Chapter 1 Solution

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Solution to Exercise 1.1

Part 1: Prove that $B_1, B_2, \ldots$ are disjoint.

Let us consider B_i and B_j where $i < j$. By the definition of B_j, any $\omega \in B_j$ satisfies $\omega \notin A_i$. On the other hand, by the definition of B_i, any $\omega \in B_i$ satisfies $\omega \in A_i$. Therefore, $\omega \notin B_i$ if $\omega \in B_j$. Similarly, $\omega \notin B_j$ if $\omega \in B_i$. Hence, $B_i \cap B_j = \emptyset$ for $i \neq j$, proving that $B_1, B_2, \ldots$ are disjoint.

Part 2: Prove that $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$.

1. **Direction 1:** $\bigcup_{i=1}^n B_i \subseteq \bigcup_{i=1}^n A_i$ For any element $\omega \in \bigcup_{i=1}^n B_i$, it must belong to exactly one B_i because of the disjoint property proved above. By the definition of B_i, this implies $\omega \in A_i$. Therefore, $\omega \in \bigcup_{i=1}^n A_i$.

2. **Direction 2:** $\bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n B_i$ For any $\omega \in \bigcup_{i=1}^n A_i$, we can find the smallest index j such that $\omega \in A_j$. If $j = 1$, then $\omega \in B_1$ by definition. If $j > 1$, $\omega \notin A_{j-1}$ (since j is the smallest index), so $\omega \in B_j$. Thus, $\omega \in \bigcup_{i=1}^n B_i$.

By both directions, we conclude $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$.

Part 3: Prove monotone decreasing behavior.

To construct disjoint sets B_i in the case of a monotone decreasing sequence of sets $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$, we define $B_1 = A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i > 1$. These B_i are disjoint by construction.

By the same logic as in Part 2, we can prove that $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$. With the disjoint property and axiom 3 (countable additivity), we reach the same conclusion for this case.

Solution to Exercise 1.11

Problem

Suppose that A and B are independent events. Show that A^c and B^c are independent events

Solution

We use complement rule in property 1.1 and union formula in lemma 1.6 in the proof below.

$$\begin{aligned}
P(A^c)P(B^c) &= (1 - P(A))(1 - P(B)) && \text{(Property 1.1)} \\
&= 1 - (P(A) + P(B) - P(AB)) && \text{(Lemma 1.6)} \\
&= 1 - P(A \cup B) && \text{(Property 1.1)} \\
&= P((A \cup B)^c) && \text{(De Morgan's Law)} \\
&= P(A^c \cap B^c)
\end{aligned}$$

Solution to Exercise 2.1

Problem

Show that $P(X = x) = F(x^+) - F(x^-)$.

Solution

Proof. With Lemma 2.15, we have $P(X = x) = F(x) - F(x^-)$, and by Theorem 2.8, $F(x) = F(x^+)$, so the result follows directly.

Now, we prove Lemma 2.15, as the textbook does not provide a proof.

Using Axiom 3 in Definition 1.5, it is clear that

$$F(x) = P(X < x) + P(X = x),$$

so we only need to prove that $P(X < x) = F(x^-)$.

For ease of discussion, let $y_1, y_2, \ldots$ be a sequence of real numbers such that

$$y_1 < y_2 < \ldots \quad \text{and} \quad \lim_{i \to \infty} y_i = x.$$

Define the sets $A_i = (-\infty, y_i]$ and $A = (-\infty, x)$.

By the definition of $F(x^-)$, we have

$$F(x^-) = \lim_{y \uparrow x} F(y) = \lim_{i \to \infty} P(X \leq y_i).$$

Our goal is to prove that

$$P(A) = \lim_{i \to \infty} P(A_i).$$

We proceed in two steps:

- **Step 1:** $\lim_{i\to\infty} P(A_i) \le P(A)$. Since $y_i < x$ for all i, we have

  $$P(X \le y_i) \le P(X < x),$$

  so taking the limit gives

  $$\lim_{i\to\infty} P(X \le y_i) \le P(X < x).$$

- **Step 2:** $P(A) \le \lim_{i\to\infty} P(A_i)$. For any $y \in A$, there exists some A_j such that $y \in A_j$. Since A_i is monotonic increasing, we have $P(A) \le P(A_i)$ for all $i \ge j$. Taking the limit, we obtain

  $$P(A) \le \lim_{i\to\infty} P(X \le y_i).$$

Thus, we conclude that

$$P(A) = \lim_{i\to\infty} P(A_i),$$

which completes the proof. □