- Difference Quotient: $\frac{f(x+h)-f(x)}{h} = \frac{f(y)-f(x)}{y-x}$
- **AL1:** (localisation principle) For any fixed $r > 0$, $\lim_{x\to x_0} f(x)$ is unaffected by points on $f(x)$ where $|x_0 - x| > r$.
- **AL2:** For $f(x) \to L$ and $C \in \mathbb{R}$, $Cf(x) \to CL$, and$C + f(x) \to C + L$
- **AL3:** (Sandwich Theorem)
- **AL4:** $f(x) \pm g(x) \to L \pm M$; **AL5:** $f(x)g(x) \to LM$
- **AL6:** $f(x)/g(x) \to L/M$, $M \neq 0$
- **AL7:** (composition property) If $g(y) \to z_0$ as $f(x) \to y_0$ then$(g \circ f)(x) \to z_0$
- $\sum_{n=0}^{N} ar^n = \frac{a(1-r^{N+1})}{1-r} \to \frac{1}{1-r}$, if $|x| < 1$
- **AIL1:** Same as AL1
- **AIL2:** if $f(x) \to \pm\infty$ then $Cf(x) \to \pm\infty$
- **AIL3:** if $f(x) \to \infty$, $g(x) \geq C \forall x$ then $f(x) + g(x) \to \infty$; **AIL4:** if $f(x) \to \pm\infty$ then $1/f(x) \to 0$
- $f : D \to \mathbb{R}$ is continuous on D if, for every $x_0 \in D$, $f(x) \to f(x_0)$ as $x \to x_0$.
- Every polynomial is continuous on $\mathbb{R}$; every rational function is continuous everywhere when the denominator isn't 0.
- $f'(x) = \lim_{h\to 0}$(Difference Quotient)
- Remainder Term: $f(x+h) = \underbrace{f(x) + hf'(x)}_{\text{Local Linear Approx}} + r(h)$ where $r(h)/h \to 0$
- Differentiability $\implies$ continuity
- Compositions of continuous functions are continuous (By AL1-7).
- Additivity: $(f+g)'(x) = f'(x) + g'(x)$
- Product Rule: $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$
- Chain Rule: $(f \circ g)'(x) = f'(g(x))g'(x)$
- Quotient Rule: $(f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
- Left and right derivatives exist. Apply left/right limits to the difference quotient.
- L'Hôpital's Rule: $f(x_0) = g(x_0) = 0$, $g'(x_0) \neq 0$ then $\frac{f(x)}{g(x)} \to \frac{f'(x_0)}{g'(x_0)}$.
- or $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$
- Parametric Curve: $\gamma'(t) = (x'(t), y'(t))$ Speed: $|\gamma'(t)|$
- **IVT:** Suppose $a < b$ and f continuous on $[a, b]$. Then for every y such that $\min(f(a),f(b)) < y < \max(f(a),f(b))$ $\exists x \in (a, b)$ st $f(x) = y$
- **EVT:** $\exists x_{\min}, x_{\max} \in [a, b]$ st $\forall x \in [a, b]$, $f(x_{\min}) \leq f(x) \leq f(x_{\max})$.
- Every strictly monotonic function is injective (1-1).
- **IFT:** $f : I \to \mathbb{R}$ and strictly monotonic then f^{-1} is continuous.
- local extremum at x_0 then $f'(x_0) = 0$. Stationary/ Turning/ Critical Point.
- **MVT:** $\exists x_0 \in (a, b)$ st $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.
- $(f^{-1})'(y) = 1/f'(f^{-1}(y))$
- $(1 + h)^u \approx 1 + uh$ if h is small.
- Every real polynomial of odd degree has a real zero (root).
- $x^y = exp(y \log x)$ and $\left(1 + \frac{x}{y}\right)^y \to e^x$ as $y \to \infty$
- Solutions of $u'' = -u \implies u(x) = A\cos(x) + B\sin(x)$
- $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
- $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
- $\frac{\sin(h)}{h} \to 1$ as $h \to 0$
- $(\sin^{-1}(x))' = 1/\sqrt{1-x^2}$; $(\cos^{-1}(x))' = -1/\sqrt{1-x^2}$
- $(\tan^{-1}(x))' = 1/(1+x^2)$
- $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$; $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$
- $\exp(w) = z \implies w = \log|z| + i(\theta + 2k\pi)$
- $\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)tanh(y)}$
- $\sinh' = \cosh$; $\tanh' = \text{sech}^2$
- $(\sinh^{-1})' = 1/\sqrt{1+x^2}$; $(\cosh^{-1})' = 1/\sqrt{x^2-1}$; $(\tanh^{-1})' = 1/(1-x^2)$
- Riemann Sums: $R = \sum_{j=1}^{n}(x_j - x_{j-1})f(c_j)$
- **MVTI:** $f(x_0)\int_a^b g = \int_a^b fg$ Consider $g = 1$
- **FTC 1:** $F(x) = \int_a^x f$; **FTC 2:** $\int_a^b f' = f(b) - f(a)$
- Gamma f$\underline{\text{n}}$: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$
- $\Gamma(\alpha+1) = \alpha(\Gamma(\alpha))$, $(a > 0)$; $\Gamma(n) = (n-1)!$, $(n \in \mathbb{N})$
- Gaussian Int: $\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$
- Arc Length:$L(\gamma) = \int_a^b |\gamma'(t)|\,dt$
- Radian Measure: $r(\phi - \theta)$, $(\theta < \phi)$
- Rev: $V = \pi\int_a^b (r(x))^2\,dx$; $A = 2\pi\int_a^b r(x)\sqrt{1 + r'(x)^2}\,dx$
- Taylor Th$\underline{\text{m}}$: $f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{1}{N!}\int_{x_0}^{x}(x-t)^N f^{(N+1)}(t)\,dt$
- Lagrange Remainder: $R_N(x) = \frac{(x-x_0)^{N+1}}{(N+1)!}f^{(N+1)}(c)$

- Fourier: $S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(\mathbf{a_n}\cos nx + \mathbf{b_n}\sin nx)$
- $\mathbf{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$; $\mathbf{b_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$
- Even: $f(-x) = f(x)$; Odd: $f(-x) = -f(x)$
- $\int_{-a}^{a}(\text{Odd}) = 0$
- $f(x^{\pm}) = \lim_{\zeta\to x^{\pm}} f(\zeta)$
- f continuous when $f(x_0) = f(x_0^-) = f(x_0^+)$
- Piecewise Continuous (PC): not Continuous for finite many points.
- PC Differentiable: PC, Diffable at most points, derivative is PC.
- Periodic Extension: $\widetilde{f}(x + 2k\pi) := f(x)$, $(k \in \mathbb{Z})$ and $-\pi \le x < \pi$
- Fourier Convergence Th$\underline{^m}$: Fourier Series of $\underline{f}$ converges to $\lim_{N\to\infty} S_N(x) = S(x) = \frac{1}{2}(\widetilde{f}(x^+) + \widetilde{f}(x^-))$
- Parseval Th$\underline{^m}$: $\frac{1}{\pi}\int_{\pi}^{\pi} f^2(x)\,dx = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$
- R.Zeta fn: $\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p\text{prime}}\frac{1}{1-p^{-s}}$, $(s > 1)$
- Riemann Conjecture: $\zeta(s)$, $(s \in \mathbb{C}) \implies \zeta(s) = 0$ at $\Re(z) = 1/2$
- $\frac{\partial f}{\partial x} = \frac{f(x+h,y)-f(x,y)}{h}$
- If $f(x, y)$ defined for open region: $f_{xy}(a, b) = f_{yx}(a, b)$
- $F(t) = f[\underline{r}(t)] = f[x(t), y(t)]$
- $F'(t) = x'(t)f_x[x(t), y(t)] + y'(t)f_y[x(t), y(t)]$
- Gradient: $\nabla f := \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j}$
- Directional Deriv: $D_{\underline{u}} f(\underline{r}_0) := \lim_{h\to 0}\frac{f(\underline{r}_0 + h\underline{u}) - f(\underline{r}_0)}{h}$
- $\frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)}$; $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$; $\left(\frac{\partial z}{\partial y}\right)_x = -\frac{F_y}{F_z}$
- Laplacian of a fn: $\Delta f \equiv \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$
- Polar Laplacian: $\Delta f \equiv^2 f = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}$
- Fubini Th$\underline{^m}$ (weak):
  $R := \{(x, y) \in \mathbb{R}^2 | a \le x \le b,\ c \le y \le d\}$,
  $\iint_R f\,dA = \int_a^b(\int_c^d f(x, y)\,dy)\,dx = \int_c^d(\int_a^b f(x, y)\,dx)\,dy$
- $R_1 = \{(x, y) \in \mathbb{R}^2 | a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$
- $R_2 = \{(x, y) \in \mathbb{R}^2 | c \le y \le d,\ \psi_1(x) \le x \le \psi_2(x)\}$
- $\iint_{R_1} f\,dA = \int_a^b(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy)\,dx$
- $\iint_{R_2} f\,dA = \int_c^d(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\,dx)\,dy$
- Centroid of region:
  $\bar{x} = \frac{1}{A(R)}\iint_R x\,dA$, $\bar{y} = \frac{1}{A(R)}\iint_R y\,dA$; $A(R) =$ Area.
- $\iint_R f(x, y)\,dxdy = \iint_S f[x(u, v), y(u, v)]|J(u, v)|\,dudv$
- $J = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$
- SP at $\nabla f(x_0, y_0) = 0$; SaddleP is a SP but not Extrema.
- Hessian Matrix: $H(x, y) := \begin{pmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{pmatrix}$
- $\Delta(x_0, y_0) = \det H(x_0, y_0)$
- $\Delta > 0, f_{xx} > 0 \implies$ minimum
  $\Delta > 0, f_{xx} < 0 \implies$ maximum $\Delta < 0 \implies$ Saddle
- Constrained SP:
  $D_{\underline{u}} f(x_0, y_0) = \underline{u} \cdot \nabla f(x_0, y_0) = 0\ \forall \underline{u}$ tangent to C
  $\underline{u} \perp \nabla f$; $\underline{u} \perp \nabla g$ $(g(x, y) = C) \implies \nabla f = \lambda \nabla g$
  3 Conditions:
  $f_x = \lambda g_x$, $f_y = \lambda g_y$, $g(x, y) = c$
- Lagrange constrained SP:
  $F(x, y; \lambda) = f(x, y) - \lambda(g(x, y) - c)$ then $\nabla F = (0, 0, 0)$
- A set of functions are linearly independent if none of the functions are linear combinations of theothers.
- Any linear combination of two solutions is a solution.
- Ansatz: A guess for the form of a solution to a problem such as a DE.
- We often use $y = e^{\lambda x}$
- If y_p is a particular solution then any other solution can be written as $y = y_p + u$ where u isthe homogeneous equation.
- Linearity: If $L[y_1] = r_1$ and $L[y_2] = r_2$ then $y := y_1 + y_2$ is a solution of$L[y] = r_1 + r_2$.
- If $\lambda = s, r \in \mathbb{R}$ then: $y = Ae^{sx} + Be^{rx}$,
  If $\lambda = p$ then: $y = (A + Bx)e^{px}$;
  If $\lambda = z, z^*$ then: $y = e^{\Re z}(A\cos\Im z + B\sin\Im z)$
- $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$
- Level surface: $\{(x, y, z) \in \mathbb{R}^3 | F(x, y, z) = c\}$
- Tangent Vector surface: $\nabla f(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0) = 0$
- Normal Vector surface: $\nabla f(\underline{x}_0) \cdot (\underline{x}) = 0$