Chapter 1 Solution

January 16, 2025

Solution to Exercise 1.1

Part 1: Prove that $B_1, B_2, \dots$ are disjoint.

Let us consider B_i and B_j where $i < j$. By the definition of B_j, any $\omega \in B_j$ satisfies $\omega \notin A_i$. On the other hand, by the definition of B_i, any $\omega \in B_i$ satisfies $\omega \in A_i$. Therefore, $\omega \notin B_i$ if $\omega \in B_j$. Similarly, $\omega \notin B_j$ if $\omega \in B_i$. Hence, $B_i \cap B_j = \emptyset$ for $i \neq j$, proving that $B_1, B_2, \dots$ are disjoint.

Part 2: Prove that $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$.

1. **Direction 1:** $\bigcup_{i=1}^n B_i \subseteq \bigcup_{i=1}^n A_i$ For any element $\omega \in \bigcup_{i=1}^n B_i$, it must belong to exactly one B_i because of the disjoint property proved above. By the definition of B_i, this implies $\omega \in A_i$. Therefore, $\omega \in \bigcup_{i=1}^n A_i$.

2. **Direction 2:** $\bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n B_i$ For any $\omega \in \bigcup_{i=1}^n A_i$, we can find the smallest index j such that $\omega \in A_j$. If $j = 1$, then $\omega \in B_1$ by definition. If $j > 1$, $\omega \notin A_{j-1}$ (since j is the smallest index), so $\omega \in B_j$. Thus, $\omega \in \bigcup_{i=1}^n B_i$.

By both directions, we conclude $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$.

Part 3: Prove monotone decreasing behavior.

To construct disjoint sets B_i in the case of a monotone decreasing sequence of sets $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$, we define $B_1 = A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i > 1$. These B_i are disjoint by construction.

By the same logic as in Part 2, we can prove that $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$. With the disjoint property and axiom 3 (countable additivity), we reach the same conclusion for this case.