- $|z| = \sqrt{z\bar{z}}$
- $\frac{\bar{z}}{|z|^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i \in \mathbb{C}$
  Proof: $z \times \frac{\bar{z}}{|z|^2} = \frac{z \times \bar{z}}{|z|^2} = 1$
- $z + \bar{z} = 2\Re(z)$; $z - \bar{z} = \Im(z)$; $\overline{z + w} = \bar{z} + \bar{w}$; $\overline{zw} = \bar{z}\bar{w}$
- $|\bar{z}| = |z|$; $|zw| = |z||w|$; $|z/w| = |z|/|w|$; $\frac{1}{w} = \frac{\bar{w}}{|w|^2}$
  Proof: $|\bar{z}|^2 = (|x - iy|)^2 = x^2 + (-y)^2 = x^2 + y^2 = |z|^2$
  $|zw|^2 = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = |z|^2|w|^2$
  $|\frac{1}{w}| = |\frac{\bar{w}}{|w|^2}| = \frac{|\bar{w}|}{|w|^2} = \frac{1}{|w|}$
- $e^{i\theta} = \cos\theta + i\sin\theta$; $e^{i\pi} = -1$; $\bar{z} = e^{-i\theta}$
  Principle value of the argument: $-\pi < \arg(z) \le \pi$
- De Moivres Thm: $wz = \rho r e^{i(\varphi+\theta)}$ pf: write wz then compound angle formulae.
- $e^z = e^x(\cos y + i \sin y)$
  $|e^{i\theta}| = 1$ if $\theta \in \mathbb{R}$; $|e^z| = e^{\Re(z)}$
  $e^z = 1 \iff z = 2\pi i k$, $k \in \mathbb{Z}$; $e^{z+2\pi i k} = e^z$
- $\cos(x) = \frac{1}{2}(e^{iz} + e^{-iz})$; $\sin(x) = \frac{1}{2i}(e^{iz} - e^{-iz})$
- Roots of unity:
  Find the exponential form of your z.
  find the nth root of $|z|$ then your argument is $(arg(z)/n + 2k\pi/n)$. Facts:
  $\sum_{j=1}^{n} \zeta_j^{(n)} = 0$; $\prod_{j=1}^{n} \zeta_j^{(n)} = (-1)^{n+1}$
- Square root:
  $a + ib = (x + iy)^2 = x^2 - y^2 + 2ixy$;
  so $a = x^2 - y^2$, $b = 2xy \implies x^4 - ax^2 - \frac{1}{4}b^2 = 0$
  let $t = x^2$ so $t = \frac{a \pm \sqrt{a^2+b^2}}{2}$. Find t_+ & solve.
  $x = \pm\sqrt{t_+}$ and $y = b/2x$
- Curves & planar regions:
  $\Re(e^{i\theta}z) = \cos(\theta)x - \sin(\theta)y = r$
  $\implies y = \cot(\theta)x - r\csc(\theta)$ Similarly $\Im(e^{i\theta}z) = s$ is a line. $|z - \omega| = r$: Circle radius r center: ω

Extra:

- **GCD**: for $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, there exists a unique $q, r \in \mathbb{Z}$ with $0 \le r < b$ such that $a = qb + r$
- $a \equiv b (\text{mod } m)$ if $m \mid (a - b)$
- To solve: $ax \equiv b(\text{mod } m)$,
  Find $d = \gcd(a, m)$
  If $d \nmid b$ then no solns
  If $d \mid b$ then write $b = b'd$ and write $d = sa + tm$
  $x = sb'$ is a soln. Let $m' = m/d$. The set of numbers congruent to $sb'(\text{mod } m')$ is the set of all solns
- Chinese remainder thm: If m_1 and m_2 are s.t. $\gcd(m_1, m_2) = 1$ then there is a unique $x(\text{mod } m_1m_2)$ that satisfies $x \equiv c_1(\text{mod } m_1)$ and $x \equiv c_2(\text{mod } m_2)$
  if $\gcd(m_1, m_2) = 1$, write $1 = sm_1 + tm_2$, then $c_1tm_2 + c_2sm_1$ is a unique soln modulo m_1m_2
- FTA: A degree n complex polyn has exactly n complex roots.
- Lagrange interpolation:
  $p(x) = \sum_{j=1}^{n+1} p_j(x)$ where $p_j(x) = y_j \prod_{k=1, k\neq j}^{n+1} \frac{x - x_k}{x_j - x_k}$
- Scalar: Number, Vector: Direction & Magnitude
- $\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}|\cos\theta$
- $\underline{u} \times \underline{v} = \det \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$
  $= i\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - j\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + k\begin{vmatrix} a_1 & b_1 \\ a_1 & b_2 \end{vmatrix}$
- Point to line: $(\underline{u} - \underline{r}) \cdot \underline{v} = 0$
- Point to plane: for $\underline{r} \cdot \underline{n} = \underline{p} \cdot \underline{n}$, $(\underline{x} - \underline{x}_0) \cdot \underline{n}$

- i is up and down, j is across.
  $\mathbb{R}^{m \times n} \implies n$ across, m down.
  To multiply a matrix, must have $n_A = m_B$.
- In a matrix product: $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$;
  Use to prove multiplication.
- pf: $(A+B)_{ij} = A_{ij} + B_{ij} = B_{ij} + A_{ij} = (B+A)_{ij}$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$; $A_{ij}^T = A_{ji}$
  $(AB)^T = B^T A^T$
- Solution of a system of linear equations:
  1) Write the problem in matrix notation
  2) Construct the "Augmented Matrix"
  3) Do "row operations" to get the matrix into Echelon Form
- Echelon forms: $\square$: real number, $\star$: non-zero "pivot"
  A) $\begin{pmatrix} \star & \square & \square \\ 0 & \star & \square \\ 0 & 0 & \star \end{pmatrix}$ B) $\begin{pmatrix} \star & \square & \square \\ 0 & \star & \square \\ 0 & 0 & 0 \end{pmatrix}$
  C) $\begin{pmatrix} \star & \square & \square \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{pmatrix}$ D) $\begin{pmatrix} \star & \square & \square \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- Reduced Echelon forms:
  A) $\begin{pmatrix} \star & 0 & 0 \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix}$ B) $\begin{pmatrix} \star & 0 & \square \\ 0 & \star & \square \\ 0 & 0 & 0 \end{pmatrix}$ C) $\begin{pmatrix} \star & \square & 0 \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{pmatrix}$
- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$
  $= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$

Extra:

- Upper triangular matrix:
  $$A^E = \begin{pmatrix} \mu_1 & \square & \dots & \square \\ 0 & \mu_2 & \square & \vdots \\ \vdots & \vdots & \ddots & \square \\ 0 & 0 & \dots & \mu_n \end{pmatrix}$$
  and, $\det(A^E) = \mu_1 \mu_2 \dots \mu_n = \prod_{i=1}^{n} \mu_i$
  Proof: show result is true for $n = 2$, then assume true for $n-1$ with $n-1 \geq 2$
- Eigenvectors & Eigenvalues example:
  $\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
  The first step is to write:
  $\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
  Which is equivalent to:
  $\begin{pmatrix} 3-\lambda & 2 \\ 2 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
  Then we must have:
  $\begin{vmatrix} 3-\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = 0$
  Which gives us $\lambda^2 - 3\lambda - 4 = 0$ so $\lambda = 4$ or $\lambda = -1$, two possible eigenvalues.
  At $\lambda = 4$: $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
  Which implies $x_1 = 2x_2$ so we have $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
  At $\lambda = -1$: $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
  Which implies $x_2 = -2x_1$ so we have $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c' \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
  where $c, c' \neq 0$ and so we have our eigenvectors.