2nd Order Circuit Example

Circuit:

$$x\ (-+)\ \to L \to R \to C \to y(t)\ (+-)\ \to (-+)x$$

KVL Equation:

$$x - L\frac{di}{dt} - Ri - y = 0$$

Using ($i = C\frac{dy}{dt}$):

$$x - LC\frac{d^2y}{dt^2} - RC\frac{dy}{dt} - y = 0$$

Examples

- **Scaling System** ((a_0 = 1, b_0 = a)):

$$y = ax$$

- **Integrator** ((a_1 = 1, b_0 = 1)):

$$\frac{dy}{dt} = x$$

- **Differentiator** ((a_0 = 1, b_1 = 1)):

$$y = \frac{dx}{dt}$$

- **Integrator with Feedback** ((a_1 = 1, a_0 = a, b_0 = 1)):

$$\frac{dy}{dt} + ay = x$$

Systems Described by Differential Equations

Many systems are described by a **Linear Constant Coefficient Ordinary Differential Equation (LCCODE):**

$$a_n\frac{d^ny}{dt^n} + \cdots + a_1\frac{dy}{dt} + a_0y(t) = b_m\frac{d^mx}{dt^m} + \cdots + b_1\frac{dx}{dt} + b_0x(t)$$

With given **initial conditions:**

$$y^{(n-1)}(0),\ \dots,\ y'(0),\ y(0)$$

- (n): the order of the system.
- (b_0, \dots, b_m, a_0, \dots, a_n): coefficients of the system.

An LCCODE describes how (x(t)), (y(t)), and their derivatives interrelate. It does **not** provide an explicit solution for (y(t)) in terms of (x(t)).

System Invertibility

A system is invertible if the input signal can be recovered from the output signal:

$$y = Fx, \quad x = F^{-1}y = F^{-1}Fx$$

Diagram:

$$x \rightarrow [F] \rightarrow y \rightarrow [F^{-1}] \rightarrow x$$

Examples:

- AM Radio transmitter and receiver.
- Multi-path echo canceler.

Example: Cruise Control

Control system diagram:

$$x \rightarrow (+)_{+} \rightarrow \text{error} \rightarrow [k] \rightarrow \text{gas} \rightarrow [H] \rightarrow y$$

$$[H][k]y \rightarrow (+)_{-}$$

Output Equation:

$$y = H(k(x - y))$$

The system can become unstable if (k) is too large:

- Positive error adds gas.
- Delay in velocity change causes speed to overshoot.
- Negative error reduces gas.
- Delay in velocity change causes speed to undershoot.

System Stability

A system is **Bounded Input Bounded Output (BIBO) stable** if:

$$|x(t)| \leq M_x < \infty \quad \implies \quad |y(t)| \leq M_y < \infty$$

Where (M_x) and (M_y) are finite positive numbers.

Time-Invariance

A system is time-invariant if a time shift in the input produces the same time shift in the output:

$$y(t) = F(x(t)) \implies y(t-\tau) = F(x(t-\tau))$$

System Memory

- A **memoryless system**: The output depends only on the **present input**.
  - Examples: Ideal amplifier, ideal gear or transmission.
- A **system with memory**: The output depends on **past or future inputs**.
  - Examples: Capacitors and inductors, springs or moving masses.
- A **causal system**: The output depends only on **past or present inputs**.

Linearity

A system (F) is linear if:

1. **Homogeneity:**

   $$F(ax) = aF(x)$$

   Where (a) is a scalar.
2. **Superposition:**

   $$F(x + \tilde{x}) = F(x) + F(\tilde{x})$$

Examples of Linear Systems: Scaling, differentiation, integration, convolution.
Examples of Nonlinear Systems: Sign detector, squaring system, comparator.

Interconnection of Systems

1. **Cascade (Series):**

   $$y = G(F(x)) = GFx$$

2. **Parallel (Sum):**

$$y = Fx + Gx$$

3. **Feedback:**

$$y = F(x - Gy)$$

Impulse Signals

- An **impulse function** (\delta(t)) is an idealization of a signal that:
  - Is very large near (t = 0).
  - Is very small away from (t = 0).
  - Has an integral of 1.

Properties:

1. $$\int_{-\infty}^{\infty} f(t)\delta(t - T)dt = f(T)$$

2. $$\int_{a}^{b} \delta(t)dt = 1 \text{ if } a < 0 < b, \text{ else } 0$$

More Complex Signals

- **Unit Triangle Function:**

$$\text{Tri}(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

- **Unit Ramp Function:**

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

  Integral of the unit step function:

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

Sinusoidal Signals

General form:

$$x(t) = A\cos(\omega t + \theta)$$

- $\omega = 2\pi f$: Radian frequency.
- $T = \frac{1}{f} = \frac{2\pi}{\omega}$: Period.

Complex Signals

- Euler's formula:

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

- Complex sinusoid:

$$Ae^{j(\omega t+\theta)} = A\cos(\omega t + \theta) + jA\sin(\omega t + \theta)$$

  Real part:

$$\mathrm{Re}(Ae^{j(\omega t+\theta)}) = A\cos(\omega t + \theta)$$