Classical Dynamics Semester 1 - Teaching weeks 1-8 - Autumn 2023

The purpose of these notes is to gather together the topics covered in lectures. The notes should be helpful for revision purposes but they are not a complete verbatim record of the lectures. In particular, not all illustrative examples are included (others will be given in lectures and problem classes).

• The lectures will be captured automatically by the University's system for you to consult later.

• Note: the module has changed since last taught in 2022/23 and previous years (when the material was spread over two modules - one in each of the Autumn and Spring Terms), which means previous materials (such as videos made in 2020) are no longer a good match to the current module content.

• If you spot any serious errors or misprints in these typed notes, please let me know.

• As the term progresses I might occasionally update the notes to take into account your comments, to make additions, or to correct misprints. You will be advised of any such changes.

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The module is about using mathematics to describe various aspects of our experiences of the everyday world using Newton's Laws of Motion and to model some of the physical laws that have been discovered over the course of many centuries. In most cases, the applications of these laws are to idealised situations (for example, considering the planets as 'point particles' when considering their motions relative to the Sun), but can be modified progressively to describe more complicated and more realistic situations. The module will also explore several formulations of dynamics especially Lagrangian and Hamiltonian formulations that are useful for future discussions of Quantum Mechanics, Quantum Field Theory and other advanced topics.

Part I contains a review of 3d vectors together with some common conventions and notation.

Part II is a collection of topics in Newtonian Mechanics, including a detailed discussion of orbits that 'particles' (for example, planets or other objects in the Solar System) follow under the influence of an inverse square law gravitational force.

Part III is devoted to Lagrangian mechanics, which is a reformulation of Newtonian mechanics that can be very useful in situations where the motion is constrained in some way (for example, a pendulum swinging in a vertical plane is constrained to move on a circle). A spinning top is an interesting example that can be analysed using these methods.

Part IV contains a description of the Hamiltonian version of mechanics, which, alongside Part III, is important for the formulation of quantum mechanics and quantum field theory.

Recommended texts

There is a list of recommended texts with the module description and I do suggest you consult these.

The following book gives a lot of physics background (the University Library has an electronic version of it). It also introduces Special Relativity, which may interest some of you:

- JR Forshaw and AG Smith, *Dynamics and Relativity*, Wiley 2009.

Part I Each of the recommended texts has a section that covers the basics of vectors (eg Chapter 1 of Smith and Smith, Chapter 1 of Forshaw and Smith, Appendix A of Kibble and Berkshire). Notation varies a little from book to book.

Part II All the recommended texts cover the basics of classical mechanics together with many examples. Since this part of the module has Newtonian Gravity as its main focus, in particular orbits in a central inverse square law force, recommended texts vary in their presentations of this (eg Chapter 4 of Kibble and Berkshire, Chapter 9 of Forshaw and Smith).

Parts III and IV Many books on Classical Mechanics will cover Lagrangian and Hamiltonian Dynamics. These include the books by Kibble and Berkshire, by Gregory, by Goldstein and others listed on the module page. The treatments vary in their complexity and detail and of necessity, since the number of lectures is limited, we will follow our own path and cover a selection of topics.

Contact

• I can be contacted by email (ec9@york.ac.uk) or via the module discussion forum on Moodle (the preferred option for technical questions). If you wish to make an appointment to see me about any issues you may have with the module, please send me an email to arrange it.

I Vectors

(See, for example, the summaries supplied in Appendix A of Kibble and Berkshire or in Chapter 1 of Smith and Smith.)

Some quantities can be described by giving a single number, for example the length of a piece of string, the number of people in a room, the temperature of a bowl of ice cream - these are called *scalars*. Other quantities of interest cannot be described by a single number, for example the velocity of an aeroplane needs the speed and direction of travel to be specified (if you are an air traffic controller knowing the speed alone is not too helpful); similarly, a force will have a magnitude but it is also important to specify the sense in which it is being applied. Quantities requiring three numbers to specify them completely are called *vectors*. Other examples are electric and magnetic fields, the position (or trajectory) of a 'particle', or the velocity field of a fluid - and you can think of many others.

There are, however, yet other quantities that are of physical significance but are not described by scalars or vectors but require even more numbers to specify them. Examples of these are: moment of inertia, stress and strain, the metric tensor, where each needs six pieces of information to specify it completely. These are called symmetric *tensors*; you will meet examples as the module progresses.

1.1 **Vectors as displacements**

As far as we are concerned 'space' is three-dimensional.

Choose an origin O and consider every other point in space as displaced from O. Represent the displacement of point A by $\mathbf{a}$, denoted by a directed straight line from O to A. The length of the line OA is referred to as the length or magnitude of the vector $\mathbf{a}$ and denoted $|\mathbf{a}|$. It is the distance from O to A.

Note: in these notes vectors will be denoted by using a boldface font. This is difficult to achieve in handwritten notes where a vector is often denoted by an underline (ie $\mathbf{a} \equiv \underline{a}$). Sometimes (meaning in some books), vectors are represented by placing an arrow above the letter, for example $\vec{a}$; this convention will not be used in these notes.

The vector $\mathbf{a} + \mathbf{b}$ represents the displacement from O obtained by starting from O and displacing to A and then finding the point displaced from A by a move parallel to the displacement of B from O. It is the same as $\mathbf{b} + \mathbf{a}$, and addition is associative

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}.$$

The vectors $\lambda\mathbf{a}$ for real λ represent all the points reachable by displacements from O along a line from O through A.

The displacement of O from A is $-\mathbf{a}$ and $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$, where $\mathbf{0}$ is the zero-vector.

Note: There is an implicit assumption that it makes sense to 'move' a vector parallel to itself; an idea that has to be examined more carefully (and modified) if space is not 'flat'.

1.2 **Scalar and vector products**

The angle between two vectors $\mathbf{a}, \mathbf{b}$, say θ, is the angle $\leq \pi$ between the pair of displacements A from O and B from O.

The *scalar* product of the two vectors is defined to be

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta = \mathbf{b} \cdot \mathbf{a}$$

If $\mathbf{a}\cdot\mathbf{b}=0$, the vectors are said to be orthogonal. Note: $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$.

The *vector* product is defined to be

$$\mathbf{a}\times\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\,\mathbf{n}\,\sin\theta=-(\mathbf{b}\times\mathbf{a}),\qquad |\mathbf{n}|=1,$$

where $\mathbf{n}$ is a unit vector orthogonal to both the vectors $\mathbf{a},\mathbf{b}$ and the three of them together form a right-handed set.

Note: the vector product is not symmetric and $\mathbf{a}\times\mathbf{a}=0$. It is also worth remembering that the magnitude of $\mathbf{a}\times\mathbf{b}$ is twice the area of a triangle two of whose sides are the displacements $\mathbf{a},\ \mathbf{b}$.

1.3 **(a) Orthonormal basis vectors**

Besides choosing an origin, it is often useful to represent vectors in terms of a right-handed set of orthogonal vectors, each of unit length (think of them as unit displacements from O along the x,y,z axes). Denote these $\mathbf{i},\mathbf{j},\mathbf{k}$, then write for any vector

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$$

where $a_1=\mathbf{a}\cdot\mathbf{i},\ a_2=\mathbf{a}\cdot\mathbf{j}$ and $a_3=\mathbf{a}\cdot\mathbf{k}$ are the components of $\mathbf{a}$ in this basis.

Then, the scalar and vector products are conveniently given in terms of components

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3,$$

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})_1&=a_2b_3-a_3b_2\\(\mathbf{a}\times\mathbf{b})_2&=-(a_1b_3-a_3b_1)\\(\mathbf{a}\times\mathbf{b})_3&=a_1b_2-a_2b_1\end{aligned}$$

To verify the latter, note that $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$, $\mathbf{k}\times\mathbf{i}=\mathbf{j}$, which follow directly from the definition of the vector product. Also, $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0,\ \mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$.

(b) Some useful conventions

If the notation is changed a bit by setting $\mathbf{e}_1=\mathbf{i}$, $\mathbf{e}_2=\mathbf{j}$, $\mathbf{e_3}=\mathbf{k}$ then the scalar products between the unit vectors can be written very succinctly:

$$\mathbf{e}_a\cdot\mathbf{e}_b=\delta_{ab}=\begin{cases}1 & a=b\\ 0 & a\neq b\end{cases},\qquad a,b=1,2,3.$$

What's more the vector product relations between the basis vectors can also be written in a single expression as follows:

$$\mathbf{e}_a\times\mathbf{e}_b=\sum_{c=1}^{3}\epsilon_{abc}\,\mathbf{e}_c,$$

where

$$\epsilon_{abc}=\begin{cases}1 & \text{if } abc \text{ is an even permutation of } 1,2,3,\\ -1 & \text{if } abc \text{ is an odd permutation of } 1,2,3,\\ 0 & \text{if } abc \text{ is not a permutation of } 1,2,3.\end{cases}$$

Within this notation a vector is written

$$\mathbf{a}=\sum_{b=1}^{3}a_b\mathbf{e}_b\equiv a_b\mathbf{e_b}$$

where the last step illustrates a convention (Einstein's summation convention) that a repeated label (in this case b) is summed over.

The components of the vector product are then captured by

$$(\mathbf{a} \times \mathbf{b})_p = \epsilon_{pqr} a_q b_r,$$

where this time q, r are summed over.

Note: there is a useful property that is worth remembering

$$\epsilon_{pqr}\epsilon_{puv} = \delta_{qu}\delta_{rv} - \delta_{qv}\delta_{ru}.$$

This is at the heart of the triple vector product given below. The identity is proved by checking all the possible choices for $q, r, u, v = 1, 2, 3$ and remembering there is an implicit sum over p.

There are other identities too (and some are on the assignment sheet). Remember the convention: repeated indices (no more than two should ever occur with the same label) are summed over. So, for example

$$\delta_{aa} = \delta_{11} + \delta_{22} + \delta_{33} = 3, \;\; \delta_{ab}\delta_{bc} = \delta_{a1}\delta_{1c} + \delta_{a2}\delta_{2c} + \delta_{a3}\delta_{3c} = \delta_{ac},$$

the latter checked by checking every possibility for $a, c = 1, 2, 3$. Noting the symmetries can be useful: for example if A is antisymmetric ($A^T = -A$, or $A_{ab} = -A_{ba}$) and S is symmetric ($S^T = S$, or $S_{ab} = S_{ba}$), then

$$\epsilon_{abc}S_{bc} = 0, \quad \delta_{ab}A_{ab} = 0,$$

as a consequence of the symmetry (write out all parts of the sums to see this). On the other hand, expressions like

$$\epsilon_{aaa}, \quad M_{abb}N_{bc}$$

do not make sense within the convention because an index is repeated three times.

1.4 **Lines and planes**

The equation

$$\mathbf{r}(t) = \mathbf{a} + t\mathbf{b},$$

where t is a real number, represents all the points which lie on a straight line through the point represented by the vector $\mathbf{a}$ with respect to O, and parallel to the vector represented by $\mathbf{b}$. The point at which two intersecting lines cross is the vector $\mathbf{r}$ common to both.

All the points on a plane through $\mathbf{a}$ and containing the independent vectors $\mathbf{b}$ and $\mathbf{c}$ are given by

$$\mathbf{r}(s, t) = \mathbf{a} + s\mathbf{b} + t\mathbf{c},$$

where s, t are real numbers. Alternatively, if $\mathbf{p}$ is a vector orthogonal to both $\mathbf{b}$ and $\mathbf{c}$ (ie normal to the plane, or parallel to $\mathbf{b} \times \mathbf{c}$), then the same plane is described by

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{p} = 0.$$

Using these definitions Euclidean geometry becomes a vector application.

1.5 **Triple products**

The *triple scalar product* defined by

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \sum_{p,q,r=1}^{3} \epsilon_{pqr}\, a_p b_q c_r = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

represents twice the volume of a prism whose sides are parallel to $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The triple scalar product vanishing is equivalent to the three vectors being coplanar.

It is worth remembering the formula for the vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

(**NB** This is not the same as $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, meaning the cross product is not associative.)

1.6 **Changing the axes**

The origin could be shifted by translating it and/or the coordinate axes could be rotated.

(a) Suppose the new origin is O' shifted from O by $\mathbf{s}$. Then, a point A represented by $\mathbf{a}$ as a displacement from O will be represented by $\mathbf{a}'$ as a displacement from O' where

$$\mathbf{a} = \mathbf{a}' + \mathbf{s}.$$

Note: the displacement of B relative to A is given by $\mathbf{b} - \mathbf{a} = \mathbf{b}' - \mathbf{a}'$.

(b) Rotating the unit vectors (keeping the origin fixed). The unit vectors $\mathbf{e}_a,\ a = 1, 2, 3$ are expressed as linear combinations of a new set $\mathbf{e}'_a,\ a = 1, 2, 3,$

$$\mathbf{e}_b = \mathbf{e}'_c R_{cb}.$$

Then

$$\delta_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b = \mathbf{e}'_c R_{ca} \cdot \mathbf{e}'_d R_{db} = \delta_{cd} R_{ca} R_{db} = (R^T R)_{ab}$$

so that the matrix representing the change of axes is orthogonal (with determinant +1 to represent a rotation preserving the orientation). Thus, we could alternatively write

$$\mathbf{e}'_b = R_{bc}\mathbf{e}_c.$$

The components of a vector are also related by the orthogonal matrix:

$$x_a \mathbf{e}_a = x_a \mathbf{e}'_b R_{ba} = x'_b \mathbf{e}'_b, \ \Rightarrow \ x'_b = R_{ba} x_a.$$

Note

$$x'_a x'_a = R_{ab} x_b R_{ac} x_c = (R^T R)_{cb} x_b x_c = x_b x_b,$$

which represents the fact that while the individual components of a vector change under a rotation of axes its magnitude does not.

Note (i) If the rotation of axes is constant (independent of time) then the unit vectors in each frame are constant and a position vector given by

$$\mathbf{r} = x'_a \mathbf{e}'_a = x_a \mathbf{e}_a$$

can be differentiated with respect to time straightforwardly, meaning the components of the velocity are given by

$$\frac{d\mathbf{r}}{dt} = \frac{dx'_a}{dt}\,\mathbf{e}'_a = \frac{dx_a}{dt}\,\mathbf{e}_a.$$

Repeating gives the components of acceleration in each frame.

Note (ii) However, if the rotation relating the two frames is not independent of time, then it becomes more interesting. Suppose the unprimed unit vectors are constant while the primed frame orthonormal unit vectors are given by

$$\mathbf{e}'_a = R_{ab}(t)\,\mathbf{e}_b.$$

Then

$$\frac{d\mathbf{e}'_a}{dt} = \frac{dR_{ab}}{dt}\,\mathbf{e}_b = \frac{dR_{ab}}{dt}\,R^T_{bc}\,\mathbf{e'}_c.$$

Now, we noted before that R is an orthogonal matrix so using that fact we see that the combination

$$\frac{dR_{ab}}{dt}\,R^T_{bc}$$

is actually antisymmetric when a, c are interchanged, which means we can write

$$\frac{dR_{ab}}{dt}\,R^T_{bc} = \epsilon_{acd}\,\omega_d,$$

and hence

$$\mathbf{v} \equiv \frac{d\mathbf{r}}{dt} = \frac{dx_a}{dt}\,\mathbf{e}_a = \frac{dx'_c}{dt}\,\mathbf{e}'_c + \omega \times \mathbf{r} \equiv \mathbf{v}' + \omega \times \mathbf{r}$$

In this expression, the first term on the right hand side is what the observer in the rotating frame would consider to be the components of velocity; the second term is there even if the components of $\mathbf{r}$ in the rotating frame are constant. Repeating the process, and assuming the vector ω is time-independent, relates the components of acceleration in the two frames:

$$\mathbf{a} \equiv \frac{d^2\mathbf{r}}{dt^2} \equiv \mathbf{a}' + 2\omega \times \mathbf{v}' + \omega \times (\omega \times \mathbf{r}).$$

In this expression the primed vectors are the acceleration and velocity viewed from the rotating frame. The unfamiliar terms give rise to interesting phenomena from the point of view of the rotating observer (and this is sometimes important because we tend to use coordinates fixed on a rotating Earth). The second term in this expression is called the Coriolis force; the third term is called the centrifugal force. Anyone who has experienced a rotating roundabout in a playground is familiar with the consequences of both.

Remark If you have wondered what a rotation matrix looks like then you could take

$$R_{ab} = \cos\theta\,\delta_{ab} + (1-\cos\theta)\,n_a n_b + \sin\theta\,\epsilon_{abc} n_c,$$

where $n_a e_a$ is a unit vector and θ is an angle. Using the properties of δ_{ab} and ϵ_{abc} it is reasonably straightforward to check that

$$(R^T R)_{ac} = R_{ba} R_{bc} = \delta_{ac}.$$

Note also that $\mathbf{n}$ is an eigenvector of the matrix R since $R_{ae} n_e = n_a$. You can think of $\mathbf{n}$ as the 'axle' around which the coordinates rotate (through an angle θ).

NB Throughout the preceding calculations the summation convention has been used (repeated indices - no more than two with the same letter - are understood to be summed over).

II **Newtonian dynamics**

2.1 (a) Kinematics: velocity and acceleration

A particle (an idealisation since real objects even if very small have a spatial extent) is represented by a point in space that moves on a trajectory denoted by $\mathbf{r}(t)$, the vector representing its position at time t relative to a specified origin. We often think of a particle's position in terms of orthogonal coordinates relative to a chosen origin:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

where the fixed (meaning time-independent) orthogonal unit vectors lie along the Cartesian axes with origin at $\mathbf{0}$.

Note: it is important to remember the components of a vector carry sufficient information to locate the particle provided the origin and axes are specified in advance (and agreed on by all parties describing the same particle).

Note: it has to be agreed among interested parties what time actually is (ie the division of measured time into seconds, minutes, hours, days, etc). In our society there has been agreement on this (GMT) for quite a while though there is a necessity to define 'local' time shifted relative to it to make life more convenient for people living at different longitudes on Earth.

(i) **Velocity**: the velocity of a particle is just the rate of change of its position with respect to time:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \lim_{\delta t\to 0} \frac{\mathbf{r}(t+\delta t) - \mathbf{r}(t)}{\delta t} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}.$$

Note, the velocity of a particle points along the tangent to its trajectory at time t.

(ii) **Acceleration**: the acceleration of a particle is just the rate of change of its velocity with respect to time:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \lim_{\delta t\to 0} \frac{\mathbf{v}(t+\delta t) - \mathbf{v}(t)}{\delta t} = \dot{\mathbf{v}} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}}.$$

Note: to calculate the derivative there is again an assumption that you can compare, for example, the vector $\mathbf{v}(t+\delta t)$ with the vector $\mathbf{v}(t)$, despite requiring a translation of $\mathbf{v}(t)$ parallel to itself to the point $\mathbf{r}(t+\delta t)$.

Note: The rules for differentiating products survive suitably adapted. For example,

$$\frac{d}{dt}(\alpha\mathbf{a}) = \dot{\alpha}\mathbf{a} + \alpha\dot{\mathbf{a}}, \quad \frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \dot{\mathbf{a}}\cdot\mathbf{b} + \mathbf{a}\cdot\dot{\mathbf{b}}, \quad \frac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \dot{\mathbf{a}}\times\mathbf{b} + \mathbf{a}\times\dot{\mathbf{b}}.$$

But be careful never to change the order of factors in a vector product.

Note: what is regarded as a point is largely a question of perspective. A cosmologist will consider galaxies as points moving under their mutual gravitational influence. Nearer home, astronomers may consider the sun and other stars as points within our galaxy again moving under their mutual gravitational attraction (after all in the night sky the stars - and galaxies - look like points to the naked eye). We might consider most objects in our solar system as points relative to the sun. If we try to describe a fluid, for most purposes we will not go down to the structure of its constituent molecules (there are far too many of them to be described individually). Nor for most purposes do we worry about the constituents of atoms (electrons, protons, neutrons), or the constituents of protons and neutrons (quarks and 'glue'), etc. Some elementary particles are indeed regarded as elementary because (so far) no constituents have been detected within them (examples are electrons, quarks, neutrinos, photons, ...). However, in any case, Newtonian mechanics is inadequate to describe gravity at a cosmological scale (you need General Relativity for that) or to describe elementary particles on a microscopic scale (you will need Quantum Field Theory for that).

If we do wish at some point to apply mechanics to an extended object then we will have to pay attention to its detailed composition and treat its constituents as points.

(b) Examples of particle trajectories

(i) Suppose

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t, \quad \mathbf{v},\ \mathbf{r}_0 \text{ are constants}$$

then

$$\dot{\mathbf{r}} = \mathbf{v}, \quad \ddot{\mathbf{r}} = 0.$$

The particle moves at constant velocity in a straight line with zero acceleration.

(ii) Suppose a particle moves in such a manner that

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t - \frac{1}{2}\mathbf{g}t^2$$

where $\mathbf{r}_0$, $\mathbf{v}_0$, $\mathbf{g}$ are constant vectors. Then,

$$\mathbf{v} = \dot{\mathbf{r}} = \mathbf{v}_0 - \mathbf{g}t, \quad \ddot{\mathbf{r}} = -\mathbf{g}.$$

In other words, the particle is moving with constant acceleration starting at $\mathbf{r}(0) = \mathbf{r}_0$ with velocity $\dot{\mathbf{r}}(0) = \mathbf{v}_0$. The shape of the trajectory is a parabola.

(iii) Suppose

$$\mathbf{r} = a(\cos(\omega t)\,\mathbf{i} + \sin(\omega t)\,\mathbf{j}),$$

where a, ω are positive constants. Then, since $|\mathbf{r}| = a$ the trajectory is a circle (centred at the origin and in a plane parallel to the unit vectors $\mathbf{i}, \mathbf{j}$), and

$$\dot{\mathbf{r}} = \omega a(-\sin(\omega t)\,\mathbf{i} + \cos(\omega t)\,\mathbf{j}), \quad \ddot{\mathbf{r}} = -\omega^2 a(\cos(\omega t)\,\mathbf{i} + \sin(\omega t)\,\mathbf{j}).$$

Note, the velocity vector is perpendicular to the position vector (that is, $\mathbf{r} \cdot \dot{\mathbf{r}} = 0$), while the acceleration vector is parallel to it, with

$$\ddot{\mathbf{r}} = -\omega^2 \mathbf{r}.$$

Note: in all these cases the trajectory can be determined by starting with the expression for the acceleration and then solving the second order differential equation. In the first two cases this is a direct integration with respect to time; in the third case, the differential equation is a vector version of 'simple harmonic motion'.

(c) Plane polar coordinates

If a particle motion is entirely on the plane $z = 0$, or more generally $z =$constant, then it can be useful to use plane polar coordinates and set

$$x = r\cos\theta \qquad y = r\sin\theta,$$

in which case

$$\mathbf{r} = r(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \equiv r\mathbf{e}_r$$

in the plane perpendicular to $\mathbf{k}$. Here $r = |\mathbf{r}|$ represents the distance of the particle from the origin. Then, setting

$$-\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \equiv \mathbf{e}_\theta, \quad \mathbf{e}_r \cdot \mathbf{e}_\theta = 0, \quad \mathbf{e}_r \times \mathbf{e}_\theta = (\cos^2\theta + \sin^2\theta)\,\mathbf{i} \times \mathbf{j} = \mathbf{k},$$

we have the following facts (which you should check for yourself):

$$\begin{aligned}
&\dot{\mathbf{e}}_r = \dot\theta \mathbf{e}_\theta, \quad \dot{\mathbf{e}}_\theta = -\dot\theta \mathbf{e}_r \\
&\dot{\mathbf{r}} = \dot r \mathbf{e}_r + r\dot\theta \mathbf{e}_\theta, \\
&|\dot{\mathbf{r}}|^2 = \dot r^2 + r^2\dot\theta^2 \\
&\ddot{\mathbf{r}} = (\ddot r - r\dot\theta^2)\mathbf{e}_r + (r\ddot\theta + 2\dot r\dot\theta)\mathbf{e}_\theta \\
&\mathbf{r} \times \dot{\mathbf{r}} = r^2\dot\theta\,\mathbf{k}
\end{aligned}$$

These expressions are especially useful (as we will see later) when analysing the motion of a particle subject to a force directed towards the origin (or away from it).

Note: in this case the coordinate system is moving with the particle (because $\mathbf{e}_r, \mathbf{e}_\theta$ depend on θ), which means the unit vectors are not (usually) constants.

Note: $\dot r \neq |\dot{\mathbf{r}}|$, rather

$$\dot r = \dot{\mathbf{r}} \cdot \mathbf{e}_r = \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r}, \text{ while } |\dot{\mathbf{r}}| = \sqrt{\dot r^2 + r^2\dot\theta^2}.$$

2.2 Newton's equation of motion

In an inertial frame (to be clarified later), a particle moves in such a way that its acceleration is proportional to the sum of all the forces acting on it (Newton's second law):

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{r}}{dt^2} \equiv m\ddot{\mathbf{r}}$$

where m is its 'mass'. This assumes the character (meaning mass) of the particle is unchanging. A more general statement is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt},$$

where $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$ is the momentum of the particle.

Note: mass is measured in kg; a force of 1 Newton is required to give a particle of mass 1 kilogram an acceleration of 1 metre/sec^2.

Note: if the forces acting on the particle sum to zero, meaning $\mathbf{F} = 0$, then the equation of motion can be integrated directly to give the trajectory

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}_0.$$

In other words, the particle moves in a straight line at a constant velocity (Newton's first law).

One aim of mechanics is to describe the trajectories of a (number of) particle(s), given information about the forces acting on it (them).

2.3 Sample forces

(a) The gravitational force between two particles of masses m_1, m_2, situated at $\mathbf{r}_1, \mathbf{r}_2$ (ie the force felt by particle 1 because of the presence of particle 2) is given by

$$\mathbf{F}_{12} = G\,m_1 m_2\,\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} = -\mathbf{F}_{21}.$$

Newton's constant G has been measured to be

$$G = 6.67408 \times 10^{-11}\, m^3 kg^{-1} s^{-2}$$

Newton's constant is very small but we live on a planet that is (by our standards) very massive ($m_E = 5.972 \times 10^{24}\,kg$) - yet light relative to the sun ($m_S = 1.989 \times 10^{30}\,kg$).

(b) The gravitational force felt by a particle (mass m) near the surface of the earth is given approximately by

$$\mathbf{F} = -mg\mathbf{k}, \text{ where } g = \frac{Gm_{\text{earth}}}{R^2_{\text{earth}}} \approx 9.8\ m/s^2.$$

(Here, $\mathbf{k}$ is a unit vector pointing vertically upwards from the Earth's surface.) Note, $R = 6371\,km$ approximately (the Earth is not quite a perfect sphere, and in any case it is quite bumpy, so this is an average). Hence, near the Earth, Newton's equation states

$$m\ddot{\mathbf{r}} = -mg\mathbf{k} \ \Rightarrow\ \ddot{\mathbf{r}} = -g\mathbf{k},$$

which is independent of the mass, a fact that has been tested very accurately (see comment below). Again, this differential equation can be integrated directly to give:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}_0 - \frac{1}{2}t^2\,g\mathbf{k}, \quad \mathbf{r}_0 = \mathbf{r}(0),\ \mathbf{v}_0 = \dot{\mathbf{r}}(0).$$

(c) A charged particle situated in an electromagnetic field $(\mathbf{E}, \mathbf{B})$, in a convention where the electric field and magnetic field have the same dimensions feels a force

$$\mathbf{F} = q\left(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}/c\right),$$

where q is the electric charge carried by the particle and c is the speed of light.

(d) Other, more mundane forces such as friction or the restoring force of a stretched string or rope have their origins at a molecular level but are usually modelled in a simple way. For example, the restoring force of a spring can (for reasonable materials and non-violent distortions) be taken to be proportional to the amount it has been stretched; air resistance might be modelled by taking it to be proportional to $|\mathbf{v}|^\alpha \mathbf{v}$, for example, where α is a constant depending on the atmospheric conditions and the composition of the object being described.

(e) If we do wish to take into account the finite size, or shape of an object (such as the Sun or Earth) then as far as gravity is concerned we would simply adapt (a) by summing up the gravitational forces between the individual 'pointlike' constituents of the objects. For example, Newton proved that provided the Sun is a constant density sphere then the gravitational force exerted by the Sun on another particle (such as the much smaller Earth - also assumed to be a constant density sphere) is exactly (a) where m_1, m_2 are the total masses of the Sun and Earth and $\mathbf{r}_1, \mathbf{r}_2$ are the positions of their centres.

Important remarks:

(i) The fact the mass appearing in Newton's equation (inertial mass) and the mass appearing in the gravitation force law (gravitational mass) are actually the same to great accuracy is an observation that has profound implications and a long history (Galileo, Newton, Eötvös, Dicke et al, ...; for recent measurements, see Schlamminger et al., Phys. Rev. Letts. 100 (041101) 2008 and Wagner et al., Classical and Quantum Gravity 29 (2012) 1840012; you should be able to find the articles easily via Google Scholar). As technology evolves, measurements of the two different masses are made to greater accuracy; if a discrepancy ever shows up it will force a re-evaluation of current thinking.

(ii) If we decide to change coordinates by making the transformation

$$\mathbf{r}' = \mathbf{r} - \mathbf{u}t, \quad t' = t,$$

where $\mathbf{u}$ is constant (this is equivalent, for example, to choosing the origin on a train moving with a constant velocity $\mathbf{u}$ in the unprimed coordinate frame), then we note that

$$m\ddot{\mathbf{r}}' = m\ddot{\mathbf{r}} = \mathbf{F}.$$

This means that provided the force does not depend explicitly on the particle's position or velocity, Newton's equation has the same form. This is the reason why a dropped object on a constant velocity train or aeroplane behaves exactly (as seen by an observer on the train) as if the train or aeroplane was at rest. A transformation of this type (assuming time is unchanged between two frames) is called a Galilean transformation; performing two transformations one after the other with parameters $\mathbf{u}_1$, $\mathbf{u}_2$ is clearly equivalent to a similar transformation with parameter $\mathbf{u}_1 + \mathbf{u}_2$, which demonstrates that the set of all Galilean transformations forms an abelian group.

In Newtonian mechanics inertial frames are related by Galilean transformations together with time-independent rotations and the assumption that time is universal (chosen to be the same in all frames).

Note: it will be useful later to consider points on a particle's trajectory as 'events' labelled by time and position $(t, \mathbf{r}(t))$ then a Galilean transformation can be thought of as a map between events

$$G: \ (t,\ \mathbf{r}(t)) \to (t',\ \mathbf{r}'(t')) = (t,\ \mathbf{r}(t) - \mathbf{u}t),$$

with $\mathbf{u}$ constant.

Note: the gravitational force between two particles depends only on the difference of their position vectors (5(a) above), which means that their equations of motion are invariant under Galilean transformations; near the Earth, the gravitational force is constant (5(b) above), which means the equation of motion of a particle moving under near-Earth gravity is also Galilean invariant. However, even if the electric and magnetic fields are constants, the Lorentz force (5(c) above) is not Galilean invariant if $B \neq 0$ and $\mathbf{u}$ is not parallel to $\mathbf{B}$ (though it is not easy to detect the difference because $(\mathbf{u} \times \mathbf{B})/c$ is very small unless $|\mathbf{u}|$ is a reasonable fraction of the speed of light).

(iii) Mass is additive. If two particles with masses m_1, m_2 coalesce then the combined particle has mass $m_1 + m_2$.

(iv) Note: it is often necessary to work within a non-inertial frame (for example, the usual coordinates we use near the Earth's surface are fixed on a rotating Earth - the non-intertial nature is demonstrated famously by Foucault's pendulum - look it up!). We will not consider such frames now. If you do want to explore them experimentally then you can use a roundabout in a playground (the observer at rest on a moving roundabout is not in an inertial frame).

2.4 **Energy**

An important concept in mechanics (or other applications of mathematics to physical phenomena) is the notion of energy. Kinetic energy for a single particle of mass m is defined by the expression

$$K = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}m|\dot{\mathbf{r}}|^2.$$

This depends on time typically but it is useful to consider how it changes with time:

$$\dot{K} = \frac{d}{dt}\left(\frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right) = m\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = \dot{\mathbf{r}} \cdot \mathbf{F},$$

where the last step uses Newton's equation of motion. Next, note that some common forces (including the gravitational force) have a very special property because they can be written as the gradient of a 'potential'. In other words, for such forces we may write

$$\mathbf{F} = -\nabla\Phi \equiv -\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\Phi = -\left(\mathbf{i}\frac{\partial\Phi}{\partial x} + \mathbf{j}\frac{\partial\Phi}{\partial y} + \mathbf{k}\frac{\partial\Phi}{\partial z}\right),$$

where the (-) sign is conventional. Note, such forces are called 'conservative'.

Then, along the particle trajectory (depending on time)

$$\dot{K} = -\dot{\mathbf{r}} \cdot \nabla\Phi = -\frac{d\Phi}{dt} \;\Rightarrow\; \frac{d}{dt}(K + \Phi) = 0,$$

which means $K + \Phi$ is constant along the trajectory. The constant is called the total energy comprising the kinetic energy K and the 'potential energy' Φ; in most circumstances the kinetic energy and potential energy are not individually constant along the trajectory. Seeking constants of motion is an important exercise.

(i) Near the Earth the gravitational force on a particle of mass m is $-mg\mathbf{k}$ and we note:

$$-mg\mathbf{k} = -mg\mathbf{k}\frac{\partial z}{\partial z} = -\nabla\Phi \text{ with } \Phi = mgz + \Phi_0,$$

and thus the total conserved energy for a particle moving on a trajectory near the surface of the Earth (subject to gravity only, ignoring friction) is

$$E = \frac{1}{2}m|\mathbf{v}|^2 + mgz + \Phi_0.$$

The constant Φ_0 rarely matters in practice and can be safely dropped (or incorporated in the left hand side by redefining E). From this we can deduce immediately that a particle dropped from rest from a height $z = h$ will hit the ground ($z = 0$) with speed $\sqrt{2gh}$.

(ii) If a particle is moving under the influence of an inverse square law force directed towards the origin,

$$\mathbf{F} = -\frac{\alpha}{|\mathbf{r}|^3}\,\mathbf{r} = -\frac{\alpha\,\hat{\mathbf{r}}}{|\mathbf{r}|^2}, \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|},$$

where α is a positive constant, then we note

$$\mathbf{F} = \nabla\left(\frac{\alpha}{|\mathbf{r}|}\right) = -\nabla\Phi \;\Rightarrow\; \Phi = -\frac{\alpha}{|\mathbf{r}|}.$$

In this case, the total conserved energy is

$$E = \frac{1}{2}m|\mathbf{v}|^2 - \frac{\alpha}{|\mathbf{r}|}.$$

Note: this already reveals a couple of secrets:

(i) If $E \geq 0$ then the particle can escape to infinity but cannot come too close to the origin $\mathbf{r} = 0$.

(ii) If $E < 0$ and finite then the particle cannot escape to infinity nor can its orbit approach too close to $\mathbf{r} = 0$.

2.5 Angular momentum

Angular momentum is defined relative to a specified point. For example, the angular momentum of a particle of mass m about the origin, O, is defined to be

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}},$$

where $\mathbf{p}$ is the momentum of the particle previously defined ($\mathbf{p} = m\dot{\mathbf{r}}$). Note

$$\dot{\mathbf{J}} = m\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \mathbf{F} \equiv \mathbf{M},$$

as a consequence of Newton's equations. The quantity $\mathbf{M}$ is called the moment of the force about O.

Important note: an interesting example is supplied by a particle moving under a force directed towards (or away from) the origin

$$\mathbf{F} = f(\mathbf{r})\mathbf{r},$$

where $f(\mathbf{r})$ is a scalar function of the components of $\mathbf{r}$. In this case $\mathbf{r} \times \mathbf{F} = f(\mathbf{r})\, \mathbf{r} \times \mathbf{r} = 0$, meaning the angular momentum $\mathbf{J}$ is constant (note the specific choice of $f(\mathbf{r})$ is unimportant for this result).

2.6 Collections of particles

Having to deal with systems of particles is commonplace; for example, the solar system contains millions of small individual rocks (and a few much larger objects called the sun, the planets, the moons, etc); for example, a rigid object is an idealisation where it is supposed the constituent particles do not move relative to each other; an elastic object is a more realistic collection of particles that move relative to each other in response to external forces but not by much and tend to return to their initial positions relative to each other when the forces are removed; a fluid is a collection of particles that can move relatively freely with respect to each other in response to external forces. The latter two situations would be better described in a continuous fashion whereas the solar system would be considered as discrete with a hierarchy of constituents defined by decreasing mass (sun, planets, moons, asteroids, etc...). Note: the sun is by far the most massive object in the solar system - its mass accounts for over 99.8% of the mass of the whole solar system.

In a discrete system with N particles, of mass m_i and positions $\mathbf{r}_i(t)$ relative to a chosen origin O, the particle i experiences a force due to each of the others, $\mathbf{F}_{ij}$, together with an external force $\mathbf{F}_i^e$. Note the particle 'i' does not feel a self-force so $\mathbf{F}_{ii} = 0$. Hence, its equation of motion will be

$$m_i \ddot{\mathbf{r}}_i = \sum_{j=1}^{N} \mathbf{F}_{ij} + \mathbf{F}_i^{(e)}, \;\; i = 1, 2, 3, \ldots$$

Newton's third law is the assumption that $\mathbf{F}_{ij} + \mathbf{F}_{ji} = 0$ for each pair $i \neq j$; or, in words, each internal inter-particle force has an equal but opposite reaction. For example the two-body gravitational force in 5(a) clearly has this property. This is useful because it allows to sum all the equations of motion to find the inter-particle forces cancel out to give

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \mathbf{F}_i^{(e)} = \mathbf{F}_{\text{total}}^{(e)}.$$

Defining the centre of mass and total mass of the system by

$$\mathbf{R} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i}, \quad M = \sum_{i=1}^{N} m_i,$$

we find

$$M\ddot{\mathbf{R}} = \mathbf{F}_{\text{total}}^{(e)}.$$

(i) If the sum of the external forces is zero then the centre of mass of the system moves with constant velocity in a straight line.

(ii) The specific motion of the centre of mass of a many particle system is a useful observation because it justifies, for example, considering the solar system as a whole moving under the influence of forces due to the presence of other star systems (including black-holes) residing in the galaxy.

(iii) For a two-body gravitational system the observation allows a complete analysis of the problem (see below). For $N \geq 3$ the system cannot be solved exactly but there are many numerical simulations to be found on the web (I have placed some links on Moodle). Note also, for realistic systems such as the solar system, the number of particles is not fixed since from time to time particles collide and coalesce, or disintegrate (and humankind have added quite a few artificial satellites over the last 60 years for communications and other purposes).

(iv) The total kinetic energy and angular momentum of a collection of N particles are given by

$$K_{\text{tot}} = \frac{1}{2}\sum_{i=1}^{N} m_i |\dot{\mathbf{r}}_i|^2, \quad \mathbf{J}_{\text{tot}} = \sum_{i=1}^{N} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i.$$

Again the centre of mass can be considered separately by setting $\mathbf{r}_i = \mathbf{R} + \mathbf{s}_i$ and noting $\sum_i m_i \mathbf{s}_i = 0$ so that

$$K_{\text{tot}} = \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{1}{2}\sum_{i=1}^{N} m_i |\dot{\mathbf{s}}_i|^2, \quad \mathbf{J}_{\text{tot}} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_{i=1}^{N} m_i \mathbf{s}_i \times \dot{\mathbf{s}}_i.$$

Note: for an N-body system with no external forces moving under mutual gravitational forces:

$$\dot{K}_{\text{tot}} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \dot{\mathbf{r}}_i \cdot \sum_{j \neq i} G m_i m_j \frac{(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} = \frac{d}{dt}\sum_{i<j} \frac{G m_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|}.$$

[Note also: we have rewritten the sum over i, j as a sum over two equal parts, then exchanged $i \leftrightarrow j$ in one part and used the following result that applies to any non-zero vector function of time:

$$|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u} \;\Rightarrow\; 2|\mathbf{u}|\,\frac{d|\mathbf{u}|}{dt} = 2\mathbf{u} \cdot \dot{\mathbf{u}} \;\Rightarrow\; \frac{d|\mathbf{u}|}{dt} = \frac{\mathbf{u} \cdot \dot{\mathbf{u}}}{|\mathbf{u}|},$$

and hence

$$\frac{d}{dt}\left(\frac{1}{|\mathbf{u}|}\right) = -\frac{1}{|\mathbf{u}|^2}\frac{\mathbf{u} \cdot \dot{\mathbf{u}}}{|\mathbf{u}|}$$

with $\mathbf{u} = \mathbf{r}_j - \mathbf{r}_i$.]

Thus the total energy

$$E = K_{\text{tot}} - \sum_{i<j} \frac{G m_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|} = \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{1}{2}\sum_{i=1}^{N} m_i |\dot{\mathbf{s}}_i|^2 - \sum_{i<j} \frac{G m_i m_j}{|\mathbf{s}_j - \mathbf{s}_i|}$$

is conserved. The third term is the total gravitational potential energy expressed as a sum over all pairs of particles.

(v) **A virial theorem** Suppose we define the 'root mean square' radius $\bar{R}$ of a collection of particles via

$$\bar{R}^2 = \frac{\sum_i m_i |\mathbf{r}_i|^2}{\sum_i m_i} \equiv \frac{D}{M}.$$

Then

$$\dot{D} = 2\sum_i m_i \mathbf{r}_i \cdot \dot{\mathbf{r}}_i, \quad \ddot{D} = 2\sum_i m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i + 2\sum_i m_i \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i.$$

Now, supposing the particles are mutually gravitating, there are no external forces, and noting the first term of $\ddot{D}$ is four times the total kinetic energy while the second term can be rewritten using Newton's equation for each particle, you will find

$$2\sum_i m_i \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i = 2\sum_i \mathbf{r}_i \cdot \sum_{j\neq i} Gm_i m_j \frac{(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} = \sum_{i\neq j} (\mathbf{r}_i - \mathbf{r}_j) \cdot Gm_i m_j \frac{(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}$$
$$= -\sum_{i\neq j} \frac{Gm_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|} = 2V_{\text{tot}}.$$

Hence

$$\frac{d^2}{dt^2}\left(\frac{1}{2}M\bar{R}^2\right) = 2K_{\text{tot}} + V_{\text{tot}} = E + K_{\text{tot}},$$

where E is the total conserved energy. Now if we also define the average kinetic energy (or potential energy) over a long time period τ by

$$< K_{\text{tot}} >= \frac{1}{\tau}\int_0^\tau K_{\text{tot}} dt$$

and suppose the quantity $\bar{R}$ does not change, we find

$$E = - < K_{\text{tot}} >, \text{ or } < V_{\text{tot}} >= -2 < K_{\text{tot}} > .$$

This fact was the basis of an analysis of the Coma cluster of galaxies by Zwicky ('On the Masses of Nebulae and of Clusters of Nebulae', F Zwicky, Astrophysical Journal, vol. 86 (1937) 217), which demonstrated that there should be some kind of 'dark matter' to account for observation. So far, 'dark matter' has not been identified directly though there are other, independent, indications that it should exist and many theories as to what it might be. (For example, see 'Particle dark matter: evidence, candidates and constraints', G Bertone, D Hooper, J Silk, Physics Reports 405 (2005) 279).

2.7 Gravitational two-body problem

If there are just two particles moving under their mutual gravitational attraction the system can be solved exactly.

The two particles m_1, m_2 are represented by the vectors $\mathbf{r}_1, \mathbf{r}_2$ with respect to a chosen origin, with equations of motion

$$m_1\ddot{\mathbf{r}}_1 = Gm_1m_2\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \quad m_2\ddot{\mathbf{r}}_2 = Gm_1m_2\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}.$$

Clearly, by adding the two equations, we verify

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0$$

and hence that the centre of mass satisfies

$$M\ddot{\mathbf{R}} = 0, \text{ where } M = m_1 + m_2, \ M\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2.$$

Putting

$$\mathbf{r}_1 = \mathbf{R} + \mathbf{s}_1, \ \mathbf{r}_2 = \mathbf{R} + \mathbf{s}_2, \ \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{s}_1 - \mathbf{s}_2$$

we find

$$m_1\mathbf{s}_1 + m_2\mathbf{s}_2 = 0, \ \mathbf{s}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2}, \quad \mathbf{s}_2 = -\frac{m_1\mathbf{r}}{m_1 + m_2}$$

and

$$\ddot{\mathbf{r}} = -\frac{GM\mathbf{r}}{|\mathbf{r}|^3}.$$

Energy From a previous result, the total energy is conserved so

$$E = \frac{1}{2}m_1 \left|\dot{\mathbf{r}}_1\right|^2 + \frac{1}{2}m_2 \left|\dot{\mathbf{r}}_2\right|^2 - \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|},$$

which in terms of $\mathbf{R}, \mathbf{r}$ is

$$E = \frac{1}{2}(m_1 + m_2)\left|\dot{\mathbf{R}}\right|^2 + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}\dot{\mathbf{r}}^2 - \frac{Gm_1m_2}{|\mathbf{r}|}.$$

Note, the first term is constant (since the centre of mass moves with constant velocity) and so the second and third terms taken together are also constant.

Angular momentum The total angular momentum with respect to the origin is

$$\mathbf{J} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2,$$

which in terms of $\mathbf{R}, \mathbf{r}$ is

$$\mathbf{J} = (m_1 + m_2)\mathbf{R} \times \dot{\mathbf{R}} + \frac{m_1m_2}{m_1 + m_2}\mathbf{r} \times \dot{\mathbf{r}},$$

and again the two terms are separately conserved.

Thus, the problem we need to solve reduces to:

$$\begin{aligned}
\ddot{\mathbf{r}} &= -\frac{G(m_1 + m_2)\mathbf{r}}{|\mathbf{r}|^3} \\
\mathcal{E} &= \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}|\dot{\mathbf{r}}|^2 - \frac{Gm_1m_2}{|\mathbf{r}|} \\
\mathbf{L} &= \frac{m_1m_2}{m_1 + m_2}\mathbf{r} \times \dot{\mathbf{r}}
\end{aligned}$$

where $\mathcal{E}$ and $\mathbf{L}$ are both constant. We can also note that both $\mathbf{r}$ and $\dot{\mathbf{r}}$ are orthogonal to the constant vector $\mathbf{L}$, which means that the solution for $\mathbf{r}(t)$ lies entirely within the plane orthogonal to $\mathbf{L}$. This observation simplifies matters greatly because it allows the use of plane polar coordinates.

Solution in polar coordinates Recalling the previous expressions (with $\mathbf{r} = 0$ being the origin)

$$\begin{aligned}
&\mathbf{r} = r\,\mathbf{e}_r, \quad \dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta, \quad \mathbf{r} \times \dot{\mathbf{r}} = r^2\dot{\theta}\,\mathbf{e}_r \times \mathbf{e}_\theta \\
&\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\mathbf{e}_\theta
\end{aligned}$$

we see that $r^2\dot{\theta} = h$ must be constant; setting $\mu = m_1m_2/(m_1 + m_2)$ then $\mu h = |\mathbf{L}|$ and $\dot{\theta} = h/r^2$. Also,

$$\mathcal{E} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\mu h^2}{r^2} - \frac{Gm_1m_2}{r},$$

and

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{GM}{r^2}.$$

[If we wished we could now regard this system as a one-dimensional problem of a particle of mass μ moving in a effective potential provided by the second and third terms of $\mathcal{E}$.]

The second equation is a differential equation for $r(t)$. To solve it, it's worth remembering that if we want the shape of the trajectory then we are really after r as a function of θ. So, the trick is to let $r = 1/u$ and think of u as a function of θ. Thus

$$\dot{r} = \dot{\theta}\frac{d}{d\theta}\left(\frac{1}{u}\right) = \frac{h}{r^2}\left(-\frac{1}{u^2}\right)\frac{du}{d\theta} = -hu', \quad \ddot{r} = -h^2u^2u'',$$

so that the equation for r becomes

$$u'' + u = \frac{GM}{h^2};$$

this is now a standard type of second order differential equation, with solution

$$u = A\cos(\theta - \theta_0) + \frac{GM}{h^2} \quad \text{or} \quad \frac{1}{r} = \frac{GM}{h^2}(1 + e\cos(\theta - \theta_0)),$$

where e, θ_0 are constants. To follow common conventions, we put $l = h^2/GM$ and choose $\theta_0 = 0$ (ie a choice of origin for θ), so that the equation for r is

$$\frac{l}{r} = 1 + e\cos\theta, \quad e > 0.$$

Note: Given the description of the orbit above, that is

$$r = \frac{l}{1 + e\cos\theta}, \quad \dot{\theta} = \frac{\sqrt{GMl}}{r^2},$$

it is a useful exercise to calculate $\dot{r}$ directly by differentiating with respect to time. Thus

$$\dot{r} = \frac{le\dot{\theta}\sin\theta}{(1 + e\cos\theta)^2} = \frac{le\sqrt{GMl}}{r^2}\,\frac{\sin\theta}{(1 + e\cos\theta)^2} = \sqrt{\frac{GM}{l}}\,e\sin\theta.$$

Then, since $|\dot{\mathbf{r}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$, we also have

$$|\dot{\mathbf{r}}|^2 = \frac{GM}{l}\left(e^2\sin^2\theta + (1 + e\cos\theta)^2\right) = \frac{GM}{l}(1 + 2e\cos\theta + e^2).$$

From this expression, we can see directly that the maximum speed on the orbit occurs when $\cos\theta = 1$ (ie when $r = r_{\min}$) while the minimum speed occurs at $\cos\theta = -1$ (ie when $r = r_{\max}$).

Using the expressions for r and $|\dot{\mathbf{r}}|$ we can calculate $\mathcal{E}$ to find

$$\mathcal{E} = \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 - \frac{\mu MG}{r} = \frac{\mu GM}{2l}\left(1 + 2e\cos\theta + e^2)\right) - \frac{\mu MG}{l}(1 + e\cos\theta) = \frac{\mu GM}{2l}\left(e^2 - 1\right).$$

Clearly, this expression does not depend on the angle θ so it is time-independent.

Remarks For the Earth-Sun system (ignoring the rest of the solar system) we note

$$\frac{m_E}{m_S} \approx 3 \times 10^{-6} \quad \text{so} \quad M \approx m_S, \ \mu \approx m_E$$

and

$$e_E = 0.017, \quad r_{max} = 152.1 \times 10^9\,m, \quad r_{min} = 147.1 \times 10^9\,m.$$

Also

$$\mathbf{s}_S = -\frac{m_E}{m_E + m_S}\,\mathbf{r} \ \Rightarrow \ |\mathbf{s}_S| \approx \frac{m_E}{m_S}\,|\mathbf{r}| \approx 450\,km.$$

So, the Sun wobbles a little relative to the Earth-Sun centre of mass. But remember, the centre of mass of the whole solar system is moving rather fast (by our standards) through the galaxy at around $230\,km/s$.

The nature of the orbit Given the value of the energy it is clear the nature of the orbit is strongly dependent on the constant e.

(i) $e = 0$: in this case $r = l$ and the orbit is a circle traversed at a constant rate since $l^2\dot{\theta} = h = \sqrt{GMl}$. If the orbit has time period T then $\dot{\theta} = 2\pi/T$ and so

$$T = \frac{2\pi}{\sqrt{GM}}\,l^{3/2} \text{ or } T^2 = \frac{4\pi^2}{GM}\,l^3.$$

(ii) $0 < e < 1$: in this case, the orbit is periodic with

$$r_{max} = r(\pi) = \frac{l}{1-e}, \quad r_{min} = r(0) = \frac{l}{1+e}.$$

For a planet-Sun system r_{max} is called the aphelion, r_{min} is called the perihelion. The distance between these two events is $r_{max} + r_{min} = 2a = 2l/(1-e^2)$.

If we prefer, we can think of the orbit with respect to Cartesian coordinates (X, Y) centred at $r = 0$ with the X-axis along the line joining r_{max} to r_{min}. Then

$$r\cos\theta = X, \quad r\sin\theta = Y$$

and

$$r^2 = X^2 + Y^2 = (l - eX)^2 = l^2 - 2elX + e^2X^2$$

which implies

$$\frac{1-e^2}{l^2}\,Y^2 + \frac{(1-e^2)^2}{l^2}\left(X + \frac{el}{1-e^2}\right)^2 = 1.$$

Defining

$$a = \frac{l}{1-e^2}, \quad b = \frac{l}{\sqrt{1-e^2}} = \sqrt{1-e^2}\,a,$$

the orbit becomes

$$\frac{Y^2}{b^2} + \frac{(X+ea)^2}{a^2} = 1,$$

which is an ellipse whose centre is located at $(X, Y) = (-ea, 0)$. The point $(0, 0)$, which corresponds to $r = 0$, is a 'focus' of the ellipse (the other focus is at $(-2ae, 0)$).

Note: Recall that Kepler had noted using contemporary data that the orbits of planets should be ellipses with the sun at one focus and Newton's derivation of this fact from the inverse square law (reproduced in modern language above) is an astonishing achievement. It is very difficult for us to place ourselves in the shoes of Newton and his contemporaries in order to realise fully just how spectacular this result is.

The orbit is periodic (relative to the centre of mass) with period T and we can relate the period to the properties of the elliptical orbit. Noting that $r^2\dot{\theta} = h$ and integrating over the whole period we have

$$hT = \int_0^T r^2\dot{\theta}\,dt = \int_0^{2\pi} r^2 d\theta = 2\pi ab = 2\pi\frac{l}{1-e^2}\,\frac{l}{\sqrt{1-e^2}}$$

and hence

$$T^2 = \frac{4\pi^2}{GM}\,a^3.$$

Note: this accords with one of Kepler's observations that T^2/a^3 is approximately the same for every planet (since for each $M \approx m_S$).

We can also work out the energy to find

$$\mathcal{E} = -\frac{Gm_1m_2}{2a}.$$

For Earth-Sun this is

$$\mathcal{E} = -\frac{6.67 \times 10^{-11} \times 1.99 \times 10^{30} \times 5.97 \times 10^{24}}{2 \times 1.5 \times 10^{11}} \approx 2.6 \times 10^{33}\,Nm, \quad (1 \text{ Nm} = 1 \text{ Joule}).$$

This is a huge energy (for comparison a one megaton hydrogen bomb is roughly equivalent to 4.2×10^{15} Joules).

Note: in the cases (i,ii), the energy $\mathcal{E}$ is negative.

(iii) $e \geq 1$: in this case, the energy $\mathcal{E}$ is positive (or zero if $e = 1$) and the orbit comes from infinity and returns there. This is clear because r diverges when $\cos\theta = -1/e$, that is when $\theta = \pm\cos^{-1}(-1/e)$. It is useful to define $a = l/(e^2 - 1),\ e > 1$ and deal with $e = 1$ separately. As before, we may use Cartesian cooordinates with origin at $r = 0$ to find, via a similar calculation,

$$\frac{(X - ea)^2}{a^2} - \frac{Y^2}{b^2} = 1,\ b = \sqrt{e^2 - 1}\,a,\ e > 1; \quad Y^2 = l^2 - 2lX,\ e = 1,$$

demonstrating that for $e > 1$ the orbit is a branch of a hyperbola centred at $(X, Y) = (ae, 0)$, while for $e = 1$ the orbit is a parabola. For the parabolic orbit the speed relative to $\mathbf{r} = 0$ is given by

$$|\dot{\mathbf{r}}|^2 = \frac{2GM}{r_{min}} \cos^2\left(\frac{\theta}{2}\right).$$

Thus the maximum speed occurs at $\theta = 0$ and is the least speed required for a particle to escape to infinity. Anything less and the particle follows an elliptic orbit; anything more and it follows a hyperbolic orbit. For this reason $\sqrt{2GM/r_{min}}$ is called the escape velocity.

Note: for the hyperbolic orbit, assuming the particle is travelling from infinity when $\theta < 0$, the angle between the incoming direction and the outgoing direction (both asymptotes of the hyperbola), called the 'scattering angle' Ψ, is given by

$$\Psi = \pi - 2\tan^{-1}(\sqrt{e^2 - 1}) = \pi - 2\cos^{-1}\left(\frac{1}{e}\right).$$

2.8 The gravitational potential revisited

We remarked earlier that Newton justified treating the Sun and planets as point particles as far as their gravitational attraction is concerned. In this section, let's consider this more carefully.

The gravitational force between two point particles of masses m_1, m_2, situated at $\mathbf{r}_1, \mathbf{r}_2$ (ie the force felt by particle 1 because of the presence of particle 2) is given by

$$\mathbf{F}_{12} = G\,m_1 m_2\,\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} = -\mathbf{F}_{21}.$$

We also note that the gravitational force can be written as the gradient of a potential:

$$\mathbf{F}_{12} = \nabla^{(1)}\left(\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\right),\ \mathbf{F}_{21} = \nabla^{(2)}\left(\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\right),$$

where the superscripts (1), (2) mean the gradient is computed with respect to the variable $\mathbf{r}_1,\ \mathbf{r}_2$, respectively. This is useful because we can calculate the gravitational potential in many circumstances, including the potential due to an extended massive body.

The situation we will consider is a point particle (mass m located at $\mathbf{r}$) experiencing a force due to a large object of volume V, total mass M, where the mass density is $\mu(\mathbf{s})$, so that the total mass and the centre of mass are defined by

$$M = \int_V \mu(\mathbf{s})\,dV, \quad M\mathbf{R} = \int_V \mu(\mathbf{s})\,\mathbf{s}\,dV,$$

where the volume integration is with respect to the variables represented by $\mathbf{s}$. The total gravitational potential in this system is given by

$$\Phi(\mathbf{r}) = Gm \int_V \frac{\mu(\mathbf{s})}{|\mathbf{r} - \mathbf{s}|}\,dV.$$

(i) Suppose the large object is a sphere of radius a with a constant mass density μ_0. Then, choosing the origin to be in the centre of the sphere (the centre of mass of the sphere because the mass density is constant), the gravitational potential is given by

$$\Phi(\mathbf{r}) = Gm\mu_0 \int_V \frac{1}{|\mathbf{r}-\mathbf{s}|}\, dV = Gm\mu_0 \int_V \frac{1}{(|\mathbf{r}|^2 - 2\mathbf{r}\cdot\mathbf{s} + |\mathbf{s}|^2)^{1/2}}\, dV.$$

Next, it is useful to use polar coordinates $s \equiv |\mathbf{s}|, \theta, \phi$, with $\mathbf{r}$ as the polar axis, and also to put $r = |\mathbf{r}|$, so that:

$$\Phi(\mathbf{r}) = Gm\mu_0 \int_V \frac{1}{(r^2 - 2rs\cos\theta + s^2)^{1/2}} s^2 \sin\theta\, ds\, d\theta\, d\phi.$$

The ϕ, θ integrals are straightforward to perform (remember $0 \le \phi \le 2\pi,\ 0 \le \theta \le \pi$), to get

$$\Phi(\mathbf{r}) = Gm\mu_0 \frac{2\pi}{r} \int_0^a \left((r^2 + 2rs + s^2)^{1/2} - (r^2 - 2rs + s^2)^{1/2}\right) sds.$$

There are two possibilities for the final step; either (a) $r > a$ or (b) $r < a$.

$$\text{(a)}: \quad \Phi(\mathbf{r}) = Gm\mu_0 \frac{2\pi}{r} \int_0^a ((r+s) - (r-s))\; sds = Gm\mu_0 \frac{2\pi}{r}\frac{2a^3}{3} \equiv \frac{GMm}{r}.$$

Assuming the Sun and other planets are constant mass density spheres, this justifies the assumption we made before.

The other case is more tricky because the integral over s needs to be split into two pieces according to whether s is greater or less than r:

$$\text{(b)}: \qquad \Phi(\mathbf{r}) = Gm\mu_0 \frac{2\pi}{r} \left(\int_0^r 2s^2 ds + \int_r^a 2rsds\right) = \frac{3GmM}{2a}\left(1 - \frac{r^2}{3a^2}\right).$$

Note: the expressions (a) and (b) agree at $r = a$. In case (b), the force is given by

$$\nabla\Phi = -\frac{GmM}{a^3}\mathbf{r}.$$

(ii) Suppose the object is neither a sphere nor of constant density. In this case, it is useful to pick the origin at the centre of mass of the object and consider the case when $|\mathbf{r}|$ is much larger than the scale of the object. In this situation, the integrand can be approximated using a vector version of Taylor's theorem. Thus

$$\frac{1}{|\mathbf{r}-\mathbf{s}|} = \frac{1}{|\mathbf{r}|} - (\mathbf{s}\cdot\nabla)\left(\frac{1}{|\mathbf{r}|}\right) + \frac{1}{2}(\mathbf{s}\cdot\nabla)(\mathbf{s}\cdot\nabla)\left(\frac{1}{|\mathbf{r}|}\right) + \ldots$$

So,

$$\Phi(\mathbf{r}) = Gm \int_V \frac{\mu(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|}\, dV = \frac{GMm}{|\mathbf{r}|} + 0 + \frac{Gm}{|\mathbf{r}|^5}\left(3r_a r_b - |\mathbf{r}|^2\delta_{ab}\right) \int_V \mu(\mathbf{s})\, s_a s_b\, dV + \ldots$$

The second term is zero because the origin has been chosen to be the centre of mass of the object. In the last term of the expression we have used an index notation and it assumes a, b are summed over as usual. Hence, the first approximation is exactly what we have been assuming and is good provided $|\mathbf{r}|$ is large. The third term could be important under special circumstances. For example, Earth does not have constant density, nor is it a perfect sphere, so nearby orbiting objects (for example the Moon, or near-Earth objects such as communications satellites), may feel the influence of this.

2.9 Rigid bodies

If (in a many particle system) the interparticle forces are such as to keep constant the interparticle distances the collection of particles is known as a rigid body. It's an idealisation, of course, but a useful one because many everyday objects are effectively inelastic in response to moderate stresses. A rigid body can be discussed effectively but first it is necessary to develop a way to describe it. As before, each point particle will be described by its position relative to the (moving) centre of mass, or, if the rigid body has a fixed point then relative to that point. Thus,

$$\mathbf{r}_i = \mathbf{R} + \mathbf{s}_i, \quad |\mathbf{s}_i| = \text{ constant.}$$

Also, we assume the interparticle forces (which we cannot actually specify precisely), $\mathbf{F}_{ij}$, are antisymmetric and proportional to the vector representing the displacement of particle i from particle j. In other words,

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \quad \mathbf{F}_{ij} \propto (\mathbf{r}_i - \mathbf{r}_j).$$

As before, the total energy and angular momentum are given by

$$K_{\text{tot}} = \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{1}{2}\sum_{i=1}^{N} m_i|\dot{\mathbf{s}}_i|^2, \quad \mathbf{J}_{\text{tot}} = M\mathbf{R}\times\dot{\mathbf{R}} + \sum_{i=1}^{N} m_i\mathbf{s}_i\times\dot{\mathbf{s}}_i.$$

Now, because the distance of each particle from the centre of mass is fixed, at each time $\mathbf{s}_i$ must be rotated relative to its starting position and the first task is to compute its time derivative. It is convenient to think about the components of $\mathbf{s}_i$ relative to axes fixed in space, and also relative to axes fixed in the body because with regard to the latter the coordinates of $\mathbf{s}_i$ are **constant**. Thus

$$\mathbf{s}_i(t) = s_{ia}(t)\,\mathbf{e}_a^{(0)} = s_{ia}^{(0)}\,\mathbf{e}_a(t), \quad \mathbf{e}_a(t) = R_{ab}(t)\,\mathbf{e}_b^{(0)}, \quad RR^T = 1, \quad \det(R) = 1.$$

Then

$$\dot{\mathbf{e}}_a(t) = \dot{R}_{ab}(t)\,\mathbf{e}_b^{(0)} = \dot{R}_{ab}(t)R_{bc}^T(t)\,\mathbf{e}_c(t).$$

Note: $\dot{R}(t)R^T(t)$ is an antisymmetric matrix that can be conveniently represented by setting

$$\left(\dot{R}(t)R^T(t)\right)_{ac} = \epsilon_{acd}\omega_d(t),$$

also implying

$$\dot{\mathbf{e}}_a(t) = \epsilon_{acd}\omega_d(t)\mathbf{e}_c(t),$$

which then implies

$$\dot{\mathbf{s}}_i = (\omega(t)\times\mathbf{s}_i(t))_c\mathbf{e}_c(t) = \omega\times\mathbf{s}_i.$$

In this expression, the components of ω, $\mathbf{s}_i$ are with respect to axes fixed in the body; the vector ω is called the (instantaneous) angular velocity of the body at time t.

Armed with this result it is then possible to evaluate the kinetic energy and angular momentum of the rigid body. Thus

$$K_{\text{tot}} = \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{1}{2}\sum_{i=1}^{N} m_i|\dot{\mathbf{s}}_i|^2, \quad \sum_{i=1}^{N} m_i|\dot{\mathbf{s}}_i|^2 = \sum_{i=1}^{N} m_i\left(|\mathbf{s}_i|^2|\omega|^2 - (\mathbf{s}_i\cdot\omega)^2\right).$$

The last part of the expression (since ω does not depend on i) can be written in terms of components (with respect to the axes fixed in the body) as

$$\sum_{i=1}^{N} m_i\left(|\mathbf{s}_i|^2|\omega|^2 - (\mathbf{s}_i\cdot\omega)^2\right) = \omega_a I_{ab}\omega_b, \quad I_{ab} = \sum_{i=1}^{N} m_i\left(s_{ic}^{(0)}s_{ic}^{(0)}\delta_{ab} - s_{ia}^{(0)}\,s_{ib}^{(0)}\right).$$

Here, we have noted again that $\dot{\mathbf{s}}_i = \omega \times \mathbf{s}_i$, and made use of the identity

$$(\omega \times \mathbf{s}_i) \cdot (\omega \times \mathbf{s}_i) = \omega \cdot (\mathbf{s}_i \times (\omega \times \mathbf{s}_i) = \omega \cdot ((\mathbf{s}_i \cdot \mathbf{s}_i)\omega - (\mathbf{s}_i \cdot \omega)\mathbf{s}_i) = |\mathbf{s}_i|^2 \, |\omega|^2 - (\mathbf{s}_i \cdot \omega)^2.$$

The quantity I_{ab} is called the inertia tensor of the rigid body relative to its centre of mass; it is symmetric ($I_{ab} = I_{ba}$) and computed with respect to axes fixed in the body with origin located at the centre of mass. Since the inertia tensor is real and symmetric, it has real eigenvalues and its eigenvectors can be chosen to be an orthogonal set. Moreover, the eigenvalues are non-negative because the quadratic form $x_a I_{ab} x_b$ is never negative. The reason for this relies on the well known inequality $|\mathbf{s}_i \cdot \mathbf{x}| \leq |\mathbf{s}_i| \, |\mathbf{x}|$ for any real vector $\mathbf{x}$. It is often useful to choose the eigenvectors as basis so that the inertia tensor is diagonal (in which case, the diagonal entries are called the principal moments of inertia and are clearly positive).

Note: if we ever needed to compute the inertia tensor with respect to axes fixed in the rigid body but displaced from the centre of mass by $\mathbf{t}$ then we just replace $\mathbf{s}_i$ by $\mathbf{s}'_i = \mathbf{s}_i - \mathbf{t}$, then use the facts that $\sum_i \mathbf{s}_i = 0, \ \sum_i m_i = M$ to deduce that

$$I'_{ab} = I_{ab} + M(|\mathbf{t}|^2 \delta_{ab} - t_a t_b).$$

In summary,

$$K_{\text{tot}} = \frac{1}{2} M |\dot{\mathbf{R}}|^2 + \frac{1}{2} \omega_a I_{ab} \omega_b.$$

Also, the total angular momentum can be rewritten in terms of the inertia tensor

$$\mathbf{J}_{\text{tot}} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_{i=1}^{N} m_i \mathbf{s}_i \times \dot{\mathbf{s}}_i, \quad \sum_{i=1}^{N} m_i \mathbf{s}_i \times \dot{\mathbf{s}}_i = \sum_{i=1}^{N} m_i \mathbf{s}_i \times (\omega \times \mathbf{s}_i).$$

The second expression can again be written using the inertia tensor to find:

$$\mathbf{J}_{\text{tot}} = M\mathbf{R} \times \dot{\mathbf{R}} + \mathbf{e}_a I_{ab} \omega_b \equiv M\mathbf{R} \times \dot{\mathbf{R}} + \mathbf{L},$$

where $\mathbf{L}$ is the contribution to the angular momentum that involves the inertia tensor.

Note: the second term is not generally proportional to the angular velocity, a fact which gives rise to the wonderful effects exhibited by rigid bodies (for example during the motion of a spinning top).

[Comment: If we decide to write

$$\mathbf{L} = \mathbf{e}_a I_{ab} \omega_b = \mathbf{I} \cdot \omega, \ \text{with } \mathbf{I} = \mathbf{e}_a I_{ab} \mathbf{e}_b,$$

(this is called a 'dyadic' notation - some books use this, which is why it is mentioned here - but we will not use it apart from this instance), then the total kinetic energy takes the simpler-looking form

$$K_{\text{tot}} = \frac{1}{2} M |\dot{\mathbf{R}}|^2 + \frac{1}{2} \omega \cdot \mathbf{I} \cdot \omega.]$$

Note also that we can compute the time derivative of $\mathbf{L}$ to find

$$\dot{\mathbf{L}} = I_{ab} \dot{\omega}_b + \dot{\mathbf{e}}_a I_{ab} \omega_b = \mathbf{e}_a I_{ab} \dot{\omega}_b + \mathbf{e}_c \epsilon_{acd} \omega_d I_{ab} \omega_b = \mathbf{e}_c \left(I_{cb} \dot{\omega}_b + \epsilon_{acd} \omega_d I_{ab} \omega_b \right).$$

Note, every time there is a repeated index there is an implied sum, as usual. If there are no external forces acting, the angular momentum of the centre of mass is separately conserved, and $\dot{\mathbf{L}} = 0$. If we decide to use axes fixed in the body for which the inertia tensor is diagonal, we have the following three differential equations for the components of ω:

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = 0, \ I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) = 0, \ I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) = 0.$$

These are known as Euler's equations. (Note: in these expressions we refer to the diagonal entries of the diagonalised inertia tensor by using $I_1 = I_{11}$, $I_2 = I_{22}$, $I_3 = I_{33}$ – these are the principal moments of inertia).

Comment: Since the inertia tensor is symmetric ($I_{ab} = I_{ba}$), we know its eigenvalues are real, its eigenvectors are orthogonal to each other and that it can be diagonalised using an orthogonal transformation (which rotates the basis vectors fixed relative to the rigid body to a basis of normalised eigenvectors also fixed relative to the rigid body - called the principal axes).

Comment: If the rigid body is subject only to near-Earth gravity (for example, it is tumbling through the air and air resistance is ignored) then each constituent particle experiences an external force

$$\mathbf{F}^{(ext)} = -m_i g\mathbf{k},$$

where $\mathbf{k}$ is a fixed vertical unit vector. Then, as we showed before, the equation of motion of the centre of mass is

$$M\ddot{\mathbf{R}} = -gM\mathbf{k},$$

and we notice by a direct calculation, that the time derivative of the total angular momentum is also given entirely by the contribution from the centre of mass, namely

$$\dot{\mathbf{J}} = -Mg\mathbf{R} \times \mathbf{k},$$

and so $\dot{\mathbf{L}} = 0$ also in that case.

Remarks

Using Euler's equations, we can check immediately a couple of things.

(a) If we multiply the first of Euler's equations by ω_1, the second by ω_2, the third by ω_3, and then add the equations together, we find the nonlinear terms cancel out to leave

$$I_1\,\dot{\omega}_1\,\omega_1 + I_2\,\dot{\omega}_2\,\omega_2 + I_3\,\dot{\omega}_3\,\omega_3 = 0,$$

which implies

$$I_1\,\omega_1^2 + I_2\omega_2^2 + I_3\,\omega_3^2 = \text{constant}.$$

(b) If we multiply the first of Euler's equations by $I_1\,\omega_1$, the second by $I_2\,\omega_2$, the third by $I_3\,\omega_3$, and then add the equations together, we find the nonlinear terms cancel out again to leave

$$I_1^2\,\dot{\omega}_1\,\omega_1 + I_2^2\,\dot{\omega}_2\,\omega_2 + I_3^2\,\dot{\omega}_3\,\omega_3 = 0,$$

which implies

$$I_1^2\,\omega_1^2 + I_2^2\omega_2^2 + I_3^2\,\omega_3^2 = \text{constant}.$$

There are also some fairly straightforward consequences in special situations.

Example

For example, you can check there is a solution to Euler's equations where one of the components of the angular velocity is non-zero and constant and the other two components are zero. To be specific, let's suppose

$$\omega_3 \neq 0, \quad \dot{\omega}_3 = 0, \quad \omega_1 = 0 = \omega_2,$$

which is clearly a solution to Euler's equations. The question to ask is: is this motion stable to small perturbations?

To explore this, we set

$$\omega_1 = \eta_1\,e^{pt},\ \omega_2 = \eta_2\,e^{pt},$$

where η_1, η_2 are small enough that all second order quantities can be ignored. Then, the third Euler equation remains the same to first order but the other two equations to first order become:

$$(I_1 p\, \eta_1 + (I_3 - I_2)\omega_3\, \eta_2)\, e^{pt} = 0, \qquad (I_2 p\, \eta_2 + (I_1 - I_3)\omega_3 \eta_1)\, e^{pt} = 0,$$

and these are compatible with a non-zero solution provided

$$\det \begin{pmatrix} pI_1 & \omega_3(I_3 - I_2) \\ \omega_3(I_1 - I_3) & pI_2 \end{pmatrix} = 0.$$

This is an equation for p giving:

$$p^2 = \frac{\omega_3^2}{I_1 I_2}(I_3 - I_2)(I_1 - I_3).$$

Now, for the solution to be stable p should be pure imaginary so that e^{pt} is periodic. Otherwise, if p is real, there will be an exponentially growing solution, indicating the original motion is unstable. Thus, for example, we deduce that if

$$I_1 > I_3 > I_2 \quad \text{or} \quad I_2 > I_3 > I_1$$

the steady motion is unstable. On the other hand, if

$$I_3 > I_2 > I_1, \quad \text{or} \quad I_1 > I_2 > I_3$$

p will be imaginary and the perturbations are stable (bounded periodic). These interesting facts can be demonstrated easily using an asymmetrical object, such as a tennis racket, matchbox, or packet of cornflakes (try it!).

Another special case, occurs when two of the principal moments of inertia are equal. Then, taking for example $I_1 = I_2$, the third of Euler's equations requires $I_3\dot{\omega}_3 = 0$, meaning that ω_3 must be constant. The other two equations then become linear differential equations for ω_1 and ω_2.

We shall return to the special case of a spinning top with one point fixed in section III.

(c) A couple of sample inertia tensors

(i) Constant density sphere - mass density ρ, radius R.

In this case, the centre of mass is at the centre of the sphere and the definition of the inertia tensor provided above becomes an integral rather than a discrete sum:

$$I_{ab} = \int_{\text{sphere}} \rho\, (\delta_{ab} x_c x_c - x_a x_b)\, dV,$$

where, for the purposes of computing the integral, we have put $\mathbf{s}_a^{(0)} = x_a$. Because of the symmetry of the sphere, all the terms where $a \neq b$ are the same and zero, while all the terms with $a = b$ are also the same but non-zero. We can calculate one of them, say $a = b = 3$, using spherical polar coordinates ($x_1 = r\sin\theta\cos\phi,\ x_2 = r\sin\theta\sin\phi,\ x_3 = r\cos\theta$):

$$I_{33} = \rho \int_{\text{sphere}} (x_1^2 + x_2^2) dV = \rho \int_0^R r^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, (r^2 \sin^2\theta) = \frac{8\pi\rho R^5}{15}.$$

Also, if the total mass of the sphere is M then

$$M = \frac{4\pi R^3 \rho}{3},$$

and hence, in terms of M and R, the moment of inertia tensor is:

$$I_{ab} = \frac{2MR^2}{5}\,\delta_{ab}.$$

Remark: Earth is approximately a sphere but it is not uniform density; so for Earth (and the Sun and other planets/moons) this is an approximate formula.

(ii) Constant density rectangular slab of material - mass density ρ, width $2l$, breadth $2m$, depth $2n$.

In this case, choose coordinates with origin at the centre, with axes parallel to the sides of the rectangle so that

$$-l \le x_1 \le l, \quad -m \le x_2 \le n, \quad -n \le x_3 \le n.$$

Note

$$M = 8\rho lmn.$$

Again, the terms in the inertia tensor I_{ab} with $a \neq b$ are zero. For example,

$$I_{12} = \rho \int_{-l}^{l} dx_1 \int_{-m}^{m} dx_2 \int_{-n}^{n} dx_3\,(-x_1 x_2) = -\rho \left[\frac{x_1^2}{2}\right]_{-l}^{l} \left[\frac{x_2^2}{2}\right]_{-m}^{n} [x_3]_{-n}^{n} = 0.$$

On the other hand, the diagonal terms are not zero, for example

$$\begin{aligned} I_1 \equiv I_{11} =& \rho \int_{-l}^{l} dx_1 \int_{-m}^{m} dx_2 \int_{-n}^{n} dx_3\,(x_2^2 + x_3^2) \\ =& \ \rho\,[x_1]_{-l}^{l} \left(\left[\frac{x_2^3}{3}\right]_{-m}^{m} 2n + 2m \left[\frac{x_3^3}{3}\right]_{-n}^{n} \right) = \frac{M}{3}(m^2 + n^2). \end{aligned}$$

Similar calculations give

$$I_2 = \frac{M}{3}(l^2 + n^2), \quad I_3 = \frac{M}{3}(l^2 + m^2).$$

Unlike the case of the sphere, if the slab is not a cube, the three principal moments of inertia will not be the same. However, the choice of coordinates we made to do the explicit computations does correspond to a set of principal axes because the moment of inertia tensor is diagonal.

III Lagrangian Dynamics

In this section we will explore a different (and quite surprising) way of formulating Newtonian mechanics. It can be very useful for formulating certain sorts of problems where the forces providing constraints do not contribute to the energy (meaning they do no work). The simplest example of this is a simple pendulum with a point mass (m) attached to a light rod whose other end is fixed. The particle is constrained to move on a circle but the force keeping it attached to the rod is always acting in a direction orthogonal to the particle's velocity (ie it does no work). Can the problem be formulated in such a way that this force of constraint never enters? (If you really wanted to know what this force was in detail - for example you want to know what stress the rod must be able to cope with - then you would calculate it from the description of the particle's motion). The formalism will also allow us to tackle some other types of problem, including a symmetric top with one point fixed.

The Lagrangian formulation is also important for quantum mechanics, especially the path integral methods estblished by Feynman, and the formulation of quantum field theory - for which the main application is the 'Standard Model' of elementary particles and their interactions.

3.1 The calculus of variations

It is useful first to introduce some new ideas and techniques.

(a) The geodesic problem: this can be formulated in several ways but we can start by writing an expression for the distance between two points (x_1, y_1), (x_2, y_2) along a curve $y(x)$ that lies entirely within the (x, y) plane. If we write $D[y]$ as the distance, we have

$$D[y] = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx, \quad y' = \frac{dy}{dx}.$$

As the choice of path is varied, the length of the path varies, and the basic question to ask is: for what choice of $y(x)$ is the length of the path minimised? We know the answer should be that the length of the path is least when $y(x)$ is a linear function of x (ie the two points are joined by a straight line). However, we could ask the same question for paths between fixed points on a different non-flat surface, such as a spherical surface, or an ellipsoidal surface, a cylinder, a cone, etc.

These are all examples of the following general set up.

Define a functional $F[y]$ by

$$F[y] = \int_{x_1}^{x_2} f(x, y, y')\, dx,$$

where, as indicated, f depends on x, y, y'. Then, ask for what functions y this is a minimum, or, more generally, stationary.

3.2 The first step - Euler-Lagrange equation

The treatment given here is intended to demonstrate the ideas but will not be rigorous (there is no time for that). First, we need to explore the functional $F[y]$ as the curve $y(x)$ is changed slightly to $y(x) + \delta y(x)$, requiring $\delta y(x_1) = 0 = \delta y(x_2)$ (meaning the variation is zero at the end points of the integral). We also assume the variation is smooth (meaning it has at least well-defined first and second derivatives on the interval $[x_1, x_2]$).

Note first:

$$\begin{aligned} f(x, y + \delta y, y' + \delta y') &= f(x, y, y') + \delta y\, \frac{\partial f}{\partial y} + \delta y'\, \frac{\partial f}{\partial y'} + O(\delta y^2, \delta y \delta y', \delta y'^2) \\ &= f(x, y, y') + \delta y \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) + \frac{d}{dx} \left[\delta y \frac{\partial f}{\partial y'} \right] + \dots, \end{aligned}$$

where the dots in the second line indicate the second order terms in the line above.

Integrating this expression, we write

$$
\begin{aligned}
F[y+\delta y]-F[y] &= \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}\right)\delta y+\frac{d}{dx}\left(\delta y\frac{\partial f}{\partial y'}\right)+\ldots\right]dx \\
&= \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}\right)\delta y+\ldots\right]dx+\left[\delta y\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} \\
&= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}\right)\delta y\,dx+\ldots\,,
\end{aligned}
$$

where we have made use of the facts $\delta y(x_1)=0=\delta y(x_2)$ to remove the last term in the second line above. As previously, the dots represent higher order terms.

As in standard one-variable calculus a stationary path will be defined as one for which the first variation vanishes for all small changes of path. Since δy is smooth and arbitrary subject to the constraints at the endpoints, we deduce that the path for which the integral is stationary should satisfy the equation:

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}=0.$$

This is called the Euler-Lagrange equation for the path y.

Example

For the geodesic problem on a plane we have

$$f(x,y,y')=\sqrt{1+y'^2},$$

which has no explicit dependence on x and y. Hence

$$\frac{\partial f}{\partial x}=0,\quad \frac{\partial f}{\partial y}=0,$$

so the Euler-Lagrange equation reduces to

$$\frac{d}{dx}\frac{\partial f}{\partial y'}=0,$$

which implies that

$$\frac{\partial f}{\partial y'}=\frac{y'}{\sqrt{1+y'^2}}=\text{constant},$$

and therefore y' is a constant, meaning $y(x)$ is a linear function of x. Inserting the endpoints with $y(x_1)=y_1,\ y(x_2)=y_2$, the appropriate solution to the problem is

$$y(x)=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)+y_1.$$

To check this is actually a minimum would require us to have a look at the next order in the expression for $F[y+\delta y]-F[y]$. It is not difficult to check (an exercise for you) that including the next term gives:

$$
\begin{aligned}
F[y+\delta y]-F[y] = &\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}\right)\delta y\,dx \\
&+\int_{x_1}^{x_2}\left[\frac{1}{2}\delta y^2\left(\frac{\partial^2 f}{\partial y^2}-\frac{d}{dx}\frac{\partial^2 f}{\partial y\partial y'}\right)+\frac{1}{2}\delta y'^2\,\frac{\partial^2 f}{\partial y'^2}\right]dx+\ldots
\end{aligned}
$$

Then, provided both terms in the quadratic form under the second integral are positive the stationary path will be a minimum. It is clearly the case for our example since the first term is zero, and evaluated at the solution

$$\frac{\partial^2 f}{\partial y'^2} = \left(1 + y'^2\right)^{-3/2} > 0.$$

3.3 Important comments

(a) If $f(x, y, y')$ is independent of y then, as we noted above, the Euler-Lagrange equation implies that for the stationary path

$$\frac{\partial f}{\partial y'} = \text{constant}.$$

(b) If $f(x, y, y')$ does not depend explicitly on x, meaning

$$\frac{\partial f}{\partial x} = 0,$$

then the Euler-Lagrange equation can be used to find another expression that must be constant. First consider the full x derivative of f (taking into account that y, y' also depend on x). Thus:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} = y'\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) + \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right).$$

Note, the last step uses the Euler-Lagrange equation. Finally, rearranging we find

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'} - f\right) = 0,$$

meaning that

$$y'\frac{\partial f}{\partial y'} - f = \text{constant}.$$

This is an important observation that will be used in future.

(c) The functional could be a functional of several independent variables. In other words, a generalisation of the above ideas can be applied to

$$F[y_1, y_2, y_3, ..., y_n] = \int_{x_1}^{x_2} f(x, y_1, y_1', y_2, y_2', \ldots, y_n, y_n')\, dx.$$

Then, the stationary value of the functional will require a collection of Euler-Lagrange equations:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0, \qquad i = 1, 2, \ldots, n.$$

If f does not in fact depend on one or more of the y_i, then for each value of i for which this is the case the corresponding Euler-Lagrange equation implies

$$\frac{\partial f}{\partial y_i} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial y_i'} = \text{constant}.$$

Also, if f does not depend explicitly on x then the Euler-Lagrange equations imply:

$$\sum_{i=1}^{n}\left(y_i'\frac{\partial f}{\partial y_i'}\right) - f = \text{constant}.$$

As we shall see, in a different context these comments will become very important.

(d) The functional could be defined by a higher dimensional integral (for example if we were seeking a minimal area surface in 3d with a specified (not necessarily planar) boundary).

For example,

$$F[u] = \int_D f(x,y,u,u_x,u_y)dxdy, \; u_x \equiv \frac{\partial u}{\partial x}, \quad u_y \equiv \frac{\partial u}{\partial y},$$

where D is a closed finite, simply connected region in the (x,y) plane, with u being prescribed on the boundary of D. Then the appropriate Euler-Lagrange equation is:

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial u_y} = 0.$$

Example: the Euler-Lagrange equation for the minimum of the functional

$$F[u] = \int_D f(x,y,u,u_x,u_y)dxdy \equiv \int_D (u_x^2 + u_y^2)\, dxdy$$

is the 2d Laplace equation:

$$u_{xx} + u_{yy} = 0.$$

It turns out that many interesting equations and problems can be converted to the problem of finding a minimum (or at least a stationary point) of a suitably constructed functional.

3.4 The principle of least action

It is a remarkable fact that the equations of motion for many dynamical systems can be derived by defining a functional of the dynamical variables called the 'action' and then demanding that the action is stationary. In other words, suppose there is a set of variables $q_i(t), \; i = 1, \ldots, N$ describing the motion of a particle, or system of particles (these could be cartesian coordinates, or angles - generically, we call them 'generalised coordinates'), and a set of associated generalised velocities $\dot{q}_i(t), \; i = 1, \ldots, N$. Thus we set

$$A[q_1, \ldots, q_N] = \int_{t_1}^{t_2} \mathcal{L}(t, q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots, q_N, \dot{q}_N)\, dt,$$

where the function $\mathcal{L}(t, q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots, q_N, \dot{q}_N)$ of the generalised coordinates and velocities is called the 'Lagrangian', and demand that $A[q_1, \ldots, q_N]$ is minimum, or at least stationary. In other words, the task is to choose the Lagrangian in such a way that the minimum of the action corresponds to the Newtonian equations of motion for the system expressed in terms of the generalised coordinates and their derivatives. Thus, using the ideas in the previous section, there will be a set of Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0, \quad i = 1, 2, \ldots, N,$$

equivalent to Newton's equations.

Definition It is useful to define a generalised momentum p_i associated with the generalised coordinate q_i by setting

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

Note, neither q_i or p_i need to be specific components of a vector; they are 'generalised' coordinates and momenta.

Remarks

As in the previous discussion in section(3.3), there are some special situations.

(a) If the Lagrangian does not depend explicitly on a generalised coordinate, q_k, say, then

$$\frac{\partial \mathcal{L}}{\partial q_k} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \text{constant}.$$

If this happens, then the coordinate q_k is called 'ignorable' and the associated generalised momentum p_k is conserved. In some circumstances these can provide useful constants of the motion.

(b) If the Lagrangian function does not depend explicitly on time then

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{dt} + \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial q_i} + \sum_{i=1}^{N} \ddot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \sum_{i=1}^{N} \dot{q}_i \left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right] + \sum_{i=1}^{N} \frac{d}{dt}\left(\dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

and using the Euler-Lagrange equations removes the first term on the right hand side. So, we find

$$\frac{d}{dt}\left(\mathcal{L} - \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0.$$

Finally, this implies

$$\sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} = \text{constant}.$$

We will discover that this is a significant fact.

Note: the conserved quantity

$$J(q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots, q_N, \dot{q}_N) \equiv \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}$$

is called the Jacobi function and is often equal to the total conserved energy of the system.

3.5 Some examples

Example 1

The simplest case is the one-dimensional example of particle (mass m) moving along the x-axis under the influence of a force that is derived from a potential $V(x)$. This means Newton's equation in terms of the particle's position at time t is:

$$m\ddot{x} = -\frac{dV}{dx}.$$

In this situation, a suitable Lagrangian is just

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x).$$

To check, we simply calculate using the Euler-Lagrange equation, noting that

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial \mathcal{L}}{\partial x} = -\frac{dV}{dx}$$

to find

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = -\frac{dV}{dx} - \frac{d}{dt}(m\dot{x}) = -\frac{dV}{dx} - m\ddot{x} = 0 \;\Rightarrow\; m\ddot{x} = -\frac{dV}{dx}.$$

The latter is precisely Newton's equation in this case.

Note, the Lagrangian does depend on $x(t)$, $\dot{x}(t)$ but it has no explicit dependence on time, which means that comment (b) above is relevant. In other words,

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \;\Rightarrow\; \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \mathcal{L} = m\dot{x}^2 - \mathcal{L} = m\dot{x}^2 - \left(\frac{1}{2}m\dot{x}^2 - V\right) = \frac{1}{2}m\dot{x}^2 + V = \text{constant} = E.$$

This is impressive because it is telling us that the quantity that is constant because $\mathcal{L}$ has no explicit dependence on time is precisely the conserved energy! This is an important observation.

Example 2

The next case is a simple pendulum: a particle of mass m is attached to one end of a light, inflexible rod of fixed length l the other end of which is fixed (at the origin). The particle is free to move under the influence of gravity in a vertical plane (say, the (x, z) plane). Although the particle is free to move in 2d it is in fact fixed to the end of an inflexible rod and therefor constrained to move on a circle of radius l. This means the problem only has one 'degree of freedom'. If we let $\mathbf{i}$, $\mathbf{k}$ be the unit vectors in the x and z axis directions respectively, and if we define θ be the angle between the rod and the unit vector $-\mathbf{k}$ then we can say, that the position, velocity and acceleration of the particle are given by:

$$\mathbf{r}(t) = l\sin\theta\,\mathbf{i} - l\cos\theta\,\mathbf{k},\ \dot{\mathbf{r}} = l\dot{\theta}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{k}),\ \ddot{\mathbf{r}} = l\ddot{\theta}(\cos\theta\,\mathbf{i} + \sin\theta)\mathbf{k} - l\dot{\theta}^2(\sin\theta\,\mathbf{i} - \cos\theta\mathbf{k})$$

and its equation of motion is:

$$m\ddot{\mathbf{r}} = -mg\mathbf{k} + \mathbf{T},$$

where $\mathbf{T}$ is the force the rod supplies to keep the particle attached to the free end. We cannot say in advance what this force is only that its function is to keep the particle on its circular path. In other words, $\mathbf{T}$ serves to supply whatever force is required to constrain the path. In effect, if we could calculate $m\ddot{\mathbf{r}}$ we would be able to calculate $\mathbf{T}$, meaning it is part of the solution!

One way of approaching this issue is simply to take the scalar product of the equation of motion with $\dot{\mathbf{r}}$, since the velocity of the particle is tangential to the circle on which the particle moves, and therefore orthogonal to $\mathbf{T}$. In other words, using the expressions for the velocity and acceleration we have

$$\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -mg\mathbf{k} \cdot \dot{\mathbf{r}} \ \Rightarrow\ ml^2\dot{\theta}\,\ddot{\theta} = -mgl\dot{\theta}\sin\theta \ \Rightarrow\ \ddot{\theta} = -\frac{g}{l}\sin\theta \quad \text{or} \quad \dot{\theta} = 0.$$

The final equation is an equation for θ (though the possibility $\dot{\theta} = 0$ is not very interesting since the particle is simply hanging at rest). Once the equation is solved for θ, the constraint force can be calculated if needed.

A Lagrangian for this system involves only the kinetic and potential energies of the particle. The kinetic energy is given by

$$K = \frac{1}{2}m|\dot{\mathbf{r}}|^2 = \frac{1}{2}ml^2\dot{\theta}^2,$$

and the potential energy is given by

$$V(\theta) = mgz = -mgl\cos\theta,$$

so we can write

$$\mathcal{L}(\theta, \dot{\theta}) = K - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta.$$

Using this Lagrangian, the Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \ \Rightarrow -mgl\sin\theta - \frac{d}{dt}(ml^2\dot{\theta}) = 0 \ \Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta.$$

This formulation allows a calculation of θ directly. Moreover, the expression for the Lagrangian has no specific dependence on t, which means, according to the arguments given above,

$$K + V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = \text{constant} = E.$$

Example 3

Consider a particle (mass m) moving in a plane subject to a force which is directed towards the origin and derived from a potential $V(r)$ that depends only on the particle's distance

r from the origin. The appropriate coordinates for this situation are plane polars $(r,\ \theta)$ and in those coordinates the particle has kinetic energy given by

$$K = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2),$$

Hence the Lagrangian will be

$$\mathcal{L}(r, \dot{r}, \theta, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r).$$

Notice, that the Lagrangian does not in fact depend on the coordinate θ and so we deduce

$$p_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant},$$

(as we found previously in the context of gravity). Also, the Euler-Lagrange equation for r is

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = mr\dot{\theta}^2 - \frac{dV}{dr} - m\ddot{r} = 0 \quad \Rightarrow \quad m\ddot{r} - mr\dot{\theta}^2 = -\frac{dV}{dr}.$$

Hence, if we put $r^2\dot{\theta} = h$, as previously, the equation for r is

$$m\ddot{r} = \frac{mh^2}{r^3} - \frac{dV}{dr}.$$

Also, the Lagrangian does not depend on t explicitly, which means

$$\dot{r}\frac{\partial \mathcal{L}}{\partial \dot{r}} + \dot{\theta}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L} = m\dot{r}^2 + mr^2\dot{\theta}^2 - \left(\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\right) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E,$$

where E is constant (the total conserved energy). Hence

$$E = \frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2}\right) + V(r).$$

3.6 Rigid bodies revisited - this time a symmetrical top with one point fixed along the symmetry axis

First, it is necessary to set up an efficient description of the time-dependent rotation matrix that relates the axes fixed in the rotating object to the fixed axes whose origin is the fixed point of the object (which does not have to be at the centre of mass).

(a) Euler angles

Suppose the fixed axes are set up so that the vertical axis is in the direction of the unit vector $\mathbf{e}_3^{(0)}$ then the other two axes are in a plane perpendicular to the vertical direction.

(i) Let R_ϕ represent an anticlockwise rotation around the axis $\mathbf{e}_3^{(0)}$ through an angle ϕ.

(ii) Under rotation (i), the axis in the direction $\mathbf{e}_1^{(0)}$ is rotated to a new position, say $\mathbf{e}_1'$ and the next step is to let R_θ represent an anticlockwise rotation around the axis $\mathbf{e}_1'$ through an angle θ.

(iii) Under rotation (ii), the axis in the direction $\mathbf{e}_3^{(0)}$ is rotated to a new position, say $\mathbf{e}_3'$ and it is useful to choose this to be the symmetry axis of the top. The next step is to let R_ψ represent an anticlockwise rotation around the axis $\mathbf{e}_3'$ through an angle ψ.

Thus, combining these, the change from the fixed axes to the principal axes of the spinning top is represented by

$$R = R_\psi\, R_\theta\, R_\phi.$$

The angles ϕ, θ, ψ are called Euler angles. For the previous discussion of R see section 2.9.

Next, we identify the three components of the angular velocity ω by calculating $\dot{R}\,R^T$, as before. Thus

$$\dot{R}\,R^T = \left(\dot{R}_\psi R_\theta R_\phi + R_\psi \dot{R}_\theta R_\phi + R_\psi R_\theta \dot{R}_\phi\right) R_\phi^T R_\theta^T R_\psi^T.$$

Thus, simplifying a little,

$$\dot{R}\,R^T = \dot{R}_\psi\,R_\psi^T + R_\psi\,\dot{R}_\theta\,R_\theta^T R_\psi^T + R_\psi R_\theta \dot{R}_\phi\,R_\phi^T R_\theta^T R_\psi^T.$$

To finish the computation requires explicit expressions for the rotation matrices. These are:

$$R_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},\ R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix},\ R_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, it is straightforward to check that

$$\dot{R}_\psi\,R_\psi^T = \begin{pmatrix} 0 & \dot\psi & 0 \\ -\dot\psi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \dot{R}_\theta\,R_\theta^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dot\theta \\ 0 & -\dot\theta & 0 \end{pmatrix},\ \dot{R}_\phi\,R_\phi^T = \begin{pmatrix} 0 & \dot\phi & 0 \\ -\dot\phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and then a direct computation of the expression for $\dot{R}\,R^T$ given above gives:

$$\dot{R}\,R^T = \begin{pmatrix} 0 & \dot\psi + \dot\phi\cos\theta & \dot\theta\sin\psi - \dot\phi\cos\psi\sin\theta \\ -\dot\psi - \dot\phi\cos\theta & 0 & \dot\theta\cos\psi + \dot\phi\sin\psi\sin\theta \\ -\dot\theta\sin\psi + \dot\phi\cos\psi\sin\theta & -\dot\theta\cos\psi - \dot\phi\sin\psi\sin\theta & 0 \end{pmatrix}.$$

Finally, we can identify the components of the angular velocity by using the results of section 2.9 since

$$\dot{R}\,R^T = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

to get

$$\omega_1 = \dot\theta\cos\psi + \dot\phi\sin\psi\sin\theta,\ \omega_2 = -\dot\theta\sin\psi + \dot\phi\cos\psi\sin\theta,\ \omega_3 = \dot\psi + \dot\phi\cos\theta.$$

(b) We can now use these expressions to calculate the kinetic energy of the symmetric top with principal moments of inertia $I_{11} = I_{22} \equiv A,\ I_{33} \equiv C$ to get

$$K = \frac{1}{2}\left(I_{11}(\omega_1^2+\omega_2^2) + I_{33}\omega_3^2\right) \equiv \frac{1}{2}\left(A(\dot\theta^2 + \dot\phi^2\sin^2\theta) + C(\dot\psi + \dot\phi\cos\theta)^2\right).$$

For a typical symmetrical top, the centre of mass is a distance l from the fixed point and situated along the symmetry axis. This means that if the mass of the top is $M$the potential energy is

$$V = Mgl\cos\theta.$$

Putting all this together, the Lagrangian for the symmetrical top is:

$$\mathcal{L} = K - V = \frac{1}{2}\left(A(\dot\theta^2 + \dot\phi^2\sin^2\theta) + C(\dot\psi + \dot\phi\cos\theta)^2\right) - Mgl\cos\theta.$$

This is the starting point for a discussion here and for several questions on the assignment sheet.

Remarks

Notice that the Lagrangian for the spinning top does not depend explicitly on t, ϕ, ψ.

(1) Independence of ψ:

$$\frac{\partial\mathcal{L}}{\partial\psi} = 0 \ \Rightarrow\ p_\psi = \frac{\partial\mathcal{L}}{\partial\dot\psi} = C(\dot\psi + \dot\phi\cos\theta) = \text{constant} \equiv a.$$

(2) Independence of ϕ:

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0 \ \Rightarrow\ p_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = A\dot\phi\sin^2\theta + C(\dot\psi + \dot\phi\cos\theta)\cos\theta = \text{constant} \equiv b.$$

and note, using (1), this implies

$$A\dot{\phi}\sin^2\theta + a\cos\theta = b,$$

and hence,

$$\dot{\phi} = \frac{b - a\cos\theta}{A\sin^2\theta}, \quad \sin\theta \neq 0.$$

Also, using (1) again:

$$\dot{\psi} = \frac{a}{C} - \dot{\phi}\cos\theta = \frac{a}{C} - \left(\frac{b - a\cos\theta}{A\sin^2\theta}\right)\cos\theta.$$

This means the initial conditions will allow a calculation of a, b and then $\dot{\phi}, \dot{\psi}$ will be determined fully once the time evolution of θ is known.

(3) Independence of t:

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \ \Rightarrow \ \dot{\phi}\, p_\phi + \dot{\psi}\, p_\psi + \dot{\theta}\, p_\theta - \mathcal{L} = \text{constant} \equiv E,$$

where E is the total energy of the spinning top. Evaluating the left hand side (though not yet taking account of the constraints) gives

$$E = \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}A\dot{\phi}^2\sin^2\theta + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 + Mgl\cos\theta.$$

Notice that the third term in this expression is constant from (1) above so

$$E = \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}A\dot{\phi}^2\sin^2\theta + \frac{a^2}{2C} + Mgl\cos\theta,$$

and the second term can be replaced using the result from (2) above to get

$$E - \frac{a^2}{2C} \equiv \hat{E} = \frac{1}{2}A\dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{2A\sin^2\theta} + Mgl\cos\theta.$$

Rearranging, we have

$$\dot{\theta}^2 = \frac{2\hat{E}}{A} - \frac{(b - a\cos\theta)^2}{A^2\sin^2\theta} - \frac{2Mgl}{A}\cos\theta.$$

As mentioned above, for given initial conditions, the conserved quantities $\hat{E}, a, b$ can be determined and we are left with a differential equation for θ. In practice, the latter is not easy to solve completely but there are some special situations where a solution can be found and others where a qualitative discussion is possible and gives insight into the motion.

It is useful to set $u = \cos\theta$, so that $\dot{u} = -\dot{\theta}\sin\theta$, because then the differential equation for θ becomes a slightly simpler looking differential equation for u

$$\dot{u}^2 = \frac{2\hat{E}}{A}(1 - u^2) - \frac{1}{A^2}(b - au)^2 - \frac{2Mgl}{A}u(1 - u^2) = \frac{2}{A}(\hat{E} - Mglu)(1 - u^2) - \frac{1}{A^2}(b - au)^2.$$

Since the left hand side of the equation is always positive, u is constrained to lie within the intervals for which the cubic polynomial on the right hand side is positive. Note, for large $u > 0$ the polynomial is positive, while for large $u < 0$ the polynomial is negative. This means the polynomial (which typically has three distinct roots $u_1 < u_2 < u_3$) will be positive for $u_1 < u < u_2$ and for $u > u_3$. The other regions $u < u_1$, $u_2 < u < u_3$ are regions where $\dot{u}^2 < 0$ and are therefore not allowed. Also, in an actual motion, since $u = \cos\theta$, $-1 \leq u \leq 1$ and we note that if $u = \pm 1$ the cubic polynomial is actually negative, which means that $-1 < u_1$ and $u_2 < 1 < u_3$. Hence, typically, the possible motion of the top is constrained so that $u_1 \leq u \leq u_2$.

The location of the three roots is strongly dependent on the initial conditions and there are several examples on the assignment sheet for you to try.

Example 1

For this example, look at the Euler-Lagrange equation for θ:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = A\ddot{\theta} - A\dot{\phi}^2 \sin\theta\cos\theta + C\dot{\phi}\sin\theta(\dot{\psi} + \dot{\phi}\cos\theta) - Mgl\sin\theta.$$

Hence

$$A\ddot{\theta} = A\dot{\phi}^2 \sin\theta\cos\theta - a\dot{\phi}\sin\theta + Mgl\sin\theta.$$

Suppose $\theta = \pi/2$, then the equation of motion for θ gives

$$0 = 0 - a\dot{\phi} + Mgl \;\Rightarrow\; \dot{\phi} = \frac{Mgl}{a},$$

and, from (1) above

$$\dot{\psi} = \frac{a}{C}.$$

This is an interesting solution that can be demonstrated by anyone by using a bicycle wheel - spinning with angular speed $\dot{\psi}$ - balanced on one finger placed under one end of the wheel's axle, and you will find the wheel precesses (because $\dot{\phi} \neq 0$).

If θ is perturbed slightly by setting $\theta = \pi/2 + \epsilon$ then

$$\sin\theta = \sin(\pi/2 + \epsilon) = \cos\epsilon \approx 1 - \frac{\epsilon^2}{2}, \;\; \cos\theta = \cos(\pi/2 + \epsilon) = -\sin\epsilon \approx \epsilon.$$

Hence, to first order in ϵ the equation of motion for θ becomes

$$A\ddot{\epsilon} = -A\dot{\phi}^2\epsilon \;\Rightarrow\; \ddot{\epsilon} = -\dot{\phi}^2\epsilon.$$

Since $\dot{\phi}^2 = (Mgl/a)^2 > 0$, the solution for ϵ is periodic, meaning the motion is stable (as you will find when you experiment with a bicycle wheel) - if slightly disturbed θ will oscillate. This oscillation is called 'nutation'.

Example 2

This is essentially the same as question III 9 on the sheet.

The top is set in motion with the initial conditions

$$\theta(0) = \frac{\pi}{2}, \;\; \dot{\psi}(0) = n, \;\; \dot{\theta}(0) = \dot{\phi}(0) = \frac{Cn}{A}.$$

This means the two constants (a, b) and $\hat{E}$ can be calculated to get:

$$a = C(\dot{\psi} + \dot{\phi}\cos\theta)|_{t=0} = Cn, \;\; b = (A\dot{\phi}\sin^2\theta + a\cos\theta)|_{t=0} = Cn,$$

and

$$\hat{E} = \left(\frac{1}{2}A\dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{2A\sin^2\theta} + Mgl\cos\theta\right)\Bigg|_{t=0} = \frac{C^2n^2}{A}.$$

Hence, using the expression for $\dot{u}^2$,

$$\begin{aligned} \dot{u}^2 &= \frac{2C^2n^2}{A^2}(1 - u^2) - \frac{C^2n^2}{A^2}(1 - u)^2 - \frac{2Mgl}{A}u(1 - u^2) \\ &= \frac{2C^2n^2}{A^2}\left[1 - u^2 - \frac{1}{2}(1 - u)^2 - \frac{MglA}{C^2n^2}\,u(1 - u^2)\right], \end{aligned}$$

and setting the dimensionless combination, for example (as in qu III 9) to be:

$$\frac{MglA}{C^2n^2} = \frac{3}{2}$$

leads to

$$\dot{u}^2 = \frac{2C^2n^2}{A^2}\left[1 - u^2 - \frac{1}{2}(1-u)^2 - \frac{3}{2}u(1-u^2)\right] = \frac{C^2n^2}{A^2}(1-u)(1-3u^2).$$

Thus in this case, the three roots (in ascending order are)

$$u_1 = -\frac{1}{\sqrt{3}},\ u_2 = \frac{1}{\sqrt{3}},\ u = 1.$$

Since $\dot{u}^2 \geq 0$ and $-1 \leq u \leq 1$, the top must be constrained to move in the region for which

$$-\frac{1}{\sqrt{3}} < \cos\theta < \frac{1}{\sqrt{3}}.$$

Note, if u is one of the roots, then $\dot{u} = 0$ and u is constant. However, none of the three roots is compatible with the initial condition $u(0) = 0$.

The differential equation for u is not easily integrated though we can write

$$\int_0^{\cos\theta} \frac{du}{\sqrt{1-u}\sqrt{1-3u^2}} = \frac{Cn}{A}\,t.$$

The integral on the left hand side can be evaluated in terms of elliptic functions, if needed.

IV **Hamiltonian Dynamics**

The idea in this section is the following.

Given a Lagrangian describing the dynamics of a system reformulate the dynamics in terms of a set of generalised coordinates $q_1, q_2, \ldots, q_N$ and their corresponding momenta $p_1, p_2, \ldots, p_N$ where

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}, \quad i = 1, 2, \ldots, N.$$

To start, the latter definitions are used to find expressions for each of the generalised velocities $\dot{q}_i$ in terms of the set of generalised coordinates and momenta (we shall see a number of examples below). Thus, we write

$$\dot{q}_i = \dot{q}_i(q_1, q_2, \ldots, p_1, p_2, \ldots) \equiv \dot{q}_i(\underline{q}, \underline{p}),$$

where the underlined q, p in the last expression is simply a short hand notation for the collection of variables $q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N$.

Next, we define the Hamiltonian by setting

$$H(\underline{q}, \underline{p}) = \sum_{i=1}^{N} p_i \dot{q}_i(\underline{q}, \underline{p}) - \mathcal{L}(\underline{q}, \underline{\dot{q}}(\underline{q}, \underline{p})).$$

4.1 **Hamilton's equations**

Next, calculate the derivatives of H with respect to each of the q's and each of the p's.

(i) Thus:

$$\frac{\partial H}{\partial q_j} = \sum_{i=1}^{N} p_i \frac{\partial \dot{q}_i}{\partial q_j} - \frac{\partial \mathcal{L}}{\partial q_j} - \sum_{i=1}^{N} \frac{\partial \dot{q}_i}{\partial q_j} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = -\frac{\partial \mathcal{L}}{\partial q_j} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = -\dot{p}_j.$$

In the above, the first and third pieces in the second term cancel because of the definition of the momenta, and the final expression follows using the Euler-Lagrange equation.

(ii)

$$\frac{\partial H}{\partial p_j} = \dot{q}_j + \sum_{i=1}^{N} p_i \frac{\partial \dot{q}_i}{\partial p_j} - \sum_{i=1}^{N} \frac{\partial \dot{q}_i}{\partial p_j} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \dot{q}_j.$$

Again, the definitions of the generalised momenta allows to cancel the second and third terms in the second term.

To summarise, we have:

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j = 1, 2, \ldots, N.$$

These are known as Hamilton's equations. They represent $2N$ first order differential equations for the dynamical variables $\underline{q}, \underline{p}$. They are symmetrical looking and useful.

When a dynamical system is expressed in terms of the sets of variables $\underline{q}, \underline{p}$, the variables are often regarded as 2N coordinates for the 'phase space' of the system. Looked at that way, each possible motion of a system is a trajectory in phase space.

4.2 **Examples**

Example (1)

Consider a particle of mass m constrained to move along the $x-$axis subject to a force $F = -dV/dx$. The Lagrangian for this set up is

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x).$$

The momentum conjugate to x is defined by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \;\Rightarrow\; \dot{x} = \frac{p}{m}.$$

Then the Hamiltonian is defined by

$$H = p\,\dot{x} - \mathcal{L} = p\dot{x} - \frac{1}{2}\dot{x}^2 + V(x) = \frac{p^2}{2m} + V(x),$$

where the last step replaces $\dot{x}$ by $\frac{p}{m}$. Then, Hamilton's equations are:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m},\; \dot{p} = -\frac{\partial H}{\partial x} = -\frac{dV}{dx}.$$

In this example, you can easily recover Newton's equation by noting

$$\ddot{x} = \frac{\dot{p}}{m} = -\frac{1}{m}\frac{dV}{dx}.$$

Example (2)

Consider a particle moving on a plane subject to a central force provided by a potential $V(r)$ is described in plane polar coordinates using the Lagrnagian

$$\mathcal{L}(r, \dot{r}, \theta, \dot{\theta}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r).$$

Then the momenta conjugate to r and θ are given by

$$\begin{aligned} p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \;\Rightarrow\; \dot{r} = \frac{p_r}{m} \\ p_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \;\Rightarrow\; \dot{\theta} = \frac{p_\theta}{mr^2}. \end{aligned}$$

Using these expressions

$$H = p_r\dot{r} + p_\theta\dot{\theta} - \mathcal{L} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r).$$

Hence Hamilton's equations are:

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0, \quad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{dV}{dr}. \end{aligned}$$

As before, we note from the first of the second pair of equations that p_θ, the momentum conjugate to θ is constant.

Example (3)

Consider a simple pendulum - a particle of mass m suspended at the end of a light, inflexible rod of length l, whose other end is fixed at the origin - constrained to move in a vertical plane. As before, θ is the angle the rod makes with the downward vertical, meaning the position of the particle is described by

$$\mathbf{r} = l(\sin\theta\,\mathbf{e}_1 - \cos\theta\,\mathbf{e}_3),$$

where $\mathbf{e}_1$ and $\mathbf{e}_3$ are two orthonormal unit vectors defining the plane, and the Lagrangian is

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta.$$

Using the Lagrangian, we have

$$p_\theta = ml^2\dot\theta \;\Rightarrow\; \dot\theta = \frac{p_\theta}{ml^2},$$

and the Hamiltonian is given by:

$$H = p_\theta\dot\theta - \mathcal{L} = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta.$$

Using H, Hamilton's equations are

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}, \quad \dot p_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta.$$

Again, if we wish, we can recover the second order equation of motion

$$\ddot\theta = \frac{1}{ml^2}\dot p_\theta = -\frac{g}{l}\sin\theta.$$

If the swing of the pendulum is small meaning $\sin\theta \approx \theta,\ \cos\theta \approx 1-\theta^2/2$ - the Hamiltonian is approximated by

$$H \approx \frac{p_\theta^2}{2ml^2} + \frac{mgl\,\theta^2}{2} - mgl.$$

The constant term is not relevant to Hamilton's equations and the Hamiltonian for a simple pendulum swinging with small amplitude is:

$$H = \frac{p_\theta^2}{2ml^2} + \frac{mgl\,\theta^2}{2}.$$

This expression, a sum of squares, is typical for a simple harmonic oscillator.

4.3 Functions on phase space

Suppose $F(t, \underline{q}, \underline{p})$ is a real function of the phase space variables

$$\underline{q} \equiv q_1, q_2, \dots, q_N \quad \underline{p} \equiv p_1, p_2, \dots, p_N.$$

Then, a basic question to ask would be: is this function constant in time? To investigate this we start by noting:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^{N}\left(\dot q_i\frac{\partial F}{\partial q_i} + \dot p_i\frac{\partial F}{\partial p_i}\right),$$

and then use Hamilton's equations to replace $\dot q_i$ and $\dot p_i$ to find

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right).$$

The sum on the right hand side is a very special object called the Poisson Bracket of F and G and often denoted $\{F, G\}$. Thus, in this notation,

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}.$$

In particular, if F has no explicit time dependence then

$$\frac{dF}{dt} = \{F, H\}.$$

Remark if H has no explicit time dependence (as is the case for most of our examples),

$$\frac{dH}{dt} = \{H, H\} \equiv 0,$$

so H is conserved (as we have noted before in a different way - as a consequence of the Lagrangian having no explicit dependence on t).

Remark if we were to take one of the q_k, or one of the p_k as examples of a function F on phase space, then we learn that

$$\dot{q}_k = \{q_k, H\} = \sum_{i=1}^{N} \left(\frac{\partial q_k}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial q_k}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \sum_{i=1}^{N} \left(\delta_{ki} \frac{\partial H}{\partial p_i} - 0 \frac{\partial H}{\partial q_i} \right) = \frac{\partial H}{\partial p_k},$$

and also that

$$\dot{p}_k = \{p_k, H\} = \sum_{i=1}^{N} \left(\frac{\partial p_k}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p_k}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \sum_{i=1}^{N} \left(0 \frac{\partial H}{\partial p_i} - \delta_{ki} \frac{\partial H}{\partial q_i} \right) = -\frac{\partial H}{\partial q_k}.$$

Of course, these are just Hamilton's equations again but seen from a different perspective.

4.4 Some properties of the Poisson Bracket

(i) **Antisymmetry**:

$$\{F, G\} = -\{G, F\}.$$

To show this, simply write out both sides.

(ii) **Jacobi identity**:

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0.$$

Note: here, H is not necessarily the Hamiltonian. To show this, write out each of the triple brackets (very carefully).

[In the lecture I went through a similar relation satisfied by the commutators of matrices F, G, H:

$$[F, [G, H]] + [H, [F, G]] + [G, [H, F]] = 0,$$

where $[F, G] = FG - GF$ and the multiplication is matrix multiplication showing how the cancellations work in that case.]

(iii) **Poisson Bracket relations between the generalised coordinates and the generalised momenta:**

$$\{q_k, q_l\} = \sum_{i=1}^{N} \left(\frac{\partial q_k}{\partial q_i} \frac{\partial q_l}{\partial p_i} - \frac{\partial q_k}{\partial p_i} \frac{\partial q_l}{\partial q_i} \right) \equiv 0.$$

$$\{p_k, p_l\} = \sum_{i=1}^{N} \left(\frac{\partial p_k}{\partial q_i} \frac{\partial p_l}{\partial p_i} - \frac{\partial p_k}{\partial p_i} \frac{\partial p_l}{\partial q_i} \right) \equiv 0.$$

$$\{q_k, p_l\} = \sum_{i=1}^{N} \left(\frac{\partial q_k}{\partial q_i} \frac{\partial p_l}{\partial p_i} - \frac{\partial q_k}{\partial p_i} \frac{\partial p_l}{\partial q_i} \right) = \sum_{i=1}^{N} (\delta_{ki}\delta_{il} - 0) \equiv \delta_{kl}.$$

These expressions are nice and memorable.

The following two properties follow directly from the properties of derivatives

(iv) For any three functions on phase space F, G, H

$$\{F, G + H\} = \{F, G\} + \{F, H\}.$$

(v) For any three functions on phase space F, G, H

$$\{F, GH\} = \{F, G\}H + G\{F, H\}.$$

[Here, H is not necessarily the Hamiltonian!]

(vi) If two functions F, G on phase space are conserved (ie their Poisson brackets with the Hamiltonian for a dynamical system are zero) then so is their Poisson bracket. This property follows from antisymmetry and the Jacobi identity:

$$\{\{F, G\}, H\} = -\{H, \{F, G\} = \{G, \{H, F\}\} + \{F, \{G, H\}\} = 0.$$

Here, H is the Hamiltonian.

(vii) **Changing coordinates on phase space**.

This means setting $q_i \to Q_i(\underline{q}, \underline{p}),\ p_i \to P_i(\underline{q}, \underline{p})$.

Suppose the new coordinates (which are by definition functions on phase space) also satisfy the canonical relations:

$$\{Q_i, Q_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}.$$

Here the Poisson brackets are computed using the q_i, p_i, as before. Then we have the following nice fact: for any pair of functions on the phase F, G, we can check directly that

$$\{F, G\}_{qp} = \{F, G\}_{QP}$$

where the subscripts are there to remind you that the Poisson bracket on the left is computed using the set of phase space coordinates $q_i, p_i,\ i = 1, \ldots, N$ and the Poisson bracket on the right is computed using the new coordinates $Q_i, P_i,\ i = 1, \ldots, N$ and F, G are expressed as functions of the new variables.

To check the details of this you will need to use the facts (chain rule):

$$\frac{\partial}{\partial q_i} = \sum_{k=1}^{N} \left(\frac{\partial Q_k}{\partial q_i} \frac{\partial}{\partial Q_k} + \frac{\partial P_k}{\partial q_i} \frac{\partial}{\partial P_k} \right), \quad \frac{\partial}{\partial p_i} = \sum_{k=1}^{N} \left(\frac{\partial Q_k}{\partial p_i} \frac{\partial}{\partial Q_k} + \frac{\partial P_k}{\partial p_i} \frac{\partial}{\partial P_k} \right),$$

and then compute $\{F, G\}_{qp}$ to find (on organising the eight different products of derivatives),

$$\begin{aligned}
\{F, G\}_{qp} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) &= \sum_{k,l} \frac{\partial F}{\partial Q_k} \frac{\partial G}{\partial Q_l} \{Q_k, Q_l\} \\
&+ \sum_{k,l} \frac{\partial F}{\partial P_k} \frac{\partial G}{\partial P_l} \{P_k, P_l\} \\
&+ \sum_{k,l} \frac{\partial F}{\partial Q_k} \frac{\partial G}{\partial P_l} \{Q_k, P_l\} \\
&+ \sum_{k,l} \frac{\partial F}{\partial P_k} \frac{\partial G}{\partial Q_l} \{P_k, Q_l\}.
\end{aligned}$$

Using the Poisson brackets of the new variables with each other, the first two terms in the sum vanish and we find:

$$\begin{aligned}
\{F, G\}_{qp} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) &= \sum_{k,l} \frac{\partial F}{\partial Q_k} \frac{\partial G}{\partial P_l} \delta_{kl} + \sum_{k,l} \frac{\partial F}{\partial P_k} \frac{\partial G}{\partial Q_l} (-\delta_{kl}) \\
&= \sum_{k} \left(\frac{\partial F}{\partial Q_k} \frac{\partial G}{\partial P_k} - \frac{\partial F}{\partial P_k} \frac{\partial G}{\partial Q_k} \right) \\
&= \{F, G\}_{QP}.
\end{aligned}$$

A change of variables that preserves the Poisson brackets among the coordinates is called a canonical change of coordinates.

Example (1) The new coordinates are related linearly to the old coordinates. An example of this, which can be useful as we will see later, is to put (using a summation convention - j is summed from 1 to N)

$$Q_i = R_{ij}q_j, \ P_i = S_{ij}p_j,$$

Where R and S are constant $N \times N$ matrices. Then we clearly have

$$\{Q_k, Q_l\} = 0, \quad \{P_k, P_l\} = 0,$$

and

$$\{Q_k, P_l\} = \sum_{i=1}^{N} \frac{\partial Q_k}{\partial q_i}\frac{\partial P_l}{\partial p_i} = \sum_{i=1}^{N} R_{ki}S_{li} = (RS^T)_{kl} = \delta_{kl} \ \Rightarrow \ R^{-1} = S^T.$$

Note, if $R = S$ then R must be an orthogonal $N \times N$ matrix or vice-versa.

Example 2 Suppose the phase space is two-dimensional with coordinates q, p then we could define new variables by setting

$$Q = q^\alpha \cos\beta p, \ P = q^\alpha \sin\beta p.$$

Then

$$\begin{aligned}\{Q, P\}_{qp} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ &= (\alpha q^{\alpha-1}\cos\beta p)(\beta q^\alpha \cos\beta p) - (-\beta q^\alpha \sin\beta p)(\alpha q^{\alpha-1}\sin\beta p) \\ &= \alpha\beta q^{2\alpha-1}(\cos^2\beta p + \sin^2\beta p) \\ &= \alpha\beta q^{2\alpha-1}.\end{aligned}$$

Thus this particular transformation is canonical only when $\alpha = 1/2$ and $\beta = 2$.

4.5 Normal modes

Suppose the Hamiltonian for a dynamical system with N degrees of freedom is a quadratic function of the coordinates together with a positive definite quadratic function of the momenta (the small amplitude pendulum has this property). This means the Hamiltonian can be written (again adopting a summation convention for the repeated indices $i, j = 1, \dots N$)

$$H = \frac{1}{2}p_i A_{ij} p_j + \frac{1}{2}q_i B_{ij} q_j,$$

where the two matrices A and B are real and symmetric (which means their eigenvalues are real and they can each be diagonalised by an orthogonal transformation; also, the eigenvalues are positive because the quadratic forms are positive). Consider the coordinates and momenta as components of N-dimensional vectors so that

$$H = \frac{1}{2}p^T A p + \frac{1}{2}q^T B q.$$

Next, write $A = R^T a R$, where a is a diagonal matrix whose eigenvalues (the diagonal elements of a) are positive, $R^T R = I$, and put $P = \sqrt{a}Rp, \ Q = \frac{1}{\sqrt{a}}Rq$ so that $q = R^T\sqrt{a}Q$. This is a canonical transformation and we now have

$$H = \frac{1}{2}P^T P + \frac{1}{2}Q^T \sqrt{a}RBR^T\sqrt{a}Q$$

The next step is to write $\sqrt{a}RBR^T\sqrt{a} = R'^T\beta R'$, where β is diagonal and $R'^T R' = I$. Further defining $P' = R'P$, and $Q' = R'Q$ we find

$$H = \frac{1}{2}P'^T P' + \frac{1}{2}Q'^T\beta Q' \equiv \frac{1}{2}\sum_{i=1}^{N}\left(P_i'^2 + \beta_i Q_i'^2\right).$$

The latter expression is a set of N independent harmonic oscillators in terms of the primed coordinates. This means the system can be solved in the primed coordinates and then transformed back to the original coordinates, if needed. Since Hamilton's equations will imply

$$\ddot{Q}'_i = -\beta_i Q'_i,$$

and if $\beta_i > 0,\ i = 1, \ldots, N$, the components of the diagonal matrix β represent a set of possible frequencies of vibration for the system. These are called 'normal modes'. All possible motions of the original system will be appropriate linear combinations of these since

$$q = R^T \sqrt{a} R'^T Q'.$$

Example

Suppose we have two particles of mass m moving on the x-axis with coordinates $x_1 \geq 0$, $x_2 \geq x_1$. The particle at x_1 is connected to the origin by a spring, and the particle at x_2 is connected to particle x_1 by an identical spring and also by a third identical spring to the point $x = 3a$ where a is the rest length of a spring. Use coordinates that represent the displacement of each particle from its rest position (at rest, the particles are at $x = a,\ x = 2a$, respectively). Ie we set $x_1 = a + q_1,\ x_2 = 2a + q_2$, the Hamiltonian for the set up is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{\kappa}{2}\left(q_1^2 + (q_2 - q_1)^2 + q_2^2\right) = \frac{1}{2m} p^T p + \frac{\kappa}{2} q^T B q,$$

where κ is the spring constant and

$$B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

It is straightforward to calculate the eigenvalues of this matrix to find $\lambda_1 = 1, \lambda_2 = 3$ and also to construct the orthogonal matrix that diagonalises B. Thus

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R^T B R = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Using this expression for R, the new coordinates are defined by $Q = Rq,\ P = Rp$ and thus

$$Q_1 = \frac{1}{\sqrt{2}}(q_1 + q_2), \quad Q_2 = \frac{1}{\sqrt{2}}(q_1 - q_2),$$

and the Hamiltonian in terms of the new coordinates is

$$H = \frac{1}{2m}\left(P_1^2 + P_2^2\right) + \frac{\kappa}{2}\left(Q_1^2 + 3Q_2^2\right).$$

This is now equivalent to two independent harmonic oscillators with frequencies

$$\sqrt{\frac{\kappa}{m}}, \quad \sqrt{\frac{3\kappa}{m}}.$$

These are the frequencies of the normal modes for this system.

You could, if you wish attempt an analysis of a similar set up with N particles along the x-axis. Or consider a situation with N particles along the x-axis $x_1 \leq x_2 \leq \ldots \leq x_N$ connected by identical springs but neither x_1 or x_N is tethered to a fixed point.

4.6 Phase portraits

Given a Hamiltonian for a system with N degrees of freedom, and using the general notation, as before, we have

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, N.$$

Notice that the points where all of $\dot{q}_i$ and $\dot{p}_i$ vanish is exactly where all the partial derivatives of H are zero. In other words, these are the stationary points of the Hamiltonian H. For a high dimensional system these may be difficult to analyse but it is useful to do so because these are the points at which the system can remain at rest. Understanding the nature of the stationary points (as maxima, minima, or saddle points) helps to understand how the system will evolve for different initial conditions and can give a qualitative picture of the evolution. It is also important to know which of the stationary points is stable/unstable to small perturbations.

Example (1)

A simple harmonic oscillator is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{\omega^2 q^2}{2}.$$

Hamilton's equations are:

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial q} = -\omega^2 q.$$

The stationary point is found for

$$\frac{\partial H}{\partial q} = \omega^2 q = 0, \quad \frac{\partial H}{\partial p} = \frac{p}{m} = 0 \;\Rightarrow (q,p) = 0.$$

Also, in any particular motion (ie for specified initial conditions - meaning a specific point in phase space (q_0, p_0)- we have

$$\frac{dp}{dq} = \frac{\dot{p}}{\dot{q}} = -\frac{m\omega^2 q}{p}, \quad p^2 + m\omega^2 q^2 = \text{constant},$$

and we can sketch a curve starting from the initial point to represent the evolution in phase space. Thus the curves $p(q)$ in the two dimensional phase space that represent specific motions $p(q_0) = p_0$ are a collection of ellipses centred at the stationary point $(0,0)$; there is no trajectory that passes through the fixed point at $(0,0)$. In the lecture a sketch of this is provided. Note, the trajectories in phase space are traversed in a 'clockwise' manner (you can see this from Hamilton's equations above because p decreases with increasing q).

Example (2)

As a second example, consider a simple pendulum but with no restriction on the amplitude. For this situation, a suitable Hamiltonian is:

$$H = \frac{p^2}{2m} - \lambda \cos q.$$

Hamilton's equations are:

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial q} = -\lambda \sin q.$$

Hence,

$$\ddot{q} = -\frac{\lambda}{m} \sin q.$$

Thus the critical points of H have $p = 0, \; \sin q = 0$, meaning they are at the points

$$(q,p) = (k\pi, 0), \; k = 0, \pm 1, \pm 2, \ldots.$$

The curves in phase space, $p(q)$ are now more elaborate and

$$\frac{dp}{dq} = \frac{-\lambda \sin q}{p}.$$

To analyse near a fixed point set $q = k\pi + \epsilon$, where ϵ is small. Then

$$\ddot{q} = \ddot{\epsilon} = -\frac{\lambda}{m} \sin(k\pi + \epsilon) = -\frac{\lambda}{m} \cos(k\pi) \sin \epsilon \approx -\frac{\lambda}{m} (-1)^k \, \epsilon.$$

Hence, for k even, near the fixed point the motion is stable and periodic, but for k odd the motion near the fixed point is unstable (because in that case the right hand side of the equation for ϵ is a positive constant times ϵ. This is reflected in the phase portrait provided in the lecture. Note that trajectories do approach the unstable fixed points and leave them, as reflected in the diagrams.

4.7 **Symmetries and conservation laws**

The relationship between symmetries and conserved quantities is uniquely associated with Emmy Noether who made fundamental contributions to the idea (published in 1918). The idea is very important in quantum field theory as a means of incorporating key properties of fundamental particles. We can examine the idea either using the Lagrangian or the Hamiltonian. Here we use the Hamiltonian.

Suppose a dynamical system is described by a set of N generalised coordinates and their conjugate momenta, and the Hamiltonian describing the system is invariant under a change of variables of the form

$$q_i \to q_i + \delta q_i, \quad p_i \to p_i + \delta p_i,$$

where $\delta q_i, \ \delta p_i$ are small changes. Thus, expanding to first order

$$H(\underline{q}, \underline{p}) = H(\underline{q} + \underline{\delta q}, \underline{p} + \underline{\delta p}) = H(\underline{q}, \underline{p}) + \sum_{i=1}^{N} \left(\delta q_i \frac{\partial H}{\partial q_i} + \delta p_i \frac{\partial H}{\partial p_i} \right).$$

Thus, using Hamilton's equations

$$0 = \sum_{i=1}^{N} \left(\delta q_i \frac{\partial H}{\partial q_i} + \delta p_i \frac{\partial H}{\partial p_i} \right) = \sum_{i=1}^{N} \left(-\delta q_i \dot{p}_i + \delta p_i \dot{q}_i \right).$$

To know precisely what this means requires examining examples because what exactly is conserved depends on the details of the symmetry.

Example (1)

Suppose we have a particle of mass m moving in 3D described by its position vector $\mathbf{r}$ subject to a central force $\mathbf{F} = -\nabla V(r), \ r = |\mathbf{r}|$. Then, the motion of the particle is described by the Hamiltonian

$$H = \frac{|\mathbf{p}|^2}{2m} + V(r).$$

If we use Cartesian coordinates $x_a, \ p_a, \ a = 1, 2, 3$ to represent the vector position and momentum then the Hamiltonian is invariant under rotations of the form (using the summation convention)

$$x'_a = R_{ab} x_b, \ p'_a = R_{ab} p_b, \quad (R^T R)_{ab} = \delta_{ab},$$

since

$$x'_a x'_a = x_a x_a, \ p'_a p'_a = p_a p_a.$$

Now make a small rotation $R = I + \delta R$, where I is the identity (ie zero rotation) and note

$$I = (I + \delta R)^T (I + \delta R) \approx I + (\delta R + \delta R^T) \Rightarrow \delta R + \delta R^T = 0,$$

meaning that δR is antisymmetric and we can write

$$\delta R_{ab} = \epsilon_{abc} \delta \omega_c,$$

where $\delta \omega_c$ is small. Then

$$\begin{aligned} x'_a &= x_a + \delta x_a = x_a + \delta R_{ab} x_b = x_a + \epsilon_{abc} \delta \omega_c x_b, \\ p'_a &= p_a + \delta p_a = p_a + \delta R_{ab} p_b = p_a + \epsilon_{abc} \delta \omega_c p_b. \end{aligned}$$

Since the Hamiltonian is invariant under the rotation, from the argument above, we have

$$0 = \sum_{a=1}^{3} \left[(-\epsilon_{abc} \delta \omega_c x_b) \, \dot{p}_a + (\epsilon_{abc} \delta \omega_c p_b) \, \dot{x}_a \right] = -\dot{\mathbf{p}} \times \mathbf{r} \cdot \delta \omega + \dot{\mathbf{r}} \times \mathbf{p} \cdot \delta \omega.$$

Rearranging the last term using the antisymmetry of the vector product - and noting that the small rotation does not depend on time - gives:

$$\delta\omega \cdot \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = 0,$$

and, then, since the small rotation is arbitrary, we deduce that angular momentum, $\mathbf{J} = \mathbf{r} \times \mathbf{p}$, is conserved.

This is an important message: rotational symmetry implies the conservation of angular momentum.

Remark

(i) We could check directly that angular momentum is conserved by computing its Poisson bracket with the Hamiltonian. Thus, using the summation convention and remembering the components of $\mathbf{J}$ are given by $J_a = \epsilon_{abc} x_b p_c$, we have:

$$\begin{aligned}
\{J_a, H\} = \frac{\partial J_a}{\partial x_d}\frac{\partial H}{\partial p_d} - \frac{\partial J_a}{\partial p_d}\frac{\partial H}{\partial x_d} &= (\epsilon_{abc}\delta_{bd}p_c)\frac{p_d}{m} - (\epsilon_{abc}x_b\delta_{cd})\frac{x_d}{r}\frac{dV}{dr} \\
&= \frac{1}{m}\,\epsilon_{adc}p_c p_d - \epsilon_{abd}x_b x_d\,\frac{1}{r}\frac{dV}{dr} \\
&\equiv 0.
\end{aligned}$$

(ii) We can also compute the Poisson brackets of two components of angular momentum to see what we find. First, note that from the definition of $\mathbf{J}$, we have

$$J_1 = \epsilon_{1bc}x_b p_c = x_2 p_3 - x_3 p_2, \quad J_2 = \epsilon_{2bc}x_b p_c = x_3 p_1 - x_1 p_3.$$

Hence (again using a summation convention),

$$\{J_1, J_2\} = \frac{\partial J_1}{\partial x_d}\frac{\partial J_2}{\partial p_d} - \frac{\partial J_1}{\partial p_d}\frac{\partial J_2}{\partial x_d} = (-p_2)(-x_1) - (x_2)(p_1) = x_1 p_2 - x_2 p_1 \equiv J_3.$$

You can repeat the calculation for the other components and you will find

$$\{J_a, J_b\} = \epsilon_{abc}J_c.$$

This is an elegant result that has great importance. It shows that the Poisson Brackets for the angular momentum components close on themselves.

In quantum mechanics, angular momentum becomes an operator $\hat{\mathbf{J}}$ and the Poisson bracket is replaced by a commutator $[\cdot,\cdot]$ divided by $i\hbar$. For the angular momentum components this means:

$$[\hat{J}_a, \hat{J}_b] = i\hbar\,\epsilon_{abc}\hat{J}_c.$$

This observation has far reaching consequences.

Example (2)

Suppose we have two particles of masses m_1, m_2, with position vectors $\mathbf{r}_1, \mathbf{r}_2$ moving in 3D under the influence of a force derived from a potential that depends only on $|\mathbf{r}_1 - \mathbf{r}_2|$, the distsnce between the particles. In this situation, the Hamiltonian is

$$H = \frac{|\mathbf{p}_1|^2}{2m_1} + \frac{|\mathbf{p}_2|^2}{2m_2} + V(|\mathbf{r}_1 - \mathbf{r}_2|).$$

The Hamiltonian has a translation symmetry, meaning it does not change, under the transformation

$$\mathbf{p}_i' = \mathbf{p}_i,\ i = 1,2, \quad \mathbf{r}_i' = \mathbf{r}_i - \mathbf{h},\ i = 1,2,$$

where $\mathbf{h}$ does not depend on time (ie $\dot{\mathbf{h}} = 0$). Then, because the Hamiltonian does not change, and adjusting the notation slightly to take account of the degrees of freedom of the two particles separately, we have for a small translation:

$$0 = \sum_1^2 (-\delta\mathbf{r}_i \cdot \dot{\mathbf{p}}_i + \delta\mathbf{p}_i \cdot \mathbf{r}_i) = -\delta\mathbf{h} \cdot \sum_1^2 \dot{\mathbf{p}}_\mathbf{i}.$$

Since $\delta\mathbf{h}$ is arbitrary and constant we deduce

$$\sum_1^2 \dot{\mathbf{p}}_\mathbf{i} = \frac{d}{dt}\left(\sum_1^2 \mathbf{p}_\mathbf{i}\right) = 0 \quad \Rightarrow \quad \sum_1^2 \mathbf{p}_\mathbf{i} = \text{constant}.$$

In other words, the total momentum is conserved.

Note, this generalises to an arbitrary number of particles whose interactions only depend on the distances between them.

Message: time-independent translation symmetry implies conservation of the total momentum.

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