Prove Beta-Binomial conjugation

$$X = \sum_{i=1}^{N} x_i$$

$$P(\theta|x) = \frac{P(x|\theta)P(\theta)}{P(x)} = \frac{P(x,\theta)}{\int_0^1 P(x,\theta)\,dx}$$

$$P(x|\theta) = \binom{n}{x} \theta (1-\theta)^{n-x}, \quad P(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1-\theta)^{b-1}$$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}, \quad \Gamma(\alpha+1) = \alpha!$$

$$= \frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)}$$

$$P(x,\theta) = P(x|\theta)P(\theta)$$

$$= \frac{\Gamma(a+b)\Gamma(n+1)}{\Gamma(a)\Gamma(b)\Gamma(x+1)\Gamma(n-x+1)} \theta^{(a-1+x)} (1-\theta)^{(b-1+n-x)}$$

$$\| \\ \gamma$$

$$= \gamma\, \theta^{(a-1+x)} (1-\theta)^{(b-1+n-x)}$$

$$a' = a + x$$
$$b' = b - x + n$$

$$= \gamma\, \theta^{(a'-1)} (1-\theta)^{(b'-1)}$$

$$P(x) = \int_0^1 P(x,\theta)\,d\theta$$

$$= \overbrace{\int_0^1 \frac{\Gamma(a+b+n)}{\Gamma(a+x)\Gamma(b-x+n)}\,\theta^{a-1+x}(1-\theta)^{(b-1+n-x)}\,d\theta}^{=1}\;\gamma\;\underbrace{\frac{\Gamma(a+x)\Gamma(b+x-n)}{\Gamma(a+b+n)}}_{=\text{beta-binomial}}$$

$$P(x,\theta) = \gamma\,\theta^{(a-1+x)}(1-\theta)^{(b-1+n-x)}$$

$$P(x) = \gamma\,\frac{\Gamma(a+x)\Gamma(b+x-n)}{\Gamma(a+b+n)}$$

$$P(x,\theta) = P(\theta|x)P(x)$$
$$= P(x|\theta)P(\theta)$$

$$P(\theta|x) = \frac{\cancel{\gamma}\,\theta^{(a-1+x)}(1-\theta)^{(b-1+n-x)}}{\cancel{\gamma}\,\dfrac{\Gamma(a+x)\Gamma(b+x-n)}{\Gamma(a+b+n)}}$$

$$= \frac{\Gamma(a+b+n)}{\Gamma(a+x)\Gamma(b+x-n)}\,\theta^{(a-1+x)}(1-\theta)^{(b-1+n-x)}$$

$$= \mathrm{Beta}(\underbrace{a+x}_{a'}, \underbrace{b-x+n}_{b'})$$

$$a' = a+x$$
$$b' = b-x+n$$