2 Question 2

```
   3 usages   Henry-wxq +1
1  def q_2(n: int, x: float) -> float:
2      """Implement a Python function with parameters x and n that (ignoring floating-point issues) returns c_n.
3
4      Precondition:
5      1. x represents a non-zero real number.
6      2. n is a natural number
7      """
8      # Since c_1 is used in both n = 1 and recursion, I will put it at the front.
9      c_1 = x + 1 / x
10
11     if n == 0:
12         # From the definition of c_n, when n is 0, return the corresponding c_0
13         return 2
14     elif n == 1:
15         # From the definition of c_n, when n is 1, return the corresponding c_1
16         return c_1
17     else:
18         """This is the recursion part. According to the discovery from hint which will be stated below, I come up with a
19         general function for c_n.
20         """
21         # Aim at returning the recursive value of c for n minus 1 after reaching the case when n equals to 1.
22         c_minus1 = q_2(n-1, x)
23         # Aim at returning the recursive value of c for n minus 2. Since we don't know whether n is an even number or
24         # an odd number, we need to add both n equals to 0 and n equals to 1 to our base case.
25         c_minus2 = q_2(n-2, x)
26         # Calculate the c_n based on the discovery.
27         c_n = c_1 * c_minus1 - c_minus2
28         return c_n
```

Figure 1: Python function for Q2-a

(a) The above code is the Python function with parameter x and n that (ignoring floating-point issues) returns c_n, the comments are both in the code above and below.

Firstly, I clearly stated the pre-conditions on x and n in a header comment, which $x \in \mathbb{R}/\{0\}$ and $n \in \mathbb{N}$.

Secondly, at line 9, I write the calculation of c_1 because it will be used in both 'elif' statement at line 14 and 'else' statement at line 17, avoiding redundancy.

Thirdly, I implemented the based case when n equals to 0 and n equals to 1 according to the definition of c_n.

Fourthly, I implemented the recursion based on the discovery from hint.

$$
\begin{aligned}
(x + \frac{1}{x}) \cdot (x^n + \frac{1}{x^n}) &= x^{n+1} + \frac{1}{x^{n-1}} + x^{n-1} + \frac{1}{x^{n+1}} \\
&= (x^{n+1} + \frac{1}{x^{n+1}}) + (x^{n-1} + \frac{1}{x^{n-1}})
\end{aligned}
$$

According to the definition of c_n, gives:

$$
\begin{aligned}
& c_1 \cdot c_n = c_{n+1} + c_{n-1} \\
\implies & c_{n+1} = c_1 \cdot c_n - c_{n-1}
\end{aligned}
$$

Thus, we generalize the above equation into: $c_n = c_1 \cdot c_{n-1} - c_{n-2}$, which is the core of our recursive part, at line 27.

Fifthly, aiming at returning the recursive value of c_{n-1} after reaching the case when n eqials to 1, I write the code line 22. Aiming at return the recursive value of c_{n-2}, I write the code at line 29. Since we don't know whether n is an even number or an odd number, we need to add both n equals to 0 and n equals to 1 to our base case at line 11 and at line 14.

Finally, we can obtain the c_n using the recursive function without use any loops, or any helper functions, nor call any exponentiation functions.

(b) To state a recurrence for the sequence c, I will start from $n = 0$, which $c_0 = x^0 + \frac{1}{x^0} = 2$. Then I will goes to $n = 1$, which $c_1 = x + \frac{1}{x}$.

Moreover, for $n \geq 2$, from the hint, we have:

$$\begin{aligned}(x + \frac{1}{x}) \cdot (x^n + \frac{1}{x^n}) &= x^{n+1} + \frac{1}{x^{n-1}} + x^{n-1} + \frac{1}{x^{n+1}} \\ &= (x^{n+1} + \frac{1}{x^{n+1}}) + (x^{n-1} + \frac{1}{x^{n-1}})\end{aligned}$$

According to the definition of c_n, gives:

$$\begin{aligned} & c_1 \cdot c_n = c_{n+1} + c_{n-1} \\ \implies & c_{n+1} = c_1 \cdot c_n - c_{n-1}\end{aligned}$$

Thus, we generalize the above equation into: $c_n = c_1 \cdot c_{n-1} - c_{n-2}$.

Therefore, we have:

$$c_n = \begin{cases} 2 & \text{for} \quad n = 0 \\ x + \frac{1}{x} & \text{for} \quad n = 1 \\ c_1 \cdot c_{n-1} - c_{n-2} & \text{for} \quad n \geq 2 \end{cases}$$

(c) If $x + \frac{1}{x}$ is an integer, then $x^n + \frac{1}{x^n}$ is an integer for each $n \in \mathbb{N}$

Assume $x + \frac{1}{x}$ is an integer.

Given statement to prove: $\forall n \in \mathbb{N},\ P(n)$, which $P(n)$: $x^n + \frac{1}{x^n}$ is an integer where $x \in \mathbb{R}/\{0\}$

Proof: We prove this by complete induction on n.

Base Case: Let $0 \leq n \leq 1$

For $n = 0$, we have $x^0 + \frac{1}{x^0} = 2$ is an integer, which P(0) is True.

By assumption, $x + \frac{1}{x}$ is an integer, which P(1) is True.

We've proved that $P(0)$ & $P(1)$ is true.

Induction Step: Let $n > 1$

Induction Hypothesis: Assume $\forall k,\ 1 \leq k < n,\ P(k)$

WTS: $P(n)$

From induction hypothesis, when $k_1 = 1,\ 1 \leq k_1 < n$, $P(1)$ is true, which $x + \frac{1}{x}$ is an integer.

From induction hypothesis, when $k_{n-1} = n-1,\ 1 \leq k_{n-1} < n$, $P(n-1)$ is true, which $x^{n-1} + \frac{1}{x^{n-1}}$ is an integer.

From induction hypothesis, when $k_{n-2} = n-2,\ 1 \leq k_{n-2} < n$, $P(n-2)$ is true, which $x^{n-2} + \frac{1}{x^{n-2}}$ is an integer.

Thus, we obtain that

$$
\begin{aligned}
&(x + \frac{1}{x}) \cdot (x^{n-1} + \frac{1}{x^{n-1}}) - (x^{n-2} + \frac{1}{x^{n-2}}) \text{ is an integer.}\\
&= x^n + \frac{1}{x^{n-2}} + x^{n-2} + \frac{1}{x^n} - x^{n-2} - \frac{1}{x^{n-2}}\\
&= x^n + \frac{1}{x^n} \text{ is an integer}
\end{aligned}
$$

I've proved that $P(n)$ is true.

To conclude, I've proved that if $x + \frac{1}{x}$ is an integer, then $x^n + \frac{1}{x^n}$ is an integer for each $n \in \mathbb{N}$.

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