3.4 The Product and Quotient Rules

The derivative of a sum of functions is the sum of the derivatives. So you might assume the derivative of a product of functions is the product of the derivatives. Consider, however, the functions $f(x) = x^3$ and $g(x) = x^4$. In this case, $\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(x^7) = 7x^6$, but $f'(x)g'(x) = 3x^2 \cdot 4x^3 = 12x^5$. Therefore, $\frac{d}{dx}(f \cdot g) \neq f' \cdot g'$. Similarly, the derivative of a quotient is *not* the quotient of the derivatives. The purpose of this section is to develop rules for differentiating products and quotients of functions.

Product Rule

Here is an anecdote that suggests the formula for the Product Rule. Imagine running along a road at a constant speed. Your speed is determined by two factors: the length of your stride and the number of strides you take each second. Therefore,

$$\text{running speed} = \text{stride length} \cdot \text{stride rate}.$$

For example, if your stride length is 3 ft per stride and you take 2 strides/s, then your speed is 6 ft/s.

Now suppose your stride length increases by 0.5 ft, from 3 to 3.5 ft. Then the change in speed is calculated as follows:

$$\begin{aligned} \textit{change} \text{ in speed} &= \text{change in stride length} \cdot \text{stride rate} \\ &= 0.5 \cdot 2 = 1 \text{ ft/s}. \end{aligned}$$

Alternatively, suppose your stride length remains constant but your stride rate increases by 0.25 stride/s, from 2 to 2.25 strides/s. Then

$$\begin{aligned} \textit{change} \text{ in speed} &= \text{stride length} \cdot \text{change in stride rate} \\ &= 3 \cdot 0.25 = 0.75 \text{ ft/s}. \end{aligned}$$

If both your stride rate and stride length change simultaneously, we expect two contributions to the change in your running speed:

$$\begin{aligned} \textit{change} \text{ in speed} &= (\text{change in stride length} \cdot \text{stride rate}) \\ &\quad + (\text{stride length} \cdot \text{change in stride rate}) \\ &= 1 \text{ ft/s} + 0.75 \text{ ft/s} = 1.75 \text{ ft/s}. \end{aligned}$$

This argument correctly suggests that the derivative (or rate of change) of a product of two functions has *two components*, as shown by the following rule.

➤ In words, Theorem 3.7 states that the derivative of the product of two functions equals the derivative of the first function multiplied by the second function, plus the first function multiplied by the derivative of the second function.

THEOREM 3.7 Product Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Proof: We apply the definition of the derivative to the function fg:

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

A useful tactic is to add $-f(x)g(x+h) + f(x)g(x+h)$ (which equals 0) to the numerator, so that

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$