Lecture 4 — Calculus II

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Morning challenge!

Take the derivative of the following functions (it is not necessary to simplify):

- $f(x) = x^3 + 6x^2 + 3$
- $f(x) = (x^3 + 5x) \times (x^2 - 2)$
- $f(x) = \frac{x^3+5x}{x^2-2}$
- $f(x) = \sqrt[5]{x^4 - 3x^2}$
- $f(x) = e^{5x^3+4x}$
- $f(x) = ax^2 + bx + c$, with a, b and c constants.

Agenda

1. Integration

2. Multivariate calculus

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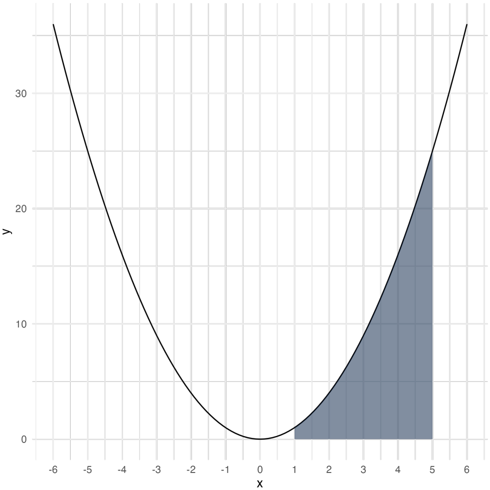
2. Multivariate calculus

What is the area under the curve?



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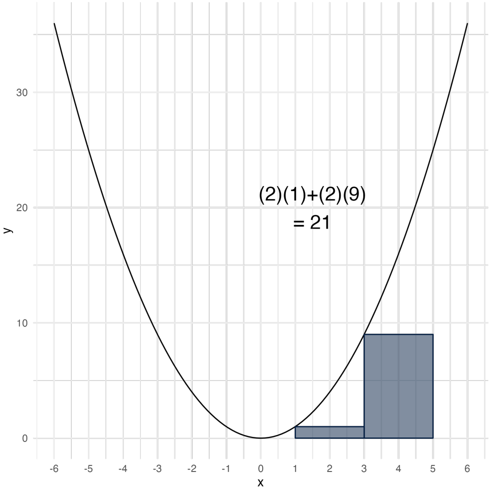

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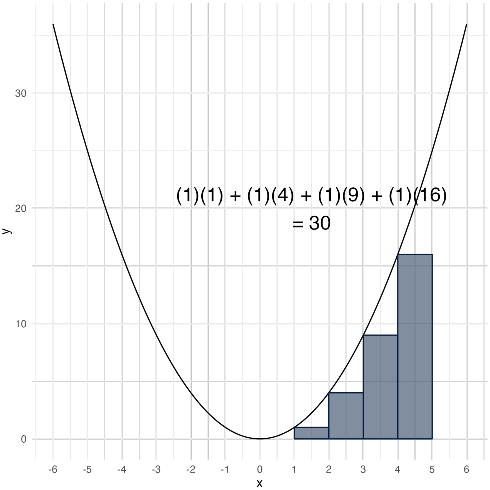

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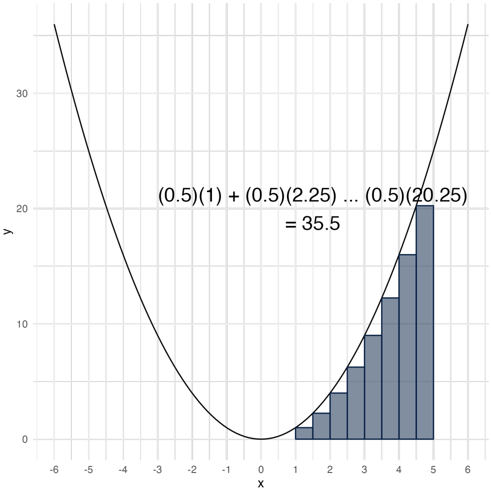

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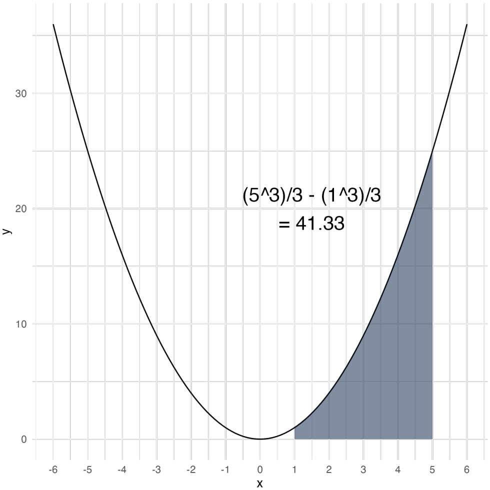
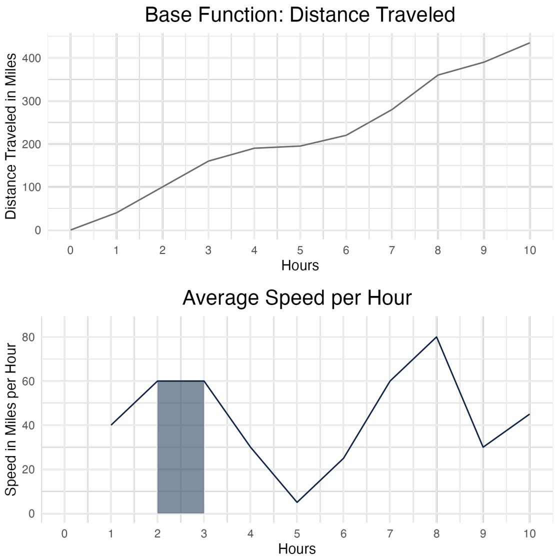



Base Function: Distance Traveled
Distance Traveled in Miles
0
100
200
300
400
0 1 2 3 4 5 6 7 8 9 10
Hours
Average Speed per Hour
Speed in Miles per Hour
0
20
40
60
80
0 1 2 3 4 5 6 7 8 9 10
Hours

Integrals

- Integrals tell us about the net effect of change, such as the net effect of a certain (changing) velocity over a certain period of time
- If the width of our rectangles is Δx and the height (value of the function) is $f(x)$, then we want to know $\lim_{\Delta \to 0} \sum_i f(x_i) \Delta x$
- This is more commonly expressed as $\int_a^b f(x) dx$
  - a and b bound the area under the curve on the left and right
  - A **Definite Integral** returns a value of the bounded area under the curve
  - $f(x)$ is the **integrand**
  - dx is the **variable of integration**
- An **Indefinite Integral** is an unbounded integral which returns the **antiderivative** function
  - When you take the derivative of the indefinite integral function, you reproduce the integrand $f(x)$

Fundamental theorem of calculus

- $\int_a^b f(x)dx = F(b) - F(a)$
- Area under the curve = Antiderivative (Upper bound) - Antiderivative (Lower bound)
- Differentiation and integration are inverse operations of one another

Reverse the rules of differentiation

- Suppose $f(x) = x^3$. What is the antiderivative $F(x)$?
- Reverse the power rule of differentiation: $(x^n)' = nx^{n-1}$
- For integration: $\int x^n dx = \frac{x^{n+1}}{n+1}$
- $F(x) = \frac{x^4}{4} + C$
- Check your answer: $\frac{dF(x)}{dx} = \frac{\cancel{4}x^3}{\cancel{4}} = x^3$ ✓✓

Do not forget the C!

- When you take the derivative of a constant, it is zero
- There is not enough information in a derivative for us to reverse engineer what that constant was in the original function
- So whenever you take an antiderivative, you must include $+C$ at the end to note that there may be a constant.
- When we took the antiderivative of x^3, we don't know if the constant should be $\frac{x^4}{4} + 1$ or $\frac{x^4}{4} + 100$
- Taking the derivative of either of those would return x^3 because the derivative of constants is 0
- Therefore, we say $F(x) = \frac{x^4}{4} + C$

Rules of integration

Table 7.1: List of Rules of Integration

Fundamental theorem of calculus	$\int_a^b f(x)dx = F(b) - F(a)$
Rules for bounds	$\int_a^b f(x)dx = -\int_b^a f(x)dx$ $\int_a^a f(x)dx = 0$ $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $c \in [a, b]$
Linear rule	$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$
Integration by substitution	$\int_a^b f(g(u))g'(u)du = \int_{g(a)}^{g(b)} f(x)dx$
Integration by parts	$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$
Power rule 1	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$
Power rule 2	$\int x^{-1}dx = \ln\|x\| + C$
Exponential rule 1	$\int e^x dx = e^x + C$
Exponential rule 2	$\int a^x dx = \frac{a^x}{\ln(a)} + C$
Logarithm rule 1	$\int \ln(x)dx = x\ln(x) - x + C$
Logarithm rule 2	$\int \log_a(x)dx = \frac{x\ln(x) - x}{\ln(a)} + C$
Trigonometric rules	$\int \sin(x)dx = -\cos(x) + C$ $\int \cos(x)dx = \sin(x) + C$ $\int \tan(x)dx = -\ln(\|\cos(x)\|) + C$
Piecewise rules	Split definite integral into corresponding pieces

More simple examples

Example 2: Calculate the area under the curve x^2 from 3 to 9
From the fundamental theorem of calculus we know that the answer is given by:

$$\int_3^9 f(x)dx = F(9) - F(3)$$

The next step is to determinate $F(x)$. Using the power rule, we know that $\int x^n dx = \frac{x^{n+1}}{n+1}$ so the antiderivative is $\frac{x^3}{3}$.
Replacing that in the fundamental theorem:

$$\int_3^9 f(x)dx = \frac{9^3}{3} - \frac{3^3}{3}$$
$$\int_3^9 f(x)dx = 243 - 9$$
$$\int_3^9 f(x)dx = 234 \checkmark\checkmark$$

Integration by substitution

- When the integral contains a complicated composite function, integration by substitution can help!
- $\int_a^b = f(g(u))g'(u)du = \int_{g(a)}^{g(b)} f(x)d(x)$
- Look in the composite function for a piece which would be the derivative of another piece
    - $g(u)$ and $g'(u)$ where u is some piece of the composite function

Example

- Calculate $\int_1^2 2x(x^2+1)^3 dx$
- Integration by Substitution: $\int_a^b = f(g(u))g'(u)du = \int_{g(a)}^{g(b)} f(x)d(x)$
- What could we use for u here? Looking for a piece of the function that also has its derivative in the function
  - $u = x^2 + 1$
  - $\frac{du}{dx} = 2x$
  - $u = x^2 + 1$
- Substituting in our original expression: $\int_1^2 u^3 du$
- However, the bounds were defined for x, not for u. So we have to adapt them to u.
  - Lower bound: $(1)^2 + 1 = 2$
  - Upper bound: $(2)^2 + 1 = 5$
- Finally, the right expression to calculate is: $\int_2^5 u^3 du$
- Final answer: $\int_2^5 u^3 du = \frac{5^4}{4} - \frac{2^4}{4} = 152.25$

Agenda

1. Integration
2. Multivariate calculus

Multivariate calculus

- Calculus with more than one variable in the function
- $f(x,y,z) = \frac{xy^2}{zy+x} - \frac{y}{x+4} + 1$
- $f(x_1,x_2,x_3) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$

Partial derivatives

Partial derivative: the instantaneous rate of change in y (marginal effect) due to one variable while holding the others constant

Treat every variable other than x as a constant, then take the derivative with respect to x.

$$f(x,y,z) = 3x^2y + zy^3 + ln(y) - x$$
$$\frac{df(x,y,z)}{dx} = 6xy - 1$$

You can take a partial derivative for any variable in the function:

$$\frac{df(x,y,z)}{dy} = 3x^2 + 3zy^2 + \frac{1}{y}$$

Key: The bottom of the derivative notation tells you which variable you will be taking the derivative with respect to!

Gradients

A vector of all possible first order derivatives.

For $f(x,y,z)$ the gradient $\nabla = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$

Simply take the first order partial derivative with respect to each variable and arrange as a vector!

$$f(x,y,z) = x^2 + 5xy + z^3$$
$$\frac{df(x,y,z)}{dx} = 2x + 5y$$
$$\frac{df(x,y,z)}{dy} = 5x$$
$$\frac{df(x,y,z)}{dz} = 3z^2$$
$$\nabla = \begin{bmatrix} 2x + 5y \\ 5x \\ 3z^2 \end{bmatrix}$$

Mixed partial derivatives

- Take the derivative first with respect to one variable, then take the second derivative with respect to another variable.
- Indicated by $\frac{d^2 f}{dxdy}$ or $\frac{\partial^2 f}{\partial x \partial y}$ or the notation f_{xy}

$$f(x,y,z) = 3x^2 y + zy^3 + ln(y) - x$$

First take the derivative with respect to x

$$\frac{df(x,y,z)}{dx} = 6xy - 1$$

Then take the derivative with respect to y

$$\frac{d^2 f(x,y,z)}{dxdy} = 6x$$

The Hessian matrix

Hessians are used in optimization of multivariate functions. They tell us how a function behaves in multiple dimensions.

$$\begin{bmatrix} \frac{d^2 f(x,y,z)}{dx^2} & \frac{d^2 f(x,y,z)}{dxdy} & \frac{d^2 f(x,y,z)}{dxdz} \\ \frac{d^2 f(x,y,z)}{dydx} & \frac{d^2 f(x,y,z)}{dy^2} & \frac{d^2 f(x,y,z)}{dydz} \\ \frac{d^2 f(x,y,z)}{dzdx} & \frac{d^2 f(x,y,z)}{dzdy} & \frac{d^2 f(x,y,z)}{dz^2} \end{bmatrix}$$

The Hessian matrix

Hessians are symmetric matrices

$$\begin{bmatrix} \frac{d^2 f(x,y,z)}{dx^2} & \frac{d^2 f(x,y,z)}{dxdy} & \frac{d^2 f(x,y,z)}{dxdz} \\ \frac{d^2 f(x,y,z)}{dydx} & \frac{d^2 f(x,y,z)}{dy^2} & \frac{d^2 f(x,y,z)}{dydz} \\ \frac{d^2 f(x,y,z)}{dzdx} & \frac{d^2 f(x,y,z)}{dzdy} & \frac{d^2 f(x,y,z)}{dz^2} \end{bmatrix}$$

And this is because $\frac{d^2 f(x,y,z)}{dxdy} = \frac{d^2 f(x,y,z)}{dydx}$

Group exercise

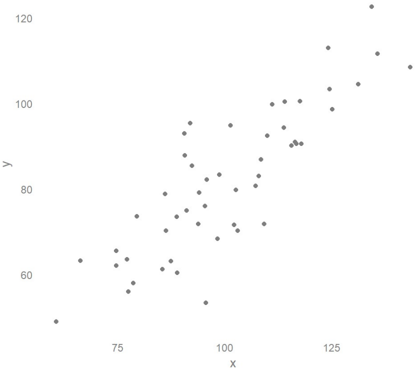
Using this function:

$$f(x,y,z) = x + y + z + x^2y^2z^2$$

Find the gradient (∇) and the Hessian.

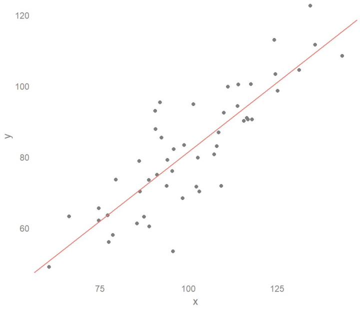
Application - Ordinary Least Squares (OLS)

- Here we have a relationship between X and Y
- The relationship isn't perfect; there is an error term ε_i
- $Y_i = \alpha + \beta X_i + \varepsilon_i$



Application - Ordinary Least Squares (OLS)

- Our goal is to identify a linear relationship between X and Y that minimizes the squared "residuals"
- Residual: the difference between the actual point Y_i and our estimation of that point $\hat{Y}_i$
- Why squared residuals? So that positive and negative residuals don't cancel out!



Deriving OLS

We want to minimize the difference between the actual point and our estimation of that point

$$min \sum (Y_i - \hat{Y}_i) \tag{1}$$

We can define our estimated point in terms of the intercept α and the slope β

$$min \sum (Y_i - (\alpha + X_i \beta))^2 \tag{2}$$

Start by taking the partial derivative with respect to α
Chain Rule: $(g(f(x))' = g'(f(x))f'(x)$

$$\frac{\partial}{\partial \alpha} = \sum (2)(Y_i - \alpha - X_i \beta)(-1) \tag{3}$$

Deriving OLS

To minimize the equation, we set the derivative equal to 0 (this will return a global minimum due to the known shape of the function)

Chain Rule: $(g(f(x))' = g'(f(x))f'(x)$

$$0 = \sum(2)(Y_i - \alpha - X_i\beta)(-1) \tag{4}$$

Essential rules of summations:

$\sum X_i = N\bar{X}$

$\sum X = NX$

$$0 = N\bar{Y} - N\alpha - \beta N\bar{X} \tag{5}$$

$$\alpha = \bar{Y} - \beta\bar{X} \tag{6}$$

Deriving OLS

Take the partial derivative of the original equation with respect to β

$$min \sum (Y_i - (\alpha + X_i \beta))^2 \tag{7}$$

$$\frac{\partial}{\partial \beta} = \sum (2)(Y_i - \alpha - X_i \beta)(-X_i) \tag{8}$$

Set the partial derivative equal to 0

$$0 = \sum X_i (Y_i - \alpha - X_i \beta) \tag{9}$$

Deriving OLS

Substitute in $\alpha = \bar{Y} - \beta\bar{X}$

$$0 = \sum X_i(Y_i - \bar{Y} - \beta\bar{X} - X_i\beta) \tag{10}$$

$$0 = \sum X_i Y_i - X_i\bar{Y} + X_i\beta\bar{X} - \beta X_i^2 \tag{11}$$

Apply distributive property of summations

$$0 = \sum X_i Y_i - \sum X_i\bar{Y} + \sum X_i\beta\bar{X} - \sum \beta X_i^2 \tag{12}$$

The terms $\sum X_i$ simplify to $N\bar{X}$, and we can pull β out in front of the summation

Deriving OLS

$$0 = \sum X_i Y_i - N\bar{X}\bar{Y} + \beta N\bar{X}\bar{X} - \beta \sum X_i^2 \tag{13}$$

To solve for β, move all of the β terms to one side of the equation

$$\beta \sum X_i^2 - \beta N\bar{X}^2 = \sum X_i Y_i - N\bar{X}\bar{Y} \tag{14}$$

$$\beta(\sum X_i^2 - N\bar{X}^2) = \sum X_i Y_i - N\bar{X}\bar{Y} \tag{15}$$

$$\beta = \frac{\sum X_i Y_i - N\bar{X}\bar{Y}}{\sum X_i^2 - N\bar{X}^2} \tag{16}$$

Deriving OLS

This formula is more conventionally written as

$$\beta = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2} \tag{17}$$

Numerator:

$$\sum X_i Y_i - \sum X_i \bar{Y} - \sum \bar{X} Y_i + \sum \bar{X}\bar{Y} \tag{18}$$

$$\sum X_i Y_i - N\bar{X}\bar{Y} \tag{19}$$

Denominator:

$$\sum X_i^2 - \sum 2X_i\bar{X} + \sum \bar{X}^2 \tag{20}$$

$$\sum X_i^2 - N\bar{X}^2 \tag{21}$$