

Introduction to Algorithms and Data Structures

Lecture 7: Classic datatypes: lists, stacks, queues

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'Lists' in general ...

We've seen that 'lists' can be implemented in several ways, e.g. via arrays or linked lists. **How might we compare these?**

Start by listing the **operations** we'd like any impl to support.

E.g. for (unsorted) lists of items of type X , might want operations

| | | |
|---------------|--|---------------------------|
| get | : $\text{int} \rightarrow X$ | # read item at given pos |
| set | : $\text{int} * X \rightarrow \text{void}$ | # write item at given pos |
| cons | : $X \rightarrow \text{void}$ | # add item at start |
| append | : $X \rightarrow \text{void}$ | # add item at end |
| insert | : $\text{int} * X \rightarrow \text{void}$ | |
| delete | : $\text{int} \rightarrow \text{void}$ | |
| length | : $\text{void} \rightarrow \text{int}$ | |

Much like an **interface** in Java.

Abstract interfaces, concrete implementations

As in Java, we can consider various concrete **implementations** of this abstract **interface**.

Further points:

- ▶ For some purposes, could consider an interface with fewer operations, or with more: e.g. **reverse** : `void → void`.
- ▶ May be other operations that make sense for specific impl's. E.g. for linked lists, 'insert/delete at current position' is useful.
- ▶ Some of our operations will be definable from others: e.g.

$$\mathbf{cons}(x) \equiv \mathbf{insert}(0, x)$$

But may wish to include **cons** in its own right: might be implementable more efficiently than general **insert**.

Implementation 1: Fixed-size arrays

Use an **array** A of some **fixed** size m .

Can store a list $L = x_0, \dots, x_{n-1}$ (where $n \leq m$)
in the first n cells of A (so $A[i] = x_i$ for each $i < n$).

Also want an integer variable n to store the value of n .

List operations are easy to implement. E.g.

get(i):

return $A[i]$

append(x):

$A[n] = x$

$n = n+1$

insert (i, x):

for $j = n-1$ downto i

$A[j+1] = A[j]$

$A[i] = x$

$n = n+1$

- ▶ **length**, **get**, **set** and **append** (when it works) take $\Theta(1)$ time.
- ▶ **cons**, **insert**, **delete** require $\Theta(n)$ time in worst case.

Lists via fixed-size arrays, ctd.

Fixed-size arrays have some strengths . . .

- ▶ Fast **get** and **set** operations – especially if we can keep the array on the stack!
- ▶ Fixed, predictable size good for memory management.
(If on stack, can reclaim space immediately on expiry.)

. . . but a major weakness . . .

- ▶ Can't cope with lists longer than pre-set limit m !
- ▶ If a computation involves a lot of lists, of unpredictable sizes, very likely we'll either **under-cater** (some array will overflow) or **over-cater** (many arrays will contain a lot of wasted space).

So not a good choice for 'general-purpose' lists.

Implementation 2: Extensible arrays

Idea is simple: if array A overflows, replace it by a bigger one!

- ▶ If memory space ‘after’ A happens to be free, cheap to do.
- ▶ But if not, may have to allocate a fresh array B , and copy contents of A into it. E.g. for some real number $r > 1$:

```
append (x):  
  if  $n = |A|$   
     $B = \text{new array } (\lceil n \times r \rceil)$   
    copy contents of  $A$  into  $B$  ( $n$  items)  
     $A = B$   
  # Now do ordinary append:  
   $A[n] = x$   
   $n = n+1$ 
```

So a ‘normal’ append takes $\Theta(1)$ time – but occasionally we may get a bad one, taking $\Theta(n)$.

Might seem ‘dirty’, but widely used in practice.

Runtime analysis is interesting ...

Amortized cost

Perhaps in some apps, even one **bad append** could be fatal.

But often, we're happy if over any long run of **appends**, the *average* time is reasonable. A bad one may be acceptable if we regard its cost as **amortized** ('spread out') over the next 100 good ones – i.e. if invested effort 'pays for itself' over time.

Does it?

Suppose array has initial capacity a , and starting from **nil** we do m **appends** in succession, expanding by factor $r > 1$ when need be.

Array size grows as a, ar, ar^2, ar^3, \dots . How many steps to reach m ? Solving ' $ar^s = m$ ' yields $s = \log_r(m/a)$ for the number of steps. An item may get copied this many times!

Since potential number of copyings of an item grows with m , might suspect 'average cost per **append**' also grows with $m \dots ??$

Let's do the sums.

Calculating amortized cost of append

Example: Suppose $a = 100$, $r = 1.1$, $m = 5000$.

Note that $1.1^{41}a < m < 1.1^{42}a$. So will need 42 expansions.

Ignoring 'rounding', number of copyings ($B[i] = A[i]$) is basically

$$100 \times (1 + 1.1 + 1.1^2 + \dots + 1.1^{41})$$

By 'sum of geometric progression' formula, this is

$$100 \times (1.1^{42} - 1)/(1.1 - 1) < 1.1m/0.1$$

So although some items get copied 42 times,

average no. of copyings per item stays below $1.1/0.1 = 11$.

In general, total number of copyings is basically at most $m(r/(r-1))$.

So **average** no. of copyings per item stays below $r/(r-1)$.

More conceptual argument

Again suppose $a = 100$, $r = 1.1$.

Imagine a copying costs 1p. Each time we do an **append**, we pay 11p into a **pension fund** to pay for future copyings.

Suppose we've just done our first expansion.

Array now has 110 cells, with 100 filled.

Next 10 **appends** pay for second expansion (110 copyings).

After second expansion, array has 121 cells, 110 filled.

Next 11 **appends** pay for third expansion (121 copyings) ...

So each **append** incurs a constant cost of 11 copyings.

Amortized cost: conclusion

So total time taken by expansion/copying is $O(m)$.

But time taken by ordinary **appends** is also clearly $O(m)$.

So may say the **amortized cost of append** is $O(1)$ per operation.

- ▶ Lists in Python are implemented like this, essentially with $r = 9/8$. Underlying arrays may also be shrunk if proportion in use dips below $1/2$. (For analysis, see CLRS 17.4.)
- ▶ Java class **ArrayList** also works like this. Precise expansion policy not prescribed, but it's required that amortized cost over a long run must be $O(1)$ per operation.

Of course, **cons**, **insert**, **delete** still take time $\Theta(n)$ in worst case (even amortized).

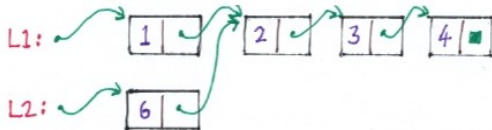
Implementation 3: Linked lists

We can also represent the lists over X using **linked lists**, where each cell contains a **key** of type X .

Clearly, for a list of length n :

- ▶ **get** and **set** have $\Theta(n)$ worst-case time (but with small 'C')
- ▶ **cons** takes $\Theta(1)$ time, **always**.
- ▶ **insert**(i, x), **delete**(i) have $\Theta(n)$ worst-case time (or $\Theta(1)$ if we've already located the cell at position $i - 1$).

Linked lists also naturally allow for **sharing** (unlike arrays):



Offline exercise: Show how the list of all 2^n binary lists of length n can be stored in $\Theta(2^n)$ space with linked list impl. (Would take $\Theta(n \cdot 2^n)$ with arrays.)

List implementations: summary

Upper bounds on runtimes (where n is length of list):

| Operation | Array impl | Linked-list impl |
|---------------|------------------------------|------------------------------|
| get | $O(1)$ | $O(n)$ |
| set | $O(1)$ | $O(n)$ |
| cons | * $O(n)$ | $O(1)$ |
| append | * $O(n)$ (amortized $O(1)$) | $O(n)$ (can make it $O(1)$) |
| insert | * $O(n)$ | $O(n)$ |
| delete | $O(n)$ | $O(n)$ |

Operations marked * may fail for fixed-array implementations, or trigger expansion for extensible-array ones.

So arrays offer fast **get/set**; linked lists offer fast **cons/append** and **insert/delete** at given position, plus sharing.

?? Is there some impl of lists for which *all* the above are 'fast' ??
Find out in Lecture 9!

Stacks and queues

Sometimes, we know that some list will only be manipulated in certain restricted ways, e.g. ...

- ▶ Elements only ever added/read/removed at front of list
(**stack** or **Last-in-first-out buffer**)
- ▶ Elements added at back, read/removed at front of list
(**queue** or **First-in-first-out buffer**)

Knowing this may affect our choice of implementation.

Interfaces for stacks and queues (of items of type X):

STACKS:

empty : $\text{void} \rightarrow \text{bool}$
push : $X \rightarrow \text{void}$
peek : $\text{void} \rightarrow X$
pop : $\text{void} \rightarrow X$

QUEUES:

empty : $\text{void} \rightarrow \text{bool}$
enqueue : $X \rightarrow \text{void}$
peek : $\text{void} \rightarrow X$
dequeue : $\text{void} \rightarrow X$

Implementing stacks

In principle, **any** impl of lists yields an impl of stacks:

But two obvious candidates:

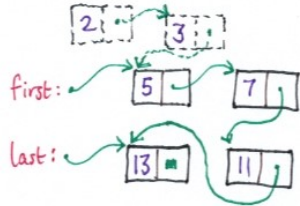
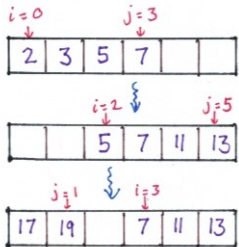
- ▶ **arrays** (growing at end)
- ▶ **linked lists** (growing at start)

| Operation | Extensible array impl | Linked list impl |
|--------------|---|------------------|
| empty | $O(1)$ | $O(1)$ |
| push | $* O(n)$ (amortized $O(1)$) | $O(1)$ |
| peek | $O(1)$ | $O(1)$ |
| pop | $O(1)$ | $O(1)$ |

Implementing queues

Impl 1: Wraparound array buffer (fixed-size/extensible)

Impl 2: Linked list with references to first and last cells



Implementing queues, ctd.

How would e.g. **enqueue** look in each case?

Wraparound array:

enqueue(x):

$j = (j+1) \bmod |A|$

 if $j = i$

 fail (or expand)

 else $A[j] = x$

Linked list:

enqueue(x):

$\text{last.next} = \text{new Cell}(x, \text{null})$

$\text{last} = \text{last.next}$

For further details, see Python Lab Sheet 3.

Situation similar to stacks:

| Operation | Wraparound array impl | Linked-list impl |
|----------------|------------------------------|------------------|
| enqueue | * $O(n)$ (amortized $O(1)$) | $O(1)$ |
| peek | $O(1)$ | $O(1)$ |
| dequeue | $O(1)$ | $O(1)$ |

Reading

Stacks and queues: CLRS chapter 10.

Table expansion / amortized analysis: CLRS section 17.4.