Lemma:

Suppose that we have a random vector $\begin{bmatrix} x_a \\ x_b \end{bmatrix}$, which follows a Gaussian distribution with mean $\begin{bmatrix} \mu_a \\ \mu_b \end{bmatrix}$ and covariance matrix $\Sigma = \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ab} & \Sigma_{bb} \end{bmatrix}$. Then the conditional distribution density function $p(x_a|x_b)$ is a Gaussian distribution as well, with mean $\mu_a + \Sigma_{ab}\Sigma_{bb}^{-1}(x_b - \mu_b)$ and covariance $\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}$.

Kalman filter:

$$\begin{cases} z_n = A z_{n-1} + w_n \sim N(0, \Gamma) & \cdots\cdots ① \\ x_n = C z_n + v_n \sim N(0, \Sigma) \end{cases}$$

Step 1: Suppose we have the filtering distribution at time $n-1$ as $p(z_{n-1}|x_1, \cdots x_{n-1}) \triangleq N(z_{n-1}|\mu_{n-1}, V_{n-1})$. With equation ①, we can get $p(z_n|x_1, \cdots, x_{n-1}) = N(z_n|A\mu_{n-1}, \underline{AV_{n-1}A^T + \Gamma})$.

Step 2: Denote $AV_{n-1}A^T + \Gamma \triangleq P_{n-1}$. Next, we consider the distribution for the vector $\begin{bmatrix} z_n|x_1, \cdots, x_{n-1} \\ x_n \end{bmatrix}$. Its mean value is $\begin{bmatrix} A\mu_{n-1} \\ CA\mu_{n-1} \end{bmatrix}$, and its covariance matrix is given as

$$\left[\begin{array}{c|c} P_{n-1} & P_{n-1}C^T \\ \hline CP_{n-1} & CP_{n-1}C^T+\Sigma \end{array}\right].$$

Base on the lemma given at the beginning, we can get

$$p(z_n \mid x_1, \cdots, x_{n-1}, x_n) = p(z_n \mid x_1, \cdots, x_{n-1} \mid x_n)$$

$$\triangleq N\left(z_n \,\middle|\, \underbrace{A\mu_{n-1} + CP_{n-1}(CP_{n-1}C^T+\Sigma)^{-1}(x_n - CA\mu_{n-1}}_{\mu_n},\ \underbrace{P_{n-1} - P_{n-1}C^T(CP_{n-1}C^T+\Sigma)^{-1}CP_{n-1}}_{V_n}\right)$$