Derived Properties of Probabilities

A^c: the set of all outcomes which are not in A.

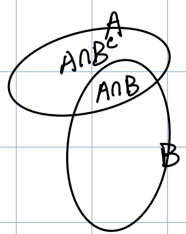
1. Fact: If A^c is the complement of A, the $P(A^c) = 1 - P(A)$.

proof: Since A and A^c are disjoint, gives.

$$P(A \cup A^c) = P(A) + P(A^c).$$

Since $P(A \cup A^c) = P(S) = 1$, $1 = P(A) + P(A^c)$. i.e.

$$P(A^c) = 1 - P(A)$$



2. Fact: For any events A and B, $P(A) = P(A \cap B) + P(A \cap B^c)$

proof: Since event $A \cap B$ and $A \cap B^c$ are disjoint, and $(A \cap B) \cup (A \cap B^c) = A$ according to the diagram.

By additivity, $P((A \cap B) \cup (A \cap B^c)) = P(A \cap B) + P(A \cap B^c)$
$= P(A)$.

当看到一个交另一个的complement时应当想到①.进行变形.

1) Rearranging also gives: $P(A \cap B^c) = P(A) - P(A \cap B)$.

2) Fact: If $A \supseteq B$, then $P(A) = P(B) + P(A \cap B^c)$.

$\hookrightarrow P(A \cap B^c) = P(A) - P(B)$.

proof: Since when $A \supseteq B$, $A \cap B = B$.

when $A \cap B^c = \phi$, $P(A') = 0$.

3. Monotonicity: If $A \supseteq B$, then $P(A) \geq P(B)$.

proof: Since $A \supseteq B$. $P(A) = P(B) + P(A \cap B^c)$; Since $P(A \cap B^c) \geq 0$, gives.

$P(A) = P(B) + P(A \cap B^c) \geq P(B) + 0 \geq P(B)$.

4. Law of Total Probability (Uncondition): Suppose $A_1, A_2, \ldots$, are a sequence (finite or infinite) of events which form a partition of S. i.e. they are disjoint ($A_i \cap A_j = \phi$ for all $i \neq j$) and $\cup_i A_i = S$. Let B be any event, gives.

$$P(B) = \Sigma_i P(A_i \cap B), \text{ which } P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots$$

proof: Since $\{A_i\}$ are disjoint and $A_i \cap B \subseteq A_i$, gives $\{A_i \cap B\}$ are disjoint

Since $\{A_i \cap B\}$ are disjoint, according to the addlitivity gives,

$$P(\cup_i (A_i \cap B)) = \Sigma_i P(A_i \cap B).$$

不能一个个与B交再加，要考虑先加再与B交.

Since $\bigcup_i A_i = S$, gives $\bigcup_i (A_i \cap B) = S \cap B = B$, gives.

$P(B) = P(\bigcup_i (A_i \cap B)) = \sum_i P(A_i \cap B)$.

5. Principle of Inclusion - Exclusion: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

proof: From the diagram gives, $A \cap B$, $A \cap B^c$, $B \cap A^c$ are disjoint, and their union is $A \cup B$, gives.

$$P(A \cup B) = P(A \cap B) + P(A \cap B^c) + P(B \cap A^c).$$

Since $P(A \cap B^c) = P(A) - P(A \cap B)$; $P(B \cap A^c) = P(B) - P(B \cap A)$, gives.

$P(A \cup B) = P(A \cap B) + (P(A) - P(A \cap B)) + (P(B) - P(B \cap A))$, gives.

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

1) When A and B are disjoint. $P(A \cup B) = P(A) + P(B)$ (as $A \cap B = \phi$).

2). Since $P(A \cap B) \geq 0$, gives. $P(A \cup B) \leq P(A) + P(B)$.

6. Subadditivity: For any sequence of events $A_1, A_2, \dots$, not neccessarily disjoint, we have $P(A_1 \cup A_2 \cup A_3 \cup \cdots) \leq P(A_1) + P(A_2) + P(A_3) + \cdots$.

will be equal when they are disjoint.

proof①: Let $B_1 = A_1$;

Let $B_2 = A_2 \cap A_1^c$

Let $B_3 = A_3 \cap (A_1 \cup A_2)^c$

$\vdots$

☆: construct a disjoint set.

Let $B_i = A_i \cap (A_1 \cap A_2 \cap \dots \cap A_{i-1})^c$, from the diagram, $\{B_i\}$ are disjoint, and $\bigcup_i B_i = \bigcup_i A_i$; $B_i \subseteq A_i$, which $P(B_i) \leq P(A_i)$.

Since $\{B_i\}$ are disjoint, $P(\bigcup_i B_i) = \sum_i P(B_i)$, which.

$P(\bigcup_i A_i) = P(\bigcup_i B_i) = P(B_1) + \cdots + P(B_i) \leq P(A_1) + \cdots + P(A_i)$.

Induction.

proof②: (for a finite number of events).

B.C. when $n=2$.

$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$ according to inclusion-exclusion.

I.S. Let $n \in \mathbb{N}$, $n \geq 2$.

I.H. Assume $P(A_1 \cup A_2 \cup \cdots \cup A_i) \leq P(A_1) + P(A_2) + \cdots + P(A_i)$ holds.

WTS. $P(A_1 \cup A_2 \cup \cdots \cup A_i \cup A_{i+1}) \leq P(A_1) + P(A_2) + \cdots + P(A_i) + P(A_{i+1})$ holds.

$P((A_1 \cup A_2 \cup \cdots \cup A_i) \cup A_{i+1}) \leq P(A_1 \cup A_2 \cup \cdots \cup A_i) + P(A_{i+1})$ acc. I.E.

From I.H. gives,

$P(A_1 \cup A_2 \cup \cdots \cup A_i \cup A_{i+1}) \leq P(A_1) \cup P(A_2) \cup \cdots \cup P(A_i) \cup P(A_{i+1})$.

$\blacksquare$