Production Function Review Sheet

Ec143, Spring 2023

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In preparing for the exam review your lecture notes, assigned course readings, problem sets and prepare answers to the questions on review sheets (such as this one). You may bring to the exam a single 8.5×11 inch sheet of paper with notes on it. No calculation aides are allowed in the exam (e.g., calculators, phones, laptops etc.). You may work in pencil or pen, however I advise the use of pencil. Please be sure to bring sufficient blue books with you to the exam if possible. Scratch paper will be provided.

[1] Consider a population of firms. A firm with capital k and labor l produces output

$$y\left(k,l;A\right) = Ak^{\alpha}l^{\beta}. \tag{1}$$

Here A is firm-specific, capturing heterogeneity in the efficiency with which different firms are able to transform capital and labor into output. Let W denote the wage rate, and R the rental price of a unit of capital, faced by the firm. We assume that a firm seeking to produce output y, while facing input prices r and w, does so in a *cost-minimizing* way; choosing $K\left(y,r,w;A\right)$ and $L\left(y,r,w;A\right)$ to solve the constrained minimization problem

$$\min_{k,l} rk + wl + \lambda\left[y - Ak^{\alpha}l^{\beta}\right]. \tag{2}$$

[a] The *derived demand* for capital and labor equals the amount of each input a firm would choose when seeking to produce y while facing input prices r and w. *Show* these derived demand schedules, under (2), equal:

$$K\left(y,r,w;A\right) = \alpha\left(\frac{y}{A}\right)^{\frac{1}{\eta}}\left(\frac{w}{r}\right)^{\frac{\beta}{\eta}}\left[\alpha^{\alpha}\beta^{\beta}\right]^{-\frac{1}{\eta}} \tag{3}$$

$$L\left(y,r,w;A\right) = \beta\left(\frac{y}{A}\right)^{\frac{1}{\eta}}\left(\frac{r}{w}\right)^{\frac{\alpha}{\eta}}\left[\alpha^{\alpha}\beta^{\beta}\right]^{-\frac{1}{\eta}}, \tag{4}$$

where $\eta = \alpha + \beta$. Briefly interpret the two demand schedules [3 to 4 sentences]. Why is η called the returns-to-scale parameter?

[b] The cost function equals

$$c\left(y,r,w;A\right) = rK\left(y,r,w;A\right) + wL\left(y,r,w;A\right). \tag{5}$$

Show that under (1) that this function equals

$$c\left(y,r,w;A\right)=\eta\left(\frac{y}{A}\right)^{\frac{1}{\eta}}r^{\frac{\alpha}{\eta}}w^{\frac{\beta}{\eta}}\left[\alpha^{\alpha}\beta^{\beta}\right]^{-\frac{1}{\eta}}. \tag{6}$$

Show that (6) is homogenous of degree one in input prices. Provide an economic explanation for this [4 to 6 sentences]? Show that the *cost shares* of capital and labor are, respectively α/η and β/η.

[c] Let $i=1,\ldots,N$ index a random sample of firms. For each firm we observe output, Y_i, the input prices W_i and R_i, and total costs C_i. We do not observe the firm-specific productivity parameter, A_i. We assume that firm behavior is governed by (1), (3), (4) and (6). Imposing the restriction that (6) is homogenous of degree one in input prices and taking logs yields

$$\ln C_i-\ln W_i=\kappa_c+\frac{1}{\eta}\ln Y_i+\frac{\alpha}{\eta}\left[\ln R_i-\ln W_i\right]-\frac{1}{\eta}\left(\ln A_i-\mathbb{E}\left[\ln A_i\right]\right)$$

for $\kappa_c=\ln\left[\eta\left[\alpha^{\alpha}\beta^{\beta}\right]^{-\frac{1}{\eta}}\right]-\frac{1}{\eta}\mathbb{E}\left[\ln A_i\right]$. Consider the linear regression of log costs onto the (logs of) output and rents minus wages:

$$\mathbb{E}^*\left[\ln C_i-\ln W_i\right|\ln Y_i,\ln R_i-\ln W_i]=k_0+c_0\ln Y_i+a_0\left[\ln R_i-\ln W_i\right]. \tag{7}$$

Is it likely that $c_0=1/\eta$ and $a_0=\alpha/\eta$? Explain (mathematical calculations are not required; 6 to 8 sentences). So far our analysis has been silent regarding how firms choose their output level. In answering this question it might be helpful to consider two cases. In one case firms *do not* choose their output level (consider an electric utility that must meet demand at regulated prices). In the second case firms *do* choose their output level.

[2] Let Y_{it} equal total capital expenditures made by firm $i=1,\ldots,N$ in year $t=1,\ldots,T$. Let X_{it} be Tobin's Q for firm i at the *beginning-of-year* t; that is

$$X_{it}=\frac{\text{Market Value of Firm}}{\text{Replacement Cost of Firm's Capital}}.$$

You posit that

$$Y_{it}=X_{it}'\beta+A_i+U_{it}.$$

The idea by this equation is that firms with high Tobin's Q will investment more since the market value of their assets is greater than their installed cost. A high value of X_{it} suggests that market is valuing some intangible or unrecorded asset of a firm. To help develop intuitions for Tobin's Q, Telsa has generally had a very high Q.

In what follows assume that all parameters are equal to their population or "true" values unless explicitly noted otherwise.

[a] Do you think β is negative, zero or positive? Why?

[b] Consider the linear predictor:

$$\mathbb{E}^*\left[A_i \middle| X_{it}\right] = \pi_0 + X_{it}'\pi_t.$$

Do you think π_t will be negative, zero or positive? Why?

[c] Assuming that at the beginning of year t stock traders know, for each firm,A_i, the past values of capital expenditure, $Y_{i1}^{t-1} = (Y_{i1}, \ldots, Y_{it-1})'$ and the history of Tobin's Q $X_{i1}^{t-1} = (X_{i1}, \ldots, X_{it-1})'$ when making trades (and hence when (indirectly) "choosing" X_{it}). Assume further that they know β, but they do not yet know U_{it} when setting X_{it}. Do these officials also know the history of investment shocks $U_{i1}^{t-1} = (U_{i1}, \ldots, U_{it-1})$ as well? Explain.

[d] Let $\mathcal{I}_{it}$ be stock traders beginning-of-year information set as described in part [c] above. Consider the assumption that

$$\mathbb{E}\left[U_{it} \middle| \mathcal{I}_{it}\right] = 0 \tag{8}$$

for $t = 1, \ldots, T$. Are government officials able to predict period t investment shocks, U_{it}, when "choosing" the beginning-of-year t Tobin's Q X_{it} under (8)? Explain? Does this seem like a reasonable assumption? Why or why not?

[e] Show that when $T = 2$ restriction (8) implies the pair of sequential moment restrictions

$$\mathbb{E}\left[U_{i1} \middle| X_{i1}, A_i\right] = 0$$
$$\mathbb{E}\left[U_{i2} \middle| X_{i1}, X_{i2}, Y_{i1}, A_i\right] = 0$$

Show that these conditions imply that $\mathbb{E}\left[U_{i1}U_{i2}\right] = 0$ (<u>HINT</u>: Use the fact that U_{i1} is a linear combination of Y_{i1}, X_{i1} and A_i and the law of iterated expectations).

[f] Now instead consider the assumption that investment shocks persist over time with

$$U_{it} = \rho U_{it-1} + V_{it}$$

where V_{it} is an unforecastable innovation with the property that

$$\mathbb{E}\left[V_{it} \middle| \mathcal{I}_{it}\right] = 0. \tag{9}$$

Compare (9) with (8) in part [d] above? Which restriction do you think is more realistic? Why?

[g] Show, for the set-up given in part [f] above, that

$$Y_{it} - \rho Y_{it-1} = \left(X_{it} - \rho X_{it-1}\right)' \beta + (1-\rho) A_i + V_{it}$$

and hence that

$$\mathbb{E}\left[Y_{it} - \rho Y_{it-1} - \left(X_{it} - \rho X_{it-1}\right)' \beta - (1-\rho) A_i \middle| \mathcal{I}_{it} \right] = 0.$$

Finally show that, for $T \geq 3$,

$$\mathbb{E}\left[Y_{it} - \rho Y_{it-1} - \left(Y_{it-1} - \rho Y_{it-2}\right) - \left[\left(X_{it} - \rho X_{it-1}\right) - \left(X_{it-1} - \rho X_{it-2}\right)\right]' \beta \middle| \mathcal{I}_{it-1} \right] = 0. \quad (10)$$

[h] You have been hired by the Public Policy Institute of California (PPIC) as research associate. As part of your new job you have been asked to study the effect of Tobin's Q on investment. Your supervisor gives you a dataset with two periods of data ($t = 1, 2$) and suggests you compute the least squares fit of $\Delta Y_{i2} = Y_{i2} - Y_{i1}$ onto a constant and $\Delta X_{i2} = X_{i1} - X_{i0}$; they claim this will "difference away" the unobserved firm-level heterogeneity A_i which may covary with X_{it}. Let b be the probability limit of the OLS coefficient on ΔX_{i2}; do you think b will equal β? Explain? Provide a sufficient condition such that b *does* equal β.

[i] One of your co-workers took the award winning class *Ec143 - Econometrics: Advanced Methods and Applications* while studying at Cal. She says you should instead construct an estimator based on the moment restriction

$$\psi_i(\rho, \beta) = \left(Y_{i3} - \rho Y_{i2} - \left(Y_{i2} - \rho Y_{i1}\right) - \left[\left(X_{i3} - \rho X_{i2}\right) - \left(X_{i2} - \rho X_{i1}\right)\right]' \beta\right) \begin{pmatrix} X_{i1} \\ X_{i2} \\ Y_{i1} \end{pmatrix} = 0.$$

She says she has investment and Tobin's Q data for a third year; so not to worry. Show that $\mathbb{E}\left[\psi_i(\rho, \beta)\right] = 0$.

[j] Your co-worker is just bubbling with ideas. She further suggest the following computation scheme: compute the linear IV fit of ΔY_{i3} onto ΔY_{i2}, ΔX_{i3} and ΔX_{i2} using Y_{i1}, X_{i1}, and X_{i2} as instruments for ΔY_{i2}, ΔX_{i3} and ΔX_{i2} (you exclude the constant in this IV fit). She claims that the coefficient on ΔY_{i2}, say $\hat{p}$, will be consistent for ρ, the coefficient on ΔX_{i3}, say $\hat{b}$, will consistent for β and the coefficient on ΔX_{i2}, say $\widehat{pb}$, will be consistent for $\rho\beta$. Assess you co-worker's claim.

[k] Next you co-worker suggests the following, more refined estimate for β,

$$\hat{\beta} = \frac{1}{2}\left(\hat{b} + \frac{\hat{pb}}{\hat{p}}\right).$$

Is this a consistent estimate for β? Will it be more or less precisely determined (asymptotically), than $\hat{b}$? Why or why not? What would you conclude if $\hat{b}$ and $\frac{\hat{pb}}{\hat{p}}$ are very different from one another?

[l] Assume that the estimate of β based upon the restriction in part [i] above is consistent. Can you say anything about the likely direction of (asymptotic) bias for the estimate based on the simple OLS fit in first differences described in part [h]?