Quantum & Continuum Dynamics

Fluid Dynamics Notes

Spring 2023–2024

Lecturer: Konstantin Ilin[1]

Recommended texts

1. D. J. Acheson, *Elementary fluid dynamics*, 1990 (available at University Library - click here).
2. J. Braithwaite, *Essential Fluid Dynamics for Scientists*, 2017 (available online - click here).
3. G. K. Batchelor, *An introduction to fluid dynamics*, 1967 (available at University Library - click here).
4. L. D. Landau and E. M. Lifshitz, *Fluid mechanics*, 1959 (or 2nd English edition, 1987). Available at University Library and also online - click here.

[1]Room G/142, email: konstantin.ilin@york.ac.uk

Contents

1 Introduction

In this part of the module we shall study fluid dynamics. You will have a chance to use many mathematical techniques from your other modules, including calculus, vector calculus and functions of a complex variable.

Fluid dynamics is a classical topic in both physics and mathematics that has numerous applications (e.g. to the dynamics of atmosphere and ocean, the weather forecast, the aerodynamics, etc.). Fluid motion is described by the Euler and Navier-Stokes equations. These are partial differential equations that are turned out to be very difficult to solve: although their study had started long time ago, it still remains one of the central topics in the modern theory of partial differential equations. These notes give a basic introduction to the dynamics of ideal (inviscid) fluids governed by the Euler equations.

Fluid dynamics studies the motion of fluids (gases and liquids, e.g. air or water). Fluids are considered as *continuous media*: all quantities which appear in the governing equations depend of time and spacial coordinates, e.g. $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity of the fluid at time t at point $\boldsymbol{x}$ in space. This is based upon the assumption that any small volume element of a fluid (a '*fluid particle*') contains huge number of molecules, and when we say an "infinitesimally small fluid element", we always mean that it is a *physically* small element, i.e. very small compared with the body of the fluid, but still containing large number of molecules. Then the fluid velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ is simply the average velocity at time t of all molecules contained in such physically small volume element centered at point $\boldsymbol{x}$.

2 Fluid flow

Notation. Let

$$(x, y, z) \quad \text{or} \quad (x_1, x_2, x_3)$$

be Cartesian coordinates in space with the unit vectors (along the axes)

$$\boldsymbol{e}_x,\ \boldsymbol{e}_y,\ \boldsymbol{e}_z \quad \text{or} \quad \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3.$$

To denote Cartesian components of vectors, we shall use the following notation

$$\boldsymbol{u} = (u_x, u_y, u_z) = (u_1, u_2, u_3),$$

which means that u_x (or u_1) is the x-component of vector $\boldsymbol{u}$, u_y (or u_2) is its y-component, and u_z (or u_3) is its z-component. u_i denotes the i-th component of $\boldsymbol{u}$ where i may take values $1, 2, 3$. In what follows we shall use both notations for components of vectors.

We shall also use $\boldsymbol{x}$ to denote the point with coordinates (x, y, z) (or (x_1, x_2, x_3)) in space.

When we use the index notation, we shall employ the *summation convention*: it will be assumed that if an expression contains a repeated index (i.e. which appears there twice), then the expression is summed up over the repeated index. For example, for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with components a_i and b_i ($i = 1, 2, 3$), the expression

$$a_i b_i \quad \text{means} \quad \sum_{i=1}^{3} a_i b_i,$$

so that we can write $\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$.

Throughout the notes, we use the following notation for partial derivatives:

$$\partial_x = \frac{\partial}{\partial x}, \quad \partial_y = \frac{\partial}{\partial y}, \quad \partial_z = \frac{\partial}{\partial z}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad \partial_x^2 = \frac{\partial^2}{\partial x^2} \quad \text{etc.}$$

Fluid flow. Mathematically, a fluid flow is described by the velocity $\boldsymbol{u}(x, y, z, t)$ (or, for brevity, $\boldsymbol{u}(\boldsymbol{x}, t)$). The fluid velocity is a vector field, which means that it is a vector whose direction and magnitude may be different at different points in space:

$$\begin{aligned} \boldsymbol{u}(x, y, z, t) &= (u_x(x, y, z, t), u_y(x, y, z, t), u_z(x, y, z, t)) \\ &= u_x(x, y, z, t)\boldsymbol{e}_x + u_y(x, y, z, t)\boldsymbol{e}_y + u_z(x, y, z, t)\boldsymbol{e}_z. \end{aligned}$$

Simple particular cases:

- Steady flow: $\partial_t \boldsymbol{u} = 0$, i.e. $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x})$.
- Two-dimensional (2D) flow: $\partial_z \boldsymbol{u} = 0$ and $u_z = 0$, i.e. $\boldsymbol{u}(\boldsymbol{x}, t) = u_x(x, y, t)\boldsymbol{e}_x + u_y(x, y, t)\boldsymbol{e}_y = (u_x(x, y, t), u_y(x, y, t))$.

A vector field can be visualised by drawing vectors attached to different points in space. For example, a sketch of the velocity field for the 2D flow $\boldsymbol{u} = (y, -x)$ is shown in Fig. 1.

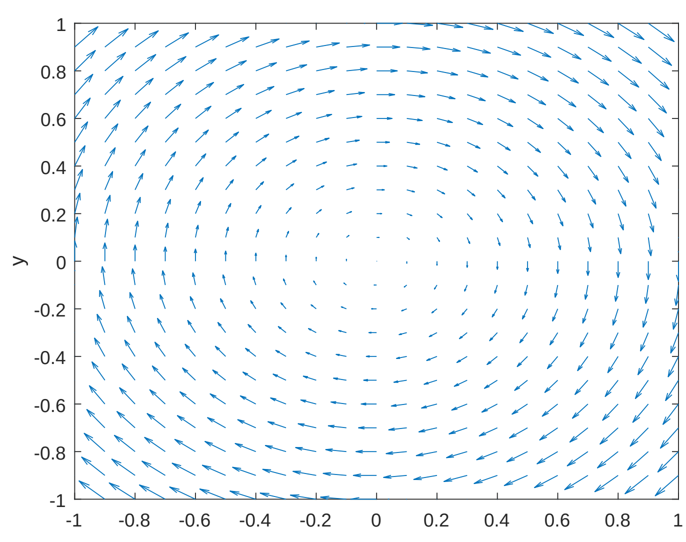


Figure 1

2.1 Pathlines

We want to answer the question: given the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, how does a fluid particle (i.e. a very small volume element of the fluid) move?

Let $\boldsymbol{x}(t)$ be the position of a fluid particle in the flow with the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ at time t. Then it must satisfy the vector ODE:

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{u}(\boldsymbol{x}(t), t). \tag{2.1}$$

Equivalently, we can write this vector ODE as a system of three scalar ODEs:

$$\begin{aligned}
\frac{dx(t)}{dt} &= u_x(x(t), y(t), z(t), t) \\
\frac{dy(t)}{dt} &= u_y(x(t), y(t), z(t), t) \\
\frac{dz(t)}{dt} &= u_z(x(t), y(t), z(t), t)
\end{aligned} \tag{2.2}$$

Suppose that the initial position of the particle is prescribed:

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \quad \text{or, equivalently,} \quad (x(0), y(0), z(0)) = (x_0, y_0, z_0)$$

Suppose also that we can solve system (2.2) with these initial conditions. As a result, we would obtain a solution in the form

$$\boldsymbol{x}(t) = \boldsymbol{X}(\boldsymbol{x}_0, t) \tag{2.3}$$

or

$$x(t) = X(x_0, y_0, z_0, t), \quad y(t) = Y(x_0, y_0, z_0, t), \quad z(t) = Z(x_0, y_0, z_0, t). \tag{2.4}$$

In Eqs. (2.3) and (2.4), we indicated that this solution corresponds to the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$. This solution gives us the pathline (i.e. the trajectory) of the fluid particle whose initial position was $\boldsymbol{x}_0$.

Example 1. Find the pathlines for the 2D flow $\boldsymbol{u} = (y, -x)$.

Solution. We solve the ODEs

$$\frac{dx(t)}{dt} = y(t), \quad \frac{dy(t)}{dt} = -x(t)$$

with the initial conditions

$$x(0) = x_0, \quad y(0) = y_0.$$

The general solution of this system is

$$\begin{aligned}
x(t) &= A \sin t + B \cos t, \\
y(t) &= -B \sin t + A \cos t.
\end{aligned}$$

Substituting these into the initial conditions, we find that $B = x_0$ and $A = y_0$. Thus, we have

$$x(t) = y_0 \sin t + x_0 \cos t, \quad y(t) = -x_0 \sin t + y_0 \cos t. \tag{2.5}$$

Equation (2.5) is the parametric equation of the pathline. Let's find its Cartesian equation (in terms of x and y only). Eliminating t from from Eq. (2.5), we get

$$x^2 + y^2 = (y_0 \sin t + x_0 \cos t)^2 + (-x_0 \sin t + y_0 \cos t)^2 = x_0^2 + y_0^2.$$

So, the pathline starting at (x_0, y_0) is a circle of radius $\sqrt{x_0^2 + y_0^2}$.

Example 2. (a) Find the pathlines for the 3D flow $\boldsymbol{u} = (u_0, k\,t, w_0)$ where u_0, k and w_0 are positive constants. (b) Sketch the pathlines for the 2D flow $\boldsymbol{u} = (u_0, k\,t)$ (i.e. when $w_0 = 0$).

Solution. (a) We need to solve the ODEs

$$\frac{dx(t)}{dt} = u_0, \quad \frac{dy(t)}{dt} = k\,t, \quad \frac{dz(t)}{dt} = w_0$$

with the initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0.$$

The general solution of this system is

$$x(t) = u_0 t + A, \quad y(t) = \frac{kt^2}{2} + B, \quad z(t) = w_0 t + C,$$

where A, B, C are arbitrary constants. Substituting these into the initial conditions, we find that $B = x_0$, $A = y_0$ and $C = z_0$. Therefore, we have the pathline starting at (x_0, y_0, z_0) is given by

$$x(t) = x_0 + u_0 t, \quad y(t) = y_0 + \frac{kt^2}{2}, \quad z(t) = z_0 + w_0 t.$$

(b) If $w_0 = 0$, we obtain the 2D flow, and the above formulae for $x(t)$ and $y(t)$ give us parametric representation (with t being the parameter) of the particle's trajectory. If we eliminate t, we find that

$$t = \frac{x - x_0}{u_0} \quad \Rightarrow \quad y - y_0 = \frac{k(x - A)^2}{2u_0^2}.$$

The last equation is the equation of parabola. So, each pathline is the right half of a parabola. Several pathlines with different (x_0, y_0) (represented by circles) are shown in Fig. 2.

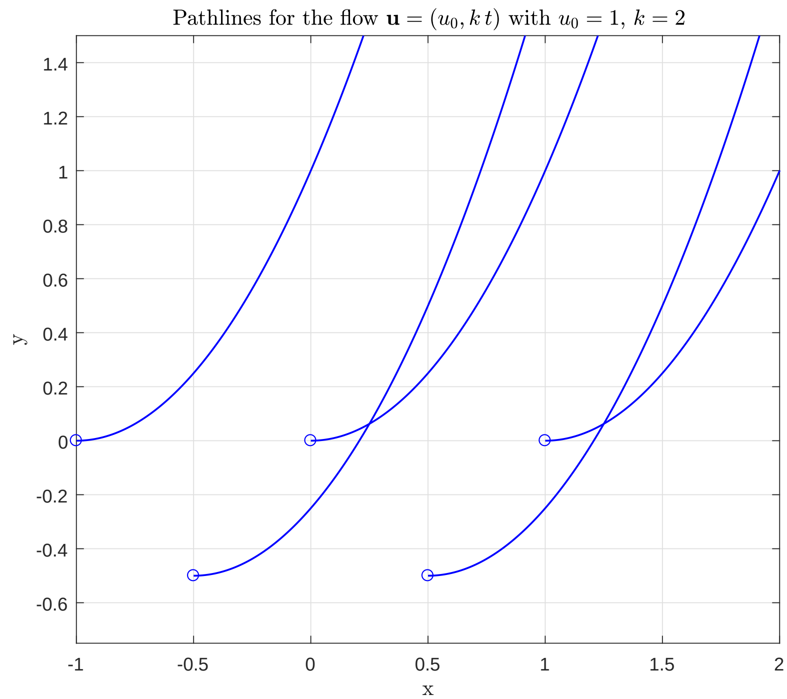


Figure 2

2.2 Streamlines

A *streamline* of a flow is a curve such that, at each point of the curve, the tangent line to it is parallel to the velocity at the same point. If the velocity field depends on time, then streamlines may be different at different moments of time. Let $\boldsymbol{u}(\boldsymbol{x}, t)$ be a velocity field. It follows from this definition that if $\boldsymbol{x}(s) = (x(s), y(s), z(s))$ is a parametric equation of a streamline (with some parameter s along the streamline), then the tangent vector to the curve, $d\boldsymbol{x}(s)/ds$, must be parallel to $\boldsymbol{u}(\boldsymbol{x}(s), t)$. This means

$$\frac{d\boldsymbol{x}(s)}{ds} = \lambda\, \boldsymbol{u}(\boldsymbol{x}(s), t)$$

where λ is a nonzero scalar which may depend on both $\boldsymbol{x}$ and t. Clearly, there is a freedom in choosing λ. The simplest choice that we shall use from now on is $\lambda = 1$. So, the streamlines are solutions of the vector ODE

$$\frac{d\boldsymbol{x}(s)}{ds} = \boldsymbol{u}(\boldsymbol{x}(s), t) \tag{2.6}$$

or

$$\begin{aligned}
\frac{dx(s)}{ds} &= u_x(x(s), y(s), z(s), t), \\
\frac{dy(s)}{ds} &= u_y(x(s), y(s), z(s), t), \\
\frac{dz(s)}{ds} &= u_z(x(s), y(s), z(s), t).
\end{aligned} \tag{2.7}$$

Note that for steady flows ($\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$) streamlines coincide with pathlines, because Eq. (2.7) will be exactly the same as Eq. (2.2), it we replace s with t.

Example 3. Find the streamlines for the 2D flow $\boldsymbol{u} = (u_0, k\,t)$ (cf. Example 2).

Solution. The streamlines are solutions of the ODE

$$\frac{dx(s)}{ds} = u_0, \qquad \frac{dy(s)}{ds} = k\,t.$$

The general solution of this system is

$$x(s) = u_0 s + A, \quad y(s) = k\,t\,s + B,$$

where A, B are arbitrary constants. If the streamline passes through the point (x_0, y_0), then

$$x(s) = x_0 + u_0 s, \quad y(s) = y_0 + k\,t\,s.$$

Eliminating s, we find the Cartesian equation of the streamlines

$$\frac{x - x_0}{u_0} = \frac{y - y_0}{kt}.$$

Hence, at a fixed t, all streamlines are straight lines parallel to the vector (u_0, kt). At $t = 0$, these are horizontal lines. In the limit as $t \to \infty$, the streamlines become vertical lines. Recall the pathlines found in Example 2 are parabolas (i.e. different from the streamlines).

3 Equations of motion for ideal fluid

3.1 Material derivative

Let $f(\boldsymbol{x}, t)$ be some quantity of interest (e.g. a component of the velocity $\boldsymbol{u}$ or the density of the fluid ρ). The rate of change of f at a fixed point in space (i.e. at fixed $\boldsymbol{x}$) is $\partial_t f(\boldsymbol{x}, t)$. What is the rate of change of f following the fluid (i.e. the rate of change of f at a fixed fluid particle as it moves with the fluid)? We denote this as Df/Dt and will refer to it as the *material derivative* of f. We have

$$\begin{aligned}
\frac{Df}{Dt} &= \frac{d}{dt} f(x(t), y(t), z(t), t) \\
&= \partial_t f(x,y,z,t) + \partial_x f(x,y,z,t)\,\frac{dx}{dt} + \partial_y f(x,y,z,t)\,\frac{dy}{dt} + \partial_z f(x,y,z,t)\,\frac{dz}{dt} \\
&= \partial_t f(x,y,z,t) + \partial_x f(x,y,z,t)\,u_x + \partial_y f(x,y,z,t)\,u_y + \partial_z f(x(t),y,z,t)\,u_z \\
&= \partial_t f(\boldsymbol{x}, t) + \boldsymbol{u} \cdot \nabla f(\boldsymbol{x}, t).
\end{aligned}$$

Thus, we have

$$\frac{Df}{Dt} = \partial_t f + \boldsymbol{u} \cdot \nabla f = \partial_t f + u_i \partial_i f$$

where $\partial_i = \partial/\partial x_i$.

In particular, $Df/Dt = 0$ means that quantity f is a constant for each fluid particle.

3.2 Acceleration of a fluid particle

The velocity of a fluid particle (whose position at time t is $\boldsymbol{x}(t)$) is $\boldsymbol{u}(\boldsymbol{x}(t), t)$. To find acceleration of this fluid particle, $\boldsymbol{a}(\boldsymbol{x}(t), t)$, we need to differentiate the velocity with respect to time. For the i-th component of $\boldsymbol{a}$, we have

$$\begin{aligned}
a_i &= \frac{d}{dt}\, u_i(x_1(t), x_2(t), x_3(t), t) \\
&= \partial_t u_i(x_1, x_2, x_3, t) + \partial_1 u_i(x_1, x_2, x_3, t)\, \frac{dx_1}{dt} + \partial_2 u_i(x_1, x_2, x_3, t)\, \frac{dx_2}{dt} + \partial_3 u_i(x_1, x_2, x_3, t)\, \frac{dx_3}{dt} \\
&= \partial_t u_i(x_1, x_2, x_3, t) + \partial_1 u_i(x_1, x_2, x_3, t)\, u_1 + \partial_2 u_i(x_1, x_2, x_3, t)\, u_2 + \partial_3 u_i(x_1, x_2, x_3, t)\, u_3 \\
&= \partial_t u_i(\boldsymbol{x}, t) + u_k\, \partial_k u_i(\boldsymbol{x}, t) = \partial_t u_i(\boldsymbol{x}, t) + (\boldsymbol{u} \cdot \nabla) u_i(\boldsymbol{x}, t).
\end{aligned}$$

Thus,

$$a_i(\boldsymbol{x}, t) = \frac{Du_i}{Dt} = \partial_t u_i + (\boldsymbol{u} \cdot \nabla) u_i \quad \text{or, in vector form} \quad \boldsymbol{a}(\boldsymbol{x}, t) = \frac{D\boldsymbol{u}}{Dt} = \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}.$$

Note that the acceleration field may be nonzero even for an unsteady steady flow. Foe example, for the velocity field $\boldsymbol{u} = (y, -x)$ (considered in Example 1), we obtain

$$\boldsymbol{a} = (\boldsymbol{u} \cdot \nabla)(y, -x) = (y\partial_x - x\partial_y)(y, -x) = (-x, -y) = -\boldsymbol{x},$$

which means that there is a nonzero acceleration directed to the origin (recall centripetal acceleration of a particle moving along a circle from Classical Dynamics).

3.3 Conservation of mass

Let $\rho(\boldsymbol{x}, t)$ be the density of the fluid, i.e. the mass of the fluid per unit volume. The density is strictly positive: $\rho(\boldsymbol{x}, t) > 0$ for all $\boldsymbol{x}$ and t. We shall deduce an equation governing the evolution of ρ from the law of conservation of mass.

Consider an arbitrary fixed (in space) volume V_0. The mass of the fluid in this volume is

$$\int_{V_0} \rho(\boldsymbol{x}, t)\, dV. \tag{3.1}$$

Here dV is the volume element (i.e. $dV = dx\, dy\, dz$). Let ∂V_0 be the boundary of V_0 and $\boldsymbol{n}$ be the outward unit normal on ∂V_0 (see Fig. 3). The mass of the fluid flowing through a surface element dS on ∂V_0 is $\rho|\boldsymbol{u} \cdot \boldsymbol{n}|\, dS$ (i.e. the absolute value of the normal component of the *mass flux density* $\rho\boldsymbol{u}$ multiplied by the area of the surface element dS). If $\boldsymbol{u}\cdot\boldsymbol{n}$ is positive, then the fluid flows out of volume V_0 through the surface element dS. If $\boldsymbol{u}\cdot\boldsymbol{n}$ is negative, then the fluid flows into V_0. So, the mass of the fluid in V_0, flowing out of V_0 though the boundary of V_0 per unit time is

$$\oint\limits_{\partial V_0} \rho\boldsymbol{u}\cdot\boldsymbol{n}\,dS.$$

If this integral is positive, this means that more fluid flows out than in per unit time; if it is negative, then more fluid flows in than out.

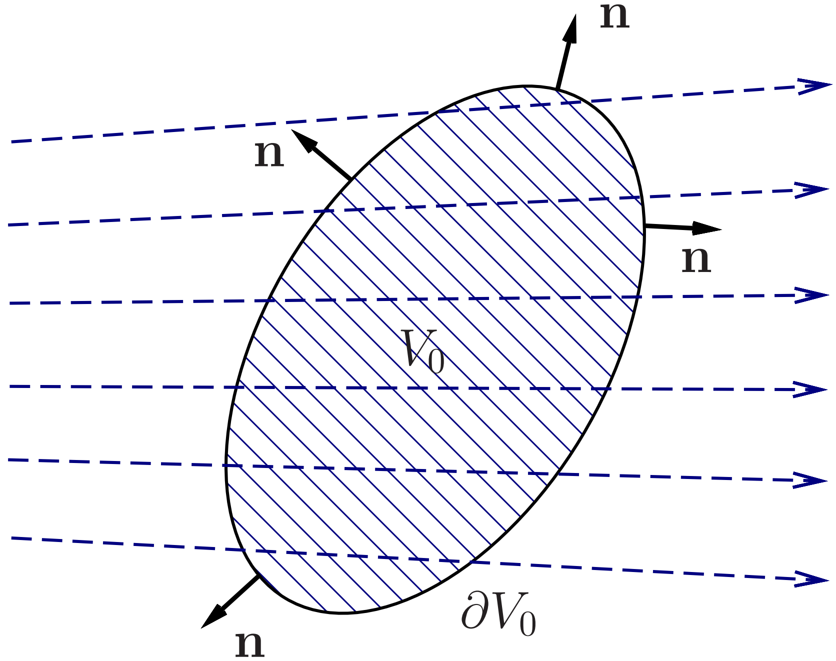


Figure 3

The rate of change of mass in volume V_0 is the derivative with respect to time of the integral (3.1). In view of to the law of conservation of mass, it must be equal to the change of mass (per unit time) due to the inflow and outflow of the fluid. Therefore, we obtain

$$\frac{d}{dt}\int\limits_{V_0}\rho\,dV = \int\limits_{V_0}\partial_t\rho\,dV = -\oint\limits_{\partial V_0}\rho\boldsymbol{u}\cdot\boldsymbol{n}\,dS. \tag{3.2}$$

Here the minus before the surface integral appears because $\boldsymbol{n}$ is chosen to be the outward normal (so that the change of mass is negative if the fluid flows out of V_0).

Divergence theorem. Now, we need to recall the divergence theorem (also known as the Gauss-Ostrogradsky theorem) of Vector Calculus. If V_0 is a region in $\mathbb{R}^3$ with boundary ∂V_0

and $\boldsymbol{F}(\boldsymbol{x})$ is a vector field defined in V_0, then

$$\int\limits_{V_0} \nabla \cdot \boldsymbol{F}\, dV = \oint\limits_{\partial V_0} \boldsymbol{F} \cdot \boldsymbol{n}\, dS. \tag{3.3}$$

Another form of the divergence theorem which we shall use later is

$$\int\limits_{V_0} \partial_i f\, dV = \oint\limits_{\partial V_0} f\, n_i\, dS \quad \text{or, in vector form} \quad \int\limits_{V_0} \nabla f\, dV = \oint\limits_{\partial V_0} f\, \boldsymbol{n}\, dS \tag{3.4}$$

where $f(\boldsymbol{x})$ is any scalar function. Formula (3.4) for $i = 1, 2, 3$ can be obtained from (3.3) with $\boldsymbol{F} = f\, \boldsymbol{e}_i$ (for $i = 1, 2, 3$).

Applying the divergence theorem to the surface integral on the right hand side of Eq. (3.2), we obtain

$$\int\limits_{V_0} \partial_t \rho\, dV = -\int\limits_{V_0} \nabla \cdot (\rho\, \boldsymbol{u})\, dV \quad \Leftrightarrow \quad \int\limits_{V_0} \left[\partial_t \rho + \nabla \cdot (\rho\, \boldsymbol{u})\right]\, dV = 0.$$

The last integral must be zero for any volume V_0. This is possible only if the integrand is zero. So, we conclude that

$$\partial_t \rho + \nabla \cdot (\rho\, \boldsymbol{u}) = 0. \tag{3.5}$$

Equation (3.5) is known as the *continuity equation*.

Note that since $\nabla \cdot (\rho\, \boldsymbol{u}) = \boldsymbol{u} \cdot \nabla \rho + \rho\, \nabla \cdot \boldsymbol{u}$, the equation of continuity can be written as

$$\partial_t \rho + \boldsymbol{u} \cdot \nabla \rho + \rho\, \nabla \cdot \boldsymbol{u} = 0 \quad \text{or} \quad \frac{D\rho}{Dt} + \rho\, \nabla \cdot \boldsymbol{u} = 0. \tag{3.6}$$

It the density of each fluid element remain constant as the fluid moves, i.e.

$$\frac{D\rho}{Dt} = 0, \tag{3.7}$$

then it follows from (3.6) that

$$\nabla \cdot \boldsymbol{u} = 0. \tag{3.8}$$

In this case, the fluid is called *incompressible* and Eq. (3.8) is known as the *incompressibility condition*. (A vector field $\boldsymbol{u}$ satisfying $\nabla \cdot \boldsymbol{u}$ is known as a *solenoidal vector field* or a *divergence-free vector field*).

In what follows, we shall mostly consider an important particular case of an incompressible fluid, namely, a *homogeneous (in the density) fluid*, i.e. a fluid whose density is constant both in space and time:

$$\rho(\boldsymbol{x}, t) = const. \tag{3.9}$$

Note that, in this particular case, Eq. (3.7) is automatically satisfied.

3.4 The equation of motion for an ideal incompressible fluid (of constant density)

The equation of motion is simply the Newton's equation: $m\mathbf{a} = \boldsymbol{F}$. Again, we consider a fixed (in space) volume V_0. We sum up the mass multiplied by the acceleration for each fluid particle (which is in V_0 at time t) over all fluid particles in the volume V_0. This yields

$$\int_{V_0} \rho\, \boldsymbol{a}\, dV = \int_{V_0} \rho\, \left[\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right]\, dV.$$

This integral represents a vector quantity. For the i-th component, we have

$$\int_{V_0} \rho\, a_i\, dV = \int_{V_0} \rho\, \left[\partial_t u_i + (\boldsymbol{u}\cdot\nabla)u_i\right]\, dV.$$

For an ideal fluid, the only force acting on the volume V_0 is a surface force due to the pressure in the surrounding fluid. Let $p(\boldsymbol{x}, t)$ be the pressure in the fluid. The total force exerted on the volume V_0 is

$$\boldsymbol{F} = -\oint_{\partial V_0} p\, \boldsymbol{n}\, dS \quad \text{or, in components,} \quad \boldsymbol{F} = -\oint_{\partial V_0} p\, n_i\, dS \;\; \text{for} \;\; i = 1, 2, 3.$$

So, the Newton's equation of motion takes the form

$$\int_{V_0} \rho\, \left[\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right]\, dV = -\oint_{\partial V_0} p\, \boldsymbol{n}\, dV.$$

Applying the divergence theorem in the form of Eq. (3.4) to the surface integral on the right hand side, we obtain

$$\int_{V_0} \rho\, \left[\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right]\, dV = -\int_{V_0} \nabla p\, dV \quad \text{or} \quad \int_{V_0} \rho\, \left[\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p\right]\, dV = 0.$$

This must hold for any volume V_0. Therefore, we conclude that the integrand in the last integral must be zero:

$$\rho\, (\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = -\nabla p. \tag{3.10}$$

Equation (3.10) is called the *Euler equation* or the *momentum equation* (as $\rho\boldsymbol{u}(\boldsymbol{x}, t)$ is the momentum of a fluid particle which is at point $\boldsymbol{x}$ at time t).

If there is an external body force[2], it can be added to the right hand side of Eq. (3.10). For example, with the standard gravitational force taken into account, Eq. (3.10) becomes

$$\rho\, (\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = -\nabla p + \rho\mathbf{g}$$

[2] A force that acts throughout the volume of a body.

or

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\,\nabla p + \mathbf{g} \tag{3.11}$$

where $\mathbf{g}$ in the (constant) gravitational acceleration. Equation (3.10) (or (3.11)) together with the incompressibility condition (3.8) are called *Euler's equations* for an ideal incompressible fluid.

Since $\boldsymbol{g}$ is a constant vector, it can be written as minus the gradient of a potential. Indeed, $\boldsymbol{g} = -\nabla\chi$ where $\chi = -\boldsymbol{x}\cdot\boldsymbol{g}$. Hence, Eq. (3.11) can also be written as

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \chi\right). \tag{3.12}$$

Let the z axis be directed vertically up, them $\boldsymbol{g} = (0, 0, -g)$ (where g is the free fall acceleration). The Euler equations for components of $\boldsymbol{u}$:

$$\partial_t u_x + (u_x\partial_x + u_y\partial_y + u_z\partial_z)\, u_x = -\frac{1}{\rho}\,\partial_x p, \tag{3.13}$$

$$\partial_t u_y + (u_x\partial_x + u_y\partial_y + u_z\partial_z)\, u_y = -\frac{1}{\rho}\,\partial_y p, \tag{3.14}$$

$$\partial_t u_z + (u_x\partial_x + u_y\partial_y + u_z\partial_z)\, u_z = -\frac{1}{\rho}\,\partial_z p - g. \tag{3.15}$$

Also, we have the incompressibility condition

$$\partial_x u_x + \partial_y u_y + \partial_z u_z = 0. \tag{3.16}$$

Thus, we have 4 equations for 4 unknowns: three components of the velocity, u_x, u_y and u_z, and pressure, p.

Remark. Note that the pressure cannot be uniquely determined from Eqs. (3.13)–(3.16), because if $\boldsymbol{u}(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t)$ represent a solution, then $\boldsymbol{u}(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t) + f(t)$ for arbitrary function f is also a solution. To determine the pressure uniquely, we heed to impose some additional condition. For example, if we consider a flow in the whole space, we may require that the pressure at infinity is a given constant: $p(\boldsymbol{x}, t) \to p_0 = const$ as $|\boldsymbol{x}| \to \infty$.

3.5 Examples

Example 4. *(Archimedes' principle).* Consider an incompressible homogeneous fluid of density ρ in the presence of the gravitational force. Suppose that a rigid body is immersed in the fluid. Both the body and the fluid are at rest. What is the force exerted by the fluid on the body?

Solution. Let D be a region occupied by the body and ∂D be its boundary. The potential of the gravitational force is $\chi = -\boldsymbol{g}\cdot\boldsymbol{x}$. Since the fluid is at rest, $\boldsymbol{u} = \boldsymbol{0}$, so that the incompressibility condition $\nabla\cdot\boldsymbol{u} = 0$ is automatically satisfied, and Eq. (3.12) yields

$$\nabla\left(\frac{p}{\rho} + \chi\right) = 0 \quad\Rightarrow\quad p = -\rho\,\chi + C = \rho\,\boldsymbol{g}\cdot\boldsymbol{x} + C$$

for an arbitrary constant C. The force exerted on the body is

$$\boldsymbol{F} = -\int\limits_{\partial D} p\,\boldsymbol{n}\,dS = -\rho\int\limits_{\partial D} (\boldsymbol{g}\cdot\boldsymbol{x} + C)\,\boldsymbol{n}\,dS$$

where $\boldsymbol{n}$ is the unit normal vector directed outward with respect to D. On using the divergence theorem (3.4), we find

$$\boldsymbol{F} = -\int\limits_{D} \nabla(\boldsymbol{g}\cdot\boldsymbol{x} + C)\,dV = -\rho\boldsymbol{g}\int\limits_{D} dV = -\rho V_D\,\boldsymbol{g}$$

where V_D is the volume of region D. This means that the force is opposite to $\boldsymbol{g}$ and its magnitude is equal to the weight of the fluid displaced by the body, which is exactly Archimedes' principle.

Example 5. Consider the steady flow of an incompressible fluid of constant density in the presence of the gravitational force $\boldsymbol{F} = -g\boldsymbol{e}_z$:

$$\boldsymbol{u} = (-\Omega y, \Omega x, 0)$$

where Ω is a constant. (This velocity field describes a fluid rotating with constant angular speed about the z axis). Show that $\boldsymbol{u}$ satisfies the Euler equations and find the pressure.

Solution. First we check whether the incompressibility condition (3.16) is satisfied. We have

$$\partial_x(-\Omega y) + \partial_y(\Omega x) + \partial_z 0 = 0.$$

So, Eq. (3.16) is satisfied. Substituting $\boldsymbol{u} = \boldsymbol{0}$ into Eqs. (3.13)–(3.15), we get

$$\begin{cases} (-\Omega y\partial_x + \Omega x\partial_y)\,(-\Omega y) = -\frac{1}{\rho}\,\partial_x p, \\ (-\Omega y\partial_x + \Omega x\partial_y)\,(\Omega x) = -\frac{1}{\rho}\,\partial_y p, \\ 0 = -\frac{1}{\rho}\,\partial_z p - g \end{cases} \quad\Rightarrow\quad \begin{cases} -\Omega^2 x = -\frac{1}{\rho}\,\partial_x p, \\ -\Omega^2 y = -\frac{1}{\rho}\,\partial_y p, \\ \rho\,g = -\partial_z p \end{cases}$$

Integrating the 3rd equation in z yields

$$p = -\rho g z + f(x,y)$$

for arbitrary $f(x,y)$. Substituting this into the first two equations, we obtain

$$\partial_x f = \rho\Omega^2 x \;\text{ and }\; \partial_y f = \rho\Omega^2 y \quad\Rightarrow\quad f = \frac{\rho\Omega^2}{2}\,(x^2+y^2) + C$$

for arbitrary constant C. This shows that the given velocity field satisfies the Euler equations and the pressure is given by

$$p = -\rho g z + \frac{\rho\Omega^2}{2}\,(x^2+y^2) + C$$

3.6 Vorticity

The vorticity $\boldsymbol{\omega}$ of a flow with the velocity field $\boldsymbol{u}$ is defined as the *curl* of $\boldsymbol{u}$ (see Vector Calculus notes, p. 28):

$$\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u} = \nabla \times \boldsymbol{u}. \tag{3.17}$$

At this stage, it is useful to recall some formulae from Vector Calculus. For any (differentiable) vector field $\boldsymbol{g} = (g_1, g_2, g_3)$:

- $(\nabla \times \boldsymbol{g})_i = \epsilon_{ijk} \partial_j g_k$ where ϵ_{ijk} is the Levi-Civita symbol:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } \ (i,j,k) \text{ is } (1,2,3), (2,3,1), (3,1,2) \\ -1 & \text{if } \ (i,j,k) \text{ is } (3,2,1), (1,3,2), (2,1,3) \\ 0 & \text{if } \ i=j, \ \text{or } j=k, \ \text{or } k=i \end{cases}$$

- Formally, $\nabla \times \boldsymbol{g}$ can be written as the determinant:

$$\nabla \times \boldsymbol{g} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ g_1 & g_2 & g_3 \end{vmatrix} = (\partial_2 g_3 - \partial_3 g_2)\, \boldsymbol{e}_1 + (\partial_3 g_1 - \partial_1 g_3)\, \boldsymbol{e}_2 + (\partial_1 g_2 - \partial_2 g_1)\, \boldsymbol{e}_3.$$

- The *curl* of a gradient and the divergence of a *curl* are zero:

$$\nabla \times (\nabla f) = \boldsymbol{0}, \quad \nabla \cdot (\nabla \times \boldsymbol{g}) = 0$$

  for any scalar function f and any vector field $\boldsymbol{g}$.

- A useful identity (valid for any vector field $\boldsymbol{g}$):

$$(\boldsymbol{g} \cdot \nabla)\boldsymbol{g} = (\nabla \times \boldsymbol{g}) \times \boldsymbol{g} + \nabla \left(\frac{\boldsymbol{g}^2}{2} \right). \tag{3.18}$$

- Another useful identity (valid for any vector fields $\boldsymbol{f}$ and $\boldsymbol{g}$):

$$\nabla \times (\boldsymbol{f} \times \boldsymbol{g}) = (\boldsymbol{g} \cdot \nabla)\boldsymbol{f} + (\nabla \cdot \boldsymbol{g})\boldsymbol{f} - (\boldsymbol{f} \cdot \nabla)\boldsymbol{g} + (\nabla \cdot \boldsymbol{f})\boldsymbol{g}. \tag{3.19}$$

Consider the steady 2D flow:

$$\boldsymbol{u} = (u_x(x,y), u_y(x,y), 0).$$

Then the vorticity is

$$\boldsymbol{\omega} = (0, 0, \omega), \quad \omega = \partial_x u_y - \partial_y u_x. \tag{3.20}$$

Simple examples:

- If $u_x(x,y) = \alpha y$ for $\alpha \in \mathbb{R}$ and $u_y(x,y) = 0$, i.e. $\boldsymbol{u} = (\alpha y, 0, 0)$ (which is a simple shear flow), then
$$\boldsymbol{\omega} = (0, 0, -\alpha).$$
- If $\boldsymbol{u} = (-\Omega y, \Omega x, 0)$ (where Ω is a constant angular velocity of the fluid particles) as in Example 5, then
$$\boldsymbol{\omega} = (0, 0, \partial_x u_y - \partial_y u_x) = (0, 0, 2\Omega).$$
This means that for this flow the vorticity is proportional to the angular velocity of the fluid particles.

3.7 The evolution of vorticity

To derive an equation which governs the evolution of vorticity, we first rewrite the Euler equation (3.12) in a more convenient form. It follows from (3.18) that

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{\omega} \times \boldsymbol{u} + \nabla \left(\frac{\boldsymbol{u}^2}{2} \right).$$

Using this, we rewrite Eq. (3.12) as

$$\partial_t \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} = -\nabla \left(\frac{p}{\rho} + \frac{\boldsymbol{u}^2}{2} + \chi \right). \tag{3.21}$$

Taking *curl* of Eq. (3.21) using the fact that *curl* of *grad* is zero, we obtain the vorticity equation:

$$\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = 0. \tag{3.22}$$

It follows from the identity (3.19), the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \cdot \boldsymbol{\omega} = 0$ (this is so because *div* of *curl* is zero) that

$$\nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}.$$

Therefore, the vorticity equation can also be written as

$$\partial_t \boldsymbol{\omega} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}. \tag{3.23}$$

Note that the pressure p does not appear in the vorticity equation (3.22) [or (3.23)], i.e. the pressure has been eliminated.

For a 2D flow

$$\boldsymbol{u} = (u_x(x,y,t), u_y(x,y,t), 0)$$

we have

$$\boldsymbol{\omega} = (0, 0, \omega(x,y,t)) \quad \text{and} \quad (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \omega(x,y,t)\, \partial_z \boldsymbol{u}(x,y,t) = 0,$$

so that Eq. (3.23) reduces to

$$\partial_t \omega + (\boldsymbol{u} \cdot \nabla)\omega = 0 \quad \text{or} \quad \frac{D\omega}{Dt} = 0. \tag{3.24}$$

Thus, *in a 2D flow, the vorticity of each fluid particle is conserved.* For a steady 2D flow, $\boldsymbol{u} = (u_x(x, y), u_y(x, y), 0)$, Eq. (3.24) simplifies to

$$(\boldsymbol{u} \cdot \nabla)\omega = 0.$$

This means that $\nabla\omega$ is perpendicular to $\boldsymbol{u}$ and, therefore, *in a steady 2D flow, the vorticity is constant on each streamline.*

3.8 Irrotational flow

It follows from Eq. (3.23) that if $\boldsymbol{\omega}(\boldsymbol{x}, 0) = \boldsymbol{0}$, then $\boldsymbol{\omega}(\boldsymbol{x}, t) = \boldsymbol{0}$ for all $t > 0$. This makes is possible to consider solutions of the Euler equations with zero vorticity.

By definition, a flow is *irrotational* (or potential) if

$$\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u} = \boldsymbol{0}$$

everywhere in the flow. If the flow domain is simply connected[3], $\operatorname{curl} \boldsymbol{u} = \boldsymbol{0}$ implies that there exist a function $\phi(\boldsymbol{x})$, known as the velocity potential, such that

$$\boldsymbol{u} = \nabla\phi. \tag{3.25}$$

So, we shall always assume that there is a velocity potential.

Note that ϕ is not uniquely defined: we can add any function of time to it without changing the velocity field.

Substituting (3.25) into the incompressibility condition, we get

$$\nabla^2\phi = 0 \tag{3.26}$$

where $\nabla^2 \equiv \partial_x^2 + \partial_y^2 + \partial_z^2$ is the *Laplace operator.* Equation (3.26) is known as the Laplace equation, and this means that the velocity potential is a solution of the Laplace equation.

It turns out that, for an irrotational flow, the momentum equation (3.21) can be integrated. Indeed, if $\boldsymbol{u} = \nabla\phi$ and hence $\boldsymbol{\omega} = \boldsymbol{0}$, Eq. (3.21) reduces to

$$\partial_t \nabla\phi = -\nabla\left(\frac{p}{\rho} + \frac{\boldsymbol{u}^2}{2} + \chi\right) \quad \text{or} \quad \nabla\left(\partial_t\phi + \frac{p}{\rho} + \frac{\boldsymbol{u}^2}{2} + \chi\right) = 0.$$

The last equation implies that $\partial_t\phi + \frac{p}{\rho} + \frac{\boldsymbol{u}^2}{2} + \chi$ must be a function of t only, say, $f(t)$. This function can be absorbed into ϕ. Therefore, this arbitrary function is usually ignored and the integral of the above equation is written in the form:

$$\partial_t\phi + \frac{p}{\rho} + \frac{\boldsymbol{u}^2}{2} + \chi = const \tag{3.27}$$

[3]A domain D is simply connected, if any simple closed curve in D can be continuously shrunk into a point while remaining in the domain.

throughout the fluid region. Equation (3.27) is sometimes referred to as the *Cauchy-Lagrange integral.*

How to find the velocity potential if we are given a velocity field $\boldsymbol{u}$ and that $\nabla \times \boldsymbol{u}$?

We need to solve three simultaneous equations for ϕ:

$$\partial_x \phi(\boldsymbol{x}) = u_x(\boldsymbol{x}), \quad \partial_y \phi(\boldsymbol{x}) = u_y(\boldsymbol{x}), \quad \partial_z \phi(\boldsymbol{x}) = u_z(\boldsymbol{x}).$$

There are many ways of solving these equations. We can find ψ from the following line integral along **any path** correcting points (x_0, y_0) and (x, y):

$$\phi(x,y) = \phi(x_0,y_0) + \int\limits_{(x_0,y_0)}^{(x,y)} \boldsymbol{u} \cdot d\boldsymbol{x}. \tag{3.28}$$

In Eq. (3.28), the value $\psi(x_0, y_0)$, as well as the starting point (x_0, y_0) itself can be chosen arbitrarily because the velocity potential is defined up to a constant (or, more precisely, a function of time).

Example 6. Show that the velocity field, given by

$$\boldsymbol{u} = (Ax + By, Bx + Cy, -(A+C)z)$$

(where A, B and C are constants), represents an irrotational flow of an ideal fluid and find the velocity potential and the pressure.

Solution. First we check whether the incompressibility condition is satisfied:

$$\partial_x(Ax + By) + \partial_y(Bx + Cy) + \partial_z(-(A+C)z) = A + C - (A+C) = 0.$$

Assuming that $\boldsymbol{u} = \nabla\phi$, we have the following system of equations for ϕ:

$$\begin{aligned} Ax + By &= \partial_x \phi, \\ Bx + Cy &= \partial_y \phi, \\ -(A+C)z &= \partial_z \phi, \end{aligned} \quad . \tag{3.29}$$

In principle, we can use Eq. (3.28). However, it is easier simply to integrate these equations one by one. Integrating the first equation in variable x, we find

$$\partial_x \phi = Ax + By \quad \Rightarrow \quad \phi = \frac{Ax^2}{2} + Bxy + f(y,z)$$

where $f(y,z)$ is an arbitrary function of variables y and z (a 'constant of integration'). Substituting the last formula for ϕ into the second equation of system (3.29), we obtain

$$\partial_y \phi = Bx + \partial_y f(y,z) = Bx + Cy \quad \Rightarrow \quad \partial_y f(y,z) = Cy \quad \Rightarrow \quad f(y,z) = \frac{Cy^2}{2} + g(z)$$

for an arbitrary $g(z)$ (another 'constant of integration'). Hence,

$$\phi = \frac{Ax^2 + Cy^2}{2} + Bxy + g(z).$$

Substituting this into the third equation of system (3.29), we get

$$\partial_z \phi = g'(z) = -(A+C)z \quad \Rightarrow \quad g(z) = -\frac{(A+C)z^2}{2} + c$$

for an arbitrary constant of integration c. Finally, we find that, up to a constant,

$$\phi = \frac{Ax^2 + Cy^2 - (A+C)z^2}{2} + Bxy.$$

To find the pressure, we use Eq. (3.27), As a result, we obtain, up to a constant,

$$p = -\rho\left(\partial_t \phi + \frac{\boldsymbol{u}^2}{2} + \chi\right) = -\rho\left(\frac{(Ax+By)^2}{2} + \frac{(Bx+Cy)^2}{2} + \frac{(A+C)^2 z^2}{2} + \chi\right).$$

3.9 Boundary conditions

What boundary conditions should be imposed on the boundary of the flow domain? The answer depends on the type of the boundary. We shall only consider two types of the boundary: (i) a fixed impermeable boundary (a rigid wall) and (ii) a free boundary.

Fixed rigid walls. For an ideal fluid in a tank with rigid impermeable walls, we need to impose only one boundary condition that there is no flow through the wall, i.e. the normal component of the velocity must be zero:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \tag{3.30}$$

on the wall.

If the impermeable wall is moving, the condition of no flow through it means that the normal velocity of the fluid at the wall must be equal to the normal velocity of the wall:

$$\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{U}_{\text{wall}} \cdot \boldsymbol{n} \tag{3.31}$$

where $\boldsymbol{U}_{\text{wall}}$ is the velocity of the wall.

Free surface. If there is an interface between two ideal fluids, two boundary condition should be imposed at the interface. In the special case when one of the fluids is much lighter than the other, its motion is usually ignored and the interface is treated as a free surface for the second fluid. For example, an interface between water and air can be treated a free surface for the water.

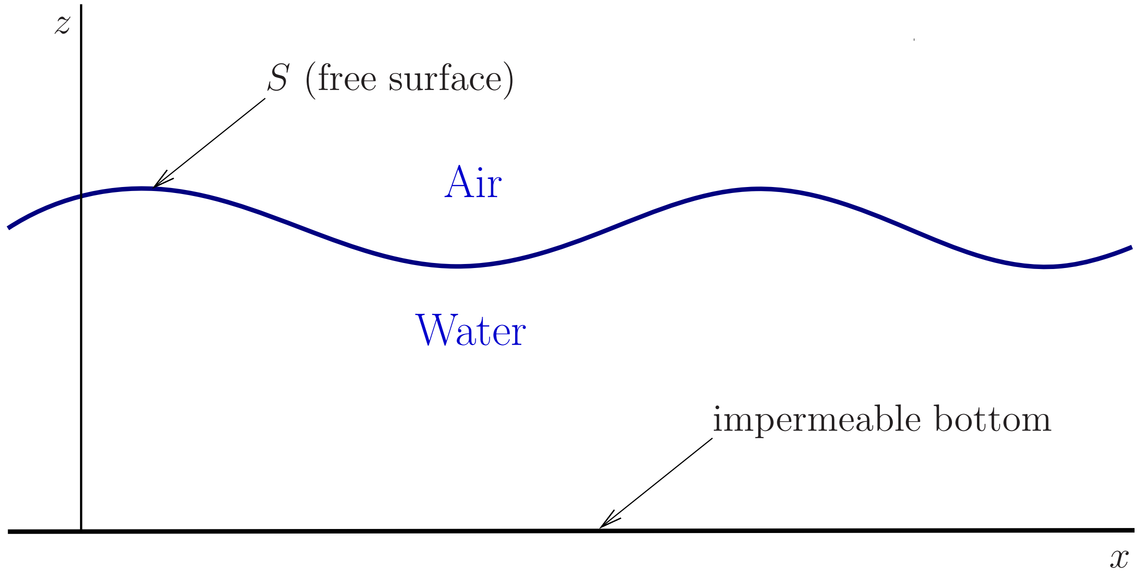


Figure 4

Let the free surface S be described by the equation

$$F(x, y, z, t) = 0. \tag{3.32}$$

We also assume that the pressure in the lighter fluid (air) is constant and equal to p_0. The following two boundary conditions are usually employed:

- *Kinematic condition* says that the free surface moves with the fluid (i.e. fluid particles that were on the free surface at $t = 0$, remain on the free surface for all $t > 0$). This means that the material derivative of $F(x, y, z, t)$ must be zero:

$$\frac{DF}{Dt} = 0 \quad \text{on} \quad S \quad \text{or} \quad \partial_t F + \boldsymbol{u} \cdot \nabla F = 0 \quad \text{at} \quad F(x, y, z, t) = 0. \tag{3.33}$$

- *Dynamic condition* says the normal components of the forces on both sides of the free surface must be balanced (so that the total force applied to the free surface is zero). This means that the pressure must be continuous across the free surface:

$$p = p_0 \quad \text{on} \quad S \quad \text{or} \quad p(x, y, z, t) = p_0 \quad \text{at} \quad F(x, y, z, t) = 0 \tag{3.34}$$

  where p is the pressure in the heavier fluid (water).

As an example, we consider an infinite layer of an ideal fluid of finite depth H over a flat bottom (as in Fig. 4). We choose the Cartesian axes such that the z-axis is vertical and directed upward, the x- and y-axes are parallel to the bottom. We choose the origin in such a way that $z = 0$ is the equation of the unperturbed (flat) free surface and $z = -H$ is the equation of the (flat) bottom. The boundary condition at $z = -H$ is the condition of no normal flow:

$$\boldsymbol{u} \cdot \boldsymbol{e}_z = 0 \quad \text{or, equivalently,} \quad u_z = 0 \quad \text{at} \quad z = -H. \tag{3.35}$$

It is convenient to describe the (perturbed) free surface using the equation

$$z = \eta(x, y, t) \quad \text{for} \quad x, y \in \mathbb{R}.$$

This means that we choose $F(x, y, z, t) = z - \eta(x, y, t)$ in Eq. (3.32). The kinematic and dynamic conditions (3.33) and (3.34) take the form

$$\partial_t \eta + u_x \partial_x \eta + u_y \partial_y \eta = u_z \quad \text{at} \quad z = \eta(x, y, t) \tag{3.36}$$

and

$$p(x, y, z, t) = p_0 \quad \text{at} \quad z = \eta(x, y, t). \tag{3.37}$$

3.10 Bernoulli's theorem

Consider a steady flow of an ideal fluid. It follows from Eq. (3.21) that

$$\boldsymbol{\omega} \times \boldsymbol{u} = -\nabla H. \tag{3.38}$$

where

$$H = \frac{p}{\rho} + \chi + \frac{\boldsymbol{u}^2}{2}. \tag{3.39}$$

Taking dot product of this equation with $\boldsymbol{u}$, we find

$$0 = -\boldsymbol{u} \cdot \nabla H \quad \text{or} \quad \boldsymbol{u} \cdot \nabla H = 0. \tag{3.40}$$

This means that H (sometimes called Bernoulli's integral) is constant along the streamlines. Thus, we have the Bernoulli theorem: *H is constant along the streamlines in a steady flow of an ideal incompressible homogeneous fluid.*

If the steady flow is irrotational, then $\boldsymbol{\omega} = 0$ in Eq. (3.38), so that $\nabla H = \mathbf{0}$. We can conclude that H is constant throughout the flow. Thus, the Bernoulli theorem for irrotational flows says that *H is constant throughout a steady irrotational flow of an ideal incompressible homogeneous fluid.*

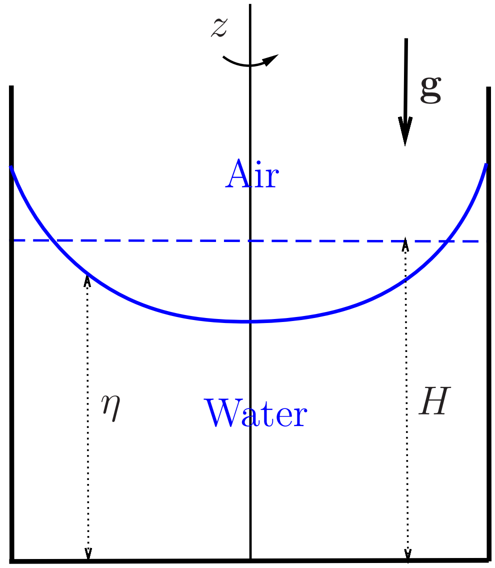


Figure 5

Example 7. (Rotating water in a bucket). Consider a cylindrical bucket filled with the water. The radius of the bucket is R, and the depth of the undisturbed water is H (see Fig. 5). Let the water be brought to a state of a rigid-body rotation with angular velocity Ω (e.g. by rotating the bucket). Then the velocity field is given by (see Example 5)

$$\boldsymbol{u} = (-\Omega y, \Omega x, 0). \tag{3.41}$$

Let the atmospheric pressure be p_A. Our aim is to find the shape of the free surface.

Solution. We have already considered the velocity field (3.41) in Example 5, where we have shown that it satisfies the Euler equation and the the pressure is given by

$$p = -\rho g z + \frac{\rho\Omega^2}{2}(x^2 + y^2) + C. \tag{3.42}$$

To find the shape of free surface, we assume that its equation is $z = \eta(x, y)$ for some function η and use the dynamic boundary condition (3.34) (which says that the pressure in the water at the free surface must be equal to the atmospheric pressure). As a result,

$$p(x, y, z) = p_A \quad \text{at} \quad z = \eta(x, y) \quad \Rightarrow \quad -\rho g\eta(x, y) + \frac{\rho\Omega^2}{2}(x^2 + y^2) + C = p_A.$$

Therefore,

$$\eta(x, y) = \frac{C - p_A}{\rho g} + \frac{\Omega^2}{2g}(x^2 + y^2). \tag{3.43}$$

Constant C can be determined from the condition that volume of the rotating water is equal to that of the undisturbed water. The latter is simply $V = \pi R^2 H$, the former is

$$V = \int\limits_0^{2\pi} d\theta \int\limits_0^R dr r \int\limits_0^{\eta(r)} dz = 2\pi \int\limits_0^R dr r\eta(r)$$

where we use polar cylindrical coordinates (r, θ, z). Using (3.43), we find

$$V = 2\pi \int_0^R dr \left(\frac{C - p_A}{\rho g} r + \frac{\Omega^2}{2g} r^3 \right) = \pi R^2 \left(\frac{C - p_A}{\rho g} + \frac{\Omega^2}{4g} R^2 \right).$$

From the two formulae for V, we obtain

$$\pi R^2 H = \pi R^2 \left(\frac{C - p_A}{\rho g} + \frac{\Omega^2}{4g} R^2 \right) \quad \Rightarrow \quad \frac{C - p_A}{\rho g} = H - \frac{\Omega^2}{4g} R^2.$$

Therefore, Eq. (3.43) takes the form

$$\eta(x, y) = H + \frac{\Omega^2}{2g} \left(x^2 + y^2 - \frac{R^2}{2} \right). \tag{3.44}$$

Thus, the surface of the water is a perfect paraboloid.

Finally, it is easy to check that the kinematic boundary condition (3.36) is also satisfied. Indeed, we have

$$\boldsymbol{u} \cdot \nabla\eta = (-\Omega y \partial_x + \Omega x \partial_y) \left[H + \frac{\Omega^2}{2g} \left(x^2 + y^2 - \frac{R^2}{2} \right) \right] = \frac{\Omega^3}{2g} (-y\partial_x + x\partial_y) \left(x^2 + y^2 \right) = 0.$$

3.11 Circulation

3.11.1 Kelvin's circulation theorem

Let C be a closed curve in the flow region. The circulation of the velocity round C is the integral

$$\Gamma = \oint_C \boldsymbol{u} \cdot d\boldsymbol{x}. \tag{3.45}$$

It turns out that the velocity circulation round any closed material curve (i.e. a curve moving with the fluid) is conserved. This property is known as *Kelvin's circulation theorem.* To prove it, we need to understand how a material line element $\delta\boldsymbol{x}$ is changing with time.

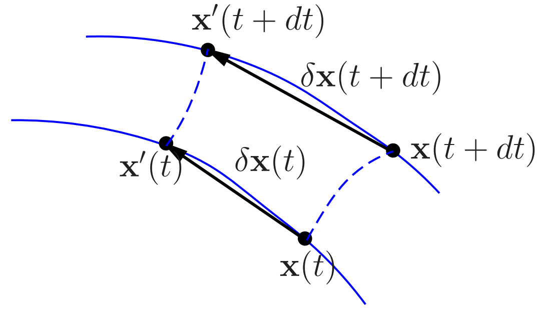


Figure 6

Consider the line element $\delta\boldsymbol{x}(t)$ connecting two fluid particles whose positions vectors at time t are $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$ (see Fig. 6). After a short time dt, these fluid particles will be at points $\boldsymbol{x}(t+dt)$ and $\boldsymbol{x}'(t+dt)$ lying on the corresponding pathlines. Hence,

$$\boldsymbol{x}(t+dt) = \boldsymbol{x}(t) + \boldsymbol{u}(\boldsymbol{x}(t),t)dt + O\left((dt)^2\right), \quad \boldsymbol{x}'(t+dt) = \boldsymbol{x}'(t) + \boldsymbol{u}(\boldsymbol{x}'(t),t)dt + O\left((dt)^2\right).$$

Since $\boldsymbol{x}'(t) = \boldsymbol{x}(t) + \delta\boldsymbol{x}(t)$, we expand $\boldsymbol{u}(\boldsymbol{x}',t)$ in Taylor's series around the point $\boldsymbol{x}$:

$$\boldsymbol{u}(\boldsymbol{x}'(t),t) = \boldsymbol{u}(\boldsymbol{x}(t)+\delta\boldsymbol{x}(t),t) = \boldsymbol{u}(\boldsymbol{x}(t),t) + (\delta\boldsymbol{x}(t)\cdot\nabla)\boldsymbol{u}(\boldsymbol{x}(t),t) + O\left(|\delta\boldsymbol{x}(t)|^2\right)$$

Then

$$\begin{aligned}
\delta\boldsymbol{x}(t+dt) &= \boldsymbol{x}'(t+dt) - \boldsymbol{x}(t+dt) \\
&= \delta\boldsymbol{x}(t) + (\boldsymbol{u}(\boldsymbol{x}(t),t) - \boldsymbol{u}(\boldsymbol{x}'(t),t))dt + O\left((dt)^2\right) \\
&= \delta\boldsymbol{x}(t) + (\delta\boldsymbol{x}(t)\cdot\nabla)\boldsymbol{u}(\boldsymbol{x}(t),t)dt + O\left(|\delta\boldsymbol{x}(t)|^2|dt|\right) + O\left((dt)^2\right).
\end{aligned}$$

Passing to the limit as $dt \to 0$, we get

$$\frac{D\delta\boldsymbol{x}}{Dt} = (\delta\boldsymbol{x}\cdot\nabla)\boldsymbol{u}(\boldsymbol{x},t) + O\left(|\delta\boldsymbol{x}|^2\right).$$

Consider now the circulation of the velocity round a closed material curve $C(t)$. Using the formula above, we obtain

$$\begin{aligned}
\frac{d\Gamma}{dt} &= \frac{d}{dt}\oint_{C(t)} \boldsymbol{u}\cdot d\boldsymbol{x} \\
&= \oint_{C(t)} \frac{D\boldsymbol{u}}{Dt}\cdot d\boldsymbol{x} + \oint_{C(t)} \boldsymbol{u}\cdot(d\boldsymbol{x}\cdot\nabla)\boldsymbol{u}.
\end{aligned}$$

Using the Euler equation (3.12) and the identity $\boldsymbol{u}\cdot(d\boldsymbol{x}\cdot\nabla)\boldsymbol{u} = (d\boldsymbol{x}\cdot\nabla)(\boldsymbol{u}^2/2)$, we find

$$\frac{d\Gamma}{dt} = -\oint_{C(t)} \nabla\left(\frac{p}{\rho} + \chi - \frac{\boldsymbol{u}^2}{2}\right)\cdot d\boldsymbol{x} = -\oint_{C(t)} d\left(\frac{p}{\rho} + \chi - \frac{\boldsymbol{u}^2}{2}\right) = 0. \tag{3.46}$$

[In (3.46), the last integral is zero because an integral over a closed curve of a total differential is zero.] Thus, the circulation is conserved: $\Gamma(t) = \Gamma(0)$.

3.11.2 Circulation in irrotational flows

Consider a closed material curve which bounds an open material surface S lying in the flow region. Then by Stokes' theorem of Vector Calculus,

$$\Gamma = \oint_{C(t)} \boldsymbol{u}\cdot d\boldsymbol{x} = \iint_{S(t)} \boldsymbol{\omega}\cdot\boldsymbol{n}\, dS.$$

It follows that if the flow is irrotational, i.e. $\boldsymbol{\omega} \equiv \boldsymbol{0}$, then $\Gamma = 0$. Does this mean that circulation is always zero in an irrotational flow? Generally, the answer is no, because the above argument is valid only if there is a surface S, which lies in the flow region and whose boundary is C. If the flow region is multiply connected, which means that there are closed curves lying in the flow region such that they cannot be continuously contracted to a point, then the circulation round such non-reducible curves may be nonzero.

For example, consider a two dimensional irrotational flow around a rigid cylinder shown in Fig. 7. The circulation of the velocity round any closed curve encircling the cylinder cannot be contracted to a point (because of the cylinder) and, therefore, the circulation may be nonzero. The value of the circulation is *a priori* unknown. What is known is that this value is the same for all such closed curves.

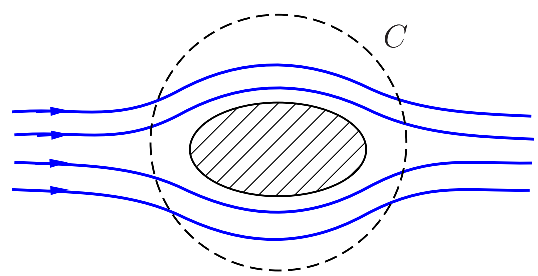


Figure 7

Example 8. Consider the two-dimensional irrotational flow around a circular cylinder of radius a centered at the origin with the velocity field

$$\boldsymbol{u} = (-\lambda y/r^2, \lambda x/r^2)$$

where $r^2 = x^2 + y^2$ and λ is a real constant. Calculate the circulation of velocity Γ round a circle of radius $R > a$ centered at the origin.

Solution. The circle can be parameterised using polar angle θ: $\boldsymbol{x} = (R\cos\theta, R\sin\theta)$. Since on the circle $\boldsymbol{u} = (-\lambda\sin\theta/R, \lambda\cos\theta/R)$, we obtain

$$\Gamma = \oint_C \boldsymbol{u}\cdot d\boldsymbol{x} = \int_0^{2\pi} \boldsymbol{u}\cdot\frac{d\boldsymbol{x}}{d\theta}\,d\theta = \int_0^{2\pi} \lambda(\sin^2\theta + \cos^2\theta)\,d\theta = 2\pi\lambda.$$

So, $\Gamma = 2\pi\lambda$. As expected, Γ doesn't depend on the radius of the circle R.

4 Two-dimensional flows

4.1 Stream function

Consider a two-dimensional incompressible flow with the velocity field $\boldsymbol{u}(\boldsymbol{x}) = (u_x(x,y), u_y(x,y))$. Incompressibility condition implies that

$$\nabla\cdot\boldsymbol{u} = \partial_x u_x + \partial_y u_y = 0.$$

This equation is automatically satisfied if

$$u_x = \partial_y \psi \quad \text{and} \quad u_y = -\partial_x \psi \quad \text{or, equivalently} \quad \boldsymbol{u} = \nabla \times (\psi \boldsymbol{e}_z) \tag{4.1}$$

for some function $\psi(x, y)$. Indeed, for any $\psi(x, y)$,

$$\partial_x u_x + \partial_y u_y = \partial_x \partial_y \psi + \partial_y(-\partial_x \psi) = 0.$$

A function $\psi(x, y)$ defined by Eq. (4.1) is called *a stream function.* A stream function is defined up to a constant because if ψ is a stream function, so is $\psi + c$ for any constant c.

The reason for the name lies in the following useful fact: *a stream function is constant along streamlines.* To check this, let's calculate the derivative of ψ in the direction of a streamline, i.e. in the direction of the velocity field (recall that the velocity field is tangent to the streamlines):

$$\boldsymbol{u} \cdot \nabla \psi = u_x \partial_x \psi + u_y \partial_y \psi = u_x(-u_y) + u_y u_x = 0. \tag{4.2}$$

Thus, the stream function gives an easy way to find streamlines: they are the curves along which the stream function is constant.

We can find ψ from the following two-dimensional line integral along any path correcting points (x_0, y_0) and (x, y):

$$\psi(x, y) = \psi(x_0, y_0) + \int\limits_{(x_0,y_0)}^{(x,y)} (-u_y, u_x) \cdot d\boldsymbol{x}. \tag{4.3}$$

In Eq. (4.3), $\psi(x_0, y_0)$ can be dropped and the starting point itself can be chosen arbitrarily because the stream function is defined up to a constant.

Example 9. Consider a two-dimensional flow with the velocity field

$$\boldsymbol{u} = (-x, y).$$

Find the streamlines for this flow.

Solution. To find the stream function, we use formula (4.3) with $(x_0, y_0) = (0, 0)$. We let $\psi(x_0, y_0) = 0$ and choose the path of integration consisting of two straight lines: the first connecting $(0, 0)$ with $(x, 0)$ and the second connecting $(x, 0)$ with (x, y). As a result, we have

$$\begin{aligned}\psi(x, y) &= \int\limits_0^x (-u_y(\tilde{x}, 0))\, d\tilde{x} + \int\limits_0^y u_x(x, \tilde{y})\, d\tilde{y} \\ &= \int\limits_0^y (-x)\, d\tilde{y} = -xy.\end{aligned}$$

So, $\psi = -xy$. Therefore, the equation of streamlines is

$$-xy = c$$

where c is an arbitrary constant. Thus, the streamlines are the coordinate axes (for $c = 0$) and the hyperbolae $y = -c/x$ (for $c \neq 0$).

Vorticity. We know that the vorticity of a two-dimensional flow has only one non-zero component (cf. Eq. (3.20)): $\boldsymbol{\omega} = \omega \boldsymbol{e}_z$ where $\omega = \partial_x u_y - \partial_y u_x$. Hence,

$$\omega = \partial_x(-\partial_x \psi - \partial_y \partial_y \psi = -\nabla^2 \psi. \tag{4.4}$$

Using (4.4), we can rewrite the 2D vorticity equation (3.24) solely in terms of ψ:

$$\partial_t \nabla^2 \psi = \{\psi, \nabla^2 \psi\} \tag{4.5}$$

where for any two functions $a(x, y)$ and $b(x, y)$,

$$\{a, b\} = \partial_x a \, \partial_y b - \partial_y a \, \partial_x b$$

is the Poisson bracket of functions a and b (recall Poisson bracket from Classical Dynamics).

For a steady flow, Eq. (4.4) reduces to

$$\partial_t \nabla^2 \psi = \{\psi, \nabla^2 \psi\} = 0 \quad \text{or, equivalently,} \quad \boldsymbol{e}_z \cdot \big(\nabla \psi \times \nabla(\nabla^2 \psi)\big) = 0.$$

The last equation means that if vectors $\nabla\psi$ and $\nabla(\nabla^2\psi)$ are nonzero, then they must be parallel in a steady flow. This, in turn, implies that there is a functional dependence between $\psi(x, y)$ and $\nabla^2\psi(x, y)$, i.e.

$$\nabla^2 \psi = F(\psi) \quad \text{or} \quad \psi = G(\nabla^2 \psi) \quad \text{or, more generally,} \quad \Phi(\psi, \nabla^2 \psi) = 0$$

for some functions F, G and Φ. Also, this means that $\nabla^2\psi$ and hence ω must be constant on each streamline.

4.2 Complex potential for two-dimensional irrotational flows

We already know that if the flow is irrotational, then $\omega = 0$, and Eq. (4.4) implies that the stream function satisfies the Laplace equation:

$$\nabla^2 \psi = 0.$$

We also know (cf. Eqs. (3.25) and (3.26)) that $\boldsymbol{u} = \nabla\phi$ and $\nabla^2\phi = 0$ where ϕ is the velocity potential. Thus, both ϕ and ψ satisfy the Laplace equation if the flow is irrotational. Also, by writing the velocity components in terms of ϕ and ψ, we obtain

$$\partial_x \phi = \partial_y \psi, \quad \partial_y \phi = -\partial_x \psi. \tag{4.6}$$

We recognise these as the Cauchy-Riemann equations. They tell us that the function $w = \phi + i\psi$ is a holomorphic function of $z = x + iy$.

By definition, the **complex potential** *of an irrotational, incompressible two-dimensional flow with velocity potential ϕ and stream function ψ is the complex-valued function*

$$w(z) = \phi(x, y) + i\psi(x, y) \quad \text{where} \quad z = x + iy. \tag{4.7}$$

Note that here we identify the (x, y) plane with the complex plane and use z to denote the complex number in that plane. This has nothing to do with the z coordinate that we used when we discussed three-dimensional flows. While in general it is a bad idea to use the same letter in the same module for different things, it is just conventional to use the letter z for both the third Cartesian coordinate and for a complex number.

For holomorphic functions, the derivative at any point, defined as

$$\frac{dw}{dz} = \lim_{|\Delta z| \to 0} \frac{\Delta w}{\Delta z}$$

does not depend on the direction from which we approach the point. So, we can consider the limit taken with Δz being parallel to the x axis (i.e. $\Delta z = \Delta x$):

$$\frac{dw}{dz} = \partial_x w = \partial_x \phi + i\partial_x \psi = u_x - iu_y. \tag{4.8}$$

If we have chosen Δz to be parallel to the y axis (i.e. $\Delta z = i\Delta y$), we would get the same result (as expected):

$$\frac{dw}{dz} = \frac{1}{i}\,\partial_y w = \frac{1}{i}\,\partial_y \phi + \partial_y \psi = \frac{1}{i}\,u_y + u_x = u_x - iu_y.$$

Thus, we can extract the velocity components from the real and imaginary parts of the derivative of the complex potential:

$$u_x = Re\left(\frac{dw}{dz}\right), \quad u_y = -Im\left(\frac{dw}{dz}\right). \tag{4.9}$$

We conclude that *any holomorphic function gives us a solution of Euler's equations, because any holomorphic function can be used as a complex potential describing an irrotational, incompressible two-dimensional flow, and such a flow is a solution of Euler's equation.*

Example 10. Let the complex potential be given by

$$w(z) = \frac{1}{2}\,z^2.$$

Find the velocity potential, stream function and the velocity field. What are the streamlines for this flow? Can we interpret this flow as a 2D flow in the upper half-plane bounded by a rigid impermeable plane at $y = 0$?

Solution. We have

$$w(z) = \frac{z^2}{2} = \frac{x^2 - y^2}{2} + ixy = \phi + i\psi \quad \Rightarrow \quad \phi = \frac{x^2 - y^2}{2}, \quad \psi = xy.$$

and

$$\frac{dw}{dz} = z = x + iy = u_x - iu_y \quad \Rightarrow \quad u_x = x, \quad u_y = -y.$$

The equations of streamlines is

$$\psi(x, y) = xy = c$$

for an arbitrary real constant c. Therefore, the streamlines are the coordinate axes (for $c = 0$) and the hyperbolae $y = c/x$ (for $c \neq 0$).

Since the x axis ($y = 0$) is a streamline, there in no flow through the line $y = 0$. So, the above velocity field can describe an irrotational 2D flow in the upper half-plane bounded by a rigid impermeable wall at $y = 0$.

Example 11. Consider the complex potential

$$w(z) = a\,e^{-i\alpha}z, \tag{4.10}$$

where $a, \alpha \in \mathbb{R}$ and $z \in \mathbb{C}$. Its derivative is just

$$\frac{dw}{dz} = a\,e^{-i\alpha} = a(cos\alpha - i\sin\alpha), \tag{4.11}$$

from which we can read off that $u_x = a\,\cos\alpha$ and $u_y = a\,\sin\alpha$. This is a constant flow with the velocity vector whose direction is at an angle α to the horizontal and whose magnitude is a.

Example 12. *(Line vortex flow.)* Consider the complex potential

$$w(z) = -\frac{i\Gamma}{2\pi}\,\ln z, \tag{4.12}$$

where $\Gamma \in \mathbb{R}$ and $z \in \mathbb{C}$. Then using the polar form of z, we have

$$w(z) = -\frac{i\Gamma}{2\pi}\,(\ln r + i\theta) = \phi + i\psi \quad \Rightarrow \quad \phi = \frac{\Gamma\theta}{2\pi}, \quad \psi = -\frac{\Gamma}{2\pi}\ln r \tag{4.13}$$

Note that the potential ϕ turned out to be a multi-valued function[4], while ψ is the good function defined everywhere except the origin. The velocity field is well defined for all z except $z = 0$. Indeed, we have (cf. Example 8)

$$\frac{dw}{dz} = -\frac{i\Gamma\bar{z}}{2\pi r^2} = -\frac{\Gamma}{2\pi r^2}\,(y - ix) \quad \Rightarrow \quad u_x = -\frac{\Gamma}{2\pi r^2}\,y, \quad u_y = \frac{\Gamma}{2\pi r^2}\,x. \tag{4.14}$$

As follows from Example 8, the velocity circulation round a circle of any radius for this flow is equal to Γ.

Remark. In polar coordinates (r, θ), the velocity field takes a much simpler form:

$$\boldsymbol{u} = \frac{\Gamma}{2\pi r}\,\boldsymbol{e}_\theta$$

where $\boldsymbol{e}_\theta = (-\sin\theta, \cos\theta)$ is the unit vector in the direction of increase of θ.

[4]This is a consequence of the fact that $\ln z$ is holomorphic only on the complex plane with 0 and negative real axis removed.

4.3 Flow around a cylinder, D'Alembert paradox

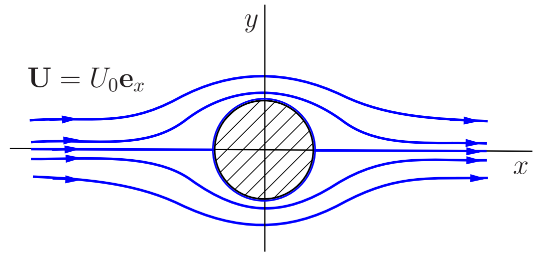


Figure 8

Consider a steady two-dimensional flow around a circular cylinder of radius R shown in Fig. 8. We assume that (i) the flow is irrotational and incompressible, (ii) far away from the cylinder the flow is uniform with a constant velocity $\boldsymbol{U} = U_0\boldsymbol{e}_x$ (with $U_0 > 0$).

We shall show that this flow is described by the complex potential

$$w(z) = Az + \bar{A}\,\frac{R^2}{z} \tag{4.15}$$

for some complex constant $A = A_r + iA_i$ and determine the value of this constant.

First, let's verify the boundary condition on the surface of the cylinder at $r = \sqrt{x^2+y^2} = R$. This condition is that there is no normal velocity on the surface or, equivalently, that the stream function $\psi = const$ at $r = R$. Using the polar form of z ($z = re^{i\theta}$), we obtain

$$w(z) = Are^{i\theta} + \bar{A}\,\frac{R^2}{re^{i\theta}} = (A_r\cos\theta - A_i\sin\theta)\left(r + \frac{R^2}{r}\right) + i\,(A_r\sin\theta + A_i\cos\theta)\left(r - \frac{R^2}{r}\right).$$

So, the velocity potential and the stream function are

$$\phi = (A_r\cos\theta - A_i\sin\theta)\left(r + \frac{R^2}{r}\right) \qquad \psi = (A_r\sin\theta + A_i\cos\theta)\left(r - \frac{R^2}{r}\right). \tag{4.16}$$

The formula for ψ implies that

$$\psi|_{r=R} = (A_r\sin\theta + A_i\cos\theta)\left(R - \frac{R^2}{R}\right) = 0.$$

Thus, the required boundary condition is satisfied.

To determine A, we employ the condition at infinity: $\boldsymbol{u} \to \boldsymbol{U} = U_0\boldsymbol{e}_x$ as $|\boldsymbol{x}| \to \infty$. On one hand, it follows from (4.15) that

$$\frac{dw}{dz} = A - \bar{A}\,\frac{R^2}{z^2} \quad \Rightarrow \quad \frac{dw}{dz} = u_x - iu_y \to A \;\text{ as }\; |z| \to \infty.$$

On the other hand, the condition at infinity says that

$$u_x \to U_0 \quad, u_y \to 0 \quad \text{as} \;\; |z| \to \infty.$$

These two limits will be consistent provided $A = U_0$ (so that A is in fact real). Finally, we have

$$w(z) = U_0 \left(z + \frac{R^2}{z} \right), \quad \frac{dw}{dz} = U_0 \left(1 - \frac{R^2}{z^2} \right). \tag{4.17}$$

Remark. Suppose that at infinity we have a uniform flow at angle α to the x axis, i.e. $\boldsymbol{u} = (U_0 \cos\alpha, U_0 \sin\alpha)$. In this case, the only necessary modification of the complex potential (4.17) is that constant $A = U_0$ should be replaced by $A = U_0 e^{-i\alpha}$ (see Example 11). As a result, we obtain

$$w(z) = U_0 e^{-i\alpha} z + U_0 e^{i\alpha} \frac{R^2}{z}. \tag{4.18}$$

D'Alembert paradox. Let's calculate the force (per unit length in the direction normal to the (x, y) plane) exerted on the cylinder by the fluid in the absence of any external body forces. First, we need to find the pressure. Since the flow is steady and irrotational, we can use Bernoulli's theorem for steady irrotational flows saying that $p/\rho + \boldsymbol{u}^2/2 = const$ throughout the fluid. So, up to an irrelevant constant,

$$p = -\rho \boldsymbol{u}^2/2 = -\frac{\rho}{2} \left| \frac{dw}{dz} \right|^2$$

Hence,

$$\boldsymbol{F} = -\oint\limits_{r=R} p \, \boldsymbol{n} \, dl = \frac{\rho}{2} \oint\limits_{r=R} \left| \frac{dw}{dz} \right|^2 \boldsymbol{n} \, dl$$

where $\boldsymbol{n}$ is the unit vector normal to the cylinder surface and directed outward from the cylinder. For a circular cylinder, $dl = R d\theta$ and $\boldsymbol{n} = \boldsymbol{x}/r = (\cos\theta, \sin\theta)$. Also, from Eq. (4.17), we have

$$\left. \left| \frac{dw}{dz} \right|^2 \right|_{z=Re^{i\theta}} = U_0^2 \left| 1 - e^{-2i\theta} \right|^2 = 2U_0^2 (1 - \cos(2\theta)).$$

Therefore,

$$F_x = \frac{\rho U_0^2}{2} \int_0^{2\pi} 2(1 - \cos(2\theta)) \, R \cos\theta \, d\theta = 0,$$

$$F_y = \frac{\rho U_0^2}{2} \int_0^{2\pi} 2(1 - \cos(2\theta)) \, R \sin\theta \, d\theta = 0.$$

Thus, both components of the force exerted on the body by the fluid are zero! It is especially surprising that there is no drag (resistance) force in the x direction. This is a manifestation of the D'Alembert paradox: namely that a steady irrotational, uniform flow of an ideal fluid past a fixed two- or three-dimensional body (of any shape) exerts no drag on the body (for a general treatment, see Acheson, section 4.13 or Batchelor, section 6.4).

4.4 Circulation and the Magnus effect

For the flow past a circular cylinder considered above, the velocity circulation round any closed curve encircling the cylinder is zero (check this!). It turns out that if there is a nonzero circulation, then there is a nonzero vertical component of the force on the cylinder. To see this, let's add the complex potential of a line vortex flow (considered in Example 12) to the complex potential (4.17). Then we have

$$w(z) = U_0\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z. \tag{4.19}$$

Hence,

$$\psi = U_0\left(r - \frac{R^2}{r}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln r. \tag{4.20}$$

As a result, we have the flow that looks like the one shown in Fig. 9.

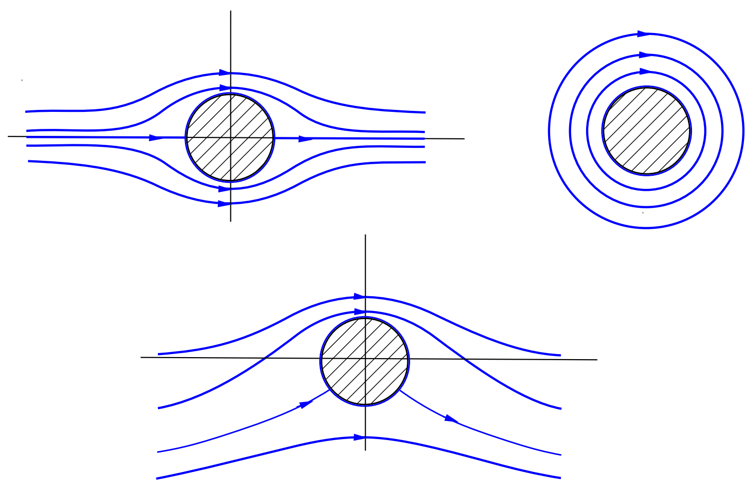


Figure 9

The azimuthal velocity on the surface of the cylinder is

$$u_\theta|_{r=R} = -\partial_r\psi|_{r=R} = -\left. U_0\left(1 + \frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}\right|_{r=R} = -2U_0\sin\theta + \frac{\Gamma}{2\pi R}. \tag{4.21}$$

If $\Gamma = 0$, there are two stagnation points (where $u_\theta|_{r=R} = 0$) at $\theta = 0$ and $\theta = \pi$. If $\Gamma \neq 0$, the positions of the stagnation points are different. From (4.21), we find

$$u_\theta|_{r=R} = 0 \quad \Rightarrow \quad \sin\theta = \frac{\Gamma}{4\pi U_0 R}.$$

This equation has solutions provided that

$$\left|\frac{\Gamma}{4\pi U_0 R}\right| \leq 1. \tag{4.22}$$

There are two stagnation points if the inequality is strict (see Fig. 9), only one stagnation point if it is the equality, and no stagnation points at all if condition (4.22) is not satisfied. In particular, if Γ is negative and we have the strict inequality (4.22), then the streamlines will look like those in Fig. 9 with stagnation points at

$$\theta = \arcsin \frac{\Gamma}{4\pi U_0 R} \quad \text{and} \quad \theta = -\pi + \arcsin \frac{\Gamma}{4\pi U_0 R}.$$

Now let's find the force exerted on the cylinder (per unit length in the direction perpendicular to the (x, y) plane). Since $u_r = 0$ at $r = R$, we have

$$\boldsymbol{u}^2|_{r=R} = u_\theta^2|_{r=R} = \left(-2U_0 \sin\theta + \frac{\Gamma}{2\pi R}\right)^2.$$

Therefore,

$$\begin{aligned} F_y &= \frac{\rho}{2} \int_0^{2\pi} \left(-2U_0 \sin\theta + \frac{\Gamma}{2\pi R}\right)^2 R \sin\theta \, d\theta \\ &= \frac{\rho}{2} \int_0^{2\pi} \left(4U_0^2 R \sin^3\theta - \frac{2U_0\Gamma}{\pi} \sin^2\theta + \frac{\Gamma^2}{4\pi^2 R} \sin\theta\right) d\theta \\ &= -\frac{U_0\Gamma}{\pi} \int_0^{2\pi} \sin^2\theta \, d\theta = -U_0\Gamma. \end{aligned}$$

Thus,

$$F_y = -U_0\Gamma. \tag{4.23}$$

Note that if $\Gamma < 0$ (this corresponds to the flow in Fig. 9) the force is positive, i.e. it is directed vertically up. That is why it is called the *lift force* (also called the *Kutta-Joukowski force*). It turns out that formula (4.23) remains valid for an arbitrary (2D) body and doesn't depend on its size, shape or orientation (for its proof in general case, see Batchelor, section 6.4).

Physical mechanism is the lift force. Since the pressure is $p = -\boldsymbol{u}^2/2 + c$ throughout the fluid, it is lower where the velocity is higher, and vice versa. Now let's look at Fig. 9. Since the flow is obtained as a sum of a symmetric (relative to the x axis) flow and a flow with circular streamlines with negative Γ, the velocity immediately above the cylinder is higher that the velocity below the cylinder. Therefore, the pressure above the cylinder is lower than the pressure below, and this produces a nonzero net force directed upwards.

Magnus effect. In the above discussion of the lift force, we have assumed that there is a circulation round the cylinder. However, it is unclear how a nonzero circulation may arise in a real flow. Moreover, in the framework of the Euler equations it is impossible to explain how circulation can be created. However, it is known that if a body rotates in a real fluid (having a non-zero viscosity), the fluid around it will begin to rotate too (at least in the vicinity of the body). So, if we consider a cylinder that rotates clockwise, its rotation will induce a negative circulation of the velocity Γ, which may be assumed to be proportional to the angular velocity Ω of the cylinder. Now if we place the cylinder in a uniform horizontal stream with the velocity $\boldsymbol{U} = U_0\boldsymbol{e}_x$ at infinity, this will produce a steady flow similar to the flow in Fig. 9). Then it is natural to expect that there will be a nonzero vertical force proportional to ΩU_0. The lift force arising in this situation is known as the *Magnus effect* that had been observed experimentally by Magnus in 1853.

4.5 Conformal mappings and flows around an elliptical cylinder and a flat plate

Conformal mappings. Let

$$\tilde{z} = f(z) \quad \text{and} \quad z = F(\tilde{z}) \tag{4.24}$$

where f and F are both holomorphic functions (with F being an inverse of f).

Suppose that $w(z)$ is the complex potential of a 2D irrotational flow in the z-plane, with $w(z) = \phi + i\psi$. Then

$$\tilde{w}(\tilde{z}) = w(F(\tilde{z}))$$

is a holomorphic function, and $\tilde{w}(\tilde{z}) = \tilde{\phi}(\tilde{x}, \tilde{y}) + i\tilde{\psi}(\tilde{x}, \tilde{y})$. Hence, $\tilde{\phi}$ and $\tilde{\psi}$ satisfy the Cauchy-Riemann equations and we have

$$\frac{d\tilde{w}}{d\tilde{z}} = \tilde{u}_{\tilde{x}} - i\tilde{u}_{\tilde{y}}, \quad \tilde{u}_{\tilde{x}} = \partial_{\tilde{x}}\tilde{\phi} = \partial_{\tilde{y}}\tilde{\psi}, \quad \tilde{u}_{\tilde{y}} = \partial_{\tilde{y}}\tilde{\phi} = -\partial_{\tilde{x}}\tilde{\psi}.$$

Thus, we have an irrotational flow in the $\tilde{z}$-plane.

Two important properties of conformal mappings:

- Let z and $\tilde{z}$ be points in the z- and $\tilde{z}$-planes related by (4.24). Then $\tilde{w}(\tilde{z}) = w(z)$, i.e. $\tilde{w}$ and w (and hence $\tilde{\phi}$ and ϕ, and $\tilde{\psi}$ and ψ). This means that streamlines are mapped to streamlines, and the circulation round any closed curve in the z-plane is the same as the circulation round the corresponding closed curve in the $\tilde{z}$-plane, i.e. *the circulation is invariant under a conformal mapping.*

- Let z_0 be a point in the z-plane and $\tilde{z}$ be the corresponding point in the $\tilde{z}$-plane, and let $f^{(n)}(z_0)$ be the first non-zero derivative of $f(z)$ at z_0. Consider a small line element δz the z-plane, originating at $z = z_0$, and the corresponding line element $\delta\tilde{z}$ in the $\tilde{z}$-plane. Expanding $f(z)$ in Taylor's series, we obtain

  $$\delta\tilde{z} = \frac{(\delta z)^n}{n} f^{(n)}(z_0) + O\left(|\delta z|^{n+1}\right).$$

  Then

  $$\arg(\delta\tilde{z}) \approx n \arg(z) + \arg\left(f^{(n)}(z_0)\right). \tag{4.25}$$

Consider now two small elements δz_1 and δz_2 with a common staring point z_0, as shown in Fig. 10. Let $\delta\tilde{z}_1$ and $\delta\tilde{z}_2$ be the corresponding elements in the $\tilde{z}$-plane. Then it follows from (4.25) that

$$\arg(\delta\tilde{z}_2) - \arg(\delta\tilde{z}_1) \approx n\left(\arg(z_2) - \arg(z_1)\right). \tag{4.26}$$

In most cases, $n = 1$ and (4.26) implies the angles between the corresponding short line elements remain unchanged.

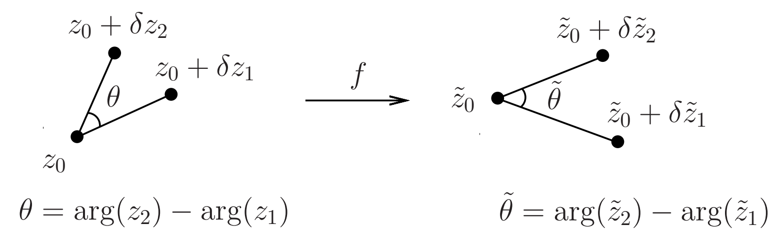


Figure 10

Flow around an elliptical cylinder. We want a conformal transformation which would map an irrotational flow around a circular cylinder to an irrotational flow around an elliptic cylinder (see Fig. 11). So, we need to make sure that it transforms a uniform flow at infinity in the z-plane into the same uniform flow at infinity in the $\tilde{z}$-plane. Let's find the velocity in the $\tilde{z}$-plane: the velocity field. We have

$$\frac{d\tilde{w}}{d\tilde{z}} = \frac{d}{d\tilde{z}}\, w(F(\tilde{z})) = \frac{dw}{dz}\frac{dz}{d\tilde{z}} = \frac{dw/dz}{d\tilde{z}/dz} = \frac{1}{f'(z)}\frac{dw}{dz}.$$

Thus, to guarantee that the velocities at infinity are the same, we must impose the condition:

$$f'(z) \to 1 \quad \text{as} \quad |z| \to \infty. \tag{4.27}$$

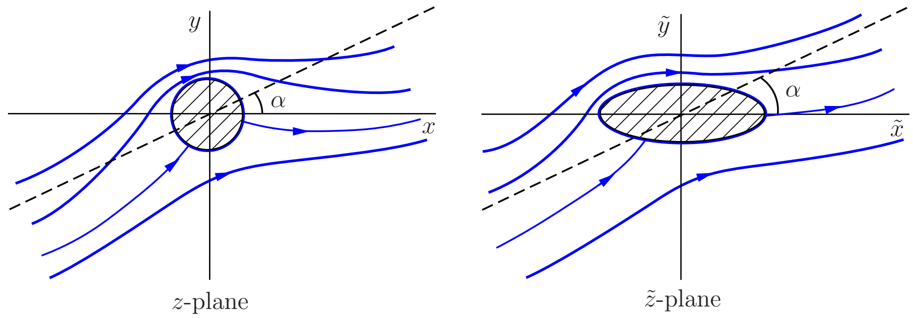
Consider the mapping (called the *Joukowski transformation*)

$$\tilde{z} = z + \frac{c^2}{z} \tag{4.28}$$

where $0 \le c \le R$. Observe that condition (4.27) is satisfied. The inverse of transformation (4.28) is given by

$$z = \frac{1}{2}\left(\tilde{z} + \sqrt{\tilde{z}^2 - 4c^2}\right). \tag{4.29}$$

This function has branching points at $\tilde{z} = \pm 2c$, and we need to make sure that it is not multi-valued. So, we cut the $\tilde{z}$-plane along the real axis between $\tilde{z} = -2c$ and $\tilde{z} = 2c$ and choose the branch of $\sqrt{\tilde{z}^2 - 4c^2}$ that behaves like $\tilde{z}$ as $|\tilde{z}| \to \infty$ (to ensure that $z \sim \tilde{z}$ for large $|\tilde{z}|$).



$$\mathbf{U} = (U_0 \cos\alpha, U_0 \sin\alpha)$$

Figure 11

Let's show that the Joukowski transformation maps a circle of radius R to an ellipse. For this circle in the z-plane, $z = Re^{i\theta}$. Then the Joukowski transformation (4.28) yields

$$\tilde{z} = R\,e^{i\theta} + \frac{c^2}{R}\,e^{-i\theta} = \left(R + \frac{c^2}{R}\right)\cos\theta + i\left(R - \frac{c^2}{R}\right)\sin\theta,$$

so that

$$\tilde{x} = \left(R + \frac{c^2}{R}\right)\cos\theta, \quad \tilde{y} = \left(R - \frac{c^2}{R}\right)\sin\theta. \tag{4.30}$$

This is a parametric equation of an ellipse. On eliminating θ, it can be rewritten as

$$\frac{\tilde{x}^2}{(R + c^2/R)^2} + \frac{\tilde{y}^2}{(R - c^2/R)^2} = 1, \tag{4.31}$$

which is the standard equation of an ellipse in the $\tilde{z}$-plane.

Substituting (4.29) into (4.18) and adding the circulation term, we obtain the complex potential of the flow past an elliptical cylinder:

$$\begin{aligned}\tilde{w}(\tilde{z}) \;=\; & U_0 e^{-i\theta}\left(\frac{\tilde{z}}{2} + \frac{1}{2}\sqrt{\tilde{z}^2 - 4c^2}\right) \\ & + U_0 e^{i\theta}\,\frac{R^2}{c^2}\left(\frac{\tilde{z}}{2} - \frac{1}{2}\sqrt{\tilde{z}^2 - 4c^2}\right) - \frac{i\Gamma}{2\pi}\ln\left(\frac{\tilde{z}}{2} + \frac{1}{2}\sqrt{\tilde{z}^2 - 4c^2}\right).\end{aligned} \tag{4.32}$$

A sketch of the streamlines for $\Gamma < 0$ is shown in Fig. 11.

5 New section (to be added)