A new minimum zone method for evaluating flatness errors

S. T. Huang,* K. C. Fan,† and John H. Wu*
*Institute of Optical Sciences, National Central University, and †Department of Mechanical Engineering, National Taiwan University, Taiwan, Republic of China

A new minimum zone method for flatness error analysis is proposed in this article. Based on the criteria for the minimum zone solution and strict rules for data exchange, a simple and rapid algorithm, called the control plane rotation scheme, is developed for the flatness analysis of a flat surface. Experimental work was performed, and some examples are given in terms of the minimum zone and least-squares solutions.

Keywords: *flatness; minimum zone method; least-squares method*

Introduction

Flat surfaces of any size, as embodied in the form of surface plates and tables, are of considerable importance in precision engineering, particularly as reference surfaces in the inspection of engineering components and as work tables for machine tools. In order to assess the flatness error of any surface plate, engineering measurements along a particular reference plane using appropriate techniques are generally made. The interpretation of the measured data has been specified using many different standards, such as BS 308: part 3: "geometrical tolerance"[1] and ANSI Y14.5M.[2] These specifications are all based on the minimum zone concept appearing in ISO/R1101,[3] which specifies form errors in general. It states that an ideal geometrical feature must be established from the actual measurements such that the maximum deviation between the ideal and the actual measurement concerned is the least possible value. The peak-to-valley distance of the deviation data from the ideal geometrical feature thus established is taken to represent the form error. The orientation of the ideal feature can be regarded as the alignment error in setting the reference plane with respect to the measured plane.

Although the least-squares method,[4] because of its simplicity in computation and the uniqueness of the solution provided, is most widely used in industry for determining flatness, it provides only an approximate solution that does not guarantee the minimum zone value. Therefore, during the last decade much research has been devoted to finding the minimum zone solutions for flatness error and other form errors using a variety of methods. Some researchers applied the numerical methods of linear programming, such as the Monte Carlo method, the simplex search and spiral search used by Murthy,[5] the revised simplex search with dual problem used by Chetwynd,[6] the minimax approximation algorithm proposed by Fukuda,[7] and the simplex search technique adopted by Shunmugam.[8] Another approach has been to find the enclosing polygon for the minimum zone solution, such as the eigen-polyhedral method proposed by Hong,[9] the convex polygon method presented by Lai,[10] and the convex hall theory given by Traband *et al.*[11] These methods are all more or less similar in their computational conception, which dynamically reveals the meaning of each search step from the distribution of the data points. The MINMAX method proposed by Fan,[12] an algorithm, uses the concept of the rotations of enclosing planes with respect to a particular contact point at each data exchange step. This technique reduces the separation of the enclosing planes step by step until the minimum zone is found. The concept is comprehensive in both the physical and engineering senses in regard to the geometrical form of the investigated data.

The methods mentioned above generally proceed initially with the random selection of data points and then follow with an iterative data exchange procedure. A longer computation time is naturally required by this kind of approach in order to reach the final minimum zone condition. Burdekin and Pahk[13] in their revision of the MINMAX method, called the "enclose tilt technique," use the least-squares result as the initial condition for the analysis of the flatness error. This method, while

Address reprint requests to K. C. Fan, Dept. of Mechanical Engineering, National Taiwan University, No. 1 Roosevelt Rd., Sec. 4, Taipei, Taiwan 10764, Republic of China.

providing an efficient method for reaching the minimum zone solution, deals with all the measured points, the so-called full field, during each data exchange process. This process can be improved by screening out unwanted data points, which makes the mathematical model simpler and the computation time even shorter.

We must first recognize that all the algorithms so far developed for the minimum zone solution of flatness error will guarantee an exact and unique solution of the minimum zone value, which must be smaller than the least-squares value. The computation times will be different depending on the complexity of the mathematical model that each algorithm employs. From an engineering point of view, in practice the ability to understand the physical meaning of the algorithm is more important than the computation time of that algorithm, the computation of each algorithm being quite fast, even when using a personal computer. In reality, some algorithms are indeed difficult to understand because they are purely numerical analyses or complex in geometrical presentation. Therefore, the simpler and clearer the algorithm, the more readily it will be accepted by an inspector who needs to know the flatness error from measured data. Practically, the best algorithm should provide not only the simplest model for analysis, but should require the least computational use of the computer. The development of such an algorithm is the main objective of this work.

This article presents a new minimum zone method for the flatness analysis of any flat surface investigated. A previous study by the investigators of straightness analysis using the control line rotation scheme (CLRS) was quite satisfactory.[14] This is an extended work from the CLRS to the control plane rotation scheme (CPRS) for the purpose of flatness evaluation. The data-exchange scheme starts with a 1-1 model based on the least-squares result. It then is followed by a 2-1 model search and continued to either the 2-2 model or the 3-1 model directed by the strict rule of the CPRS, which is developed based on the criteria of the minimum zone solution. With only a few steps of data exchange in 2-2 model or 3-1 model iteration, the minimum zone solution can be obtained easily. A handy method using the CPRS technique without the use of a computer is also included.

Flatness analysis

Minimum zone criteria

The criteria for the minimum zone solution of flatness error have already been verified and adopted for use in numerous studies.[12,13,15,16] The following conditions must be met in the final stages: (1) At least four points must be in contact with the two enclosing parallel planes in the form of a 3-1 model (three points on the upper plane, one point on the lower plane, or vice versa) or a 2-2 model (two points on the upper plane, two points on the lower plane); (2) for the case of a 3-1 model, when projected onto the upper or lower plane, that single contact point must be on the inside of the triangle formed by the other three points as shown in *Figure 1*; or (3) for the case of a 2-2 model, when projected onto the upper or lower plane, the line which is linked by those two contact points of the upper plane must be intersected by the line linked by the other two contact points of the lower plane as shown in *Figure 2*. The distance between any two such enclosing planes defines the minimum zone of flatness error. The corresponding mean plane is regarded as the best-fit plane. The contact points described here are called the control points and the corresponding enclosing planes are called the control planes. An interesting phenomenon should be noted here. When considering a side view of the deviation data from the direction generated by any two control points of the same control plane (as shown in *Figures 3 and 4*), the distribution of data points will be directly coincident with the minimum zone criteria of straightness, that is, three control points in upper-lower-upper sequence or vice versa.[13,14] This shows the correlation of minimum zone solutions for straightness and flatness. The mathematical models used to find the solution for minimum zone flatness are described in the following sections.

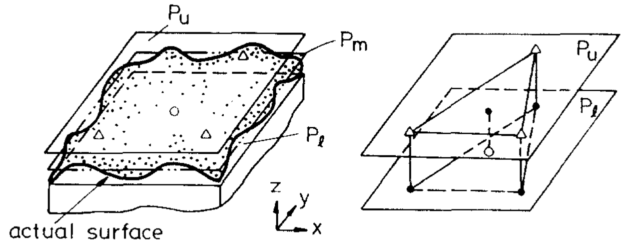


Figure 1 The minimum zone condition of the 3-1 model

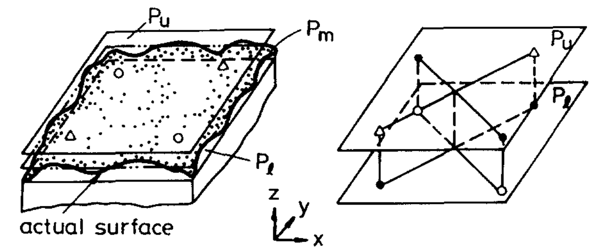


Figure 2 The minimum zone condition of the 2-2 model

Mathematical models

1-1 Model. Before searching for a best-fit plane, it is best to find the least-squares plane as the initial

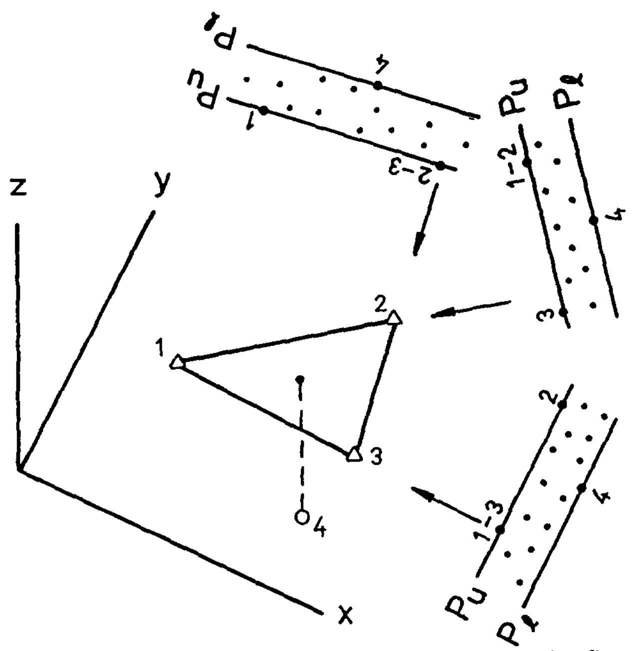


Figure 3 Relationship between the 3-1 model flatness criterion and the straightness criterion

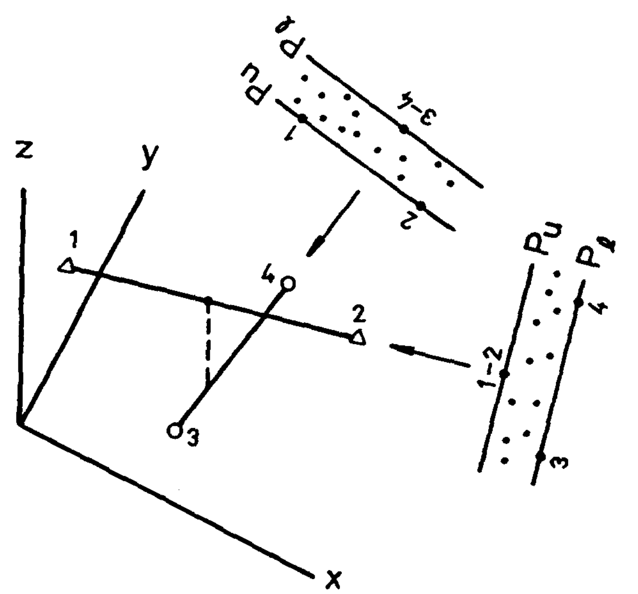


Figure 4 Relationship between the 2-2 model flatness criterion and the straightness criterion

condition of the search because in most cases it is close to the best-fit plane.

Let the equation of the least-squares plane be

$$Z = aX + bY + c \tag{1}$$

where a, b, and c are coefficients. Using the least-squares method, the following sum is minimized:

$$E = \sum_{i=1}^{N} (Z_i - aX_i - bY_i - c)^2 \tag{2}$$

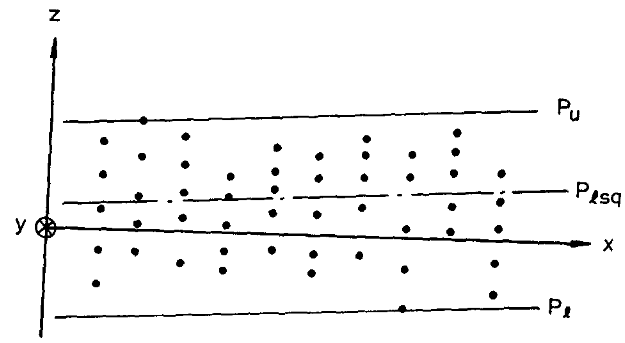


Figure 5 Construction of the 1-1 model from the least-squares plane

where (X_i, Y_i, Z_i) $(i = 1, \ldots, N)$ is the measured data. Applying the variational principle, three equations can be obtained as,

$$\sum_{i=1}^{N} (Z_i - aX_i - bY_i - c)X_i = 0 \tag{3}$$

$$\sum_{i=1}^{N} (Z_i - aX_i - bY_i - c)Y_i = 0 \tag{4}$$

$$\sum_{i=1}^{N} (Z_i - aX_i - bY_i - c) = 0 \tag{5}$$

Therefore, three variables, a, b, and c, can be solved by the modified Gauss elimination method. Thus, the deviation δZ_i of the flatness data from the least-squares plane can be defined as

$$\delta Z_i = Z_i - aX_i - bY_i - c \quad \text{for } i = 1, 2, \ldots, N \tag{6}$$

where N denotes the total number of data points. The highest of the data points with respect to this least-squares plane is now defined as the upper control point, and the lowest is defined as the lower control point. A 1-1 model is thus established in such a way that an upper control plane is generated from the upper control point and a lower control plane from the lower control point, with both planes being parallel to the least-squares plane. *Figure 5* illustrates a side view of deviation data from the y axis. The equations of the two control planes enclosing these data can be expressed as

$$Z_u = aX + bY + c_u \quad \text{(upper control plane)} \tag{7}$$

$$Z_l = aX + bY + c_l \quad \text{(lower control plane)} \tag{8}$$

where a, b, c_u, and c_l are four coefficients to be determined.

In order to define the two parallel planes of unique slope that enclose all data points, it is necessary to find at least four control points. Each of four unknown unique coefficients must be determined. The third and the fourth control points must be found. A very strict rule of search scheme is pro-

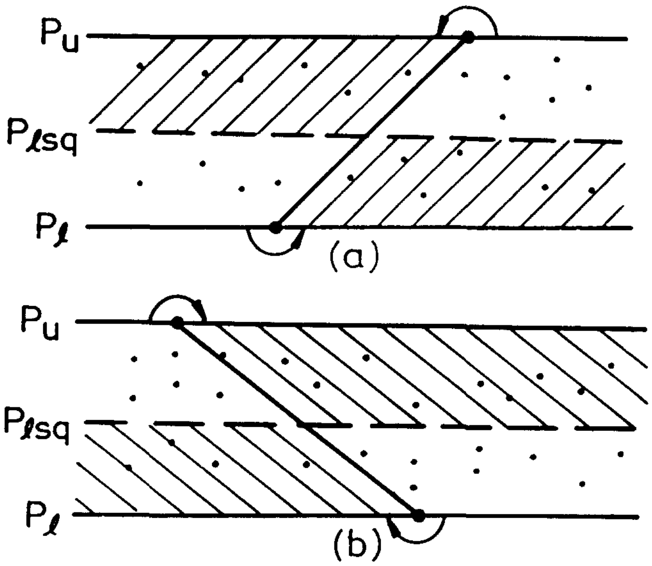


Figure 6 CPRS rule in half-field for 1-1 model: (a) lower-upper condition; (b) upper-lower condition

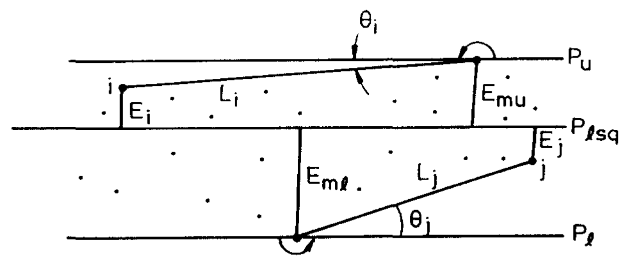


Figure 7 CPRS search from 1-1 model to 2-1 model

posed in the following sections to ensure that the solution is concise, rapid, and exact.

CPRS for 2-1 model. From the 1-1 model, two control points in association with two control planes are obtained by the least-squares method. In order to find the third control point, the CPRS is introduced. This CPRS is similar in nature to the CLRS, which was used for straightness analysis.[14] The only difference is that instead of concerning line data and control lines, the CPRS deals with surface data and control planes.

Consider a side view of deviation data with respect to the least-squares plane. There are only two 1-1 model conditions: lower-upper and upper-lower positions of the control points, as shown in *Figure 6a and b,* respectively. Each control plane will be rotated in the specified direction to find the third control point. This point, together with the current two control points, will most likely form the upper-lower-upper or lower-upper-lower condition. Furthermore, if the space between two such control planes is defined as the full field, during the rotation each control plane will eventually find its own first new contact point, which must be located within the specified quarter field. In other words, only the points within a specified quarter field will be intersected by a particular control plane for the determination of a new contact point. This entire CPRS process deals with data points in the half field only, as shown by the shaded parts in *Figure 6.* In the computer algorithm, those unwanted points can be screened out automatically based on this concept, saving half of the computational time required by a full-field search.[12,13]

During the rotation of a particular control plane, any point within the corresponding quarter field may become the first contact point, depending on its position. Because each point within such a field will correspond to a rotation angle of the control plane, the first contact point must be the one having the smallest angle with respect to the control plane. In the case of a search in the upper quarter field (*Figure 7*), if we let E_{mu} be the deviation of the upper control point from the reference plane, and E_i the deviation of point i from the reference plane, the angle of rotation of the control plane from its initial position to the position as it contacts with point i is

$$\theta_i = \sin^{-1}\left(\frac{E_{mu} - E_i}{L_i}\right) \tag{9}$$

where L_i is the distance from the upper control point to point i. The first contact point I within this quarter field will be found by

$$\theta_I = \text{minimum}\{\theta_i\} \tag{10}$$

Similarly, in the lower quarter-field search, the angle of rotation of the lower control plane from its initial position to the position where it contacts any point j in this field is

$$\theta_j = \sin^{-1}\left(\frac{E_{ml} - E_j}{L_j}\right) \tag{11}$$

where E_{ml} represents the deviation from the lower control point to the reference plane, E_j the deviation from point j to the reference plane, and L_j the distance from the lower control point to point j.

The first contact point J within this quarter field will be found by

$$\theta_J = \text{minimum}\{\theta_j\} \tag{12}$$

Although each control plane may generate its own first point of contact during its own CPRS, only one contact point is eventually needed to form the 2-1 model. Therefore, selection between points I and J will be judged by the smaller value of θ_I and θ_J.

CPRS from 2-1 model to 3-1 or 2-2 model. To find the fourth control point from the 2-1 model, consider a side view of deviation data from the direction linked by those two control points on the same control plane (e.g., a side view from $P_1 - P_2$ as shown in

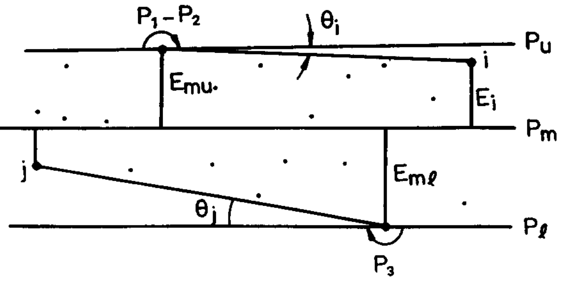


Figure 8 CPRS search for the fourth control point

Figure 8). The CPRS method for the 2-1 model can be applied again. The fourth control point (P_4) is also found in the half-field zone as described in the 2-1 model.

At this stage four control points in association with two control planes have been found conforming to the first criterion of the minimum zone solution. Only two possible conditions will have occurred in terms of these four control points, either in the form of the 3-1 model or the 2-2 model.

If the final conditions at the current stage meet the second or third criterion of the minimum zone solution, as shown in *Figures 1–4*, the job is completed. The coefficients (a, b, c) of the mean plane, as expressed in Equation 1, can now be computed according to the current model as follows.

3-1 model: Let points $P_1(x_1,y_1,z_1)$, $P_2(x_2,y_2,z_2)$, and $P_3(x_3,y_3,z_3)$, which are not on a single line, be control points on the upper control plane, and $P_4(x_4,y_4,z_4)$ be that on the lower control plane, as shown in *Figures 1 and 3*. The coefficients a, b, and c of the mean plane can be expressed as

$$a = \frac{(z_2 - z_3)(y_3 - y_4) - (z_3 - z_4)(y_2 - y_3)}{D_1} \tag{13a}$$

$$b = \frac{(x_2 - x_3)(z_3 - z_4) - (x_3 - x_4)(z_2 - z_3)}{D_1} \tag{13b}$$

$$c = \frac{(z_1 + z_2) - a(x_1 + x_2) - b(y_1 + y_2)}{2} \tag{13c}$$

where $D_1 = (x_2 - x_3)(y_3 - y_4) - (x_3 - x_4)(y_2 - y_3)$.

2-2 model: Let points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ be control points on the upper control plane, and $P_3(x_3,y_3,z_3)$ and $P_4(x_4,y_4,z_4)$ be those on the lower control plane, as shown in *Figures 2 and 4*. The coefficients a, b, and c of the mean plane can be expressed as

$$a = \frac{(y_1 - y_2)(z_3 - z_4) - (y_3 - y_4)(z_1 - z_2)}{D_2} \tag{14a}$$

$$b = \frac{(x_3 - x_4)(z_1 - z_2) - (x_1 - x_2)(z_3 - z_4)}{D_2} \tag{14b}$$

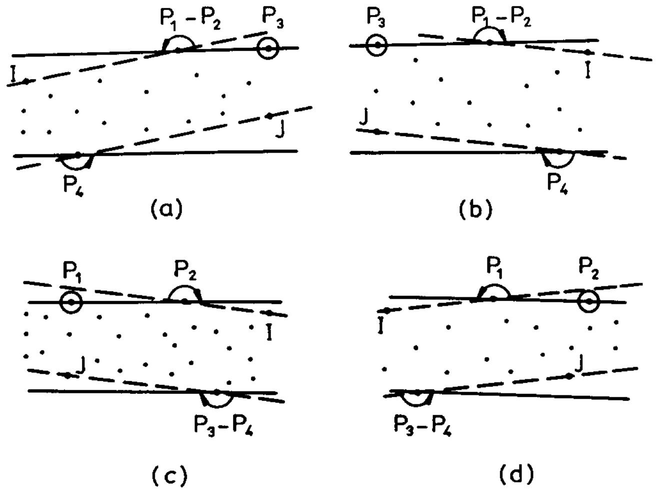


Figure 9 CPRS search for 2-2 model and 3-1 model: (a) lower-upper-upper condition in 3-1 model; (b) upper-upper-lower condition in 3-1 model; (c) upper-upper-lower condition in 2-2 model; (d) lower-upper-upper condition in 2-2 model

$$c = \frac{(z_2 + z_3) - a(x_2 + x_3) - b(y_2 + y_3)}{2} \tag{14c}$$

where $D_2 = (x_3 - x_4)(y_1 - y_2) - (x_1 - x_2)(y_3 - y_4)$

The equations of the two corresponding control planes can also be obtained by Equations 7 and 8, where

$$c_u = z_1 - ax_1 - by_1 \tag{15}$$

$$c_l = z_3 - ax_3 - by_3 \tag{16}$$

The minimum zone flatness error is then determined by the distance between these two control planes.

However, if these four control points cannot meet the second or third criterion of the minimum zone solution, further CPRS procedures will be required.

CPRS for 2-2 and 3-1 models. When the four control points of the current 2-2 or 3-1 model do not meet the minimum zone criterion, four possible conditions will occur, as illustrated in *Figure 9*. As indicated earlier, when projecting the measured data onto a particular side plane, the distribution of control points will form the upper-lower-upper or lower-upper-lower sequence. It is apparent that none of the four existing conditions as shown in *Figure 9* will meet this requirement. The principle of the data-exchange scheme, the CPRS method, is to search for a new control point to replace one of the current control points so that not only the separation of the new control planes can be lessened but that the above-mentioned criterion also can be met. The rule of CPRS is therefore both strict and essential. The direction of rotation of each control plane with respect to the corresponding control point (or points) has been specified in *Figure 9* for each individual case. It is obvious that during the rotation the point to be discarded must be the one that violates the minimum zone criterion, as circled in *Figure 9.* The remaining three control points will form a 2-1 model, and the CPRS method for a 2-1 model can be applied. This procedure is iterated until the minimum zone criterion is reached.

Algorithm of the computer program. The algorithm of the computer program for the CPRS technique can now be summarized as follows:

1. Read the data (X_i, Y_i, Z_i)
2. Determine the flatness data via the least-squares method
3. Construct the 1-1 model
4. Apply the CPRS rule to form a new 2-1 model
5. Apply the CPRS rule from the 2-1 model to a 3-1 or 2-2 model
6. Iterate the 3-1 or 2-2 model with the CPRS rule until the second or third minimum zone criterion is met
7. Output the result

Examples

There are several instruments available on the market that can measure flatness of a flat surface, such as the electronic level, autocollimator, laser interferometer, CMM, Fizeau or twyman-green interferometer, and so on. Experiments can be performed easily using appropriate instrumentation for any surface being investigated. Some of the results of this work are presented here. To demonstrate the simplicity and clarity of the CPRS method from a geometrical viewpoint, the first example, using simple data points, will be described in a step-by-step manual approach. Other examples having more data points are solved by the microcomputer.

Example 1: Step-by-step manual CPRS approach

The unique significance of the CPRS method when compared with all other existing methods[4–13] is its strict rule of control-plane rotation. The least-squares plane is only a reference for the construction of the 1-1 model. Because this method can efficiently provide the optimum strategy for data exchange, the reference plane can even be any other plane passing through the data points. The CPRS process can also be used without the computer.

Figure 10 illustrates a series of procedures performed manually with a pair of triangles and a pen on graph paper with respect to the data points given in example 1.[15] The level data of this example are listed in Figure 10a, where the number noted on each lower left corner indicates the serial number of the corresponding data. Consider a side view of all data points from direction Y, as shown in *Figure 10b.* A reference plane (P_m) can be drawn by linking the two end points (1, 9), which is deemed the simplest way to construct the reference plane. The corresponding control points (8, 4) and control planes (P_u,P_l) of the 1-1 model are determined accordingly. *Figure 10c* shows the way to obtain the third control point (6), which appears to have a smaller angle (θ_6) than that of point 1 (θ_1) after CPRS. The new 2-1 model is thus established.

To find the fourth control point, consider a side view of the data points from the direction linking control points 4 and 6 and then use the CPRS again, as shown in *Figure 10d.* It is obvious that point 7 is the correct one that, together with points 4, 6, and 8, form a new 3-1 model. Unfortunately, this 3-1 model does not meet the second criterion of the minimum zone solution. Consider a side view from the direction linking control points 6 and 7, as

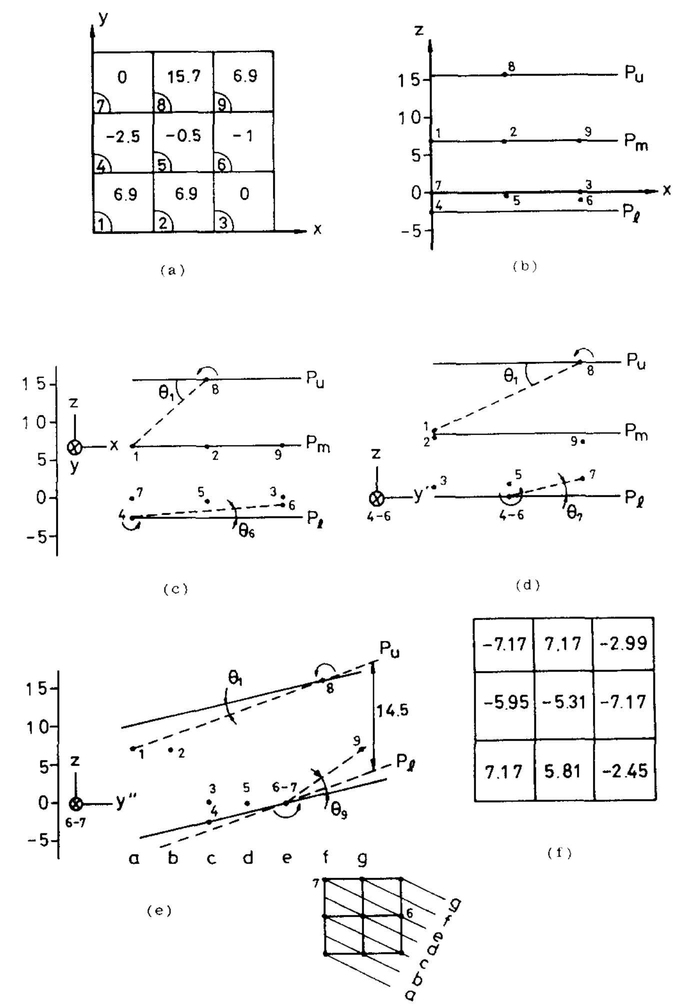


Figure 10 Graphical procedure of the CPRS search (step length, 10 cm; error, μm): (a) raw data; (b) 1-1 model construction; (c) 2-1 model construction; (d) 3-1 model construction; (e) data exchange from 3-1 model to 2-2 model; (f) final flatness data

2	5	6	8	9
5	7	8	9	12
6	7	8	9	11
7	7	6	7	9
7	6	6	6	8

(a)

-2.43	-0.70	-0.98	-0.25	-0.52
0.52	1.25	0.98	0.70	2.43
1.48	1.20	0.93	0.66	1.39
2.43	1.16	-1.11	-1.39	-0.66
2.39	0.11	-1.16	-2.43	-1.70

(b)

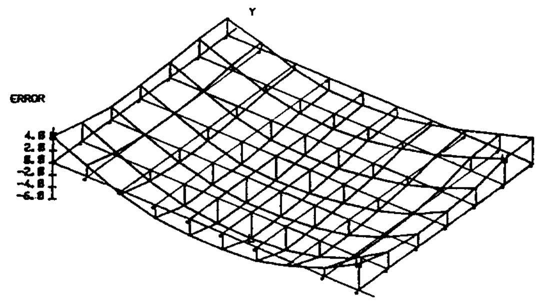
Figure 11 Example 2 (step length, 10 cm; error, μm): (a) level datum; (b) minimum zone data

10.71	4.20	-2.69	-8.67	-13.63	-18.26	-22.04	-25.43	-28.58	-31.71
8.81	1.90	-5.00	-11.01	-16.19	-20.70	-24.50	-28.11	-30.99	-33.68
6.31	-0.82	-7.77	-14.03	-19.10	-23.86	-27.90	-31.66	-34.67	-37.13
4.01	-3.17	-10.22	-16.65	-21.86	-26.55	-30.64	-34.52	-37.59	-39.62
1.91	-5.24	-12.29	-18.53	-24.09	-28.91	-33.22	-37.05	-39.99	-41.96
0.80	-6.65	-13.93	-20.56	-26.03	-31.11	-35.53	-39.16	-41.96	-43.41
0.00	-7.38	-14.66	-21.10	-26.53	-31.42	-35.87	-39.24	-42.08	-43.93

(a)

2.79	1.17	-0.84	-1.95	-2.02	-1.77	-0.67	0.82	2.55	4.29
2.94	0.91	-1.11	-2.24	-2.53	-2.17	-1.08	0.18	2.19	4.38
2.49	0.24	-1.83	-3.20	-3.39	-3.28	-2.44	-1.31	0.56	2.98
2.24	-0.06	-2.23	-3.78	-4.10	-3.92	-3.13	-2.12	-0.31	2.53
2.19	-0.07	-2.24	-3.61	-4.29	-4.22	-3.66	-2.60	-0.66	2.24
3.13	0.56	-1.84	-3.59	-4.18	-4.38	-3.91	-2.67	-0.59	2.85
4.38	1.88	-0.52	-2.07	-2.63	-2.63	-2.21	-0.69	1.35	4.38

(b)



(c)

Figure 12 Example 3 (step length, 10 cm; error, μm): (a) level datum; (b) minimum zone data; (c) minimum zone surface profile

shown in *Figure 10e.* An application of the CPRS will now discard point 4 and result in a new control point (point 1) that, together with points 6, 7, and 8, will form a new 2-2 model. This model conforms to the third criterion of the minimum zone solution. Thus, the minimum zone solution of this example is finally obtained with respect to four control points (1 and 8, 6 and 7), and two control planes (P_u,P_l) in the form of a 2-2 model. The exact flatness error, estimated from the diagram to be 14.5, is computed as 14.34, as shown in *Figure 10f* by Equations 14–16.

Examples 2 and 3: CPRS by computation

In example 1, an efficient procedure demonstrates the simplicity and clarity of the proposed CPRS method. However, it can only apply in the case of simple data points. For most of the surfaces being investigated, the data points are large, and the use of a computer for analysis is necessary.

Example 2 is a set of points from Murthy and Abdin.[5] *Figure 11a* tabulates the level datum of example 2. With only one CPRS process, the minimum zone solution was readily found. Four control points (6 and 20, 4 and 21) in the form of a 2-2 model can be clearly seen in *Figure 11b,* with a flatness error of 4.864.

A granite surface plate in the laboratory was calibrated by the use of an electronic level that was linked to an IBM PC/AT via an A/D converter. The measured level datum, listed in *Figure 12a,* was adopted for use by example 3. The minimum zone condition was found in the form of a 3-1 model as shown in *Figure 12b and c,* in numerical and graphical forms, respectively.

Table 1 shows a comparison of the results of these three examples by the CPRS and least-squares methods. It is evident that the minimum zone solution always yields smaller errors.

Conclusions

A new and comprehensive method for the minimum zone solution of flatness error is proposed in

Table 1 Comparisons of the least-squares and the minimum zone results of the three given examples

Coefficient	Example 1		Example 2		Example 3	
	LSQ	CPRS	LSQ	CPRS	LSQ	CPRS
a	0.03	0.14	0.10	0.13	−0.48	−0.49
b	0.15	0.37	0.00	−0.01	0.22	0.21
c	1.88	−0.27	5.24	4.61	−5.82	−4.38
Flatness	16.48	14.34	5.90	4.86	9.21	8.76

LSQ, least-squares method; CPRS, minimum zone method; step length, 10 cm; error unit, micron

this article. This technique, due to its strict CPRS rules for data exchange, provides a concise and rapid means for reaching the minimum zone solution. From many applications, it is apparent that only one or two CPRS searches are required to reach a final solution. This method is certainly a useful tool for flatness analysis.

In practice, although the step length being selected does not affect the computed flatness errors, it does affect the calibrated grade of the inspected surface. The selection of a proper step length relative to the size of the surface plate is therefore very important in practical use.

References

1 "Geometrical tolerance," *BS* 1972, **308,** part 3
2 Dimensioning and tolerancing for engineering drawings, ANSI Y14.5M, 1982
3 Technical drawings—geometrical tolerancing, ISO/R1101, 1983
4 Miller, M. *Engineering Dimensional Metrology,* London, Edward Arnold Co., 1962
5 Murthy, T. S. R. and Abdin, S. Z. "Minimum zone evaluation of surfaces," *Int J Mach Tool Des Res* 1980, **20,** 123–136
6 Chetwynd, D. G. Applications of linear programming to engineering metrology. *Proc Instn Mech Eng* 1985, **199,** 93–100
7 Fukuda, M. and Shimokohbe, A. Algorithm for form evaluation methods for minimum zone and least squares. Proceedings of the International Symposium on Metrology for Quality Production, Tokyo, 1984, pp. 197–202
8 Shunmugam, M. S. "Comparison of linear and normal deviations of forms of engineering surfaces," *Prec Eng* 1987, **9,** 96–102
9 Hong, J. T. An algorithm for flatness calculation from geometrical viewpoint, M.Sc. thesis, National Taiwan University, 1987
10 Lai, K. and Wang, J. A computational geometry approach to geometric tolerancing. 16th NAMRC, 1988, pp. 376–379
11 Traband, M. T., Joshi, S., Wysk, R. A. and Cavalier, T. M. "Evaluation of straightness and flatness tolerances using minimum zone," *Manufacturing Rev* 1989, **2,** 189–195
12 Fan, K. C. Computer-aided calibration of the accuracy performance of NC machine tools, Ph.D. thesis, Univ. of Manchester Inst. of Science and Technology, Manchester, UK, 1984
13 Burdekin, M. and Pahk, H. J. The application of a microcomputer to the on-line calibration of the flatness of engineering surfaces. *Proc Instn Mech Eng* 1989, **203,** 127–137
14 Huang, S. T., Fan, K. C. and Wu, J. H. "A new minimum zone method for evaluating straightness errors" (submitted for publication)
15 "Surface plate," *BS* 1983, **817**
16 Li, C. *Fundamentals of the Interchangeability and Measurement Technology.* Peking, Metrology Publisher, 1984