

(a) Prove that $\gamma_{\mathcal{B}}$ maps $\operatorname{Ker} S$ onto $\operatorname{null}[S]^{\mathcal{A}}_{\mathcal{B}}$ and $\gamma_{\mathcal{A}}$ maps imS onto $col[S]^{\mathcal{A}}_{\mathcal{B}}$. [2]

(b) Conclude that $\operatorname{null}[S]^{\mathcal{A}}_{\mathcal{B}} \cong (\ker S)$ and $col[S]^{\mathcal{A}}_{\mathcal{B}} \cong imS$ are isomorphisms.

(c) Write a statement that connects rank and nullity of $[S]^{\mathcal{A}}_{\mathcal{B}}$ to the kernel and image of S.

[2] Recall the definitions of nullspace and the column space of a matrix from MAT 223.

Q1. (a) $[I_2]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$[A]_{\mathcal{E}} = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$

(b) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Thus, $[I_2]_B = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

$A = 3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus, $[A]_B = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

(c) $C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$, which.

$[A] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(d) We need to express each basis vector of B in terms of the basis C.

Therefore $C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$

(e) We need to express each basis vector of B in terms of basis C and then in terms of the basis $\mathcal{E}$. And then arrange the result as columns of a matrix.

Thus, $D = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$

(f) To find a matrix F s.t. $F[B]_B = [B]_{\mathcal{E}}$ for all B in $M_{2\times 2}$. we need to express each basis vector of B in terms of the basis $\mathcal{E}$ and then arrange the results as columns of a matrix.

We have $F = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$

(g) We can transform a matrix in $M_{2\times 2}$ to its coordinate with respect to C. using D; then transform those coordinates to their co-ordinates with respect to the basis $\mathcal{E}$ using F. then transform those coordinates back to a matrix in $M_{2\times2}$ using C.

$$M_{2\times2} \xrightarrow{D} \mathbb{R}^4 \xrightarrow{F} \mathbb{R}^4 \xrightarrow{C} M_{2\times2}.$$

Q2. (a) To find the change of basis matrix from $\mathcal{E}$ to B,

$e_1 = \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix} = \frac{1}{2}b_1 - \frac{1}{2}b_2.$

$e_2 = \begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix} = \frac{1}{2}b_1 + \frac{1}{2}b_2$

$C_1 = \begin{bmatrix}[e_1]_B & [e_2]_B\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$ which is the change of basis from $\mathcal{E}$ to B.

$b_1 = \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = e_1 + e_2.$

$b_2 = \begin{bmatrix}-1\\1\end{bmatrix} = -\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = -e_1 + e_2$

$C_2 = \begin{bmatrix}[b_1]_{\mathcal{E}} & [b_2]_{\mathcal{E}}\end{bmatrix} = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ which is the change of basis from B to $\mathcal{E}$.

C_1 and C_2 are inverse matrix which.

$$C_1 = C_2^{-1}$$

(b) • $[\vec{v}]_B \xrightarrow{C_2} [\vec{v}]_{\mathcal{E}}.$

• $T: \mathbb{R}^2 \to \mathbb{R}^2$

• $[T\vec{v}]_{\mathcal{E}} \xrightarrow{C_1} [T\vec{v}]_B$

Therefore, the lin. trans. $S = C_1 \cdot T \cdot C_2.$

(c) To find $[T]^B_B$, we need to compute T applied to each basis vector of B.

$T(b_1) = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\-1\end{bmatrix}.$

$T(b_2) = T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\end{bmatrix}.$

$[T(b_1)]_B = \begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - 2\begin{bmatrix}-1\\1\end{bmatrix} = b_1 - 2b_2$

$[T(b_2)]_B = \begin{bmatrix}1\\-1\end{bmatrix} = -\begin{bmatrix}-1\\1\end{bmatrix} = 0b_1 - b_2$

Therefore. $[T]^B_B = \begin{bmatrix}1 & 0\\ -2 & -1\end{bmatrix}.$

(d). We have $[T(b_1)]_B = \begin{bmatrix}1\\-2\end{bmatrix}.$ $[T(b_2)]_B = \begin{bmatrix}0\\-1\end{bmatrix}.$

Therefore $\left[\left[T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\right]_B, \left[T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)\right]_B\right] = \begin{bmatrix}1 & 0\\ -2 & -1\end{bmatrix}$, which confirm the answer for (c).

Q3. (a). ① Let $\vec{v} \in Ker(S)$, which $S(\vec{v}) = 0$.

Thus $r_A(S(\vec{v})) = 0$, which gives $S(\vec{v})$ is in $null(S)^A$.

According to $r_B : V \to F^n$ denote the coordinate isomorphisms.

$r_B(\vec{v}) = [\vec{v}]_B$, since $S(\vec{v}) = 0$, we have $r_A(S(\vec{v})) = [S(\vec{v})]_A = 0$

Therefore, $[\vec{v}]_B \in null[S]^A_B$.

Hence r_B maps $Ker S$ onto $null[S]^A_B$.

② Let w be in $im S$.

Then, $\exists\, \vec{v} \in V$ s.t. $S(\vec{v}) = w$

According to definition of r_A and r_B.

we have $r_A(\vec{w}) = [\vec{w}]_A = [S\vec{v}]_A = [S]^A_B [r_B(\vec{v})]$.

Therefore, $[\vec{w}]_A$ is in $col[S]^A_B$.

Hence r_A maps $im S$ onto $col[S]^A_B$.

(b). Since r_B maps $Ker S$ onto $null[S]^A_B$, we have

$\dim(Ker S) = \dim(null[S]^A_B)$. by rank-nullity theorem.

Also, r_A maps $im S$ onto $col[S]^A_B$, gives.

$\dim(im S) = \dim(col[S]^A_B)$.

Therefore, $null[S]^A_B$ and $col[S]^A_B$ are isomorphic to $Ker S$ and $im S$, respectively.

(c). $rank([S]^A_B) = \dim(col[S]^A_B) = \dim(im S)$

$nullity([S]^A_B) = \dim(null[S]^A_B) = \dim(Ker S)$.