Chapter 2

Mathematical Models of Physical Systems

It is part of being human to try to understand the world, natural and human-designed, and our way of understanding something is to formulate a model of it. All of physics—Kepler's laws of planetary motion, Newton's laws of motion, his law of gravity, Maxwell's laws of electromagnetism, the Navier-Stokes equations of fluid flow, and so on—all are models of how things behave. Control engineering is an ideal example of the process of modelling and design of models.

In this chapter we present a common type of model called state equations. The nonlinear form is

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

and the linear form is

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Here u, x, y are vectors that are functions of time. The dot, $\dot{x}$, signifies derivative with respect to time. The components of the vector $u(t)$ are the inputs to the system; an input is an independent signal. The components of $y(t)$ are the outputs, the dependent signals we are interested in. And the vector $x(t)$ is the state; its purpose and meaning will be developed in the chapter. The four symbols A, B, C, D are matrices with real coefficients. These equations are linear, because $Ax + Bu$ and $Cx + Du$ are linear functions of (x, u), and time-invariant, because the coefficient matrices A, B, C, D do not depend on time.

2.1 Block diagrams

The importance of block diagrams in control engineering cannot be overemphasized. One could easily argue that you don't understand your system until you have a block diagram of it. This section teaches how to draw block diagrams.

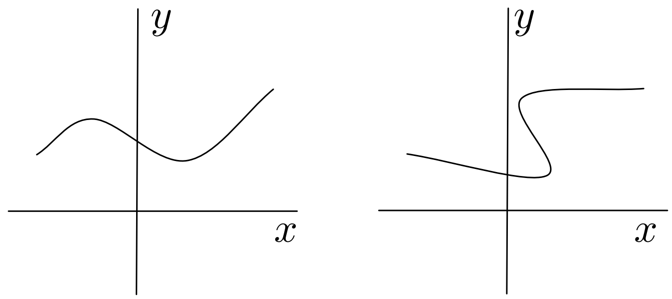


Figure 2.1: The curve on the left is the graph of a function; the curve on the right is not.

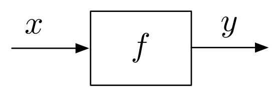


Figure 2.2: The block diagram for the function $y = f(x)$.

1. *Block diagram of a function.* Let us recall what a function is. If X and Y are two sets, a **function** from X to Y is a rule that assigns to each element of X a unique element of Y. The terms function, mapping, and transformation are synonymous. The notation

   $$f : X \longrightarrow Y$$

   means that$X$ and Y are sets and f is a function from X to Y. We typically write $y = f(x)$ for a function. To repeat, for each x there must be a unique y such that $y = f(x)$; $y_1 = f(x)$ and $y_2 = f(x)$ with $y_1 \neq y_2$ is not allowed.[1] Now let f be a function $\mathbb{R} \longrightarrow \mathbb{R}$. This means that f takes a real variable x and assigns a real variable y, written $y = f(x)$. So f has a graph in the (x, y) plane. For f to be a function, every vertical line must intersect the graph in a unique point, as shown in Figure 2.1. Figure 2.2 shows the block diagram of the function $y = f(x)$. Thus a box represents a function and the arrows represent variables; the input is the independent variable, the output the dependent variable.

2. *Electric circuit example.* Figure 2.3 shows a simple RC circuit with voltage source u. We consider u to be an independent variable and the capacitor voltage y to be a dependent variable. Thinking of the circuit as a system, we view u as the input and y as the output. Let us review how we could compute the output knowing the input. The familiar circuit equation is

   $$RC\dot{y} + y = u.$$

   Since this is a first-order differential equation, given a time $t > 0$, to compute the voltage $y(t)$ at that time we would need an initial condition, say $y(0)$, together with the input voltage $u(\tau)$

[1] The term **multivalued function** is sometimes used if two different values are allowed, but we shall not use that term.

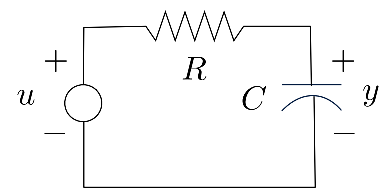


Figure 2.3: An RC circuit.

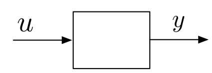


Figure 2.4: Block diagram of RC circuit.

for τ ranging from 0 to t. We could write this symbolically in the form

$$y = f(u, y(0)),$$

which says that the signal y is a function of the signal u and the initial voltage $y(0)$. A block diagram doesn't usually show initial conditions, so for this circuit the block diagram would be Figure 2.4. Inside the box we could write the differential equation or the transfer function, which you might remember is $1/(RCs + 1)$. Here's a subtle point: In the block diagram should the signals be labelled u and y or $u(t)$ and $y(t)$? Both are common, but the second may suggest that y at time t is a function of u only at time t and not earlier. This, of course, is false.

3. *Mechanics example.* The simplest vehicle to control is a cart on wheels. Figure 2.5 depicts the situation where the cart can move only in a straight line on a flat surface. There are two arrows in the diagram. One represents a force applied to the cart; this has the label u, which is a force in Newtons. The direction of the arrow is just a reference direction that we are free to choose. With the arrow as shown, if u is positive, then the force is to the right; if u is negative, the force is to the left. The second arrow, the one with a barb on it, depicts the position of the center of mass of the cart measured from a stationary reference position. The symbol y

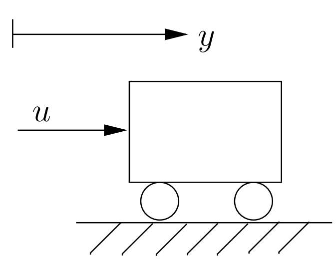


Figure 2.5: A simple vehicle.

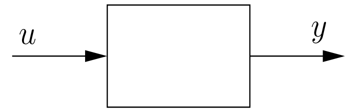


Figure 2.6: Block diagram of a cart.

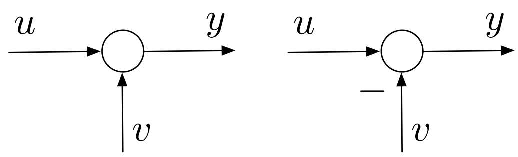


Figure 2.7: The first summing junction stands for $y = u + v$; the second for $y = u - v$.

stands for position in meters. This is a **schematic diagram**, not a block diagram, because it doesn't say which of u, y causes the other. Newton's second law tells us that there's a mathematical relationship between u and y, namely, $u = M\ddot{y}$, that is, force equals mass times acceleration. We take the viewpoint that u is an independent variable, and thus it is viewed as an input. Recall that $M\ddot{y} = u$ is a second-order differential equation. Given the forcing term $u(t)$, we need two initial conditions, say, position $y(0)$ and velocity $\dot{y}(0)$, to be able to solve for $y(t)$. More specifically, for a fixed time $t > 0$, in order to compute $y(t)$ we would need $y(0)$ and $\dot{y}(0)$ and also $u(\tau)$ over the time range from $\tau = 0$ to $\tau = t$. So symbolically we have

$$y = f(u, y(0), \dot{y}(0)).$$

Again, we leave the initial conditions out of the block diagram to get Fig. 2.6. Inside the box we could put the differential equation or the transfer function, which is $1/(Ms^2)$—the Laplace transform of y divided by the Laplace transform of u. If you don't see this, don't worry—we'll do transfer functions in detail later.

4. *Summing junctions.* Block diagrams also may have summing junctions, as in Figure 2.7. Also, we may need to allow a block to have more than one input, as in Figure 2.8. This means that y is a function of u and v, $y = f(u, v)$.

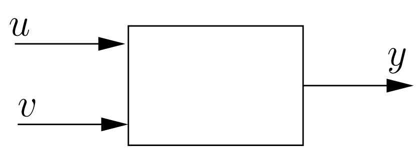


Figure 2.8: Two inputs.

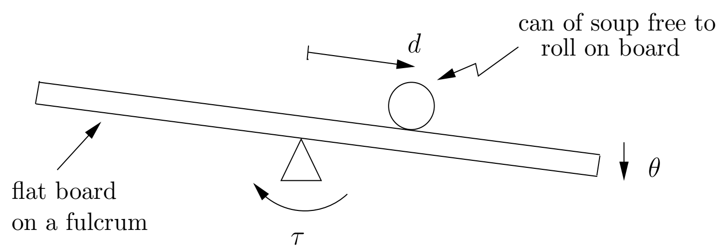


Figure 2.9: A can rolls on a board.

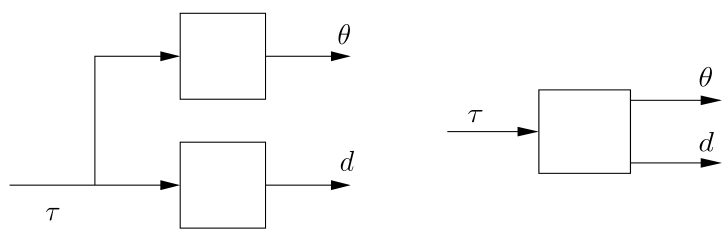


Figure 2.10: One input, two outputs.

5. *Multi-output example.* Figure 2.9 shows a can rolling on a see-saw. Suppose a torque τ is applied to the board. Let θ denote the angle of tilt and d the distance of roll. Then both θ and d are functions of τ. The block diagram could be either of the ones in Figure 2.10.

6. *Water tank.* Figure 2.11 shows a water tank. The arrow indicates water flowing out. Try to draw the block diagram; the answer is in the footnote.[2]

7. *Cart and motor drive.* This is a harder example. Consider a cart with a motor drive, a DC motor that produces a torque. See Figure 2.12. The input is the voltage u to the motor, the output the cart position y. We want the model from u to y. To model the inclusion of a motor, draw the free body diagram in Figure 2.13. Moving from right to left, we have a force

[2] There are no input signals. Assuming the geometry of the tank is fixed and considering there is an initial time, say $t = 0$, the flow rate out at any time $t > 0$ depends uniquely on the height of water at time $t = 0$. Therefore, if we let $y(t)$ denote the flowrate out, the block diagram would be a single box, no input, one output labelled y.

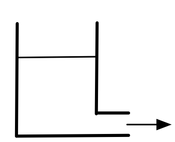


Figure 2.11: Water tank.

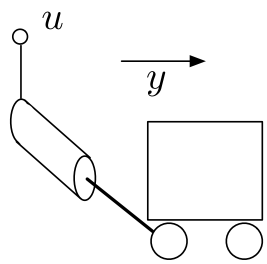


Figure 2.12: A cart with a motor drive.

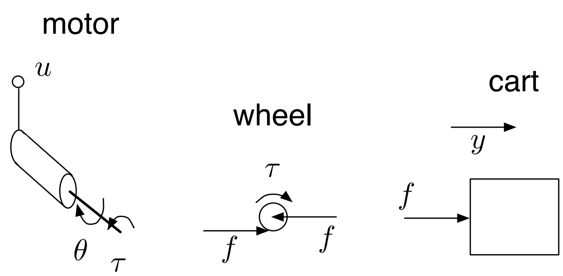


Figure 2.13: Free body diagram.

f on the cart via the wheel through the axle:

$$M\ddot{y} = f.$$

For the wheel, an equal and opposite force f appears at the axle and a horizontal force occurs where the wheel contacts the floor. If the inertia of the wheel is negligible, the two horizontal forces are equal. Finally, there is a torque τ from the motor. Equating moments about the axle gives

$$\tau = fr,$$

where r is the radius of the wheel. Now we turn to the motor. The electric circuit equation is

$$L\frac{di}{dt} + Ri = u - v_b,$$

where v_b is the back emf. The torque produced by the motor satisfies

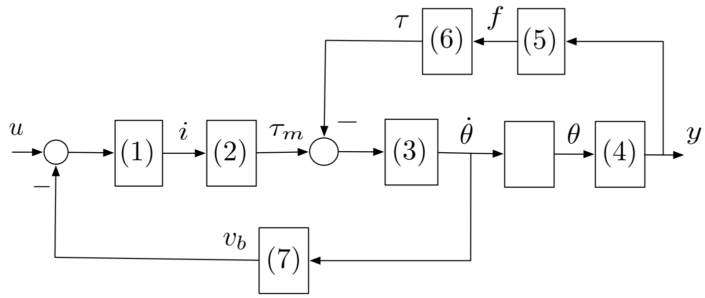
$$\tau_m = Ki.$$

Newton's second law for the motor shaft gives

$$J\ddot{\theta} = \tau_m - \tau.$$

Then the back emf is

$$v_b = K_b\dot{\theta}.$$



(1) $L\dfrac{di}{dt} + Ri = u - v_b$ (4) $y = r\theta$ (7) $v_b = K_b\dot{\theta}$

(2) $\tau_m = Ki$ (5) $f = M\ddot{y}$

(3) $J\dfrac{d}{dt}\dot{\theta} = \tau_m - \tau$ (6) $\tau = rf$

Figure 2.14: The cart and motor drive block diagram.

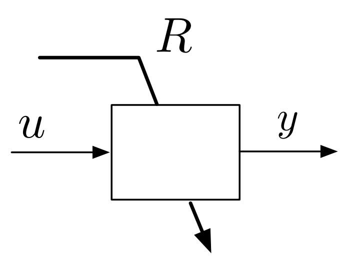


Figure 2.15: RC circut where R can be set externally.

Finally, the relationship between shaft angle and cart position is

$$y = r\theta.$$

Combining all this gives the block diagram of Figure 2.14.

8. Occasionally we might want to indicate that a parameter is a variable, or that it can be set at different values. For example, suppose that in the RC circuit in Figure 2.3 several values of R are available for us to select. The the value R ican be selected from an exteral power (us), and we can indicate this by an arrow going through the box as in Figure 2.15.

9. *Summary.* A block diagram is composed of arrows, boxes, and summing junctions. The arrows represent signals. The boxes represent systems or system components; mathematically they are functions that map one or more signals to one or more other signals. An exogenous input to a block diagram, e.g., u in Figure 2.4, is an independent variable. Other signals are dependent variables.

2.2 State equations

State equations are fundamental to the subject of dynamical systems. In this section we look at a number of examples. We begin with the notion of linearity.

1. *The concept of linearity.* To say that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is linear means the graph is a straight line through the origin; there's only one straight line that is not allowed—the y-axis. Thus $y = ax$ defines a linear function for any real constant a; the equation defining the y-axis is $x = 0$. The function $y = 2x + 1$ is not linear—its graph is a straight line, but not through the origin. In your linear algebra course you were taught that a **linear function**, or **linear transformation**, is a function f from a vector space $\mathcal{X}$ to another (or the same) vector space $\mathcal{Y}$ having the property

   $$f(a_1x_1 + a_2x_2) = a_1f(x_1) + a_2f(x_2)$$

   for all vectors x_1, x_2 in $\mathcal{X}$ and all real numbers[3] a_1, a_2. If the vector spaces are $\mathcal{X} = \mathbb{R}^n$, $\mathcal{Y} = \mathbb{R}^m$, and if f is linear, then it has the form $f(x) = Ax$, where A is an $m \times n$ matrix. Conversely, every function of this form is linear. This is a useful fact, so let us record it as the next item. We emphasize that $\mathbb{R}^n$ denotes the vector space of n-dimensional column vectors, and so the basis is fixed.

2. *Characterization of a linear function.* If A is an $m \times n$ real matrix and f is the function $f(x) = Ax$, then f is linear. Conversely, if f is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$, then there is a unique $m \times n$ matrix A such that $f(x) = Ax$.

3. *Proof.* Suppose A is given and $f(x) = Ax$. Then f is linear because

   $$A(a_1x_1 + a_2x_2) = a_1Ax_1 + a_2Ax_2.$$

   Conversely, suppose f is linear. We are going to build the matrix A column by column. Let e_1 denote this vector of dimension n:

   $$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

   Take the first column of A to be $f(e_1)$. Take the second column to be $f(e_2)$, where

   $$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

[3] This definition assumes that the vector spaces are over the field of real numbers.

And so on. Then certainly $f(x) = Ax$ for x equal to any of the vectors $e_1, e_2, \ldots$. But a general x has the form

$$x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = a_1 e_1 + \cdots + a_n e_n.$$

By linearity,

$$f(x) = f(a_1 e_1 + \cdots + a_n e_n) = a_1 f(e_1) + \cdots + a_n f(e_n).$$

But $a_1 f(e_1) + \cdots + a_n f(e_n)$ can be written as a matrix times a vector:

$$\begin{bmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

And this product is exactly Ax.

4. *Extension.* This concept of linear function extends beyond vectors to signals. In this book a signal is a function of time. For example, consider a capacitor, whose constitutive law is

$$i = C\frac{dv}{dt}.$$

Here, i and v are not constants, or vectors—they are functions of time. If we think of the current signal i as a function of the voltage signal v, then the function is linear. This is because

$$C\frac{d(a_1 v_1 + a_2 v_2)}{dt} = a_1 C\frac{dv_1}{dt} + a_2 C\frac{dv_2}{dt}.$$

On the other hand, if we try to view v as a function of i, then we have a problem, because we need, in addition, an initial condition $v(0)$ (or at some other initial time) to uniquely define v, not just i. Let us set $v(0) = 0$. Then v can be written as the integral of i like this:

$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau.$$

This *does* define a linear function $v = f(i)$.

5. *Terminology.* In control engineering, the system to be controlled is termed the **plant**.

6. *Example.* Figure 2.16 shows a cart on wheels, driven by a force u and subject to air resistance. Typically air resistance creates a force depending on the velocity, $\dot{y}$; let us say this force is a possibly nonlinear function $D(\dot{y})$. Assuming M is constant, Newton's second law gives

$$M\ddot{y} = u - D(\dot{y}).$$

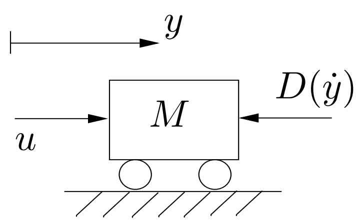


Figure 2.16: Cart on wheels.

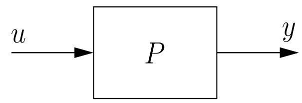


Figure 2.17: Block diagram of the cart.

This is a single second-order differential equation. It will be convenient to put it into two simultaneous first-order equations by defining two so-called state variables, in this example position and velocity:

$$x_1 := y, \quad x_2 := \dot{y}.$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}u - \frac{1}{M}D(x_2) \\ y &= x_1. \end{aligned}$$

These equations can be combined into

$$\dot{x} = f(x, u) \tag{2.1}$$

$$y = h(x), \tag{2.2}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f(x,u) = \begin{bmatrix} x_2 \\ \frac{1}{M}u - \frac{1}{M}D(x_2) \end{bmatrix}, \quad h(x) = x_1.$$

The function f is nonlinear if D is, while h is linear in view of Section 2.2, Paragraph 2 and

$$h(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Equations (2.1) and (2.2) constitute a state equation model of the system. The block diagram is shown in Figure 2.17. Here P is a possibly nonlinear system, u (applied force) is the input, y (cart position) is the output, and

$$x = \begin{bmatrix} \text{cart pos'n} \\ \text{cart velocity} \end{bmatrix}$$

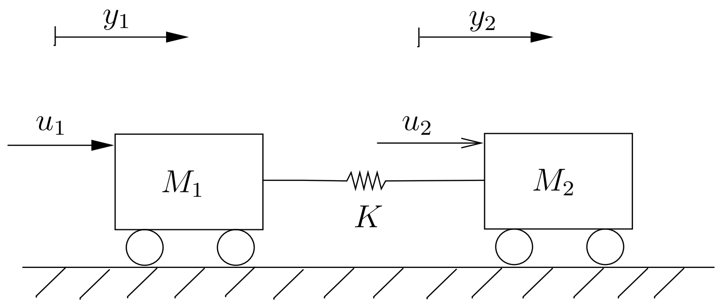


Figure 2.18: A 2-input, 2-output system.

is the state of P. (We'll define state later.) As a special case, suppose the air resistance is a linear function of velocity:

$$D(x_2) = D_0 x_2, \ D_0 \text{ a constant.}$$

Then f is linear:

$$f(x,u) = \begin{bmatrix} x_2 \\ \frac{1}{M}u - \frac{1}{M}D(x_2) \end{bmatrix} \quad f(x,u) = Ax + Bu, \quad A := \begin{bmatrix} 0 & 1 \\ 0 & -D_0/M \end{bmatrix}, \ B := \begin{bmatrix} 0 \\ 1/M \end{bmatrix}.$$

Defining $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, we get the state model

$$\dot{x} = Ax + Bu, \quad y = Cx. \tag{2.3}$$

This model is linear; it is also time-invariant because the matrices A, B, C do not vary with time. Thus the model is linear, time-invariant (LTI).

7. *Notation.* It is convenient to write vectors sometimes as column vectors and sometimes as n-tuples, i.e., ordered lists. For example

$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x = (x_1, x_2).$$

We shall use both.

8. *General statement.* An important class of models is

$$\dot{x} = f(x,u), \quad y = h(x,u),$$

where u, x, y are vectors that are functions of time, that is, $u(t), x(t), y(t)$. This model is nonlinear if f and/or h is nonlinear, but it is time-invariant because neither f nor h depends directly on time. Denote the dimensions of u, x, y by, respectively, m, n, p.

9. *Example.* An example where $m = 2, n = 4, p = 2$ is shown in Figure 2.18. For practice you should get the state equation, by taking

$$u = (u_1, u_2), \quad x = (y_1, \dot{y}_1, y_2, \dot{y}_2), \quad y = (y_1, y_2).$$

You will get equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned}$$

where A is 4×4, B is 4×2, C is 2×4, and D is 2×2.

10. *Example.* A time-varying example is the cart where the mass is decreasing with time because fuel is being used up (or because the cart is on fire). Newton's second law in this case is that force equals the rate of change of momentum, which is mass times velocity. So we have

$$\frac{d}{dt}Mv = u, \quad \dot{y} = v.$$

Expanding the first derivative gives

$$\dot{M}v + M\dot{v} = u, \quad \dot{y} = v.$$

Taking the state variables as usual to be $x_1 = y$, $x_2 = v$, we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{\dot{M}}{M}x_2 + \frac{1}{M}u \\ y &= x_1. \end{aligned}$$

In vector form:

$$\dot{x} = A(t)x + B(t)u, \quad y = Cx$$

$$A(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{\dot{M}(t)}{M(t)} \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ -\frac{1}{M(t)} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The matrices A and B are functions of t because M is a function of t. This model is linear time-varying.

11. *Terminology.* Explanation of the meaning of the **state of a system**: The state x at time t should encapsulate all the system dynamics up to time t, that is, no additional prior information is required. More precisely, the concept for x to be a state is this: For any t_0 and t_1, with $t_0 < t_1$, knowing $x(t_0)$ and knowing the input $\{u(t) : t_0 \leq t \leq t_1\}$, we can compute $x(t_1)$, and hence the output $y(t_1)$.

12. *Passive circuit.* The customary state variables are inductor currents and capacitor voltages. For a mechanical system the customary state variables are positions and velocities of all masses. The reason for this choice is illustrated as follows. Consider Figure 2.19, a cart with no external applied force. The differential equation model is $M\ddot{y} = 0$, or equivalently, $\ddot{y} = 0$.

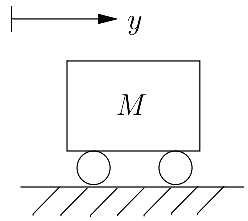


Figure 2.19: A cart with no forces.

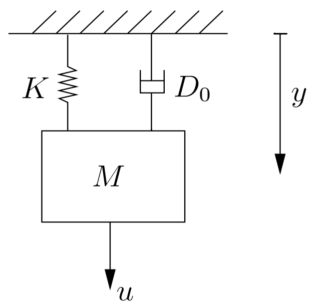


Figure 2.20: Mass-spring-damper.

To solve the equation we need two initial conditions, namely, $y(0)$ and $\dot{y}(0)$. So the state x could not simply be the position y, nor could it simply be the velocity $\dot{y}$. To determine the position at a future time, we need both the position and velocity at a prior time. Since the equation of motion, $\ddot{y} = 0$, is second order, we need two initial conditions, implying we need a 2-dimensional state vector.

13. *Another mechanical example.* Figure 2.20 shows a mass-spring-damper system. The rest length of the spring is y_0. Figure 2.21 shows the free-body diagram. The dynamic equation is

$$M\ddot{y} = u + Mg - K(y - y_0) - D_0\dot{y}.$$

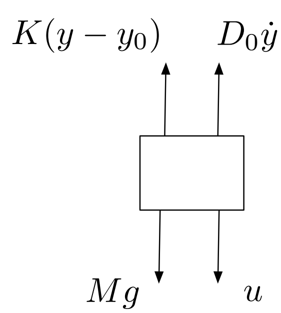


Figure 2.21: Free-body diagram.

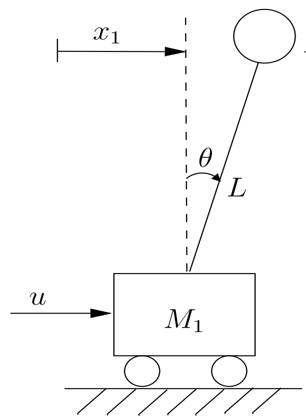


Figure 2.22: Cart-pendulum.

We take position and velocity as state variables,

$$x = (x_1, x_2), \quad x_1 = y, \quad x_2 = \dot{y},$$

and then we get the equations

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{M}u + g - \frac{K}{M}x_1 + \frac{K}{M}y_0 - \frac{D_0}{M}x_2 \\
y &= x_1.
\end{aligned}$$

These equations have the form

$$\dot{x} = Ax + Bu + c, \quad y = Cx, \tag{2.4}$$

where the matrices A, B, C, and the vector c are

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D_0}{M} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ g + \frac{K}{M}y_0 \end{bmatrix}.$$

The constant vector c is known, and hence is taken as part of the system rather than as a signal. Equations (2.4) have the form

$$\dot{x} = f(x, u), \quad y = h(x),$$

where f is not linear, because of the presence of c, while h is linear.

14. *A favourite toy control problem.* The problem is to get a cart automatically to balance a pendulum, as shown in Figure 2.22. The cart can move in a straight line on a horizontal table. The position of the cart is x_1 referenced to a stationary point. The pendulum, modeled as a point mass on the end of a rigid rod, is attached to a small rotary joint on the cart, so that the pendulum can fall either way but only in the direction that the cart can move. There

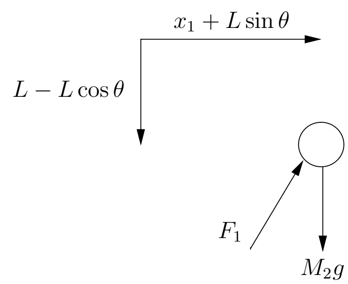


Figure 2.23: Free-body diagram of the pendulum.

is a drive mechanism that produces a force u on the cart. The figure shows the pendulum falling forward. Obviously, the cart has to speed up to keep the pendulum balanced, and the control problem is to design something that will produce a suitable force. That "something" is a controller, and how to design it is one of the topics in this book. The natural state is

$$x = (x_1, x_2, x_3, x_4) = (x_1, \theta, \dot{x}_1, \dot{\theta}).$$

We bring in a free body diagram, Figure 2.23, for the pendulum. The position of the ball is shown in a rectangular coordinate system with two axes: one is horizontally to the right; the other is vertically down. The axes intersect at an origin defined by the conditions $x_1 = 0$ and $\theta = 0$. Newton's law for the ball in the horizontal direction is

$$M_2 \frac{d^2}{dt^2}(x_1 + L\sin\theta) = F_1 \sin\theta$$

and in the vertical direction (down) is

$$M_2 \frac{d^2}{dt^2}(L - L\cos\theta) = M_2 g - F_1 \cos\theta.$$

The horizontal forces on the cart are u and $-F_1 \sin\theta$. Thus

$$M_1 \ddot{x}_1 = u - F_1 \sin\theta.$$

These are three equations in the four signals x_1, θ, u, F_1. We have to eliminate F_1. Use the identities

$$\frac{d^2}{dt^2}\sin\theta = \ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta, \quad \frac{d^2}{dt^2}\cos\theta = -\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta$$

to get

$$M_2\ddot{x}_1 + M_2 L\ddot{\theta}\cos\theta - M_2 L\dot{\theta}^2 \sin\theta = F_1 \sin\theta \tag{2.5}$$

$$M_2 L\ddot{\theta}\sin\theta + M_2 L\dot{\theta}^2\cos\theta = M_2 g - F_1\cos\theta \tag{2.6}$$

$$M_1\ddot{x}_1 = u - F_1\sin\theta. \tag{2.7}$$

We can eliminate F_1: Add (2.5) and (2.7) to get

$$(M_1 + M_2)\ddot{x}_1 + M_2 L\ddot{\theta}\cos\theta - M_2 L\dot{\theta}^2 \sin\theta = u;$$

multiply (2.5) by $\cos\theta$, (2.6) by $\sin\theta$, add, and cancel M_2 to get

$$\ddot{x}_1 \cos\theta + L\ddot{\theta} - g\sin\theta = 0.$$

Collect the latter two equations as

$$\begin{bmatrix} M_1 + M_2 & M_2 L\cos\theta \\ \cos\theta & L \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u + M_2 L\dot{\theta}^2 \sin\theta \\ g\sin\theta \end{bmatrix}.$$

Solve:

$$\ddot{x}_1 = \frac{u + M_2 L\dot{\theta}^2 \sin\theta - M_2 g \sin\theta\cos\theta}{M_1 + M_2 \sin^2\theta}$$

$$\ddot{\theta} = \frac{-u\cos\theta - M_2 L\dot{\theta}^2 \sin\theta\cos\theta + (M_1 + M_2) g\sin\theta}{L(M_1 + M_2\sin^2\theta)}.$$

Finally, in terms of the state variables we have

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 = \frac{u + M_2 L x_4^2 \sin x_2 - M_2 g \sin x_2 \cos x_2}{M_1 + M_2 \sin^2 x_2}$$
$$\dot{x}_4 = \frac{-u\cos x_2 - M_2 L x_4^2 \sin x_2 \cos x_2 + (M_1 + M_2) g \sin x_2}{L(M_1 + M_2 \sin^2 x_2)}.$$

Again, these have the form

$$\dot{x} = f(x, u).$$

We might take the output to be

$$y = \begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = h(x).$$

The system is highly nonlinear; as you would expect, it can be approximated by a linear system for $|\theta|$ small enough, say less than $5°$.

15. *Water tank.* Figure 2.24 shows water flowing into a tank in an uncontrolled way, and water flowing out at a rate controlled by a valve: The signals are x, the height of the water, u, the area of opening of the valve, and d, the flowrate in. Let A denote the cross-sectional area of the tank, assumed constant. Then conservation of mass gives

$$A\dot{x} = d - (\text{flow rate out}).$$

Also

$$(\text{flow rate out}) = (\text{const}) \times \sqrt{\Delta \mathrm{p}} \times (\text{area of valve opening}),$$

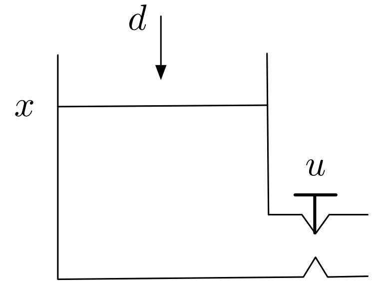


Figure 2.24: Water tank.

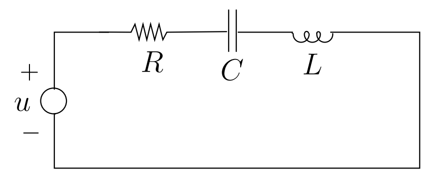


Figure 2.25: *RLC* circuit.

where Δp denotes the pressure drop across the valve, this being proportional to x. Thus

$$\text{(flow rate out)} = c\sqrt{x}u$$

and hence

$$A\dot{x} = d - c\sqrt{x}u.$$

The state equation is therefore

$$\dot{x} = f(x, u, d) = \frac{1}{A}d - \frac{c}{A}\sqrt{x}u.$$

16. *Exclusions.* Not all systems have state models of the form

$$\dot{x} = f(x, u), \quad y = h(x, u).$$

One example is the differentiator: $y = \dot{u}$. A second is a time delay: $y(t) = u(t-1)$. Finally, there are PDE models, e.g., the vibrating violin string with input the bow force.

17. *Another electric circuit example.* Consider the *RLC* circuit in Figure 2.25. There are two energy storage elements, the inductor and the capacitor. It is natural to take the state variables to be voltage drop across C and current through L: Figure 2.26. Then KVL gives

$$-u + Rx_2 + x_1 + L\dot{x}_2 = 0$$

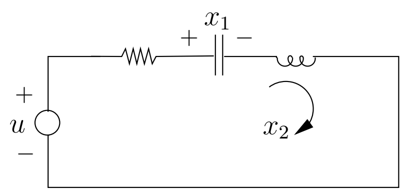


Figure 2.26: State variables.

and the capacitor equation is

$$x_2 = C\dot{x}_1.$$

Thus

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}.$$

18. *Higher order differential equations.* Finally, consider a system with input $u(t)$ and output $y(t)$ and differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_0 u.$$

The coefficients a_i, b_0 are real numbers. We can put this in the state equation form as follows. Define the state

$$x = (x_1, x_2, \ldots, x_n) = \left(y, \dot{y}, \frac{d^2y}{dt^2}, \ldots, \frac{d^{n-1}y}{dt^{n-1}} \right).$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_{n-1}x_n - \cdots - a_0 x_1 + b_0 u. \end{aligned}$$

The state model for these equations is

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

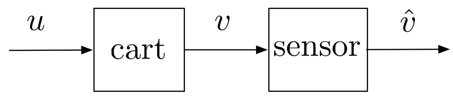


Figure 2.27: A cart with a sensor to measure velocity.

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The case

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u$$

is somewhat trickier. This has a state model if and only if $n \geq m$. We shall return to this in the next chapter. Briefly, one gets the transfer function from u to y and then gets a state model from the transfer function.

19. *Summary of state equations.* Many systems can be modeled in the form

$$\dot{x} = f(x,u), \quad y = h(x,u),$$

where u, x, y are vectors: u is the input, x the state, and y the output. This model is nonlinear if either f or h is not linear. However, the model is time invariant because neither f nor h has t as an argument. The linear time-invariant (LTI) case is

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

The functions $f(x,u) = Ax + Bu$ and $h(x,u) = Cx + Du$ are linear.

2.3 Sensors and actuators

In this section we briefly discuss sensors and actuators.

1. *Sensors.* Temperatures, positions, currents, forces, and so on can be measured by instruments called **sensors**. A common problem for motion control of mobile robots is to sense the forward velocity of the robot. This can be done by an optical rotary sensor. If placed on a robot's wheel, this gives a fixed number of voltage pulses for each revolution of the wheel, so by counting the number of pulses per second, you have a measurement of speed. If there are two wheels on the ends of an axle and each wheel has a rotary sensor, the two wheel turning rates can be used to determine the heading angle and forward speed.

   Suppose the speed of a cart is measured in this way. With v denoting the velocity of the cart, the output of the sensor measuring the speed would usually be denoted $\hat{v}$. See Figure 2.27.

   For a variety of reasons, all measurements have errors. Notice that physically $\hat{v}$ may be a voltage. Although a sensor is a dynamical system itself and therefore could be modelled, it is common to model it simply as a device that adds noise; see Figure 2.28. The noise can never be known exactly, so control engineers assume some generic noise signal using common sense and past experience. For example, one might take random white noise with a certain mean and a certain variance.

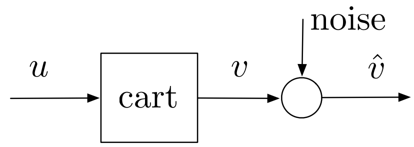


Figure 2.28: Position measurement modelled via additive noise.

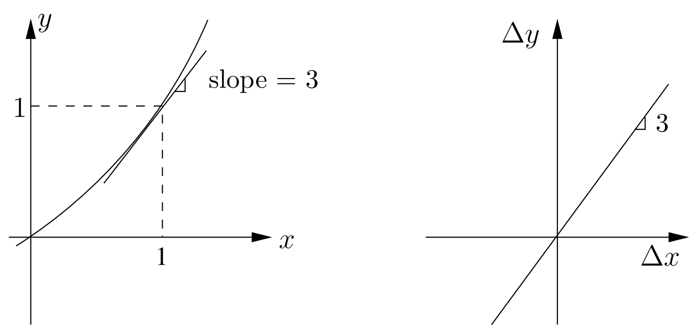


Figure 2.29: Graph of the nonlinear function $y = f(x)$ and its linearization at the point x_0.

2. *Actuators.* A sensor is typically at the output of the plant. At the input may be an actuator. Example: Paragraph 7 in Section 2.1 studies a cart with a motor drive. The force on the cart has to be produced by something, a motor in this case, and this motor is an example of an **actuator**. The actuator may be modelled or not. For example, in the motor-cart system, if the time-constant of the motor is much smaller than the fastest time-constant of the cart, then the dynamics of the motor can be neglected.

2.4 Linearization

Recap: Many systems can be modeled by nonlinear state equations of the form

$$\dot{x} = f(x, u), \quad y = h(x, u),$$

where u, x, y are vectors. There might be disturbance inputs present, but for now we suppose they are lumped into u. There are techniques for controlling nonlinear systems, but that is an advanced subject. Fortunately, many systems can be linearized about an equilibrium point. In this section we see how to do this.

1. *Example.* Linearizing just a function, not a dynamical system model. Let us linearize the function $y = f(x) = x^3$ about the point $x_0 = 1$. Figure 2.29 shows how to do it. At $x = x_0$, the value of y is $y_0 = f(x_0) = 1$. If x varies in a small neighbourhood of x_0, then y varies in a small neighbourhood of y_0. The graph of f near the point (x_0, y_0) can be approximated by the tangent to the curve, as shown in the left-hand figure. The slope of the tangent is the

derivative of f at x_0, $f'(x_0) = 3$. Therefore,

$$\frac{\Delta y}{\Delta x} \approx f'(x_0) = 3.$$

For the linearized function, we merely replace the approximation symbol by an equality:

$$\Delta y = 3\Delta x.$$

Notice that

$$\Delta y = y - y_0, \quad \Delta x = x - x_0.$$

So the linearized function approximates the nonlinear one in the neighbourhood of the point where the derivative is evaluated. Obviously, this approximation gets better and better as $|\Delta x|$ gets smaller and smaller.

2. *Extension.* The method extends to a function $y = f(x)$, where x and y are vectors. Then the derivative is the Jacobian matrix, which we shall denote by $f'(x_0)$.

3. *Example.* Consider the function

$$y = f(x), \quad x = (x_1, x_2, x_3), \quad y = (x_1 x_2 - 1, x_3^2 - 2x_1 x_3).$$

Let us linearize f at the point $x_0 = (1, -1, 2)$. The linearization is

$$\Delta y = A\Delta x,$$

where A equals the Jacobian of f at the point x_0. The element in the i^{th} row and j^{th} column is the partial derivative $\partial f_i / \partial x_j$, where f_i is the i^{th} element of f (i.e., y_i). So we have

$$\begin{aligned} f'(x_0) &= \left. \begin{bmatrix} x_2 & x_1 & 0 \\ -2x_3 & 0 & 2x_3 - 2x_1 \end{bmatrix} \right|_{x_0} \\ &= \begin{bmatrix} -1 & 1 & 0 \\ -4 & 0 & 2 \end{bmatrix}. \end{aligned}$$

Thus the linearization of $y = f(x)$ at x_0 is $\Delta y = A\Delta x$, where

$$A = f'(x_0) = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 0 & 2 \end{bmatrix}.$$

4. *Summary of the preceding two examples.* Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and x_0 a vector in $\mathbb{R}^n$. Assume f is continuously differentiable at the point x_0. Then the linearization of the equation $y = f(x)$ at the point x_0 is the equation $\Delta y = A\Delta x$, where $A = f'(x_0)$, the Jacobian of f at x_0. The variables are related by

$$x = x_0 + \Delta x, \quad y = f(x_0) + \Delta y.$$

We will find it useful to relate the block diagram for the equation $y = f(x)$ and the block diagram for the equation $\Delta y = A\Delta x$. These are shown in Figure 2.30. Because x_0 and $f(x_0)$ are known, we should regard the block diagram on the right as having just the input x and just the output y.

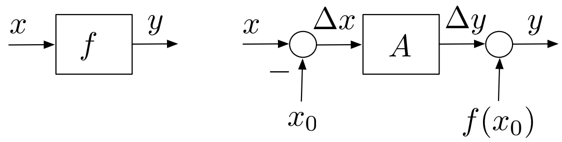


Figure 2.30: Block diagram of $y = f(x)$ and its linearization.

5. *Extension.* Now we consider the case $y = f(x, u)$, where u, x, y are all vectors, of dimensions m, n, p, respectively. We want to linearize at the point (x_0, u_0). We can combine x and u into one vector $v = \begin{bmatrix} x \\ u \end{bmatrix}$ of dimension $n + m$. Then we have the situation in the preceding example, $y = f(v)$. The Jacobian of f is an $n \times (n+m)$ matrix. Define

$$v_0 = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \quad \begin{bmatrix} A & B \end{bmatrix} = f'(v_0).$$

In this way we get the linearization

$$\begin{aligned} \Delta y &= f'(v_0)\Delta v \\ &= \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta u \end{bmatrix} \\ &= A\Delta x + B\Delta u. \end{aligned}$$

6. *Linearization of the differential equation*

$$\dot{x} = f(x, u).$$

First, **assume** there is an equilibrium point, that is, a constant solution $x(t) \equiv x_0, u(t) \equiv u_0$. This is equivalent to saying that $0 = f(x_0, u_0)$. Now consider a nearby solution:

$$x(t) = x_0 + \Delta x(t), \quad u(t) = u_0 + \Delta u(t).$$

We have

$$\begin{aligned} \dot{x}(t) &= f[x(t), u(t)] \\ &= f(x_0, u_0) + A\Delta x(t) + B\Delta u(t) + \text{higher order terms}, \end{aligned}$$

where

$$\begin{bmatrix} A & B \end{bmatrix} = f'(x_0, u_0).$$

Since $\dot{x} = \dot{\Delta x}$ and $f(x_0, u_0) = 0$, we have the linearized equation to be

$$\dot{\Delta x} = A\Delta x + B\Delta u.$$

Similarly, the output equation $y = h(x, u)$ linearizes to

$$\Delta y = C\Delta x + D\Delta u,$$

where

$$\begin{bmatrix} C & D \end{bmatrix} = h'(x_0, u_0).$$

7. *Summary.* To linearize the system $\dot{x} = f(x,u),\ y = h(x,u)$, select, if one exists, an equilibrium point, that is, constant vectors x_0, u_0 such that $f(x_0, u_0) = 0$. If the functions f and h are continuously differentiable at this equilibrium, compute the Jacobians $\begin{bmatrix} A & B \end{bmatrix}$ and $\begin{bmatrix} C & D \end{bmatrix}$ of f and h at the equilibrium point. Then the linearized system is

$$\dot{\Delta x} = A\Delta x + B\Delta u,\ \Delta y = C\Delta x + D\Delta u.$$

This linearized system is a valid approximation of the nonlinear one in a sufficiently small neighbourhood of the equilibrium point. How small a neighbourhood? There is no simple answer.

8. *Example of the cart-pendulum.* See page 21. An equilibrium point

$$x_0 = (x_{10}, x_{20}, x_{30}, x_{40}), \quad u_0$$

satisfies $f(x_0, u_0) = 0$, i.e.,

$$x_{30} = 0$$

$$x_{40} = 0$$

$$u_0 + M_2 L x_{40}^2 \sin x_{20} - M_2 g \sin x_{20} \cos x_{20} = 0$$

$$-u_0 \cos x_{20} - M_2 L x_{40}^2 \sin x_{20} \cos x_{20} + (M_1 + M_2) g \sin x_{20} = 0.$$

Multiply the third equation by $\cos x_{20}$ and add to the fourth:

$$-M_2 g \sin x_{20} \cos^2 x_{20} + (M_1 + M_2) g \sin x_{20} = 0.$$

Factor the left-hand side:

$$(\sin x_{20})(M_1 + M_2 \sin^2 x_{20}) = 0.$$

The right-hand factor is positive; it follows that $\sin x_{20} = 0$ and therefore x_{20} equals 0 or π, that is, the pendulum is straight down or straight up. Thus the equilibrium points are described by

$$x_0 = (\text{arbitrary}, 0 \text{ or } \pi, 0, 0), \quad u_0 = 0.$$

We have to choose $x_{20} = 0$ (pendulum up) or $x_{20} = \pi$ (pendulum down). Let us take $x_{20} = 0$. Then the Jacobian computes to

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{M_2}{M_1} g & 0 & 0 \\ 0 & \frac{M_1+M_2}{M_1}\frac{g}{L} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ -\frac{1}{LM_1} \end{bmatrix}.$$

We just applied the general method of linearizing. For this example, there's actually a faster way, which is to approximate $\sin\theta = \theta,\ \cos\theta = 1$ in the original equations.

2.5 Interconnections of linear subsystems

Frequently, a system is made up of components connected together in some arrangement. This raises the question, if we have state models for components, how can we assemble them into a state model for the overall system?

1. *Review.* This section involves some matrix algebra. Let us summarize what you need to know. If we have two vectors, x_1 and x_2, of dimensions n_1 and n_2, we can stack them as a vector of dimension $n_1 + n_2$. Of course, we can stack them either way:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ or } \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}.$$

Then, two state equations

$$\dot{x}_1 = A_1 x_1 + B_1 u_1$$
$$\dot{x}_2 = A_2 x_2 + B_2 u_2$$

can be combined into one state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Be careful: If you stack x_2 above x_1, the matrices will be different. In the matrix

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

the two zeros are themselves matrices of all zeros. Usually, we don't care how many rows or columns they have, but actually their sizes can be deduced. For example, if x_1 has dimension n_1 and x_2 has dimension n_2, then the sizes of zero blocks must be as shown here:

$$\begin{bmatrix} A_1 & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & A_2 \end{bmatrix}.$$

This is because the upper-right zero must have the same number of rows as A_1 and the same number of columns as A_2; likewise for the lower-left zero.

Finally, multiplication of a block vector by a block matrix, assuming the dimensions are correct, works just as if the blocks were 1×1. For example, in the product

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we multiply just as though A_1, A_2, x_1, x_2 were scalars:

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 x_1 \\ A_2 x_2 \end{bmatrix}.$$

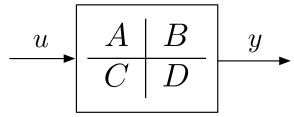


Figure 2.31: Block diagram of the state model $\dot{x} = Ax + Bu, \quad y = Cx + Du$.

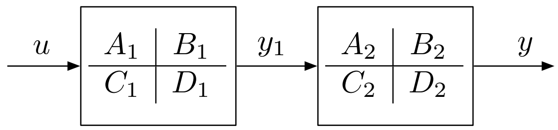


Figure 2.32: Series connection.

2. *Notation.* There is a very handy way to embed a state model into a block diagram. Suppose we have the state model

$$\dot{x} = Ax + Bu \tag{2.8}$$
$$y = Cx + Du. \tag{2.9}$$

The input is u, the output is y, and x is the state. A block diagram for this component has u on the input arrow and y on the output arrow. The system in the box is modeled by the state equations. It is convenient to encode these equations into the block diagram as in Figure 2.31. The symbol $\begin{array}{c|c} A & B \\ \hline C & D \end{array}$ in the block is just an abbreviation for equations (2.8), (2.9).

3. *Example, series connection.* Figure 2.32 shows a series connection of two subsystems. We want to get a state model from u to y. Write the state equations for the two blocks:

$$\dot{x}_1 = A_1 x_1 + B_1 u$$
$$\dot{x}_2 = A_2 x_2 + B_2 y_1.$$

Let us take the combined state to be

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then the combined preceding two equations become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} y_1.$$

The intermediate signal y_1, being the output of the left-hand block, equals $C_1 x_1 + D_1 u$, or, in terms of the combined state and u,

$$y_1 = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_1 u. \tag{2.10}$$

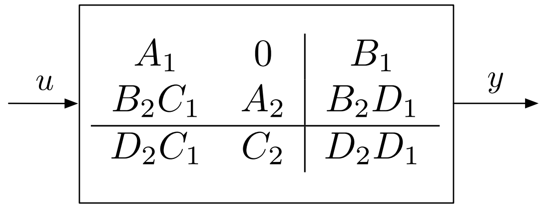


Figure 2.33: Series connection as a combined system.

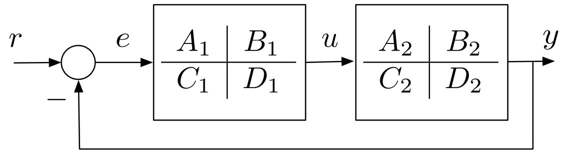


Figure 2.34: Feedback connection.

Substituting this into the preceding equation gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} u.$$

Then, the system output y is

$$\begin{aligned} y &= C_2x_2 + D_2y_1 \\ &= \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2y_1. \end{aligned}$$

Substituting in y_1 from (2.10) gives

$$y = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2D_1u.$$

The combined state equations can be written in the standard form

$$\dot{x} = Ax + Bu, \; y = Cx + Du,$$

where

$$A = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}$$

$$C = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix}, \; D = D_2D_1.$$

We conclude that the block diagram Figure 2.32 can be simplified to Figure 2.33.

4. *Example, feedback connection.* Figure 2.34 shows a feedback arrangement. Feedback is introduced later in the book, but here we merely want to do some manipulations with state models and block diagrams. Specifically, we want to derive a state model from r to y. The derivation is simpler under the assumption $D_2 = 0$, so we make that assumption. The block diagram has two blocks. Write the state equations for the two blocks:

$$\begin{aligned}\dot{x}_1 &= A_1 x_1 + B_1 e\\ \dot{x}_2 &= A_2 x_2 + B_2 u.\end{aligned}$$

Combine:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix}. \tag{2.11}$$

Write the equations for e and u in terms of r, x_1, x_2:

$$\begin{aligned}e &= r - C_2 x_2\\ u &= C_1 x_1 + D_1 e\\ &= C_1 x_1 + D_1 r - D_1 C_2 x_2.\end{aligned}$$

Combine:

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} 0 & -C_2 \\ C_1 & -D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I \\ D_1 \end{bmatrix} r.$$

Substitute into (2.11) and simplify:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & -B_1 C_2 \\ B_2 C_1 & A_2 - B_2 D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} r.$$

Then the output y is

$$y = C_2 x_2 = \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We therefore have the combined model

$$\dot{x} = Ax + Br, \quad y = Cx$$

where

$$A = \begin{bmatrix} A_1 & -B_1 C_2 \\ B_2 C_1 & A_2 - B_2 D_1 C_2 \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & C_2 \end{bmatrix}.$$

We conclude that the block diagram Figure 2.34 can be simplified to Figure 2.35.

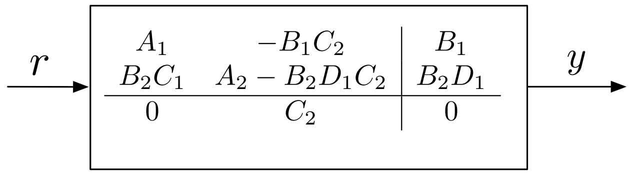


Figure 2.35: Feedback connection as a combined system.

2.6 "Nothing is as abstract as reality"

The title of this section is a quote from the artist Giorgio Morandi.

Mathematical models of physical things are approximations of reality. Consider, for example, a real swinging pendulum and a mathematical model of it. The mathematical model involves a parameter L, the length, whereas the real pendulum does not have a real length at the subatomic scale. The length L is an attribute of an idealized pendulum. Going past this issue of what L is, the mathematical model assumes perfect rigidity of the pendulum. From what we know about reality, nothing is perfectly rigid; that is, rigidity is a concept, an approximation to reality. So if we wanted to make our model "closer to reality," we could allow some elasticity by adopting a partial differential equation model and we may thereby have a better approximation. But no model is real. There could not be a sequence $M_0, M_1, \dots$ of models that are better and better approximations of reality and such that M_k converges to reality. If M_k does indeed converge, the limit is a model, and no model is real.

The only sensible question is, what do we mean by a "good model," or, if we have two models, how can we say which is better? We can test our model against the real thing. That is, we can do several tests on the real thing, perform the same test on the model, and compare the resulting measured data with the simulated data. If the two sets of data are close, and if the measuring instruments are reasonably accurate, then we can say that the model is quite good.

For more along these lines, see the article "What's bad about this habit," N. D. Mermin, *Physics Today*, May 2009, pages 8, 9.

2.7 Problems

1. Consider an electric circuit consisting of, in series, a voltage source supplying $u(t)$ volts, a resistor, an inductor, and a battery of 10 V. Take the state to be the current. Find the state equation $\dot{x} = f(x, u)$. Find all equilibria and linearize about one of them. Hint: The circuit would be linear were it not for the battery.

2. Let x and y be vectors and A a matrix. Consider the block diagram in Figure 2.36. According to the diagram, the matrix A must partition into two blocks: $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$. Then $y = A_1x + A_2y$. When does this equation define a function from x to y?

   Solution The equation $y = A_1x + A_2y$ defines a function from x to y if and only if for every x there exists a unique y satisfying $(I - A_2)y = A_1x$. For the vector y to be unique, the matrix $I - A_2$ must be invertible. In this case, $y = (I - A_2)^{-1}A_2x$. This equation defines a

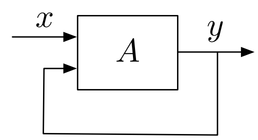


Figure 2.36: Feedback in a matrix system.

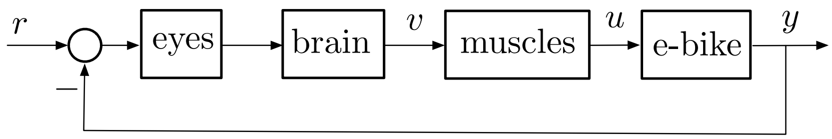


Figure 2.37: E-bike problem.

function from x to y. Thus the answer is, Figure 2.36 defines a function if and only if $I - A_2$ is invertible (equivalently, 1 is not an eigenvalue of A_2).

3. Imagine you are riding an e-bike at constant speed along a street. You are steering the e-bike to control the direction you are going. Draw the block diagram.

   Solution The e-bike, with you sitting on it, is a system component with input u, your force turning the handlebars, and output y, the direction of motion of the e-bike. The force u comes from your arm muscles, with input v an electrical voltage generated by your brain. The input to your brain is the error as seen by your eyes, the error between the desired direction r you want to go and the actual direction. The resulting block diagram is shown in Figure 2.37.

4. Modeling a bicycle is harder than you might think. Imagine a rider on a bike and the bike on the road. Take the overall output to be the bicycle position (in an (x, y)-plane). What's the overall input? The rider can apply forces to the pedals, so they are inputs; so is the torque applied to the steering wheel; and so is a leaning torque applied by the rider's muscles. Try getting a block diagram where the bicycle is one block and the person another, and there may be more.

5. There are two cars and a road. One car is to be driven by a person along the road continuously in one direction. The second car is required to follow the first at a fixed distance, but without a human driver. Suppose the second car has a camera that can see the first car, some mechanism to steer and speed up and slow down, and a computer with a program in it. The program computes a rule to steer and to speed up or slow down accordingly. The system may work well or not—we're not interested in that aspect. Draw a block diagram of this system.

6. Suppose you have a car with a GPS navigation system. The system has a screen showing a map and there's an arrow on the map showing where your car is. As the car moves, the map evolves so that the arrow stays in the middle of the screen. Draw a block diagram of this setup.

7. Consider a force-feedback joystick connected to a laptop, with a person applying a force to the joystick. Suppose the laptop is connected to another laptop through the Internet. This

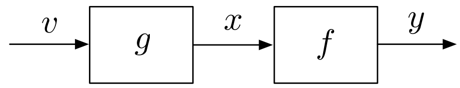


Figure 2.38: Composition of two functions.

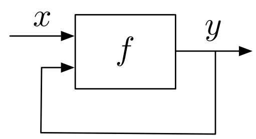


Figure 2.39: Feedback around a function.

second laptop is in a loop controlling a cart. Finally, the cart may bump into an obstacle. This is a telerobotic architecture: The first laptop and the joystick are the master manipulator, the second laptop and the cart are the slave. Let's say the system should have the following capabilities: When the person applies a force to the joystick, the remote cart should move appropriately; when the cart hits the obstacle, the force should be reflected back to the person.

Let us first model the joystick. It is a DC motor and the relevant variables are the voltage to the field windings, the torque that is generated by the magnetic field and applied to the shaft, the torque applied to the shaft by the person, and the shaft angle. Continue modeling in this way and get a block diagram.

8. If the composition $f \circ g$ is defined, i.e., the co-domain of g is contained in the domain of f, then the block diagram is as shown in Figure 2.38. Here v is the overall input and y the overall output of the system composed of f and g combined in the order shown. So far we haven't said what v, x, y represent. For example, they could be real numbers, in which case f and g are functions $\mathbb{R} \longrightarrow \mathbb{R}$. Give examples of nonlinear f, g and find the system from v to y.

   Solution Let f be defined by $y = x^2 + 1$ and g be defined by $x = \sin(v)$. Then

   $$y = x^2 + 1 = \sin^2 v + 1.$$

9. Figure 2.39 is more interesting system. Obviously this represents the equation $y = f(x, y)$. That is, f is a function of two variables, and we've attempted to define a new function by the equation $y = f(x, y)$. For the block diagram to be well-defined, that is, to represent a function, it must be true that for every x there exists a unique y such that $y = f(x, y)$. Give an example of f where the block diagram defines a function and one where it does not.

   Solution Let $f(x, y) = 2x - 3y$. Then we can solve $y = f(x, y)$ to get $y = x/2$. On the other hand, if we take f to be $f(x, y) = x + y$, then the equation $y = f(x, y)$ becomes $y = x + y$, which is not solvable for y.

10. Why does the block diagram in Figure 2.40 not define a function?

    Solution There is no input, i.e., independent variable.

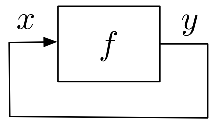


Figure 2.40: Is this a function?

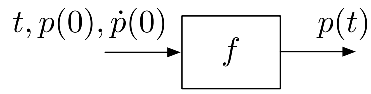


Figure 2.41: Problem about throwing a baseball.

11. A baseball is thrown and we want to model the ball's motion while in flight. Drawing a block diagram means determining what the variables are and how they depend on each other. Let us neglect the ball's rotation and think of it as a point mass. One variable is therefore the position of the ball, with respect to, say, a coordinate system fixed to the earth. Let us denote position at time t by $p(t)$, a three-dimensional vector. What does $p(t)$ depend on? The force of gravity, but that is fixed, not a variable, and therefore not an input. The position of the ball depends obviously on the force with which it was thrown, where it was thrown from, and what direction it was thrown. Equivalently, $p(t)$ depends on the initial position and velocity, $p(0), \dot{p}(0)$, and the time t. Thus the block diagram is Figure 2.41. The input is a vector of 7 real numbers and the output is a vector of 3 real numbers. The equation is

$$p(t) = f(t, p(0), \dot{p}(0)),$$

What parameters does f depend on?

Solution The function f depends only on the mass of the ball and the gravity constant g.

12. Consider two carts connected by a spring. A force u is applied to one of the carts. Let the positions of the carts be y_1, y_2 and suppose we designate y_2 to be the overall output. A common way to model this is via free-body diagrams and Newton's second law. Letting v denote the force applied to cart 1 via the spring, we get equations

$$\begin{aligned} M_1\ddot{y}_1 &= u - v \\ M_2\ddot{y}_2 &= v \\ v &= K(y_1 - y_2). \end{aligned}$$

Let us assume zero initial conditions: $y_1(0), \dot{y}_1(0), y_2(0), \dot{y}_2(0)$ all zero. Then the equations have the general form

$$\begin{aligned} y_1 &= f_1(u, v) \\ y_2 &= f_2(v) \\ v &= f_3(y_1, y_2). \end{aligned}$$