# Lecture 5 — Probability

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8/18/23

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3. For $x = (1,2,3,4,5)$ and $y = (1.5,4,4,9,14)$
   1. Calculate $\beta$ and $\alpha$ and for the OLS regression line using the formulas:
   $\hat{\beta} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y}))}{\sum_{i=1}^{n}(X_i - \overline{X})^2}$
   $\hat{\alpha} = \overline{Y} - \hat{\beta}\overline{X}$

# Agenda

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- Frequentists care about truth, whereas Bayesians care about prediction and perspective
- In academia, we tend to focus more on Frequentist statistics as it is more relevant to causal identification

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- If I have already flipped a coin, but not looked at it yet, what is the probability that that specific coin is heads-side up?
  - The bayesian will still say 50%, but the frequentist will say it's no longer a matter of probability

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- **Events**: composed of one or more outcomes
- **Sample Space**: the set of all possible outcomes
- **Random Events**: events that are probabilistic (as opposed to deterministic
    - When we say "random" here, we mean probabilistic
    - Can identify causal processes that alter the probability, but not causal processes that guarantee the event will occur

# Calculating Probability:

$$Pr(e) = \frac{\text{No. of outcomes in event e}}{\text{No. of outcomes in sample space}} \tag{1}$$

e.g. What is the probability of getting a tail when you flip a coin?

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- **Collective Exhaustivity:** Every possible event fits into one of the categories
- **Conditional events:**: The probability of one event occurring is affected by whether another event occurs

# Bayes theorem:

The *economist* (2004) offers the following illustration of Bayes' rules:
*The canonical example is to imagine that a precarious newborn observes his first sunrise, and wonders whether the sun will rise again or not. He assigns equal prior probability to both possible outcomes, and represents this by placing one white and one black marble into a bag. The following day, when the sun rises, the child places another white marble in the bag. The probability that a marble plucked randomly from the bag will be white (i.e., the child's degree of belief in future sunrises) has thus gone from a half to two-thirds. After sunrise the next day, the child adds another white marble, and the probability (and thus the degree of belief) goes from two-thirds to three-quarters. And so on. Gradually, the initial belief that the sun is just as likely to rise as not to rise each morning is modified to become a near-certainty that sun will always rise.*

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- $Pr(A \cap B) = Pr(B|A)Pr(A) = Pr(A|B)Pr(B)$
  - The probability of $A$ and $B$ happening is equal to the conditional probability of $B$ given $A$ times the unconditional probability of $A$ (or vice versa)
  - The probability of someone voting and donating to a candidate is equal to the probability of a donor voting times the probability of donating

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- When $A$ and $B$ are independent: $Pr(A \cap B) = Pr(A)Pr(B)$
- Example:
  - Pr(voting) $= 0.5$
  - Pr(donating) $= 0.2$
  - If voting is independent of donating, Pr(voting $\cap$ donating) $= 0.5 \times 0.2 = 0.10$
  - If Pr(voting|donating) $= 0.9$, Pr(voting $\cap$ donating) $= 0.9 \times 0.2 = 0.18$

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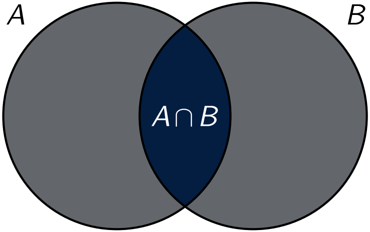
- When $A$ and $B$ are mutually exclusive: $Pr(A \cap B) = 0$ because $Pr(A|B) = Pr(B|A) = 0$.
- $A \cup B = Pr(A) + Pr(B) - Pr(A \cap B)$
  - The probability of A or B happening is equal to the sum of their unconditional probabilities minus the probability of both happening
  - Using our example from before, Pr(voting $\cup$ donating) $=$ $0.5 + 0.2 - 0.18 = 0.52$
- We will define $\sim A$ as "not $A$".

# Subtract the intersection so you don't double-count it



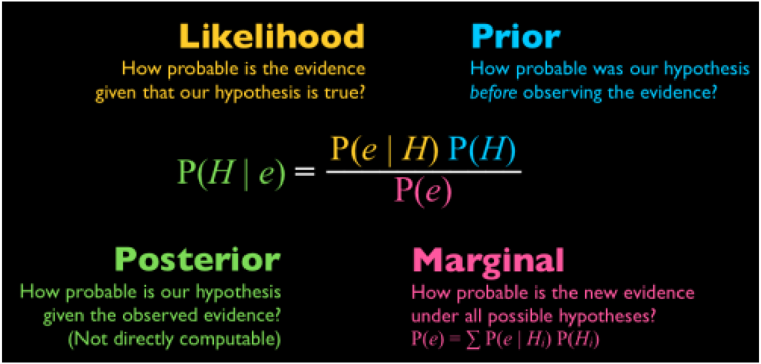
# Bayes theorem:

$$Pr(B|A) = \frac{Pr(A|B)Pr(B)}{P(A)} \tag{2}$$

What is equivalent to:

$$Pr(B|A) = \frac{Pr(A|B)Pr(B)}{P(A|B)Pr(B) + Pr(A| \sim B)Pr(\sim B)} \tag{3}$$

# Bayes theorem



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- 1,000 people have symptoms, and take a test. 50 of them are actually sick.
- 45 people test positive, but 5 of those are false positives
- 955 people test negative, but 10 of those are false negatives

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- $Pr(Positive) = 45/1000 = 0.045$
- $Pr(Sick) = 50/1000 = 0.050$
- $Pr(Positive|Sick) = \frac{(0.89)(0.045)}{0.050} = 0.80$

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- $Pr(Sick) = 50/1000 = 0.050$
- $Pr(Positive|Sick) = \frac{(0.89)(0.045)}{0.050} = 0.80$
- **Main lesson:** $Pr(Positive|Sick) \neq Pr(Sick|Positive)$

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- Switching doors will, statistically, increase your chances of winning. What?!

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    - If you picked incorrectly the first time, you will always win if you switch
- The start of the game is random: you had a $\frac{1}{3}$ chance of guessing right, and a $\frac{2}{3}$ chance of guessing wrong.
    - Chances of winning if you switch: $(1.00 \times 0.66) + (0.00 \times 0.33)$

# Odds

The odds of an event happening are defined as the ratio of the probability of the event occurring divided by the probability of the event not occurring. Example: The odds of rolling a four on a dice are $1:5$.

$$Odds = \frac{Pr(A)}{1 - Pr(A)} \tag{4}$$

# Odds Ratios

- Odds ratios: the ratio of the odds of an event occurring under the condition $A$ and the odds of that event occurring under the condition $B$

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- $\frac{13040/58860}{4890/86390} = 3.91$

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    - There is a true relationship between Democracy and Conflict, and we can estimate that relationship with tools such as linear regression
    - Our estimates will have a "confidence interval"
    - The difference in vote choice between registered Democrats and registered Republicans is *statistically significant*, in that it was unlikely to have occurred by chance

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- We are aiming at an unknown, but constant, parameter. We'll rarely hit it straight on, but if we try many times, our 95% confidence interval will contain the truth 95% of the time. 5% of the time, we'll miss completely, and our confidence interval won't contain the truth.
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  - Confidence intervals do not have to be set at 95%, but that is conventional
- A confidence interval should ***NOT*** be interpreted as "there is a 95% chance that this particular interval contains the true parameter"
  - We live in a frequentist world with constant but unknown parameters; a specific confidence interval either contains the truth or it doesn't

# Confidence Interval Demonstration in R

# Combinations and permutations

**Combinations:** Choose $k$ objects from a set of $n$ objects when the order **does not** matter.

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**Draws with Replacement:** Choose an object $k$ times from a set of $n$ objects when the order **does** matter and you **replace** the object each time

$$\binom{n}{k} = n^k$$

# Review of Factorials

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- $0! = 1$ (One possible way to arrange a dataset of zero numbers)
- $\frac{n!}{(n-k)!} = n \times (n-1) \cdots \times (n-k+1)$
  - $\frac{5!}{(5-3)!} = 5 \times 4 \times 3$
  - $\frac{5 \times 4 \times 3 \cancel{\times 2 \times 1}}{\cancel{2 \times 1}}$

# Why do these formulas work?

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- We use factorials because each time you select an object you have one fewer to choose from. If $n = 5$, the first object you draw has 5 possibilites, the second 4, and so on
- Dividing by $(n-k)!$ reduces the number of permutations based on how many objects you're choosing.
- We only need to consider the first $k$ elements of the factorial in the numerator, because you are only choosing $k$ objects from the set $n$

# Why do these formulas work?

**Combinations:** Choose $k$ objects from a set of $n$ objects when the order **does not** matter.

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- Logic of this is that you can arrange a set of $k$ numbers in $k!$ factorial ways
- So if we divide by $k!$, we're left with only one permutation of each original combination

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- We don't need factorials because replacement means our number of options isn't reduced
- If I have 5 objects, there are 5 possibilities for my first draw. **If I replace that object**, there are still 5 possibilities for my second draw.

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- Dividing by $(5-3)!$ reduces this numerator to $5 \times 4 \times 3$, because we only want combinations of three people: 60 permutations
- Additionally dividing by 3! eliminates duplicated (reordered) combinations
    - Example: I can arrange the numbers 3,4,5 in 3! (6) different ways: 3,4,5 ; 3,5,4 ; 4,3,5 ; 4,5,3 ; 5,3,4 ; 5,4,3
    - But these are not original combinations; we have six times as many permutations as unique combinations
    - Dividing by 3! (6) solves this: 10 unique combinations

# Combinations Example:

A, B, C, D, E

1. A, B, C
2. A, B, D
3. A, B, E
4. A, C, D
5. A, C, E
6. A, D, E
7. B, C, D
8. B, C, E
9. B, D, E
10. C, D, E

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12 PhD students decide to host a thanksgiving potluck:

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How many different ways can the students be divided into these groups?

$$\binom{12}{3} \cdot \binom{9}{5} \cdot \binom{4}{3} \cdot \binom{1}{1} = \frac{12!}{3!9!} \cdot \frac{9!}{5!4!} \cdot \frac{4!}{3!1!} \cdot \frac{1!}{1!0!} = 220 \cdot 125 \cdot 4 \cdot 1 = 110,880$$

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  - After 10 tries: $1-(1-0.1)^10 \approx 0.65$

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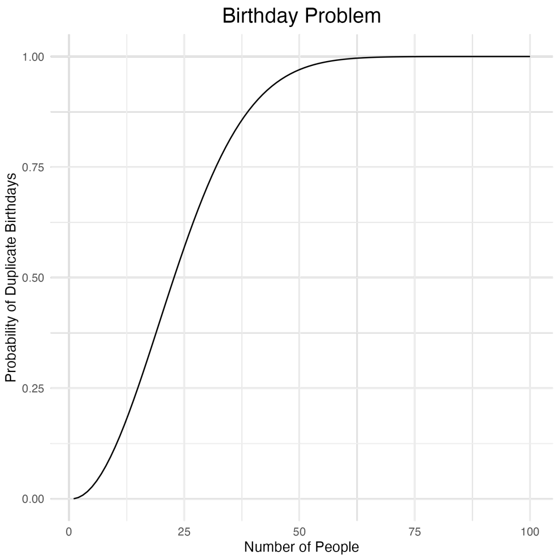
- $1 - (\frac{365!}{(365-k)!} \div 365^k)$ gives us the share of possible birthday combinations with at least one duplicate


Birthday Problem
Probability of Duplicate Birthdays
1.00
0.75
0.50
0.25
0.00
0
25
50
75
100
Number of People

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- A random variable is **realized** when it takes on any specific value from the set of possible values.
- We say that random variables are **stochastic**: Their probability distribution can be analyzed statistically, but cannot be precisely predicted

# Discrete versus continuous

- Remember from the first day:
- Discrete: finite, countable
- Continuous: infinite, uncountable

# Sample distribution

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- **Relative Frequency Distribution**: a frequency distribution represented as the share of observations/cases

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| | Dog | Cat | Total |
|---|---|---|---|
| Male | 42 | 10 | 52 |
| Female | 9 | 39 | 48 |
| Total | 51 | 49 | 100 |

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- Probability of any one value (or range of values) is between 0 and 1
- Sum of probabilities for the full range of values equals 1

# Probability Mass Function

For discrete values, we use the Probability Mass Function (effectively a bar

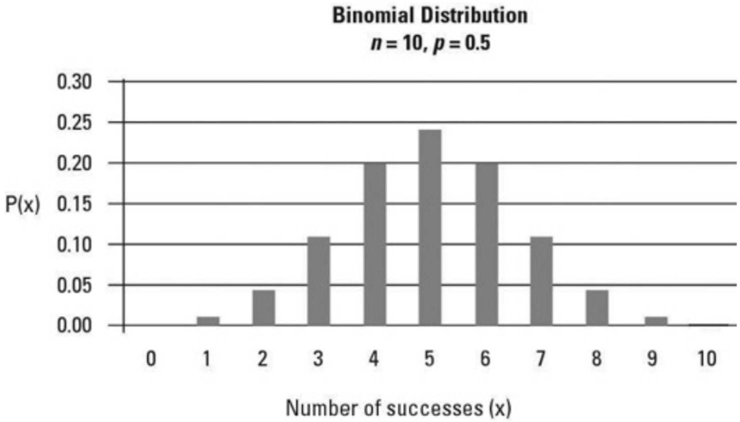
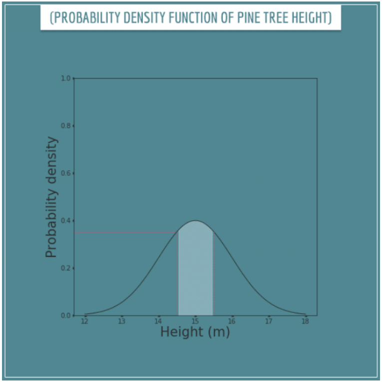



chart Binomial distribution: ten trials with $p = 0.5$.

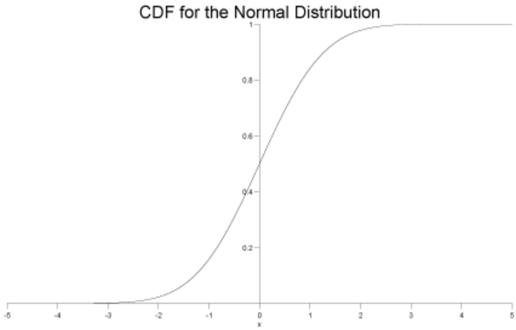
# Probability Density Function

- For continuous variables, we use the Probability Density Function
- To find the likelihood that an observation will be between two given values, we look at the area under the curve (integration)



# Cumulative Density Function

The integral, or antiderivative, of the Probability Density Function

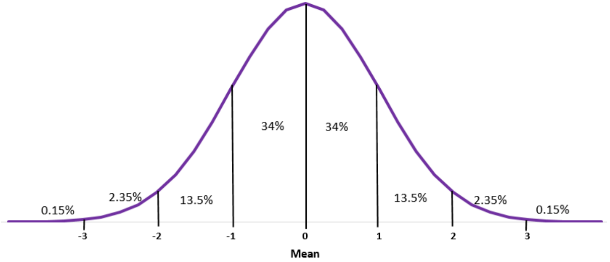


# Normal/Gaussian Distribution (the bell curve)

Important features:

- This distribution is symmetric around the center.
- Standard deviations are measures of how spread out the observations are: 68% of values are within one SD, 95% are within two, 99.7% are within three.

Very typical in political science!



# Other types of distributions:

- Poisson
- Binomial
- Negative binomial
- t distribution
- F distribution
- Exponential distribution
- Gamma distribution
- And many more...