# Lecture 2 — Linear Algebra

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# Equations and Unknowns

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- Equations: linear functions (e.g. $x^2 + y = 5$)
- Substitution: solve for one variable — in terms of the others — and plug the result into other equations
- Elimination: manipulate equations by applying operations to both the left and right hand side, and combine equations to reduce them

# Substitution Example

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$$\begin{aligned} 3(5-y) - 2y &= 5 \\ 15 - 5y &= 5 \\ -5y &= -10 \\ y &= 2 \end{aligned}$$

- Substitute $y$ for 2 in equation (1)

$$\begin{aligned} x + 2 &= 5 \\ x &= 3 \end{aligned}$$

# Elimination Example

$$2x - y + 3z = 9 \tag{3}$$

$$x + 4y - 5z = -6 \tag{4}$$

$$x - y + z = 2 \tag{5}$$

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- Transform equation (4) so that, when summed with equation (3), $x$ is eliminated

$$-2(x + 4y - 5z) = -2(-6)$$
$$-2x - 8y + 10z = 12$$

- Transform equation (5) so that, when summed with equation (3), $x$ is eliminated

$$-2(x - y + z) = -2(2)$$
$$-2x + 2y - 2z = -4$$

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$$\begin{aligned} 2x - y + 3z &= 9 \\ -2x + 2y - 2z &= -4 \\ y + z &= 5 \end{aligned}$$

# Elimination Example

- Use our new equations to eliminate y

$$
\begin{aligned}
-9y + 13z &= 21 \\
9(y+z) &= 9(5) \\
22z &= 66 \\
z &= 3
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- Use $z = 3$ to solve for $y$ through substitution

$$
\begin{aligned}
-9y + 13(3) &= 21 \\
-9y &= -18 \\
y &= 2
\end{aligned}
$$

# Elimination Example

- Use $y = 2$ and $z = 3$ to solve for $x$

$$\begin{aligned} 2x - (2) + 3(3) &= 9 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

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  - A dataset is an example of many more equations (observations) than unknowns (variables)
- Linear algebra helps us solve these complications

# Vector and scalars

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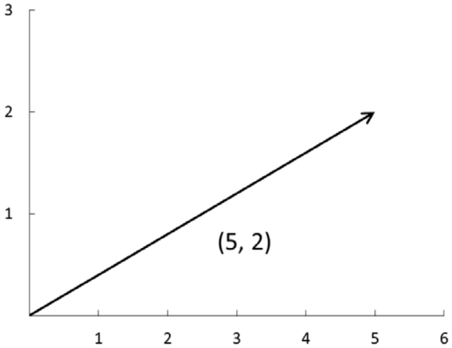


Figure: Vector (5,2)

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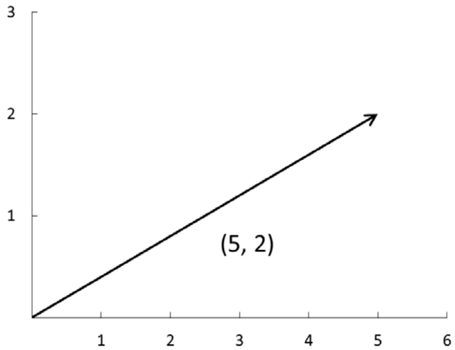


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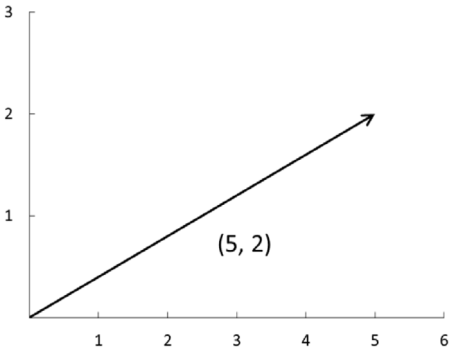


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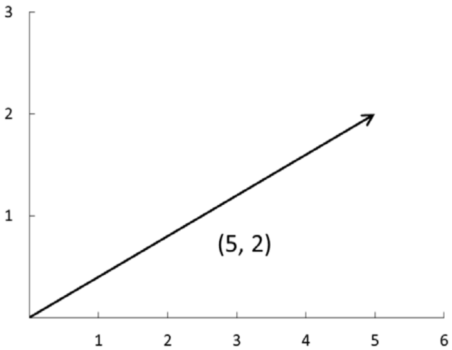


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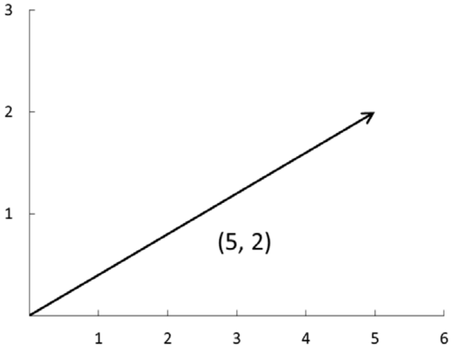


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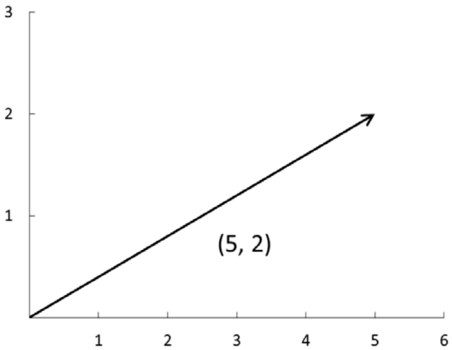


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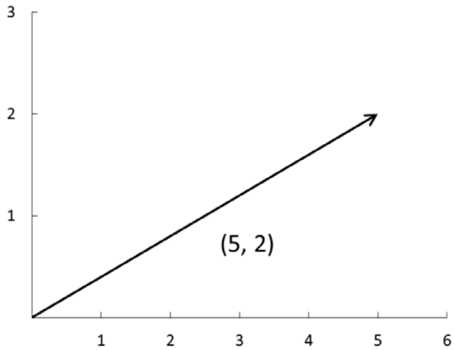


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- Length (norm) of a vector can be solved using the Pythagorean theorem: $a^2 + b^2 = c^2$
- For Vector (5,2) length = $\sqrt{5^2 + 2^2} \approx 5.39$
- This can be expanded to $\|a\| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$

# Vector addition / subtraction

- **Vector sums**

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

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- **Vector differences**
$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{bmatrix}$$

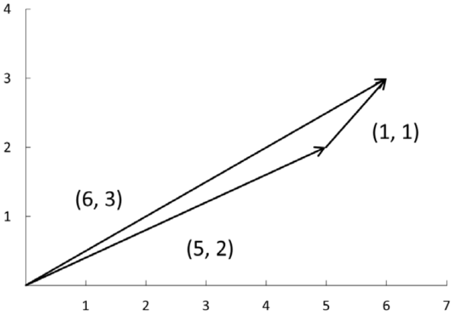


Figure: Vector addition: (5,2)+(1,1)

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$$c \cdot \vec{a} = c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} c \cdot a_1 \\ c \cdot a_2 \\ c \cdot a_3 \end{bmatrix}$$

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- **Dot product (or scalar product)**

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1 b_1 + a_2 b_2$$

This requires that the two vectors be of equal dimension.

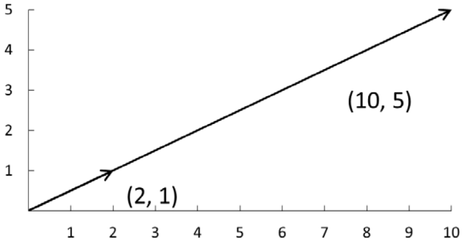


Figure: scalar multiplication: 5a where a=(2,1)

# Dot Product Example

$$\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$$

$$54 + 40 + 28 = 122$$

# Matrices

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Example:

$$A_{3x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

# Important types of matrices

**Zero matrix:**

$$A_{3x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Diagonal matrix:**

$$A_{3x3} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

**Identity matrix:**

$$I_{1x1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Symmetric matrix:**

$$A_{3x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

**Lower triangular matrix:**

$$A_{3x3} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

**Upper triangular matrix:**

$$I_{1x1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

# Matrix transposition

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Example:

$$A_{2x3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
$$A^T_{3x2} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

# Matrix addition, subtraction and scalar multiplication

**Matrix addition and subtraction:** simply add/subtract each corresponding element!

$$A_{3\times 3} \pm B_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

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**Scalar multiplication:**

$$5 \times A = \begin{bmatrix} 5 \times a_{11} & 5 \times a_{12} & 5 \times a_{13} \\ 5 \times a_{21} & 5 \times a_{22} & 5 \times a_{23} \\ 5 \times a_{31} & 5 \times a_{32} & 5 \times a_{33} \end{bmatrix}$$

# Matrix Addition Example

$$\begin{bmatrix} 6 & 7 & 8 \\ 5 & 6 & 0 \\ 5 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 3 \\ 1 & 4 & 1 \\ 0 & 8 & 7 \end{bmatrix}$$

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$$\begin{bmatrix} 6 & 10 & 11 \\ 6 & 10 & 1 \\ 5 & 8 & 11 \end{bmatrix}$$

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  - Left multiplication: multiply by the matrix on the left
  - Right multiplication: multiply by the matrix on the right

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- Repeat this until all values of matrix $C$ are filled in

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- Notation: determinant of matrix $A$ is represented with $|A|$
- Calculation: "difference of the diagonal products"

# Determinant

Determinant for a $2 \times 2$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Is given by:

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Example:

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$|B| = (1 \cdot 4) - (2 \cdot 3) = 4 - 6 = -2$$

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- **Minor:** determinant of a *submatrix*
- **Submatrix** of a matrix value is what's left over when you eliminate the row and column of that value

# Determinants of larger matrices

- Finding a specific Minor $M_{11}$ of Matrix $A$
- Take the submatrix of $a_{11}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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- Take the submatrix of $a_{11}$
- The Minor is the the determinant of the $2 \times 2$ submatrix

$$A = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = a_{22}a_{33} - a_{23}a_{32}$$

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- So now we know how to find **submatrices**, which we take the determinants of to get **Minors**, which are made positive or negative as **cofactors**
- **Laplace Expansion:** determinant of a matrix bigger than $2 \times 2$ is the sum of products of each element and its *cofactor* for *any one* row or column
- Since we can use any row or column, we need to calculate 3 cofactors for a $3 \times 3$ matrix, multiply them by their respective element in the matrix, and sum the results

# Full Example

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

# Submatrix for Minor $M_{11}$

$$A = \begin{bmatrix} \cancel{1} & \cancel{4} & \cancel{3} \\ \cancel{1} & 2 & 0 \\ \cancel{2} & 3 & 1 \end{bmatrix}$$

# Calculate Determinant

$$A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$(2 \cdot 1) - (0 \cdot 3) = 2$$

# Determine cofactor sign

- The sign of this cofactor $C_{11}$ will be positive ($C_{11} = 2$)

$$\begin{bmatrix} a_{11}(+) & a_{12}(-) & a_{13}(+) \\ a_{21}(-) & a_{22}(+) & a_{23}(-) \\ a_{31}(+) & a_{32}(-) & a_{33}(+) \end{bmatrix}$$

# Cofactor $C_{12}$

$$A = \begin{bmatrix} \cancel{1} & \cancel{4} & \cancel{3} \\ 1 & \cancel{2} & 0 \\ 2 & \cancel{3} & 1 \end{bmatrix}$$

$$M_{12} = (1 \cdot 1) - (0 \cdot 2) = 1$$

- The sign of this cofactor will be negative ($C_{12} = -1$)

# Cofactor $C_{13}$

$$A = \begin{bmatrix} \cancel{1} & \cancel{4} & \cancel{3} \\ 1 & 2 & \cancel{0} \\ 2 & 3 & \cancel{1} \end{bmatrix}$$

$$M_{13} = (1 \cdot 3) - (2 \cdot 2) = -1$$

- The sign of this cofactor will be positive ($C_{13} = -1$)

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**The determinant is equal to -5**
This result is replicable for *any* row or column

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- And generally, by this expression: $A^{-1} = \frac{1}{|A|} C^T$

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- Final step is to multiply $C^T$ by $\frac{1}{|A|}$

# Example: Inverting $2 \times 2$ Matrix

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- What is $A \times A^{-1}$?

# Example: Inverting $3 \times 3$ Matrix

- Same steps apply
- Challenge will be to create the matrix of cofactors ($C$)

# Start by calculating each Minor

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ -6 & -2 & 2 \end{bmatrix}$$

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$M_{11} = \begin{vmatrix} 4 & 3 \\ -2 & 2 \end{vmatrix} =$
$(4 \times 2) - (3 \times -2) = 14$

# Start by calculating each Minor

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# Start by calculating each Minor

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$(0 \times 2) - (3 \times -6) = 18$

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$(0 \times -2) - (4 \times -6) = 24$

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$M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} =$
$(1 \times 4) - (2 \times 0) = 4$

# This gives us the Cofactor Matrix $C$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ -6 & -2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 14 & -18 & 24 \\ -6 & 8 & -10 \\ 2 & -3 & 4 \end{bmatrix}$$

# This gives us the Cofactor Matrix $C$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ -6 & -2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 14 & -18 & 24 \\ -6 & 8 & -10 \\ 2 & -3 & 4 \end{bmatrix}$$

- Determinant $|A|$ is the dot product of *any* row or column of elements and their respective cofactors
- Using row 1, $|A| = (1)(14) + (2)(-18) + (1)(24) = 2$

Take the transpose of the Cofactor Matrix $C^T$, multiply by inverse determinant $\frac{1}{|A|}$

$$C = \begin{bmatrix} 14 & -18 & 24 \\ -6 & 8 & -10 \\ 2 & -3 & 4 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 14 & -6 & 2 \\ -18 & 8 & -3 \\ 24 & -10 & 4 \end{bmatrix}$$

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$$C^T = \begin{bmatrix} 14 & -6 & 2 \\ -18 & 8 & -3 \\ 24 & -10 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} C^T = \frac{1}{2} \begin{bmatrix} 14 & -6 & 2 \\ -18 & 8 & -3 \\ 24 & -10 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -3 & 1 \\ -9 & 4 & -\frac{3}{2} \\ 12 & -5 & 2 \end{bmatrix}$$

# Did it work?

$$A \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ -6 & -2 & 2 \end{bmatrix} \times A^{-1} \begin{bmatrix} 7 & -3 & 1 \\ -9 & 4 & -\frac{3}{2} \\ 12 & -5 & 2 \end{bmatrix}$$

# Did it work?

$$A \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ -6 & -2 & 2 \end{bmatrix} \times A^{-1} \begin{bmatrix} 7 & -3 & 1 \\ -9 & 4 & -\frac{3}{2} \\ 12 & -5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Matrix and vector properties

Table 12.2: Matrix and Vector Transpose Properties

| | |
|---|---|
| Inverse | $(A^T)^T = A$ |
| Additive property | $(A+B)^T = A^T + B^T$ |
| Multiplicative property | $(AB)^T = B^T A^T$ |
| Scalar multiplication | $(cA)^T = cA^T$ |
| Inverse transpose | $(A^{-1})^T = (A^T)^{-1}$ |
| If $A$ is symmetric | $A^T = A$ |

Table 12.3: Matrix Determinant Properties

| | |
|---|---|
| Transpose property | $\det(A) = \det(A^T)$ |
| Identity matrix | $\det(I) = 1$ |
| Multiplicative property | $\det(AB) = \det(A)\det(B)$ |
| Inverse property | $\det(A^{-1}) = \frac{1}{\det(A)}$ |
| Scalar multiplication $(n \times n)$ | $\det(cA) = c^n \det(A)$ |
| If $A$ is triangular or diagonal | $\det(A) = \prod_{i=1}^n a_{ii}$ |

Table 12.4: Matrix Inverse Properties

| | |
|---|---|
| Inverse | $(A^{-1})^{-1} = A$ |
| Multiplicative property | $(AB)^{-1} = B^{-1}A^{-1}$ |
| Scalar multiplication $(n \times n)$ | $(cA)^{-1} = c^{-1}A^{-1}$ if $c \neq 0$ |

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Examples:

- Suppose $v_1 = (1, 3)$ and $v_2 = (3, 9)$. These vectors are not linearly independent because $3v_1 - v_2 = 0$
- Suppose $v_1 = (1, 3)$ and $v_2 = (2, 9)$. These vectors are linearly independent because the only $a_i$ that allow $a_iv_1 - a_iv_2 = 0$ are $a_i = 0$

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Linearly independent matrices have non-zero determinants, can be inverted
Linearly dependent matrices have a determinant of zero, cannot be inverted. In statistics, this is called *multicollinearity*

# Matrix rank

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  - If the determinant is zero, they are dependent, cannot be inverted
  - Linear dependence is called *multicollinearity* in a statistics context

# Systems of equations

How determined is your system of equations?

- Uniquely determined
  - Same number of equations and variables to solve for.
  - Yields one unique solution
- Overdetermined
  - More equations than unknowns
  - Equations may be contradictory
- Undetermined
  - More unknown than equations
  - Can occur if equations are not linearly independent - each equation must give us new information if we want to solve the system.
  - Infinite number of possible solutions

# Systems of Equations as Matrices

$$\begin{aligned} 2x - y + 3z &= 9 \\ x + 4y - 5z &= -6 \\ x - y + z &= 2 \end{aligned}$$

becomes...

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix}$$

# Solving systems of equations - Matrix inversion

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4. Multiply $A^{-1}$ by the vector of constants ($c$)

# Solving systems of equations - Matrix inversion

Steps:

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2. Check the determinant of $A$. If it is non-zero, can be inverted.
3. Calculate $A^{-1}$
   - Inverse = 1/determinant times adjoint matrix
   - Adjoint matrix = transpose of the matrix composed of the determinants of each minor
4. Multiply $A^{-1}$ by the vector of constants ($c$)
   - Why? We want to isolate matrix $x$, so we need to multiply both sides of the equation by the inverse of matrix $A$ (divide both sides by matrix $A$)

# Matrix inversion example

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix}$$

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- Take the determinant $|A|$ using the Laplace Expansion
  - Pick a row or column
  - Find the determinants of three submatrices
  - Multiply each element of our chosen row/column by those determinants, and sum
- $|A| = -11$
- Non-zero, so we're good to continue!

# Matrix inversion example

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -5 \\ 1 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix}$$

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- Find the determinants of each minor (we've done 3 already for the determinant!)

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$$C^T = \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix}$$

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- Multiply the inverse determinant $\frac{1}{|A|}$ by the adjoint matrix $C^T$

# Matrix inversion example

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- Calculate the inverse matrix $A^{-1}$
- Multiply the inverse determinant $\frac{1}{|A|}$ by the adjoint matrix $C^T$

$$A^{-1} = \frac{1}{-11} \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix}$$

# Matrix inversion example

Last step: Multiply $A^{-1}$ by the vector of constants ($c$)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -1 & -2 & -7 \\ -6 & -1 & 13 \\ -5 & 1 & 9 \end{bmatrix} \times \begin{bmatrix} 9 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

# Overdetermined systems of equations

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- **Overdetermined**: more linearly independent equations than unknowns, so there is no solution (multiple contradicting solutions)

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$$
\begin{aligned}
0 &= 0m + b\\
2 &= 1m + b\\
1 &= 2m + b\\
5 &= 3m + b\\
3 &= 4m + b\\
2 &= 5m + b\\
4 &= 6m + b\\
5 &= 7m + b\\
5 &= 8m + b
\end{aligned}
$$

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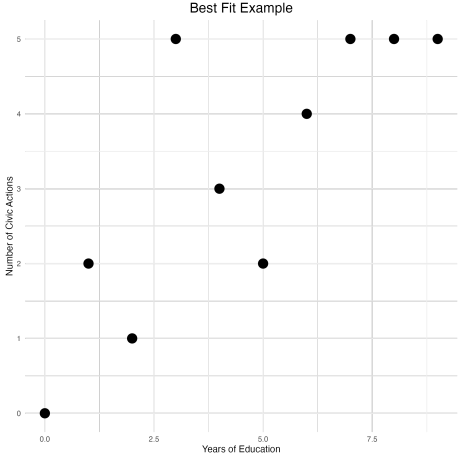
- Cannot solve this the traditional way, as there is no one "perfect" solution
- $$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \\ 6 & 1 \\ 7 & 1 \\ 8 & 1 \\ 9 & 1 \end{bmatrix} \times \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 5 \\ 3 \\ 2 \\ 4 \\ 5 \\ 5 \\ 5 \end{bmatrix}$$
- We'll call these matrices $X$, $\hat{\beta}$, and $Y$

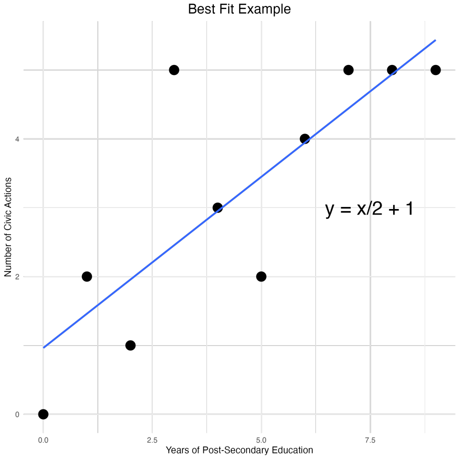
# This is an overdetermined problem

The matrix on the last slide represents a series of



# No line goes through all points, calculate the "best fit"

This “best fit line” is an “Ordinary Least Squares regression line”



# Solving for Best Fit line with matrices

- We can find the slope $m$ and intercept $b$ of the best fit line using matrix algebra!

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- We can find the slope $m$ and intercept $b$ of the best fit line using matrix algebra!
- $X^T X\hat{\beta} = X^T Y$ provides a best fit solution to $X\hat{\beta} = Y$
- $\hat{\beta} = (X^T X)^{-1} X^T Y$

# Solving for Best Fit line with matrices

- If we multiply matrix $X^T \times X$, we get $\begin{bmatrix} 285 & 45 \\ 45 & 10 \end{bmatrix}$
- How do we take the inverse of that?
  - Determinant $|X^T X| = (285 * 10) - (45 * 45) = 825$

# Solving for Best Fit line with matrices

- If we multiply matrix $X^T \times X$, we get $\begin{bmatrix} 285 & 45 \\ 45 & 10 \end{bmatrix}$
- How do we take the inverse of that?
  - Determinant $|X^T X| = (285 * 10) - (45 * 45) = 825$
  - Adjoint matrix $C^T = \begin{bmatrix} 10 & -45 \\ -45 & 285 \end{bmatrix}$

# Solving for Best Fit line with matrices

- If we multiply matrix $X^T \times X$, we get $\begin{bmatrix} 285 & 45 \\ 45 & 10 \end{bmatrix}$
- How do we take the inverse of that?
  - Determinant $|X^T X| = (285 * 10) - (45 * 45) = 825$
  - Adjoint matrix $C^T = \begin{bmatrix} 10 & -45 \\ -45 & 285 \end{bmatrix}$
  - $\frac{1}{825} \begin{bmatrix} 10 & -45 \\ -45 & 285 \end{bmatrix} = \begin{bmatrix} \frac{10}{825} & \frac{-45}{825} \\ \frac{-45}{825} & \frac{285}{825} \end{bmatrix}$
- Multiply by the product of $C^T$ and $Y = \begin{bmatrix} 185 \\ 32 \end{bmatrix}$

# Solving for Best Fit line with matrices

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# Solving for Best Fit line with matrices

- $\hat{\beta} = (X^T X)^{-1} X^T Y$
- $\hat{\beta} = \begin{bmatrix} \frac{10}{825} & \frac{-45}{825} \\ \frac{-45}{825} & \frac{285}{825} \end{bmatrix} \times \begin{bmatrix} 185 \\ 32 \end{bmatrix} = \begin{bmatrix} \approx 0.5 \\ \approx 1.0 \end{bmatrix}$

# No line goes through all points, calculate the "best fit"

This "best fit line" is an "Ordinary Least Squares regression line"