Lecture 1 — Notation, Functions, and Limits

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8/14/23

Today's Agenda

1. Introductions
2. Math Camp Logistics and schedule
3. Variables and measurement
4. Algebra
5. Functions
6. Series, sequences, and limits

Week's Agenda

- Lecture 1: Notation, functions and limits
- Lecture 2: Linear Algebra
- Lecture 3: Calculus 1 - Derivatives
- Lecture 4: Calculus 2 - Integrals and multivariate calculus
- Lecture 5: Probability

Schedule

	Monday 14th	Tuesday 15th	Wednesday 16th	Thursday 17th	Friday 18th
9:30 - 12:00	Notation, functions, and limits	Linear algebra	Calculus 1	Calculus 2	Probability
12:00 - 12:10	Break	Break	Break	Break	Break
12:10 - 1:00	Problem Set	Problem Set	Problem Set	Lunch Break	LaTeX
1:00 - 2:00	Lunch Break	Lunch Break	Lunch Break	Problem Set	Lunch Break
2:00 - 2:30	Review PS	Review PS	Review PS	Review PS	Q&A with Prof Klasnja
2:30 - 3:30	Software Installation	R - Day 1	R - Day 2	R - Day 3	Q&A Continued

Goals of Math Camp

- Familiarity
- Recognition
- Confidence

What is quantitative methods?

- Ultimate goal is to improve our understanding of the world
- We use quantitative methods to test theories regarding the relationships between concepts
- The goal of a "causal claim" is rarely achieved in full, but we can make compelling arguments
- Qualitative methods can answer "how" and "why" questions

Variables and Constants

- **Theory:** A set of statements that involve concepts.
- **Concept:** abstract ideas used to understand the world
    - Participation, voting, democracy, war
- **Measure:** an operational indicator of a concept
- **Variable:** A concept or measure that takes different values in a given set.
- **Constant:** A concept or measure that has a single value for a given set.

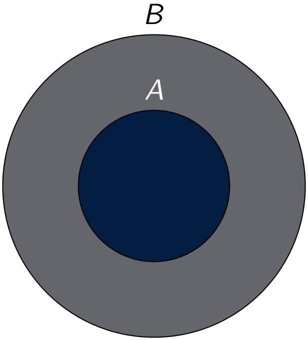
Sets and Sample Spaces

- A *set* is a collection of elements.
- **Common sets:** Natural numbers ($\mathbb{N}$), Integers ($\mathbb{Z}$), Rational numbers ($\mathbb{Q}$), Real Numbers ($\mathbb{R}$), etc.
- A set can be:
  - **Finite or infinite:** $\mathbb{Z}$ is infinite, but all the integers from 1 to 10 is finite.
  - **Countable or uncountable:** a countable set is one whose each of its element can be associated with a natural number (or an integer).
  - **Bounded or unbounded:** A bounded set has finite size (but may have infinite elements).
- Some important sets that we are going to use as political scientists:
  - **Solution set:** a set that contains all solutions for an equation
  - **Sample space:** a set that contains all values that a variable can take.

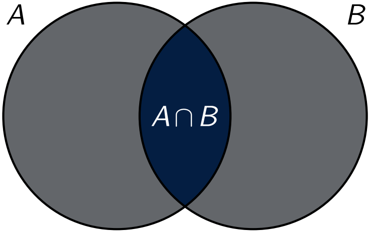
Unions and Intersections

- Much as sets contain elements, they also can contain, and be contained by, other sets.
- Notation:
  - $A \subset B$: "A is a **proper subset** of B" implies that set B contains all the elements in A, plus at least one more
  - $A \subseteq B$: "A is a **subset** of B". In this case, it allows A and B to be the same.
- **Intersection:** $A \cap B$. The set of elements common to two sets.
- **Union:** $A \cup B$. The set that contains all elements in both sets.
- **Mutually exclusive sets:** the intersection is the empty set.

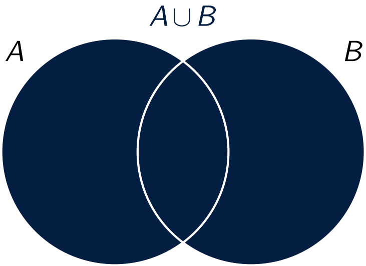
$A \subset B$ (Proper Subset)



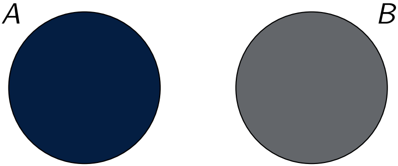
$A \cap B$ (Intersection)



$A \cup B$ (Union)



$A \cap B = \emptyset$ (Mutually Exclusive)



Levels of Measurement

- Categorical
  - **Nominal**: No mathematical relationship; difference of kind.
    - Occupation, colors, political party, gender
  - **Ordinal:** Meaningful order, no meaningful "distance" between categories.
    - Ideology, agree/disagree, education level
- Numerical
  - **Interval**: Meaningful order and distance between values.
    - Age, budget, polity scores, vote share, feeling thermometer
    - Discrete: countable
    - Continuous: non-countable
    - Ratio: meaningful zero (re-scaled thermometer)

Why it Matters

- Visualizaton
  - Categorical: pie charts, bar charts
  - Numerical (especially Continuous): histograms, box plots
- Analysis
  - Numerical data can use summary statistics (mean, median, standard deviation)
  - Categorical - Categorical : cross-tabulation
  - Categorical - Continuous : difference in means
  - Continuous - Continuous : scatter plots, regression

Basic properties of arithmetic

- For variables that stand for real numbers or integers, these properties will always hold:
- **Associative properties:**
  - $(a+b)+c=a+(b+c)$
  - $(a\cdot b)\cdot c=a\cdot(b\cdot c)$
- **Commutative properties:**
  - $a+b=b+a$
  - $a\cdot b=b\cdot a$
- **Distributive properties:**
  - $a\cdot(b+c)=ab+ac$
- **Identity properties:**
  - $a+0=a$
  - $a\cdot 1=a$
- **Inverse properties (for real numbers not integers):**
  - $(-a)+a=0$
  - $a^{-1}\cdot a=1$

Order of operations

- **Order of operations - PEMDAS**:
  - Parentheses ()
  - Exponents (^{x})
  - Multiplication ($\cdot$)
  - Division ($\div$)
  - Addition ($+$)
  - Subtraction ($-$)

Inequalities

- All pairs of real numbers have exactly one of the following relations: $x = y$, $x > y$, or $x < y$.
- Solving inequalities is similar to solving equations but there are a few extra properties
- Adding any number to each side of these relations will not change them; this includes inequalities.
- Multiplication:
  - If a is positive and $x > y$, then $ax > ay$.
  - If a is negative and $x > y$, then $ax < ay$.
- Division:
  - If a is positive and $x > y$, then $\frac{x}{a} > \frac{y}{a}$.
  - If a is negative and $x > y$, then $\frac{x}{a} < \frac{y}{a}$.

Exponents, logarithms, and roots

- Exponential: x^a
- Roots: $\sqrt[a]{x}$
  - All roots can be expressed as fractional exponents
  - $\sqrt[a]{x} = x^{1/a}$
- Logarithm: inverse of exponents $log_a x$
  - To what power would you need to raise a to to get x?
- Natural log: log base e
  - $e \approx 2.7183$
  - Useful properties in calculus
  - Taking the natural log of a variable allows us to interpret relationships in terms of percentage changes

Exponent, logarithms and root rules

Exponent rules

- $x^a \cdot x^b = x^{a+b}$
- $x^a \cdot z^a = (xz)^a$
- $(x^a)^b = x^{ab}$
- $\frac{x^a}{x^b} = x^{a-b}$
- $\frac{x^a}{z^a} = (\frac{x}{z})^a$

Logarithm rules

- $log(x_1 \cdot x_2) = log(x_1) + log(x_2)$ for $x_1, x_2 > 0$
- $log(\frac{x_1}{x_2}) = log(x_1) - log(x_2)$ for $x_1, x_2 > 0$
- $log(x^b) = b \cdot ln(x)$ for $x > 0$

Exponent, logarithms and root rules

Root rules

- $\sqrt{x} = x^{\frac{1}{2}}$
- $\sqrt[n]{x} \cdot \sqrt[n]{z} = \sqrt[n]{xz}$
  for $n > 1$
- $\frac{\sqrt[n]{x}}{\sqrt[n]{z}} = \sqrt[n]{\frac{x}{z}}$
  for $n > 1$

Functions and its characteristics

- Functions provide a **specific** description of the association or relationship between two (or among several) concepts (in theoretical work) or variables (in empirical work)
- Functions assign one element of the range to an element of the domain (one x is assigned to one y)
- Noted as $f(x) : A \rightarrow B$ or "f maps A into B"
- A is the *domain*, or set of possible x values.
- B is the *codomain*, or set of possible y values.
- To be a function, there must only be one unique value in its range (y) for each value in its domain (x)

Graph examples

In which of these graphs can we observe a function?

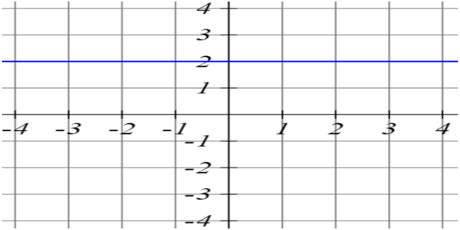


Figure: a)

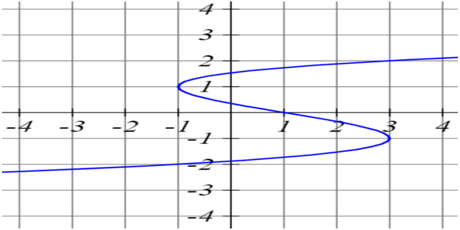
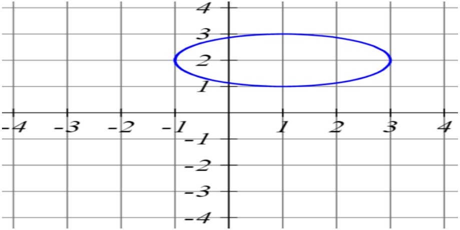
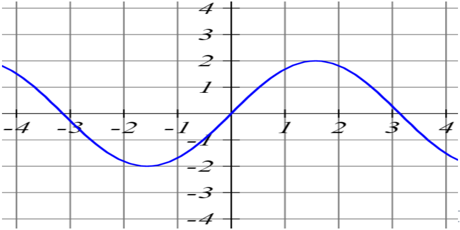


Figure: b)





Graph examples

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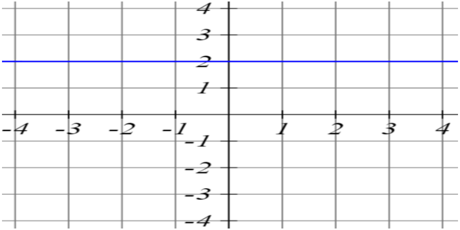

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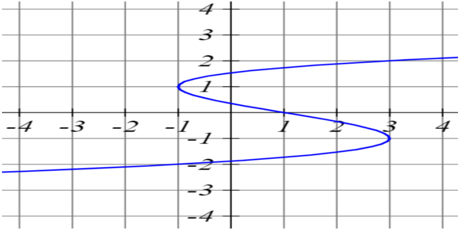
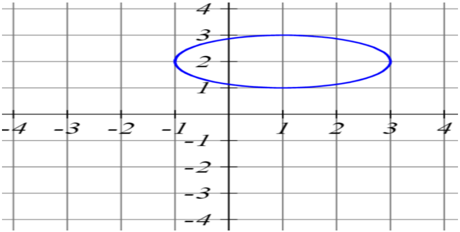
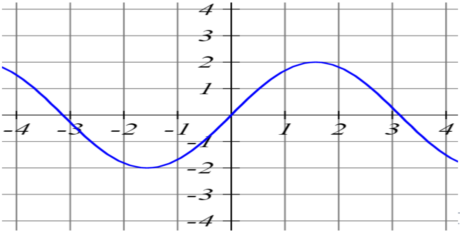


Figure: b)





Function composition

- We can chain multiple functions using function composition.
  - This is written either as $g \circ f(x)$ or $g(f(x))$.
  - It is read as "g composed with f" or "g of f of x"
  - Generally, $g \circ f(x) \neq f \circ g(x)$
- Example: Suppose $f(x) = 2x$ and $g(x) = x^3$
  - $g \circ f(x) = (2x)^3 = 8x^3$
  - $f \circ g(x) = 2(x^3) = 2x^3$

Examples of functions of one variable - linear equation

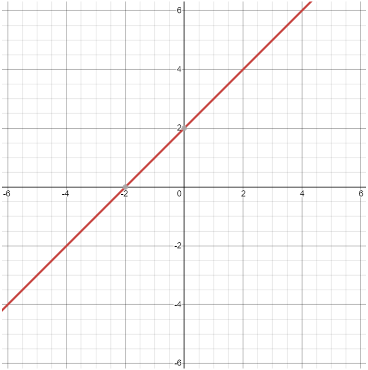


Figure: y=2+x

- This is the classic linear equation $y = a + bx$ or $y = mx + n$
- a and b are constants and x is the variable.
- a is the intercept and b is the slope of the line, or the amount that y changes given a one-unit increase in x.

Examples of functions of one variable - Quadratic function

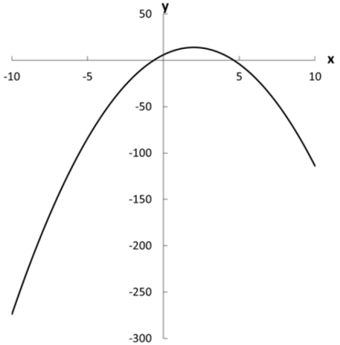
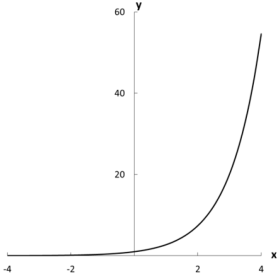


Figure: $y = -2x^2 + 8x + 6$

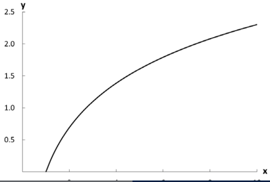
- This is the classical quadratic function: $f(x) = ax^2 + bx + c$ or $f(x) = \alpha + \beta_1 x + \beta_2 x^2$
- If we set $a > 0$ ($\beta_2 > 0$) we get a curve shaped like an U (a convex parabola).
- If we set $a < 0$ ($\beta_2 < 0$) we get a curve shaped like an inverse U (a concave parabola).

Exponent, logarithms and roots - Graphs

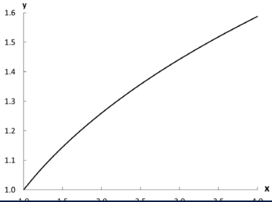
Exponential function



Logarithmic function



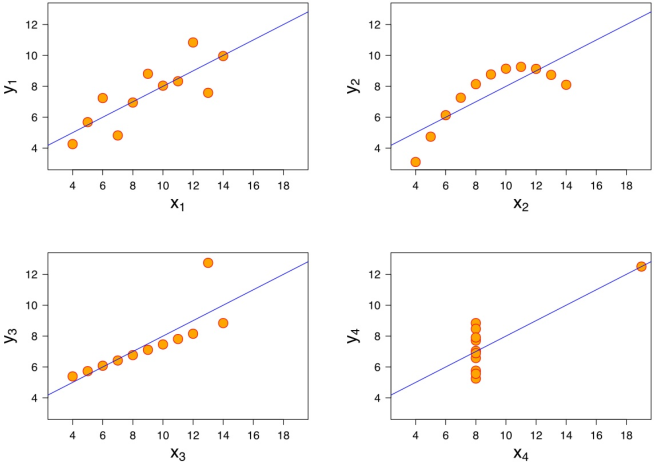
Root function



Functions and Theorizing

- Functions are important for theorizing about politics
- Most quantitative social science boils down to how two variables relate!
- Our examples suggest a precise function between x and y
- In statistics, you will *theorize* the functional form of the relationship
- Statistical methods will *estimate* the most appropriate parameters; "a line of best fit"

Anscombe's Quartet



Solving Equations

1. Isolate the variable you are looking for
2. Combine like terms
3. Factor and cancel
4. Operate on both sides of the equation
5. Check your answer

Special products

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(x+a)(x+b) = x^2 + (a+b)x + ab$
4. $(a+b)(a-b) = a^2 - b^2$
5. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+ac+bc)$
6. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Factoring

Factoring: rearranging terms in an equation to make it simpler

$$\frac{3x^4+3x^3-6x^2}{6x^2+12x} \tag{1}$$

$$\frac{3x^2(x^2+x-2)}{6x(x+2)} \tag{2}$$

$$\frac{3x^2(x+2)(x-1)}{6x(x+2)} \tag{3}$$

$$\frac{3x^2(x-1)}{6x} \tag{4}$$

$$\frac{x(x-1)}{2} \tag{5}$$

Quadratics and Special Products

Quadratic polynomials can be factored into the product of two terms: $(x \pm ?) \times (x \pm ?)$

$$x^2 - 8x + 16 = 9 \tag{6}$$

$$(x \pm ?) \times (x \pm ?) = x^2 - 8x + 16$$

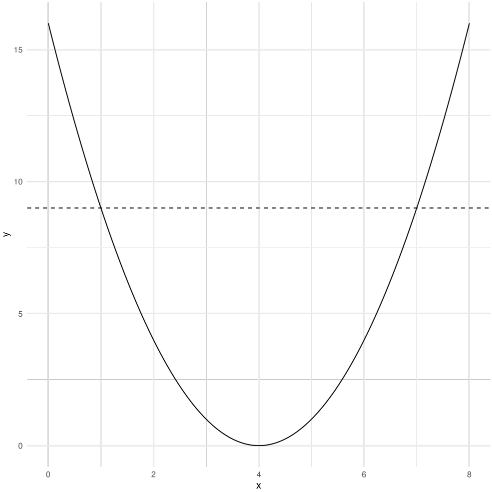
$$(x-4)^2 = 9 \tag{7}$$

$$(x-4) = \pm 3 \tag{8}$$

$$x = 4 \pm 3 \tag{9}$$

$$x = 1 \text{ OR } x = 7$$

Why two solutions?



Another example

$$x^2 + 7x = -12 \tag{10}$$

$$x^2 + 7x + 12 = 0 \tag{11}$$

How can we use special products here? Find a and b

$$(x+a)(x+b) = x^2 + (a+b)x + ab \tag{12}$$

$$(x+3)(x+4) = 0 \tag{13}$$

Either x + 3 = 0 OR x + 4 = 0

$x = -3$ OR $x = -4$

Quadratic Formula

Useful if special products can't easily be found!

$$3x^2 + 8x - 13 = 0 \tag{14}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{15}$$

Quadratic Formula

$$3x^2 + 8x - 13 = 0 \tag{16}$$

$$x = \frac{-8 \pm \sqrt{8^2 - (4 * 3 * 13)}}{2 * 3} \tag{17}$$

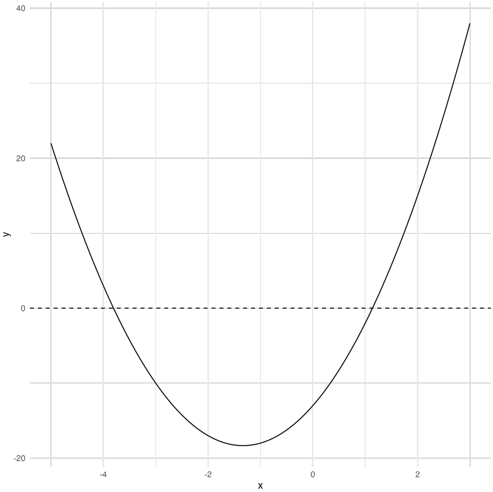
$$x = \frac{-8 + \sqrt{220}}{6} \tag{18}$$

$$x \approx 1.14 \tag{19}$$

$$x = \frac{-8 - \sqrt{220}}{6} \tag{20}$$

$$x \approx -3.81 \tag{21}$$

Checking our work graphically



Sequences and series

- A *sequence* is an ordered list of numbers
  - A sequence can be infinite, such as 1,2,3,4...
  - Or a sequence can be finite, such as 5,10,15,20,25
- A *series* is the sum of a sequence.
  - Typically noted as $\sum_{i=1}^{N} x_i$ which means add the terms in the sequence beginning at x_1 and stopping at x_n.
  - For an infinite sequence, $N = \infty$

Limits

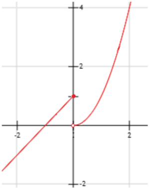
- *Limits help us describe the behavior of a sequence, series, or function as it approaches a given value.*
  - A sequence/series/function *converges* if it has a finite limit.
  - A sequence/series/function *diverges* if it has no limit or the limit is $\pm\infty$
- The limit of a sequence is the number L such that as we approach infinity, x_i gets arbitrarily close to L. Noted as: $\lim_{i\to\infty} x_i = L$
- Example of the limit of a sequence:
  - The limit of the sequence $\{i\}_{i=1}^{\infty}$ does not have an "endpoint" and approaches infinity, so it diverges.
  - The limit of the sequence $\{\frac{3}{10^i}\}_{i=1}^{\infty}$ approaches zero as $i \to \infty$, so it converges.

Limits

- The limit of a series is similar, but you are not looking for an "endpoint". In this case, you are looking for the sum of all elements in an infinite sequence. For example:
  - $\lim_{n\to\infty}\sum_{i=1}^{n} i = \infty$ So, this series is divergent.
  - $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{1}{2^i}$. This series converges. Where?
- For a function $y = f(x)$, the limit is the value of y that the function tend towards as small steps are taken towards a value $x = c$
- Piecewise function: the function (relationship between x and y) is different for different parts of the domain

Piecewise Function Example

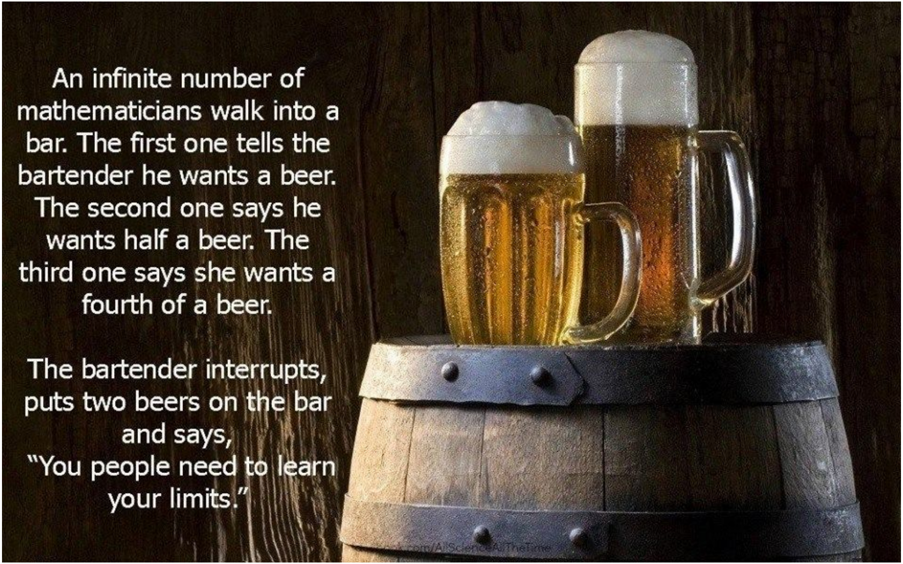
Example: Estimate the value of the following limits: $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ for the following function:



$$f(x) = \begin{cases} x+1 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x+1 = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$$


An infinite number of mathematicians walk into a bar. The first one tells the bartender he wants a beer. The second one says he wants half a beer. The third one says she wants a fourth of a beer.
The bartender interrupts, puts two beers on the bar and says, "You people need to learn your limits."

Limit Example

- The function $f(x) = \frac{1}{x-5}$ is undefined at which value of x?
- The function $f(x)$ has a limit at $x = 5$
- What is the value of $\lim_{x \to 5} f(x)$
- Does that value change whether we approach the limit from negative or positive?
- Functions are **continuous** if they do not have sudden breaks, and **discontinuous** otherwise

Math Camp Exercises - Day 1

1. M&S Chapter 1: Exercises 3, 5
2. M&S Chapter 2: Exercises 16, 17, 19, 20, 21, 22, 26, 29
3. M&S Chapter 3: Exercises 2, 3
4. M&S Chapter 4: Exercises 3, 4, 5, 6, 9