

Lecture 1 — Notation, Functions, and Limits

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Today's Agenda

- 1 Introductions
- 2 Math Camp Logistics and schedule
- 3 Variables and measurement
- 4 Algebra
- 5 Functions
- 6 Series, sequences, and limits

Week's Agenda

- Lecture 1: Notation, functions and limits
- Lecture 2: Linear Algebra
- Lecture 3: Calculus 1 - Derivatives
- Lecture 4: Calculus 2 - Integrals and multivariate calculus
- Lecture 5: Probability

Schedule

	Monday 14th	Tuesday 15th	Wednesday 16th	Thursday 17th	Friday 18th
9:30 - 12:00	Notation, functions, and limits	Linear algebra	Calculus 1	Calculus 2	Probability
12:00 - 12:10	Break	Break	Break	Break	Break
12:10 - 1:00	Problem Set	Problem Set	Problem Set	Lunch Break	LaTeX
1:00 - 2:00	Lunch Break	Lunch Break	Lunch Break	Problem Set	Lunch Break
2:00 - 2:30	Review PS	Review PS	Review PS	Review PS	Q&A with Prof Klasnja
2:30 - 3:30	Software Installation	R - Day 1	R - Day 2	R - Day 3	Q&A Continued

Goals of Math Camp

- Familiarity
- Recognition
- Confidence

What is quantitative methods?

- Ultimate goal is to improve our understanding of the world
- We use quantitative methods to test theories regarding the relationships between concepts
- The goal of a “causal claim” is rarely achieved in full, but we can make compelling arguments
- Qualitative methods can answer “how” and “why” questions

Variables and Constants

- **Theory:** A set of statements that involve concepts.
- **Concept:** abstract ideas used to understand the world
 - Participation, voting, democracy, war
- **Measure:** an operational indicator of a concept
- **Variable:** A concept or measure that takes different values in a given set.
- **Constant:** A concept or measure that has a single value for a given set.

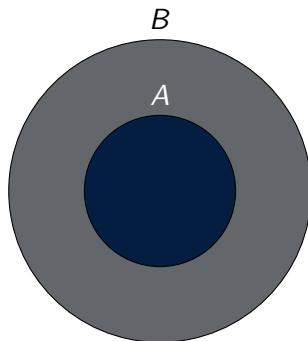
Sets and Sample Spaces

- A *set* is a collection of elements.
- **Common sets:** Natural numbers (\mathbb{N}), Integers (\mathbb{Z}), Rational numbers (\mathbb{Q}), Real Numbers (\mathbb{R}), etc.
- A set can be:
 - **Finite or infinite:** \mathbb{Z} is infinite, but all the integers from 1 to 10 is finite.
 - **Countable or uncountable:** a countable set is one whose each of its element can be associated with a natural number (or an integer).
 - **Bounded or unbounded:** A bounded set has finite size (but may have infinite elements).
- Some important sets that we are going to use as political scientists:
 - **Solution set:** a set that contains all solutions for an equation
 - **Sample space:** a set that contains all values that a variable can take.

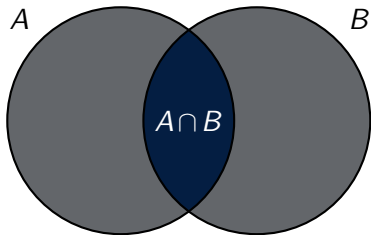
Unions and Intersections

- Much as sets contain elements, they also can contain, and be contained by, other sets.
- Notation:
 - $A \subset B$: “A is a **proper subset** of B” implies that set B contains all the elements in A, plus at least one more
 - $A \subseteq B$: “A is a **subset** of B”. In this case, it allows A and B to be the same.
- **Intersection:** $A \cap B$. The set of elements common to two sets.
- **Union:** $A \cup B$. The set that contains all elements in both sets.
- **Mutually exclusive sets:** the intersection is the empty set.

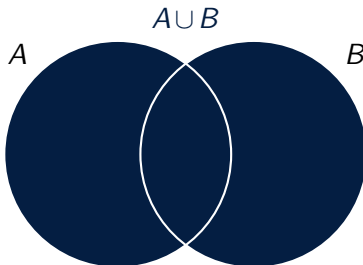
$A \subset B$ (Proper Subset)



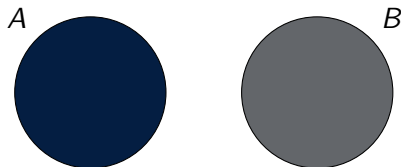
$A \cap B$ (Intersection)



$A \cup B$ (Union)



$A \cap B = \emptyset$ (Mutually Exclusive)



Levels of Measurement

- Categorical
 - **Nominal:** No mathematical relationship; difference of kind.
 - Occupation, colors, political party, gender
 - **Ordinal:** Meaningful order, no meaningful “distance” between categories.
 - Ideology, agree/disagree, education level
- Numerical
 - **Interval:** Meaningful order and distance between values.
 - Age, budget, polity scores, vote share, feeling thermometer
 - Discrete: countable
 - Continuous: non-countable
 - Ratio: meaningful zero (re-scaled thermometer)

Why it Matters

- Visualization
 - Categorical: pie charts, bar charts
 - Numerical (especially Continuous): histograms, box plots
- Analysis
 - Numerical data can use summary statistics (mean, median, standard deviation)
 - Categorical - Categorical : cross-tabulation
 - Categorical - Continuous : difference in means
 - Continuous - Continuous : scatter plots, regression

Basic properties of arithmetic

- For variables that stand for real numbers or integers, these properties will always hold:
- **Associative properties:**
 - $(a + b) + c = a + (b + c)$
 - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- **Commutative properties:**
 - $a + b = b + a$
 - $a \cdot b = b \cdot a$
- **Distributive properties:**
 - $a \cdot (b + c) = ab + ac$
- **Identity properties:**
 - $a + 0 = a$
 - $a \cdot 1 = a$
- **Inverse properties (for real numbers not integers):**
 - $(-a) + a = 0$
 - $a^{-1} \cdot a = 1$

Order of operations

- **Order of operations - PEMDAS:**
 - Parentheses ($()$)
 - Exponents (x)
 - Multiplication (\cdot)
 - Division (\div)
 - Addition ($+$)
 - Subtraction ($-$)

Inequalities

- All pairs of real numbers have exactly one of the following relations:
 $x = y$, $x > y$, or $x < y$.
- Solving inequalities is similar to solving equations but there are a few extra properties
- Adding any number to each side of these relations will not change them; this includes inequalities.
- Multiplication:
 - If a is positive and $x > y$, then $ax > ay$.
 - If a is negative and $x > y$, then $ax < ay$.
- Division:
 - If a is positive and $x > y$, then $\frac{x}{a} > \frac{y}{a}$.
 - If a is negative and $x > y$, then $\frac{x}{a} < \frac{y}{a}$.

Exponents, logarithms, and roots

- Exponential: x^a
- Roots: $\sqrt[a]{x}$
 - All roots can be expressed as fractional exponents
 - $\sqrt[a]{x} = x^{1/a}$
- Logarithm: inverse of exponents $\log_a x$
 - To what power would you need to raise a to to get x ?
- Natural log: log base e
 - $e \approx 2.7183$
 - Useful properties in calculus
 - Taking the natural log of a variable allows us to interpret relationships in terms of percentage changes

Exponent, logarithms and root rules

Exponent rules

- $x^a \cdot x^b = x^{a+b}$
- $x^a \cdot z^a = (xz)^a$
- $(x^a)^b = x^{ab}$
- $\frac{x^a}{x^b} = x^{a-b}$
- $\frac{x^a}{z^a} = \left(\frac{x}{z}\right)^a$

Logarithm rules

- $\log(x_1 \cdot x_2) = \log(x_1) + \log(x_2)$
for $x_1, x_2 > 0$
- $\log\left(\frac{x_1}{x_2}\right) = \log(x_1) - \log(x_2)$
for $x_1, x_2 > 0$
- $\log(x^b) = b \cdot \ln(x)$
for $x > 0$

Exponent, logarithms and root rules

Root rules

- $\sqrt{x} = x^{\frac{1}{2}}$
- $\sqrt[n]{x} \cdot \sqrt[n]{z} = \sqrt[n]{xz}$
for $n > 1$
- $\frac{\sqrt[n]{x}}{\sqrt[n]{z}} = \sqrt[n]{\frac{x}{z}}$
for $n > 1$

Functions and its characteristics

- Functions provide a **specific** description of the association or relationship between two (or among several) concepts (in theoretical work) or variables (in empirical work)
- Functions assign one element of the range to an element of the domain (one x is assigned to one y)
- Noted as $f(x) : A \rightarrow B$ or “ f maps A into B ”
- A is the *domain*, or set of possible x values.
- B is the *codomain*, or set of possible y values.
- To be a function, there must only be one unique value in its range (y) for each value in its domain (x)

Graph examples

In which of these graphs can we observe a function?

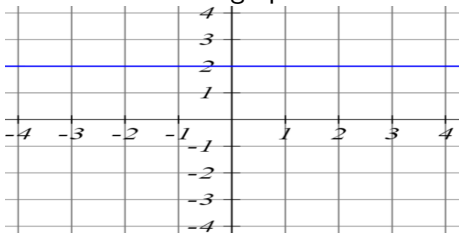


Figure: a)

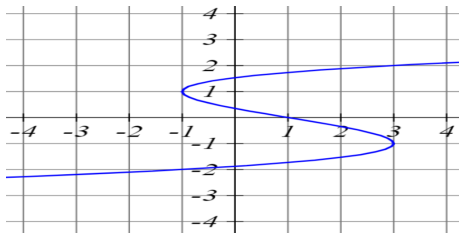
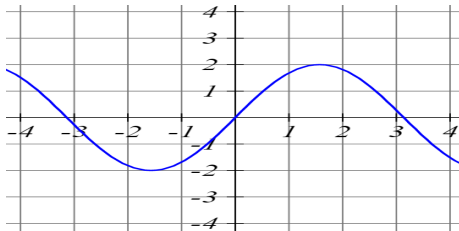
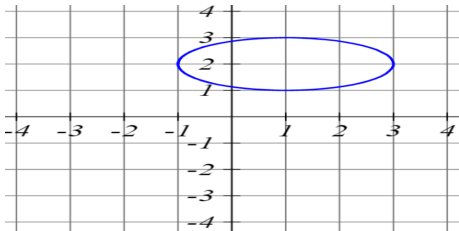


Figure: b)



Graph examples

In which of these graphs can we observe a function?

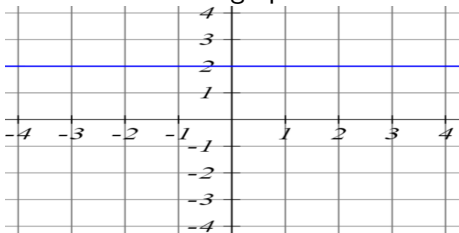


Figure: a)

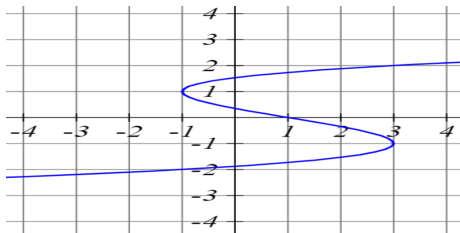
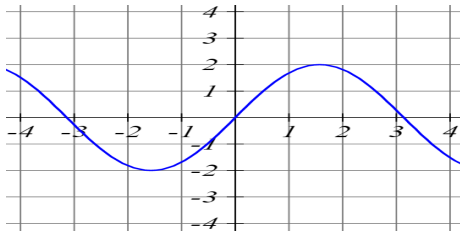
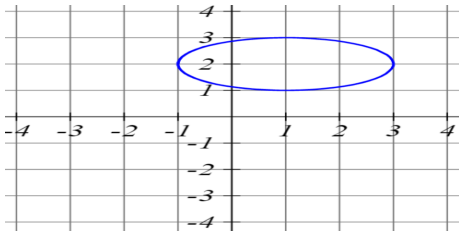


Figure: b)



Function composition

- We can chain multiple functions using function composition.
 - This is written either as $g \circ f(x)$ or $g(f(x))$.
 - It is read as “ g composed with f ” or “ g of f of x ”
 - Generally, $g \circ f(x) \neq f \circ g(x)$
- Example: Suppose $f(x) = 2x$ and $g(x) = x^3$
 - $g \circ f(x) = (2x)^3 = 8x^3$
 - $f \circ g(x) = 2(x^3) = 2x^3$

Examples of functions of one variable - linear equation

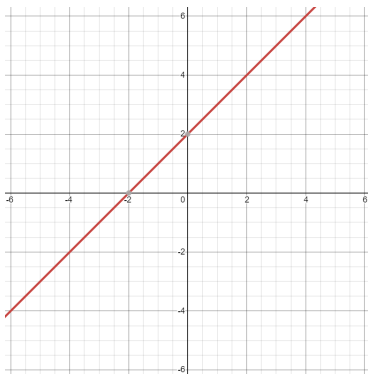


Figure: $y=2+x$

- This is the classic linear equation $y = a + bx$ or $y = mx + n$
- a and b are constants and x is the variable.
- a is the intercept and b is the slope of the line, or the amount that y changes given a one-unit increase in x .

Examples of functions of one variable - Quadratic function

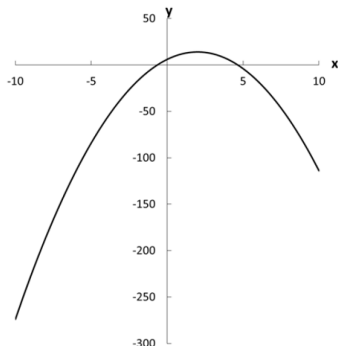
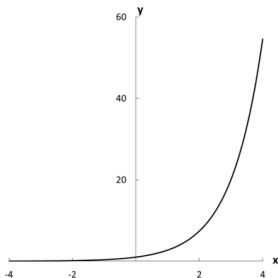


Figure: $y = -2x^2 + 8x + 6$

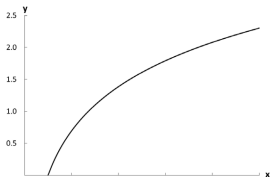
- This is the classical quadratic function:
 $f(x) = ax^2 + bx + c$ or
 $f(x) = \alpha + \beta_1x + \beta_2x^2$
- If we set $a > 0$ ($\beta_2 > 0$) we get a curve shaped like an U (a convex parabola).
- If we set $a < 0$ ($\beta_2 < 0$) we get a curve shaped like an inverse U (a concave parabola).

Exponent, logarithms and roots - Graphs

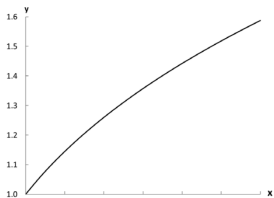
Exponential function



Logarithmic function



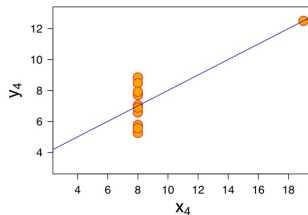
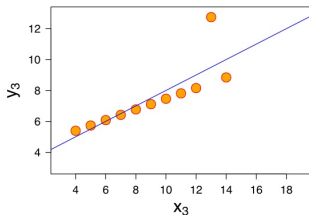
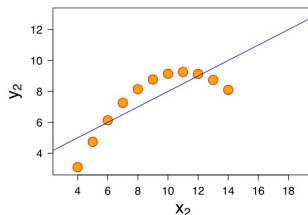
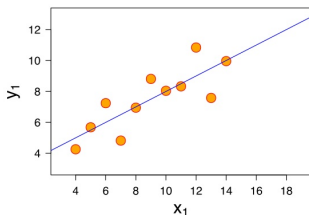
Root function



Functions and Theorizing

- Functions are important for theorizing about politics
- Most quantitative social science boils down to how two variables relate!
- Our examples suggest a precise function between x and y
- In statistics, you will *theorize* the functional form of the relationship
- Statistical methods will *estimate* the most appropriate parameters; “a line of best fit”

Anscombe's Quartet



Solving Equations

- 1 Isolate the variable you are looking for
- 2 Combine like terms
- 3 Factor and cancel
- 4 Operate on both sides of the equation
- 5 Check your answer

Special products

$$\textcircled{1} (a+b)^2 = a^2 + 2ab + b^2$$

$$\textcircled{2} (a-b)^2 = a^2 - 2ab + b^2$$

$$\textcircled{3} (x+a)(x+b) = x^2 + (a+b)x + ab$$

$$\textcircled{4} (a+b)(a-b) = a^2 - b^2$$

$$\textcircled{5} (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+ac+bc)$$

$$\textcircled{6} (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Factoring

Factoring: rearranging terms in an equation to make it simpler

$$\frac{3x^4 + 3x^3 - 6x^2}{6x^2 + 12x} \quad (1)$$

$$\frac{3x^2(x^2 + x - 2)}{6x(x + 2)} \quad (2)$$

$$\frac{3x^2(x + 2)(x - 1)}{6x(x + 2)} \quad (3)$$

$$\frac{3x^2(x - 1)}{6x} \quad (4)$$

$$\frac{x(x - 1)}{2} \quad (5)$$

Quadratics and Special Products

Quadratic polynomials can be factored into the product of two terms:

$$(x \pm ?) \times (x \pm ?)$$

$$x^2 - 8x + 16 = 9 \quad (6)$$

$$(x \pm ?) \times (x \pm ?) = x^2 - 8x + 16$$

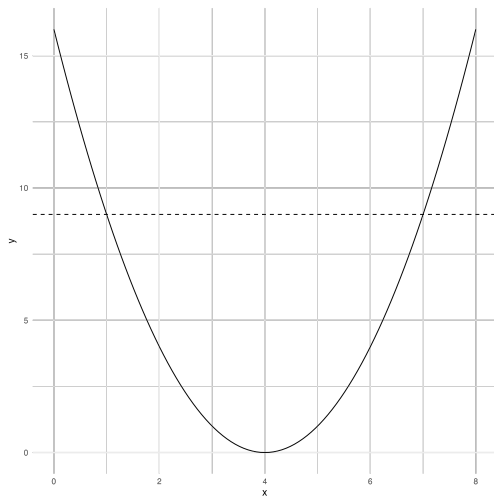
$$(x - 4)^2 = 9 \quad (7)$$

$$(x - 4) = \pm 3 \quad (8)$$

$$x = 4 \pm 3 \quad (9)$$

$$x = 1 \text{ OR } x = 7$$

Why two solutions?



Another example

$$x^2 + 7x = -12 \quad (10)$$

$$x^2 + 7x + 12 = 0 \quad (11)$$

How can we use special products here? Find a and b

$$(x + a)(x + b) = x^2 + (a + b)x + ab \quad (12)$$

$$(x + 3)(x + 4) = 0 \quad (13)$$

$$\begin{aligned} \text{Either } x + 3 &= 0 \text{ OR } x + 4 = 0 \\ x &= -3 \text{ OR } x = -4 \end{aligned}$$

Quadratic Formula

Useful if special products can't easily be found!

$$3x^2 + 8x - 13 = 0 \quad (14)$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (15)$$

Quadratic Formula

$$3x^2 + 8x - 13 = 0 \quad (16)$$

$$x = \frac{-8 \pm \sqrt{8^2 - (4 * 3 * -13)}}{2 * 3} \quad (17)$$

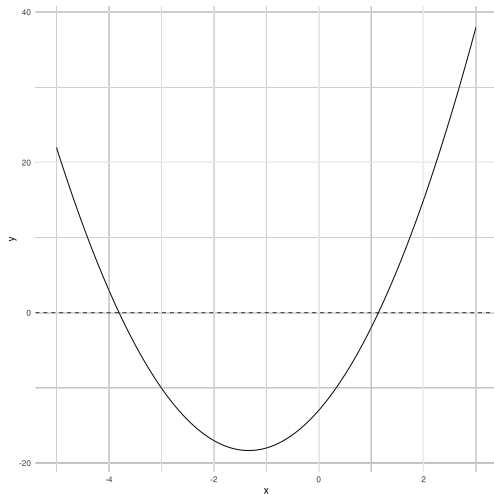
$$x = \frac{-8 + \sqrt{220}}{6} \quad (18)$$

$$x \approx 1.14 \quad (19)$$

$$x = \frac{-8 - \sqrt{220}}{6} \quad (20)$$

$$x \approx -3.81 \quad (21)$$

Checking our work graphically



Sequences and series

- A *sequence* is an ordered list of numbers
 - A sequence can be infinite, such as 1,2,3,4...
 - Or a sequence can be finite, such as 5,10,15,20,25
- A *series* is the sum of a sequence.
 - Typically noted as $\sum_{i=1}^N x_i$ which means add the terms in the sequence beginning at x_1 and stopping at x_n .
 - For an infinite sequence, $N = \infty$

Limits

- *Limits help us describe the behavior of a sequence, series, or function as it approaches a given value.*
 - A sequence/series/function *converges* if it has a finite limit.
 - A sequence/series/function *diverges* if it has no limit or the limit is $\pm\infty$
- The limit of a sequence is the number L such that as we approach infinity, x_i gets arbitrarily close to L . Noted as: $\lim_{i \rightarrow \infty} x_i = L$
- Example of the limit of a sequence:
 - The limit of the sequence $\{i\}_{i=1}^{\infty}$ does not have an “endpoint” and approaches infinity, so it diverges.
 - The limit of the sequence $\{\frac{3}{10^i}\}_{i=1}^{\infty}$ approaches zero as $i \rightarrow \infty$, so it converges.

Limits

- The limit of a series is similar, but you are not looking for an “endpoint”. In this case, you are looking for the sum of all elements in an infinite sequence. For example:
 - $\lim_{n \rightarrow \infty} \sum_{i=1}^n i = \infty$ So, this series is divergent.
 - $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2^i}$. This series converges. Where?
- For a function $y = f(x)$, the limit is the value of y that the function tend towards as small steps are taken towards a value $x = c$
- Piecewise function: the function (relationship between x and y) is different for different parts of the domain

Piecewise Function Example

Example: Estimate the value of the following limits: $\lim_{x \rightarrow 0^-} f(x)$ and $\lim_{x \rightarrow 0^+} f(x)$ for the following function:



$$f(x) = \begin{cases} x+1 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x + 1 = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 = 0$$

An infinite number of mathematicians walk into a bar. The first one tells the bartender he wants a beer. The second one says he wants half a beer. The third one says she wants a fourth of a beer.

The bartender interrupts, puts two beers on the bar and says, "You people need to learn your limits."



Limit Example

- The function $f(x) = \frac{1}{x-5}$ is undefined at which value of x ?
- The function $f(x)$ has a limit at $x = 5$
- What is the value of $\lim_{x \rightarrow 5} f(x)$
- Does that value change whether we approach the limit from negative or positive?
- Functions are **continuous** if they do not have sudden breaks, and **discontinuous** otherwise

Math Camp Exercises - Day 1

- ① M&S Chapter 1: Exercises 3, 5
- ② M&S Chapter 2: Exercises 16, 17, 19, 20, 21, 22, 26, 29
- ③ M&S Chapter 3: Exercises 2, 3
- ④ M&S Chapter 4: Exercises 3, 4, 5, 6, 9