Example 5.1.1

Let

$$f(t) = \begin{cases} 0, & -\pi < t \le 0, \\ t, & 0 \le t < \pi. \end{cases}$$

To find the Fourier coefficients, evaluate:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\int_0^{\pi} t\,dt = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} t\cos(nt)\,dt = \frac{1}{\pi}\left[\frac{t\sin(nt)}{n} + \frac{\cos(nt)}{n^2}\right]_0^{\pi} = \frac{\cos(n\pi)-1}{n^2\pi} = \frac{(-1)^n - 1}{n^2\pi}$$

Note: (L = \pi), and substitution (t = u), (\cos(nt) = dv) was used for integration by parts.

Dirichlet's Theorem

For the interval ([-L, L]), if the function (f(t)) satisfies the following conditions:

1. It is single-valued.
2. It is bounded.
3. It has at most a finite number of maxima and minima.
4. It has only a finite number of discontinuities (piecewise continuous).
5. (f(t + 2L) = f(t)) for values of (t) outside of ([-L, L]).

Then,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right)$$

converges to (f(t)) as (N \to \infty) at values of (t) where (f(t)) is continuous. At points of discontinuity, it converges to:

$$\frac{1}{2}\left[f(t^-) + f(t^+)\right]$$

where (f(t^-)) is the value of the function infinitesimally to the left of (t), and (f(t^+)) is the value of the function infinitesimally to the right of (t).

5.1 Fourier Series

The Fourier series is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right)$$

- The function converges between ([-L, L]) (a period of (2L)), with possible exceptions at discontinuities and endpoints.
- The **fundamental periodic function** ((n=1)) defines the base frequency, while the **harmonics** are terms with frequencies that are integer multiples of the fundamental.
- **Goal:** Find (a_n) and (b_n) for (f(t)).

First Attempt

Integrate the terms from (-L) to (L):

- The (\cos) and (\sin) terms disappear due to their oscillations resulting in zero:

$$\frac{1}{2L} \int_{-L}^{L} f(t)\, dt = \frac{a_0}{2}$$

Thus, (a_0) is twice the mean value of (f(t)) over one period.

Further Evaluation

Multiply the equation by (\cos\left(\frac{m \pi t}{L}\right)) and integrate over ([-L, L]):

$$\int_{-L}^{L} f(t) \cos\left(\frac{m\pi t}{L}\right) dt = \int_{-L}^{L} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right) \right] \cos\left(\frac{m\pi t}{L}\right) dt$$

- The (a_0) and (b_n) terms vanish due to orthogonality, and (a_n) vanishes when (m \neq n).

$$\int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt = a_n \int_{-L}^{L} \cos^2\left(\frac{n\pi t}{L}\right) dt$$

Using:

$$\int \cos^2(x)\,dx = \int \frac{1}{2}[1+\cos(2x)]\,dx$$

Thus,

$$\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt = a_n L$$

Final Formulation

Using the same method for (a_n), we find (b_n):

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

To generalize for an arbitrary interval ([\tau, \tau + 2L]):

$$a_0 = \frac{1}{L}\int_{\tau}^{\tau+2L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{\tau}^{\tau+2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$b_n = \frac{1}{L}\int_{\tau}^{\tau+2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

Questions to Consider

1. What types of functions can be represented by Fourier series?
2. What happens at points of discontinuity?

To answer these, refer to Dirichlet's theorem.