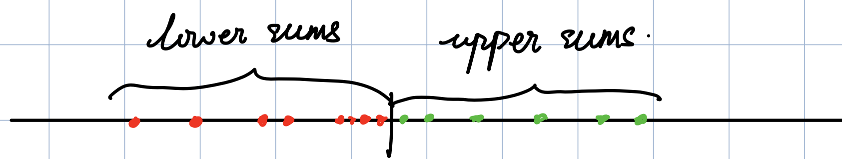
Integrability

1. Integrable

① Let f be a bounded function on $[a,b]$.

When $\underline{I}_a^b(f) = \overline{I}_a^b(f)$, we say that f is integrable on $[a,b]$ and that

$$\int_a^b f(x)\,dx = \underline{I}_a^b(f) = \overline{I}_a^b(f)$$

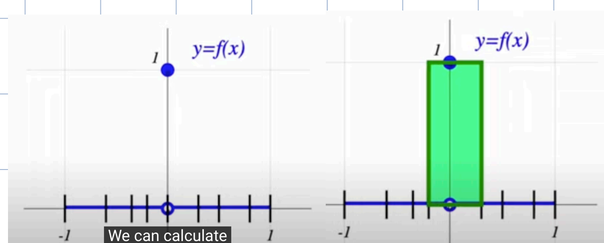


e.g. $f(x) = \begin{cases} 1 & \text{if } x=0. \\ 0 & \text{if } x \neq 1. \end{cases}$ on $[-1,1]$?

$\underline{I}_{-1}^{1}(f) = \sup\{\text{lower sums of } f\} = \sup\{0\} = 0$

$\overline{I}_{-1}^{1}(f) = \inf\{\text{upper sums of } f\} = \inf(0, 2] = 0.$

$2\times1.$



②

If f is a continuous function on $[a,b]$, then f is integrable on $[a,b]$

2. Non-integrable

Let f be a bounded function on $[a,b]$.

When $\underline{I}_a^b(f) < \overline{I}_a^b(f)$, we say f is non-integrable on $[a,b]$ and $\int_a^b f(x)\,dx$ is undefined.

Dirichlet Function

e.g. $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ on $[0,1]$?

Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[0,1]$.
inf of g on every subinterval $[x_{i-1}, x_i]$ is 0.
sup of g on every subinterval $[x_{i-1}, x_i]$ is 1.
$L_P(f) = 0$; $U_P(f) = 1$.
$\underline{I}_0^1(g) = \sup\{\text{lower sums of } g\} = \sup\{0\} = 0$
$\overline{I}_0^1(g) = \inf\{\text{upper sums of } g\} = \inf\{1\} = 1$.
Since $\underline{I}_0^1(g) \neq \overline{I}_0^1(g)$, g is not integrable on $[a,b]$.