L3 Summary

For an input $x(t)$, the output of a linear system is given by the superposition integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau$$

If the system is also time-invariant, the result is a convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

The response of an LTI system is completely characterized by its impulse response $h(t)$.

Another expression for the superposition integral can be found by substituting $\tau = t - \tau_1$, then $d\tau = -d\tau_1$ and $\tau_1 = t - \tau$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau_1)h(t-(t-\tau_1))d(-\tau_1)$$

System Equation

The system equation relates the outputs of a system to its inputs. Example from last time: the system described by the block diagram:

$$x \to (+)_+ \to \text{integral} \to y,\ y \cdot a \to (+)_-$$

This results in:

$$y' = x - ay$$

where initial conditions specify a solution.

Solutions for the System Equation

The output consists of two components:

1. **Zero-input response**: The output when $x(t) = 0$.
2. **Zero-state response**: The output of the system with initial conditions set to zero.

For a linear system H, the zero-input response is zero due to homogeneity: if $y = F(ax) = aF(x)$, then $a = 0$ implies $y = 0$.

Example: Solving for Voltage Across a Capacitor

Find $y(t)$ for an arbitrary input voltage $x(t)$, given $y(0) = Y_0$.

Circuit:

- $x(t) \to (-+)$, $i(t) \to R \to C \to y(t) \to (-+)x(t)$.

Using KVL:

$$x(t) = Ri(t) + y(t)$$

Using $i(t) = C\frac{dy}{dt}$:

$$x(t) = RC\frac{dy}{dt} + y(t)$$

Zero-input case:

$$RC\frac{dy}{dt} + y(t) = 0$$

Using the first-order ODE solution:

$$\frac{dy}{dt} + p(t)y = q(t),\ \mu = e^{\int P(t)}$$

We get:

$$y(t) = Ae^{-t/RC}$$

For arbitrary input $x(t)$:

$$RC\left[A'(t)e^{-t/RC} - \frac{1}{RC}A(t)e^{-t/RC}\right] + A(t)e^{-t/RC} = x(t)$$

Simplifying:

$$A'(t) = x(t) \cdot \frac{1}{RC}e^{t/RC}$$

$$A(t) = \int_0^t x(\tau)\frac{1}{RC}e^{\tau/RC}d\tau + A(0)$$

Given $y(t) = A(t)e^{-t/RC}$:

$$y(t) = \left[\int_0^t x(\tau)\frac{1}{RC}e^{\tau/RC}d\tau + A(0)\right]e^{-t/RC}$$

At $t = 0$, $y(0) = Y_0 \implies A(0) = Y_0$.

$$y(t) = \underbrace{\int_0^t x(\tau)\frac{1}{RC}e^{-(t-\tau)/RC}d\tau}_{\text{Zero-state response}} + \underbrace{Y_0 e^{-t/RC}}_{\text{Zero-input response}}$$

Impulse Response

The impulse response $h(t)$ is:

$$h(t) = \frac{1}{RC}e^{-t/RC}$$

The response of this system to an input $x(t)$ is:

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t x(\tau)\frac{1}{RC}e^{-(t-\tau)/RC}d\tau$$

This matches the zero-state solution found earlier.

Output of an LTI System

An LTI system H transforms the signal $x(t)$:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta_\tau(t)d\tau$$

Apply H:

$$y(t) = H\left(\int_{-\infty}^{\infty} x(\tau)\delta_\tau(t)d\tau\right)$$

Resulting in:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)H(\delta_\tau(t))$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau$$

For time-invariant systems:

$$h_\tau(t) = h(t - \tau)$$

Time-Invariance

For a time-invariant system H:

$$h_\tau(t) = h(t - \tau)$$

Linearity

A system S is linear if:

1. **Homogeneity:** If $y = Sx$, then $ay = S(ax)$.
2. **Superposition:** If $y_1 = Sx_1$ and $y_2 = Sx_2$, then $y_1 + y_2 = S(x_1 + x_2)$.

Thus:

$$ay_1 + by_2 = S(ax_1 + bx_2)$$

Extended Linearity:

- **Summation:**

$$\sum_n a_n y_n = S\left(\sum_n a_n x_n\right)$$

- **Integration:**

$$\int_{-\infty}^{\infty} a(\tau)y(t-\tau)d\tau = S\left(\int_{-\infty}^{\infty} a(\tau)x(t-\tau)d\tau\right)$$

Impulse Response

The impulse response of a linear system, (h_τ), is the output of the system at time (t) to an impulse at time (\tau). For a system (H), this can be written as:

$$h_\tau = H(\delta_\tau)$$

Notes:

1. A positive (τ) causes a forward shift of the signal and defines the sampling range of ($t - \tau$).
2. On the left-hand side, (t) represents a specific value of time, while on the right-hand side, (t) varies over all real numbers (e.g., all time (t)). Despite the impulse nature, the specific value of (t) on the left depends on this variation.

The equation:

$$h_\tau(t) = H(\delta_\tau(t))$$

can be confusing.

- At (\tau = 0):

$$h = h_0 = H(\delta_0)$$

- Using (t = 2) for evaluation:

$$h(2) = H(\delta(2))$$

  implies (h(2) = H(0)), which is **incorrect**.

A better notation for the process is:

$$h(t, \tau) = h(2, 0) - h(t, 0)$$