Lecture 2 — Linear Algebra

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Equations and Unknowns

- A system of equations has n equations and n unknowns
- Unknowns: variables (e.g. x, y, z, etc.)
- Equations: linear functions (e.g. $x^2 + y = 5$)
- Substitution: solve for one variable — in terms of the others — and plug the result into other equations
- Elimination: manipulate equations by applying operations to both the left and right hand side, and combine equations to reduce them

Substitution Example

$$x + y = 5 \quad (1)$$
$$3x - 2y = 5 \quad (2)$$

- What operation can we perform on both sides of equation 1 to solve for x in terms of y?
- Substitute x for $5 - y$ in equation (2)

$$\begin{aligned} 3(5 - y) - 2y &= 5 \\ 15 - 5y &= 5 \\ -5y &= -10 \\ y &= 2 \end{aligned}$$

- Substitute y for 2 in equation (1)

$$\begin{aligned} x + 2 &= 5 \\ x &= 3 \end{aligned}$$

Elimination Example

$$2x - y + 3z = 9 \tag{3}$$
$$x + 4y - 5z = -6 \tag{4}$$
$$x - y + z = 2 \tag{5}$$

- Transform equation (4) so that, when summed with equation (3), x is eliminated

$$-2(x + 4y - 5z) = -2(-6)$$
$$-2x - 8y + 10z = 12$$

- Transform equation (5) so that, when summed with equation (3), x is eliminated

$$-2(x - y + z) = -2(2)$$
$$-2x + 2y - 2z = -4$$

Elimination Example

$$\begin{aligned} 2x - y + 3z &= 9 \\ -2x - 8y + 10z &= 12 \\ -2x + 2y - 2z &= -4 \end{aligned}$$

- Sum equations to eliminate x

$$\begin{aligned} 2x - y + 3z &= 9 \\ -2x - 8y + 10z &= 12 \\ -9y + 13z &= 21 \end{aligned}$$

$$\begin{aligned} 2x - y + 3z &= 9 \\ -2x + 2y - 2z &= -4 \\ y + z &= 5 \end{aligned}$$

Elimination Example

- Use our new equations to eliminate y

$$\begin{aligned} -9y + 13z &= 21 \\ 9(y+z) &= 9(5) \\ 22z &= 66 \\ z &= 3 \end{aligned}$$

- Use $z = 3$ to solve for y through substitution

$$\begin{aligned} -9y + 13(3) &= 21 \\ -9y &= -18 \\ y &= 2 \end{aligned}$$

Elimination Example

- Use $y = 2$ and $z = 3$ to solve for x

$$\begin{aligned} 2x - (2) + 3(3) &= 9 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

More complex systems of equations

- Substitution and elimination work great for simple systems
- But what if things get more complex?
  - Systems that are not **uniquely determined**: yields one unique solution
  - Some systems are **underdetermined**: more unknowns than equations, so there are an infinite number of possible solutions
  - Other systems are **overdetermined**: more equations than unknowns, so there is no solution (multiple contradicting solutions)
  - A dataset is an example of many more equations (observations) than unknowns (variables)
- Linear algebra helps us solve these complications

Vector and scalars

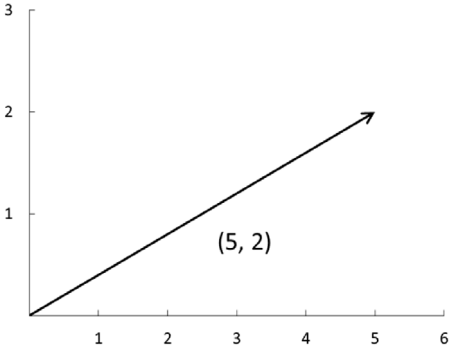


Figure: Vector (5,2)

- A *scalar* is a single number/element
- A *vector* is a list of numbers (scalars) in some order
- Useful to think of vectors as an arrow in n-dimensional space
- Length (norm) of a vector can be solved using the Pythagorean theorem: $a^2 + b^2 = c^2$
- For Vector (5,2) length = $\sqrt{5^2 + 2^2} \approx 5.39$
- This can be expanded to $\|a\| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$

Vector addition / subtraction

- **Vector sums**
$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$
- **Vector differences**
$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{bmatrix}$$

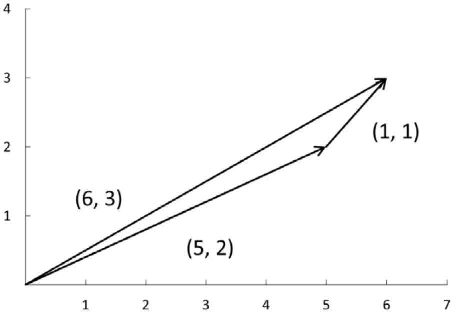


Figure: Vector addition: (5,2)+(1,1)

Scalar multiplication and dot product

- **Scalar multiplication**
  $$c \cdot \vec{a} = c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} c \cdot a_1 \\ c \cdot a_2 \\ c \cdot a_3 \end{bmatrix}$$
- **Dot product (or scalar product)**
  $$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1 b_1 + a_2 b_2$$
  This requires that the two vectors be of equal dimension.

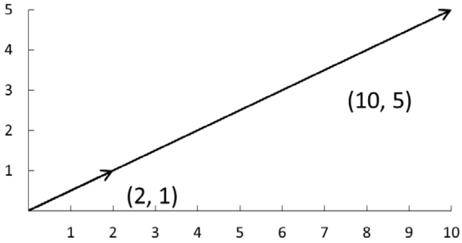


Figure: scalar multiplication: 5a where a=(2,1)

Dot Product Example

$$\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$$

$$54 + 40 + 28 = 122$$

Matrices

A *matrix* is a rectangular table of numbers or variables that are arranged in a specific order in rows and columns

- They can vary in size from a few columns and rows to hundreds of thousands of rows and columns. A dataset is a matrix.
- The size of a matrix is known as its dimensions and is expressed in terms of how many rows, *n*, and columns, *m*, it has, written as *nxm* (read "n by m").

Example:

$$A_{3x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Important types of matrices

Zero matrix:

$$A_{3x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Diagonal matrix:

$$A_{3x3} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Identity matrix:

$$I_{1x1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric matrix:

$$A_{3x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Lower triangular matrix:

$$A_{3x3} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Upper triangular matrix:

$$I_{1x1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Matrix transposition

The *transpose* switches the rows and columns of the matrix.
The first row becomes the first column, and so on
Example:

$$A_{2x3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
$$A^T_{3x2} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Matrix addition, subtraction and scalar multiplication

Matrix addition and subtraction: simply add/subtract each corresponding element!

$$A_{3\times3} \pm B_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$A_{3\times3} \pm B_{3\times3} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix}$$

Scalar multiplication:

$$5 \times A = \begin{bmatrix} 5 \times a_{11} & 5 \times a_{12} & 5 \times a_{13} \\ 5 \times a_{21} & 5 \times a_{22} & 5 \times a_{23} \\ 5 \times a_{31} & 5 \times a_{32} & 5 \times a_{33} \end{bmatrix}$$

Matrix Addition Example

$$\begin{bmatrix} 6 & 7 & 8 \\ 5 & 6 & 0 \\ 5 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 3 \\ 1 & 4 & 1 \\ 0 & 8 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 10 & 11 \\ 6 & 10 & 1 \\ 5 & 8 & 11 \end{bmatrix}$$

Matrix multiplication

- In order to multiply two matrices, the number of columns in the first matrix must match the number of rows in the second matrix. e.g. $A_{n\times m} \cdot B_{m\times p}$
- This will result in a matrix of dimensions $n \times p$
- Therefore, $A \times B$ will not result in the same matris as $B \times A$
  - Left multiplication: multiply by the matrix on the left
  - Right multiplication: multiply by the matrix on the right

Matrix multiplication

- Say we multiply matrix A by matrix B to get matrix C
- The value in Row 1 Column 1 of matrix C (C_{11}) is equal to the dot product of Row 1 of matrix A and Column 1 of matrix B
- The row vector is "rotated" so that we can take the dot product with the column vector
- Repeat this until all values of matrix C are filled in

Matrix multiplication - example

Suppose $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$

$A \times B$?

- $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} = (2 \times 2) + (1 \times 5) = 9 = \begin{bmatrix} 9 & - \\ - & - \end{bmatrix}$
- $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} = (2 \times 4) + (1 \times 3) = 11 = \begin{bmatrix} 9 & 11 \\ - & - \end{bmatrix}$
- $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} = (3 \times 2) + (2 \times 5) = 16 = \begin{bmatrix} 9 & 11 \\ 16 & - \end{bmatrix}$
- $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} = (3 \times 4) + (2 \times 3) = 18 = \begin{bmatrix} 9 & 11 \\ 16 & 18 \end{bmatrix}$

$$A \times B = \begin{bmatrix} 9 & 11 \\ 16 & 18 \end{bmatrix}$$

Determinant

- **Determinant:**
- The determinant of a matrix is a commonly used function that converts the matrix into a scalar.
- Only defined for a *square* matrix (same number of rows as columns)
- A matrix with a non-zero determinant is *nonsingular* and can be *inverted* which is important for solving systems of equations
- Notation: determinant of matrix A is represented with $|A|$
- Calculation: "difference of the diagonal products"

Determinant

Determinant for a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Is given by:

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Example:

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$|B| = (1 \cdot 4) - (2 \cdot 3) = 4 - 6 = -2$$

Determinants of larger matrices

- For 3×3 (or more dimensions) matrices, we can use the **Laplace expansion**.
- **Laplace Expansion:** determinant of a matrix bigger than $2 times 2$ is the sum of products of each element and its *cofactor* for *any* row or column
- **Cofactor:** a series of *Minors* with positive/negative signs according to the position of the element in the matrix
- **Minor:** determinant of a *submatrix*
- **Submatrix** of a matrix value is what's left over when you eliminate the row and column of that value

Determinants of larger matrices

- Finding a specific Minor M_{11} of Matrix A
- Take the submatrix of a_{11}

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Determinants of larger matrices

- Finding a specific Minor M_{11} of Matrix A
- Take the submatrix of a_{11}

$$A = \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix}$$

Determinants of larger matrices

- Finding a specific Minor M_{11} of Matrix A
- Take the submatrix of a_{11}
- The Minor is the the determinant of the 2×2 submatrix

$$A = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = a_{22}a_{33} - a_{23}a_{32}$$

Determinants of larger matrices

- Cofactor: the signed minor of an element. Alternates positive/negative like so:

$$\begin{bmatrix} a_{11}(+) & a_{12}(-) & a_{13}(+) \\ a_{21}(-) & a_{22}(+) & a_{23}(-) \\ a_{31}(+) & a_{32}(-) & a_{33}(+) \end{bmatrix}$$

Determinants of larger matrices

- So now we know how to find **submatrices**, which we take the determinants of to get **Minors**, which are made positive or negative as **cofactors**
- **Laplace Expansion:** determinant of a matrix bigger than 2×2 is the sum of products of each element and its *cofactor* for *any one* row or column
- Since we can use any row or column, we need to calculate 3 cofactors for a 3×3 matrix, multiply them by their respective element in the matrix, and sum the results

Full Example

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Submatrix for Minor M_{11}

$$A = \begin{bmatrix} \cancel{1} & \cancel{4} & \cancel{3} \\ \cancel{1} & 2 & 0 \\ \cancel{2} & 3 & 1 \end{bmatrix}$$

Calculate Determinant

$$A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$(2 \cdot 1) - (0 \cdot 3) = 2$$

Determine cofactor sign

- The sign of this cofactor C_{11} will be positive ($C_{11} = 2$)

$$\begin{bmatrix} a_{11}(+) & a_{12}(-) & a_{13}(+) \\ a_{21}(-) & a_{22}(+) & a_{23}(-) \\ a_{31}(+) & a_{32}(-) & a_{33}(+) \end{bmatrix}$$

Cofactor C_{12}

$$A = \begin{bmatrix} \cancel{1} & \cancel{4} & \cancel{3} \\ 1 & \cancel{2} & 0 \\ 2 & \cancel{3} & 1 \end{bmatrix}$$

$$M_{12} = (1 \cdot 1) - (0 \cdot 2) = 1$$

- The sign of this cofactor will be negative ($C_{12} = -1$)

Cofactor C_{13}

$$A = \begin{bmatrix} \cancel{1} & \cancel{4} & \cancel{3} \\ 1 & 2 & \cancel{0} \\ 2 & 3 & \cancel{1} \end{bmatrix}$$

$$M_{13} = (1 \cdot 3) - (2 \cdot 2) = -1$$

- The sign of this cofactor will be positive ($C_{13} = -1$)

Bringing it all together

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

- Our *elements* are 1, 4, and 3
- Our *cofactors* are 2, -1, and -1
- Multiply each element by its respective cofactor, and sum

$$|A| = 1(2) + 4(-1) + 3(-1) = -5$$

The determinant is equal to -5
This result is replicable for *any* row or column

Inverse Matrix

- Square matrices are invertible if the determinant is non-zero.
- If the determinant of the matrix is zero, then it is *singular* and cannot be inverted.
- A matrix multiplied by its inverse returns the identity matrix: $A \times A^{-1} = A^{-1} \times A = I$
- For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
- The inverse (A^{-1}) is given by the following expression: $A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$
- And generally, by this expression: $A^{-1} = \frac{1}{|A|} C^T$

Deriving Inverse Matrix for 2×2

- Why does $C^T = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$
- C^T is the transpose of the matrix of cofactors of A. It is called also the adjoint Matrix of A: adj(A)
- Remember: a cofactor is the signed determinant of the submatrix

Deriving Inverse Matrix for 2×2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- Submatrix of a_{11} is just a_{22}, which makes the Minor also just a_{22}
- $M_{11} = a_{22}$
- $M_{12} = a_{21}$
- $M_{21} = a_{12}$
- $M_{22} = a_{11}$

Deriving Inverse Matrix for 2×2

- Matrix of cofactors $C = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$
- Transposed matrix of cofactors $C^T = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$
- Final step is to multiply C^T by $\frac{1}{|A|}$

Example: Inverting 2×2 Matrix

- Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- Determinant $|A| = (1 \cdot 4) - (2 \cdot 3) = -2$
- Matrix of Cofactors $C = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$
- Adjoint Matrix $C^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$
- Inverse Matrix $A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$
- What is $A \times A^{-1}$?

Example: Inverting 3×3 Matrix

- Same steps apply
- Challenge will be to create the matrix of cofactors (C)

Start by calculating each Minor

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ -6 & -2 & 2 \end{bmatrix}$$

$M_{11} = \begin{vmatrix} 4 & 3 \\ -2 & 2 \end{vmatrix} =$
$(4 \times 2) - (3 \times -2) = 14$

$M_{12} = \begin{vmatrix} 0 & 3 \\ -6 & 2 \end{vmatrix} =$
$(0 \times 2) - (3 \times -6) = 18$

$M_{13} = \begin{vmatrix} 0 & 4 \\ -6 & -2 \end{vmatrix} =$
$(0 \times -2) - (4 \times -6) = 24$

$M_{21} = \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} =$
$(2 \times 2) - (1 \times -2) = 6$

$M_{22} = \begin{vmatrix} 1 & 1 \\ -6 & 2 \end{vmatrix} =$
$(1 \times 2) - (1 \times -6) = 8$

$M_{23} = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} =$
$(2 \times 3) - (1 \times 4) = 10$

$M_{31} = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} =$
$(2 \times 3) - (1 \times 4) = 2$

$M_{32} = \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} =$
$(1 \times 3) - (1 \times 0) = 3$

$M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} =$
$(1 \times 4) - (2 \times 0) = 4$

This gives us the Cofactor Matrix C

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ -6 & -2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 14 & -18 & 24 \\ -6 & 8 & -10 \\ 2 & -3 & 4 \end{bmatrix}$$

- Determinant $|A|$ is the dot product of *any* row or column of elements and their respective cofactors
- Using row 1, $|A| = (1)(14) + (2)(-18) + (1)(24) = 2$

Take the transpose of the Cofactor Matrix C^T, multiply by inverse determinant $\frac{1}{|A|}$

$$C = \begin{bmatrix} 14 & -18 & 24 \\ -6 & 8 & -10 \\ 2 & -3 & 4 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 14 & -6 & 2 \\ -18 & 8 & -3 \\ 24 & -10 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} C^T = \frac{1}{2} \begin{bmatrix} 14 & -6 & 2 \\ -18 & 8 & -3 \\ 24 & -10 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -3 & 1 \\ -9 & 4 & -\frac{3}{2} \\ 12 & -5 & 2 \end{bmatrix}$$

Did it work?

$$A \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ -6 & -2 & 2 \end{bmatrix} \times A^{-1} \begin{bmatrix} 7 & -3 & 1 \\ -9 & 4 & -\frac{3}{2} \\ 12 & -5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix and vector properties

Table 12.2: Matrix and Vector Transpose Properties

Inverse	$(A^T)^T = A$
Additive property	$(A+B)^T = A^T + B^T$
Multiplicative property	$(AB)^T = B^T A^T$
Scalar multiplication	$(cA)^T = cA^T$
Inverse transpose	$(A^{-1})^T = (A^T)^{-1}$
If A is symmetric	$A^T = A$

Table 12.3: Matrix Determinant Properties

Transpose property	$\det(A) = \det(A^T)$
Identity matrix	$\det(I) = 1$
Multiplicative property	$\det(AB) = \det(A)\det(B)$
Inverse property	$\det(A^{-1}) = \frac{1}{\det(A)}$
Scalar multiplication $(n \times n)$	$\det(cA) = c^n \det(A)$
If A is triangular or diagonal	$\det(A) = \prod_{i=1}^{n} a_{ii}$

Table 12.4: Matrix Inverse Properties

Inverse	$(A^{-1})^{-1} = A$
Multiplicative property	$(AB)^{-1} = B^{-1}A^{-1}$
Scalar multiplication $(n \times n)$	$(cA)^{-1} = c^{-1}A^{-1}$ if $c \neq 0$

Linear Independence

A set of vectors is *linearly independent* if we cannot write any vector in the set as a combination of other vectors in the set.
So the only way for $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$, is if every scalar multiplier is zero.
Examples:

- Suppose $v_1 = (1,3)$ and $v_2 = (3,9)$. These vectors are not linearly independent because $3v_1 - v_2 = 0$
- Suppose $v_1 = (1,3)$ and $v_2 = (2,9)$. These vectors are linearly independent because the only a_i that allow $a_iv_1 - a_iv_2 = 0$ are $a_i = 0$

Linearly independent matrices have non-zero determinants, can be inverted
Linearly dependent matrices have a determinant of zero, cannot be inverted. In statistics, this is called *multicollinearity*

Matrix rank

- The **rank** of this matrix is the maximum number of linearly independent rows (or columns)
- The main question here is: **How many rows (or columns) of the matrix give us new information?**
- We can test linear independence by taking the determinant of a matrix:
  - If the determinant is non-zero, the vectors are linearly independent, can be inverted
  - If the determinant is zero, they are dependent, cannot be inverted
  - Linear dependence is called *multicollinearity* in a statistics context

Systems of equations

How determined is your system of equations?

- Uniquely determined
  - Same number of equations and variables to solve for.
  - Yields one unique solution
- Overdetermined
  - More equations than unknowns
  - Equations may be contradictory
- Undetermined
  - More unknown than equations
  - Can occur if equations are not linearly independent - each equation must give us new information if we want to solve the system.
  - Infinite number of possible solutions

Systems of Equations as Matrices

$$\begin{aligned} 2x - y + 3z &= 9 \\ x + 4y - 5z &= -6 \\ x - y + z &= 2 \end{aligned}$$

becomes...

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix}$$

Solving systems of equations - Matrix inversion

Steps:

1. Arrange the equations in the format $Ax = c$
2. Check the determinant of A. If it is non-zero, can be inverted.
3. Calculate A^{-1}
   - Inverse = 1/determinant times adjoint matrix
   - Adjoint matrix = transpose of the matrix composed of the determinants of each minor
4. Multiply A^{-1} by the vector of constants (c)
   - Why? We want to isolate matrix x, so we need to multiply both sides of the equation by the inverse of matrix A (divide both sides by matrix A)

Matrix inversion example

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix}$$

- Take the determinant $|A|$ using the Laplace Expansion
  - Pick a row or column
  - Find the determinants of three submatrices
  - Multiply each element of our chosen row/column by those determinants, and sum
- $|A| = -11$
- Non-zero, so we're good to continue!

Matrix inversion example

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix}$$

- Calculate the adjoint matrix C^T
- Find the determinants of each minor (we've done 3 already for the determinant!)
- Create the matrix of cofactors, switching the signs of the determinants as appropriate
- Transpose that matrix of cofactors

$$C^T = \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix}$$

Matrix inversion example

$$C^T = \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix}$$

- Calculate the inverse matrix A^{-1}
- Multiply the inverse determinant $\frac{1}{|A|}$ by the adjoint matrix C^T

$$A^{-1} = \frac{1}{-11} \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix}$$

Matrix inversion example

Last step: Multiply A^{-1} by the vector of constants (c)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix} \times \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Overdetermined systems of equations

- **Overdetermined**: more linearly independent equations than unknowns, so there is no solution (multiple contradicting solutions)
- Cannot solve this the traditional way, as there is no one "perfect" solution

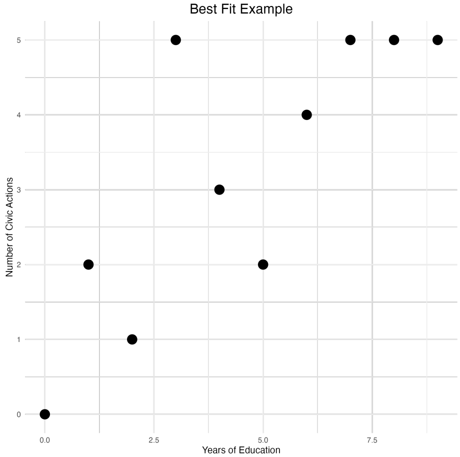
$$
\begin{aligned}
0 &= 0m + b \\
2 &= 1m + b \\
1 &= 2m + b \\
5 &= 3m + b \\
3 &= 4m + b \\
2 &= 5m + b \\
4 &= 6m + b \\
5 &= 7m + b
\end{aligned}
$$

Overdetermined systems of equations

- **Overdetermined**: more linearly independent equations than unknowns, so there is no solution (multiple contradicting solutions)
- Cannot solve this the traditional way, as there is no one "perfect" solution
- $$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \\ 6 & 1 \\ 7 & 1 \\ 8 & 1 \\ 9 & 1 \end{bmatrix} \times \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 5 \\ 3 \\ 2 \\ 4 \\ 5 \\ 5 \\ 5 \end{bmatrix}$$
- We'll call these matrices X, $\hat{\beta}$, and Y

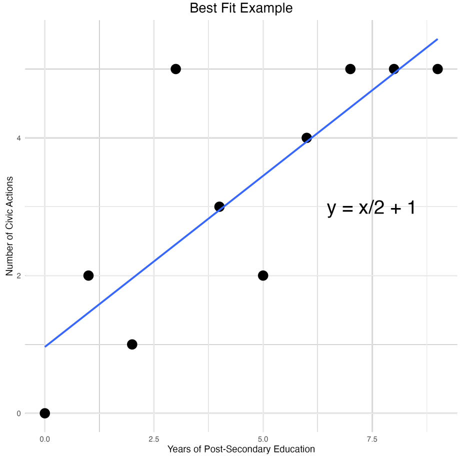
This is an overdetermined problem

The matrix on the last slide represents a series of



No line goes through all points, calculate the "best fit"

This "best fit line" is an "Ordinary Least Squares regression line"



Solving for Best Fit line with matrices

- We can find the slope m and intercept b of the best fit line using matrix algebra!
- $X^T X\hat{\beta} = X^T Y$ provides a best fit solution to $X\hat{\beta} = Y$
- $\hat{\beta} = (X^T X)^{-1} X^T Y$

Solving for Best Fit line with matrices

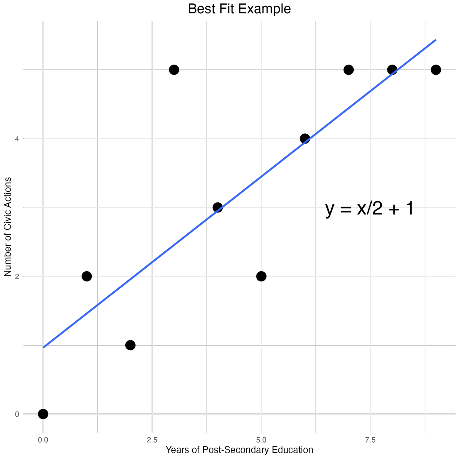
- If we multiply matrix $X^T \times X$, we get $\begin{bmatrix} 285 & 45 \\ 45 & 10 \end{bmatrix}$
- How do we take the inverse of that?
  - Determinant $|X^T X| = (285 * 10) - (45 * 45) = 825$
  - Adjoint matrix $C^T = \begin{bmatrix} 10 & -45 \\ -45 & 285 \end{bmatrix}$
  - $\frac{1}{825} \begin{bmatrix} 10 & -45 \\ -45 & 285 \end{bmatrix} = \begin{bmatrix} \frac{10}{825} & \frac{-45}{825} \\ \frac{-45}{825} & \frac{285}{825} \end{bmatrix}$
- Multiply by the product of C^T and $Y = \begin{bmatrix} 185 \\ 32 \end{bmatrix}$

Solving for Best Fit line with matrices

- $\hat{\beta} = (X^T X)^{-1} X^T Y$
- $\hat{\beta} = \begin{bmatrix} \frac{10}{825} & \frac{-45}{825} \\ \frac{-45}{825} & \frac{285}{825} \end{bmatrix} \times \begin{bmatrix} 185 \\ 32 \end{bmatrix} = \begin{bmatrix} \approx 0.5 \\ \approx 1.0 \end{bmatrix}$

No line goes through all points, calculate the "best fit"

This "best fit line" is an "Ordinary Least Squares regression line"



Problem Set