2^{nd}-order Non-Homogeneous ODE

1. Core points:

1). $Y_1(t) - Y_2(t) = C_1 y_1(t) + C_2 y_2(t)$. ← General solⁿ to Homo.

2). $y = C_1 y_1(t) + C_2 y_2(t) + Y(t)$. ← General solⁿ to non-Homo

2. Solⁿ to $ay'' + by' + cy = g(t)$.

1). Steps:

①. 解对应 homo.

②. 找到 non-homo 的一个解 $(y_p(t))$: Method of Undetermined Coefficient. ← 特定情况下使用.

→ 根据 $g(t)$ 组成方式来确定 $y_p(t)$ 组成方式.

☆ $g(t)$ 为单独 Exp. / Tri. / poly: 直接写对应.

☆ $g(t)$ 为三种相加/减: 对应相加/减

☆ $g(t)$ 为三种乘: 对应相乘. → 遇到除法用 variation.

→ 根据 $g(t)$ 形式来确定 $y_p(t)$ 形式

☆. $g(t)$ 有 e^{ax}.

(i) e^{at} not a homo solⁿ: $y_p(t) = Ae^{at}$.

(ii) e^{at} is a homo solⁿ: $y_p(t) = A \cdot t \cdot e^{at}$

(iii) e^{at} and $t \cdot e^{at}$ are both homo solⁿ: $y_p(t) = A \cdot t^2 \cdot e^{at}$.

$g(t)$ 中 e^{ax} 部分. e^{ax} 等于几个 homo 的解就乘 t 的几次方.

☆ $g(t)$ 有 $e^{\alpha t}\sin(\beta t)$ or $e^{\alpha t}\cos(\beta t)$.

若只有 sin / cos 为特殊情况, 即 $\alpha = 0$. $r = \pm\beta i$.

(i) $e^{\alpha t}\sin(\beta t)$ or $e^{\alpha t}\cos(\beta t)$. 为 not homo solⁿ: $y_p(t) = e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t))$.

(ii) $e^{\alpha t}\sin(\beta t)$ or $e^{\alpha t}\cos(\beta t)$. 为 homo solⁿ: $y_p(t) = t \cdot e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t))$.

因 complex root 不可能相等, 不然变成 IR 3.

☆ $g(t)$ 有 poly

(i). poly: $a_n t^{\boxed{n}} + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$; then $y_p(t) = A_n t^{\boxed{n}} + \cdots + A_1 t + A_0$.

最高次相等.

e.g. $y'' - 3y' - 4y = 2\sin t$.

①Solve Homo solⁿ.

Solve for homo: $y'' - 3y' - 4y = 0$.

$\Rightarrow r^2 - 3r - 4 = 0. \quad \Rightarrow (r-4)(r+1) = 0.$

$r_1 = 4.\ r_2 = -1$; where. $y_1(t) = e^{4t}$ $y_2(t) = e^{-t}$; $g(t)$ is not sol^n.

② Based on $g(t)$, write $y_p(t)$.

$g(t) = 2\sin t$, which $\alpha = 0$. $\beta = 1$.

Let $y_p(t) = e^{\alpha t} \cdot (A\cos(\beta t) + B\sin(\beta t))$. $= A\cos t + B\sin t$.

③ Calculate $y_p'(t)$, $y_p''(t)$ and substitute to solve.

$y_p'(t) = -A\sin t + B\cos t$; $y_p''(t) = -A\cos t - B\sin t$.

$-A\cos t - B\sin t - 3(-A\sin t + B\cos t) - 4\cdot(A\cos t + B\sin t) = 2\sin t$.

$\Rightarrow \sin t \cdot (-B + 3A - 4B) + \cos t(-A - 3B - 4A) = 2\sin t$.

$$\begin{cases} 3A - 5B = 2. \\ -5A - 3B = 0. \end{cases} \Rightarrow \begin{cases} A = \frac{3}{17}. \\ B = -\frac{5}{17} \end{cases}$$

$\therefore y_p(t) = \frac{3}{17}\cos t - \frac{5}{17}\sin t$.

$y(t) = C_1 e^{4t} + C_2 e^{-t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t$.

e.g. $y'' + 2y' = 3 + 4\sin(2t)$.

e.g. $y'' + y = 3\sin(2t) + t\cos(2t)$.

3. Variation of Parameters.（第2个用来找 $y_p(t)$）.

1) $y_p(t) = -y_1(t)\int \frac{y_2(t)\cdot g(t)}{W[y_1,y_2](t)}dt + y_2(t)\cdot\int \frac{y_1(t)\cdot g(t)}{W[y_1,y_2](t)}dt$.

2) General sol$^{\underline{n}}$: $y(t) = u_1y_1 + u_2y_2$; $u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}dt + C_1$.

$$u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}dt + C_2.$$

e.g. $y'' + 4y = 8\tan(t)$. for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

① 找 y_1, y_2, W.

$r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i \qquad \alpha = 0.\ \beta = 2.$

$\therefore y_1(t) = \cos(2t).\ y_2(t) = \sin(2t).$

$\therefore W[y_1,y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t$

$= 2.$

② 找 u_1, u_2.

$$u_1 = -\int \frac{y_2\cdot g}{W}dt = -\int \frac{\sin 2t\cdot 8\cdot\tan t}{2}dt = -4\int 2\sin t\cos t\cdot\frac{\sin t}{\cos t}dt$$

$$= -4\int 2\sin^2 t\,dt = -4\int 1-\cos 2t\,dt$$

$$= -4(t - \tfrac{1}{2}\sin 2t) + C$$

$$= 2\sin 2t - 4t.$$

$$u_2 = \int \frac{y_1\cdot g}{W}dt = \int \frac{\cos 2t\cdot 8\cdot\tan t}{2}dt = 4\int(2\cos^2 t - 1)\cdot\frac{\sin t}{\cos t}dt$$

$$= 4\int 2\cos t\sin t - \frac{\sin t}{\cos t}dt$$

$$= 4\left(\int \sin(2t)\,dt + \int \frac{1}{u}du\right)$$

$$= 4\cdot\left(-\tfrac{1}{2}\cos(2t) + \ln|\cos t|\right).$$

$$= -2\cos(2t) + 4\ln|\cos t|.$$

$\therefore y(t) = u_1y_1 + u_2y_2 = \cdots$