$\underline{p(x_t | y_{1:t})}$

$\underline{p(x_{1:t} | y_{1:t})}$

$$p(x_t | y_{1:t}) = p(x_t | y_{1:t-1}, y_t)$$

$$= \frac{p(y_t|x_t)\, p(x_t | y_{1:t-1})}{p(y_t | y_{1:t-1})}$$

$$= \frac{p(y_t|x_t) \int p(x_t|x_{t-1})\, p(x_{t-1}|y_{1:t-1})\, dx_{t-1}}{p(y_t|y_{1:t-1})} \qquad (*)$$

Assume we have the particle approximation for $p(x_{t-1} | y_{1:t-1}) \doteq \sum_{i=1}^{N} w_{t-1}^{i}\, \delta(x_{t-1} - \tilde{x}_{t-1}^{i})$. Plug this formula into equation (*), we get

$$p(x_t | y_{1:t}) \doteq \frac{p(y_t|x_t)}{p(y_t|y_{1:t-1})} \sum_{i=1}^{N} w_{t-1}^{i}\, p(x_t | \tilde{x}_{t-1}^{i}).$$

Sampling proposal: $q(x_t, a_t | y_{1:t}) = v_{t-1}^{a_t}\, q(x_t | \tilde{x}_{t-1}^{a_t}, y_t)$

Target distribution: $p(x_t, a_t | y_{1:t}) \propto w_{t-1}^{a_t}\, p(x_t|\tilde{x}_{t-1}^{a_t})\, p(y_t|x_t)$

$\Rightarrow$ Unnormalized Weight: $$\bar{w}_t^{i} = \frac{w_{t-1}^{a_t^i}\, p(x_t^i | \tilde{x}_{t-1}^{a_t^i})\, p(y_t|x_t^i)}{v_{t-1}^{a_t^i}\, q(x_t^i | \tilde{x}_{t-1}^{a_t^i}, y_t)}$$

$$= \frac{w_{t-1}^{a_t^i}\, p(y_t | \tilde{x}_{t-1}^{a_t^i})\, p(x_t^i | y_t, \tilde{x}_{t-1}^{a_t^i})}{v_{t-1}^{a_t^i}\; q(x_t^i | \tilde{x}_{t-1}^{a_t^i}, y_t)} =$$

i) if $v_{t-1}^i = w_{t-1}^i$, for $i = 1 \sim N$, and $q(x_t^i \mid x_{t-1}^{a_t^i}, y_t) = \dots$

$\dots\ p(x_t^i \mid x_{t-1}^{a_t^i})$, then $\bar{w}_t^i = p(y_t \mid x_t^i)$.

ii) if $v_{t-1}^i = w_{t-1}^i \cdot \underline{p(y_t \mid \tilde{x}_{t-1}^i)}$, $q(x_t^i \mid \tilde{x}_{t-1}^i, y_t) = \underline{p(x_t^i \mid \tilde{x}_{t-1}^i, y_t)}$,

then $\bar{w}_t^i = 1$.

Note:

Δ proof of $p(\underbrace{x_t^i}_{B} \mid \underbrace{x_{t-1}^{a_t^i}}_{A})\, p(\underbrace{y_t}_{C} \mid \underbrace{x_t^i}_{B}) = p(\underbrace{y_t}_{C} \mid \underbrace{\tilde{x}_{t-1}^{a_t^i}}_{A})\, p(\underbrace{x_t^i}_{B} \mid \underbrace{y_t, \tilde{x}_{t-1}^{a_t^i}}_{A,C})$

$$\Leftrightarrow \quad p(B \mid A)\, p(C \mid B) = p(C \mid A) \cdot p(B \mid A, C)$$

$$\Leftrightarrow \quad \frac{p(A,B)}{\cancel{p(A)}} \cdot \frac{p(B,C)}{p(B)} = \frac{\cancel{p(A,C)}}{\cancel{p(A)}} \cdot \frac{p(A,B,C)}{\cancel{p(A,C)}}$$

$$\Leftrightarrow \quad p(C \mid B) = p(C \mid A, B) \leftarrow \boxed{\text{this is given by the Markov property of the System!}}$$