Introduction to Algorithms and Data Structures

Lecture 2: Inefficient vs. efficient algorithms

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Inefficient vs. efficient algorithms

Goal of lecture: Introduce some simple and important examples of algorithms, illustrating the difference between ‘efficient’ and ‘inefficient’ solutions.

Problem 1 (toy example): Given a large decimal whole number n, compute n mod 9 (i.e. the *remainder* on dividing n by 9).

Method A: Do division by school method; note the remainder.

$$\begin{array}{rl} & \,4\ 1\ 6\ 4\ 7\ 3\ 5\ 2\ 5\ 3\ 7\ 7\ 2\ 8\ 9 \\ 9\ \big) & \overline{3\ \ 7^1 4^5 8^4 2^6 6^3 1^4 7^2 2^4 8^3 3^6 9^6 5^2 6^8 0^8 7} \quad \text{rem } 6 \end{array}$$

This is the ‘obvious’ method: no cleverness involved.

Alternative method

Method B: Add the digits of n to get a new number n'.
Do the same to n'; repeat till we get down to a single digit d,
If $d = 9$, answer is 0; otherwise answer is d.

E.g. for $n = 3748261728395607$:

$$\begin{aligned} 3+7+4+8+2+6+1+7+2+8+3+9+5+6+0+7 &= 78 \\ 7+8 &= 15 \\ 1+5 &= 6 \end{aligned}$$

Why does this work?

E.g. for a 4-digit number written $abcd$:

$$\begin{aligned} 1000a + 100b + 10c + d &= (999a + a) + (99b + b) + (9c + c) + d \\ &\equiv a + b + c + d \pmod 9 \end{aligned}$$

Comparing methods A and B

Advantages of method B:

- Faster, at least for humans (though not spectacularly so).
- More flexible: can add digits in whatever order you like.
- Can be parallelized (you add first 8 digits, I'll add last 8).

Moral: Using mathematical insight, improvements over the 'obvious' algorithm may be possible.

Improved algorithm may be 'non-obvious' and may need justifying.

Modular exponentiation

Problem 2: Given (large) whole numbers a, n, m, compute $a^n \bmod m$. E.g. $2^{10} \bmod 17 = 1024 \bmod 17 = 4$.

Believe it or not, this problem is absolutely fundamental to modern cryptosystems (e.g. RSA, as explained in DMP course).

Method A: Literally compute a^n, then reduce modulo m.

- If e.g. $a = 3$, $n = 12345678901234567890 1234$, then a^n won't even fit in memory.
- In any case, working with very big numbers is time-consuming.

Modular exponentiation, continued

Method B: Start from a.
Do $(n-1)$ multiplications by a, but *reduce* mod m *each time*.
Works because:

$$(x \times y) \bmod m \; = \; ((x \bmod m) \times (y \bmod m)) \bmod m$$

E.g. for 2^{10} mod 17:

$$2 \times 2 = 4, \quad 4 \times 2 = 8, \quad 8 \times 2 = 16, \quad 16 \times 2 = 32 \equiv 15, \quad 15 \times 2 = 30 \equiv 13,$$
$$13 \times 2 = 26 \equiv 9, \quad 9 \times 2 = 18 \equiv 1, \quad 1 \times 2 = 2, \quad 2 \times 2 = 4.$$

- Now numbers never get bigger than am.
- But still impractical if $n = 12345678901234567890 1234$.

Fast modular exponentiation

Method C: Notice that it's easy to compute $e = a^n \bmod m$ if we've already computed $d = a^{\lfloor n/2 \rfloor} \bmod m$:

- If n is even, take $e = (d \times d) \bmod m$.
- If n is odd, take $e = (d \times d \times a) \bmod m$.

This suggests the following recursive algorithm:

```
Expmod (a,n,m):             # Computes $a^n$ mod $m$
    if n=0 then return 1
    else
        d = Expmod (a,$\lfloor n/2 \rfloor$,m)
        if n is even
            return (d $\times$ d) mod m
        else return (d $\times$ d $\times$ a) mod m
```

(Example of pseudocode: informal mix of programming constructs, math notation, and English. Useful for expressing algorithms in a readable way.)

Example of Method C

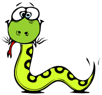
```
**Expmod** (a,n,m):
    if n=0 then return 1
    else
        d = **Expmod** (a,$\lfloor n/2 \rfloor$,m)
        if n is even
            return (d $\times$ d) mod m
        else return (d $\times$ d $\times$ a) mod m
```

Imagine each evaluation of **Expmod** is done by a different ‘person’:

Us to A:	What’s **Expmod** (2,10,17) ?
A to B:	What’s **Expmod** (2,5,17) ?
B to C:	What’s **Expmod** (2,2,17) ?
C to D:	What’s **Expmod** (2,1,17) ?
D to E:	What’s **Expmod** (2,0,17)?
E to D:	1
D to C:	$1 \times 1 \times 2$ mod 17 = 2
C to B:	2×2 mod 17 = 4
B to A:	$4 \times 4 \times 2$ mod 17 = 15
A to us:	15×15 mod 17 = 4.

This is feasible even when a, n, m are large (say $\sim$ 1000 digits).

Some Python experiments



Time in milliseconds to compute $3^n \bmod 2n$ (on my laptop):

n	Method A	Method B	Method C	min(A,B)/C
10	.0027	.0051	.0055	0.49
100	.0041	.0134	.0071	0.58
1000	.0092	.1229	.0076	1.21
10000	.133	2.41	.010	13.3
100000	2.81	10.46	.016	176.
1000000	101.	127.	.017	5941.
10000000	3986.	1168.	.017	68700.
100000000	149000	11150	.019	587000.
1000000000	crashed	213000	.022	9680000.
10^{100}	—	—	.656	—

Key idea: It's not just that my Python program for C is the best. Rather, the algorithm itself is vastly, fundamentally superior.

How do we make this idea precise?

Digression: Primality testing

Fermat's little theorem: If n is prime and $0 < a < n$, then $a^{n-1} \bmod n = 1$. [Proved in DMP.]

Application: Let n be the following 270-digit number.

```
41202343698665954385553136533257594817981169984432798284545
56264338764455652484261980988704231618418792614202471888694
92560931776375033421130982397485150944909106910269861031862704
```

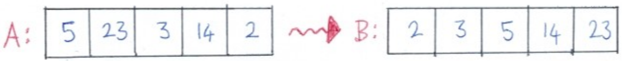
My Python 'Program C' takes <5 ms to discover that $2^{n-1} \bmod n = \cdots \neq 1$. Conclusion: n is not prime.

So what are its factors? If you knew, you'd be (slightly) famous. This is RSA-896, not yet cracked. ($75,000 prize sadly withdrawn!)

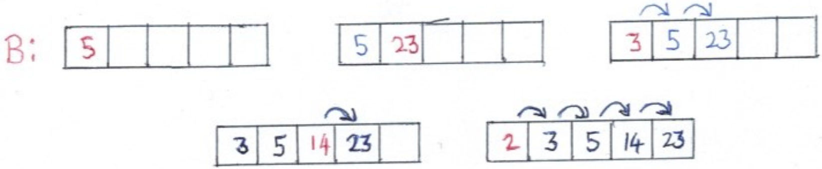
Note: This isn't a perfect primality test: a few non-primes (e.g. 561) masquerade as primes. But for big numbers, error probability is very small — and by refining the test, can be made even smaller (Miller-Rabin test).

Insert-sort

Problem 3: Given an array *A* containing *n* whole numbers, construct an array *B* containing the same *n* numbers in non-decreasing order.

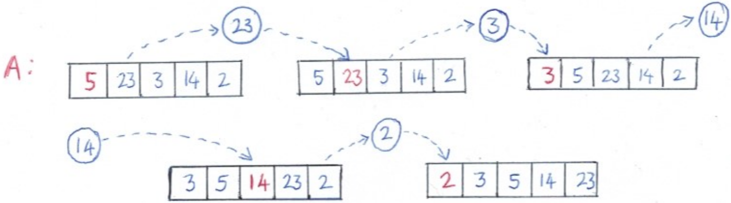


Method A ('obvious'): Go through elements of *A* one by one. Copy them into *B*, filling *B* from left to right, and inserting each element in its correct position.



Insert-sort: in-place version

Actually, don't need a separate array *B*: can do everything within *A* itself (in-place sorting). Just need to be able to hold one number 'in our hand' at any given time.

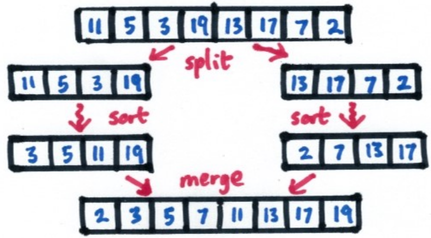


In pseudocode:

```
InsertSort(A):
    for i = 1 to |A|−1        # |A| means size of A
        x = A[i]
        j = i−1
        while j ≥ 0 and A[j] > x
            A[j+1] = A[j]
            j = j−1
        A[j+1] = x
```

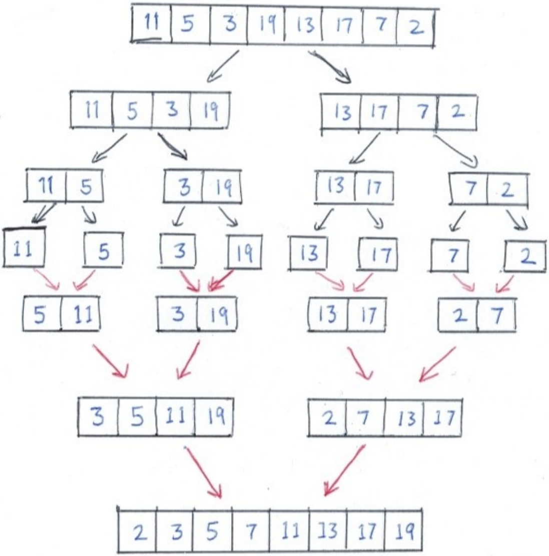
Merge-sort

Method B (less obvious). Split A into two halves.
Sort these separately, then merge the results.



```
Merge (B,C):
    allocate D of size |B| + |C|
    i = j = 0
    for k = 0 to |D|−1
        if B[i] < C[j]        # Convention: ∞ if index out of range
            D[k] = B[i], i = i+1
        else
            D[k] = C[j], j = j+1
    return D
```

Merge-sort: Recursive application of Merge



Merge-sort continued

A recursive sorting algorithm:

```
MergeSort (A,m,n):      # sorts A[m], A[m+1], . . . , A[n−1]
                        # returning result in an array D of size n−m
    if n−m = 1
        return [ A(m) ]
    else
        p = ⌊(m+n)/2⌋
        B = MergeSort (A,m,p)
        C = MergeSort (A,p,n)
        D = Merge (B,C)
        return D

MergeSortAll (A):
    return MergeSort (A,0,|A|)
```

Python again

Time in milliseconds to sort a list of length n.
(Entries were random whole numbers $< n^2$.)

n	Insert-sort	Merge-sort	Speedup
10	.023	.068	0.34
100	.97	.74	1.31
1000	69.8	7.9	8.84
10000	8210.	76.2	107.
100000	906000.	1080.	839.
1000000	–	13300.	
10000000	–	158000.	
100000000	–	2619000.	

So merge-sort seems fundamentally superior (as regards runtime).
Again, how can we make this precise?
And why is merge-sort so much better? What's going on here?
Will explore this next time.

Reading

- Insert sort: CLRS 2.1
- Merge sort: CLRS 2.3
- Modular exponentiation: CLRS 31.6, second half
- [RSA challenge numbers: good Wikipedia pages.]

Today's music: Fryderyk Chopin, *Minute Waltz*