Lecture 3 — Calculus I

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8/16/23

Agenda

1. Differentiation
2. Optimization

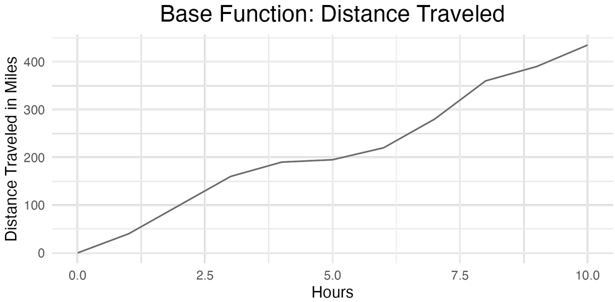
Morning challenge!

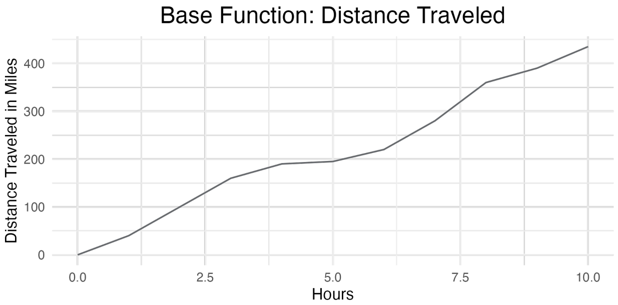
- Find $C = \begin{bmatrix} 4 & 2 \\ -\frac{2}{3} & -1 \end{bmatrix} + \begin{bmatrix} 3 & 43 \\ -4 & 3 \end{bmatrix}$
- Suppose $A = \begin{bmatrix} 3 & 6 \\ -4 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} -10 & -4 \\ 5 & 2 \end{bmatrix}$. Find AB and BA. Is $AB = BA$?
- Find x, y and z: $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & x \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 2 & 7 & 1 \\ 8 & 2 & 8 \\ 1 & 8 & y \end{bmatrix} = \begin{bmatrix} z & 55 & 19 \\ 51 & 89 & 59 \\ 57 & 66 & 60 \end{bmatrix}$
- Suppose $A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$. Find A^{-1}

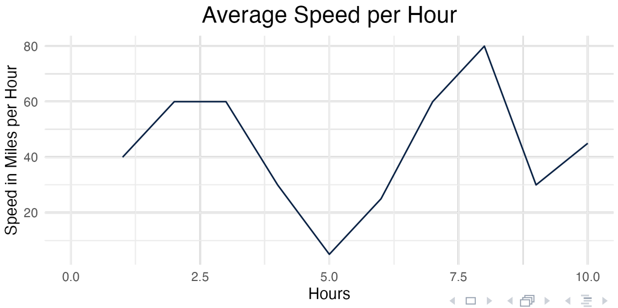
Agenda

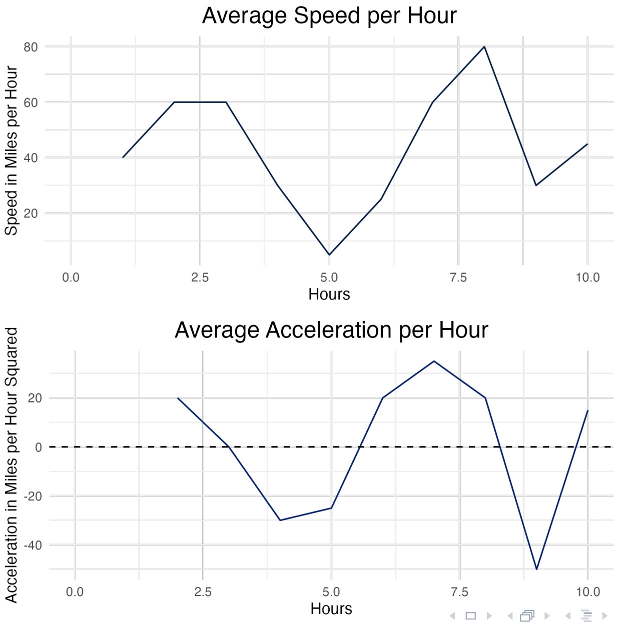
1. Differentiation

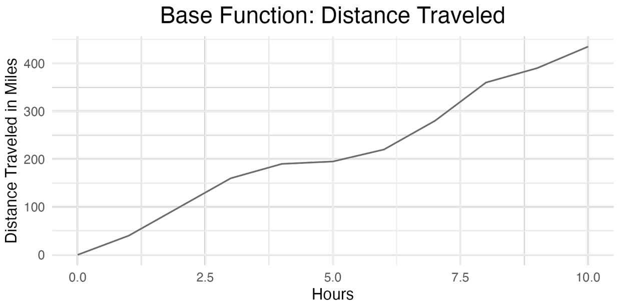
2. Optimization

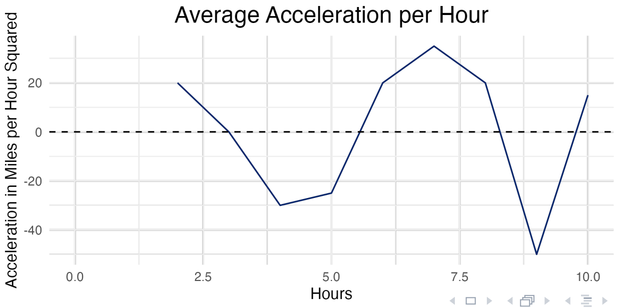

Base Function: Distance Traveled
Distance Traveled in Miles
400
300
200
100
0
0.0
2.5
5.0
7.5
10.0
Hours

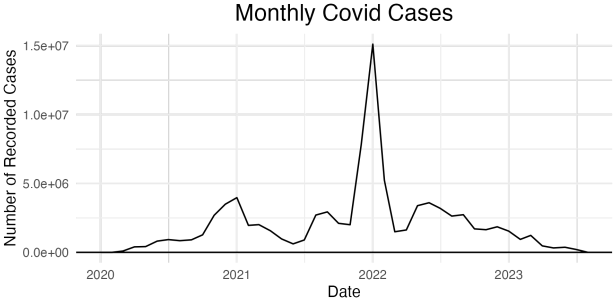

Base Function: Distance Traveled
Distance Traveled in Miles
400
300
200
100
0
0.0
2.5
5.0
7.5
10.0
Hours

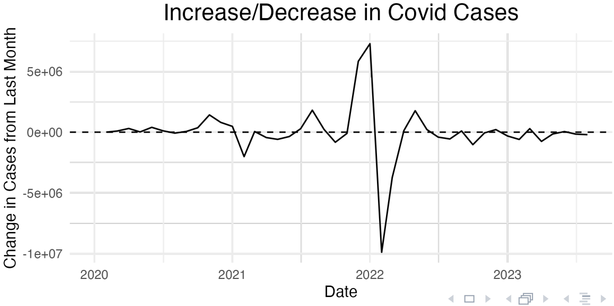

Average Speed per Hour
Speed in Miles per Hour
80
60
40
20
0.0
2.5
5.0
7.5
10.0
Hours


Average Speed per Hour
Speed in Miles per Hour
80
60
40
20
0.0
2.5
5.0
7.5
10.0
Hours
Average Acceleration per Hour
Acceleration in Miles per Hour Squared
20
0
-20
-40
0.0
2.5
5.0
7.5
10.0
Hours


Base Function: Distance Traveled
Distance Traveled in Miles
400
300
200
100
0
0.0
2.5
5.0
7.5
10.0
Hours


Average Acceleration per Hour
Acceleration in Miles per Hour Squared
20
0
-20
-40
0.0
2.5
5.0
7.5
10.0
Hours


Monthly Covid Cases
Number of Recorded Cases
1.5e+07
1.0e+07
5.0e+06
0.0e+00
2020
2021
2022
2023
Date


Increase/Decrease in Covid Cases
Change in Cases from Last Month
5e+06
0e+00
-5e+06
-1e+07
2020
2021
2022
2023
Date

Types of changes and derivatives

Types of changes:

- Discrete change: "first difference"; a measure of change in a variable across two *discrete* moments in time
- First difference $= x_{t+1} - x_t$
- Percentage change $= \frac{x_{t+1} - x_t}{x_t}$

Aggregate Heavy Weapons, China

-

Year	Total	First Difference	Percentage Change
1995	37095	NA	NA
1996	35747	-1348	-3.6%
1997	36910	1163	3.3%
1998	37032	122	0.3%
1999	36494	-538	-1.5%
2000	31435	-5059	-13.9%
2001	34281	2846	9.1%

Types of changes and derivatives

Types of changes:

- Instantaneous change: a measure of change in a variable at a single point (or moment in time)
- A *derivative* is the *instantaneous rate of change* of a function.
- Notation: the derivative of a function is represented as $\frac{dy}{dx}$ or $f'(x)$
- Calculated using *limits*

Discrete Change: Secant Lines

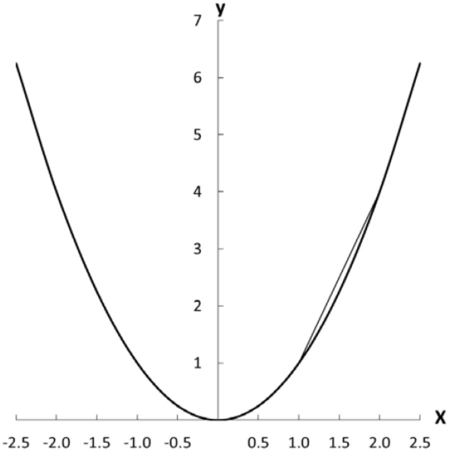


Figure 5.1: Graph of $y = x^2$ with Secant Line

- Discrete change is computed as $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$
- Secant line: line connecting two discrete points on a function
- Discrete change is the slope of the secant line

Instantaneous Change: Tangent Lines

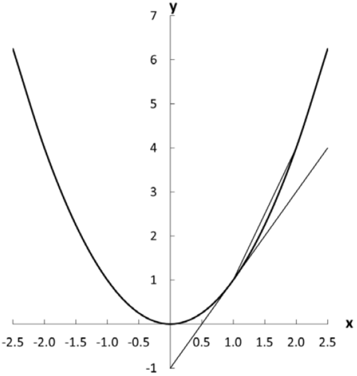


Figure 5.2: Graph of $y = x^2$ with Tangent Line

- Tangent line: a line that touches the function at one point, and has a slope equal to the instantaneous rate of change of the function at that point
- Our discrete equation is undefined for a single point: $m = \frac{f(x_1)-f(x_1)}{x_1-x_1} = \frac{y_1-y_1}{x_1-x_1}$
- Calculate the discrete change for smaller and smaller differences (h) between x_2 and x_1
- $\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Example

Calculate $f'(x)$ using the definition of the derivative:

$$f(x) = x^3 - 16x + 7$$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{((x+h)^3 - 16(x+h) + 7) - (x^3 - 16x + 7)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{((x^3 + 3x^2h + 3xh^2 + h^3) - 16x - 16h + 7) - (x^3 - 16x + 7)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 16h}{h}$$

$$f'(x) = \lim_{h\to 0} 3x^2 + 3xh + h^2 - 16$$

$$f'(x) = 3x^2 - 16$$

Five minutes practice

Calculate $f'(x)$ using the definition of the derivative:

$$f(x) = x^2 - 4x + 3$$

Five minutes practice!

Calculate $f'(x)$ using the definition of the derivative:

$$f(x) = x^2 - 4x + 3$$

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{((x+h)^2 - 4(x+h) + 3) - (x^2 - 4x + 3)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{((x^2 + 2xh + h^2) - 4x - 4h + 3) - (x^2 - 4x + 3)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2 - 4h}{h}$$

$$f'(x) = \lim_{h \to 0} 2x + h - 4$$

$$f'(x) = 2x - 4$$

Rules of differentiation

Table 6.1: List of Rules of Differentiation

Sum rule	$(f(x)+g(x))' = f'(x)+g'(x)$
Difference rule	$(f(x)-g(x))' = f'(x)-g'(x)$
Multiply by constant rule	$f'(ax) = af'(x)$
Product rule	$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$
Quotient rule	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain rule	$(g(f(x)))' = g'(f(x))f'(x)$
Inverse function rule	$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$
Constant rule	$(a)' = 0$
Power rule	$(x^n)' = nx^{n-1}$
Exponential rule 1	$(e^x)' = e^x$
Exponential rule 2	$(a^x)' = a^x(\ln(a))$
Logarithm rule 1	$(\ln(x))' = \frac{1}{x}$
Logarithm rule 2	$(\log_a(x))' = \frac{1}{x(\ln(a))}$
Trigonometric rules	$(\sin(x))' = \cos(x)$
	$(\cos(x))' = -\sin(x)$
	$(\tan(x))' = 1+\tan^2(x)$
Piecewise rules	Treat each piece separately

Rules of differentiation Cont

Table 6.1: List of Rules of Differentiation

Sum rule		$(f(x)+g(x))' = f'(x)+g'(x)$
Difference rule		$(f(x)-g(x))' = f'(x)-g'(x)$
Multiply by constant rule		$f'(ax) = af'(x)$
Product rule	Tricky Differentiation	$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$
Quotient rule		$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain rule		$(g(f(x)))' = g'(f(x))f'(x)$
Inverse function rule		$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$
Constant rule		$(a)' = 0$
Power rule		$(x^n)' = nx^{n-1}$
Exponential rule 1	Easy Differentiation	$(e^x)' = e^x$
Exponential rule 2		$(a^x)' = a^x(\ln(a))$
Logarithm rule 1		$(\ln(x))' = \frac{1}{x}$
Logarithm rule 2		$(\log_a(x))' = \frac{1}{x(\ln(a))}$
Trigonometric rules		$(\sin(x))' = \cos(x)$
		$(\cos(x))' = -\sin(x)$
		$(\tan(x))' = 1+\tan^2(x)$
Piecewise rules		Treat each piece separately

The Power Rule

- $(x^n)' = nx^{n-1}$
- $f(x) = x^3$
- $f'(x) = 3x^2$

The Product Rule

- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- Sum the derivative of the first part, multiplied by the second part, and the derivative of the second part, multiplied by the first part
- $f(x) = (x^2+1) \times (x^3+2)$
- $f'(x) = (2x)(x^3+2) + (x^2+1)(3x^2)$
- $f'(x) = 2x^4 + 4x + 3x^4 + 3x^2$
- $f'(x) = 5x^4 + 3x^2 + 4x$

The Quotient Rule

- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
- $f(x) = \frac{x^2+1}{x^3+2}$
- $f'(x) = \frac{(2x)(x^3+2) - (x^2+1)(3x^2)}{(x^3+2)(x^3+2)}$
- $f'(x) = \frac{(2x^4+4x) - (3x^4+3x^2)}{x^6+4x^3+4}$
- $f'(x) = \frac{-x^4 - 3x^2 + 4x}{x^6+4x^3+4}$

The Chain Rule

- $(g(f(x))' = g'(f(x))f'(x)$
- $f(x) = (3x+1)^2$
- $f'(x) = (2(3x+1)) \times 3$
- $f'(x) = 18x+6$

Examples

- $f(x) = 3x^2$
- $f(x) = \sqrt{x}$
- $f(x) = 2x^9 + x^2 + 8$
- $f(x) = \frac{x^2+5}{x^3+1}$
- $f(x) = (x+5)(x^3+x^2+2)$
- $f(x) = (4x^2+2x+1)^3$
- $f(x) = ln(2x^4 - x^3 + 3x^2 - 3x)$

Agenda

1. Differentiation
2. Optimization

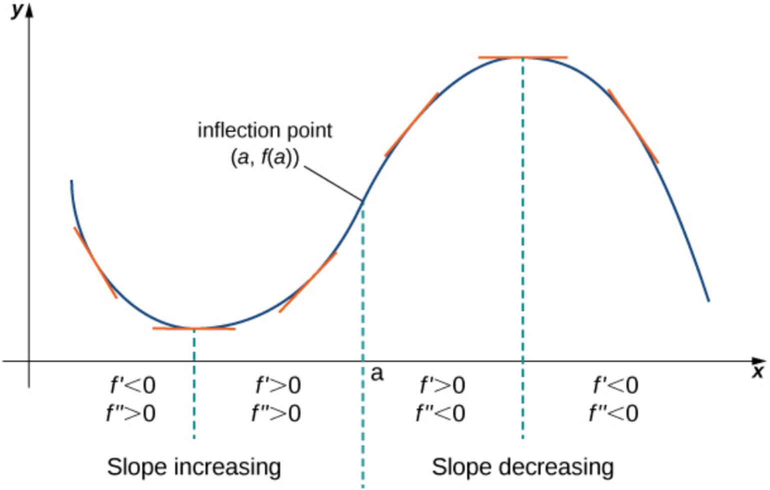
What is optimization?

- A method for finding all the extrema of a function.
- In political science and economics, several approaches assume that (political) agents want to maximize/minimize an objective function. For example:
  - Maximization of Utility
  - Minimization of Risk
  - Maximization of Welfare
  - Maximization of the survival probability
  - Minimization of errors (for calculating best fit liens)

Extreme value theorem

- A real-valued function that is continuous on a closed and bounded interval [a,b] must hit both its global maximum and minimum on that interval, at least once each
- A high point is called a *maximum*
- A low point is called a *minimum*
- An extrema is local whenever it is the largest (or smallest) value of the function over some interval of values in the domain of a function (over some interval on the x-axis). Think of them as "valleys" and "peaks".
- A global extremum is the highest (or lowest) point on the function across the full domain

Derivatives and the "shape" of a function



How to optimize!

1. Take the derivative of $f(x)$ to get $f'(x)$
2. First derivative test: Set $f'(x) = 0$ and solve for x^* (critical points).
3. Take the derivative of $f'(x)$ to get $f''(x)$
4. Second derivative test: Calculate $f''(x^*)$
   - If $f''(x^*) > 0$, x^* is a local minimum.
   - If $f''(x^*) < 0$, x^* is a local maximum.
   - If $f''(x^*) = 0$, x^* may be an inflection point.
5. Substitute each x^* into $f(x)$ to get (x, y) for each point.
6. If the function is bounded, check the value of $f(x)$ at each bound
7. Compare the values of $f(x)$ and $f(bounds)$ to find global min/max.

Inflection point or extrema?

1. If $f'(x^*) = 0$ and $f''(x^*) = 0$, you should continue taking the derivative until $f^n(x^*) =$ a nonzero number.
    - If $n =$odd number then x^* is an inflection point, not and extremum.
    - If $n =$even number continue to step 2.
2. Calculate $f^n(x^*)$:
    - If $f^n(x^*) > 0$, the point is a local minimum.
    - If $f^n(x^*) < 0$, the point is a local maximum.

Example optimization!

Find the extrema of the following equation:

$$f(x) = x^3 - 3x^2 + 7, x \in [-4, 4]$$

Also, graph the function.

Example optimization!

1. Take the derivative of $f(x)$ to get $f'(x)$.

$$f(x) = x^3 - 3x^2 + 7$$
$$f'(x) = 3x^2 - 6x$$

2. First derivative test: Set $f'(x) = 0$ and solve for x^*:

$$f'(x) = 3x^2 - 6x = 0$$
$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$

Therefore, we will have $x_1^* = 0$ and $x_2^* = 2$

Example optimization!

3. Take the derivative of $f'(x)$ to get $f''(x)$.

$$f'(x) = 3x^2 - 6x$$
$$f''(x) = 6x - 6$$

4. Second derivative test: calculate $f''(x^*)$
   - For $x_1^* = 0$: $f''(0) = 6(0) - 6 = -6$.
     If $f''(x^*) < 0$, x^* is a local maximum.
   - For $x_2^* = 2$: $f''(2) = 6(2) - 6 = 6$.
     If $f''(x^*) > 0$, x^* is a local minimum.

Example optimization!

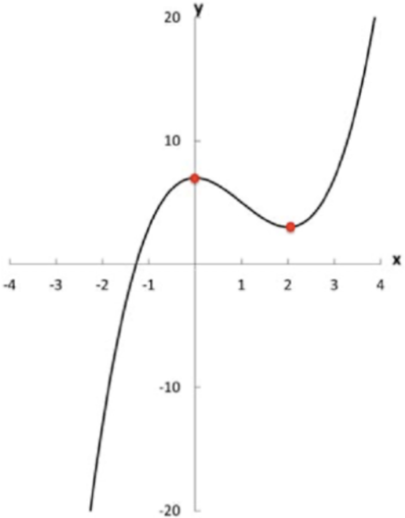
5. Substitute each x^* into $f(x)$ to get (x,y) for each point:
Remember that $f(x) = x^3 - 3x^2 + 7$

- For $x_1^* = 0$: $f(0) = 0^3 - 3(0)^2 + 7 = 7.$
  Local maximum at (0,7).
- For $x_2^* = 2$: $f(2) = 2^3 - 3(2)^2 + 7 = 3.$
  Local minimum at (2,3).

6. If the function is bounded, check the value of $f(x)$ at each bound:
Remember that $f(x) = x^3 - 3x^2 + 7, , x \in [-4, 4]$

- Lower bound:$f(-4) = (-4)^3 - 3(-4)^2 + 7 = -105$
  Global minimum at (-4, -105)
- Upper bound:$f(4) = (4)^3 - 3(4)^2 + 7 = 23$
  Global maximum at (4,23).

Example optimization!



7. Compare the values of $f(x^*)$ and $f(bounds)$ to find global min/max:
    - Global minimum at (-4,-105)
    - Local maximum at (0,7)
    - Local minimum at (2,3)
    - Global maximum at (4,23)