2nd Order Circuit Example

A circuit with the following components:
$x\ (-+)\ \rightarrow L \rightarrow R \rightarrow C \rightarrow y(t)\ (+-)\ \rightarrow (-+)x$

KVL:

$$x - L\frac{di}{dt} - Ri - y = 0$$

Using $i = C\frac{dy}{dt}$,

$$x - LC\frac{d^2y}{dt^2} - RC\frac{dy}{dt} - y = 0$$

Examples

- **Scaling system** ($a_0 = 1,\ b_0 = a$)

$$y = ax$$

- **Integrator** ($a_1 = 1,\ b_0 = 1$)

$$\frac{dy}{dt} = x$$

- **Differentiator** ($a_0 = 1,\ b_1 = 1$)

$$y = \frac{dx}{dt}$$

- **Integrator with feedback** ($a_1 = 1,\ a_0 = a,\ b_0 = 1$)

$$\frac{dy}{dt} + ay = x$$

Systems Described by Differential Equations

Many systems are described by a *linear constant coefficient ordinary differential equation (LCCODE)*:

$$a_n \frac{d^n y}{dt^n} + \cdots + a_1 \frac{dy}{dt} + a_0 y(t) = b_m \frac{d^m x}{dt^m} + \cdots + b_1 \frac{dx}{dt} + b_0 x(t)$$

With given *initial conditions*:

$$y^{(n-1)}(0),\ \ldots,\ y'(0),\ y(0)$$

- **n**: order of the system.
- $b_0, \ldots, b_m, a_0, \ldots, a_n$: coefficients of the system.

An LCCODE gives an implicit description of a system:

- Describes how $x(t)$, $y(t)$, and their derivatives interrelate.
- Does not provide an explicit solution for $y(t)$ in terms of $x(t)$.

System Invertibility

A system is invertible if the input signal can be recovered from the output signal:

$$y = Fx, \quad x = F^{-1}y = F^{-1}Fx$$

Diagram:

$$x \to [F] \to y \to [F^{-1}] \to x$$

Note: $F^{-1}F = I$, the identity operator.

Examples:

- AM Radio transmitter and receiver.
- Multi-path echo canceler.

Example: Cruise Control

Control feedback system:

$$x \to (+)_+ \to \text{error} \to [k] \to \text{gas} \to [H] \to y$$
$$[H][k]y \to (+)_-$$

Output:

$$y = H(k(x - y))$$

The system can become unstable if k is too large (depending on H):

- Positive error adds gas.
- Delayed velocity change causes speed to overshoot.
- Negative error reduces gas.
- Delayed velocity change causes speed to undershoot.

System Stability

- Stability is critical for most engineering applications.
- A system is **Bounded Input Bounded Output (BIBO) stable** if:

  $$|x(t)| \leq M_x < \infty$$

  always results in:

  $$|y(t)| \leq M_y < \infty,$$

  where M_x and M_y are finite positive numbers.

Time-Invariance

A system is time-invariant if a time shift in the input produces the same time shift in the output.

For a system F,

$$y(t) = F(x(t))$$

implies that:

$$y(t-\tau) = F(x(t-\tau))$$

for any time shift τ.

System Memory

- A system is **memoryless** if the output depends only on the **present input**.
  - Example: Ideal amplifier.
- A system with **memory** has an output that depends on **past or future inputs**.

- Example: Energy storage elements like capacitors and inductors.
- A **causal system** has an output that depends only on **past or present inputs**.

Linearity

A system F is linear if:

1. **Homogeneity**: $F(ax) = aF(x)$, where a is a scalar.
2. **Superposition**: $F(x + \tilde{x}) = F(x) + F(\tilde{x})$.

Interconnection of Systems

1. **Cascade (series):**

$$y = G(F(x)) = GFx$$

   Diagram: $x \to [F] \to [G] \to y$.
2. **Sum (parallel):**

$$y = Fx + Gx$$

   Diagram: $x \to [F] \to (+),\ x \to [G] \to (+) \to y$.
3. **Feedback:**

$$y = F(x - Gy)$$

   Diagram: $x \to (+)_+ \to [F] \to y,\ [G]y \to (+)_- \to (+)_+$.

Example: Integrator with Feedback

Input to the integrator is $x - ay$, where a is the feedback coefficient:

$$\int_0^t (x(\tau) - ay(\tau))d\tau = y(t)$$

Derivative:

$$x - ay = \frac{dy}{dt}$$

Block Diagram

- Lines with arrows denote signals.
- Boxes denote systems; arrows show inputs and outputs.

Examples:

1. **Scaling system:**

$$y(t) = ax(t)$$

   - Amplifier: $|a| > 1$.
   - Attenuator: $|a| < 1$.
   - Inverting: $a < 0$.
   - a: gain or scale factor.

2. **Time shift system:**

$$y(t) = x(t - T)$$

   - Delay system: $T > 0$.
   - Predictor system: $T < 0$.

3. **Convolution system:**

$$y(t) = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) d\tau$$

   where $h(\tau)$ is a given function.