Functions of a Complex Variable

Henning Bostelmann[1]
Department of Mathematics, University of York,
Heslington, York, YO10 5DD, UK

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[1]These notes are based on earlier ones by my colleagues, Roger Colbeck and Benoit Vicedo.

Contents

Chapter 1

Complex sequences and series

Before getting to the main topic of this module, we will go over a few important things. If you are not confident with using complex numbers, you should revise this topic now – being able to use and manipulate complex numbers quickly and with confidence is essential for this module.

1.1 Review of complex numbers

First let us recall some simple facts about complex numbers:

1. A complex number $z \in \mathbb{C}$ can be written as $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$, and $\mathrm{i}^2 = -1$. The real part of z is $x = \mathrm{Re}(z)$ and the imaginary part $y = \mathrm{Im}(z)$.

2. The polar form of z is $z = r\,\mathrm{e}^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta)$ with $r > 0$ and $\theta \in \mathbb{R}$. Thus $z = x + \mathrm{i}y$ for $x = r\cos\theta$ and $y = r\sin\theta$.

3. The modulus (or absolute value) of z is $|z| = \sqrt{x^2 + y^2} = r$. This is the distance between 0 and z.

   The equation $|z| = r$ is an equation for the circle of radius r centred at 0.

   Note that $|\,\mathrm{e}^{\mathrm{i}\theta}\,| = 1$ for $\theta \in \mathbb{R}$.

4. The argument of z is $\arg(z) = \theta$, the anticlockwise angle between the positive real axis and the line connecting 0 to z (note that $\arg(0)$ is undefined). Because $\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(\theta+2n\pi)}$ for any integer n, to make arg a well-defined function we have to choose a set of values for this to take. Common choices are $-\pi < \arg(z) \leq \pi$ and $0 \leq \arg(z) < 2\pi$. The choice $-\pi < \arg(z) \leq \pi$ is usually referred to as the *principal argument* and is the one we will use in this course (unless otherwise stated).

5. The complex conjugate of z is $\overline{z} = x - \mathrm{i}y$.

Note that $z\overline{z} = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2 = |z|^2$. Thus, $|z|^2 = z\overline{z}$.

For $z, w \in \mathbb{C}$, $|zw| = |z|\,|w|$ and $|z/w| = |z|/|w|$ if $w \neq 0$. These follow from $|zw|^2 = zw\overline{zw} = z\overline{z}w\overline{w} = |z|^2|w|^2$.

Complex numbers also have a geometric interpretation in the complex plane:

1. Addition and subtraction in the complex plane are identical to these operations for two-dimensional vectors.

   Therefore $|z_1 - z_2|$ is the distance between z_1 and z_2 in the complex plane.

2. To multiply and divide write $w = R\,\mathrm{e}^{\mathrm{i}\varphi}$ and $z = r\,\mathrm{e}^{\mathrm{i}\theta}$. Then $wz = Rr\,\mathrm{e}^{\mathrm{i}(\varphi+\theta)}$, and so:

   - $w \mapsto wz$ is dilation by $|z|$ and rotation through angle θ.
   - $w \mapsto w/z$ is dilation by $1/|z|$ and rotation through angle $-\theta$.

1.1.1 Inequalities

Lemma 1.1. *For $z \in \mathbb{C}$, we have $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$.*

Proof. Let $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$. Then $|z| = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = |x| = |\operatorname{Re} z|$. Similarly, $|z| \geq |y| = |\operatorname{Im} z|$. □

Corollary 1.2. *For $z \in \mathbb{C}$ we have $-|z| \leq \operatorname{Re} z \leq |z|$ and $-|z| \leq \operatorname{Im} z \leq |z|$.*

Lemma 1.3 (Triangle inequality)**.** *For z_1, $z_2 \in \mathbb{C}$ we have $|z_1 + z_2| \leq |z_1| + |z_2|$.*

Proof. Consider

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2 \\
&= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\overline{z_2}) \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1\overline{z_2}| && \text{using Corollary 1.2} \\
&= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\
&= (|z_1| + |z_2|)^2 ,
\end{aligned}$$

from which the claim follows by taking square roots. □

Theorem 1.4 (Generalized triangle inequality)**.** *If z_1, z_2,...,$z_n \in \mathbb{C}$, then*

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n| .$$

Proof. This follows by repeated application of the triangle inequality:

$$\begin{aligned}
|z_1 + z_2 + \cdots + z_{n-1} + z_n| &\leq |z_1 + z_2 + \cdots + z_{n-1}| + |z_n| \\
&\leq |z_1 + z_2 + \cdots + z_{n-2}| + |z_{n-1}| + |z_n| \\
&\leq |z_1| + |z_2| + \cdots + |z_n| .
\end{aligned}$$ □

Thinking geometrically, the generalized triangle inequality is fairly obvious.

Corollary 1.5 (Reverse triangle inequality)**.** *For $z_1, z_2 \in \mathbb{C}$ we have $\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$.*

Proof. By the triangle inequality $|z_1| = |(z_1 + z_2) - z_2| \leq |z_1 + z_2| + |z_2|$, which rearranges to $|z_1| - |z_2| \leq |z_1 + z_2|$.

Since the right hand side is symmetric it also holds that $|z_2| - |z_1| \leq |z_1 + z_2|$ and hence, since $\big||z_1| - |z_2|\big|$ is either $|z_1| - |z_2|$ or $|z_2| - |z_1|$, the claim follows. □

1.2 Complex sequences

Let (z_n) denote the sequence of numbers $z_1, z_2, z_3, \ldots$ (or sometimes $z_0, z_1, z_2, \ldots$).

The definition of convergence of a complex sequence is based on that for real sequences, which we now recall

Definition 1.6. For a real sequence (x_n), we say that x_n *converges* to x as n tends to infinity, (or "$x_n \to x$ as $n \to \infty$" for short) iff $|x_n - x| \to 0$ as $n \to \infty$. An alternative way of writing this is $\lim\limits_{n\to\infty} x_n = x$.

This definition seems a bit circular; in fact, it defines the convergence of a general real sequence to any limit using the convergence of a sequence of nonnegative reals, $a_n = |x_n - x|$, to zero. From the Calculus module, we're familiar with that $a_n \to 0$ heuristically means. More quantitatively, this was defined in the Real Analysis module: $\rho_n \to 0$ is shorthand for saying that for every real $\varepsilon > 0$, there exists a number $N \in \mathbb{N}$ such that for all $n > N$, we have $a_n < \varepsilon$.

Definition 1.7. For a complex sequence (z_n), we say that z_n *converges* to z as n tends to infinity (or "$z_n \to z$ as $n \to \infty$" for short) iff $|z_n - z| \to 0$ as $n \to \infty$. An alternative way of writing this is $\lim_{n\to\infty} z_n = z$.

Thus a complex sequence has a limit if and only if there exists a complex number z such that the *real* sequence (a_n) defined by $a_n := |z_n - z|$ converges to 0 as n tends to ∞.

Example 1. If $z \in \mathbb{C}$ with $|z| < 1$, then $\lim_{n\to\infty} z^n = 0$.

Proof. First note that if $r \in \mathbb{R}$ with $|r| < 1$ then $\lim_{n\to\infty} r^n = 0$. (See Lemma A.1.)

Then let $r = |z|$ and note that $\lim_{n\to\infty} |z^n - 0| = \lim_{n\to\infty} |z|^n = \lim_{n\to\infty} r^n = 0$, Thus, $\lim_{n\to\infty} z^n = 0$. □

We recall the sandwich theorem (see Theorem A.2) which states that for real sequences (a_n) and (b_n) if $0 \le a_n \le b_n$ for all sufficiently large n, and if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$. Here "for all sufficiently large n" means "for all $n > N$ for some $N \in \mathbb{N}$".

Lemma 1.8. *For a complex sequence (z_n), $\lim_{n\to\infty} z_n = z$ if and only if $\lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} z$ and $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} z$.*

Proof. First we show that if $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} \operatorname{Re} z = \operatorname{Re} z$ and $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} z$. By definition $\lim_{n\to\infty} |z_n - z| = 0$. Using $0 \le |\operatorname{Re} z_n - \operatorname{Re} z| = |\operatorname{Re}(z_n - z)| \le |z_n - z|$ and the Sandwich Theorem, we find $\lim_{n\to\infty} |\operatorname{Re} z_n - \operatorname{Re} z| = 0$. A similar argument shows $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} z$.

Conversely, if $\lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} z$ and $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} z$ then given $\varepsilon > 0$ let N be such that for all $n > N$ we have $|\operatorname{Re} z_n - \operatorname{Re} z| < \varepsilon/\sqrt{2}$ and $|\operatorname{Im} z_n - \operatorname{Im} z| < \varepsilon/\sqrt{2}$. Thus, for $n > N$ we have

$$|z_n - z| = \sqrt{|\operatorname{Re} z_n - \operatorname{Re} z|^2 + |\operatorname{Im} z_n - \operatorname{Im} z|^2} < \sqrt{\tfrac{1}{2}\varepsilon^2 + \tfrac{1}{2}\varepsilon^2} = \varepsilon$$

as required. □

Remark 1.9. If we were thinking of $\mathbb{C}$ as being a real two-dimensional vector space, then the definition of a limit would be that the vector converges to the limit component-wise.

This lemma implies that the algebra of limits is the same as for real sequences.

Theorem 1.10 (Algebra of limits for complex sequences)**.** *If $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$, then*

1. $\lim_{n\to\infty} (z_n + w_n) = z + w$,
2. $\lim_{n\to\infty} z_n w_n = zw$,
3. *If $w \ne 0$, then* $\lim_{n\to\infty} z_n/w_n = z/w$,
4. *For any $c \in \mathbb{C}$, we have* $\lim_{n\to\infty} cz_n = cz$.

Proof of Property 2. Let $z_n = x_n + \mathrm{i}y_n$, $w_n = u_n + \mathrm{i}v_n$, $z = x + \mathrm{i}y$, and $w = u + \mathrm{i}v$, with $x_n, y_n, u_n, v_n, x, y, u, v \in \mathbb{R}$.

By Lemma 1.8, $\lim\limits_{n\to\infty} z_n = z$ and $\lim\limits_{n\to\infty} w_n = w$ implies $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$, $\lim\limits_{n\to\infty} u_n = u$ and $\lim\limits_{n\to\infty} v_n = v$. Now,

$$\begin{aligned} z_n w_n &= (x_n + \mathrm{i}y_n)(u_n + \mathrm{i}v_n) \\ &= (x_n u_n - y_n v_n) + \mathrm{i}(y_n u_n + x_n v_n). \end{aligned}$$

Using the analogous result for real sequences (see Theorem A.3) we have

$$\lim_{n\to\infty} (x_n u_n - y_n v_n) = xu - yv, \quad \text{and} \quad \lim_{n\to\infty} (y_n u_n + x_n v_n) = yu + xv.$$

Thus,

$$\lim_{n\to\infty} \mathrm{Re}(z_n w_n) = xu - yv = \mathrm{Re}(zw), \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Im}(z_n w_n) = yu + xv = \mathrm{Im}(zw).$$

Using Lemma 1.8 again, this implies $\lim\limits_{n\to\infty} z_n w_n = zw$. □

Some of the remaining proofs are left as exercises, as is the proof of the following lemma.

Lemma 1.11. *If $\lim\limits_{n\to\infty} z_n = z$, then $\lim\limits_{n\to\infty} |z_n| = |z|$ and $\lim\limits_{n\to\infty} \bar{z}_n = \bar{z}$.*

1.3 Complex series

A series is the sum of the terms in a sequence.

Definition 1.12. A complex series $\sum_{n=0}^{\infty} z_n$ is said to be *convergent* if there exists a $Z \in \mathbb{C}$ such that

$$\lim_{N\to\infty} \sum_{n=0}^{N} z_n = Z$$

(the series is then said to *converge to* Z). Note that the notation $\sum_{n=0}^{\infty} z_n$ is short for $\lim\limits_{N\to\infty} \sum_{n=0}^{N} z_n$.

Definition 1.13. A complex series $\sum_{n=0}^{\infty} z_n$ is said to be *absolutely convergent* if the real series $\sum_{n=0}^{\infty} |z_n|$ is convergent.

The following theorem is analogous to a standard result in real analysis (see Theorem A.7). The proof uses the bounded monotone convergence theorem (see Theorem A.6).

Theorem 1.14. *If $\sum_{n=0}^{\infty} |z_n| = Z$ for some real Z, then there exists $Z' \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} z_n = Z'$. (That is, absolute convergence implies convergence for series[1].)*

Proof. Let $z_n = x_n + \mathrm{i}y_n$, with $x_n, y_n \in \mathbb{R}$, so that

$$\sum_{n=0}^{N} z_n = \sum_{n=0}^{N} x_n + \mathrm{i} \sum_{n=0}^{N} y_n.$$

Let $S_N := \sum_{n=0}^{N} |x_n|$. We have $|x_n| = |\mathrm{Re}\, z_n| \le |z_n|$, so

$$0 \le S_N \le \sum_{n=0}^{N} |z_n| \le \sum_{n=0}^{\infty} |z_n| = Z.$$

[1]But not of course for sequences, where it is the other way round (cf. Lemma 1.11).

Thus, (S_N) is a bounded and monotonically increasing real sequence. Hence, by the bounded monotone convergence theorem (Theorem A.6) we conclude that $\lim\limits_{N\to\infty} S_N = \sum_{n=0}^{\infty}|x_n|$ exists.

Since absolutely convergent *real* series are convergent (see Theorem A.7), we conclude that $\sum_{n=0}^{\infty} x_n$ converges. Thus, the sequence (X_N) of partial sums $X_N := \sum_{n=0}^{N} x_n$ converges.

Define $Y_N := \sum_{n=0}^{N} y_n$. The convergence of the sequence (Y_N) can be proved similarly.

Now, if $Z_N := \sum_{n=0}^{N} z_n$, then $Z_N = X_N + \mathrm{i}Y_N$. Since (X_N) and (Y_N) converge, (Z_N) converges (Lemma 1.8). Hence there exists $Z' \in \mathbb{C}$ such that $\lim\limits_{N\to\infty} Z_N = \sum_{n=0}^{\infty} z_n = Z'$. $\square$

Lemma 1.15. *Let $\sum_{n=0}^{\infty} z_n$ be an absolutely convergent series. Then the following inequality holds*

$$\left|\sum_{n=0}^{\infty} z_n\right| \le \sum_{n=0}^{\infty} |z_n|.$$

Proof. Since $\sum_{n=0}^{\infty} z_n$ is absolutely convergent, $\sum_{n=0}^{\infty}|z_n| = Z$ for some $Z \in \mathbb{R}$. Define $S_N := \sum\limits_{n=0}^{N} z_n$. By the triangle inequality, $|S_N| \le \sum\limits_{n=0}^{N}|z_n| \le \sum\limits_{n=0}^{\infty}|z_n| = Z$ for all N. Then also $\lim_{n\to\infty}|S_N| \le Z$ (Theorem A.3(5)). But the left-hand side is equal to $|\lim_{n\to\infty} S_N|$ (Lemma 1.11), noting that we know existence of $\lim_{n\to\infty} S_N$ from Theorem 1.14. $\square$

For the *real* series $\sum_{n=0}^{\infty}|z_n|$, various convergence tests are available. Since the convergence of $\sum_{n=0}^{\infty}|z_n|$ means absolute convergence of $\sum_{n=0}^{\infty} z_n$, these tests can be used as tests for absolute convergence of $\sum_{n=0}^{\infty} z_n$.

Lemma 1.16 (Comparison test)**.** *Let (z_n) and (w_n) be complex sequences. If $|z_n| \le |w_n|$ for all $n > N_0$ for some $N_0 \in \mathbb{N}$ and if $\sum_{n=0}^{\infty} w_n$ is absolutely convergent, then $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.*

Lemma 1.17 (Ratio Test)**.** *Let (z_n) be a complex sequence. If $\lim\limits_{n\to\infty}|z_{n+1}/z_n| = r$, then $\sum_{n=0}^{\infty} z_n$ is absolutely convergent if $r < 1$ and divergent if $r > 1$ or $r = \infty$. [If $r = 1$ then this test is inconclusive.]*

Example 2. Show that $\sum\limits_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$ if $|z| < 1$ and that $\sum_{n=0}^{\infty} z^n$ diverges if $|z| \ge 1$.

Solution. If $|z| \ge 1$, then the sequence (z^n) does not converge to 0, so $\sum_{n=0}^{\infty} z^n$ cannot be convergent (see exercises). Now, suppose $|z| < 1$. If $S_N = 1 + z + \cdots + z^N$, then $(1-z)S_N = 1 - z^{N+1}$, i.e.

$$1 + z + \cdots + z^N = \sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1-z}.$$

Since $z^{N+1} \to 0$ as $N \to \infty$ (cf. Example 1), using the algebra of limits we find

$$\sum_{n=0}^{\infty} z^n = \lim_{N\to\infty}\sum_{n=0}^{N} z^n = \lim_{N\to\infty}\frac{1-z^{N+1}}{1-z} = \frac{1}{1-z}.$$

Remark 1.18. This is consistent with the ratio test. We have

$$\lim_{n\to\infty}\left|\frac{z^{n+1}}{z^n}\right| = |z|.$$

So if $|z| < 1$, then the series $\sum_{n=0}^{\infty} z^n$ converges and if $|z| > 1$ this series diverges.

Absolutely convergent series behave "nicely" when forming combinations, etc.; for example, the following two theorems can quite easily be deduced from the bounded monotone convergence theorem.

Theorem 1.19. *If $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ are absolutely convergent series, then so is*

$$\sum_{n=0}^{\infty} (\alpha z_n + \beta w_n) \qquad \text{for any } \alpha, \beta \in \mathbb{C}\,.$$

Theorem 1.20. *If (z_n) is a sequence such that $\sum_{n=0}^{\infty} z_n$ is an absolutely convergent series, then any subsequence*[2] *also produces an absolutely convergent series.*

Remark 1.21. The same is not necessarily true for *conditionally* convergent series (i.e., series that converge, but do not converge absolutely). For instance, the series $1 - 1/2 + 1/3 - 1/4 + 1/5 - \ldots$ is conditionally convergent, but $1 + 1/3 + 1/5 + \ldots$ does not converge.

1.4 Limits and convergence for complex-valued functions

Proceeding from complex sequences, we now discuss limits of complex-valued functions. If $f : \mathbb{R} \to \mathbb{C}$ is a complex-valued function of a *real* variable, we proceed completely analogously: We say that $f(t) \to z$ as $t \to t_0$ iff $|f(t) - z| \to 0$ in this limit, referring only to convergence of *real-valued* functions (as known from previous modules); it is equivalent (much like in Lemma 1.8) that $\operatorname{Re} f(t) \to \operatorname{Re} z$ and $\operatorname{Im} f(t) \to \operatorname{Im} z$. Building on these definitions, we can also introduce derivatives of f by t, and $\frac{d}{dt} f = \frac{d}{dt} \operatorname{Re} f + i \frac{d}{dt} \operatorname{Im} f$ as expected. For $f : \mathbb{R}^2 \to \mathbb{C}$ we can likewise consider partial derivatives; again these act on real and imaginary parts separately.

In this module, however, we want to study complex-valued functions of a *complex* variable, i.e., functions that map from $\mathbb{C}$ to $\mathbb{C}$. For example, $f(z) = z^2$, which gives $f(2) = 4$, $f(3\mathrm{i}) = -9$, $f(2 + \mathrm{i}) = 3 + 4\mathrm{i}$, $f(\sqrt{2} + \sqrt{3}\,\mathrm{i}) = -1 + 2\sqrt{6}\,\mathrm{i}$, etc. We want to explain how convergence works for these complex functions, for which we first need some preparations.

Definition 1.22. An *open disc* of radius $r > 0$ centred at $a \in \mathbb{C}$ is defined by

$$\mathrm{D}(a; r) := \{z \in \mathbb{C} : |z - a| < r\}\,,$$

and a *punctured disc* of radius $r > 0$ centred at $a \in \mathbb{C}$ is defined by

$$\mathrm{D}'(a; r) := \{z \in \mathbb{C} : 0 < |z - a| < r\}\,.$$

The crucial thing is that points with $|z - a| = r$ are excluded from both discs and $z = a$ is excluded from the punctured disc.

Definition 1.23. A set $S \subseteq \mathbb{C}$ is said to be *open* iff for all $a \in S$ there is an open disc $\mathrm{D}(a; r)$ for some $r > 0$ such that $\mathrm{D}(a; r) \subseteq S$.

Example. The set $S = \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) \leq 1,\ 0 < \operatorname{Im}(z) < 1\}$ is not open because the point $1 + \mathrm{i}/2$ is in the set, but there is no open disc around this point that is in S. [An open disc of radius $r > 0$ about this point must contain $1 + \varepsilon + \mathrm{i}/2 \notin S$ for $0 < \varepsilon < r$.]

Below we consider functions that map from open subsets of the complex numbers to complex numbers, i.e., $f : G \to \mathbb{C}$, where $G \subseteq \mathbb{C}$ is open, unless otherwise stated.

[2] Recall that a subsequence is a sequence with some of its terms deleted.

Definition 1.24. We say that $f(z)$ *converges to* c as z tends to a ("$f(z) \to c$ as $z \to a$" or "$\lim_{z\to a} f(z) = c$" for short) if $|f(z) - c|$ converges to 0 as $|z - a|$ tends to 0, which means that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - c| < \varepsilon$ whenever z satisfies $0 < |z - a| < \delta$.

Remark 1.25. Note that $|f(z) - c|$ is a real valued function, so Definition 1.24 is using the definition of convergence of real valued functions.

Remark 1.26. Putting $z = a$ is not allowed, since $0 < |z - a|$ is a strict inequality. In particular, this means that a need not be part of the set G, the domain of f. The δ in the definition is thus the radius of a punctured disc centred on a.

Remark 1.27 (Important)**.** The variable z can approach a in many different ways on $\mathbb{C}$. If $\lim\limits_{z\to a} f(z) = c$, $f(z)$ must tend to c *irrespective of the way z approaches a.*

For example, for *real* t tending to zero, we have

- $\lim\limits_{t\to 0} f(a + t) = c$ (limit in the real direction).
- $\lim\limits_{t\to 0} f(a + \mathrm{i}t) = c$ (limit in the imaginary direction).
- $\lim\limits_{t\to 0} f(a + \alpha t) = c$ for any non-zero $\alpha \in \mathbb{C}$ (limit along a generic straight line).
- $\lim\limits_{t\to 0} f(a + t\, \mathrm{e}^{2\pi \mathrm{i}t}) = c$ (limit along a spiral).
- (And uncountably many more examples).

Example 3. The real function $f(x) = \mathrm{e}^{-1/x^2}$ converges to 0 as x tends to 0, but the complex function $f(z) = \mathrm{e}^{-1/z^2}$ has no limit as z tends to 0 (since as z tends to 0 along the imaginary axis $f(\mathrm{i}y) = \mathrm{e}^{1/y^2}$ and this becomes arbitrarily large as y tends to 0).

One might imagine that the limit being the same along any straight line is sufficient for the existence of the limit, but this is **not** the case. An explicit counterexample is the function $f(z) = x^2y/(x^4 + y^2)$, where $z = x + \mathrm{i}y)$ with $x, y \in \mathbb{R}$. For this function the limits as $z \to 0$ on any straight line through $z = 0$ exist and are equal, but $\lim_{z\to 0} f(z)$ does not exist, as can be seen by considering the limit along $y = x^2$.

Definition 1.28. The function f is said to be *continuous* at $a \in G$ if and only if $\lim\limits_{z\to a} f(z) = f(a)$.

Example 4. Show that the function $\mathbb{C} \to \mathbb{C} : f(z) = \bar{z}$ is continuous at $z = z'$ for every $z' \in \mathbb{C}$.

Solution. We need to show that $\lim\limits_{z\to z'} \bar{z} = \bar{z}'$. Using the definition of convergence, this is equivalent to showing that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\bar{z} - \bar{z}'| < \varepsilon$ whenever $0 < |z - z'| < \delta$. If we take $\delta = \varepsilon$, this follows immediately from $|\bar{z} - \bar{z}'| = |z - z'|$.

The algebra of limits work as in the real case.

Theorem 1.29 (Algebra of limits for complex functions)**.** *If $\lim\limits_{z\to a} f(z) = c$ and $\lim\limits_{z\to a} g(z) = d$, then*

- $\lim\limits_{z\to a} (\alpha\, f(z) + \beta\, g(z)) = \alpha c + \beta d$ *for* $\alpha, \beta \in \mathbb{C}$
- $\lim\limits_{z\to a} f(z)\, g(z) = cd$
- $\lim\limits_{z\to a} f(z)/g(z) = c/d$ *provided* $d \neq 0$.

It is also worth noting that $\lim\limits_{z\to a} f(z) = c$ is equivalent to $\lim\limits_{z\to a} \operatorname{Re} f(z) = \operatorname{Re} c$ and $\lim\limits_{z\to a} \operatorname{Im} f(z) = \operatorname{Im} c$.

Chapter 2

Differentiation of complex functions

2.1 Complex Differentiability

Definition 2.1. Let $G \subseteq \mathbb{C}$ be open. A complex-valued function $f : G \to \mathbb{C}$ is said to be *differentiable* at $z \in G$ if

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \text{ exists.}$$

Remark 2.2 (This is important)**.** For f to be differentiable, this limit must be the same irrespective of the way $h \in \mathbb{C}$ approaches 0 (cf. Remark 1.27).

Example 5. Let $f(z) = z^2$. Then

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h \to 0} \frac{2zh + h^2}{h} = \lim_{h \to 0} (2z + h) = 2z.$$

Hence f is differentiable at all $z \in \mathbb{C}$ and $f'(z) = 2z$.

Example 6. Let $f(z) = z^n$, $n \in \mathbb{N}$. Then f is differentiable for all $z \in \mathbb{C}$ and $f'(z) = nz^{n-1}$.

To prove this, first note that for $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

(This can be confirmed by multiplying out the right-hand side.) Thus letting $a = z + h$ and $b = z$,

$$\frac{f(z+h) - f(z)}{h} = \frac{(z+h)^n - z^n}{h} = (z+h)^{n-1} + (z+h)^{n-2}z + \cdots + (z+h)z^{n-2} + z^{n-1}.$$

Each term on the right-hand side tends to z^{n-1} as $h \to 0$ and there are n of them. Hence

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = nz^{n-1}.$$

Thus, f is differentiable at all $z \in \mathbb{C}$ and $f'(z) = nz^{n-1}$.

There are, however, many functions that appear "smooth" (say, in a plot) but which are *not* complex differentiable. The prime example is:

Example 7. The function $f(z) = \bar{z}$ is nowhere differentiable.

This is seen as follows: We have

$$\frac{f(z+h)-f(z)}{h}=\frac{\overline{z+h}-\bar{z}}{h}=\frac{\bar{h}}{h}.$$

But the right-hand side does not converge as $h \to 0$ in the sense of our definition: Taking t real, for $h=t$ we have $\bar{h}/h=1$, and for $h=it$ we have $\bar{h}/h=\overline{it}/it=-1$; both expressions converge as $t \to 0$ in the real numbers, but they converge to *different values*. Hence f is not differentiable at any $z \in \mathbb{C}$.

Example 8. The function $f(z)=|z|^2=z\overline{z}$ is differentiable only at $z=0$.

We have

$$\frac{f(z+h)-f(z)}{h}=\frac{(z+h)(\overline{z}+\overline{h})-z\overline{z}}{h}=\frac{h\overline{z}+z\overline{h}+h\overline{h}}{h}=z\frac{\overline{h}}{h}+\overline{z}+\overline{h}=: r(z,h).$$

If we take the limit as h tends to 0 along the real axis by letting $h=t\in\mathbb{R}$, then

$$\lim_{t\to 0} r(z,t)=\lim_{t\to 0}(z+\overline{z}+t)=z+\overline{z}.$$

On the other hand, if we take the limit as h tends to 0 along the imaginary axis by letting $h=\mathrm{i}t$, $t\in\mathbb{R}$, then

$$\lim_{t\to 0} r(z,\mathrm{i}t)=\lim_{t\to 0}\left(z\cdot\frac{-\mathrm{i}t}{\mathrm{i}t}+\overline{z}-\mathrm{i}t\right)=\lim_{t\to 0}\left(-z+\overline{z}-\mathrm{i}t\right)=-z+\overline{z}.$$

For f to be differentiable, these two limits must agree, so the only possible place for differentiability is $z=0$. At $z=0$, we have

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\overline{h}=0\,.$$

[This is a special case of Example 4.] Hence $f'(0)$ exists and f is differentiable only at 0 with $f'(0)=0$.

2.1.1 Rules for differentiation

The standard rules for differentiation (product rule, quotient rule, chain rule) all hold for complex functions too.

Lemma 2.3 (Differentiation rules)**.** *Let $G\subseteq\mathbb{C}$ be open and f and g be complex-valued functions $G\to\mathbb{C}$ that are differentiable at z then:*

1. *The function $\alpha f+\beta g$ with $\alpha,\beta\in\mathbb{C}$ is differentiable at z with derivative $\alpha f'(z)+\beta g'(z)$ [the derivative is* linear*].*
2. *The function fg is differentiable at z with derivative $f'(z)\,g(z)+f(z)\,g'(z)$ [the derivative obeys the* product rule*].*
3. *If $g(z)\neq 0$, the function f/g is differentiable at z with derivative*

$$\frac{f'(z)\,g(z)-f(z)\,g'(z)}{g(z)^2}$$

[the derivative obeys the quotient rule*].*

Lemma 2.4 (The chain rule). *Let $G_1, G_2 \subseteq \mathbb{C}$ be open and $g : G_1 \to G_2$ be differentiable at z and $f : G_2 \to \mathbb{C}$ be differentiable at $g(z)$. The composite function $f \circ g$ defined by $(f \circ g)(z) := f(g(z))$ is differentiable at z and its derivative there is given by*

$$(f \circ g)'(z) = f'(g(z))\, g'(z).$$

Example 9. We have already shown that z^n is differentiable with derivative nz^{n-1} for all $z \in \mathbb{C}$. Using linearity we can conclude that any polynomial

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots c_N z^N$$

is differentiable at all $z \in \mathbb{C}$ and its derivative is

$$p'(z) = c_1 + 2c_2 z + \cdots N c_N z^{N-1}.$$

Example 10. Using the quotient rule, we conclude that any rational function $p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials of z, is differentiable at all $z \in \mathbb{C}$ except where $q(z) = 0$ and its derivative is

$$\frac{p'(z)\, q(z) - p(z)\, q'(z)}{q(z)^2}.$$

2.2 Holomorphic functions

Definition 2.5. Let $G \subseteq \mathbb{C}$ be open. A function f of a complex variable is said to be *holomorphic* on G if f is differentiable at all $z \in G$.

Definition 2.6. A function f of a complex variable is said to be *holomorphic at a point* $a \in \mathbb{C}$ if f is differentiable for all z in the open disc $\mathrm{D}(a; r)$ for some $r > 0$.

Being holomorphic at a point is different from being differentiable at a point; being holomorphic at a point means the function is differentiable in an open set containing that point.

As an example, recall (from Example 8) that the function $f(z) = |z|^2$ is differentiable at $z = 0$ only. It is hence not holomorphic at $z = 0$ (nor anywhere else).

2.3 The Cauchy-Riemann equations

In this subsection we identify the complex plane with the xy-plane through $z = x + \mathrm{i}y$.

Theorem 2.7. *Let $f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$ where u and v are real. If f is differentiable at $z = x + \mathrm{i}y$, then*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(These equations are called the Cauchy-Riemann equations.)

Proof. If f is differentiable at $z = x + \mathrm{i}y$, then by definition

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \tag{2.1}$$

must exist. By taking the limit in the real direction, with $h = t$, $t \in \mathbb{R}$, we find

$$\lim_{t \to 0} \frac{f(x + t + \mathrm{i}y) - f(x + \mathrm{i}y)}{t} = \frac{\partial}{\partial x} f(x + \mathrm{i}y).$$

On the other hand, if we take the limit (2.1) in the imaginary direction, with $h = \mathrm{i}t$, $t \in \mathbb{R}$, we find

$$\lim_{t\to 0} \frac{f(x + \mathrm{i}(y + t)) - f(x + \mathrm{i}y)}{\mathrm{i}t} = -\mathrm{i}\frac{\partial}{\partial y} f(x + \mathrm{i}y). \tag{2.2}$$

Since f is differentiable at $z = x + \mathrm{i}y$, these limits must agree. Hence

$$\frac{\partial}{\partial x} f(x + \mathrm{i}y) = -\mathrm{i}\frac{\partial}{\partial y} f(x + \mathrm{i}y).$$

Substituting $f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$ into this equation we find

$$\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = -\mathrm{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Comparing the real and imaginary parts gives the Cauchy-Riemann equations. □

Example 11. Let $f(z) = z^2$ (which we have already shown is differentiable for all $z \in \mathbb{C}$). Now, $f(x + \mathrm{i}y) = (x + \mathrm{i}y)^2 = (x^2 - y^2) + 2\mathrm{i}xy$. Thus, the functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ must satisfy the Cauchy-Riemann equations everywhere. Indeed

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Example 12. We have previously shown (Example 8) that $f(z) = |z|^2$ is differentiable only at $z = 0$. Now, since $f(x + \mathrm{i}y) = x^2 + y^2$, we have $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$, and so

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

Hence, for the Cauchy-Riemann equations to be satisfied, we must have $(x, y) = (0, 0)$, i.e. $z = 0$. This implies that $f(z) = |z|^2$ cannot be differentiable except at $z = 0$. This agrees with the previous conclusion.

Remark 2.8. Note that the Cauchy-Riemann equations can be written

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) f(x + \mathrm{i}y) = 0.$$

Remark 2.9. The Cauchy-Riemann equations can also be written as

$$\frac{\partial}{\partial \overline{z}} f(z) = 0$$

where z and $\overline{z}$ are treated as two separate variables. To see this, we perform a change of variables from $(z, \overline{z})$ to (x, y) with $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/(2\mathrm{i})$:

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \overline{z}} + \mathrm{i}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \overline{z}}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + \mathrm{i}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right),$$

which is zero if and only if the Cauchy-Riemann equations are satisfied.

It is important that the function is written in terms only of z and $\overline{z}$ to use this. For example, if using $z = x + \mathrm{i}y$, we have a function $x(x + \mathrm{i}y)$ we need to rewrite it as $(z + \overline{z})z/2$ before trying to compute $\frac{\partial}{\partial \overline{z}}$.

Example 13. Taking the example of $f(z) = |z|^2$ again, we see that $f(z) = z\overline{z}$, and so

$$\frac{\partial}{\partial \overline{z}}(z\overline{z}) = z$$

and this vanishes only when $z = 0$, as before.

If two real functions $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ satisfy the Cauchy-Riemann equations, this is not quite enough to conclude that the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(x+\mathrm{i}y) = u(x,y)+\mathrm{i}v(x,y)$ is complex differentiable. However, adding the requirement that the partial derivatives are continuous *is* sufficient.

Theorem 2.10. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ have continuous first partial derivatives with respect to x and y. If they satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \textit{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

then $f : \mathbb{C} \to \mathbb{C}$ given by $f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$ is (complex) differentiable.

The theorem can be proved using first-order Taylor expansion of u and v. Since you haven't necessarily seen the precise form of Taylor's theorem in two variables, I'm skipping the proof, but it is here that continuity of the first partial derivatives enters.

Remark 2.11. If u, v fulfill the Cauchy-Riemann equations and are *twice* continuously differentiable, then one finds

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2}$$

where continuity of the second derivatives is needed to exchange their order. Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Functions u fulfilling this equation (the *Laplace equation*) are called *harmonic.* Since the same argument holds for v, we have almost (up to the extra differentiability assumption) shown that *all holomorphic functions are harmonic.*

Remark 2.12. It turns out that holomorphic functions are infinitely differentiable, hence the extra smoothness assumptions in the previous statments are actually not necessary. We will see this later in Theorem 5.3.

Chapter 3

Contour integration

3.1 Integrals of complex functions of a real variable

If $f : \mathbb{R} \to \mathbb{C}$, let $f(t) = u(t) + \mathrm{i}v(t)$ with u and v real functions, then

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + \mathrm{i}\int_a^b v(t)\,\mathrm{d}t$$

so for functions with a real input and a complex output, all the normal rules and results from standard calculus work (simply apply these rules to the real and imaginary parts of the output separately).

Theorem 3.1 (Fundamental Theorems of Calculus)**.** *Let $[a, b] \subseteq \mathbb{R}$.*

1. *If $f : [a, b] \to \mathbb{C}$ is continuous, then for $t \in (a, b)$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(s)\,\mathrm{d}s = f(t)\,.$$

2. *If $f : [a, b] \to \mathbb{C}$ is continuous and if $f' : (a, b) \to \mathbb{C}$ is continuous, then*

$$\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a)\,.$$

Lemma 3.2. *If $f : [a, b] \to \mathbb{C}$ is an integrable function, then*

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \leq \int_a^b |f(t)|\,\mathrm{d}t.$$

Proof. Because the function is integrable we can write

$$\int_a^b f(t)\,\mathrm{d}t = r\,\mathrm{e}^{\mathrm{i}\theta} \qquad \text{with } r > 0 \text{ and } \theta \in \mathbb{R}.$$

Since $|\,\mathrm{e}^{\mathrm{i}\theta}\,| = 1$ for all $\theta \in \mathbb{R}$ we have

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| = r = \mathrm{e}^{-\mathrm{i}\theta}\int_a^b f(t)\,\mathrm{d}t\,.$$

But this is real, by construction. Hence it equals its real part, so

$$\begin{aligned}\left|\int_a^b f(t)\,\mathrm{d}t\right| &= \mathrm{Re}\left(\int_a^b \mathrm{e}^{-\mathrm{i}\theta} f(t)\,\mathrm{d}t\right) \\ &= \int_a^b \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}\theta} f(t)\right)\,\mathrm{d}t && \text{(by linearity of integration)} \\ &\leq \int_a^b |\,\mathrm{e}^{-\mathrm{i}\theta} f(t)|\,\mathrm{d}t && \text{(since } \mathrm{Re}\,w \leq |w| \text{ for all } w \in \mathbb{C}) \\ &= \int_a^b |f(t)|\,\mathrm{d}t && (|\,\mathrm{e}^{-\mathrm{i}\theta}\,| = 1 \text{ for all } \theta \in \mathbb{R}),\end{aligned}$$

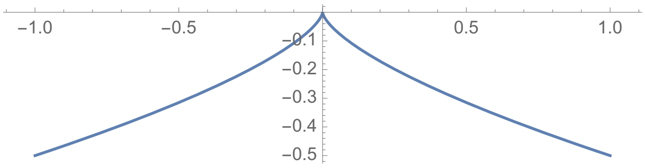
as required. □

Remark 3.3. Note that Lemma 3.2 is analogous to the generalized triangle inequality $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$ (Theorem 1.4) and that the generalized triangle inequality can also be proven following a similar argument (replace the integral with a sum).

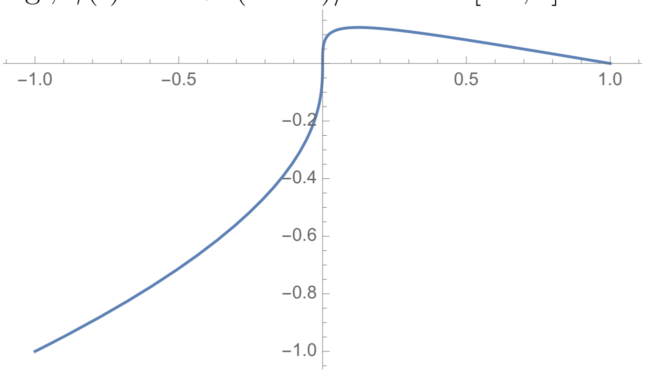
3.2 Curves on the complex plane

Definition 3.4. A *path* γ is a continuous function from some interval $[a, b] \subseteq \mathbb{R}$ to $\mathbb{C}$. A path γ is said to be *smooth* if γ' exists and is continuous on (a, b). A path is said to be *regular* if, in addition to being smooth, $\gamma'(t)$ is nonzero for all $t \in (a, b)$.

Remark 3.5. Note that some paths are smoother than they look. For example, consider $\gamma(t) = t^3 - \mathrm{i}t^2/2$ for $t \in [-1, 1]$. This has $\gamma'(t) = 3t^2 - \mathrm{i}t$ which is continuous, hence $\gamma(t)$ is smooth. Plotting it gives



which doesn't look very smooth. However, because $\gamma'(t) = 0$ at $t = 0$ the path is not regular. Basically, the path can be smooth through the apparent kink because the gradient is zero there. [If this path represented the motion of a particle, the particle would come to rest at the origin before setting off in another direction.] If we consider another path that is smooth and regular, e.g., $\gamma(t) = t^3 + \mathrm{i}(t - t^2)/2$ for $t \in [-1, 1]$ then there is no longer a kink:

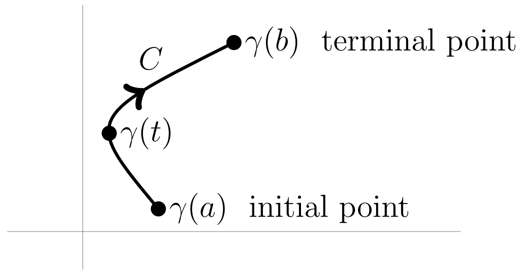


Definition 3.6. A *curve* is a set of points that can be parameterized either by a smooth path, or by connecting finitely many smooth paths.

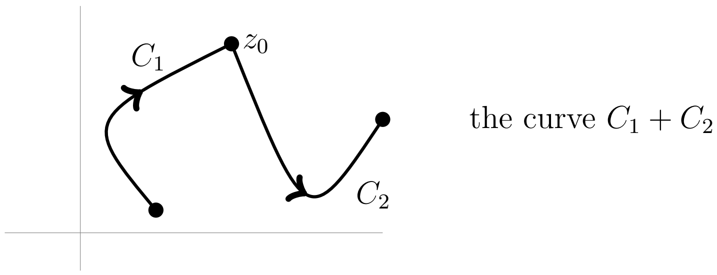
Remark 3.7. While there are many ways to parameterize a given curve, a path comes with a particular parameterization. Note also that other sources may use different definitions of these terms.

Remark 3.8. Because there are many ways to parameterize the same curve, your way may different from someone else's (or a given solution). When coming up with a parameterization it is always a good idea to check that you have the right endpoints and are going in the right direction.

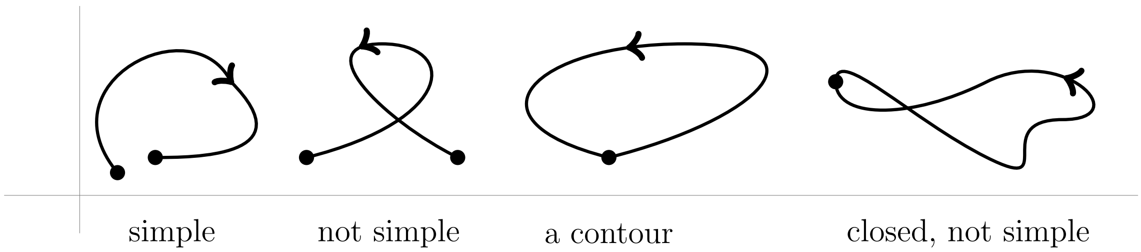
In this section we will (except where stated otherwise) take curves to be oriented, i.e., to have a direction which can be indicated by an arrow. The following diagram shows a curve parameterized by $\gamma : [a,b] \to \mathbb{C}$ together with its initial and terminal points:



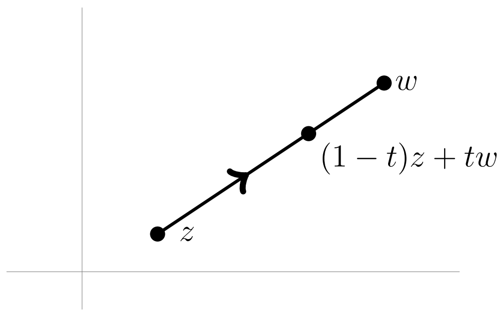
If a curve C_1 ends at $z_0 \in \mathbb{C}$ and a curve C_2 starts at z_0, then they can be connected to form a new curve $C_1 + C_2$:



Definition 3.9. A curve C is called *simple* iff it does not intersect itself. A curve C is called *closed* iff its initial and terminal points coincide. A *contour* is a simple and closed curve.

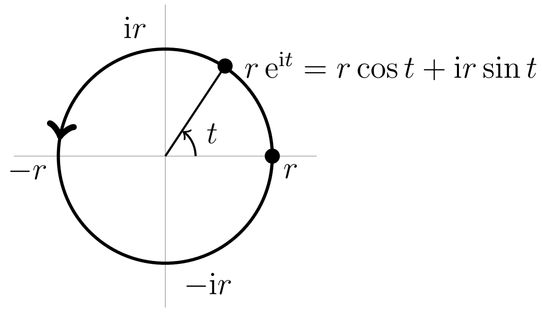


Example 14. The line segment from z to w, denoted $[z,w]$, can be parameterized by $\gamma(t) = (1-t)z + tw$, $0 \le t \le 1$. (Note that $\gamma(0) = z$ and $\gamma(1) = w$.)



Remark 3.10. This same line can also be parameterized by $\gamma_1(t) = (1 - t/5)z + tw/5$ for $0 \le t \le 5$, or by $\gamma_2(t) = (1 - 9t^2)z + 9t^2 w$ for $0 \le t \le 1/3$, or (infinitely many) other ways. Note that to define a parameterization we always need to state the range of values the parameters take.

Example 15. A circle of radius r centred at 0 can be parameterized by $\gamma(t) = r\,\mathrm{e}^{\mathrm{i}t}$, $0 \le t \le 2\pi$ and is a contour.



Let a curve C be parameterized by a smooth path $\gamma : [a, b] \to \mathbb{C}$. If we write $\gamma(t) = x(t) + \mathrm{i}y(t)$ with $x(t), y(t) \in \mathbb{R}$, then $\gamma'(t) = x'(t) + \mathrm{i}y'(t)$. Thus,

$$|\gamma'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

The length of C is (using standard calculus)

$$L_C := \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t = \int_a^b |\gamma'(t)|\,\mathrm{d}t.$$

We restate this as a lemma.

Lemma 3.11. *The length of a curve C parameterized by a smooth path $\gamma : [a, b] \to \mathbb{C}$ is*

$$L_C = \int_a^b |\gamma'(t)|\,\mathrm{d}t.$$

3.3 Contour integrals in the complex plane

In this section we consider how to generalize the integral

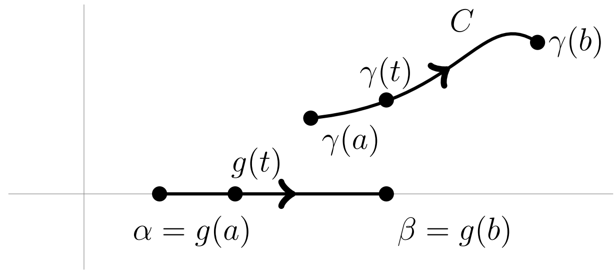
$$\int_\alpha^\beta f(x)\,\mathrm{d}x$$

of a function f of a real variable to an integral of a function of a complex variable along a curve on $\mathbb{C}$.

Consider the familiar case of integration by substitution. If $g : [a, b] \to [\alpha, \beta]$ is a one-to-one function with $g(a) = \alpha$ and $g(b) = \beta$, then, by letting $x = g(t)$, we have $\mathrm{d}x = g'(t)\,\mathrm{d}t$ and

$$\int_\alpha^\beta f(x)\,\mathrm{d}x = \int_a^b f(g(t))\,g'(t)\,\mathrm{d}t. \tag{3.1}$$

The function g can be regarded to be parameterizing the straight line segment $[\alpha, \beta]$ on $\mathbb{C}$. Then we can generalise Eq. (3.1) to <u>define</u> an integral along a curve in $\mathbb{C}$ parameterized by $\gamma : [a, b] \to \mathbb{C}$.



Definition 3.12. The *line integral* of a function f of a complex variable along a curve C on $\mathbb{C}$ parameterized by a smooth regular path $\gamma : [a,b] \to \mathbb{C}$ is defined by

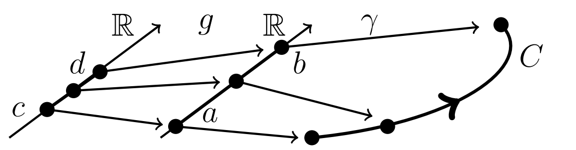
$$\int_C f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t.$$

The line integral of f along a curve $C = C_1 + C_2 + \cdots + C_N$, where $C_1, C_2, \ldots, C_N$ can each be parameterized by smooth regular paths, is defined by

$$\int_C f(z)\,\mathrm{d}z := \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z + \cdots + \int_{C_N} f(z)\,\mathrm{d}z.$$

Theorem 3.13. *The integral $\int_C f(z)\,\mathrm{d}z$ does not depend on how the curve C is parameterized as long as the orientation is the same.*

Proof of a special case (non-examinable). If $\gamma : [a,b] \to \mathbb{C}$ is a smooth and regular path parameterizing a curve C on $\mathbb{C}$, and $g : [c,d] \to [a,b]$ is a one-to-one onto function differentiable on (c,d), then the path $\tilde{\gamma} := \gamma \circ g : [c,d] \to \mathbb{C}$ also parameterizes the curve C. (Note $(\gamma \circ g)(t) = \gamma(g(t))$.)



The integral $\int_C f(z)\,\mathrm{d}z$ using the parameterization $\tilde{\gamma}$ is

$$\begin{aligned}
\int_c^d f(\tilde{\gamma}(t))\,\tilde{\gamma}'(t)\,\mathrm{d}t &= \int_c^d f(\gamma(g(t)))\,(\gamma \circ g)'(t)\,\mathrm{d}t \\
&= \int_c^d f(\gamma(g(t)))\,\gamma'(g(t))\,g'(t)\,\mathrm{d}t && \text{(chain rule)} \\
&= \int_a^b f(\gamma(s))\,\gamma'(s)\,\mathrm{d}s && \text{(let } s = g(t) \text{ so } \mathrm{d}s = g'(t)\,\mathrm{d}t\text{),}
\end{aligned}$$

which is $\int_C f(z)\,\mathrm{d}z$ using the parameterization γ. (Note $a = g(c)$ and $b = g(d)$.) □

To upgrade this to a full proof we would, *inter alia,* need to show that any smooth and regular path $\tilde{\gamma}$ parameterizing C can be written as $\gamma \circ g$ where g has the form mentioned above.

Remark 3.14. By a similar argument the length L_C of C is also independent of the parameterization of the curve.

Notation 3.15. The curve obtained by reversing the direction of the curve C is denoted $-C$.

Lemma 3.16. *For a curve C on $\mathbb{C}$,*

$$\int_{-C} f(z)\,\mathrm{d}z = -\int_C f(z)\,\mathrm{d}z.$$

Proof. If $\gamma : [a,b] \to \mathbb{C}$ parameterizes C, then $\tilde{\gamma} : [-b,-a] \to \mathbb{C}$ defined by $\tilde{\gamma}(t) = \gamma(-t)$ parameterizes $-C$. Then

$$\int_{-C} f(z)\,\mathrm{d}z = \int_{-b}^{-a} f(\gamma(-t))\,\frac{\mathrm{d}}{\mathrm{d}t}\gamma(-t)\,\mathrm{d}t.$$

By the substitution $s=-t$, we have $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(-t) = -\frac{\mathrm{d}}{\mathrm{d}s}\gamma(s) = -\gamma'(s)$ and $\mathrm{d}t = -\mathrm{d}s$. Hence

$$\int_{-C} f(z)\,\mathrm{d}z = \int_{b}^{a} f(\gamma(s))\,\gamma'(s)\,\mathrm{d}s = -\int_{a}^{b} f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t = -\int_{C} f(z)\,\mathrm{d}z\,. \qquad \square$$

Example 16. If C is the straight line segment from i to 1 then it can be parameterized by $\gamma(t) = (1-t)\cdot \mathrm{i} + t\cdot 1 = \mathrm{i} + (1-\mathrm{i})t$ with $0 \le t \le 1$. We have $\gamma'(t) = 1-\mathrm{i}$, and

$$\begin{aligned}\int_C z^2\,\mathrm{d}z &= \int_0^1 \left[(\mathrm{i}+(1-\mathrm{i})t\right]^2 (1-\mathrm{i})\,\mathrm{d}t = (1-\mathrm{i})\int_0^1 \left[-1+2(1+\mathrm{i})t - 2\mathrm{i}t^2\right]\mathrm{d}t \\ &= (1-\mathrm{i})\left[-t+(1+\mathrm{i})t^2 - \frac{2}{3}\mathrm{i}t^3\right]_0^1 = \frac{1}{3}(1+\mathrm{i}).\end{aligned}$$

Example 17. If C_1 is the straight line segment from $-r$ to r and C_2 is the upper semicircle of radius r centred at 0, then

$$\int_{C_1} |z|^2\,\mathrm{d}z = \int_{-r}^{r} x^2\,\mathrm{d}x = \frac{2}{3}r^3.$$

The curve C_2 is parameterized by $\gamma(t) = r\,\mathrm{e}^{\mathrm{i}t}$ with $0 \le t \le \pi$ with $\gamma'(t) = \mathrm{i}r\,\mathrm{e}^{\mathrm{i}t}$, so

$$\int_{C_2} |z|^2\,\mathrm{d}z = \int_0^{\pi} r^2\cdot \mathrm{i}r\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t = r^3(\mathrm{e}^{\pi\mathrm{i}} - \mathrm{e}^0) = -2r^3.$$

Hence

$$\int_{C_1+C_2} |z|^2\,\mathrm{d}z = \frac{2}{3}r^3 - 2r^3 = -\frac{4}{3}r^3.$$

Notation 3.17. If the curve C is closed, we often write the line integral as $\oint_C f(z)\,\mathrm{d}z$.

Remark 3.18. The previous example showed that $\oint_{C_1+C_2} |z|^2\,\mathrm{d}z \neq 0$.
When you come back to review this after studying more of the course, you should note that this is not in contradiction with Cauchy's theorem because $|z|^2$ is not holomorphic anywhere (cf. Example 8).

Example 18 (This example is important)**.** Let C be a circle of radius r centered at the origin, parameterized by $\gamma(t) = r\,\mathrm{e}^{\mathrm{i}t}$ with $0 \le t \le 2\pi$. Thus $\gamma'(t) = \mathrm{i}r\,\mathrm{e}^{\mathrm{i}t}$. Then,

$$\oint_C z^n\,\mathrm{d}z = \int_0^{2\pi} r^n\,\mathrm{e}^{\mathrm{i}nt}\cdot \mathrm{i}r\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t = \mathrm{i}r^{n+1}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n+1)t}\,\mathrm{d}t.$$

1. If $n \neq -1$, $n \in \mathbb{Z}$, then $\oint_C z^n\,\mathrm{d}z = \mathrm{i}r^{n+1}\left[\frac{\mathrm{e}^{\mathrm{i}(n+1)t}}{\mathrm{i}(n+1)}\right]_0^{2\pi} = \mathrm{i}r^{n+1}\frac{\mathrm{e}^{2\pi(n+1)\mathrm{i}} - 1}{\mathrm{i}(n+1)} = 0.$
2. If $n=-1$, then $\oint_C z^{-1}\,\mathrm{d}z = \mathrm{i}r^0\int_0^{2\pi}\mathrm{d}t = 2\pi\mathrm{i}$, i.e.,

$$\oint_C \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i}.$$

In summary, $$\oint_C z^n \,\mathrm{d}z = \begin{cases} 0 & \text{if } n \in \mathbb{Z} \text{ with } n \neq -1 \\ 2\pi\mathrm{i} & \text{if } n = -1. \end{cases}.$$

Definition 3.19. Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$ be a function. If $F'(z) = f(z)$ for all $z \in G$, then $F : G \to \mathbb{C}$ is called an *antiderivative* (or *primitive*) of f on G.

Theorem 3.20 (Fundamental Theorem of Calculus for Line Integration in $\mathbb{C}$)**.** *Let $G \subseteq \mathbb{C}$ be an open set and suppose F is an antiderivative of f on G. If C is a curve in G with initial point z_0 and final point z_1, then*

$$\int_C f(z)\,\mathrm{d}z = F(z_1) - F(z_0).$$

Proof. If C is a single curve parameterized by $\gamma : [a, b] \to \mathbb{C}$, then $\gamma(a) = z_0$ and $\gamma(b) = z_1$, and

$$\begin{aligned} \int_C f(z)\,\mathrm{d}z &= \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t = \int_a^b F'(\gamma(t))\,\gamma'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t))\,\mathrm{d}t \\ &= F(\gamma(b)) - F(\gamma(a)) = F(z_1) - F(z_0), \end{aligned}$$

where we have used the second part of the Fundamental theorem of Calculus (Theorem 3.1). If, on the other hand, $C = C_1 + C_2$ where C_1 is a curve with initial and terminal points z_0 and z_2, and C_2 is a curve with initial and terminal points z_2 and z_1, then

$$\begin{aligned} \int_C f(z)\,\mathrm{d}z &= \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z \\ &= F(z_1) - F(z_2) + F(z_2) - F(z_0) \\ &= F(z_1) - F(z_0). \end{aligned}$$

This generalizes for any finite number of piecewise smooth pieces, as required. □

Corollary 3.21. *If f has an antiderivative F in G, then for any closed curve C in G,*

$$\oint_C f(z)\,\mathrm{d}z = 0.$$

Proof. We have $\oint_C f(z)\,\mathrm{d}z = F(z_1) - F(z_0) = 0$ because $z_0 = z_1$. □

Example 19. Revisiting Example 16, since $z^3/3$ is an antiderivative of z^2 on $\mathbb{C}$, if we integrate z^2 along a straight line segment from i to 1, we have

$$\int_C z^2\,\mathrm{d}z = \left[\frac{z^3}{3}\right]_{z=\mathrm{i}}^{z=1} = \frac{1}{3}(1+\mathrm{i}).$$

Example 20. We can also revisit Example 17. We found for the contour $C_1 + C_2$

$$\oint_{C_1+C_2} |z|^2\,\mathrm{d}z = -\frac{4}{3}R^3 \neq 0.$$

Hence the function $f(z) = |z|^2 = z\bar{z}$ has no antiderivative in any region containing $C_1 + C_2$.

Example 21. We now go back to Example 18.

1. In the case $n \in \mathbb{Z}$, $n \neq -1$ the function z^n has an antiderivative $z^{n+1}/(n+1)$ on $\mathbb{C}$ if $n = 0, 1, 2, \cdots$ and on $\mathbb{C}^*$ if $n = -2, -3, \ldots$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The contour C (the circle of radius r centred at 0) is in $\mathbb{C}^* \subseteq \mathbb{C}$. Hence we can apply Corollary 3.21 to conclude that $\displaystyle\oint_C z^n \, \mathrm{d}z = 0$ as we found.

2. In the case $n = -1$, we saw that $\displaystyle\oint_C \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i} \neq 0$. It follows that there is no antiderivative of $1/z$ valid in the whole of $\mathbb{C}^*$ (or, indeed any region containing C). For more information on this case, see the additional remark at the end of this chapter (Remark 3.24).

Lemma 3.22 (The Estimation Lemma). *If $|f(z)| \leq M$ on C and if L_C is the length of C, then*

$$\left| \int_C f(z) \, \mathrm{d}z \right| \leq M L_C.$$

Proof. If $\gamma : [a, b] \to \mathbb{C}$ parameterizes C, then

$$\left| \int_C f(z) \, \mathrm{d}z \right| = \left| \int_a^b f(\gamma(t)) \, \gamma'(t) \, \mathrm{d}t \right| \leq \int_a^b |f(\gamma(t))| \, |\gamma'(t)| \, \mathrm{d}t \leq M \int_a^b |\gamma'(t)| \, \mathrm{d}t = M L_C \,,$$

as required (where we used Lemmas 3.2 and 3.11). □

Remark 3.23. As a word of warning, note that it is *not* the case that

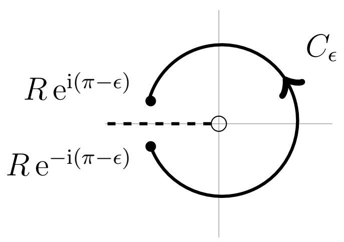
$$\left| \int_C f(z) \, \mathrm{d}z \right| \leq \int_C |f(z)| \, \mathrm{d}z \qquad \text{(WRONG)}$$

for any path γ parameterizing C. In fact, this estimate wouldn't even formally make sense, since the right-hand side is not real in general!

Make sure you understand how this warning relates to Lemma 3.2.

Remark 3.24 (Additional remark about $\oint_C \frac{\mathrm{d}z}{z}$ for enthusiasts). In fact, the function $\log z$ (see the exercises) is an antiderivative of $1/z$ on $\mathbb{C}_c$, where $\mathbb{C}_c = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ is the cut plane with the negative real axis removed. $\mathbb{C}_c$ does not contain the circle C of radius r centred at the origin because of the cut.

Consider the curve C_ϵ, $\epsilon > 0$, parameterized by $\gamma(t) = R\,\mathrm{e}^{\mathrm{i}t}$, $-\pi + \epsilon \leq t \leq \pi - \epsilon$.



The curve C_ϵ is in $\mathbb{C}_c$, in which $\log z$ is an antiderivative of $1/z$. Hence, by the Fundamental Theorem of Calculus for Line Integration in $\mathbb{C}$ (Theorem 3.20), we have

$$\int_{C_\epsilon} \frac{\mathrm{d}z}{z} = [\log z]_{z=R\,\mathrm{e}^{-\mathrm{i}(\pi-\epsilon)}}^{z=R\,\mathrm{e}^{\mathrm{i}(\pi-\epsilon)}} = [\log R + \mathrm{i}(\pi - \epsilon)] - [\log R + \mathrm{i}(-\pi + \epsilon)] = 2(\pi - \epsilon)\mathrm{i}.$$

We can consider the curve C parameterized by $\gamma(t) = R\,\mathrm{e}^{\mathrm{i}t}$, $-\pi \leq t \leq \pi$, as the $\epsilon \to 0$ limit of C_ϵ. Hence

$$\oint_C \frac{\mathrm{d}z}{z} = \lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{\mathrm{d}z}{z} = \lim_{\epsilon \to 0} 2(\pi - \epsilon)\mathrm{i} = 2\pi\mathrm{i}.$$

Chapter 4

Cauchy's Theorem

We know from the Fundamental Theorem of Calculus for Line Integration in $\mathbb{C}$ (Theorem 3.20) that if a function f has an antiderivative in G, then the integral of f around any contour in G is zero (see Corollary 3.21). However it is not always obvious whether a given function has an antiderivative in G. Remarkably, it is also sufficient for f to be differentiable on G; this is the main substance of Cauchy's theorem. Before getting to this we need a little more background material.

4.1 Deformation of curves

Definition 4.1. Let $G \subseteq \mathbb{C}$ be open. Let C_0 and C_1 be two curves in G parameterized by γ_0 and γ_1, which are both maps from $[0,1]$ to G and that both have initial point z_0 and terminal point z_1. We say that C_0 and C_1 are homotopic to each other in G, if there exists a map $\phi : [0,1] \times [0,1] \to G$ such that

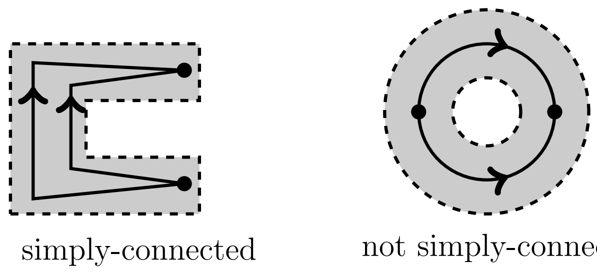
1. ϕ is continuous
2. $\phi(0,t) = \gamma_0(t)$
3. $\phi(1,t) = \gamma_1(t)$
4. For every $0 \leq s \leq 1$, the curve C_s parameterized by $\gamma_s(t) := \phi(s,t)$ is piecewise smooth, contained in G and has initial point z_0 and terminal point z_1.

In other words, C_0 is homotopic to C_1 in G if C_0 can be "continuously deformed" to C_1 without leaving the set G. To get the idea, if we imagine an elastic band whose ends are fixed at z_0 and z_1, then C_0 is homotopic to C_1 if the band with shape C_0 can be deformed to the shape C_1 without leaving the set G (and without breaking the band).

Remark 4.2. If C_0 and C_1 are both closed (in which case any point on each curve can be taken as both the initial and terminal points of that curve) then we can extend the definition of homotopic, changing the final condition in Definition 4.1 to

4. For every $0 \leq s \leq 1$, the curve C_s parameterized by $\gamma_s(t) := \phi(s,t)$ is piecewise smooth, contained in G and closed.

We can use this definition to make precise a notion of a set "without any holes", called a simply-connected set.



Definition 4.3. An open set $G \subseteq \mathbb{C}$ is said to be *simply-connected* if all closed curves C are homotopic in G to a null curve N, which is a "curve" consisting of a single point.

In other words, within a simply connected set we can continuously deform all closed curves to a single point within G. Equivalently, a set is simply-connected if all curves with the same initial and terminal points can be continuously deformed to each other in G.

Example 22. The set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, on which $1/z$ is holomorphic, is not simply connected.

4.2 Cauchy's Theorem

To prove this we will use Green's theorem from vector calculus, which allows us to swap integrals along contours with integrals over the areas they enclose.

Theorem 4.4 (Green's theorem)**.** *Suppose R is some open subset of $\mathbb{R}^2$ and $g : R \to \mathbb{R}^2$ is a function with continuous first partial derivatives. Then*

$$\oint_C \underline{g}(\underline{x}) \cdot \mathrm{d}\underline{x} = \int_R \operatorname{curl} \underline{g} \, \mathrm{d}A \,,$$

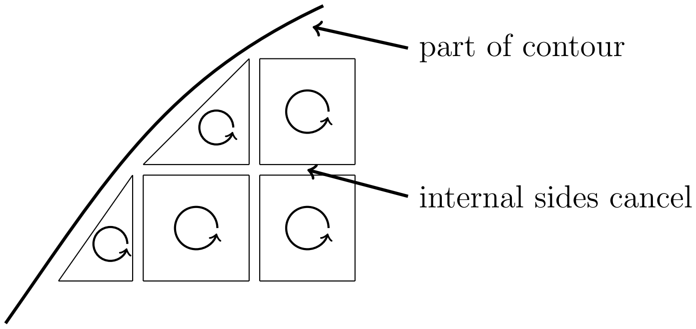
where C is the boundary of the set R. This can be equivalently written in terms of $u : R \to \mathbb{R}$ and $v : R \to \mathbb{R}$ with $g(x, y) = (u(x, y),\, v(x, y))$ as

$$\oint_C (u \, \mathrm{d}x + v \, \mathrm{d}y) = \int_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y \,,$$

Theorem 4.5 (Cauchy's theorem)**.** *Let f be holomorphic on an open and simply connected set $G \subseteq \mathbb{C}$. Then for any contour C contained in G*

$$\oint_C f(z) \, \mathrm{d}z = 0.$$

Remark 4.6. We will give a proof in the (apparently) special case that f has continuous partial derivatives. I say "apparently" because it turns out that all holomorphic functions have this property. However, we don't know that yet and we will need Cauchy's theorem to prove it. The proof below can be upgraded to a version without this assumption (this was first done by Goursat and hence Cauchy's theorem is sometimes called the Cauchy-Goursat theorem). Although we will not go through the details in this course, the idea is roughly as follows. Since linear functions have continuous partial derivatives, Green's theorem implies Cauchy's theorem for linear functions (see the proof below). To upgrade this to arbitrary differentiable functions, the idea is to show that differentiability implies the existence of an arbitrarily good linear approximation at any point. We can thus divide up region R into arbitrarily small rectangles/triangles and treat each as its own contour, as shown:



Because the integrals along the internal sides cancel out, the sum of the integrals over all these contours gives an arbitrarily good approximation of $\oint_C f(z)\,\mathrm{d}z$ as we decrease the size of the rectangles/triangles. We can also use Cauchy's theorem for the linear approximations on each small rectangle/triangle to show that these integrals are zero. It remains to check that all the limits work in the right way to establish the general form of Cauchy's theorem stated above.

Proof of Cauchy's theorem assuming f has continuous partial derivatives. We use $f(x+\mathrm{i}y) = u(x,y)+\mathrm{i}v(x,y)$ where x, y, u and v are real to give

$$\begin{aligned}\oint_C f(z)\,\mathrm{d}z &= \oint_C (u(x,y)+\mathrm{i}v(x,y))\,(\mathrm{d}x+\mathrm{i}\,\mathrm{d}y)\\ &= \oint_C (u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y) + \mathrm{i}\oint_C (v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y)\end{aligned}$$

Assuming that f has continuous partial derivatives, we can use Green's theorem to give

$$\oint_C f(z)\,\mathrm{d}z = \int_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + \mathrm{i}\int_R \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y\,,$$

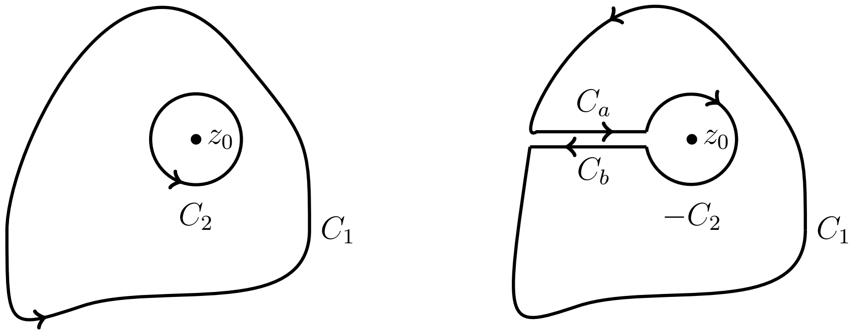
where R is the region of G enclosed by the curve C. Because f is holomorphic on G, the Cauchy-Riemann equations are satisfied throughout R. The right hand side of the above equation is thus zero. $\square$

Remark 4.7 (Important remark)**.** Note that $f(z) = 1/z$ has $f'(z) = -1/z^2$ for $z \neq 0$ and is hence holomorphic on $\mathbb{C}^*$. The set $\mathbb{C}^*$ is not simply connected and since the circle $C(0,r)$ of radius r centred at 0 encloses $z = 0$, the Cauchy-Riemann equations do not hold throughout $R = \mathrm{D}(0;r)$. Thus, the fact that the integral $\displaystyle\oint_{C(0,r)} \frac{1}{z}\,\mathrm{d}z$ is equal to $2\pi\mathrm{i}$ (cf. Example 18) does not contradict Cauchy's theorem. Note also that for any contour C that does not enclose $z = 0$ Cauchy's theorem implies that $\displaystyle\oint_C \frac{1}{z}\,\mathrm{d}z = 0$.

Remark 4.8. An alternative way to state Cauchy's theorem is that if a contour C can be continuously deformed to a null curve while remaining in a region in which f is holomorphic, then $\displaystyle\oint_C f(z)\,\mathrm{d}z = 0$.

Remark 4.9. In addition, Cauchy's theorem implies that if C_1 can be continuously deformed to give C_2 within a region in which f is holomorphic, then $\displaystyle\oint_{C_1} f(z)\,\mathrm{d}z = \oint_{C_2} f(z)\,\mathrm{d}z$.

The argument behind this can be seen from the following diagram, in which (for simplicity) we suppose a function that is holomorphic except at z_0:



In the limit that C_a and C_b are on top of each other (but in opposite directions), the contour on the right is composed of C_1, C_a, $-C_2$ and $C_b(= -C_a)$. This contour encloses a region in which f is holomorphic, so the integral over this contour is zero. Since the integrals along C_a and C_b cancel we have

$$\oint_{C_1} f(z)\,\mathrm{d}z = \oint_{C_2} f(z)\,\mathrm{d}z\,.$$

This is often used to turn arbitrary contours into circles, which are generally easier to integrate around: for any contour C that encloses z_0 (with the same orientation as $C(z_0, r)$) we have $\displaystyle\oint_C f(z)\,\mathrm{d}z = \oint_{C(z_0,r)} f(z)\,\mathrm{d}z$.

Remark 4.10. Cauchy's theorem has, in a sense, a converse: If f is continuous function on some open set $G \subset \mathbb{C}$ such that $\oint_C f(z)dz = 0$ for every contour C in G, then f is holomorphic in G. This is the content of Morera's theorem (Theorem B.20 in the appendix).

4.3 Winding number

Consider a smooth and closed curve C that does not pass through $z \in \mathbb{C}$ (and is not necessarily simple). The winding number of C about z, denoted $\mathrm{n}(C, z)$, tells us how many times C wraps around z in the anticlockwise direction. We saw earlier that

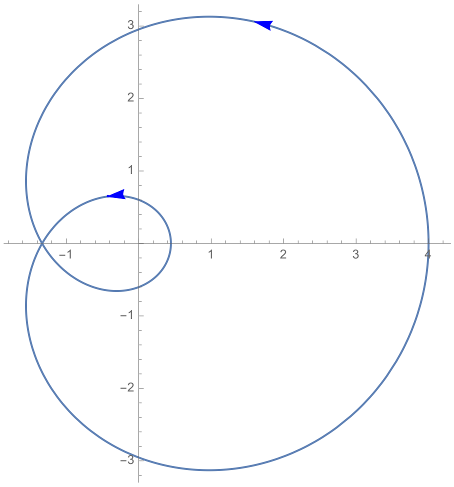
$$\oint_{C'} \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i}$$

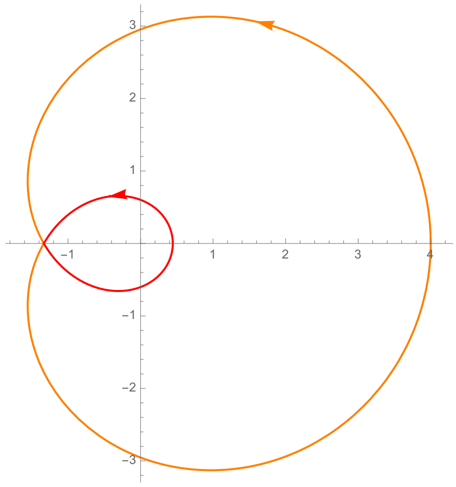
for C' as the contour defined by taking an anticlockwise circle parameterized by $\gamma(t) = r\,\mathrm{e}^{\mathrm{i}t}$ with $0 \le t \le 2\pi$. Because of Cauchy's theorem, and the fact that $1/z$ is holomorphic on $\mathbb{C}^*$, the same is true for any contour that circles the origin in an anticlockwise direction. This motivates the following definition.

Definition 4.11. Consider a smooth and closed curve C that does not pass through $z \in \mathbb{C}$ (and is not necessarily simple). The winding number of C about z, denoted $\mathrm{n}(C, z)$ is defined by

$$\mathrm{n}(C, z) := \frac{1}{2\pi\mathrm{i}} \oint_C \frac{\mathrm{d}w}{w - z}\,.$$

We can understand why this definition makes sense by using Cauchy's theorem. Consider the closed curve on the left which circles the origin twice.





We can split this into two at the point of intersection, as shown on the right. The integral of $1/w$ around the blue curve is equal to the sum of the integrals of $1/w$ around the red (inner) and orange (outer) curves. [Note that both these curves are smooth because they can be described by a smooth path that starts and finishes at their point of intersection.] Both the red and orange curves have identical integrals of $1/w$ because they can be deformed to one another without crossing $w = 0$: each evaluates as $2\pi\mathrm{i}$. The winding number of the blue curve about $w = 0$ is hence 2.

By extending this idea, we can see that the winding number satisfies the following:

- if $\mathrm{n}(C, z) \geq 1$, then the curve C winds around z anticlockwise $\mathrm{n}(C, z)$ times;
- if $\mathrm{n}(C, z) \leq -1$, then the curve C winds around z clockwise $|\,\mathrm{n}(C, z)|$ times;
- if $\mathrm{n}(C, z) = 0$, then the curve C does not wind around z.

Example 23. Consider a smooth and closed curve C parameterized by

$$\gamma(t) = z + r(t)\,\mathrm{e}^{\mathrm{i}\theta(t)}, \quad a \leq t \leq b$$

with $r(t) > 0$, $\theta(t) \in \mathbb{R}$, and $\gamma(t)$ differentiable. [In fact, any smooth curve not passing through z has such a parameterization.]

Since C is closed, we have $r(b)\,\mathrm{e}^{\mathrm{i}\theta(b)} = r(a)\,\mathrm{e}^{\mathrm{i}\theta(a)}$. This implies

$$r(b) = r(a) \text{ and } \theta(b) - \theta(a) = 2k\pi, \; k \in \mathbb{Z}.$$

We show now that k is the winding number. Since

$$\gamma'(t) = r'(t)\,\mathrm{e}^{\mathrm{i}\theta(t)} + \mathrm{i}r(t)\theta'(t)\,\mathrm{e}^{\mathrm{i}\theta(t)} = r(t)\,\mathrm{e}^{\mathrm{i}\theta(t)} \left[\frac{r'(t)}{r(t)} + \mathrm{i}\theta'(t)\right]$$

we have

$$\begin{aligned}
\mathrm{n}(C, z) &= \frac{1}{2\pi\mathrm{i}} \oint_C \frac{\mathrm{d}w}{w - z} = \frac{1}{2\pi\mathrm{i}} \int_a^b \frac{1}{r(t)\,\mathrm{e}^{\mathrm{i}\theta(t)}} r(t)\,\mathrm{e}^{\mathrm{i}\theta(t)} \left[\frac{r'(t)}{r(t)} + \mathrm{i}\theta'(t)\right] \mathrm{d}t \\
&= \frac{1}{2\pi\mathrm{i}} \int_a^b \left[\frac{r'(t)}{r(t)} + \mathrm{i}\theta'(t)\right] \mathrm{d}t = \frac{1}{2\pi\mathrm{i}} \left[\log r(t) + \mathrm{i}\theta(t)\right]_a^b \\
&= \frac{1}{2\pi} \left[\theta(b) - \theta(a)\right] = k
\end{aligned}$$

Example 24. Find the winding number around the point $z = 1$ of the curve C parameterized by

$$\gamma(t) = 1 + \cos(\pi t^2) + \mathrm{i}\sin(\pi t^2)$$

for $0 \leq t \leq 2$.

Solution. Note first that this is a closed curve, since $\gamma(0) = 2 = \gamma(2)$. We can write $\gamma(t) = 1 + \mathrm{e}^{\mathrm{i}\pi t^2}$, so we have that $\gamma'(t) = 2\pi\mathrm{i}t\,\mathrm{e}^{\mathrm{i}\pi t^2}$ and thus

$$\mathrm{n}(C,1) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{1}{w-1}\,\mathrm{d}w = \frac{1}{2\pi\mathrm{i}}\int_0^2 \frac{1}{\mathrm{e}^{\mathrm{i}\pi t^2}} 2\pi\mathrm{i}t\,\mathrm{e}^{\mathrm{i}\pi t^2}\,\mathrm{d}t = \int_0^2 t\,\mathrm{d}t = \left[\tfrac{1}{2}t^2\right]_0^2 = 2$$

so the winding number is 2.

The following theorem was mentioned informally earlier. We state it for completeness.

Theorem 4.12. *If two closed curves C_1 and C_2 are homotopic to each other in $\mathbb{C} \setminus \{z\}$, then $\mathrm{n}(C_1, z) = \mathrm{n}(C_2, z)$.*

Proof. The function $f(w) = 1/(w-z)$ is holomorphic on $\mathbb{C} \setminus \{z\}$, so from Cauchy's theorem

$$\oint_{C_1} \frac{\mathrm{d}w}{w-z} = \oint_{C_2} \frac{\mathrm{d}w}{w-z}$$

and hence $\mathrm{n}(C_1, z) = \mathrm{n}(C_2, z)$. □

Chapter 5

Power series

In this section of the course we will present some results on power series mostly without proof. If you are interested in the theory behind these results, you can consult Appendix B.

5.1 Taylor's theorem

The complex version of Taylor's theorem can be stated in a form that looks similar to the real version.

Theorem 5.1 (Taylor's Theorem)**.** *If $f : \mathrm{D}(z_0; r) \to \mathbb{C}$ is holomorphic for some $r > 0$ then $f(z)$ has a power series valid on this disc. In other words there exist complex numbers $\{a_n\}$ such that for $z \in \mathrm{D}(z_0; r)$*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \,.$$

Furthermore, these coefficients can be computed as $a_n = \frac{f^{(n)}(z_0)}{n!}$.

Thus, for the existence of a valid power series for a complex function at a point, the function must be holomorphic at this point and the Taylor series is valid on the open disc centred on the point on which the function is holomorphic.

Remark 5.2 (non-examinable)**.** Note that this form of Taylor's theorem connects to the validity of real Taylor series. For instance, consider the real Taylor series about $x = 0$ of $f_R : \mathbb{R} \to \mathbb{R}$ with $f_R(x) = (1 + x^2)^{-1}$. Using the Binomial expansion, this is

$$\begin{aligned} f_R(x) &= 1 + (-1)x^2 + (-1)(-2)(x^2)^2/2! + \ldots \\ &= 1 - x^2 + x^4 - x^6 + \ldots \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n} \,. \end{aligned}$$

The ratio needed in the ratio test is $\lim_{n\to\infty} \left|(-1)^{n+1} x^{2(n+1)}/(-1)^n x^{2n}\right| = x^2$, showing that the series converges for $|x| < 1$, diverges for $|x| > 1$ and the test is inconclusive for $x = \pm 1$.

Using the complex form of the same function, i.e., extending f_R to the complex numbers without changing its behaviour on the reals (by defining $f_C : \mathbb{C} \mapsto \mathbb{C}$, with $f_C(z) = (1 + z^2)^{-1}$) we can understand why the Taylor series of $f_R(x)$ cannot be valid for $|x| > 1$: the complex function $f_C(z)$ is not defined when $1 + z^2 = 0$, i.e., for $z = \pm\mathrm{i}$ and $f_C(z)$ diverges at these points. Thus, the largest open disc about $z = 0$ on which $f(z)$ is holomorphic is $\mathrm{D}(0; 1)$.

It turns out that the radius of convergence of a Taylor series is equal to the distance from z_0 to the nearest pole of the function.

5.2 Differentiation of power series

An important property of holomorphic functions is that they have arbitrarily many derivatives.

Theorem 5.3. *Let $G \subseteq \mathbb{C}$ be open and $f : G \to \mathbb{C}$ be holomorphic. Then f' is holomorphic on G.*

If we have a power series form of a holomorphic function about some point, we can find the power series of the derivative by term-by-term differentiation.

Lemma 5.4. *Suppose $f(z)$ is holomorphic on some open disc $\mathrm{D}(z_0; r) \subseteq \mathbb{C}$ ($r > 0$) and has a power series about z_0 that is valid on this disc, i.e., we can write $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ on $\mathrm{D}(z_0; r)$, then f has derivative $f'(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}$ on $\mathrm{D}(z_0; r)$.*

Remark 5.5. It appears that there is a simple proof proof of the above along the following lines. Let $f_N(z) := \sum_{n=1}^{N} a_n(z - z_0)^n$. Then, since $f(z) = \lim_{N\to\infty} f_N(z)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z) = \frac{\mathrm{d}}{\mathrm{d}z} \lim_{N\to\infty} f_N(z)\,.$$

For any finite N, we can exchange the derivative and the sum, so that $\frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=1}^{N} a_n(z - z_0)^n = \sum_{n=1}^{N} na_n(z - z_0)^{n-1}$ and so $\lim_{N\to\infty} \frac{\mathrm{d}f_N}{\mathrm{d}z} = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}$. The result would therefore hold if we could exchange the limit and the derivative, i.e., if it were true that

$$\frac{\mathrm{d}}{\mathrm{d}z} \lim_{N\to\infty} g_N(z) = \lim_{N\to\infty} \frac{\mathrm{d}}{\mathrm{d}z} g_N(z)\,.$$

Unfortunately, this is not true in general (although it does hold for power series).

A detailed proof of Lemma 5.4 is given in the appendix (Lemma B.22)

Remark 5.6. An important consequence of Lemma 5.4 is that any function with power series about z_0 that is valid on $\mathrm{D}(z_0; r)$ is holomorphic on this disc. This is like a converse of Taylor's theorem (Theorem 5.1).

Remark 5.7. If a function is defined by a power series about z_0, we can check whether it is valid on $\mathrm{D}(z_0; r)$ by testing its convergence, e.g., using the techniques from Section 1.3.

Example 25. Show that the power series

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$$

is holomorphic on $\mathbb{C}$.

Solution. Because of Lemma 5.4, this follows if the power series converges absolutely for all $z \in \mathbb{C}$. We can use the ratio test (Lemma 1.17) to prove this. For any $z \in \mathbb{C}$ we have

$$\lim_{n\to\infty} \left| \frac{z^{2(n+1)}}{(2(n+1)+1)!} \Big/ \frac{z^{2n}}{(2n+1)!} \right| = \lim_{n\to\infty} \frac{|z|^2}{(2n+3)(2n+2)} = 0.$$

Since this is smaller that 1, the series converges for all $z \in \mathbb{C}$.

5.3 The exponential and trigonometric functions for complex variables

Earlier in the course we defined $\exp(x+\mathrm{i}y) = \mathrm{e}^x(\cos(y)+\mathrm{i}\sin(y))$ and then $\cos(z) = \frac{1}{2}(\exp(\mathrm{i}z)+\exp(-\mathrm{i}z))$ and $\sin(z) = \frac{1}{2\mathrm{i}}(\exp(\mathrm{i}z) - \exp(-\mathrm{i}z))$. However, in light of Taylor's theorem we can take alternative definitions via the following power series:

$$\begin{aligned}
\exp(z) &:= \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots \\
\cos(z) &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \dots \\
\sin(z) &:= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots
\end{aligned}$$

We can then check that these converge absolutely for all $z \in \mathbb{C}$ (e.g., using the ratio test as in Example 25). Furthermore, these turn out to be holomorphic everywhere (in fact all absolutely convergent power series are holomorphic). We can then see that the properties we know and love about these functions hold.

For example, using Lemma 5.4, we can differentiate term by term to see that $\exp(z)$ differentiates to itself, that $\cos(z)$ differentiates to $-\sin(z)$ and that the derivative of $\sin(z)$ is $\cos(z)$. Furthermore, $\exp(z)\exp(w) = \exp(z+w)$ can be established by considering $f(z) = \exp(z)\exp(u-z)$. Using the product rule we find $f'(z) = \exp(z)\exp(u-z) - \exp(z)\exp(u-z) = 0$, hence $f(z)$ is constant on $\mathbb{C}$. Thus $f(z) = f(0) = \exp(u)$. Choosing $u = z + w$ then gives the claimed relation. We can also take $w = -z$ to give $\exp(-z) = 1/\exp(z)$.

We see that $\exp(\mathrm{i}z) = \cos(z) + \mathrm{i}\sin(z)$ directly from the power series (since the series are absolutely convergent, their terms can be freely rearranged), and hence $\cos(z) = \frac{1}{2}(\exp(\mathrm{i}z) + \exp(-\mathrm{i}z))$ and $\sin(z) = \frac{1}{2\mathrm{i}}(\exp(\mathrm{i}z) - \exp(-\mathrm{i}z))$.

Chapter 6

Cauchy's residue theorem

By Cauchy's theorem, if f is holomorphic in some *simply connected* region G, then $\oint_C f(z)dz = 0$ for any closed contour C in G. An example where G is not simply connected is $f(z) = \frac{c}{z-z_0}$ with some $z_0, c \in \mathbb{C}$, and $G = \mathbb{C}\backslash\{z_0\}$. From Example 18 and Remark 4.9, we can deduce that $\oint_C f(z)dz = 2\pi ic$ if C is a contour going anticlockwise around z_0. We aim to generalize this formula to any function f which is holomorphic except at a finite number of poles inside C. This leads to a powerful technique for computing contour integrals, called the Residue Theorem. Before formulating it, we will however need to define carefully what we mean by a pole or zero of a holomorphic function.

For convenience in this chapter we will sometimes consider open discs (or punctured open discs) rather than arbitrary open subsets of $\mathbb{C}$. Recall that the punctured open disc centred on $a \in \mathbb{C}$ of radius $r > 0$ is denoted $\mathrm{D}'(a;r) := \{z \in \mathbb{C} : 0 < |z-a| < r\}$.

6.1 Zeros and poles of holomorphic functions

Suppose f is holomorphic on an open set $G \subseteq \mathbb{C}$.

Definition 6.1. We say f has a *zero* at $z_0 \in G$ if $f(z_0) = 0$. We say f has a zero of order $k \in \mathbb{N}$ at z_0 if

$$f(z_0) = f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0 \qquad \text{and} \qquad f^{(k)}(z_0) \neq 0.$$

The set of all zeros of f (within G) is denoted $\mathcal{Z}(f)$. A zero of order 1 is called a *simple* zero.

Example 26.

- $f(z) = z$ has a simple zero at $z = 0$.
- $f(z) = (z - 3 + 4\mathrm{i})^2$ has a zero of order 2 at $z = 3 - 4\mathrm{i}$.
- $f(z) = \mathrm{e}^z(z^2 - 1)$ has simple zeros at $z = 1$ and $z = -1$.

Remark 6.2. Because we can differentiate power series term by term, the order of a zero at z_0 is the lowest power of $(z - z_0)$ in its power series about z_0.

We now define poles, which are in a sense the reciprocal of zeros.

Definition 6.3. If f is holomorphic on $\mathrm{D}'(z_0;r)$ for some $r > 0$ and $z_0 \in \mathbb{C}$, but not at z_0, then z_0 is called an *isolated singularity* of f. We classify isolated singularities as follows. We say z_0 is

- *a removable singularity* if $\lim_{z\to z_0} f(z)$ exists;

- *a pole* if, for some $k \in \mathbb{N}$, $\lim_{z\to z_0}(z-z_0)^k f(z)$ exists and is non-zero (k is called the *order* of the pole);
- *an essential singularity* if neither of the above is true.

Poles of order $1, 2, 3, \ldots$ are also known as *simple*, *double*, *triple*, ... poles.

Example 27. Some examples of the above:

- The function $f(z) = \frac{z^2-1}{z+1}$ has a removable singularity at $z = -1$ because, for $z \neq -1$ we have $f(z) = z - 1$. Thus, if we extend f by taking $f(-1) = -2$ we can remove the singularity.
- The function $f(z) = \frac{z^2}{(z-\mathrm{i})^2}$ has a pole of order 2 at $z = \mathrm{i}$, since $\lim_{z\to\mathrm{i}}(z-\mathrm{i})^2 f(z) = -1$.
- $f(z) = \mathrm{e}^{1/z}$ has an essential isolated singularity at $z = 0$ because $\lim_{z\to 0} z^k \mathrm{e}^{1/z}$ does not exist for any $k \in \mathbb{N}$. To see this, note that the real limit $\lim_{x\to 0} x^k \mathrm{e}^{1/x} = \lim_{n\to\infty} \mathrm{e}^n / n^k$ does not exist (cf. Lemma A.4)

Theorem 6.4. *Suppose f is holomorphic on $\mathrm{D}'(z_0; r)$ for some $r > 0$ and that z_0 is either a pole of order k or a removable singularity, in which case we set $k = 0$. Then there are complex numbers $(a_n)_{n\geq -k}$ such that*

$$f(z) = \sum_{n=-k}^{\infty} a_n (z-z_0)^n$$

for all $z \in \mathrm{D}'(z_0; r)$, where the series converges absolutely. If z_0 is a pole of f, then $a_{-k} \neq 0$.

Proof. Define g on $\mathrm{D}(z_0; r)$ by

$$g(z) = \begin{cases} (z-z_0)^{k+1} f(z) & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0 \end{cases}$$

Because f is holomorphic on $\mathrm{D}'(z_0; r)$, so is g. It turns out that g is also differentiable at z_0: if $z \neq z_0$ then

$$\lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z\to z_0} (z-z_0)^k f(z) = a \tag{6.1}$$

for some $a \in \mathbb{C}$, with $a \neq 0$ if z_0 is a pole of f (from the definition of a pole of order k). Thus, g is holomorphic on $\mathrm{D}(z_0; r)$ and so we can use Taylor's Theorem (Theorem 5.1) to write

$$g(z) = \sum_{n=1}^{\infty} b_n (z-z_0)^n$$

for $z \in \mathrm{D}(z_0; r)$ (the series starts at $n = 1$, because $g(z_0) = 0$, so the constant term is zero). Hence for $z \in \mathrm{D}'(z_0; r)$ we have

$$f(z) = \sum_{n=1}^{\infty} b_n (z-z_0)^{n-k-1} = \sum_{n=-k}^{\infty} a_n (z-z_0)^n$$

where $a_n = b_{n+k+1}$.

Finally, note that (6.1) implies that $a = g'(z_0)$. Differentiating the power series of g term-by-term we find $g'(z_0) = b_1$. Hence, $a_{-k} = b_1 = a$ which is not equal to 0 if z_0 is a pole of f. ☐

Remark 6.5. The case $k = 0$ shows that removable singularities can in fact be "removed": Defining $f(z_0) := a_0$, the extended f is given by a Taylor series, and hence holomorpic in the disc $D(z_0, r)$ (not only in the punctured disc).

Definition 6.6. The expansion

$$f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n$$

in the previous theorem is called the *Laurent expansion* of f about z_0. The two parts of the series

$$\sum_{n=-k}^{-1} a_n (z - z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

are called the *principal part* and the *holomorphic part* of the expansion. The numbers (a_n) are called *Laurent coefficients*, and are determined by f, because they are the Taylor coefficients of $(z - z_0)^k f(z)$.

Example 28. To find the Laurent series of $z^{-3}\cos(z)$ at $z = 0$ we divide the Taylor series of $\cos(z)$ by z^3 term-by-term:

$$\frac{\cos(z)}{z^3} = \frac{1}{z^3}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) = \frac{1}{z^3} - \frac{1}{2z} + \frac{z}{24} - \frac{z^3}{720} + \cdots$$

Thus $z^{-3}\cos(z)$ has a pole of order 3 at $z = 0$. The principal part of the Laurent series is

$$\frac{1}{z^3} - \frac{1}{2z}$$

and the holomorphic part is $z/24 - z^3/720 + \ldots$, or more formally

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+4)!}.$$

Definition 6.7. Suppose $G \subseteq \mathbb{C}$ is open. A function $f : G \to \mathbb{C}$ is said to be *meromorphic* on G if for some set of isolated points $\{z_0, z_1, \ldots\}$ at which f has poles (or removable singularities), it is holomorphic on $G \setminus \{z_0, z_1, \ldots\}$.

Some examples to illustrate the definition:

- The function $f(z) = \frac{\cos(z)}{z(z-1)}$ is meromorphic on $\mathbb{C}$, with isolated poles at 0 and 1.
- The function $f(z) = 1/\sin(z)$ is meromorphic on $\mathbb{C}$ as well: although it has an infinite number of poles (at $z = n\pi$ for $n \in \mathbb{Z}$), these are all isolated.
- The function $f(z) = 1/\sin(\frac{1}{z})$ is not meromorphic on $\mathbb{C}$: the poles accumulate near $z = 0$ and so are not isolated: there is no open set around $z = 0$ in which this function is holomorphic.

Clearly all holomorphic functions are also meromorphic (with emtpy pole set). Another large class of examples is given by ratios of holomorphic functions: if g and h are holomorphic on (for simplicity) some disc G, and h is not identical to zero, then the function $f(z) = g(z)/h(z)$ is holomorphic on $G \setminus \mathcal{Z}(h)$, hence meromorphic on G. In fact, let $z_0 \in \mathcal{Z}(h)$ be a zero of h, let $k \geq 1$ be its order, and let ℓ be the order of z_0 as a zero of g (with $\ell = 0$ if $g(z_0) \neq 0$). Then f has, at z_0,

- a pole of order $k - \ell$, if $k > \ell$,
- a removable singularity if $k \leq \ell$, which is even a zero of order $\ell - k$ if $k < \ell$.

We have implicitly used here that the zeros of h are isolated and have finite order; this can be shown from Taylor's theorem, but we won't do thathere.

6.2 Residues

The coefficient a_{-1} in the Laurent expansion of f around the point z_0 is called the *residue* of f at z_0 and denoted

$$a_{-1} = \operatorname{res}(f, z_0) = \operatorname{res}(f(z), z = z_0) = \operatorname*{res}_{z=z_0} f(z).$$

Example 29. The residue of $z^{-3}\cos(z)$ at $z = 0$ is $-1/2$ as that is the coefficient of $1/z$ in the principal part of the Laurent expansion found in the Example 28.

The significance of the residue is that for sufficiently small $r > 0$, if $C(z_0, r)$ is a positively oriented (i.e., anticlockwise) circle of radius r about z_0 (sufficiently small means small enough that $f_{\text{hol}}(z)$ exists for $|z - z_0| \le r$) then

$$\begin{aligned} \oint_{C(z_0,r)} f(z)\,\mathrm{d}z &= \oint_{C(z_0,r)} (f_{\text{prin}}(z) + f_{\text{hol}}(z))\,\mathrm{d}z \\ &= \sum_{n=-k}^{-1} a_n \oint_{C(z_0,r)} (z - z_0)^{-n}\,\mathrm{d}z + 0 = 2\pi\mathrm{i}a_{-1} = 2\pi\mathrm{i}\operatorname{res}(f, z_0)\,, \end{aligned} \tag{6.2}$$

where we have used Example 18, i.e.,

$$\oint_{C(0,r)} z^n\,\mathrm{d}z = \begin{cases} 2\pi\mathrm{i} & n = -1 \\ 0 & n \ne -1 \end{cases}, \text{ from which } \oint_{C(z_0,r)} (z - z_0)^n\,\mathrm{d}z = \begin{cases} 2\pi\mathrm{i} & n = -1 \\ 0 & n \ne -1 \end{cases}$$

follows directly. Thus, if we can calculate residues, we can also compute the values of certain integrals.

Some hints for calculating residues:

- It is often convenient to manipulate known Taylor series in order to obtain residues, particularly for poles of higher order.

- Use big-O notation, where $O((z-z_0)^m)$ indicates terms of order m. Essentially this means in an absolutely convergent series, all the terms $(z - z_0)^n$ for $n \ge m$ can be swallowed up into $O((z - z_0)^m)$. More formally, if $r(z) = O((z - z_0)^m)$ then there exists a constant $K > 0$ such that $|r(z)| < K|z - z_0|^m$ for all z sufficiently near to z_0.

- Series can be multiplied as normal, bundling any $(z - z_0)^k$ terms with $k \ge m$ into an $O((z - z_0)^m)$ term.

- Make use of the binomial expansion, which says that

  $$(1 + w)^n = 1 + nw + \frac{n(n-1)}{2!}w^2 + \frac{n(n-1)(n-2)}{3!}w^3 + O(w^4) \tag{6.3}$$

  for $|w| < 1$ and $n \in \mathbb{R}$. This can be used to find expansions of expressions like $(1 + f(z))^n$ about a zero z_0 of f, by inserting the Taylor expansion of f about z_0 for w in the right-hand side.

It is important to keep track of the order to which each expansion is made to ensure that no contributions are missed.

Example 30. Find the residue of $f(z) = \cos^2(z)/\sin^3(z)$ at $z = 0$.

Solution. Expanding numerator and denominator, using known power series gives

$$\begin{aligned} f(z) &= \frac{(1-\frac{1}{2}z^2+O(z^4))^2}{(z-\frac{1}{6}z^3+O(z^5))^3} \\ &= \frac{1}{z^3}\left(1-\frac{1}{6}z^2+O(z^4)\right)^{-3}\left(1-\frac{1}{2}z^2+O(z^4)\right)^2 . \end{aligned}$$

we then use the binomial expansion (cf. (6.3)) twice, once with $n=-3$ and $w=-\frac{1}{6}z^2+O(z^4)$ and once with $n=2$ and $w=-\frac{1}{2}z^2+O(z^4)$ to give

$$= \frac{1}{z^3}\left(1+\frac{3}{6}z^2+O(z^4)\right)\left(1-z^2+O(z^4)\right)$$

multiplying out the series, with any term with a power of z above 3 going into the big-O term we find

$$\begin{aligned} &= \frac{1}{z^3}\left(1+\frac{1}{2}z^2-z^2+O(z^4)\right) \\ &= \frac{1}{z^3}-\frac{1}{2z}+O(z) \end{aligned}$$

so the residue is $-1/2$.

Another useful way to compute residues is via the following result.

Lemma 6.8. *If f is meromorphic and z_0 is a pole of f of order k, then*

$$\operatorname{res}(f,z_0) = \begin{cases} \lim_{z\to z_0}(z-z_0)f(z) & \text{if } k=1 \text{ (simple pole)} \\ \lim_{z\to z_0}\frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\left[(z-z_0)^k f(z)\right] & \text{if } k\geq 2. \end{cases}$$

Proof. Since f is meromorphic with a pole of order k at z_0, f has a Laurent expansion on $\mathrm{D}'(z_0;r)$ for some $r>0$:

$$f(z) = \frac{a_{-k}}{(z-z_0)^k}+\cdots+\frac{a_{-2}}{(z-z_0)^2}+\frac{\operatorname{res}(f,z_0)}{z-z_0}+a_0+a_1(z-z_0)+\cdots$$

with $a_{-k}\neq 0$. Then

$$(z-z_0)^k f(z) = a_{-k}+\cdots+a_{-2}(z-z_0)^{k-2}+\operatorname{res}(f,z_0)(z-z_0)^{k-1}+a_0(z-z_0)^k+\cdots$$

If $k=1$, the result follows immediately by taking the limit $z\to z_0$ on both sides. If $k\geq 2$, then

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\left[(z-z_0)^k f(z)\right] = (k-1)!\operatorname{res}(f,z_0)+k!\,a_0(z-z_0)+\cdots ,$$

so

$$\lim_{z\to z_0}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\left[(z-z_0)^k f(z)\right] = (k-1)!\operatorname{res}(f,z_0). \qquad \square$$

Example 31. We can use this to compute

$$\operatorname*{res}_{z=0}(z^{-3}\cos(z)) = \lim_{z\to 0}\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(z^3\frac{\cos(z)}{z^3}\right) = -\frac{1}{2}\cos(0) = -\frac{1}{2},$$

which agrees with what we found in Example 29.

6.3 Cauchy's Residue Theorem

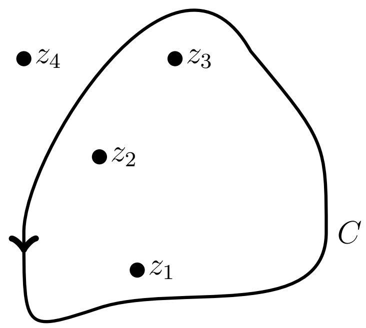
We now arrive at the most important theorem of this course (and one of the most important of all mathematics):

Theorem 6.9 (Cauchy's Residue Theorem). *Let f be meromorphic on a simply-connected open set G, containing a positively-oriented contour C that does not meet any of the poles of f (i.e., the poles are either inside or outside C). Then*

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{m} \operatorname{res}(f, z_k)$$

where $z_1, \ldots, z_m$ are all of the poles enclosed by C.

As an illustration, consider the following contour for which $m = 3$ (because z_4 is outside C):



Proof. Let $C^{\text{int}} \subseteq G$ be the open subset of G enclosed by C, let $g_k(z)$ be the principal part of the Laurent expansion of f about z_k and let $h_k(z)$ be the holomorphic part. Define

$$\tilde{H}(z) := f(z) - \sum_{k=1}^{m} g_k(z)\,.$$

By construction, $\tilde{H}$ has removable singularities at $z_1, \ldots, z_m$, so it can be extended to a function $H(z)$ that is holomorphic on all of C^{int} (as well as on C itself).

[In more detail, note that for each k the function $g_k(z)$ is holomorphic except at $z = z_k$. Thus, we can remove the singularities of $\tilde{H}(z)$ by taking $H(z)$ to be equal to $\tilde{H}(z)$ for $z \in C^{\text{int}} \setminus \{z_1, \ldots, z_m\}$, and by defining $H(z_k) = h_k(z_k) - \sum_{j=1, j\neq k}^{m} g_j(z_k)$ for each $k = 1, \ldots, m$.]

It follows that $\oint_C f(z)\,\mathrm{d}z = \underbrace{\oint_C H(z)\,\mathrm{d}z}_{=0\ \text{(Cauchy)}} + \sum_{k=1}^{m}\oint_C g_k(z)\,\mathrm{d}z = \sum_{k=1}^{m}\oint_C g_k(z)\,\mathrm{d}z$. Equation (6.2) then gives

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{m}\operatorname{res}(g_k, z_k) = 2\pi\mathrm{i}\sum_{k=1}^{m}\operatorname{res}(f, z_k)\,,$$

where the last equality follows from g_k being the principal part of the Laurent series of f at z_k, so the residue of g_k at $z = z_k$ is the residue of f at $z = z_k$. $\square$

Remark 6.10. Because poles of meromorphic functions are isolated, there are only ever finitely many poles enclosed by a finite contour.

Remark 6.11. If C is *negatively* oriented (i.e., clockwise), then $\displaystyle\oint_C f(z)\,\mathrm{d}z = -2\pi\mathrm{i}\sum_{k=1}^{m}\operatorname{res}(f, z_k)$.

The residue theorem provides a bag of tricks for performing integrals. This is the business of 'residue calculus' (next section).

Example 32. Let C be any positively oriented contour in $\mathbb{C}$ enclosing $z = 0$. Compute

$$\oint_C \frac{\cos z}{z^3}\,\mathrm{d}z.$$

Solution. The integrand has a triple pole at $z = 0$ and is holomorphic elsewhere. By Cauchy's residue theorem,

$$\oint_C \frac{\cos z}{z^3}\,\mathrm{d}z = 2\pi\mathrm{i}\operatorname*{res}_{z=0}(z^{-3}\cos(z)) = -\pi\mathrm{i}$$

because the residue is $-1/2$ (as we showed in Example 29).

Example 33. We can use complex methods to evaluate real integrals. As an example, we will show how to find

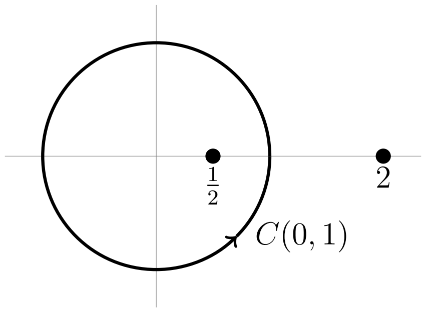
$$I = \int_0^{2\pi} \frac{\mathrm{d}t}{5 - 4\cos(t)}.$$

We substitute $\cos(t) = (\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t})/2$ to give

$$I = \int_0^{2\pi} \frac{\mathrm{d}t}{5 - 2(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t})} = \mathrm{i}\int_0^{2\pi} \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t}{2(\mathrm{e}^{\mathrm{i}t})^2 - 5\,\mathrm{e}^{\mathrm{i}t} + 2}.$$

The circle $C(0,1)$ is parameterized by $\gamma(t) = \mathrm{e}^{\mathrm{i}t}$, $0 \le t \le 2\pi$ with $\gamma'(t) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}t}$, so this integral is the contour integral

$$I = \mathrm{i}\oint_{C(0,1)} \frac{\mathrm{d}z}{2z^2 - 5z + 2} = \mathrm{i}\oint_{C(0,1)} \frac{\mathrm{d}z}{(z-2)(2z-1)} = \frac{\mathrm{i}}{2}\oint_{C(0,1)} \frac{\mathrm{d}z}{(z-2)(z-\frac{1}{2})}.$$



The point $\frac{1}{2}$ is the only pole of the function $\dfrac{1}{(z-2)(z-\frac{1}{2})}$ enclosed by $C(0,1)$, which is positively oriented. This point is a simple pole. Hence

$$I = \frac{\mathrm{i}}{2}\cdot 2\pi\mathrm{i}\operatorname{res}\left(\frac{1}{(z-2)(z-\frac{1}{2})}, z = \tfrac{1}{2}\right) = -\pi\lim_{z\to\frac{1}{2}}\frac{1}{z-2} = \frac{2\pi}{3}.$$

Example 34. Let $f(z) = \dfrac{\mathrm{e}^{z^2}}{(z-2)^3(z-1)^2 z}$. Find $\displaystyle\oint_{C(0,4)} f(z)\,\mathrm{d}z$.

Solution. The function has a simple pole at 0, a double pole at 1 and a triple pole at 2.

For the simple pole at $z = 0$,

$$\operatorname*{res}_{z=0}(f(z)) = \lim_{z\to 0} z f(z) = \left. \frac{\mathrm{e}^{z^2}}{(z-2)^3(z-1)^2} \right|_{z=0} = \frac{-1}{8}$$

and for the double pole at $z = 1$,

$$\operatorname*{res}_{z=1}(f(z)) = \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z} \big((z-1)^2 f(z)\big) = \left. \frac{2\,\mathrm{e}^{z^2}}{(z-2)^3} - \frac{3\,\mathrm{e}^{z^2}}{(z-2)^4 z} - \frac{\mathrm{e}^{z^2}}{(z-2)^3 z^2} \right|_{z=1} = -4\,\mathrm{e}$$

and for the triple pole at $z = 2$,

$$\begin{aligned}\operatorname*{res}_{z=2}(f(z)) &= \lim_{z\to 2} \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} \big((z-2)^3 f(z)\big) = \lim_{z\to 2} \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{\mathrm{e}^{z^2}}{(z-1)^2 z}\right) = \lim_{z\to 2} \frac{-1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{e}^{z^2}(1-3z-2z^2+2z^3)}{(z-1)^3 z^2}\right) \\ &= \left. \frac{\mathrm{e}^{z^2}(1-4z+5z^2+6z^3-3z^4-4z^5+2z^6)}{(z-1)^4 z^3} \right|_{z=2} = \frac{13\,\mathrm{e}^4}{8}\,.\end{aligned}$$

Therefore integrating around the positively oriented circle centred at $z = 0$ with radius 4, which encloses all the poles, we have

$$\oint_{C(0,4)} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\left(\operatorname*{res}_{z=0}(f(z)) + \operatorname*{res}_{z=1}(f(z)) + \operatorname*{res}_{z=2}(f(z))\right) = 2\pi\mathrm{i}\left(\frac{-1}{8} - 4\,\mathrm{e} + \frac{13\,\mathrm{e}^4}{8}\right).$$

Chapter 7

Calculus of residues

7.1 Principal value integrals

In ordinary calculus on $\mathbb{R}$, the integral $\displaystyle\int_0^\infty f(x)\,\mathrm{d}x$ is shorthand for $\displaystyle\lim_{R\to\infty}\int_0^R f(x)\,\mathrm{d}x$, while the integral $\displaystyle\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ requires more care because of the two infinite tail regions, each of which should tend to infinity independently. The *principal value integral* is defined by

$$⨍_{-\infty}^{\infty} f(x)\,\mathrm{d}x := \lim_{R\to\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x$$

and in this module we will always interpret integrals over $\mathbb{R}$ in this way.

To see why this is important, note that (for example)

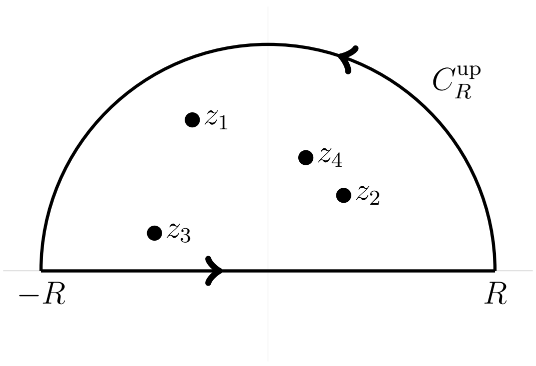
$$\int_{-\infty}^{\infty}\frac{2x}{x^2+1}\,\mathrm{d}x = \lim_{S,R\to\infty}\int_{-S}^{R}\frac{2x}{x^2+1}\,\mathrm{d}x$$

is not defined (the indefinite integral is $\log(x^2+1)$, which tends to ∞ as $x\to\infty$), whereas

$$⨍_{-\infty}^{\infty}\frac{2x}{x^2+1}\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R}\frac{2x}{x^2+1}\,\mathrm{d}x = 0.$$

Definition 7.1. The open upper half-plane is denoted $\mathbb{H} := \{z\in\mathbb{C} : \operatorname{Im} z > 0\}$, and the upper half-plane including the real axis is $\overline{\mathbb{H}} := \{z\in\mathbb{C} : \operatorname{Im} z \geq 0\}$.

Let C_R^{up} be the upper semicircle of radius R centred at 0. Suppose f is meromorphic on the upper half-plane including the real line, i.e., on $\overline{\mathbb{H}}$. (Since $\overline{\mathbb{H}}$ is not open, f must be meromorphic on an open set containing $\overline{\mathbb{H}}$.) Suppose f has poles in $\mathbb{H}$ at $z_1, z_2, \ldots, z_m$. Then, for large enough R all the poles in $\mathbb{H}$ are enclosed by C_R^{up} joined with the straight line segment $[-R, R]$.



By Cauchy's Residue Theorem (Theorem 6.9) we have

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^{\mathrm{up}}} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{m} \operatorname{res}(f, z_k) \tag{7.1}$$

since the contour is positively oriented. If (and it is something that needs to be proved)

$$\lim_{R\to\infty} \int_{C_R^{\mathrm{up}}} f(z)\,\mathrm{d}z = 0$$

then by taking the limit $R \to \infty$ of (7.1), we find

$$\int_{-\infty}^{\infty} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{m} \operatorname{res}(f, z_k).$$

This method of computing principal value integrals is sometimes referred to as 'closing the contour in the upper half-plane'.

So, what we need to do is

1. Check $\displaystyle\int_{C_R^{\mathrm{up}}} f(z)\,\mathrm{d}z \to 0$ as $R \to \infty$.
2. Compute the residues at the poles in $\mathbb{H}$.

To make this a practical technique, we need some methods for showing that the integral over C_R^{up} tends to zero as $R \to \infty$.

Defining $M_R = \max_{z\in C_R^{\mathrm{up}}} |f(z)|$ then by the estimation lemma

$$\left|\int_{C_R^{\mathrm{up}}} f(z)\,\mathrm{d}z\right| \leq \pi R M_R$$

so if $RM_R \to 0$ as $R \to \infty$ then the integral over C_R^{up} tends to 0 as $R \to \infty$. The following lemma gives a useful way of bounding $|f(z)|$ on C_R^{up}.

Lemma 7.2. *Let $p(z)$ and $q(z)$ be polynomials of degrees m and n, respectively. Then there exist positive constants A and R_0 such that if $|z| \geq R_0$, then*

$$\left|\frac{p(z)}{q(z)}\right| \leq A|z|^{m-n}.$$

Proof. By assumption $|p(z)/z^m|$ and $|q(z)/z^n|$ have nonzero limits as $|z| \to \infty$. Thus

$$B := \lim_{|z|\to\infty} \left|z^{n-m}\frac{p(z)}{q(z)}\right| \neq 0$$

and so for any $A > B$ there exists R_0 such that

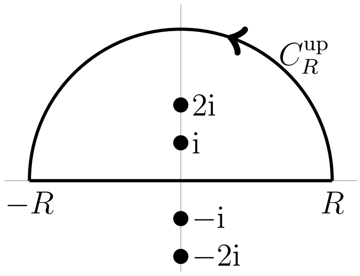
$$\left|z^{n-m}\frac{p(z)}{q(z)}\right| \leq A$$

for all $|z| \geq R_0$. □

Example 35. Use a suitable contour to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)(x^2+4)}.$$

Solution. Let $f(z) = \dfrac{1}{(z^2+1)(z^2+4)} = \dfrac{1}{(z+\mathrm{i})(z-\mathrm{i})(z+2\mathrm{i})(z-2\mathrm{i})}$, which has simple poles at $z = \pm\mathrm{i}$ and $z = \pm 2\mathrm{i}$. We form a contour by closing in the upper half plane:



The poles $z = \mathrm{i}, 2\mathrm{i}$ are enclosed by the contour, so we find their residues

$$\begin{aligned}
\operatorname{res}(f,\mathrm{i}) &= \left.\frac{1}{(z+\mathrm{i})(z^2+4)}\right|_{z=\mathrm{i}} = \frac{1}{6\mathrm{i}},\\
\operatorname{res}(f,2\mathrm{i}) &= \left.\frac{1}{(z^2+1)(z+2\mathrm{i})}\right|_{z=2\mathrm{i}} = -\frac{1}{12\mathrm{i}}.
\end{aligned}$$

Hence

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^{\mathrm{up}}} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\left(\frac{1}{6\mathrm{i}} - \frac{1}{12\mathrm{i}}\right) = \frac{\pi}{6}.$$

Now using Lemma 7.2 we have that there exists a positive real A such that

$$|f(z)| = \frac{1}{|(z^2+1)(z^2+4)|} \le \frac{A}{|z|^4} = \frac{A}{R^4}$$

if z is on C_R^{up} (so that $|z| = R$) and R is sufficiently large. Hence, using the estimation lemma (Lemma 3.22) we have

$$\left|\int_{C_R^{\mathrm{up}}} f(z)\,\mathrm{d}z\right| \le \frac{A}{R^4}\cdot \pi R = \frac{A\pi}{R^3}$$

The right-hand side tends to 0 as $R \to \infty$. Hence

$$\lim_{R\to\infty}\int_{C_R^{\mathrm{up}}} f(z)\,\mathrm{d}z = 0 \quad \text{from which we find} \quad \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{\pi}{6}.$$

Remark 7.3. We could also have closed in the lower half plane. As an exercise, you might like to check that the same answer is obtained. Be careful about the signs, noting that the contour is clockwise when closing in the lower half plane.

Remark 7.4. This integral can also be evaluated by a real-calculus method:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)(x^2+4)} &= \frac{1}{3}\int_{-\infty}^{\infty}\left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right)\mathrm{d}x\\
&= \frac{1}{3}\left[\tan^{-1}x - \frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-\infty}^{\infty} = \frac{1}{3}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi}{6}.
\end{aligned}$$

7.2 Integrals of the form $\int_{-\infty}^{\infty} g(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x$

In this section we consider how to compute integrals of this form where $k \in \mathbb{R}$ and g is meromorphic on $\mathbb{C}$ with no poles on the real axis and at most finitely many poles in total. Such integrals are important (we will see why in the next chapter).

Writing $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$, if $k \in \mathbb{R}$ then we have $\mathrm{e}^{\mathrm{i}kz} = \mathrm{e}^{\mathrm{i}kx - ky}$. This implies $|\,\mathrm{e}^{\mathrm{i}kz}\,| = \mathrm{e}^{-ky} = \mathrm{e}^{-k\operatorname{Im} z}$ and hence

1. if $k \geq 0$ and if z is in $\overline{\mathbb{H}}$, then $|\,\mathrm{e}^{\mathrm{i}kz}\,| \leq 1$
2. if $k \leq 0$ and if z is not in $\mathbb{H}$, then $|\,\mathrm{e}^{\mathrm{i}kz}\,| \leq 1$.

If $k \geq 0$ we close the contour in the upper half-plane, defining $M_R = \max_{z \in C_R^{\mathrm{up}}} |g(z)|$ where C_R^{up} is the upper semicircle of radius R, centred at 0. By the estimation lemma

$$\left| \int_{C_R^{\mathrm{up}}} g(z)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}z \right| \leq \pi R M_R$$

so if $RM_R \to 0$ as $R \to \infty$ then the integral over C_R^{up} tends to 0 as $R \to \infty$, in which case

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = 2\pi\mathrm{i} \sum_{k=1}^{m} \operatorname*{res}_{z=z_k} \left(g(z)\,\mathrm{e}^{\mathrm{i}kz}\right),$$

where $z_1, \ldots, z_m$ are the poles of g in the upper half-plane.

On the other hand, if $k \leq 0$, we have

$$|\,\mathrm{e}^{\mathrm{i}kz}\,| = \mathrm{e}^{|k|\,\mathrm{Im}\,z}$$

which is exponentially growing for $z \in \mathbb{H}$, so we cannot use our previous argument for the semicircular contour in $\mathbb{H}$. However, we do have $|\,\mathrm{e}^{\mathrm{i}kz}\,| \leq 1$ in the *lower* half-plane, which suggests that we close the contour in the *lower* half-plane. In this case we define $M_R = \max_{z \in C_R^{\mathrm{low}}} |g(z)|$ where C_R^{low} is the lower semicircle of radius R, centred at 0. By the estimation lemma

$$\left| \int_{C_R^{\mathrm{low}}} g(z)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}z \right| \leq \pi R M_R$$

so if $RM_R \to 0$ as $R \to \infty$ then the integral over C_R^{low} tends to 0 as $R \to \infty$, which gives

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = -2\pi\mathrm{i} \sum_{k=1}^{m} \operatorname*{res}_{z=z_k} \left(g(z)\,\mathrm{e}^{\mathrm{i}kz}\right)$$

where $z_1, \ldots, z_m$ are the poles of g in the *lower* half-plane.

Note the minus sign here, which comes from the contour being negatively oriented when closing using the lower semicircle.

7.2.1 Jordan's lemma

There are some cases in which the above estimation procedure doesn't work, but for which we can still conclude that the integral around either C_R^{up} or C_R^{low} is zero: this is the subject of Jordan's lemma. More precisely, it allows us to cover the case that RM_R does not tend to 0 as R tends to ∞ provided $k \neq 0$ and M_R tends to 0 as R tends to ∞. In effect, it gains us a factor of $1/R$ by doing the estimation more carefully.

Lemma 7.5 (Jordan's Lemma)**.** *Let g be meromorphic on $\mathbb{C}$ with at most finitely many poles in total.*

1. *Suppose $k > 0$ and let $M_R = \max_{z \in C_R^{up}} |g(z)|$ where C_R^{up} is the upper semicircle of radius R, centred at 0. If $M_R \to 0$ as $R \to \infty$ then*

$$\lim_{R \to \infty} \int_{C_R^{up}} g(z)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}z = 0\,.$$

2. *Suppose $k < 0$ and let $M_R = \max_{z \in C_R^{low}} |g(z)|$ where C_R^{low} is the lower semicircle of radius R, centred at 0. If $M_R \to 0$ as $R \to \infty$ then*

$$\lim_{R\to\infty} \int_{C_R^{low}} g(z)\,\mathrm{e}^{ikz}\,\mathrm{d}z = 0\,.$$

Remark 7.6. It is important that $k \neq 0$, and that the conditions are checked in the correct half-plane.

Proof. (Not examinable) For the first case, we parameterize the upper semicircle C_R^{up} by $\gamma(t) = R\,\mathrm{e}^{\mathrm{i}t}$, $0 \le t \le \pi$. Thus $\gamma'(t) = \mathrm{i}R\,\mathrm{e}^{\mathrm{i}t}$, and

$$\begin{aligned}\int_{C_R^{\mathrm{up}}} g(z)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}z &= \int_0^\pi g(R\,\mathrm{e}^{\mathrm{i}t}) \exp\left[\mathrm{i}kR(\cos(t)+\mathrm{i}\sin(t))\right] \mathrm{i}R\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t \\ &= \mathrm{i}R\int_0^\pi g(R\,\mathrm{e}^{\mathrm{i}t})\,\mathrm{e}^{\mathrm{i}kR\cos(t)}\,\mathrm{e}^{-kR\sin(t)}\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t.\end{aligned}$$

Hence, recalling that $|\,\mathrm{e}^{\mathrm{i}\theta}\,| = 1$ if $\theta \in \mathbb{R}$, and using $|g(R\,\mathrm{e}^{\mathrm{i}t})| \le M_R$ we find

$$\left|\int_{C_R^{\mathrm{up}}} g(z)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}z\right| \le RM_R \int_0^\pi \mathrm{e}^{-kR\sin(t)}\,\mathrm{d}t.$$

Since $\sin(t)$ is symmetric about $t = \pi/2$, the contributions to this integral from $[0,\pi/2]$ and from $[\pi/2,\pi]$ are equal.[1] Hence

$$\left|\int_{C_R^{\mathrm{up}}} g(z)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}z\right| \le 2RM_R \int_0^{\pi/2} \mathrm{e}^{-kR\sin(t)}\,\mathrm{d}t.$$

Since $\sin(t) \ge 2t/\pi$ for $0 \le t \le \pi/2$ (this is fairly obvious by drawing the graphs[2]) we have

$$\mathrm{e}^{-kR\sin(t)} \le \mathrm{e}^{-2kRt/\pi} \quad \text{for} \quad 0 \le t \le \pi/2.$$

Hence

$$\left|\int_{C_R^{\mathrm{up}}} g(z)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}z\right| \le 2RM_R \int_0^{\pi/2} \mathrm{e}^{-2kRt/\pi}\,\mathrm{d}t = \frac{\pi M_R}{k}\left(1-\mathrm{e}^{-kR}\right) \le \frac{\pi M_R}{k}.$$

As $M_R \to 0$ as $R \to \infty$, we have

$$\lim_{R\to\infty}\int_{C_R^{\mathrm{up}}} g(z)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}z = 0.$$

The proof of the second case is similar – it is worth going through it to check that you have understood the argument. □

[1] Writing it out in full,

$$\begin{aligned}\int_0^\pi \mathrm{e}^{-kR\sin(t)}\,\mathrm{d}t &= \int_0^{\pi/2} \mathrm{e}^{-kR\sin(t)}\,\mathrm{d}t + \int_{\pi/2}^{\pi} \mathrm{e}^{-kR\sin(t)}\,\mathrm{d}t = \int_0^{\pi/2} \mathrm{e}^{-kR\sin(t)}\,\mathrm{d}t + \int_0^{\pi/2} \mathrm{e}^{-kR\sin(\pi-s)}\,\mathrm{d}s \qquad (t=\pi-s) \\ &= \int_0^{\pi/2} \mathrm{e}^{-kR\sin(t)}\,\mathrm{d}t + \int_0^{\pi/2} \mathrm{e}^{-kR\sin(s)}\,\mathrm{d}s = 2\int_0^{\pi/2} \mathrm{e}^{-kR\sin(t)}\,\mathrm{d}t.\end{aligned}$$

[2] More formally, check that $f(t) = \sin(t) - 2t/\pi$ has a single local maximum in $(0,\pi/2)$; then because $f(0) = f(\pi/2) = 0$, it follows that $f \ge 0$ on $[0,\pi/2]$. This is known as Jordan's inequality.

Corollary 7.7. *Let g be a complex function with no poles on $\mathbb{R}$. If g satisfies the conditions for Jordan's lemma for $k > 0$ then*

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{i=1}^{m} \operatorname{res}(g(z)\,\mathrm{e}^{\mathrm{i}kz}, z_i)$$

where $z_1, \ldots, z_m$ are the poles of g in the upper *half-plane. Likewise, if g satisfies the conditions for Jordan's lemma for $k < 0$ then*

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = -2\pi\mathrm{i}\sum_{i=1}^{m} \operatorname{res}(g(z)\,\mathrm{e}^{\mathrm{i}kz}, z_i)$$

where $z_1, \ldots, z_m$ are the poles of g in the lower *half-plane.*

7.3 Examples

Example 36. Show that for real k

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{x^2+1}\,\mathrm{d}x = \pi\,\mathrm{e}^{-|k|}$$

Solution. Let $f(z) = \mathrm{e}^{\mathrm{i}kz}\,g(z)$ where $g(z) = 1/(z^2+1) = 1/((z+\mathrm{i})(z-\mathrm{i}))$.

First consider the case $k \geq 0$. On the circle of radius R we have $z = R\,\mathrm{e}^{\mathrm{i}\theta}$ and hence $\left|\mathrm{e}^{\mathrm{i}kz}\right| = \left|\mathrm{e}^{\mathrm{i}kR\cos(\theta)}\,\mathrm{e}^{-kR\sin(\theta)}\right| = \mathrm{e}^{-kR\sin(\theta)}$. In the upper half plane, $0 \leq \theta \leq \pi$ so $\sin(\theta) \geq 0$ and hence $|\,\mathrm{e}^{\mathrm{i}kz}\,| \leq 1$.

Furthermore, there exists a constant $A > 0$ such that $|g(R\,\mathrm{e}^{\mathrm{i}\theta})| \leq A/R^2$ for sufficiently large R (cf. Lemma 7.2). Hence, the estimation lemma gives $\left|\int_{C_R^{\mathrm{up}}} f(z)\,\mathrm{d}z\right| \leq \frac{A}{R^2}\pi R$ so that $\lim_{R\to\infty}\int_{C_R^{\mathrm{up}}} f(z)\,\mathrm{d}z = 0$.

[Note that because there exists a constant $A > 0$ such that $|g(R\,\mathrm{e}^{\mathrm{i}\theta})| \leq A/R^2$ for sufficiently large R and so if $M_R = \max_{z\in C_R^{\mathrm{up}}} |g(z)|$ we have $M_R \to 0$ as $R \to \infty$ we could also have used Jordan's lemma, but here there is no need.]

Only the pole at $z = \mathrm{i}$ is in the upper half plane, so we compute the residue there:

$$\operatorname{res}(f, \mathrm{i}) = \lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{\mathrm{e}^{\mathrm{i}|k|z}}{(z-\mathrm{i})(z+\mathrm{i})} = \frac{1}{2\mathrm{i}}\,\mathrm{e}^{-|k|}\,.$$

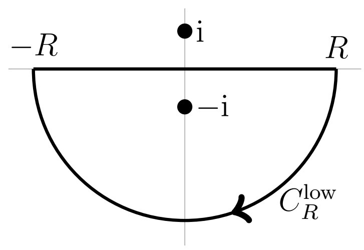
Hence

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^{\mathrm{up}}} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\operatorname{res}(f,\mathrm{i}) = \pi\,\mathrm{e}^{-|k|}\,.$$

Thus, by taking the limit $R \to \infty$ we find

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}|k|x}}{x^2+1}\,\mathrm{d}x = \pi\,\mathrm{e}^{-|k|}\,.$$

If $k \leq 0$ we can close the contour in the lower half-plane.



Note that this contour is *negatively* oriented, in order that we can proceed along the real portion in the usual direction, from $-R$ to R. For sufficiently large R, by a similar argument to the case $k \geq 0$ we have $\lim_{R\to\infty} \int_{C_R^{\text{low}}} f(z)\,\mathrm{d}z = 0$. Thus, taking the limit as $R \to \infty$ in

$$\int_{-R}^{R} \frac{\mathrm{e}^{\mathrm{i}kx}}{x^2+1}\,\mathrm{d}x + \int_{C_R^{\text{low}}} \frac{\mathrm{e}^{\mathrm{i}kz}}{z^2+1}\,\mathrm{d}z = -2\pi\mathrm{i}\,\mathrm{res}\left(\frac{\mathrm{e}^{\mathrm{i}kz}}{z^2+1}, z=-\mathrm{i}\right) = -2\pi\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}k(-\mathrm{i})}}{-2\mathrm{i}} = \pi\,\mathrm{e}^{k} = \pi\,\mathrm{e}^{-|k|}$$

(note the leading minus sign on the RHS due to the clockwise contour) gives

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{x^2+1}\,\mathrm{d}x = \pi\,\mathrm{e}^{-|k|}\,.$$

Putting together the two cases,

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kx}}{x^2+1}\,\mathrm{d}x = \pi\,\mathrm{e}^{-|k|}$$

for any real k.

Example 37. Using the above, and taking the real part it follows that for $k \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2+1}\,\mathrm{d}x = \pi\,\mathrm{e}^{-|k|}\,.$$

We can also obtain this using $\cos(kx) = (\mathrm{e}^{\mathrm{i}kx} + \mathrm{e}^{-\mathrm{i}kx})/2$ and using the above to do each part separately.

Remark 7.8. One might imagine going straight for $\cos(kx)/(x^2+1)$ without using the complex exponential. This does not work because if we consider $|\cos(kz)/(z^2+1)|$ on the semicircle of radius R we find that it becomes arbitrarily large as R increases. In more detail, recall from the first set of exercises that $|\cos(x+\mathrm{i}y)|^2 = \cosh^2(y) - \sin^2(x)$ (for x and y real). Thus, for $z = R\,\mathrm{e}^{\mathrm{i}\theta}$ we have $|\cos(kz)|^2 = \cosh^2(kR\sin\theta) - \sin^2(kR\cos\theta)$, which grows like $\cosh^2(kR)$ for $\theta = \pm\pi/2$. Thus, $|\cos(kz)/(z^2+1)|$ grows like $\cosh(kR)/R^2$ for these values of θ and hence can be made arbitrarily large as R increases rendering us unable to use the estimation lemma. Fortunately, this problem is avoided by decomposing the cosine and closing in a different direction for each part.

Example 38. Use contour integration to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(kx)}{x}\,\mathrm{d}x$ for $k > 0$.

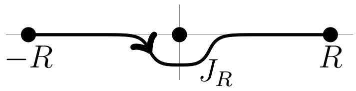
Solution. This looks similar to the previous example, except that if we write $\sin(kz)$ in terms of exponentials, we will be left with integrals like $\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kz}/z\,\mathrm{d}z$ which passes straight through the simple pole at $z = 0$. However, because the singularity of $\sin(kz)/z$ at $z = 0$ is removable we can consider integrating $f(z) = k\,\mathrm{sinc}(kz)$ where

$$\mathrm{sinc}(z) := \begin{cases} \sin(z)/z & z \neq 0 \\ 1 & z = 0 \end{cases},$$

which is holomorphic. Using Cauchy's theorem, the integral of $f(z)$ is the same for any curve from $-R$ to R. Thus,

$$\int_{-R}^{R} \frac{\sin(kx)}{x}\,\mathrm{d}x = \int_{[-R,R]} f(z)\,\mathrm{d}z = \int_{J_R} f(z)\,\mathrm{d}z = \int_{J_R} \frac{\sin(kz)}{z}\,\mathrm{d}z\,,$$

where J_R is some other curve that passes below $z = 0$.

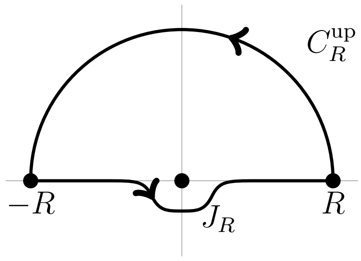


Because this curve passes below $z = 0$, we can now consider splitting the sine function, so that

$$\int_{-R}^{R} \frac{\sin(kx)}{x}\,\mathrm{d}x = \frac{1}{2\mathrm{i}}\left(\int_{J_R} \frac{\mathrm{e}^{\mathrm{i}kz}}{z}\,\mathrm{d}z - \int_{J_R} \frac{\mathrm{e}^{-\mathrm{i}kz}}{z}\,\mathrm{d}z\right).$$

We tackle each term separately.

For the first, we close in the upper half-plane, setting C_R^{up} to be the upper semicircle of radius R centred at 0 as shown



Therefore

$$\int_{J_R} \frac{\mathrm{e}^{\mathrm{i}kz}}{z}\,\mathrm{d}z + \int_{C_R^{\mathrm{up}}} \frac{\mathrm{e}^{\mathrm{i}kz}}{z}\,\mathrm{d}z = 2\pi\mathrm{i}\ \mathrm{res}\left(\frac{\mathrm{e}^{\mathrm{i}kz}}{z}, z = 0\right) = 2\pi\mathrm{i}.$$

We have

$$\lim_{R\to\infty} \int_{C_R^{\mathrm{up}}} \frac{\mathrm{e}^{\mathrm{i}kz}}{z}\,\mathrm{d}z = 0$$

by Jordan's lemma with $g(z) = 1/z$. This is because $k > 0$, $|g(z)| = 1/R$ if z is on C_R^{up} and $1/R \to 0$ as $R \to \infty$. Hence

$$\lim_{R\to\infty} \int_{J_R} \frac{\mathrm{e}^{\mathrm{i}kz}}{z}\,\mathrm{d}z = 2\pi\mathrm{i}. \tag{7.2}$$

For the other integral we can use a similar argument, but closing in the lower half plane, to give

$$\lim_{R\to\infty} \int_{J_R} \frac{\mathrm{e}^{-\mathrm{i}kz}}{z}\,\mathrm{d}z = 0.$$

[For the details, see the exercises.]

Therefore,

$$\int_{-\infty}^{\infty} \frac{\sin(kx)}{x}\,\mathrm{d}x = \frac{1}{2\mathrm{i}} \lim_{R\to\infty}\left(\int_{J_R} \frac{\mathrm{e}^{\mathrm{i}kz}}{z}\,\mathrm{d}z - \int_{J_R} \frac{\mathrm{e}^{-\mathrm{i}kz}}{z}\,\mathrm{d}z\right) = \frac{1}{2\mathrm{i}} 2\pi\mathrm{i} = \pi.$$

Remark 7.9. Note that in this example we need Jordan's lemma; straightforward application of the estimation lemma doesn't work.

Remark 7.10. Note that we cannot take the imaginary part of (7.2) to directly give the answer. [Exercise: Why?]

Remark 7.11. Note that for $k < 0$ we can write $k' = -k$ to give

$$\int_{-\infty}^{\infty} \frac{\sin(kx)}{x}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\sin(-k'x)}{x}\,\mathrm{d}x = -\int_{-\infty}^{\infty} \frac{\sin(k'x)}{x}\,\mathrm{d}x = -\pi.$$

Hence, for any $k \in \mathbb{R}$ we have $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(kx)}{x}\,\mathrm{d}x = \pi\,\mathrm{sgn}(k)$, where sgn is the sign function

$$\mathrm{sgn}(k) = \begin{cases} 1 & k > 0 \\ 0 & k = 0 \\ -1 & k < 0 \end{cases}.$$

Chapter 8

Fourier transforms

8.1 Motivation and definition

We begin with a reminder about something closely related to Fourier transforms: *Fourier series.* These provide a way of expanding a (sufficiently nice) function with period L in terms of sines and cosines with evenly spaced frequencies:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right) , \tag{8.1}$$

where the *Fourier coefficients* are given by

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2n\pi x}{L}\right) \mathrm{d}x \qquad \text{for } \ n \geq 0 \qquad \text{and} \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n\pi x}{L}\right) \mathrm{d}x \qquad \text{for } \ n > 0. \end{aligned}$$

If a function is not periodic, but is defined on some finite interval (e.g. $[0, L]$), then a periodic extension can be considered and described by a Fourier series. If f is nice enough (for instance if it satisfies the Dirichlet conditions[1]), then the series converges to the function wherever the function is continuous, and otherwise converges to the average of the left and right limits of the function at a discontinuity. Thus, strictly, we should write

$$\frac{1}{2}(f_+(x) + f_-(x)) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right) ,$$

where $f_\pm(x) = \lim\limits_{x'\to x\pm} f(x')$ are the left and right limits (rather than having "$f(x)$" on the left hand side).

Fourier series have limitations: they cannot be used for functions that extend over $\mathbb{R}$, or that involve a continuum of frequencies (for example a glissando on a trombone, or the Doppler shifted sound of a passing siren). Fourier transforms allow us to deal with these limitations.

To introduce them, it is useful to work with the complex form of the Fourier series in which we write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, \mathrm{e}^{2\mathrm{i}n\pi x/L} \qquad \text{with} \qquad c_n = \frac{1}{L}\int_0^L f(x)\,\mathrm{e}^{-2\mathrm{i}n\pi x/L} \, \mathrm{d}x \quad (n \in \mathbb{Z}).$$

[1] Namely, f is absolutely integrable over one period and has at most a finite number of maxima, minima and jump discontinuities over one period.

This is just a convenient relabelling: by writing cos and sin in terms of complex exponentials, we see that the trigonometric coefficients are related to the complex ones by

$$a_n = c_n + c_{-n} \qquad b_n = \mathrm{i}(c_n - c_{-n}) \qquad (n \geq 0)$$

and also that

$$\begin{aligned}
&\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right) \\
&= c_0 + \sum_{n=1}^{\infty}\left((c_n + c_{-n})\cos\left(\frac{2n\pi x}{L}\right) + \mathrm{i}(c_n - c_{-n})\sin\left(\frac{2n\pi x}{L}\right)\right) \\
&= c_0 + \sum_{n=1}^{\infty}\left(c_n \,\mathrm{e}^{2\mathrm{i}n\pi x/L} + c_{-n}\,\mathrm{e}^{-2\mathrm{i}n\pi x/L}\right) \\
&= \sum_{n=-\infty}^{\infty} c_n \,\mathrm{e}^{2\mathrm{i}n\pi x/L}.
\end{aligned}$$

The *Fourier transform* provides a related representation, but as an integral, rather than a sum.

Definition 8.1. The *Fourier transform* of $f : \mathbb{R} \to \mathbb{C}$ is a function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

We also write $\mathcal{F}[f](k)$ and $\mathcal{F}[f(x)](k)$ for $\hat{f}(k)$. The *inverse Fourier transform* of $g : \mathbb{R} \to \mathbb{C}$ is

$$\check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k$$

and is also written $\mathcal{F}^{-1}[g](x)$ or $\mathcal{F}^{-1}[g(k)](x)$.

Remark 8.2. The exact sense in which the inverse transform is a genuine inverse of the transform requires elucidation (see Theorem 8.6 below).

Remark 8.3. The Fourier transform and its inverse are closely related. From the formulae it follows that $\mathcal{F}^{-1}[g(k)](x) = \mathcal{F}[g(k)](-x)$. This can be useful to relate properties of Fourier transforms to analogous properties of their inverses. Be careful when using this because of the switch of usual variables on the right hand side.

The idea of Fourier transforms is to analyse a function f as an integral of terms with different frequencies k, with the transform $\hat{f}(k)$ indicating the relative contribution of each frequency k. Thus, if $|\hat{f}(k)|$ is peaked near some $k = k_0$, then f has a strong contribution of frequency k. This is the basis of signal processing (e.g., k_0 may be the frequency of some unwanted noise that we wish to process away).

Example 39. Find the Fourier transform of the function $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(x) = \begin{cases} \mathrm{e}^{ax} & x \leq 0 \\ 0 & x > 0 \end{cases} \qquad \text{with } \operatorname{Re}(a) > 0.$$

Solution. Because f is defined in a 'piecewise' fashion, we have to split up the integral into pieces

$$\begin{aligned}\hat{f}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \underbrace{f(x)}_{=\mathrm{e}^{ax}}\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \underbrace{f(x)}_{=0}\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \mathrm{e}^{(a-\mathrm{i}k)x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\frac{1}{a-\mathrm{i}k}\end{aligned}$$

for all $k \in \mathbb{R}$. (The computation of the inverse transform of this function is a nice application of contour integration methods – a similar computation is on the exercises.)

Remark 8.4. The Fourier transform can be regarded as the $L \to \infty$ limit of the Fourier series.

We have stated above the definition of a Fourier transform and its inverse, but without giving conditions on when these exist. In order to state this we need the concept of absolute integrability.

Definition 8.5. A function $f : \mathbb{R} \to \mathbb{C}$ is *absolutely integrable* if $\displaystyle\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x$ exists.

Example 40. $f(x) = \sin(x)/(1+x^2)$ is absolutely integrable, but $f(x) = 1/(x+\mathrm{i})$ is not.

Absolute integrability is a sufficient condition for a function to have a Fourier transform and inverse Fourier transform.

Theorem 8.6 (Fourier inversion theorem)**.** *Suppose $f : \mathbb{R} \to \mathbb{C}$ is absolutely integrable, and continuous except at (at most) finitely many points, at which f nonetheless has finite left and right limits and one-sided derivatives.*[2] *Then*

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k = \frac{1}{2}\left(f(x+) + f(x-)\right) \qquad \text{for all } x \in \mathbb{R}.$$

In particular, if f is continuous, so $f(x) = \frac{1}{2}(f(x+) + f(x-))$ for all x, then

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k$$

for all x and the inverse Fourier transform is a genuine inverse to the Fourier transform.

Remark 8.7. There are stronger versions of the inversion theorem, but these are beyond the scope of this course. We also won't give a proof in this course; for completeness, a proof is provided in Appendix C, but it is not examinable.

Remark 8.8. There are other conventions for Fourier transforms! A common alternative is to define the transform without the factor of $1/\sqrt{2\pi}$ and to put a $1/(2\pi)$ in the inverse transform, i.e., to define

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-ikx}\,\mathrm{d}x \qquad \check{g}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k)\,\mathrm{e}^{ikx}\,\mathrm{d}k\,.$$

Some of the relations for our definitions need additional factors of 2π for other definitions, e.g., Remark 8.3.

We won't use these alternative definitions in this course, but it is worth being aware of them when you are reading other resources.

[2] f has one-sided derivatives at x if $\displaystyle\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$ or $\displaystyle\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}$ exist.

8.2 Examples

Example 41. The Fourier transform of $f(x) = \begin{cases} 1 & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$ is

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x.$$

If $k \neq 0$ we may integrate to find

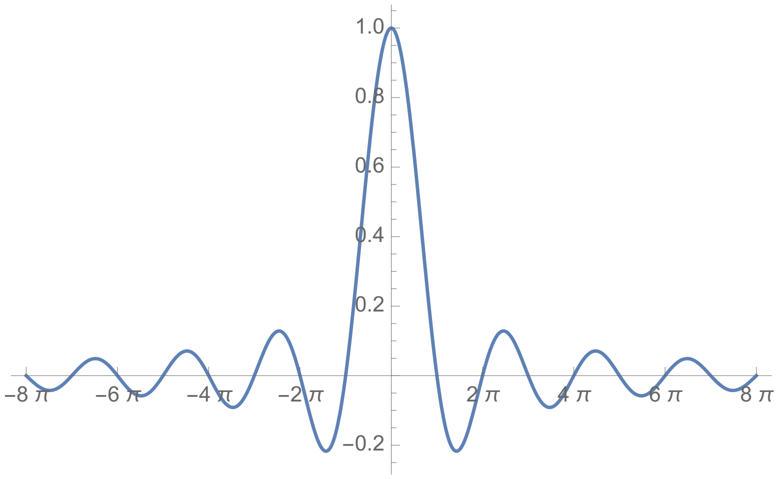
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{\mathrm{e}^{-\mathrm{i}kx}}{-\mathrm{i}k} \right]_{-1}^{1} = \frac{\mathrm{e}^{-\mathrm{i}k} - \mathrm{e}^{\mathrm{i}k}}{-\sqrt{2\pi}k\mathrm{i}} = \sqrt{\frac{2}{\pi}} \frac{\sin(k)}{k}$$

while if $k = 0$ we cannot divide by k, so calculate separately:

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}0x}\, \mathrm{d}x = \frac{2}{\sqrt{2\pi}}.$$

Thus, $\hat{f}(k) = \sqrt{2/\pi}\,\mathrm{sinc}(k)$, where sinc is the function defined in Example 38.

Remark 8.9. Our original function f vanishes outside a bounded region, and is discontinuous. By contrast, $\hat{f}(k)$ is proportional to $\mathrm{sinc}(k)$ which is smooth and decays slowly to 0:



$\mathrm{sinc}(k)$

This is a typical feature of Fourier transforms: decay and smoothness properties are exchanged. In fact, $\hat{f}$ is better than smooth: it is a holomorphic function on all of $\mathbb{C}$, which is always true for functions f that are zero except on a finite interval.

Example 42. Use the Fourier inversion theorem to find the inverse transform of $\hat{f}$ from Example 41.

Solution. The inversion theorem states that the inverse transform of the Fourier transform is the average of the left and right limits of the function at every point. Since the function is continuous except at $|x| = 1$, the inverse transform of $\hat{f}$ is equal to the function for $|x| \neq 1$. At both $x = -1$ and $x = 1$, the average of the left and right limits of f is $1/2$. Hence, we have

$$\mathcal{F}^{-1}[\hat{f}](x) = \int_{-\infty}^{\infty} \frac{\mathrm{sinc}(k)}{\pi}\, \mathrm{e}^{ikx}\, \mathrm{d}k = \begin{cases} 0 & x < -1 \\ 1/2 & x = -1 \\ 1 & -1 < x < 1 \\ 1/2 & x = 1 \\ 0 & x > 1 \end{cases},$$

Example 43. Let's see that this agrees with what we get by doing the integral directly:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-K}^{K}\sqrt{\frac{2}{\pi}}\operatorname{sinc}(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k &= \int_{-K}^{K}\frac{\sin(k)}{\pi k}(\cos(kx)+\mathrm{i}\sin(kx))\,\mathrm{d}k \\
&= \int_{-K}^{K}\frac{\sin(k)\cos(kx)}{\pi k}\,\mathrm{d}k && (\sin(k)\sin(kx)/k \text{ is odd}) \\
&= \int_{-K}^{K}\frac{\sin(k(x+1))}{2\pi k}\,\mathrm{d}k - \int_{-K}^{K}\frac{\sin(k(x-1))}{2\pi k}\,\mathrm{d}k && (\text{trig identities})
\end{aligned}$$

so taking $K\to\infty$, we can use $\displaystyle\int_{-\infty}^{\infty}\frac{\sin(kx)}{k}\,\mathrm{d}k = \pi\operatorname{sgn}(x)$ (this is Remark 7.11 with x and k exchanged) to give

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{2}\left(\operatorname{sgn}(x+1) - \operatorname{sgn}(x-1)\right).$$

Carefully collecting all the cases, we can see that this agrees with the claim using the inversion theorem.

Example 44. Find $\mathcal{F}[\mathrm{e}^{-a|x|}](k)$ for $a>0$.

$$\begin{aligned}
\mathcal{F}[\mathrm{e}^{-a|x|}](k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\mathrm{e}^{(a-\mathrm{i}k)x}\,\mathrm{d}x + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\mathrm{e}^{-(a+\mathrm{i}k)x}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a-\mathrm{i}k}+\frac{1}{a+\mathrm{i}k}\right) = \sqrt{\frac{2}{\pi}}\frac{a}{k^2+a^2}
\end{aligned}$$

We have already seen in Example 36 that $\displaystyle\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kx}}{\pi(k^2+1)}\,\mathrm{d}k = \mathrm{e}^{-|x|}$ for all $x\in\mathbb{R}$. We can use this to check the inversion of our result:

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{a\,\mathrm{e}^{\mathrm{i}kx}}{\pi(k^2+a^2)}\,\mathrm{d}k &= \int_{-\infty}^{\infty}\frac{a\,\mathrm{e}^{\mathrm{i}mx'}}{\pi a^2(m^2+1)}a\,\mathrm{d}m && (k=am;\ x=x'/a) \\
&= \mathrm{e}^{-|x'|} = \mathrm{e}^{-a|x|},
\end{aligned}$$

which agrees with the inversion theorem.

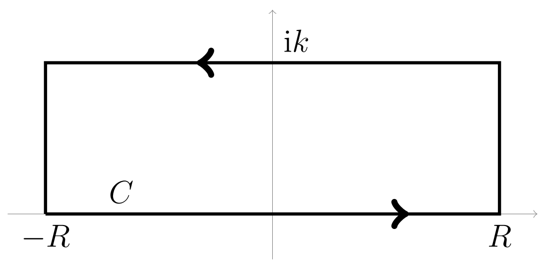
Example 45. Because the formulae for transform and inverse transform are closely related (cf. Remark 8.3), the result of the previous example immediately gives that for $a>0$ we have

$$\mathcal{F}[a/(x^2+a^2)](k) = \mathcal{F}^{-1}[a/(x^2+a^2)](-k) = \sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-a|-k|} = \sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-a|k|}.$$

Example 46 (The Gaussian)**.** To find $\mathcal{F}[\mathrm{e}^{-x^2/2}](k)$, first observe that

$$\mathcal{F}[\mathrm{e}^{-x^2/2}](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-x^2/2-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-k^2/2}\int_{-\infty}^{\infty}\mathrm{e}^{-(x+\mathrm{i}k)^2/2}\,\mathrm{d}x \quad (\text{completing a square}) \tag{8.2}$$

Now consider the rectangular contour C with vertices R, $R+\mathrm{i}k$, $-R+\mathrm{i}k$ and $-R$ (taking $k>0$):



Because the integrand is holomorphic, Cauchy's theorem gives $\oint_C \mathrm{e}^{-z^2/2}\,\mathrm{d}z = 0$. For the two vertical sections, we use the estimation lemma (Lemma 3.22):

1. On the vertical line between R and $R + \mathrm{i}k$ we have $z = R + \mathrm{i}t$ for $0 \le t \le k$. Thus, $|\,\mathrm{e}^{-z^2/2}\,| = |\,\mathrm{e}^{-(R+\mathrm{i}t)^2/2}\,| = |\,\mathrm{e}^{-R^2/2 - \mathrm{i}t + t^2/2}\,| = \mathrm{e}^{-R^2/2+t^2/2} \le \mathrm{e}^{-R^2/2+k^2/2}$. Thus, by the estimation lemma, the absolute value of the integral on this line is at most $k\,\mathrm{e}^{-R^2/2+k^2/2}$, which tends to zero as R tends to ∞.

2. On the vertical line between $-R$ and $-R + \mathrm{i}k$ we have $z = -R + \mathrm{i}t$ for $0 \le t \le k$. Thus, $|\,\mathrm{e}^{-z^2/2}\,| = |\,\mathrm{e}^{-(-R+\mathrm{i}t)^2/2}\,| = |\,\mathrm{e}^{-R^2/2 + \mathrm{i}t + t^2/2}\,| = \mathrm{e}^{-R^2/2+t^2/2} \le \mathrm{e}^{-R^2/2+k^2/2}$. Thus, by the estimation lemma, the absolute value of the integral on this line is at most $k\,\mathrm{e}^{-R^2/2+k^2/2}$, which tends to zero as R tends to ∞.

The case $k < 0$ can be treated in the same way, and hence it follows that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-(x+\mathrm{i}k)^2/2}\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi}\,, \tag{8.3}$$

where we have used a standard result for the last part.[3]

Plugging this into (8.2) gives the result: $\mathcal{F}[\mathrm{e}^{-x^2/2}](k) = \mathrm{e}^{-k^2/2}$.

The Gaussian is an eigenvector (usually called an eigenfunction in this context) of the Fourier transformation map, with eigenvalue 1.

8.3 Properties of Fourier transforms

8.3.1 Basic properties

For functions $f, g : \mathbb{R} \to \mathbb{C}$, whenever the relevant Fourier transforms exist, they obey:

- **Linearity.** $\mathcal{F}[\alpha f + \beta g](k) = \alpha\mathcal{F}[f](k) + \beta\mathcal{F}[g](k)$ for $\alpha, \beta \in \mathbb{C}$.

- **Dilation.** $\mathcal{F}[f(x/a)](k) = |a|\,\mathcal{F}[f(x)](ak)$, for any $a \in \mathbb{R} \setminus \{0\}$.

  Written another way: If $g(x) = f(x/a)$ (with $a \in \mathbb{R} \setminus \{0\}$) then $\hat{g}(k) = |a|\hat{f}(ak)$.

- **Complex conjugation.** $\mathcal{F}[\overline{f}](k) = \overline{\mathcal{F}[f](-k)}$. Equivalently, $\hat{\overline{f}}(k) = \overline{\hat{f}(-k)}$.

- **Translation.** $\mathcal{F}[f(x-a)](k) = \mathrm{e}^{-\mathrm{i}ak}\,\mathcal{F}[f(x)](k)$ for any $a \in \mathbb{R}$.
  Equivalently, if $g(x) = f(x-a)$ for $a \in \mathbb{R}$, then $\hat{g}(k) = \mathrm{e}^{-\mathrm{i}ak}\,\hat{f}(k)$.

- **Phase-shift.** $\mathcal{F}[\mathrm{e}^{\mathrm{i}ax}\,f(x)](k) = \mathcal{F}[f](k-a)$ for any $a \in \mathbb{R}$.
  Equivalently, $\hat{g}(k) = \hat{f}(k-a)$ if $g(x) = \mathrm{e}^{\mathrm{i}ax}\,f(x)$.

These are simple exercises using the definitions. As an example, we prove the translation property.

[3] To compute the positive constant $A = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2}\,\mathrm{d}x$, note that $A^2 = \int_{-\infty}^{\infty}\mathrm{d}y \int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-(x^2+y^2)/2}$. Converting to plane polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $A^2 = \int_0^{\infty}\int_0^{2\pi} r\,\mathrm{e}^{-r^2/2}\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\left[-\mathrm{e}^{-r^2/2}\right]_0^{\infty} = 2\pi$.

Proof of translation property. For $g(x) = f(x-a)$,

$$\begin{aligned}\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')\, \mathrm{e}^{-\mathrm{i}k(x'+a)}\, \mathrm{d}x' \qquad (x' = x-a) \\ &= \frac{\mathrm{e}^{-\mathrm{i}ka}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')\, \mathrm{e}^{-\mathrm{i}kx'}\, \mathrm{d}x' = \mathrm{e}^{-\mathrm{i}ka}\, \hat{f}(k)\,.\end{aligned}$$

Thus, $\hat{g}(k) = \mathrm{e}^{-\mathrm{i}ak}\, \hat{f}(k)$ as required. $\square$

Proofs for some of the remaining properties can be found on the exercises/problem class questions.

Remark 8.10. Note that due to its relation to the transform (cf. Remark 8.3) the inverse transform has similar properties and that translation in normal space becomes phase-shift in Fourier space and *vice versa*. For example, from $\mathcal{F}[f(x/a)](k) = |a|\,\mathcal{F}[f(x)](ak)$, we have

$$\mathcal{F}^{-1}[g(k/a)](x) = \mathcal{F}[g(k/a)](-x) = |a|\,\mathcal{F}[g(k)](-ax) = |a|\,\mathcal{F}^{-1}[g(k)](ax)\,.$$

[We could also directly prove this from the definition of $\mathcal{F}^{-1}$.]

Example 47. From Example 46, we can now derive a slightly more general result, namely that for real $b > 0$, we have $\mathcal{F}[\frac{1}{\sqrt{b}}\, \mathrm{e}^{-x^2/2b}](k) = \mathrm{e}^{-bk^2/2}$. To see this, simply use the dilation property with $a = \sqrt{b}$:

$$\mathcal{F}[\mathrm{e}^{-x^2/2b}/\sqrt{b}](k) = \frac{1}{\sqrt{b}}\mathcal{F}[\mathrm{e}^{-(x/a)^2/2}](k) = \frac{a}{\sqrt{b}}\mathcal{F}[\mathrm{e}^{-x^2/2}](ap) = \mathrm{e}^{-bp^2/2}\,.$$

The value of b tells us something about the width of the Gaussian: as b increases, the Gaussian $\mathrm{e}^{-x^2/2b}/\sqrt{b}$ gets wider. This example shows that the Fourier transform of a wide Gaussian is a narrow Gaussian and vice versa.

It is a general feature that wide functions have narrow Fourier transforms and vice versa. This is the basis of the uncertainty principle in quantum mechanics.

Example 48. Find the Fourier transform of $g(x) = \begin{cases} \sin(x) & x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$.

Solution. Note that $g(x) = \dfrac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}\right) f(x)$, where $f(x) = \begin{cases} 1 & x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$ has transform $\hat{f}(k) = \sqrt{2/\pi}\,\mathrm{sinc}(k)$ (see Example 41). Using the phase-shift and linearity properties, we obtain

$$\hat{g}(k) = \frac{1}{2\mathrm{i}}\left(\hat{f}(k-1) - \hat{f}(k+1)\right) = \frac{-\mathrm{i}}{\sqrt{2\pi}}\left(\mathrm{sinc}(k-1) - \mathrm{sinc}(k+1)\right).$$

Example 49. The Fourier transform of $h(x) = \mathrm{e}^{-a|x-1|}$ is $\hat{h}(k) = \sqrt{\dfrac{2}{\pi}}\dfrac{a\,\mathrm{e}^{-\mathrm{i}k}}{a^2+k^2}$ by the translation property and Example 44.

8.3.2 Further properties of Fourier transforms

Lemma 8.11 (Fourier transforms of derivatives)**.** *Suppose f is differentiable, obeys $f(x) \to 0$ as $|x| \to \infty$, and both f and f' have Fourier transforms. Then*

$$\mathcal{F}[f'](k) = \mathrm{i}k\mathcal{F}[f](k).$$

Proof. Integrating by parts for some $R > 0$ we have

$$\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f'(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[f(x)\,\mathrm{e}^{-\mathrm{i}kx} \right]_{-R}^{R} + \frac{\mathrm{i}k}{\sqrt{2\pi}} \int_{-R}^{R} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$$

Thus, taking the limit of large R, $\mathcal{F}[f'](k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f'(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \mathrm{i}k\mathcal{F}[f](k)$, as required. □

Corollary 8.12. *If f is n times differentiable and $f, f', \ldots, f^{(n)}$ all have Fourier transforms and tend to 0 as $|x|$ tends to ∞, then*

$$\mathcal{F}[f^{(n)}](k) = (\mathrm{i}k)^n \mathcal{F}[f](k)\,. \tag{8.4}$$

Proof. This follows by iterating the previous result n times. □

Example 50. For $f(x) = a/(a^2 + x^2)$ with $a > 0$, $\mathcal{F}[f](k) = \sqrt{\pi/2}\,\mathrm{e}^{-a|k|}$. Thus

$$f'(x) = \frac{-2ax}{(a^2 + x^2)^2} \qquad \text{has Fourier transform} \qquad \mathcal{F}[f'](k) = \mathrm{i}k\sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-a|k|}$$

(this could be computed by contour integration, but it would be messy: note the double poles at $\pm\mathrm{i}a$). Similarly,

$$\mathcal{F}[2a(3x^2 - a^2)/(a^2 + x^2)^3](k) = \mathcal{F}[f''](k) = -k^2\sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-a|k|}$$

We now consider differentiation of a Fourier transforms. Consider $\hat{f}'(k) = \dfrac{\mathrm{d}}{\mathrm{d}k} \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$. For sufficiently nice functions (for example, if both f and xf are absolutely integrable), we can exchange the differentiation and integration. For such functions

$$\hat{f}'(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(-\mathrm{i}x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = -\mathrm{i}\mathcal{F}[xf](k).$$

Thus $\hat{f}'(k) = \mathcal{F}[-\mathrm{i}xf(x)](k)$, or, equivalently, $\mathcal{F}[xf(x)](k) = \mathrm{i}\hat{f}'(k)$.
This can be iterated, to give the following.

Lemma 8.13 (Differentiation of a Fourier transform)**.** *If $x^s f(x)$ is absolutely integrable for each $s = 0, 1, \ldots, n$ then $\hat{f}(k)$ is n-times differentiable and*

$$\hat{f}^{(s)}(k) = (-\mathrm{i})^s \mathcal{F}[x^s f(x)](k) \qquad \textit{equivalently,} \qquad \mathcal{F}[x^s f(x)](k) = \mathrm{i}^s \hat{f}^{(s)}(k)$$

for all $k \in \mathbb{R}$ and $s = 0, 1, \ldots, n$.

Remark 8.14. These properties will be useful later when we use Fourier transforms to solve differential equations.

8.3.3 Plancherel's theorem

Theorem 8.15 (Plancherel's theorem). *Let f be square-integrable, i.e., $\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x < \infty$, then its Fourier transform, $\hat{f}$, is square-integrable and*

$$\int_{-\infty}^{\infty} |\hat{f}(k)|^2 \mathrm{d}k = \int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x. \tag{8.5}$$

More generally, if f and g are square-integrable then

$$\int_{-\infty}^{\infty} \overline{\hat{f}(k)} \, \hat{g}(k) \, \mathrm{d}k = \int_{-\infty}^{\infty} \overline{f(x)} \, g(x) \, \mathrm{d}x. \tag{8.6}$$

Remark 8.16. Plancherel's theorem is the Fourier transform analogue of Parseval's theorem for Fourier series, the latter stating that for a function $f(x)$ with the form of (8.1) we have

$$\int_0^L |f(x)|^2 \, \mathrm{d}x = \frac{L}{2} \left(\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \right).$$

Because of this analogy, Plancherel's theorem is sometimes called Parseval's theorem for Fourier transforms (especially in physics).

Proof sketch. Equation (8.5) can be obtained from (8.6) for $g = f$, so it suffices to show (8.6). We have

$$\begin{aligned}
\int_{-\infty}^{\infty} \overline{\hat{f}(k)} \, \hat{g}(k) \mathrm{d}k &= \int_{-\infty}^{\infty} \overline{\hat{f}(k)} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \, \mathrm{e}^{-ikx} \, \mathrm{d}x \right) \mathrm{d}k \\
&= \int_{-\infty}^{\infty} g(x) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\hat{f}(k) \, \mathrm{e}^{-ikx}} \, \mathrm{d}k \right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} g(x) \, \overline{f(x)} \, \mathrm{d}x
\end{aligned}$$

where we have exchanged order of integration and used the inversion theorem. (These manipulations are valid for sufficiently nice f and g – and if you study more analysis in the future, you will find out that this is sufficient to prove the result in general.) □

Example 51. We have seen previously (Example 41) that

$$f(x) = \begin{cases} 1 & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases} \quad \text{has Fourier transform} \quad \hat{f}(k) = \sqrt{\frac{2}{\pi}} \, \mathrm{sinc}(k).$$

Plancherel's theorem tells us $\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(k)}{k^2} \, \mathrm{d}k = \int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \int_{-1}^{1} \mathrm{d}x = 2,$ and hence

$$\int_{-\infty}^{\infty} \frac{\sin^2(k)}{k^2} \, \mathrm{d}k = \pi.$$

(This can be verified by contour integral methods (as in one of the exercises), but it's fairly tricky.)

Remark 8.17. Plancherel's theorem has a nice interpretation in terms of signals: if $V(x)$ is the voltage level at time x, then the total energy conveyed by the signal is

$$E = \frac{1}{R} \int_{-\infty}^{\infty} |V(x)|^2 \, \mathrm{d}x$$

(where R is the resistance of the load, e.g., a speaker – those who have studied physics may remember that power $= VI = V^2/R$, where I is the current, and that energy is the time integral of power). Plancherel's theorem tells us that the energy can also be expressed as

$$E = \frac{1}{R}\int_{-\infty}^{\infty} |\hat{V}(k)|^2 \, \mathrm{d}k$$

which leads to the idea that $\frac{1}{R}|\hat{V}(k)|^2 \delta k$ is the energy associated with frequency range $[k, k+\delta k]$.

8.3.4 The Convolution Theorem

The next important property of Fourier transforms is called the convolution theorem. It gives us an expression for the Fourier transform of a product of two functions in terms of the original Fourier transforms. Before getting to this, we need to introduce convolutions.

A convolution is a way of turning two functions into another. Suppose f and g are absolutely integrable. Then we may define a new function

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x')\, g(x - x')\, \mathrm{d}x'$$

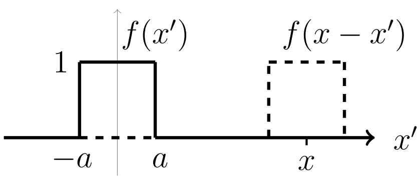
which is called the *convolution* of f and g. Convolutions have some nice properties, e.g.,

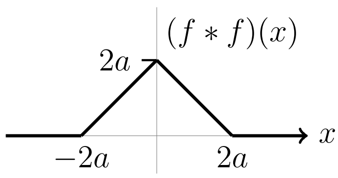
$$f * g = g * f \quad \text{and} \quad f * (g * h) = (f * g) * h \quad \text{(exercises)}.$$

Example 52. Consider the convolution of $f(x)$ with itself, where $f(x) = \begin{cases} 1 & x \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$.

It is clear that $(f * f)(x) = 0$ for $x - a \geq a$ and for $x + a \leq -a$, i.e., for $|x| \geq 2a$. In between we have

$$\begin{aligned} (f * f)(x) &= \int_{\max(-a,x-a)}^{\min(a,x+a)} \mathrm{d}x = \min(a, x+a) - \max(-a, x-a) \\ &= \begin{cases} a - (x - a) = 2a - x & 0 \leq x \leq 2a \\ x + a - (-a) = 2a + x & -2a \leq x \leq 0 \end{cases} \end{aligned}$$





Remark 8.18. Convolutions are useful in statistics. For two continuous random variables, X_1 and X_2, taking values over the reals and distributed according to $P_{X_1}(x)$ and $P_{X_2}(x)$ the distribution of the sum of the random variables, $Y = X_1 + X_2$ is $P_Y(y) = \int_{-\infty}^{\infty} P_{X_1}(x) P_{X_2}(y - x)\, \mathrm{d}x = (P_{X_1} * P_{X_2})(y)$.

Theorem 8.19 (Convolution theorem)**.** *If f and g are absolutely integrable, then*

$$\mathcal{F}[f * g](k) = \sqrt{2\pi}\, \hat{f}(k)\, \hat{g}(k) \qquad \forall k \in \mathbb{R}$$

i.e., the transform of a convolution is the product of the transforms multiplied by $\sqrt{2\pi}$.

Proof sketch. We have

$$\begin{aligned}\mathcal{F}[f*g](k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x')\,g(x-x')\,\mathrm{d}x'\right)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x')\,g(x-x')\,\mathrm{d}x\right)\mathrm{d}x'\end{aligned}$$

where we have exchanged the order of integration without asking questions

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx'} f(x')\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k(x-x')}\,g(x-x')\,\mathrm{d}x\right)\mathrm{d}x' \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx'} f(x')\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx''}\,g(x'')\,\mathrm{d}x''\right)\mathrm{d}x' \qquad (x''=x-x') \\ &= \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k). \end{aligned}$$

□

Example 53. Find the Fourier transform of $(f*f)$ where f is the function in Example 52.

Solution. We first find the Fourier transform of f:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\left[\frac{\mathrm{e}^{-\mathrm{i}kx}}{-\mathrm{i}k}\right]_{x=-a}^{x=a} = \sqrt{\frac{2}{\pi}}\frac{\sin(ka)}{k} \quad (k\neq 0)$$

The case $k=0$ is readily incorporated by writing this as $\sqrt{2/\pi}\,a\,\mathrm{sinc}(ka)$. Thus, $\mathcal{F}[f*f](k) = \sqrt{2\pi}(\hat{f}(k))^2 = \sqrt{2\pi}\dfrac{2a^2\,\mathrm{sinc}^2(ka)}{\pi} = 2\sqrt{\dfrac{2}{\pi}}\,a^2\,\mathrm{sinc}^2(ka)$.

Example 54. Let $f(x)$ be the function in Example 52 and $g(x) = 1/(1+x^2)$. In Examples 53 and 45, we found

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}}\,a\,\mathrm{sinc}(ka) \qquad \text{and} \qquad \hat{g}(k) = \sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-|k|}\,.$$

Thus

$$\begin{aligned}(f*g)(x) &= \int_{-a}^{a}\frac{1}{1+(x-x')^2}\,\mathrm{d}x' \qquad\qquad \text{use } x' = x+\tan\theta \\ &= \tan^{-1}(a-x)+\tan^{-1}(a+x)\end{aligned}$$

which would be difficult to transform directly. However, using the convolution theorem we find

$$\mathcal{F}[(f*g)(x)](k) = \sqrt{2\pi}\hat{f}(k)\,\hat{g}(k) = \sqrt{2\pi}\,\mathrm{e}^{-|k|}\,a\,\mathrm{sinc}(ka).$$

Example 55. Using $\mathcal{F}[(x^2+a^2)^{-1}](k) = \sqrt{\pi/2}\,\mathrm{e}^{-a|k|}\,/a$ for $a>0$, find an absolutely integrable continuous function f obeying

$$\int_{-\infty}^{\infty}\frac{f(x')}{(x-x')^2+a^2}\,\mathrm{d}x' = \frac{1}{x^2+b^2}\,,\quad 0<a<b\,.$$

Solution. Note that the equation can be written as $(f*g_a)(x) = g_b(x)$, where $g_a(x) = \frac{1}{x^2+a^2}$. Since g_a is absolutely integrable as is (by requirement) f, we may apply the convolution theorem to find

$$\sqrt{2\pi}\hat{f}(k)\,\hat{g}_a(k) = \hat{g}_b(k)$$

Using the relation in the question gives

$$\sqrt{2\pi}\hat{f}(k)\sqrt{\frac{\pi}{2}}\frac{\mathrm{e}^{-a|k|}}{a} = \sqrt{\frac{\pi}{2}}\frac{\mathrm{e}^{-b|k|}}{b}\,.$$

Thus,

$$\hat{f}(k) = \frac{a}{b\sqrt{2\pi}}\,\mathrm{e}^{-(b-a)|k|} = \frac{a(b-a)}{\pi b}\sqrt{\frac{\pi}{2}}\frac{\mathrm{e}^{-(b-a)|k|}}{b-a} = \frac{a(b-a)}{\pi b}\mathcal{F}[g_{b-a}](k)\,.$$

We deduce that $f(x) = \dfrac{a(b-a)}{\pi b(x^2+(b-a)^2)}$ is a solution.

Chapter 9

Application to differential equations

An important application of Fourier transforms is in solving differential equations. In this chapter, techniques for doing so will be illustrated with examples.

9.1 Ordinary differential equations

9.1.1 A second order ODE with constant coefficients

We consider the differential equation

$$f''(x) - a^2 f(x) = g(x)$$

for constant $a > 0$.

The general solution to the homogeneous equation (i.e., with $g(x) \equiv 0$) is readily found to be

$$f(x) = A\,\mathrm{e}^{ax} + B\,\mathrm{e}^{-ax}$$

for constants A, B. The two independent solutions to the homogeneous equation decay to zero either as $x \to \infty$ or as $x \to -\infty$.

Here we use Fourier methods to find a specific solution of the inhomogeneous equation ($g(x) \not\equiv 0$), assuming that f and g are nice enough that the formulae for transforms of derivatives apply.

We apply the Fourier transform to our ODE using $\mathcal{F}[f'](k) = \mathrm{i}k\,\hat{f}(k)$ which we proved earlier (under the assumption that f is differentiable and obeys $f(x) \to 0$ as $|x| \to \infty$, and that both f and f' have Fourier transforms — cf. Lemma 8.13). This gives

$$-k^2\,\hat{f}(k) - a^2\,\hat{f}(k) = \hat{g}(k),$$

and so

$$\hat{f}(k) = -\frac{\hat{g}(k)}{k^2 + a^2}.$$

Note that this gives a unique solution: by using Fourier methods we have implicitly imposed boundary conditions that the solution f should decay as $x \to \pm\infty$ (so that the transform integrals exist).

The only remaining problem is that we need to invert the transform, i.e., to compute

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-\hat{g}(k)\,\mathrm{e}^{\mathrm{i}kx}}{k^2 + a^2}\,\mathrm{d}k.$$

To assist with this we can write $\hat{f}(k) = \hat{g}(k)\,\hat{h}(k)$, where $\hat{h}(k) = -1/(k^2 + a^2)$. Using the convolution theorem (Theorem 8.19) we see that $\sqrt{2\pi}\,f(x) = (g * h)(x)$.

Example 44 gives that $\mathcal{F}[\mathrm{e}^{-a|x|}](k) = \sqrt{2/\pi}\, a/(k^2 + a^2)$ for $a > 0$. Therefore, $h(x) = -\sqrt{\pi/2}\,\mathrm{e}^{-a|x|}/a$. We can hence write

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x')\, h(x - x')\, \mathrm{d}x' \\ &= \frac{-1}{2a} \int_{-\infty}^{\infty} g(x')\ \mathrm{e}^{-a|x-x'|}\ \mathrm{d}x', \end{aligned}$$

as the solution for general g (provided g has a Fourier transform).

For a more explicit example, let us specialize the function $g(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$, for which the convolution integral can be computed explicitly: We have

$$f(x) = \frac{-1}{2a} \int_0^1 \mathrm{e}^{-a|x-x'|}\ \mathrm{d}x'.$$

We divide into three cases. For $x \le 0$ and $0 \le x' \le 1$, we have $-|x - x'| = x - x'$ and so

$$f(x) = \frac{-1}{2a} \int_0^1 \mathrm{e}^{a(x-x')}\ \mathrm{d}x' = \frac{1}{2a^2}\,\mathrm{e}^{ax} \left[\mathrm{e}^{-ax'}\right]_{x'=0}^{x'=1} = \frac{1}{2a^2}\,\mathrm{e}^{ax}(\mathrm{e}^{-a} - 1).$$

Similarly, for $x \ge 1$ and $0 \le x' \le 1$, we have $-|x - x'| = -(x - x')$ and so

$$f(x) = \frac{-1}{2a} \int_0^1 \mathrm{e}^{-a(x-x')}\ \mathrm{d}x' = \frac{1}{2a^2}\,\mathrm{e}^{-ax}(1 - \mathrm{e}^{a}).$$

In between we have $-|x - x'| = \begin{cases} -(x - x') & x' \le x \\ x - x' & x' > x \end{cases}$. Thus,

$$f(x) = \frac{-1}{2a} \left(\int_0^x \mathrm{e}^{-a(x-x')}\ \mathrm{d}x' + \int_x^1 \mathrm{e}^{a(x-x')}\ \mathrm{d}x' \right) = \frac{1}{2a^2} \left(\mathrm{e}^{a(x-1)} + \mathrm{e}^{-ax} - 2 \right).$$

Bringing everything together, we have

$$f(x) = \begin{cases} \frac{1}{2a^2}\,\mathrm{e}^{ax}(\mathrm{e}^{-a} - 1) & x < 0 \\ \frac{1}{2a^2}\left(\mathrm{e}^{a(x-1)} + \mathrm{e}^{-ax} - 2\right) & 0 \le x \le 1 \\ \frac{1}{2a^2}\,\mathrm{e}^{-ax}(1 - \mathrm{e}^{a}) & x > 1 \end{cases}.$$

Note that this looks almost like it satisfies the homogeneous equation. However, there is one region where it does not, where the -2 appears for $0 \le x \le 1$. Note also how the solution uses the solutions of the homogeneous equation to patch together the solutions for the different regions in such a way as to satisfy the (implicit) boundary conditions that $f(x) \to 0$ as $|x| \to \infty$, and such that $f(x)$ is continuous.

9.1.2 Hermite's equation

Consider the differential equation

$$f''(x) + x f'(x) + f(x) = 0 \tag{9.1}$$

where f is such that $f(x)$, $xf(x)$ $f'(x)$, $xf'(x)$ and $f''(x)$ all have Fourier transforms, and $f(x), f'(x) \to 0$ as $|x| \to \infty$. We show how solutions to this can be obtained using Fourier methods.

Taking the Fourier transform and using Lemmas 8.11 and 8.13 yields

$$\begin{aligned}0 &= \mathcal{F}[f''](k) + \mathcal{F}[xf'(x)](k) + \mathcal{F}[f](k) = -k^2\hat{f}(k) + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}k}\mathcal{F}[f'](k) + \hat{f}(k) \\ &= (1-k^2)\hat{f}(k) + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}k}(\mathrm{i}k\hat{f}(k)) = (1-k^2)\hat{f}(k) - \frac{\mathrm{d}}{\mathrm{d}k}(k\hat{f}(k)) = (1-k^2)\hat{f}(k) - k\hat{f}'(k) - \hat{f}(k)\end{aligned}$$

i.e., $\hat{f}'(k) = -k\hat{f}(k)$. This is a simple first order differential equation which we can solve to give $\hat{f}(k) = A\,\mathrm{e}^{-k^2/2}$ for some constant A.

We can then find $f(x)$ by taking the inverse transform. In this case we know the answer (from Example 46) to be $f(x) = A\,\mathrm{e}^{-x^2/2}$.

Remark 9.1. We expect to find two independent solutions to the Hermite equation, but here have only one. The reason is that only one has the decay properties that are tacitly assumed when using Fourier methods (i.e., $f(x) \to 0$ as $|x| \to \infty$).

9.2 Partial differential equations

Fourier transforms can be used to solve many partial differential equations. We illustrate the general approach taking as examples the heat equation and the wave equation.

9.2.1 Solving the heat equation with Fourier methods

The heat equation is

$$\frac{\partial T}{\partial t} = \eta\frac{\partial^2 T}{\partial x^2}$$

where $T(t,x)$ is the temperature at time $t \geq 0$ at position $x \in \mathbb{R}$ along a rod of infinite length. Here $\eta > 0$ is a constant called the *thermal diffusivity.* (For aluminium, $\eta = 8\times 10^{-5}\mathrm{m}^2\mathrm{s}^{-1}$; for nylon, $\eta = 9\times 10^{-8}\mathrm{m}^2\mathrm{s}^{-1}$).

Remark 9.2. In a previous course, you may have used Fourier series to solve the heat equation for a rod with finite length. Use of Fourier transforms allows us to cover the extension to infinite length.

One solution method is to take a Fourier transform in x for each fixed t, i.e., defining

$$\hat{T}(t,k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} T(t,x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

Differentiating under the integral,

$$\begin{aligned}\frac{\partial \hat{T}}{\partial t}(t,k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial T}{\partial t}(t,x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \eta\frac{\partial^2 T}{\partial x^2}(t,x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \eta(\mathrm{i}k)^2\hat{T}(t,k) \\ &= -\eta k^2\hat{T}(t,k),\end{aligned}$$

where we have used the heat equation and the standard result on the Fourier transform of a derivative. All these steps are valid, provided that T solves the heat equation and is sufficiently smooth and, with its derivatives, decays to 0 as $|x| \to \infty$.

The original equation now becomes a first order partial differential equation in which only the derivative with respect to t enters. This may be solved for each fixed k:

$$\hat{T}(t,k) = A(k)\,\mathrm{e}^{-\eta k^2 t},$$

where the constant $A(k) = \hat{T}(0,k)$ can depend on k. Thus

$$\hat{T}(t,k) = \hat{T}(0,k)\,\mathrm{e}^{-\eta k^2 t}\,. \tag{9.2}$$

Inverting the transform, we obtain an integral representation for the solution

$$T(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{T}(0,k)\,\mathrm{e}^{-\eta k^2 t + \mathrm{i}kx}\;\mathrm{d}k.$$

We may notice that for $t > 0$ the transform $\hat{T}(t,k)$ decays rapidly in k ($\hat{T}(0,k)$ will be bounded). This means that $T(t,x)$ is smooth in x: the heat equation instantly smooths out any discontinuities in the initial heat distribution $T(0,x)$.

We can do more. Noting that (9.2) is a product, using the convolution theorem (Theorem 8.19) we can write

$$T(t,x) = \frac{1}{\sqrt{2\pi}}(T_0 * G_t)(x)$$

where $T_0(x) = T(0,x)$ is the initial temperature distribution and $G_t(x)$ has Fourier transform

$$\mathcal{F}[G_t(x)](k) = \mathrm{e}^{-\eta k^2 t}\,.$$

Using Example 47 we have that

$$G_t(x) = \frac{\mathrm{e}^{-x^2/(4\eta t)}}{\sqrt{2\eta t}}$$

has the required property. Writing out the convolution, this gives a second representation for the solution in terms of the initial temperature profile.

$$T(t,x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\eta t}} T(0,x')\,\mathrm{e}^{-(x-x')^2/(4\eta t)}\;\mathrm{d}x'.$$

9.2.2 Solving the wave equation with Fourier methods

The wave equation is

$$\frac{\partial^2\Phi}{\partial t^2} = c^2\frac{\partial^2\Phi}{\partial x^2}$$

where $\Phi(t,x)$ is the displacement at time $t \geq 0$ at position $x \in \mathbb{R}$ on a rope of infinite length (or the electromagnetic field or ...). Here $c > 0$ is a constant, the wave speed.

As in the case of the heat equation, we take the Fourier transform in x for fixed t, i.e.,

$$\hat{\Phi}(t,k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(t,x)\,\mathrm{e}^{-\mathrm{i}kx}\;\mathrm{d}x.$$

Differentiating under the integral,

$$\begin{aligned}
\frac{\partial^2\hat{\Phi}}{\partial t^2}(t,k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2\Phi}{\partial t^2}(t,x)\,\mathrm{e}^{-\mathrm{i}kx}\;\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c^2\frac{\partial^2\Phi}{\partial x^2}(t,x)\,\mathrm{e}^{-\mathrm{i}kx}\;\mathrm{d}x \\
&= c^2(\mathrm{i}k)^2\hat{\Phi}(t,k) \\
&= -c^2k^2\hat{\Phi}(t,k)\,,
\end{aligned}$$

where we have used the wave equation and the standard result on the Fourier transform of a derivative. All these steps are valid, provided that Φ solves the wave equation and is sufficiently smooth and, with its derivatives, decays to 0 as $|x| \to \infty$.

Thus, for each k we have a second order differential equation in t, which we can solve using techniques from previous courses. For fixed k, we assume a solution of the form $\hat{\Phi}(t,k) = \mathrm{e}^{\lambda t}$ so that $\lambda^2 \,\mathrm{e}^{\lambda t} = -c^2 k^2 \,\mathrm{e}^{\lambda t}$, i.e., $\lambda = \pm \mathrm{i}ck$. Thus, emphasizing that the coefficients depend on k, we have

$$\hat{\Phi}(t,k) = \hat{A}(k)\,\mathrm{e}^{\mathrm{i}ckt} + \hat{B}(k)\,\mathrm{e}^{-\mathrm{i}ckt}\,.$$

From this we can obtain an integral representation of the solution,

$$\begin{aligned}\Phi(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\hat{A}(k)\,\mathrm{e}^{\mathrm{i}ckt} + \hat{B}(k)\,\mathrm{e}^{-\mathrm{i}ckt})\,\mathrm{e}^{\mathrm{i}kx}\;\mathrm{d}k \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\hat{A}(k)\,\mathrm{e}^{\mathrm{i}k(x+ct)} + \hat{B}(k)\,\mathrm{e}^{\mathrm{i}k(x-ck)})\,\mathrm{d}k \\ &= A(x+ct) + B(x-ct)\,,\end{aligned}$$

where $A(x)$ and $B(x)$ are the inverse Fourier transforms of $\hat{A}(k)$ and $\hat{B}(k)$. This is D'Alembert's form of the solution to the wave equation. The A part moves to the left at speed c, while the B part moves to the right. To determine A and B we need boundary conditions.

For example, the rope might begin at $t = 0$ at rest with shape $s(x)$. In this case we have

$$\begin{aligned}s(x) &= \Phi(0,x) = A(x) + B(x) \\ 0 &= \frac{\partial \Phi}{\partial t}(0,x) = cA'(x) - cB'(x)\,.\end{aligned}$$

From the second of these we have $A(x) - B(x) = d$ for some constant d, and hence $A(x) = (s(x)+d)/2$ and $B(x) = (s(x)-d)/2$. The overall solution is then

$$\Phi(t,x) = \frac{1}{2}\left(s(x+ct) + s(x-ct)\right)\,.$$

Thus, an initial shape neatly splits into two such shapes of half the size, one of which travels to the left and the other to the right, both with speed c.

Appendix A

Real analysis facts

This appendix covers material from real analysis that most of you will have seen before. For those on particular joint programmes, you may not have seen these, so the relevant information is here.

Lemma A.1. *If $r \in \mathbb{R}$, $|r| < 1$, then* $\lim_{n\to\infty} r^n = 0$.

Proof. Using Definition 1.6, we need to show that for all $\varepsilon > 0$ there exists N such that $|r^n - 0| < \varepsilon$ (or equivalently $|r|^n < \varepsilon$) for all $n > N$. Since $|r| < 1$, for any $a > 0$, $a|r| < a$ and hence $|r|^{n+1} < |r|^n$. Thus, if $|r|^N < \varepsilon$, then $|r|^n < \varepsilon$ for $n > N$. If we choose N to be any integer above $\log \varepsilon / \log |r|$ then $|r|^N = \mathrm{e}^{N \log |r|} < \mathrm{e}^{\log \varepsilon} = \varepsilon$. It hence follows that $\lim_{n\to\infty} r^n = 0$. □

Theorem A.2 (Sandwich Theorem (Special Case))**.** *For real sequences (a_n) and (b_n), if $0 \leq a_n \leq b_n$ for all sufficiently large n, and if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Proof. That $\lim_{n\to\infty} b_n = 0$ means that for all $\delta > 0$ there exists $N_0(\delta)$ such that $|b_n| < \delta$ for all $n \geq N_0(\delta)$. Furthermore, that $0 \leq a_n \leq b_n$ for all sufficiently large n means that there exists N_1 such that $0 \leq a_n \leq b_n$ for all $n \geq N_1$, and hence such that $|a_n| = a_n$ and $|b_n| = b_n$

Thus, for any $\varepsilon > 0$, we can choose $N = \max[N_0(\varepsilon), N_1]$, so that $|a_n| = a_n \leq b_n = |b_n| < \varepsilon$ for all $n > N$. Thus, $\lim_{n\to\infty} a_n = 0$. □

Example. Since $0 \leq \frac{1}{n}\cos^2(\pi/n) \leq 1/n$, and since $\lim_{n\to\infty} 1/n = 0$, by the Sandwich Theorem we have that $\lim_{n\to\infty} \frac{1}{n}\cos^2(\pi/n) = 0$.

Theorem A.3 (Algebra of limits for real sequences)**.** *Let (x_n) and (y_n) be real sequences. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

1. $\lim_{n\to\infty} (x_n + y_n) = x + y$,
2. $\lim_{n\to\infty} x_n y_n = xy$,
3. *If $y \neq 0$, then* $\lim_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{x}{y}$,
4. *For $c \in \mathbb{R}$,* $\lim_{n\to\infty} c x_n = cx$.
5. *If $x_n \leq y_n$ for all n, then $x \leq y$.*

Proof of Point 1. Since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, for all $\varepsilon > 0$ there exists N such that $|x_n - x| < \varepsilon/2$ and $|y_n - y| < \varepsilon/2$ for all $n > N$. Thus, for $n > N$ we have $|x_n + y_n - x - y| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, as required. □

Proof of Point 5. Suppose that $x > y$. With $\varepsilon := \frac{x-y}{2} > 0$, we have

$$x = (x - x_n) + \underbrace{(x_n - y_n)}_{\leq 0} + (y_n - y) + y \leq |x - x_n| + |y - y_n| + y < 2\varepsilon + y$$

if we choose n large enough for ε. But this means $x < y + (x - y) = x$, a contradiction. $\square$

Lemma A.4 (Convergence of real sequences). *In the following table of real sequences, if (a_n) is higher up the table than (b_n) then $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ unless both (a_n) and (b_n) are bounded. (Here, x is always a real number and q a rational.)*

$$\begin{array}{rll}
\text{tend to } \infty & \left\{\begin{array}{l} n^n \\ n! \\ x^n \\ n^q \end{array}\right. & \begin{array}{l} \\ \\ (x > 1) \\ (q > 0) \end{array} \\
\text{remain bounded} & \left\{\begin{array}{l} x \\ (-1)^n \\ x^{1/n} \to 1 \\ n^{q/n} \\ (1 + \frac{x}{n})^n \to \mathrm{e}^x \end{array}\right. & \begin{array}{l} \\ \text{(does not converge)} \\ (x > 0) \\ \\ \\ \end{array} \\
\text{tend to } 0 & \left\{\begin{array}{l} n^{-q} \\ x^n \\ \frac{1}{n!} \\ n^{-n} \end{array}\right. & \begin{array}{l} (q > 0) \\ (|x| < 1) \\ \\ \\ \end{array}
\end{array}$$

Together with Theorem A.3, this lemma can be used to derive many other useful results. One example is the following.

Corollary A.5. *Let P be a polynomial of degree r and Q be a polynomial of degree $s > r$, then $\lim_{n\to\infty} \frac{P(n)}{Q(n)} = 0$.*

Proof. Since P is a polynomial of degree r we can write $P(n) = a_r n^r + a_{r-1} n^{r-1} + \ldots + a_1 n + a_0$, where $\{a_i\}$ are real coefficients and $a_r \neq 0$. Likewise, $Q(n) = b_s n^s + b_{s-1} n^{s-1} + \ldots + b_1 n + b_0$. Thus,

$$\begin{aligned}
\frac{P(n)}{Q(n)} &= \frac{a_r n^r + a_{r-1} n^{r-1} + \ldots + a_1 n + a_0}{b_s n^s + b_{s-1} n^{s-1} + \ldots + b_1 n + b_0} \\
&= \frac{a_r n^{r-s} + a_{r-1} n^{r-1-s} + \ldots + a_1 n^{1-s} + a_0 n^{-s}}{b_s + b_{s-1} n^{-1} + \ldots + b_1 n^{1-s} + b_0 n^{-s}}
\end{aligned}$$

Since $r < s$, the limit of the numerator is 0 (from Lemma A.4 and Point 1 of Theorem A.3) and, likewise, the limit of the denominator is b_s. Then, using Point 3 of Theorem A.3, we have $\lim\limits_{n\to\infty} \frac{P(n)}{Q(n)} = 0$. $\square$

Theorem A.6 (Bounded Monotone Convergence Theorem).

1. *If a real sequence (x_n) satisfies $x_{n+1} \geq x_n$ for all n, and if it is bounded (that is, there exists a B such that $x_n \leq B$ for all n), then (x_n) converges. (A real sequence converges if it is monotonically increasing and bounded from above.)*

2. *If a real sequence (x_n) satisfies $0 \leq x_{n+1} \leq x_n$ for all n, then (x_n) converges. (A real sequence converges if it is monotonically decreasing and bounded below by zero.)*

Some common real series:

1. The *geometric series* $\sum_{n=1}^{\infty} x^n$ converges to $\frac{1}{1-x}$ if and only if $|x| < 1$.
2. A *telescopic series* is one that can be written $\sum_{n=1}^{\infty} x_n$ with $x_n = a_n - a_{n+1}$. This means that $\sum_{n=1}^{N} x_n = a_1 - a_{N+1}$. Example: since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, we have $\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$, and so $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.
3. The *harmonic series* is $\sum_{n=1}^{\infty} \frac{1}{n}$. It diverges.

Theorem A.7. *Let (x_n) be a real sequence. If $\sum_{n=0}^{\infty} |x_n|$ converges, then $\sum_{n=0}^{\infty} x_n$ converges. (That is, for a real series absolute convergence implies convergence.)*

Theorem A.8 (Leibniz alternating series test)**.** *Let (x_n) be a real sequence with $x_n \geq 0$ and $\lim_{n\to\infty} x_n = 0$, then $\sum_{n=0}^{\infty} (-1)^n x_n$ converges.*

Hence, although the harmonic series diverges, the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges.

Appendix B

Cauchy's Integral Formula and Taylor's theorem

This appendix contains more detail about the complex power series and their properties; to develop this, one first proves Cauchy's integral formula, a simplified version of what we shall later get to know as the Residue Theorem. As a byproduct, one obtains a proof of the famous Fundamental Theorem of Algebra. The material of this appendix is *not examinable*.

B.1 Cauchy's Integral Formula

This says that the value of a holomorphic function at a particular point is determined by its values on a circle around the point.

Theorem B.1 (Cauchy's Integral Formula). *Let $G \subseteq \mathbb{C}$ be open, $z \in G$ and $f : G \to \mathbb{C}$ be holomorphic on G. For any $r > 0$ such that $C(z,r) \cup D(z;r) \subset G$ (that is, $r > 0$ is sufficiently small that the closed disc of radius r centered at z is entirely contained within G) we have*

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{C(z,r)} \frac{f(w)}{w-z} \, \mathrm{d}w.$$

[Recall that $C(z,r)$ is the anticlockwise oriented circle of radius r centred at z.]

Remark B.2. Note that we need f to be holomorphic on both the contour and the region it encloses.

Proof. Note that the integrand $f(w)/(w-z)$ (as a function of w) is holomorphic on $G \setminus \{z\}$.

The contour C can be parameterized by setting $w = \gamma(t) = z + r\,\mathrm{e}^{\mathrm{i}t}$ with $0 \le t \le 2\pi$. Using $\gamma'(t) = \mathrm{i}r\,\mathrm{e}^{\mathrm{i}t}$ we have

$$\frac{1}{2\pi \mathrm{i}} \oint_{C(z,r)} \frac{f(w)}{w-z} \, \mathrm{d}w = \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \frac{f(z + r\,\mathrm{e}^{\mathrm{i}t})}{r\,\mathrm{e}^{\mathrm{i}t}} \mathrm{i}r\,\mathrm{e}^{\mathrm{i}t} \, \mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} f(z + r\,\mathrm{e}^{\mathrm{i}t}) \, \mathrm{d}t.$$

Now note that

$$\begin{aligned} \left| \frac{1}{2\pi \mathrm{i}} \oint_{C(z,r)} \frac{f(w)}{w-z} \, \mathrm{d}w - f(z) \right| = \left| \frac{1}{2\pi} \int_0^{2\pi} \left(f(z + r\,\mathrm{e}^{\mathrm{i}t}) - f(z) \right) \mathrm{d}t \right| &\le \frac{1}{2\pi} \int_0^{2\pi} \left| f(z + r\,\mathrm{e}^{\mathrm{i}t}) - f(z) \right| \, \mathrm{d}t \\ &\le \max_t \left| f(z + r\,\mathrm{e}^{\mathrm{i}t}) - f(z) \right| \end{aligned}$$

By Cauchy's theorem, the radius r can be chosen to be arbitrarily small without affecting the value of the integral. For this to hold as $r \to 0$, the left hand side must be equal to zero. □

Theorem B.3 (Cauchy's Integral Formula for simply-connected domains)**.** *If f is holomorphic on a simply connected domain $G \subset \mathbb{C}$ and C is a closed curve contained in G, then for any $z \in G$ such that $z \notin C$,*

$$\mathrm{n}(C, z) f(z) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(w)}{w - z} \, \mathrm{d}w.$$

Proof. Since the region is simply-connected, and $f(w)/(w-z)$ is holomorphic on $G \setminus \{z\}$, then the curve is homotopic in G to a circle winding around z exactly $\mathrm{n}(C, z)$ times. □

Remark B.4. Since we take G to be simply-connected, if C is contained within G then so is the area it encloses.

B.2 Liouville's Theorem

Theorem B.5 (Liouville's Theorem)**.** *If f is holomorphic on (the whole of) $\mathbb{C}$ and bounded, then f is constant.*

Proof. By Cauchy's Integral Formula, so long as $R > |z|$, then

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{C(0,R)} \frac{f(w)}{w - z} \, \mathrm{d}w$$

and so

$$f(z) - f(0) = \frac{1}{2\pi \mathrm{i}} \oint_{C(0,R)} \left(\frac{1}{w - z} - \frac{1}{w} \right) f(w) \, \mathrm{d}w = \frac{z}{2\pi \mathrm{i}} \oint_{C(0,R)} \frac{f(w)}{w(w - z)} \, \mathrm{d}w.$$

That f is bounded means that there exists an $M \in \mathbb{R}$ such that $|f(w)| \leq M$ for all $w \in \mathbb{C}$. If $R > 2|z|$ then for all $w \in C(0, R)$ by the reverse triangle inequality we have $|w - z| \geq |w| - |z| > R/2$ so $|w(w - z)| > R^2/2$ and so

$$\left| \frac{f(w)}{w(w - z)} \right| < \frac{2M}{R^2}.$$

The length of $C(0, R)$ is $2\pi R$, and thus by the Estimation Lemma (Lemma 3.22),

$$|f(z) - f(0)| \leq \frac{|z|}{2\pi} \left| \oint_{C(0,R)} \frac{f(w)}{w(w - z)} \, \mathrm{d}w \right| < |z| \frac{2M}{R^2} R = \frac{2|z|M}{R}.$$

We must have $f(z) = f(0)$ to satisfy this inequality in the limit of large R (since the right-hand-side becomes arbitrarily small). □

B.3 Fundamental Theorem of Algebra

Theorem B.6 (Fundamental Theorem of Algebra)**.** *If $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ is a non-constant polynomial then $p(z) = 0$ for some $z \in \mathbb{C}$.*

Proof. Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/p(z)$ is holomorphic on $\mathbb{C}$ with derivative $-p'(z)/p(z)^2$.

The function $1/p(z)$ is also bounded because if $|z| = R$ and R is large, then

$$\left| \frac{1}{p(z)} \right| \approx \frac{1}{|c_n| R^n}$$

and hence $1/p(z) \to 0$ as $|z| \to \infty$, and this is bounded on all of $\mathbb{C}$. Hence, by Liouville's Theorem (Theorem B.5), we conclude that $1/p(z)$ is constant, which implies that $p(z)$ is constant, which is a contradiction. Hence $p(z) = 0$ for some $z \in \mathbb{C}$. □

Remark B.7. Since polynomials can be factored over $\mathbb{C}$, having found one root, α say, the linear term $(z-\alpha)$ can be factored out leaving a remainder polynomial of degree $n-1$. Applying this theorem repeatedly, it shows that every degree n polynomial has exactly n (possibly repeated) complex roots.

B.4 Taylor's theorem

Theorem B.8 (Taylor's Theorem)**.** *Let $R > 0$, $f : \mathrm{D}(z_0; R) \to \mathbb{C}$ be holomorphic, r be such that $0 < r < R$ and $z \in \mathrm{D}(z_0; r)$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

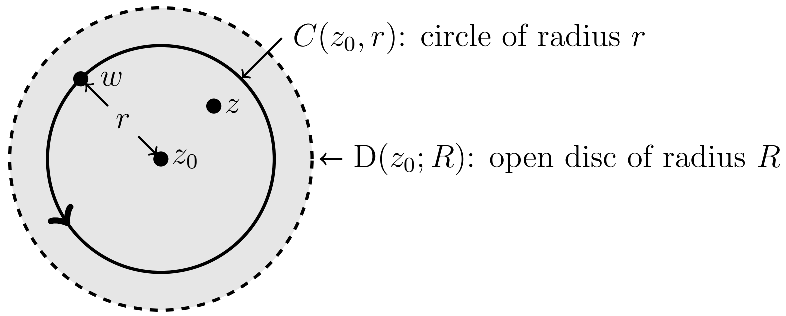
with

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w - z_0)^{n+1}} \, \mathrm{d}w.$$

Remark B.9. The form $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ appearing in the above expression is called a *power series* of $f(z)$ about z_0.

Remark B.10. Note that Cauchy's Integral Formula tells us that a_0 equals $f(z_0)$.

The domain where $f(z)$ is holomorphic, $\mathrm{D}(z_0; R)$, the circle of integration, $C(z_0, r)$, and the point at which the function is being evaluated, z, are all shown in this diagram:



Proof. Using Cauchy's integral formula for $f(z)$ we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{w - z} \, \mathrm{d}w = \frac{1}{2\pi \mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w - z_0) - (z - z_0)} \, \mathrm{d}w \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{w - z_0} \left(\frac{1}{1 - \dfrac{z - z_0}{w - z_0}} \right) \mathrm{d}w \end{aligned}$$

where we have added and subtracted z_0 in the denominator of the middle equation. Using the formula for the geometric series

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1 - y}$$

(which holds so long as $|y| < 1$) and that $|z - z_0| < |w - z_0|$ gives

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{w - z_0} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^n} \, \mathrm{d}w.$$

The series is absolutely convergent, so may be interchanged with the (finite) integral (this is a special case of Fubini's theorem, though see below for a direct justification). Thus

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi \mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w-z_0)^{n+1}} \, \mathrm{d}w \right] (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w-z_0)^{n+1}} \, \mathrm{d}w,$$

as required. □

Remark B.11. This interchange of the integral and infinite sum can be justified by splitting the infinite sum up into two pieces:

$$\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^n} = \sum_{n=0}^{N} \frac{(z-z_0)^n}{(w-z_0)^n} + \sum_{n=N+1}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^n}$$

and the interchange of the integral and the *finite* sum is justified by linearity.

Since $f(w)/(w-z_0)$ is holomorphic on $C(z_0,r)$ (that is, it is not divergent anywhere on the circle $C(z_0,r)$), it is bounded there as a function of w. This means there exists an $M \in \mathbb{R}$ such that

$$\left| \frac{f(w)}{w-z_0} \right| \leq M$$

for all w on $C(z_0,r)$. Also, if $\varepsilon = \left| \frac{z-z_0}{w-z_0} \right|$ then $\varepsilon < 1$ since $|z-z_0| < |w-z_0|$. Hence,

$$\left| \sum_{n=N+1}^{\infty} \left(\frac{z-z_0}{w-z_0} \right)^n \right| \leq \sum_{n=N+1}^{\infty} \left| \frac{z-z_0}{w-z_0} \right|^n = \sum_{n=N+1}^{\infty} \varepsilon^n = \frac{\varepsilon^{N+1}}{1-\varepsilon}$$

Since the length of $C(z_0,r)$ is $2\pi r$, we find by the Estimation Lemma (Lemma 3.22) that

$$\frac{1}{2\pi} \left| \oint_{C(z_0,r)} \frac{f(w)}{w-z_0} \sum_{n=N+1}^{\infty} \left(\frac{z-z_0}{w-z_0} \right)^n \right| \leq rM \frac{\varepsilon^{N+1}}{1-\varepsilon}$$

and since $\varepsilon < 1$, this tends to 0 as $N \to \infty$.

Remark B.12. The previous remark allows one to estimate the error in truncation the Taylor series at some finite point.

B.5 Cauchy's Integral Formula for derivatives

From Taylor's theorem we can get another important result: Cauchy's Integral Formula for Derivatives, which shows that for complex functions differentiation and integration are connected.

Theorem B.13 (Cauchy's Integral Formula for Derivatives)**.** *Let f be holomorphic on $\mathrm{D}(z_0;R)$ with $R > 0$, and let $0 < r < R$. The n^{th} derivative of f at z_0 is given by*

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w-z_0)^{n+1}} \, \mathrm{d}w.$$

Remark B.14. Note that we can use this theorem to give an alternative expression for the co-efficients in Taylor's theorem. Taking the expression for a_n from Theorem 5.1 we have

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w-z_0)^{n+1}} \,\mathrm{d}w = \frac{f^{(n)}(z_0)}{n!}\,. \tag{B.1}$$

(See also Remark B.10.)

We won't give a full proof of this here. However, we can see that it appears to be doing the right thing by using Cauchy's integral formula and switching the differential with the integral (when no-one is looking):

$$\frac{\mathrm{d}^n}{\mathrm{d}z_0^n} f(z_0) = \frac{\mathrm{d}^n}{\mathrm{d}z_0^n} \frac{1}{2\pi\mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{w-z_0}\,\mathrm{d}w = \frac{1}{2\pi\mathrm{i}} \oint_{C(z_0,r)} \frac{\mathrm{d}^n}{\mathrm{d}z_0^n} \frac{f(w)}{w-z_0}\,\mathrm{d}w = \frac{n!}{2\pi\mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w\,.$$

To elevate this to a proof we would need to justify the exchange of the differential and integral. Instead, we offer a proof for the first derivative (this proof can be inductively extended to higher derivatives – the next case is on one of the exercises).

Proof of Theorem B.13 in the case of a single differentiation, i.e., $n = 1$. Using Cauchy's integral formula we have

$$\begin{aligned}
\frac{f(z_0+h)-f(z_0)}{h} &= \frac{1}{2h\pi\mathrm{i}} \oint_{C(z_0,r)} f(w)\left(\frac{1}{w-z_0-h} - \frac{1}{w-z_0}\right)\mathrm{d}w \\
&= \frac{1}{2\pi\mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w-z_0-h)(w-z_0)}\,\mathrm{d}w \\
&= \frac{1}{2\pi\mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w-z_0)^2}\,\mathrm{d}w + \frac{h}{2\pi\mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w-z_0-h)(w-z_0)^2}\,\mathrm{d}w\,,
\end{aligned} \tag{B.2}$$

where in the last line we have used

$$\frac{1}{(w-z_0-h)(w-z_0)} - \frac{1}{(w-z_0)^2} = \frac{w-z_0-(w-z_0-h)}{(w-z_0-h)(w-z_0)^2} = \frac{h}{(w-z_0-h)(w-z_0)^2}$$

(the reason we chose to do this is because the first term in (B.2) is the answer we're looking for). We require that the last term in (B.2) tends to 0 as $h \to 0$. To see this, choose h such that $|h| < r/2$ so that $|w - z_0 - h| \geq |w - z_0| - |h| > r/2$ for all $w \in C(z_0, r)$ where we have used the reverse triangle inequality (Corollary 1.5). Since f is continuous on $C(z_0, r)$, there exists a constant M such that $|f(w)| \leq M$ for all $w \in C(z_0, r)$. Hence, $\left|\frac{f(w)}{(w-z_0-h)(w-z_0)^2}\right| \leq \frac{2M}{r^3}$ on $C(z_0, r)$.

We can then use the estimation lemma (Lemma 3.22) to give

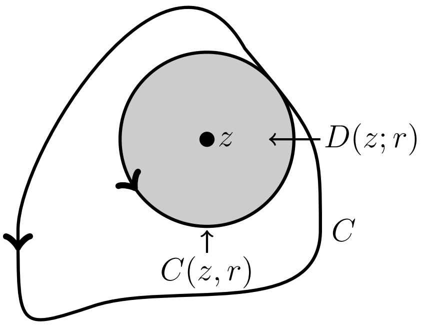
$$\left| \frac{f(z_0+h)-f(z_0)}{h} - \frac{1}{2\pi\mathrm{i}} \oint_{C(z_0,r)} \frac{f(w)}{(w-z_0)^2}\,\mathrm{d}w \right| \leq \frac{|h|}{2\pi}\frac{2M}{r^3} 2\pi r = \frac{2|h|M}{r^2}\,,$$

which tends to zero as $h \to 0$. □

Remark B.15. By using what we know about homotopy, we can generalize Cauchy's Integral Formula (for Derivatives) in the following way. Let C be a positively-oriented contour (that is, a simple (i.e. non-intersecting) closed curve with an anticlockwise orientation) such that f is holomorphic on and inside C. Then for any z strictly enclosed by C,

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \oint_C \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w.$$

To see this, choose $r > 0$ so that $\mathrm{D}(z; r)$ is contained within the area enclosed by C as shown:



Since f is holomorphic on and inside C, f is also holomorphic on the boundary of the disc, $C(z,r)$, as well as throughout. Hence we can continuously deform C to $C(z,r)$ so that

$$\frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w = \frac{n!}{2\pi \mathrm{i}} \oint_{C(z,r)} \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w\,.$$

This is equal to $f^{(n)}(z)$ by Cauchy's Integral Formula for Derivatives.

An important consequence of Cauchy's Integral Formula for Derivatives is that holomorphic functions have arbitrarily many derivatives.

Theorem B.16. *Let $G \subseteq \mathbb{C}$ be open and $f : G \to \mathbb{C}$ be holomorphic. Then f' is holomorphic on G.*

Proof. This is similar to the proof of Theorem B.13 in the case $n = 1$ given above and is left as an exercise (see the relevant question on the exercise sheet). □

Corollary B.17. *Let $G \subseteq \mathbb{C}$ be open and $f : G \to \mathbb{C}$ be holomorphic. Then f can be differentiated infinitely many times on G.*

Proof. Apply the previous theorem repeatedly. □

Remark B.18. Although we have shown that a holomorphic function can be differentiated arbitrarily many times, we have not shown rigorously that the derivative is equal to the expression given by Cauchy's formula (except for the first derivative in the proof of Theorem B.13 and the second derivative (the exercise related to Theorem 5.3)). Later in the course it will become clear that the general formula can be derived as a consequence of Cauchy's residue theorem (Theorem 6.9; see the exercises).

B.6 Morera's Theorem

We can use Theorem B.16 to derive Morera's theorem, which is in a sense a converse of Cauchy's theorem. We will use a result from vector calculus in its proof.

Lemma B.19. *Let $U \subseteq \mathbb{R}^2$ be open and $\underline{g} : U \to \mathbb{R}^2$ have continuous first partial derivatives. There exists $F : U \to \mathbb{R}$ such that $\underline{g} = \nabla F$ if and only if for every closed curve C in U we have*

$$\oint_C \underline{g}(\underline{x}) \cdot \mathrm{d}\underline{x} = 0\,. \qquad \text{(B.3)}$$

Theorem B.20 (Morera's theorem)**.** *Let $G \subseteq \mathbb{C}$ be open and $f : G \mapsto \mathbb{C}$ be a continuous function. If for every contour C in G we have*

$$\oint_C f(z)\,\mathrm{d}z = 0$$

then f is holomorphic on G.

Proof. We begin by noting that we can rephrase (B.3) as

$$\oint_C (u\,\mathrm{d}x + v\,\mathrm{d}y) = 0\,,$$

by writing $\underline{g}(x, y) = (u(x, y),\, v(x, y))$. Then, by writing $f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$ we have

$$\oint_C f(z)\,\mathrm{d}z = \oint_C (u(x, y)\,\mathrm{d}x - v(x, y)\,\mathrm{d}y) + \mathrm{i}\oint_C (v(x, y)\,\mathrm{d}x + u(x, y)\,\mathrm{d}y)\,.$$

If the right hand side is zero for any C then both its real and imaginary parts are. Using Lemma B.19 for the real part gives that $(u, -v) = \nabla F_R$ for some $F_R : G \to \mathbb{R}$. Likewise, $(v, u) = \nabla F_I$ for some $F_I : G \to \mathbb{R}$. It follows that

$$\frac{\partial F_R}{\partial x} = u = \frac{\partial F_I}{\partial y} \quad \text{and} \quad \frac{\partial F_R}{\partial y} = -v = -\frac{\partial F_I}{\partial x}\,.$$

Thus, $F(x + \mathrm{i}y) := F_R(x, y) + \mathrm{i}F_I(x, y)$ satisfies the Cauchy-Riemann equations, and, since u and v are continuous (because g is differentiable) in G, Theorem 2.10 tells us that F is complex differentiable in G (hence holomorphic). Furthermore, F has derivative f. To see this, we can take its derivative in the real direction (since F is holomorphic, the direction doesn't matter). We have

$$\frac{\mathrm{d}F}{\mathrm{d}z} = \frac{\partial F(x + \mathrm{i}y)}{\partial x} = \frac{\partial F_R}{\partial x} + \mathrm{i}\frac{\partial F_I}{\partial x} = u + \mathrm{i}v = f(z).$$

That f is holomorphic on G then follows because the derivative of a holomorphic function is holomorphic (cf. Theorem B.16). $\square$

Remark B.21. As a check, we see that we get the same answer by differentiating in the imaginary direction (cf. Eq. 2.2):

$$\frac{\mathrm{d}F}{\mathrm{d}z} = -\mathrm{i}\frac{\partial F(x + \mathrm{i}y)}{\partial y} = -\mathrm{i}\frac{\partial F_R}{\partial y} + \frac{\partial F_I}{\partial y} = u + \mathrm{i}v = f(z).$$

B.7 Differentiation of power series

If we have a power series form of a holomorphic function about some point, we can find the power series of the derivative by term-by-term differentiation.

Lemma B.22. *Suppose $f(z)$ is holomorphic on some open disc $\mathrm{D}(z_0; r) \subseteq \mathbb{C}$ $(r > 0)$ and has a power series about z_0 that is valid on this disc, i.e., we can write $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ on $\mathrm{D}(z_0; r)$, then f has derivative $f'(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}$ on $\mathrm{D}(z_0; r)$.*

Proof. This follows directly from the form of the Taylor coefficients we derived using Cauchy's Integral Formula for Derivatives, i.e., from Remark B.14. Let us write $f'(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$. Using (B.1) we have $b_n = f^{(n+1)}(z_0)/n! = (n+1)a_{n+1}$. Thus, $f'(z) = \sum_{n=0}^{\infty}(n+1)a_{n+1}(z - z_0)^n$. This is readily seen to be equivalent to the expression given in the lemma (e.g., by substituting $n' = n + 1$). $\square$

Remark B.23. The lemma essentially says that one can find the derivative of a power series by exchanging the derivative with the summation. This is clear for finite sums, but far from trivial for *infinite* sums. In fact, it is false for series in general: As a counterexample, consider the series of real functions

$$h_N(x) = \begin{cases} 0 & x \leq 0 \\ Nx & 0 < x < 1/N \\ 1 & x \geq 1/N \end{cases}$$

for which $\lim_{N\to\infty} h_N$ is the unit step function. Then let $g_N(x) = \int_{-\infty}^{x} h_N(x')\,\mathrm{d}x'$. Since h_N is continuous, $\frac{\mathrm{d}}{\mathrm{d}x} g_N(x)$ is always defined. However, $\lim_{N\to\infty} g_N(x)$ is not differentiable at $x = 0$ (it is the function equal to 0 for $x < 0$ and equal to x for $x \geq 0$).

Appendix C

A proof of the Fourier inversion theorem

In this appendix, we give a proof of the Fourier inversion theorem. Strictly speaking we give a *partial* proof, because our argument relies on the following famous lemma that we will *not* prove here.

Lemma (Riemann–Lebesgue lemma). *If f is absolutely integrable then $\hat{f}$ is continuous and $\hat{f}(k) \to 0$ as $k \to \infty$.*

Roughly the idea is that as k becomes large, the $\mathrm{e}^{\mathrm{i}kx}$ in the Fourier transform integral oscillates so rapidly as to create cancellations, so the transform tends to zero.

Now we prove the following part of Theorem 8.6:

Theorem C.1. *If f is piecewise continuous and absolutely integrable on $\mathbb{R}$ then*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k = \frac{f(x+) + f(x-)}{2}$$

at each x for which the one-sided derivatives of f exist.

Proof. By the translation property,

$$I_K(x') := \frac{1}{\sqrt{2\pi}} \int_{-K}^{K} \mathcal{F}[f](k)\, \mathrm{e}^{\mathrm{i}kx'}\, \mathrm{d}k = \frac{1}{\sqrt{2\pi}} \int_{-K}^{K} \mathcal{F}[g](k)\, \mathrm{d}k,$$

where $g(x) = f(x + x')$. In order to perform the integral, we would like to write $\mathcal{F}[g](k)$ in terms of a derivative of something and proceed as follows. As f has one-sided derivatives at s, the function

$$h(x) = \begin{cases} (g(x) - g(0+)\, \mathrm{e}^{-\epsilon x})/x & x > 0 \\ (g(x) - g(0-)\, \mathrm{e}^{\epsilon x})/x & x < 0 \end{cases}$$

is continuous everywhere except $x = 0$ and is absolutely integrable for any $\epsilon > 0$. (Note that $h(0\pm) = g'(0\pm) \pm \epsilon$.) Clearly $xh(x)$ is also absolutely integrable, so $\mathcal{F}[h]$ is differentiable (cf. Lemma 8.13) and

$$\mathrm{i}\mathcal{F}[h]'(k) = \mathcal{F}[xh(x)](k) = \mathcal{F}[g](k) - g(0+)\mathcal{F}[\Theta(x)\, \mathrm{e}^{-\epsilon x}](k) - g(0-)\mathcal{F}[\Theta(-x)\, \mathrm{e}^{\epsilon x}](k),$$

where $\Theta(x)$ is the **Heaviside function**

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

Thus

$$\mathcal{F}[g](k) = \mathrm{i}\mathcal{F}[h]'(k) + \frac{g(0+)}{\sqrt{2\pi}(\epsilon + \mathrm{i}k)} + \frac{g(0-)}{\sqrt{2\pi}(\epsilon - \mathrm{i}k)}$$

and, integrating,

$$I_K(x') = \frac{\mathrm{i}\,(\mathcal{F}[h](K) - \mathcal{F}[h](-K))}{\sqrt{2\pi}} + \int_{-K}^{K} \frac{g(0+)}{2\pi(\epsilon + \mathrm{i}k)}\,\mathrm{d}k + \int_{-K}^{K} \frac{g(0-)}{2\pi(\epsilon - \mathrm{i}k)}\,\mathrm{d}k\,.$$

By the Riemann–Lebesgue lemma, $\mathcal{F}[h](\pm K) \to 0$ as $K \to \infty$, while

$$\frac{1}{2\pi}\int_{-K}^{K} \frac{1}{\epsilon \pm \mathrm{i}k}\,\mathrm{d}k = \frac{1}{2\pi}\int_{0}^{K} \left(\frac{1}{\epsilon + \mathrm{i}k} + \frac{1}{\epsilon - \mathrm{i}k}\right)\mathrm{d}k = \frac{1}{2\pi}\int_{0}^{K} \frac{2\epsilon}{\epsilon^2 + k^2}\,\mathrm{d}k \to \frac{1}{2}$$

as $K \to \infty$. Thus

$$\lim_{K\to\infty} I_K(x') = \frac{g(0+) + g(0-)}{2} = \frac{f(x'+) + f(x'-)}{2}$$

so

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k = \frac{f(x+) + f(x-)}{2}$$

(provided we interpret the LHS as a principal value integral). □

This establishes Theorem 8.6 at least in the case where the function f has half-sided derivatives at *all* points (not only at the discontinuities); this is as far as we want to go at this point.