Problem Set 4

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Q2:

(a).

```python
# (a)
# Precondition: x is in R, y is in N
# Postcondition: return x to the power of y(return 1 if x = y = 0)
new *
def Pow1(x, y) -> float:
    """
    Precondition: x is in R, y is in N
    Postcondition: return x to the power of y(return 1 if x = y = 0)

    :param x: x is in R, which is a real number
    :param y: y is in N, which is a natural number
    :return: x to the power of y
    """
    # Precondition: xi is in R, yi is in N, zi is in R
    # Postcondition: return x to the power of y(return 1 if x = y = 0)
    new *
    def r(xi, yi, zi) -> float:
        """
        Precondition: xi is in R, yi is in N, zi is in R, zi times xi to the yi equals x to the power of y
        Postcondition: return x to the power of y(return 1 if x = y = 0)
        Initial Arguments: xi = x, yi = y, zi = 1

        :param xi: xi is in R, which is a real number
        :param yi: yi is in N, which is a natural number
        :param zi: xi is in R, which is a real number
        :return: return x to the power of y
        """
        if yi == 0:
            return zi
        if yi % 2 == 1:
            zi *= xi
        return r(xi*xi, yi//2, zi)
    return r(x, y, zi: 1)
```

Precondition: $y_i \in \mathbb{N}$, $x_i, z_i \in \mathbb{R}$, $z_i \cdot x_i^{y_i} = x^y$.

(b)

```python
# (b)
# Precondition: x is in R, y is in N
# Postcondition: return x to the power of y (return 1 if x = y = 0)
new *
def Pow2(x, y) -> float:
    """
    Precondition: x is in R, y is in N
    Postcondition: return x to the power of y (return 1 if x = y = 0)

    :param x: x is in R, which is a real number
    :param y: y is in N, which is a natural number
    :return: x to the power of y (return 1 if x = y = 0)
    """
    xi, yi, zi = x, y, 1
    return r(xi, yi, zi)

# Precondition: xi is in R, yi is in N, zi is in R
# Postcondition: return zi time xi to the power of yi
new *
def r(xi, yi, zi) -> float:
    """
    Precondition: xi is in R, yi is in N, zi is in R
    Postcondition: return zi time xi to the power of yi

    :param xi: xi is in R, which is a real number
    :param yi: yi is in N, which is a natural number
    :param zi: zi is in R, which is a real number
    :return: zi times xi to the power of yi
    """
    if yi == 0:
        return zi
    if yi % 2 == 1:
        zi *= xi
    return r(xi * xi, yi // 2, zi)
```

WTS: correctness of function r.

Let the predicate $P(n)$: If the precondition listed above holds, the function $r(x_i, n, z_i)$ where $n = y_i$ will terminate and return $z_i \cdot x_i^{y_i}$, satisfying the postcondition.

I'll prove by complete induction on n.

Base Case: $n=0$.

When $n=0$, since $n=y_i$, gives $y_i=0$.

Since $y_i==0$, the function goes into the first 'if' statement and return z_i.

Since $z_i = z_i \cdot x_0^0 = z_i \cdot x_0^{y_0}$, the function terminates and satisfy the post condition.

Inductive Step: Let $n \in \mathbb{N}$, $n \geq 0$.

Inductive Hypothesis: $\forall k \in \mathbb{N}$, $0 \leq k < n$, $P(k)$.

WTS: $P(n)$, where $n = y_i$.

Case 1: n is even, which y_i is even.

Since y_i is even, $y_i // 2 = \frac{y_i}{2} < y_i$.

Since $0 \leq \frac{y_i}{2} < y_i = n$, by I.H., gives, the return is $z_i \cdot (x_i^2)^{\frac{y_i}{2}}$, which is $z_i \cdot x_i^{y_i}$, gives the function terminates and satisfy the postcondition as it doesn't go to the 'if' branch and calls for $r(x_i^2, y_i // 2, z_i)$.

Case 2: n is odd, which y_i is odd.

Since y_i is odd, $y_i // 2 = \lfloor \frac{y_i}{2} \rfloor = \frac{y_i - 1}{2} < y_i = n$.

Since $0 \leq \frac{y_i - 1}{2} < n$, and if $y_i = 1$, which $\frac{y_i - 1}{2} = 0$, the base case is shown correct, by I.H., since it calls $r(x_i^2, y_i // 2, z_i \cdot x_i)$, which returns $x_i \cdot z_i \cdot (x_i^2)^{\frac{y_i - 1}{2}} = z_i \cdot x_i \cdot (x_i)^{y_i - 1} = z_i \cdot x_i^{y_i}$, satisfies the postcondition and the function terminates.

Thus, $P(y_i)$ is correct when $P(n)$ is correct.

(c)

```python
# (c)
# Precondition: x is in R, y is in N
# Postcondition: return x to the power of y (return 1 if x = y = 0)
2 usages    new *
def PowR(x, y) -> float:
    """
    Precondition: x is in R, y is in N
    Postcondition: return x to the power of y (return 1 if x = y = 0)

    :param x: x is in R, which is a real number
    :param y: y is in N, which is a natural number
    :return: x to the power of y (return 1 if x = y = 0)
    """
    if y == 0:
        return 1
    if y % 2 == 0:
        return PowR(x*x, y//2)
    else:
        return x * PowR(x*x, y//2)
```

WTS: correctness of PowR

Let the Predicate $P(n)$: If the precondition stated above holds, the function PowR(x, n), where $n = y$ will terminate and return x^y.

I'll prove by apply complete induction on n.

Base Case: $n = 0$.

Since $n = 0$, $n = y$, gives $y = 0$.

From line 85. the function will proceed into the first 'if' statement, where the return will be 1, the function terminates.

Since $1 = x^0 = x^y$ when $y=1$. gives the function satisfies the postcondition I've shown the base case is True.

Inductive Step. Let. $n \in \mathbb{N}$, $n \geq 0$.

Inductive Hypothesis: Let $k \in \mathbb{N}$, $0 \leq k < n$, $P(k)$. holds.

WTS: $P(n)$ holds.

Case 1: n is even, which y is even.

Since y is even. $y // 2 = \lfloor \frac{y}{2} \rfloor = \frac{y}{2}$

Since y is even. $y \% 2 == 0$. it will call $PowR(x^2, \frac{y}{2})$.

Since $y \in \mathbb{N}$ (in the precondition), $\frac{y}{2} \in \mathbb{N}$, also, $\frac{y}{2} < y$, as $y \geq 0$.

By I.H., PowR will return $(x^2)^{\frac{y}{2}} = x^y$ which satisfies the postcondition and the function terminates.

Case 2: n is odd. which y is odd.

Since y is odd, $y // 2 = \lfloor \frac{y}{2} \rfloor = \frac{y-1}{2}$. and $y \% 2 == 1$ which goes into the 'else' statement. calling $x \cdot PowR(x^2, \frac{y-1}{2})$.

When $y=1$. $\frac{y-1}{2} = 0$, which calling the $x \cdot PowR(x^2, 0)$ is the base case, gives $x \cdot 1 = x = x^1 = x^y$ as $y=1$.

Otherwise, since $\frac{y-1}{2} < y$. and $y \in \mathbb{N}$. gives $\frac{y-1}{2} \in \mathbb{N}$, by I.H. gives the return will be $(x^2)^{\frac{y-1}{2}} = x^{y-1}$.

Thus, times the x, gives $x \cdot x^{y-1} = x^y$ satisfies the postcondition and the function terminates.

Therefore. $P(y)$ holds, which $P(n)$ holds. ∎