Smart Search using Constraints

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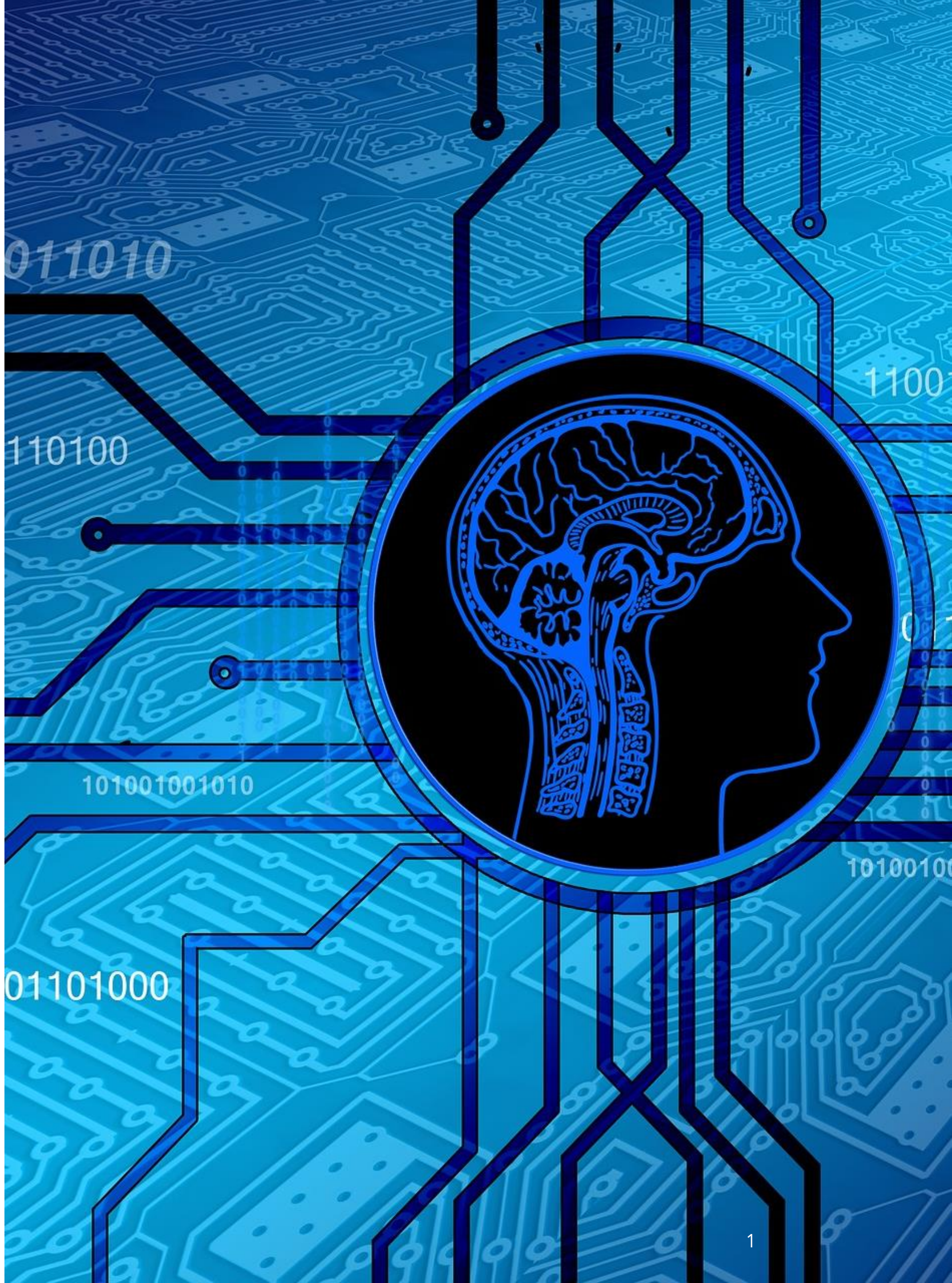
Informatics 2D: Reasoning and Agents
Lecture 5



Constraint satisfaction problems (CSPs)

State

- Set of *variables* X_i with *values* from *domain* D_i

Actions

- *Assign* a value to a variable

Goal test

- A set of *constraints* specifying allowable combinations of values for subsets of variables

Path cost

- None

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Simple example of a *formal representation language*.

Allows useful *general-purpose* algorithms with more power than standard search algorithms.

Structure of a CSP

- A set of **variables**: $X=\{X_1,\dots X_n\}$
- A set of **domains**: $D=\{D_1,\dots D_n\}$
  - each domain D_i is a set of possible values for variable X_i
- A set of **constraints** C that specify acceptable combinations of values.
  - Each $c \in C$ consists of:
    - a **scope** – tuple of variables (neighbours) involved in the constraint
    - a **relation** that defines the values that the variables can take

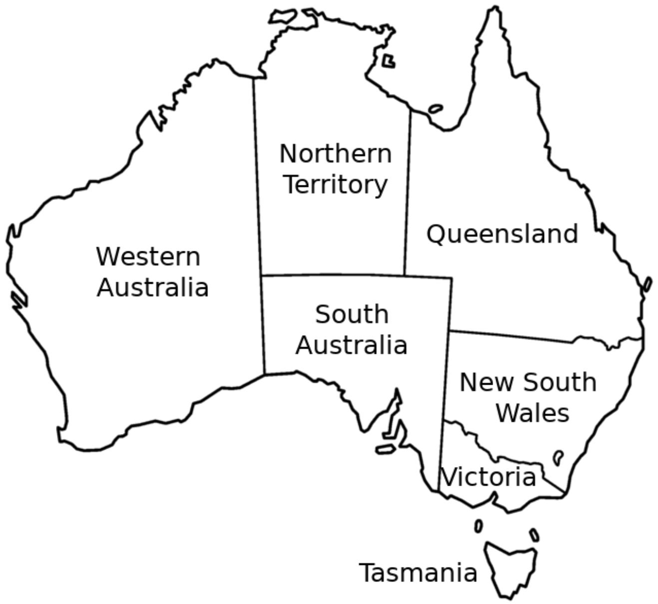
Example: Map-Colouring

Variables: $\{WA, NT, Q, NSW, V, SA, T\}$

Domains: D_i = {red, green, blue}

Constraints: adjacent regions must have different colours,

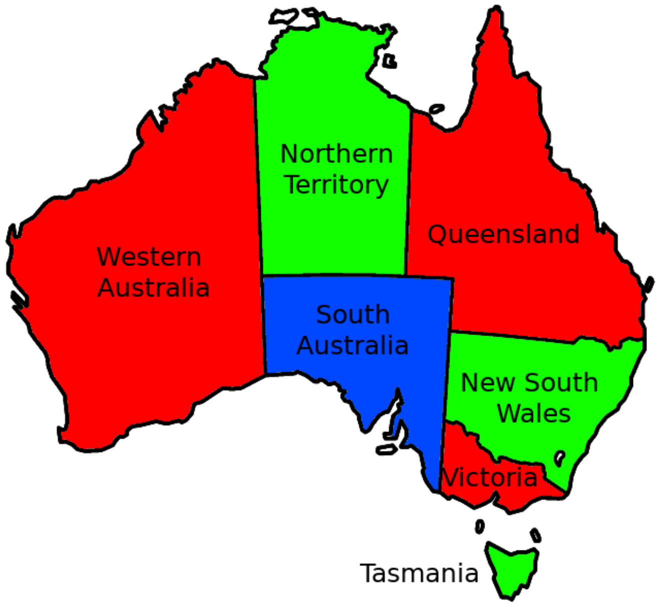
- e.g. WA ≠ NT,
- or (WA,NT) ∈ {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)}.



Example: Map-Colouring

Solutions are *complete* and *consistent* assignments,

- e.g. WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green.



Constraint graph

Binary CSP: each constraint relates two variables.

Constraint graph: nodes are variables, arcs are constraints.

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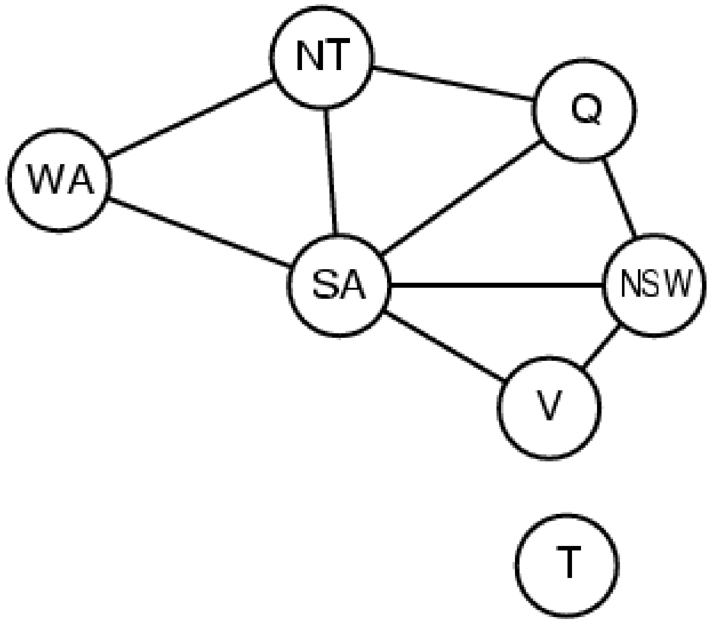


Varieties of CSPs

Discrete variables:

- finite domains:
  - n variables, domain size d → $O(d^n)$, complete assignments.
  - e.g. Boolean CSPs, incl. Boolean satisfiability (NP-complete).
- infinite domains:
  - integers, strings, etc.
  - e.g. job scheduling, variables are start/end days for each job.
  - need a constraint language, e.g. $StartJob_1 + 5 \leq StartJob_3$.

Continuous variables:

- e.g. start/end times for Hubble Space Telescope observations.
- linear constraints solvable in polynomial time by linear programming.

Varieties of constraints

Unary constraints involve a single variable,
- e.g. SA ≠ green.

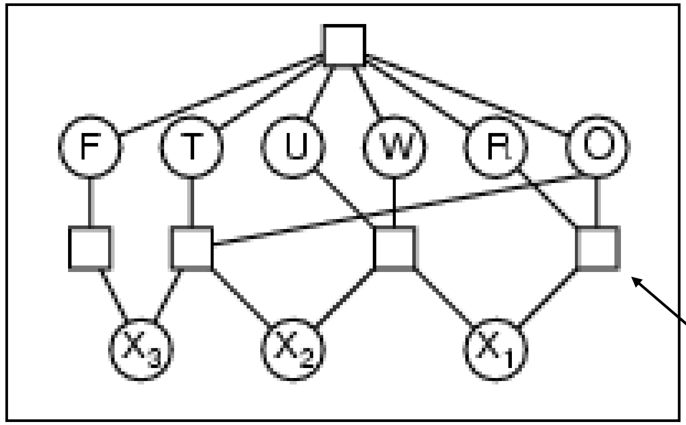
Binary constraints involve pairs of variables,
- e.g. SA ≠ WA.

Higher-order constraints involve 3 or more variables,
- e.g. crypt-arithmetic column constraints.

Global constraints involve an arbitrary number of variables

Example: Crypt-arithmetic

```
   T W O
 + T W O
 -------
 F O U R
```



constraint hypergraph

Hypernode (n-ary constraint)

Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2 X_3.$

Domains: $\{0,1,2,3,4,5,6,7,8,9\}$.

Constraints:

- $Alldiff\ (F,T,U,W,R,O)$ Global constraint
- $O + O = R + 10 \cdot X_1$
- $X_1 + W + W = U + 10 \cdot X_2$
- $X_2 + T + T = O + 10 \cdot X_3$
- $X_3 = F,\ T \neq 0,\ F \neq 0$

Real-world CSPs

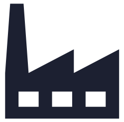
Assignment problems

e.g. who teaches what class.

Timetabling problems

e.g. which class is offered when and where.

Transportation scheduling



Factory scheduling

Notice that many real-world problems involve real-valued variables.

Search in CSPs

Standard search formulation (incremental)

Let's start with the straightforward approach, then adapt it.

States are defined by the values assigned so far.

Initial state: the empty assignment { }.

Successor function: assign a value to an unassigned variable that does not conflict with current assignment
→ fail if no legal assignments.

Goal test: the current assignment is complete.

- For a CSP with *n variables*, every solution appears at depth *n*
→ use depth-first search!

Backtracking search

Variable assignments are *commutative*,

- e.g. [WA = red then NT = green] same as [NT = green then WA = red].

Only need to consider assignments to a single variable at each node

$b = d$ and there are d^n leaves

Depth-first search for CSPs with single-variable assignments is called *backtracking* search.

Backtracking search is the basic uninformed algorithm for CSPs.

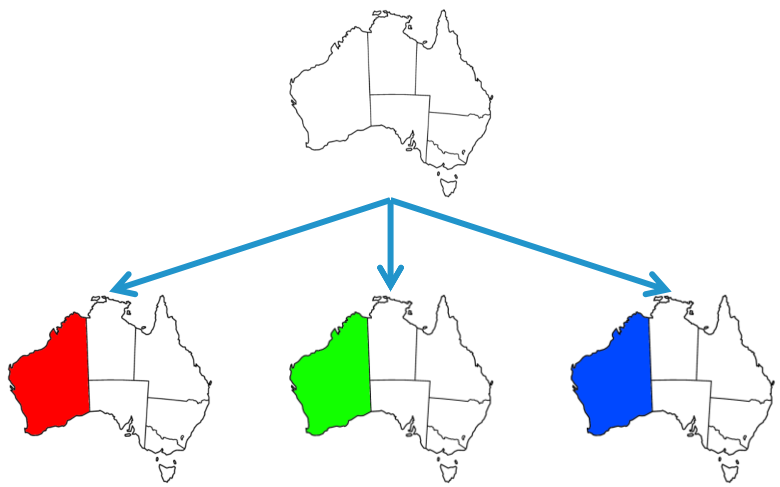
Can solve n-queens for $n \approx 25$.

Backtracking search

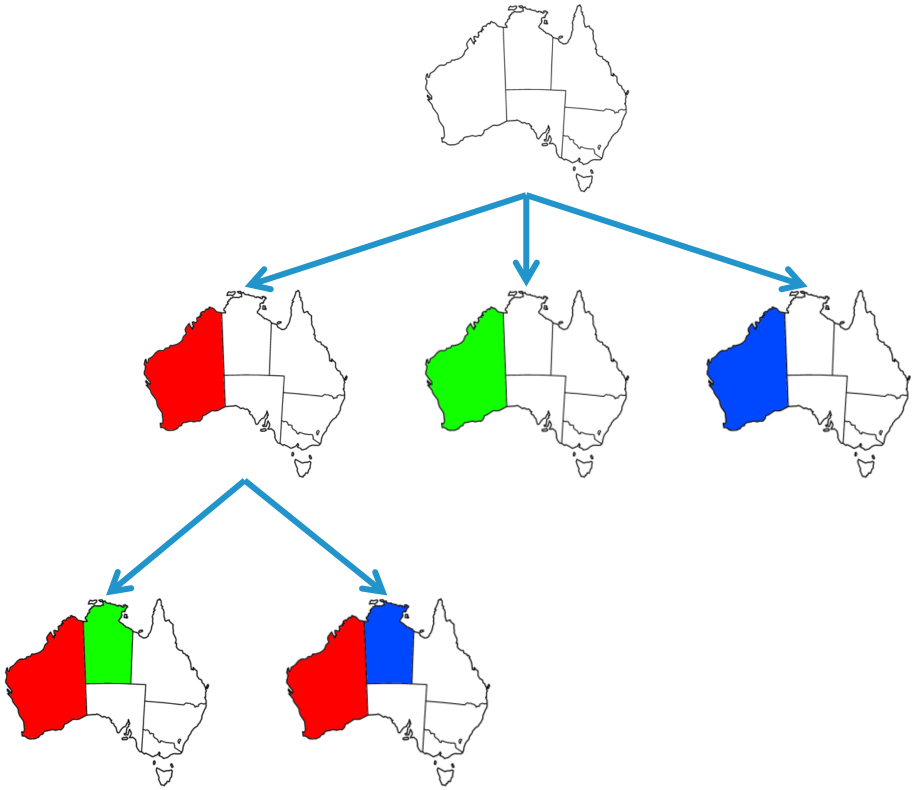
```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
    return BACKTRACK({ }, csp)

function BACKTRACK(assignment, csp) returns a solution, or failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment then
            add {var = value} to assignment
            inferences ← INFERENCE(csp, var, value)
            if inferences ≠ failure then
                add inferences to assignment
                result ← BACKTRACK(assignment, csp)
                if result ≠ failure then
                    return result
        remove {var = value} and inferences from assignment
    return failure
```

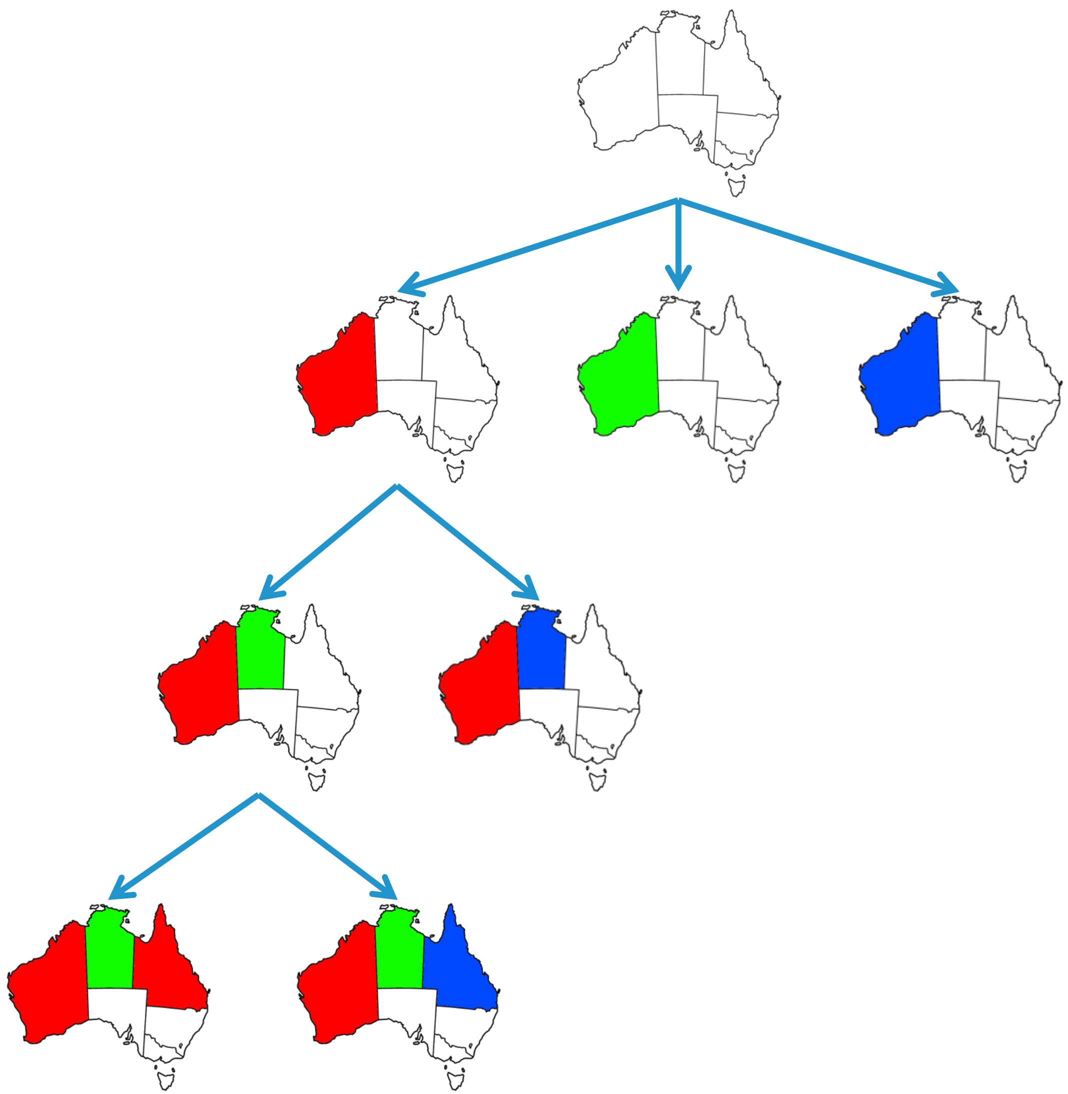
Backtracking example



Backtracking example



Backtracking example



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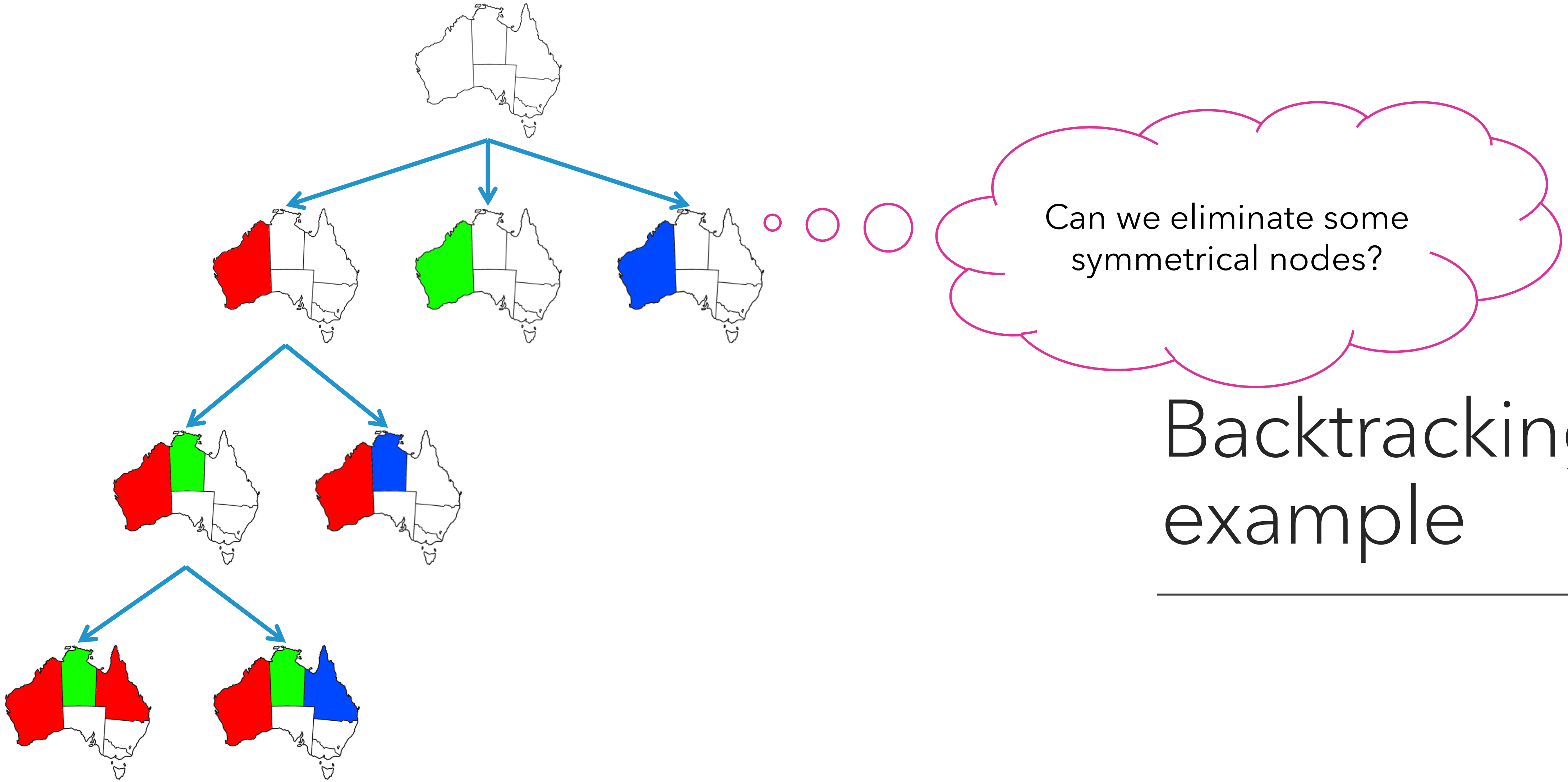

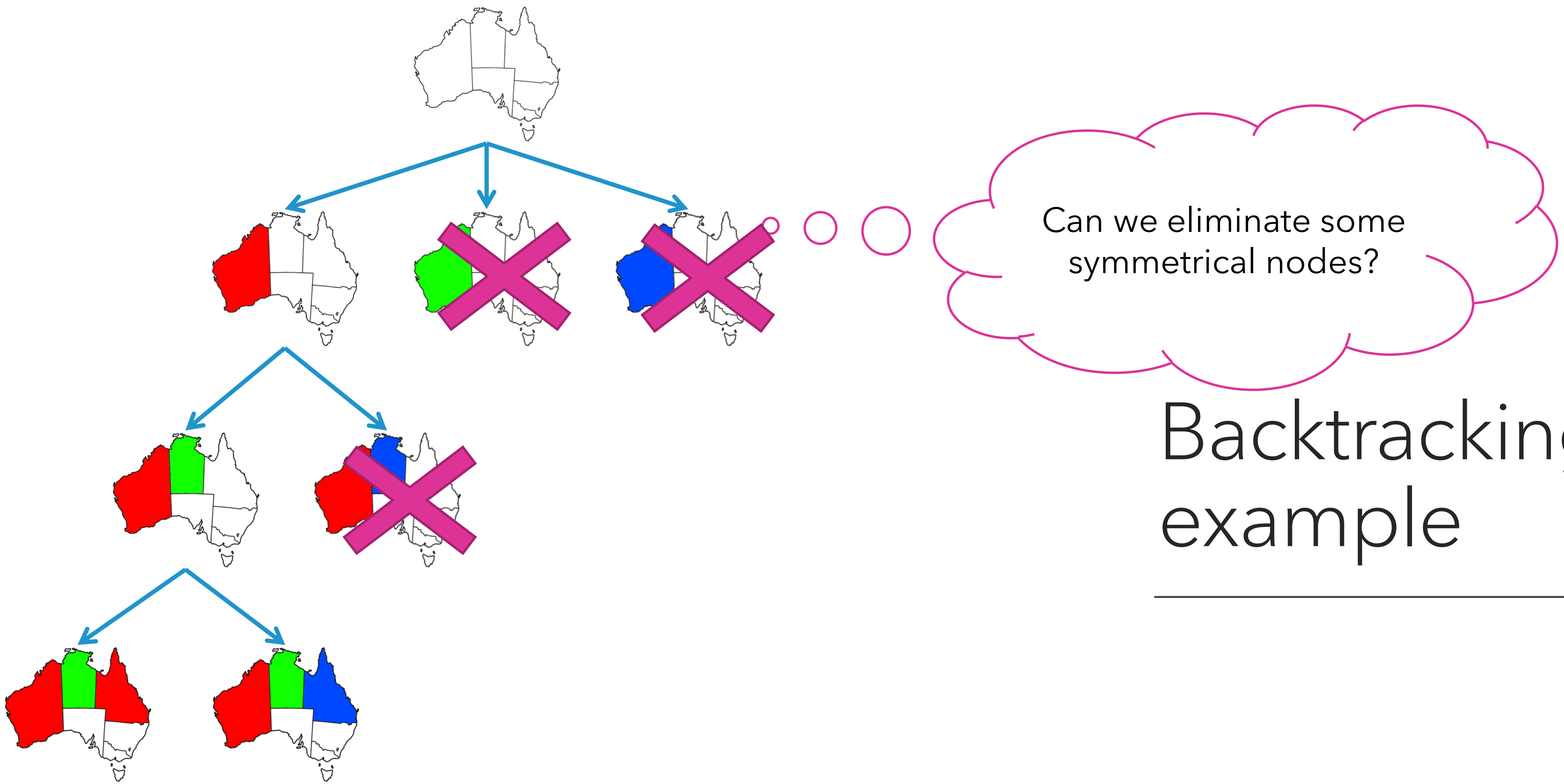

Backtracking example

Smart Search in CSPs

… or how to improve from backtracking

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           if result ≠ failure then
              return result
     remove {var = value} and inferences from assignment
  return failure
```

Improving backtracking efficiency

General-purpose methods can give huge gains in speed:

- Which variable should be assigned next?
  - **SELECT-UNASSIGNED-VARIABLE**
- Then, in what order should its values be tried?
  - **ORDER-DOMAIN-VALUES**
- What inferences should be performed at each step of the search?
  - **INFERENCE**
- Can we detect inevitable failure early?

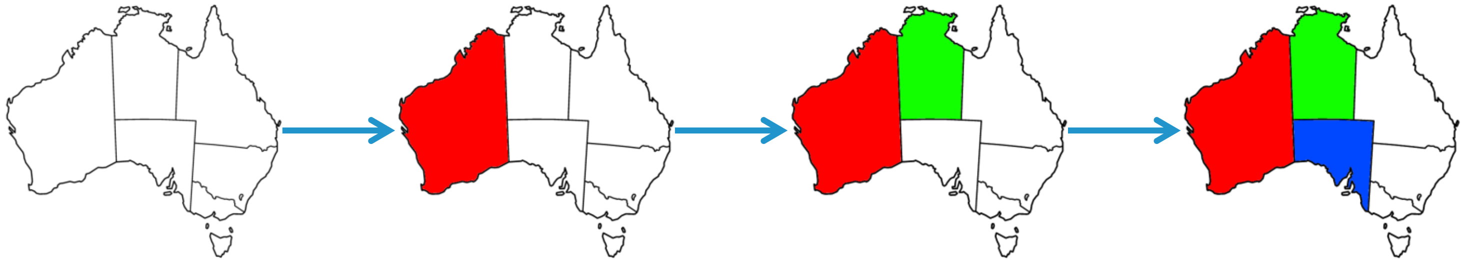
Most constrained variable

$$var \leftarrow \textsc{Select-Unassigned-Variable}(csp)$$

Most constrained variable:

- *choose the variable with the fewest legal values.*

a.k.a. *minimum-remaining-values* (MRV) heuristic.



Most constraining variable

Tie-breaker among most constrained variables.

Most constraining variable:

- *choose the variable with the most constraints on remaining variables – thus reducing branching.*

a.k.a. degree heuristic

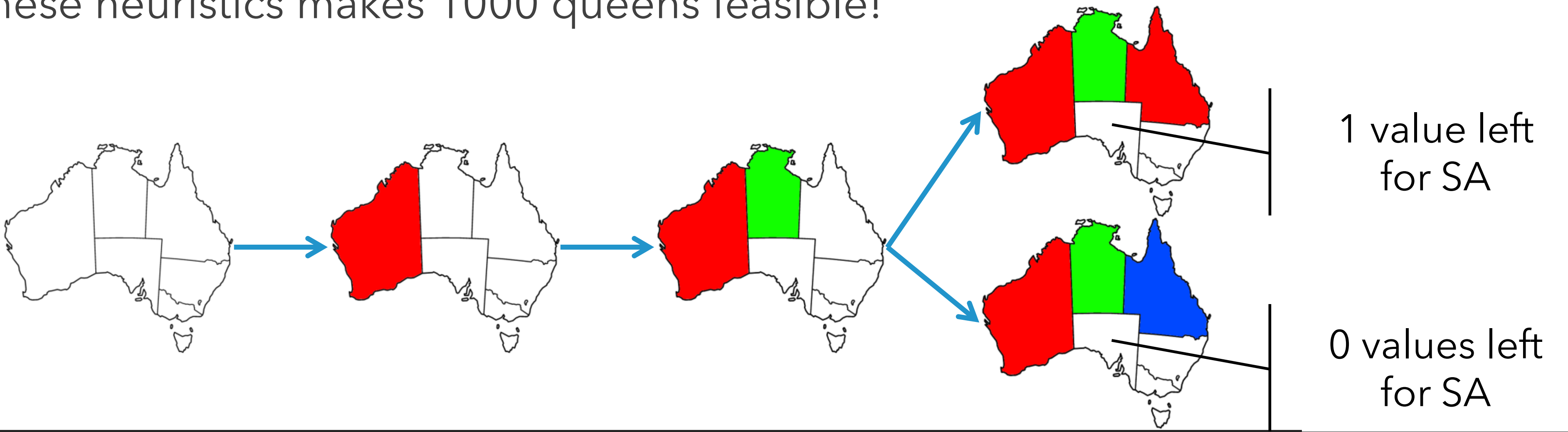
Least constraining value

```
for value in ORDER-DOMAIN-VALUES(var, assignment, csp)
```

Least constraining value:

- *given a variable, choose the value that rules out the fewest values in the remaining variables.*

Combining these heuristics makes 1000 queens feasible!



Inference: Forward checking

Idea:

- Keep track of remaining legal values for unassigned variables.
- Terminate search when any variable has no legal values.

WA	NT	Q	NSW	V	SA	T

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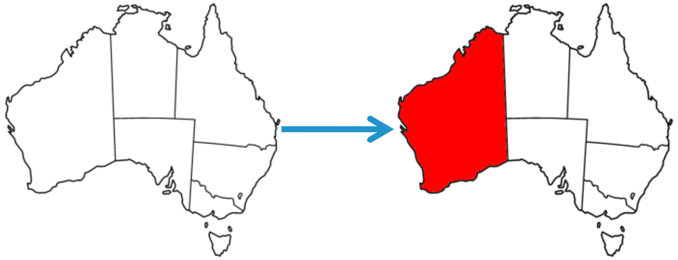

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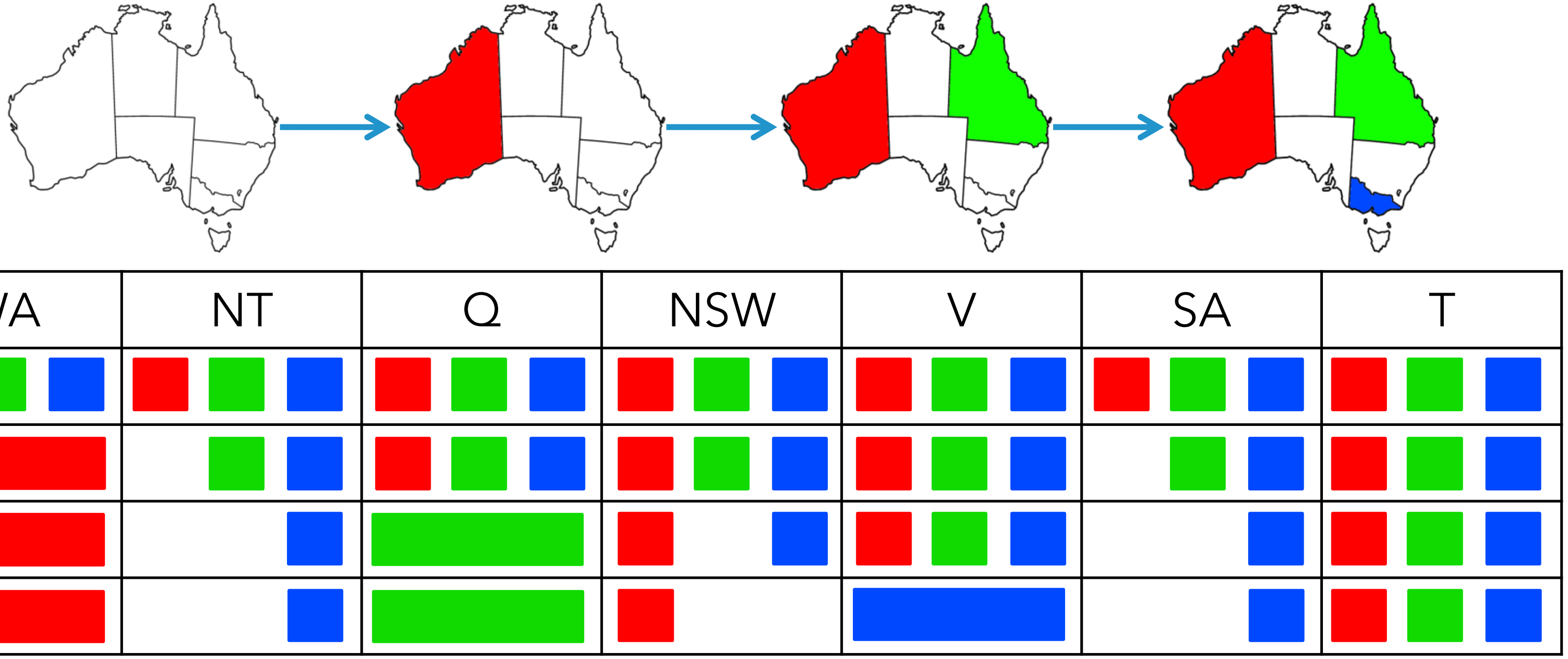


Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

NT and SA cannot both be blue!

Constraint propagation repeatedly enforces constraints locally.

Arc consistency

Simplest form of propagation makes each arc consistent.

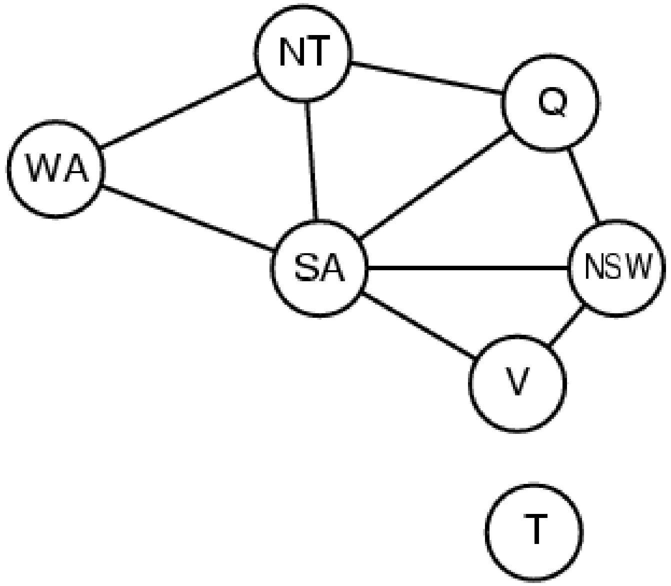
X →*Y* is consistent iff for every value *x* of in the domain of *X* there is some allowed *y in the domain of Y*.

Is there a value for X that makes the domain of Y empty?

Can be run as a preprocessor or after each assignment.

Start with all directed arcs from the graph (18 here):

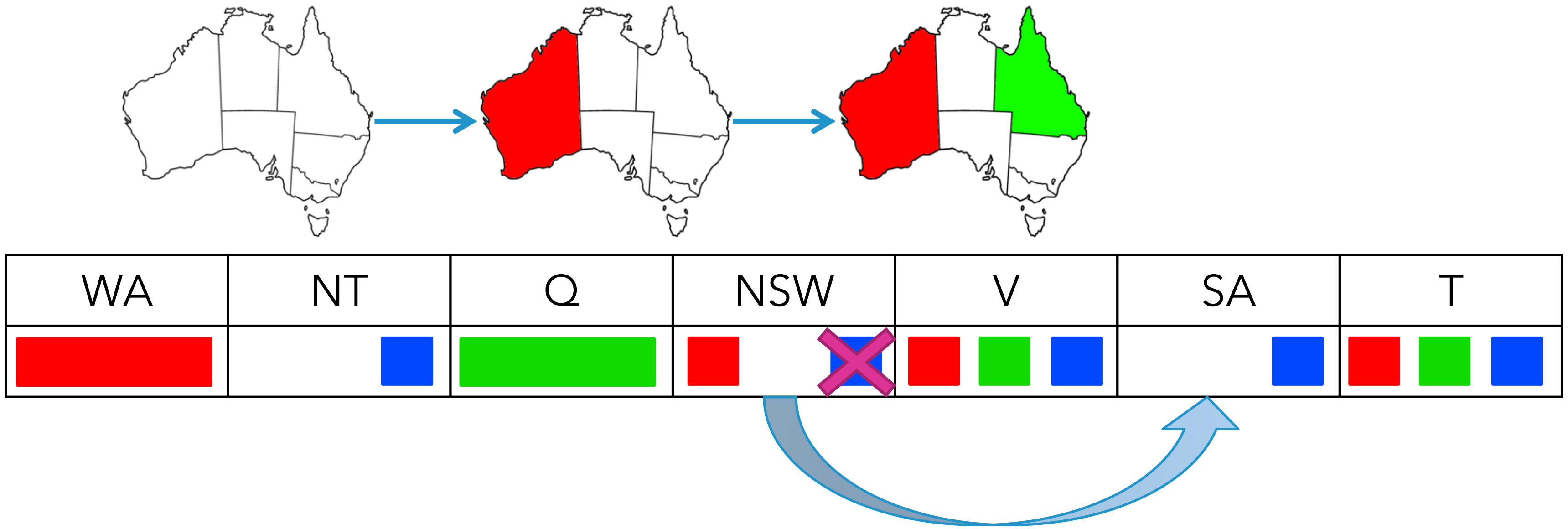
WA→NT, WA→SA, NT→WA, NT→SA, NT→Q, Q→NT, Q→SA, Q→NSW, SA→WA, SA→NT, SA→Q, SA→NSW, SA→V, NSW→Q, NSW→SA, NSW→V, V→SA, V→NSW



Arc consistency

X→Y : *Is there a value for X that makes the domain of Y empty?*

e.g. NSW → SA



Arc consistency

X→Y : *Is there a value for X that makes the domain of Y empty?*

e.g. NSW → SA

Once a value is removed, add all arcs pointing to X back in the queue!

WA	NT	Q	NSW	V

Domain of NSW became smaller, so some arcs may have become *inconsistent!*

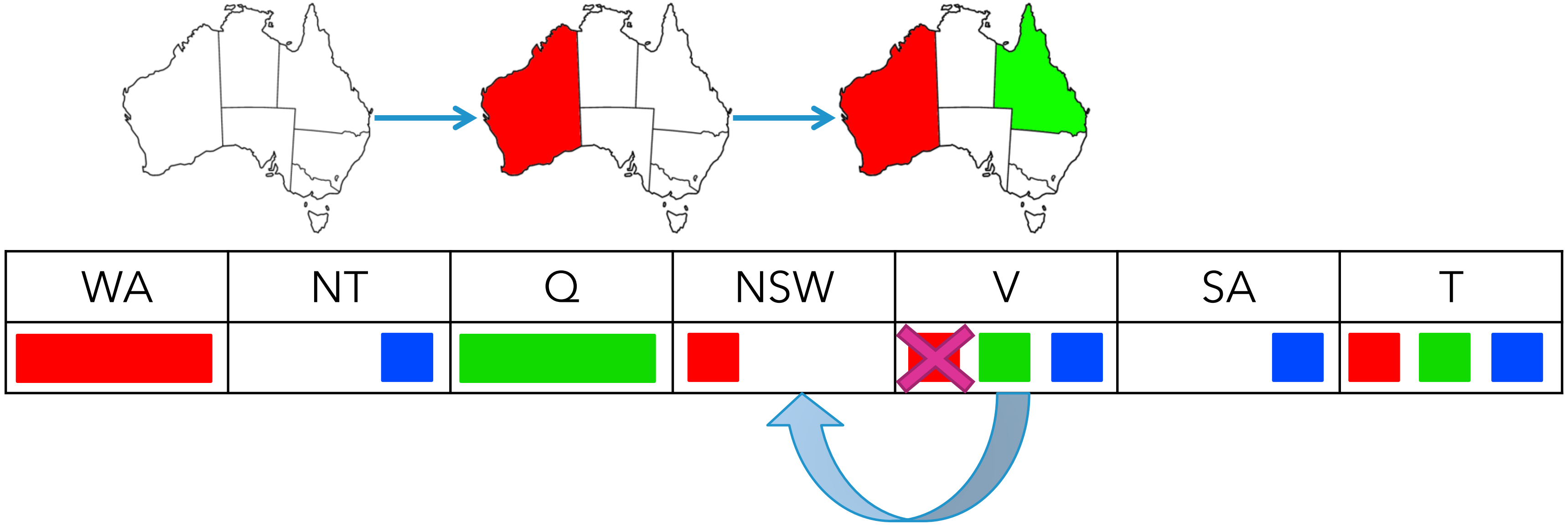
Arc consistency

Add:
V→NSW
SA→NSW
Q→NSW

X→Y : *Is there a value for X that makes the domain*

e.g. NSW → SA

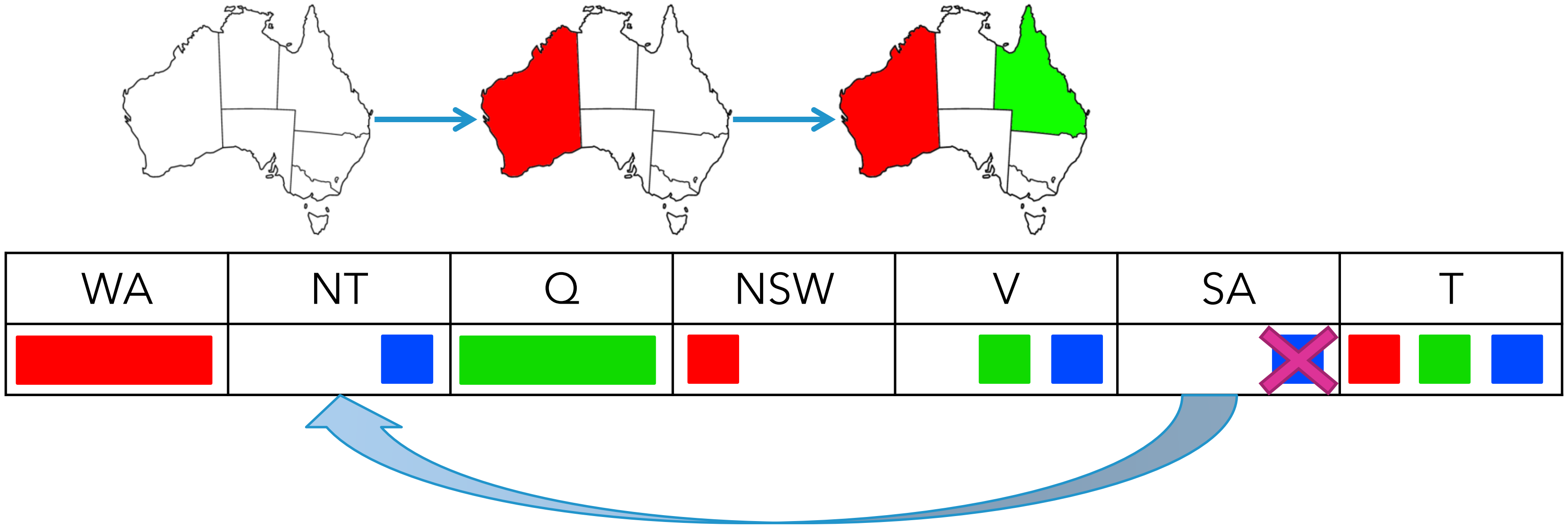
Once a value is removed, add all arcs pointing to X back in the queue!



Arc consistency

X→Y : *Is there a value for X that makes the domain of Y empty?*

Eventually check SA→NT



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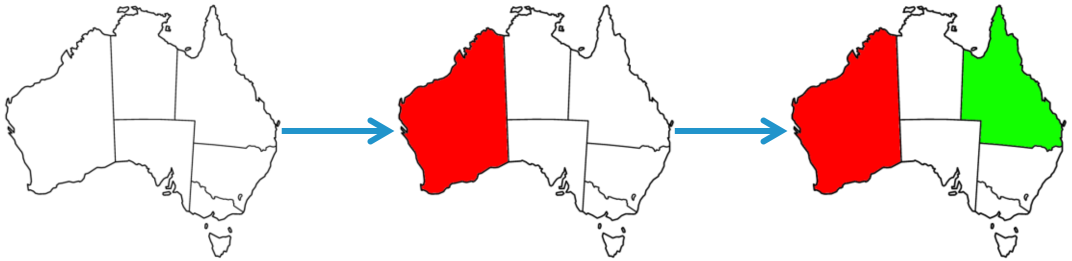

WA	NT	Q	NSW	V	SA	T
					Fail!	

Arc consistency detects failure earlier than forward checking.

Arc consistency algorithm AC-3

function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
  inputs: *csp*, a binary CSP with components (X, D, C)
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REVISE(*csp*, X_i, X_j) **then** — Make Xi arc-consistent with respect to Xj
      if size of $D_i = 0$ **then return** *false* — No consistent value left for Xi so fail
      for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
        add (X_k, X_i) to *queue* — Since revision occurred, add all neighbours of Xi for consideration (or reconsideration)
  return *true*

---

function REVISE(*csp*, X_i, X_j) **returns** true iff we revise the domain of X_i
  revised $\leftarrow$ *false*
  for each x **in** D_i **do**
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j **then**
      delete x from D_i
      revised $\leftarrow$ *true*
  return *revised*

Time complexity: $O(cd^3)$, where d is maximum size of each domain and c is the number of binary constraints (arcs).

Summary

CSPs are a special kind of problem:

- states defined by values of a fixed set of variables
- goal test defined by constraints on variable values

Backtracking = depth-first search with one variable assigned per node

Variable ordering and value selection heuristics help significantly

Forward checking prevents assignments that guarantee later failure

Constraint propagation (e.g. arc consistency) does additional work to constrain values and detect inconsistencies

Why?

CSPs are prevalent in modern computation.

Examples mentioned in this lecture.

Particularly: resource allocation, planning & scheduling, automated configuration, puzzles/games.

More complex problem formulations exist: e.g. Distributed Constraint Optimisation Problems (DCOPs).

Other solutions exist too: e.g. genetic algorithms, optimization