HW1

Exercise 3

Q1

As $R_1 - \mu$ has a zero mean distribution, all moments with odd orders are zero. Therefore, we have,

$$\begin{aligned}\gamma = \mathbf{E}[R_1^3] &= E[(R_1 - \mu + \mu)^3] \\ &= \mathbf{E}[(R_1 - \mu)^3 + 3(R_1 - \mu)^2\mu + 3(R_1 - \mu)\mu^2 + \mu^3] \\ &= 3\mu\mathbf{E}[(R_1 - \mu)^2] + \mu^3 \\ &= 3\mu Var[R_1 - \mu] + \mu^3 \\ &= \mu^3 + 3\mu\sigma^2\end{aligned}$$

Q2

(a) Since $\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$ has the distribution of $\mathbf{N}(\mu, \sigma^2/n)$, similarly to Q1, we can derive $\mathbf{E}[\bar{R}^3] = \mu^3 + 3\mu\frac{\sigma^2}{n}$. So the bias is $\mathbf{E}[\hat{\gamma} - \gamma] = -\frac{n-1}{n}\mu\sigma^3$.

(b) $\hat{\gamma}$ is not consistent. Since $\bar{R} \sim N(\mu, \frac{\sigma^2}{n})$, we have,

$$\begin{aligned}\Pr[|\bar{R}^3 - (\mu^3 + 3\mu\frac{\sigma^2}{n})| \geq \epsilon] &= 1 - \Pr[|\bar{R}^3 - (\mu^3 + 3\mu\frac{\sigma^2}{n})| \leq \epsilon] \\ &= 1 - \Phi(\sqrt{n}\frac{(\mu^3 + 3\mu\frac{\sigma^2}{n} + \epsilon)^{\frac{1}{3}} - \mu}{\sigma^2}) \\ &\quad + \Phi(\sqrt{n}\frac{(\mu^3 + 3\mu\frac{\sigma^2}{n} - \epsilon)^{\frac{1}{3}} - \mu}{\sigma^2}) \\ &\to 1 - \Phi(\sqrt{n}\frac{(\mu^3 + \epsilon)^{\frac{1}{3}} - \mu}{\sigma^2}) + \Phi(\sqrt{n}\frac{(\mu^3 - \epsilon)^{\frac{1}{3}} - \mu}{\sigma^2}) \\ &\to 1 - \Phi(\infty) + \Phi(-\infty) \\ &= 1 - 1 + 0 = 0, \textit{ as } n \to \infty \textit{ with fixed } \epsilon\end{aligned}$$

So $\hat{\gamma}$ converges to $\mu^3 + 3\mu\frac{\sigma^2}{n} \to \mu^3$, so it is not consistent to the estimated parameter $\gamma = \mu^3 + 3\mu\sigma^3$.

Q3

Since we have $\mathbf{E}[R_1R_2R_3] = \mu^3$ and $\mathbf{E}[\hat{\gamma}] = \mu^3 + \frac{3\mu\sigma^2}{n}$, we have $3\mu\sigma^2/n = \mathbf{E}[\hat{\gamma}] - \mathbf{E}[R_1R_2R_3]$.Therefore, we can choose $n\hat{\gamma} - (n-1)R_1R_2R_3$ as the unbias estimator, whose mean is exactly μ^3.

Q4

(a) Since $\mathbf{E}[\tilde{\gamma} - \gamma] = n * \frac{1}{n}\mathbf{E}[R_1^3] - \gamma = 0$, the bias is 0.

(b) $\tilde{\gamma}$ is consistent. Using LLT, $\tilde{\gamma} \overset{p}{\sim} \mathbf{E}[R_1^3] = \gamma$. So it's consistent.

Q5

Since the minimal sufficient statistics for normal distributions are $\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$ and $\overline{R^2} = \frac{1}{n}\sum_{i=1}^{n} R_i^2$. And they are also complete statistics. According to the Rao-Blackwell, we only need to find the conditional expection of an unbiased estimator by setting the two statistics as the condition. Therefore $\gamma_{UVME} = \mathbf{E}[\tilde{\gamma}|\bar{R}, \overline{R^2}]$. In the following, we use T to denote the condition. We have,

$$
\begin{aligned}
\mathbf{E}[\tilde{\gamma}|T] &= \mathbf{E}[\frac{1}{n}\sum_{i=1}^{n} R_i^3|T] \\
&= \mathbf{E}[\frac{1}{n}\sum_{i=1}^{n}(R_i - \bar{R} + \bar{R})^3|T] \\
&= \mathbf{E}[\frac{1}{n}\sum_{i=1}^{n}[(R_i - \bar{R})^3 + 3(R_i - \bar{R})^2\bar{R} \\
&\quad + 3(R_i - \bar{R})\bar{R}^2 + \bar{R}^3]|T]
\end{aligned}
$$

By using symmmetry of the conditional distribution, one can prove that all (conditional) moments of $R_i - R$ which have odd orders are zero. Therefore, we have,

$$
\begin{aligned}
\mathbf{E}[\tilde{\gamma}|T] &= \mathbf{E}[\frac{1}{n}\sum_{i=1}^{n}[3(R_i - \bar{R})^2\bar{R} + \bar{R}^3]|T] \\
&= \mathbf{E}[\frac{1}{n}\sum_{i=1}^{n}[3R_i^2\bar{R} - 6R_i\bar{R}^2 + 3\bar{R}^3 + \bar{R}^3]|T] \\
&= \mathbf{E}[\frac{3}{n}\bar{R}\sum_{i=1}^{n} R_i^2 - 2\bar{R}^3|T] \\
&= 3\bar{R}\overline{R^2} - 2(\bar{R})^3
\end{aligned}
$$

Exercise 4

Load dataset

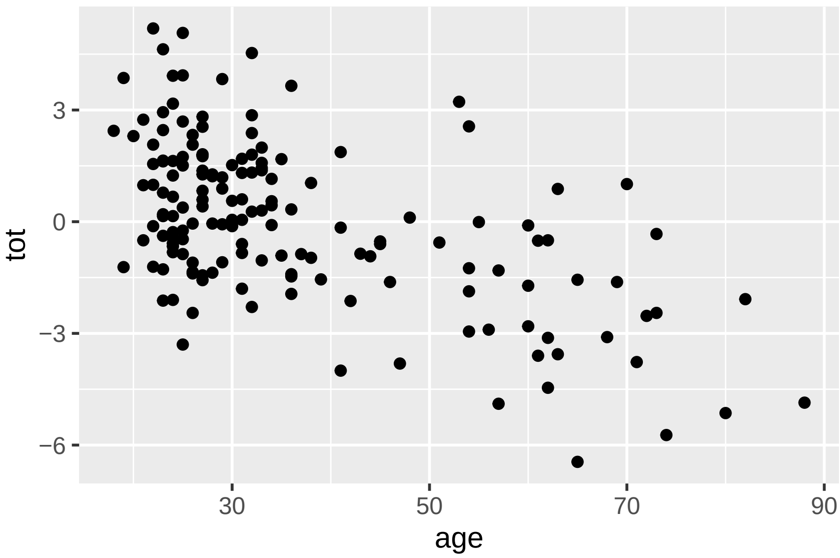
```r
lines <- readLines("kidney.txt")
```

```r
numbers_vec <- lapply(lines,
  function (line)  stringr::str_extract_all(line, "[-]?\\d+[\\.]?\\d*")) %>%
  unlist(recursive = FALSE) %>%
  Filter(f = function(x) length(x) == 3) %>%
  Map(f = function(x) lapply(x, as.numeric))
```

```r
df <- do.call(rbind.data.frame, numbers_vec)
colnames(df) <- c("id", "age", "tot")
rownames(df) <- df$id
```

Q1

```r
library(ggplot2)
scatterPlot <- ggplot(df, mapping = aes(x=age, y=tot)) + geom_point()
scatterPlot
```



The scatter plot shows that "age" and "tot" have a negative relationship and it could be fitted with a linear model.

Q2

I would choose tot as the response.

Q3

```r
Corr = cor(df$age, df$tot)
Corr
```

```
## [1] -0.5718387
```

Negative sign, since the correlation is false.

Q4

α denotes the expected value when the input is 0, while β denotes how much the response will change if the input is increased or decreased by 1.

Q5

```r
linearModel <- lm(tot ~ age, data = df)
summary(linearModel)
```

```
##
## Call:
## lm(formula = tot ~ age, data = df)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -4.2018 -1.3451  0.0765  1.0719  4.5252
```

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  2.860027   0.359565   7.954 3.53e-13 ***
## age         -0.078588   0.009056  -8.678 5.18e-15 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.801 on 155 degrees of freedom
## Multiple R-squared:  0.327,  Adjusted R-squared:  0.3227
## F-statistic: 75.31 on 1 and 155 DF,  p-value: 5.182e-15
```

They are statistically significant with a very small p-value.

Q6

I would use the geometry to intepret these two parameters. In linear algebra, the estimates provide the optimal group of parameters to project a high-dimension vector to a two-dimension plane with minimal loss.

Q7

```r
#prediction <- function(age) linearModel$coefficients * age + linearModel$
beta <- as.numeric(linearModel$coefficients["age"])
alpha <- as.numeric(linearModel$coefficients["(Intercept)"])
predict <- function (age) alpha + beta * age
```

```r
predict(100)
```

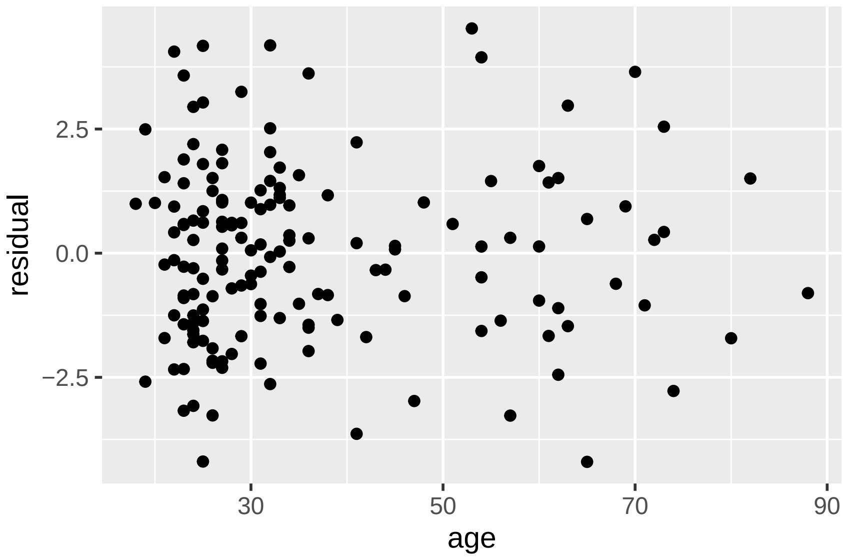
```
## [1] -4.998815
```

The prediction seems reasonable.

Q8

```r
res_df <- df %>%
  dplyr::mutate(prediction = predict(df$age)) %>%
  dplyr::mutate(residual = tot - prediction)
```

```r
ggplot(res_df) + geom_point(aes(x=age, y=residual))
```



The plot shows that the residuals are randomly distributed around 0, so it's reasonable to have the i.i.d assumption.

Q9

```
minus <- function(x, y) max(y,x) - min(y,x)
betaIntNormal <- Reduce(minus, confint(linearModel)[2,])
betaIntAsym <- Reduce(minus, confint.default(linearModel)[2, ])
```

The asymptotic one seems to have a shorter interval.

Q10

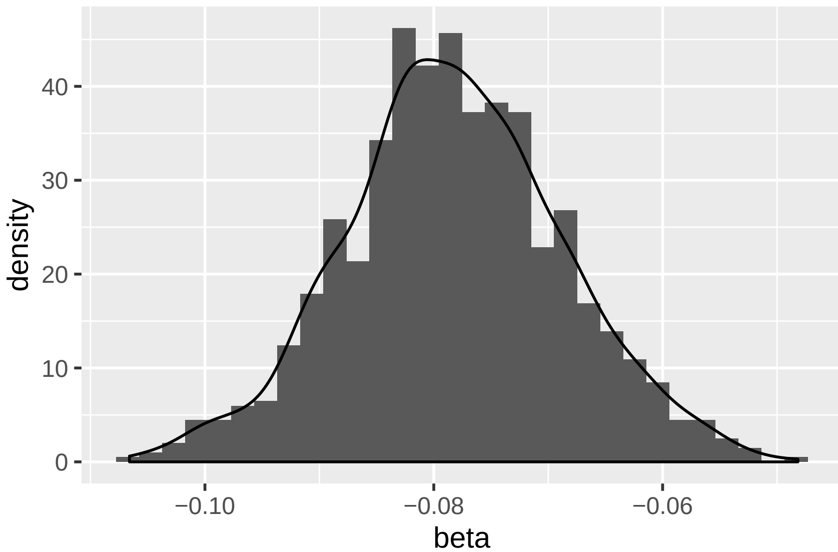
```
boot.stat <- function(data, indices){
data <- data[indices, ] # select cases in bootstrap sample
mod <- lm(tot ~ age, data=data) # refit model
coef(mod)["age"] # return coefficient vector
}
```

```
set.seed(12345) # for reproducibility
df.boot <- boot::boot(data=df, statistic=boot.stat, R=1000)
```

```
bootResult <- as.data.frame(df.boot$t) %>%
  dplyr::rename(beta=V1)
```

```
ggplot(bootResult) +
  geom_histogram(aes(x=beta,y=..density..)) +
  geom_density(aes(x=beta, y=..density..))
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```




```
confInts <- boot::boot.ci(df.boot)
```

```
## Warning in boot::boot.ci(df.boot): bootstrap variances needed for
## studentized intervals
```

```
basicInt <- Reduce(minus, confInts$basic[4:5])
percentInt <- Reduce(minus, confInts$percent[4:5])
```

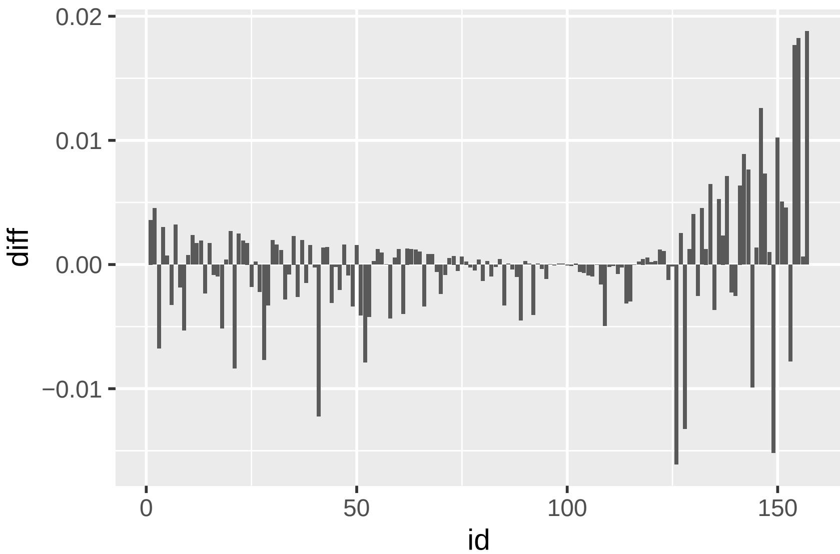
The bootstrap interval is larger.

Q11

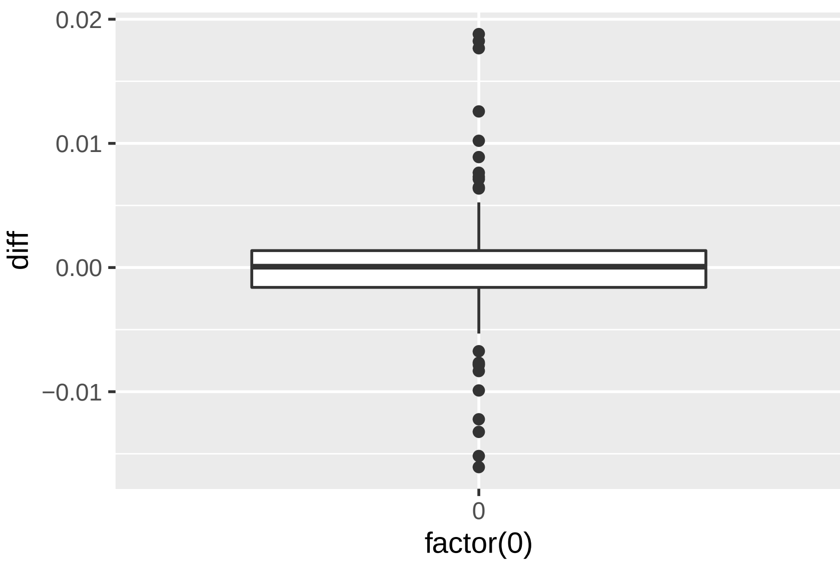
```
LeaveOneOutCorr <- function (idx)  {
  df_tmp <- df %>% dplyr::filter(id != idx)
  cor(df_tmp$age, df_tmp$tot) - Corr
}
```

```
corr_diff <- unlist(Map(LeaveOneOutCorr, seq.int(1, nrow(df))))
df_lou <- df %>% dplyr::mutate(diff=corr_diff)
```

```
ggplot(df_lou) + geom_col(aes(x=id, y=diff))
```

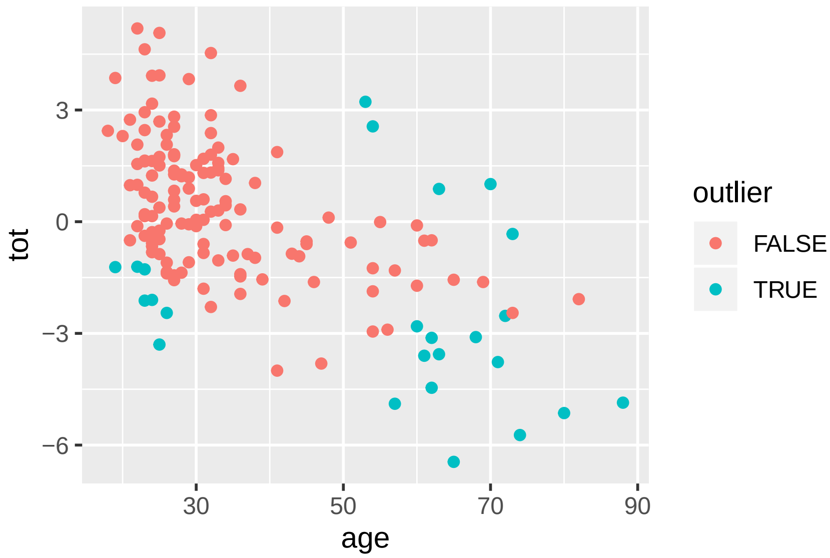



```
ggplot(df_lou, aes(x=factor(0), y=diff)) + geom_boxplot()
```




```
outlierDetect <- function(corrVal) {
  ifelse (abs(corrVal) > 0.005, TRUE, FALSE)
}
df_outlier <- df_lou %>% dplyr::mutate(outlier=outlierDetect(diff))
```

```
ggplot(df_outlier) + geom_point(aes(x=age, y=tot, color=outlier))
```



There are some data points which are more influential than others. In the above plot, they are marked as "outlier"s with special leave-one-out differences.

Bonus Question

Q1

The author want to answer the question whether techno-scientific findings are inevitable or not by fitting the findings dataset using a Poisson model. The optimal parameter chosen after experiments tends to show that techno-scientific discoveries are not inevitable and highly depends on luck. I think for me, the choice of Poisson distribution seems reasonable, since techno-scientific findings are odd and can happen at a low probability. And Poisson distribution is quite suitable for modeling the probability of rare events happening.

Q2

Since there are no data for singleton and no-findings in the dataset, so using a truncated model will not give weird expected values for $k = 0$ or $k = 1$.

Q3

Suppose $X \sim Poisson(\mu)$, then we can derive the expectation and variance of Y using $\mathbf{E}[X]$ and $Var[X]$. We have,

$$\begin{aligned}\mathbf{E}[X] &= \mu \\ &= \sum_{k=0}^{\infty} k \frac{e^{-\mu}\mu^k}{k!} \\ &= \mu e^{-\mu} + \sum_{k=2}^{\infty} k \frac{e^{-\mu}\mu^k}{k!}\end{aligned}$$

If we denote $C = \frac{1}{1-e^{-\mu}-\mu e^{-\mu}}$, we have,

$$\begin{aligned}\mathbf{E}[Y] &= C \sum_{k=2}^{\infty} k \frac{e^{-\mu}\mu^k}{k!} \\ &= C(\mu - \mu e^{-\mu})\end{aligned}$$

Similarly, we can derive $Var[Y]$ with the help of $\mathbf{E}[X]$ and $\mathbf{E}[X^2]$. We have,

$$Var[Y] = C\mu(2\mu - e^{-\mu}) - C^2\mu^2(1 - e^{-\mu})^2$$

Q4 The data can be saved as a dataframe as below,

```
tbl1 <- data.frame(
  k = seq.int(2, 9),
  count = c(179, 51, 17, 6, 8, 1, 0, 2)
)
tbl1
```

```
##   k count
## 1 2   179
## 2 3    51
## 3 4    17
## 4 5     6
## 5 6     8
## 6 7     1
## 7 8     0
## 8 9     2
```

Then the log likelihood function can be derived as,

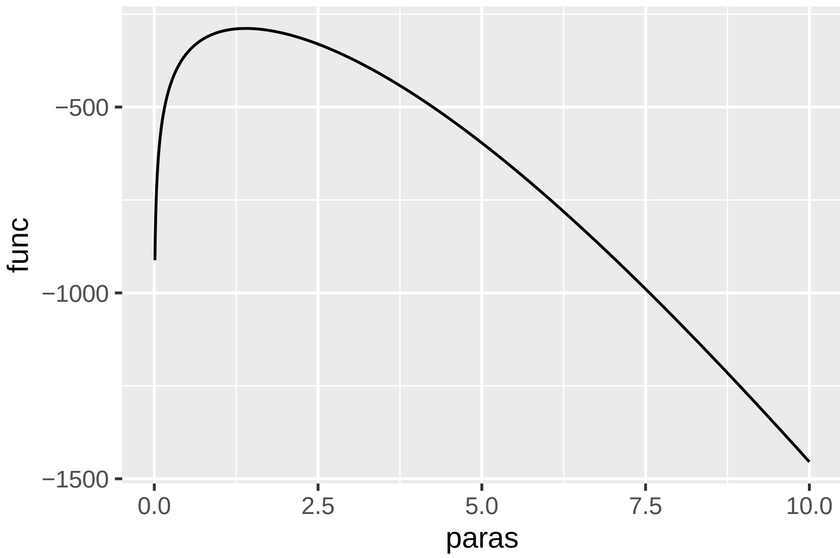
$$\begin{aligned}\log L &= \log(\prod_{k=2}^{9}(C\frac{e^{-\mu}\mu^k}{k!})^{COUNT_k}) \\ &= \sum_{k=2}^{9} COUNT_k \log(\frac{e^{-\mu}\mu^k}{k!})\end{aligned}$$

```
logL <- function (mu) {
  prob_dist <- function(x) {
    exp(-mu) * mu^x / factorial(x) /
      (1 - exp(-mu) - mu * exp(-mu))
  }
  data_ <- tbl1 %>%
    dplyr::mutate(prob=prob_dist(k)) %>% # likelihood
    dplyr::mutate(likelihood=count*log(prob)) # log-likelihood
  sum(data_$likelihood) # sum them up
}
```

And the log likelihood can be plotted as below,

```
plot_curve <- function(pars, f) {
 df_curve <- data.frame(
  paras = pars,
  func = unlist(lapply(pars, f))
 )
 ggplot(df_curve, aes(x=paras, y=func)) + geom_line()
}

plot_curve(10^seq.int(-2, 1, 0.01), logL)
```



Q5

The algorithm I choose is "BFGS", implemented in "optimx:optimx" function. Since it's a convex and nonlinear optimization problem as plotted above, this algorithm will converge shortly. The results and code are shown below,

```
opt <- optim(as.vector(c(1)), method = "BFGS", fn=function(x) {-logL(x)}, gr = NULL)
opt$par
```

```
## [1] 1.398391
```

Q6

Since the given distribution follows the regularity conditions, the asymptotic dsitribution would be a normal distribution,

$$\sqrt{n}(\hat{\mu}^{MLE} - \mu_0) \to N(0, I(\mu_0)^{-1})$$

where the fisher information can be calculated as,

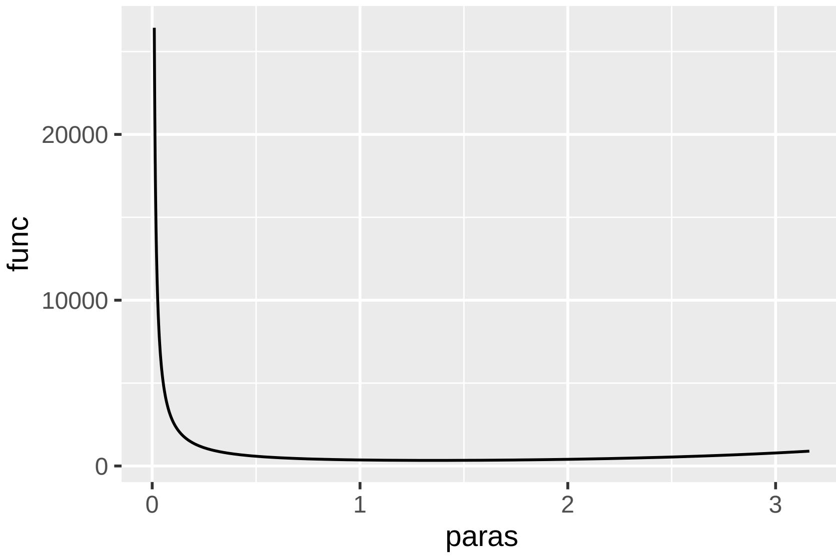
$$I(\mu_0) = -\mathbf{E}[\frac{\partial^2 \log(L)}{\partial \mu^2}] = \frac{n}{\mu} + \frac{n(\mu + e^{-\mu} - 1)}{e^{-\mu}(e^{\mu} - 1 - \mu)^2}$$

```
fisher <- function(mu) {
  n <- 264
  n / mu + n*(mu+exp(-mu)-1)/exp(-mu)/(exp(-mu)-1-mu)^2
}
optimize(fisher, lower=0, upper=10)
```

```
## $minimum
## [1] 1.35379
##
## $objective
## [1] 337.4855
```

Also, from the curve below, we can notice that the curve of fisher information around the MLE or optimal μ is quite flat. Therefore, we use MLE to calculate fisher information, which is 337.8257851.

```r
plot_curve(10^seq.int(-2, 0.5, 0.01), fisher)
```



Q7

Given the asymptotic distribution given by Q6, we have the confidence interval as,

$$\mu \in [\mu_{ML} - 1.96\frac{I(\mu_{ML})^{-1}}{\sqrt{n}}, \mu_{ML} + 1.96\frac{I(\mu_{ML})^{-1}}{\sqrt{n}}] = [1.39803, 1.39875]$$

Q8

It seems like a reasonable choice since different groups of majors have quite different value of μ as mentioned in the paper. But it would be hard to evaluate the mathematical properties of this estimator.

Q9

Our ML estimator is 1.3983907, which is quite similar to the result ($\mu = 1.4$) given by the paper. Both of them can show evidence that the techno-scientific findings are not inevitable.