Introduction to Algorithms and Data Structures

Lecture 3: Asymptotics: o and ω

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Outline

Goal of Lectures 3,4,5:

- Introduce asymptotic analysis, the core mathematical theory used in this course. Centres around a certain **'Gang of Five'**:

$$o \qquad O \qquad \Theta \qquad \Omega \qquad \omega$$

- Apply this theory to **InsertSort** and **MergeSort**.

Purpose of the theory: Way of making precise, quantitative statements about efficiency properties of *algorithms themselves*. (E.g. What do *all* implementations of MergeSort have in common?)

Note: These ideas may take a while to master – don't worry!

This lecture: In what sense is MergeSort 'fundamentally faster' than InsertSort? o and ω.

Comparing runtimes for InsertSort and MergeSort

Take some specific implementations of **InsertSort** and **MergeSort**.

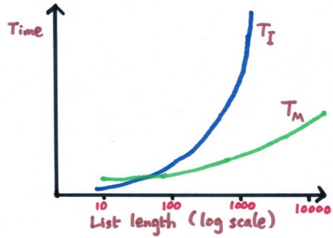
Broadly, we want to consider ...

$$
\begin{aligned}
T_I(n) &= \text{time taken by } \textbf{InsertSort} \text{ on a list of length } n \text{ (in ms)} \\
T_M(n) &= \text{time taken by } \textbf{MergeSort} \text{ on a list of length } n
\end{aligned}
$$

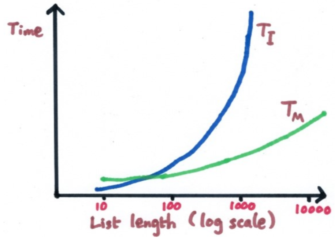
Which list of length n? Time may vary widely between lists!

Will come back to this. For now, take $T_I(n)$, $T_M(n)$ to be the worst-case (i.e. maximum) times for a list of length n.

Could then plot a graph (schematic only):



Comparing T_I and T_M



How can we capture our intuition 'T_I grows much faster than T_M'?

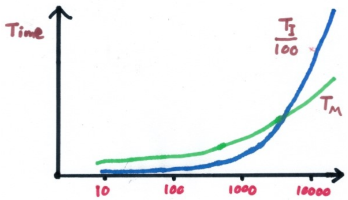
Attempt 1: $\forall n.\ T_M(n) < T_I(n)$.
Not true! We've seen that for *small n*, **InsertSort** is faster. Really want to say that **MergeSort** is *eventually* faster.

Attempt 2: $\exists N. \forall n \geq N.\ T_M(n) < T_I(n)$.
True. E.g. $N = 100$ would do here.
But doesn't capture the essential difference ...

Comparing growth rates

Attempt 3: Idea is that we expect that *any* impl of **MergeSort** will eventually beat *any* impl of **InsertSort**.

E.g. suppose we gave **InsertSort** an unfair advantage by running it on a machine 100 times faster.
Even $T_I(n)/100$ would eventually overtake $T_M(n)$:



In symbols: $\exists N. \forall n \geq N. \; T_M(n) < 0.01 \, T_I(n)$.
(E.g. $N = 100000$ would do here.)

Question: What if we replaced 0.01 by 0.0001? Or by 0.000001?

Growth rates and 'little o'

Intuition (will justify later): For *any* handicap factor c, however close to zero, $cT_I(n)$ will eventually break out and overtake T_M:

$$\forall c > 0.\ \exists N.\ \forall n \geq N.\ T_M(n) < cT_I(n)$$

We express this by saying T_M is $o(T_I)$. Can read this as: 'T_M is slower-growing than' or 'asymptotically smaller than T_I'.

In general, we say f is $o(g)$ if

$$\forall c > 0.\ \exists N.\ \forall n \geq N.\ f(n) < cg(n)$$

(Here $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$, c ranges over $\mathbb{R}$, and N, n range over $\mathbb{N}$.)

Equivalent to saying $g(n)/f(n) \to \infty$ as $n \to \infty$ (if $f : \mathbb{N} \to \mathbb{R}_{>0}$).

o-notation: Simple examples

Will come back to **InsertSort** and **MergeSort** later.
Meanwhile, some simpler examples of o.

Example 1: Is it true that n^2 is $o(n^3)$? **YES!**

Informal justification: The ratio n^3/n^2 is n, which (trivially!) tends to ∞ as n tends to ∞.

Rigorous justification: Want to show that the o formula is satisfied:

$$\forall c > 0.\ \exists N.\ \forall n \geq N.\ n^2 < cn^3$$

Suppose we're given some $c > 0$. Need to pick a suitable N.

Take any $N > 1/c$. Then for all $n \geq N$, we have

$$cn^3 \ = \ cn.n^2 \ \geq \ cN.n^2 \ > \ c(1/c)n^2 \ = \ n^2$$

(Idea: If $n > 1/c$, the extra factor n will compensate for the c.)

Examples of o-notation, continued

Example 2: Is it true that $100\sqrt{n}$ is $o(n)$? **YES!**

Informal justification: The ratio $n/(100\sqrt{n})$ is $\sqrt{n}/100$, which tends to ∞ as n tends to ∞.

Rigorous justification: Want to show that the o formula is satisfied:

$$\forall c > 0.\ \exists N.\ \forall n \geq N.\ 100\sqrt{n} < cn$$

Suppose we're given some $c > 0$. Need to pick a suitable N.

Take any $N > 10000/c^2$. Then for all $n \geq N$, we have

$$cn \ = \ c\sqrt{n}\sqrt{n} \ \geq \ c\sqrt{N}\sqrt{n} \ > \ c(100/c)\sqrt{n} \ = \ 100\sqrt{n}$$

How did we pick that $10000/c^2$?

E.g. by working backwards from the requirement $n/(100\sqrt{n}) > 1/c$.

Examples of o-notation, continued

Example 3: Is it true that $n + 1000000$ is $o(6n)$? **NO!**

Informal justification: Even though the ratio $6n/(n+1000000)$ continues to increase as n tends to ∞, it never exceeds 6, so doesn't tend to ∞.

Rigorous justification: Want to show the *negation* of the o formula:

$$\neg\ (\forall c > 0.\ \exists N.\ \forall n \geq N.\ n + 1000000 < c.6n)$$

which is equivalent to

$$\exists c > 0.\ \forall N.\ \exists n \geq N.\ n + 1000000 \geq c.6n$$

We can take $c = 1/7$. It's then true for *any* $n \geq 0$ that

$$n + 1000000\ >\ n\ \geq\ 6n/7\ =\ c.6n$$

So it's clear that $\forall N. \exists n \geq N.\ n + 1000000 \geq c.6n$ (given N, can just take $n = N$).

What is '$o(g)$' officially?

Officially, $o(g)$ is a set: namely, the set of all f that 'are $o(g)$'.

$$o(g) \;=\; \{f : \mathbb{N} \to \mathbb{R}_{\geq 0} \;\mid\; \forall c > 0.\, \exists N.\, \forall n \geq N.\, f(n) < cg(n)\}$$

So, 'f is $o(g)$' technically means $f \in o(g)$.

Common convention: Write '$o(g)$' to mean 'some (unspecified) function in the set $o(g)$'. E.g.

$$f \;=\; o(g)\,, \qquad\qquad f(n) = 3n^2 + o(n)$$

Needs care: e.g. $n = o(n^2)$ and $2n = o(n^2)$ don't imply $n = 2n$!

But many useful laws are valid, e.g.

$$o(g) + o(g) \;=\; o(g)$$

which strictly means 'if $f \in o(g)$ and $f' \in o(g)$, then $f + f' \in o(g)$'.

(Exercise if you like maths: Prove this from the definition of o.)

Reducing clutter using o

Asymptotic notation is useful when we're only interested in the broad headlines of how some function behaves.

E.g. Can read $3n^2 + o(n)$ as '$3n^2$ plus small change.'

Reduces clutter and simplifies calculations!

Example: How does the following behave for large n?

$$(3n + 5\sqrt{n} + 17 \lg n)\,(4n + (\sqrt{n}\,/\lg n) + 12)$$

(In this course, lg means logarithm to base 2.)

Rather than expanding this in full, can reason as follows:

$$\begin{aligned}(3n + o(n) + o(n))(4n + o(n) + o(n)) &= (3n + o(n))(4n + o(n)) \\ &= 12n^2 + o(3n^2) + o(4n^2) + o(n^2) \\ &= 12n^2 + o(n^2)\end{aligned}$$

(where every step can be rigorously justified).

Some key points

- Saying $f = o(g)$ gives just the main headlines of how f and g are related: 'In the limit, f is vanishingly small relative to g'. Often, this is all we care about.
- $f = o(g)$ makes a robust statement about f, g.
  E.g. unaffected by scaling: $f = o(g) \Leftrightarrow 3f = o(0.2g)$.
- So can expect that e.g. '$T_M = o(T_I)$' will remain true for *any* implementations of **MergeSort**/**InsertSort**.
- Use of o can reduce clutter and simplify calculations.
- But without sacrificing mathematical rigour: '$f = o(g)$' has a precisely defined meaning.

General advice: Sketch graphs to understand what's going on!

And finally: ω

ω is dual to o. Recall that $f = o(g)$ means:

$$\forall c > 0.\ \exists N.\ \forall n \geq N.\ f(n) < cg(n)$$

('f is asymptotically smaller than / grows slower than g').

By contrast, read $f = \omega(g)$ as saying:
'f is asympotically larger than / grows faster than g').

Formal definition: f is $\omega(g)$ if

$$\forall C > 0.\ \exists N.\ \forall n \geq N.\ f(n) > Cg(n)$$

('However much we scale g up by, f will eventually overtake it.')

For purpose of comparing f and g, scaling g 'up' by C has same effect as scaling f 'down' by $c = 1/C$. So easy to show:

$$f = \omega(g) \text{ if and only if } g = o(f)$$

(Compare: $x > y$ if and only if $y < x$.)

We'll tend to use o more than ω.

Next time: O, Ω, Θ.
(Most presentations start with these!)

Reading for Lectures 3 and 4:
CLRS Chapter 3, GGT Sections 3.3, 3.4

Today's music:
J.S. Bach, *Toccata in D minor*.