Metric Spaces Notes

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1 suggested reading

- **Metric Spaces (Springer Undergraduate Mathematics Series)**, Mícheál Ó Searcóid [1]
- **Metric Sapces**, Satish & Harkrishan [2]
- **Guide to Analysis**, F. Mary Hart [3]
- **How to think about Analysis**, Lara Alcock [4]
- **Numbers Sequences & Series**, Kieth E. Hirst [5]

2 Introduction to Metric Spaces

2.1 Whats a Metric?

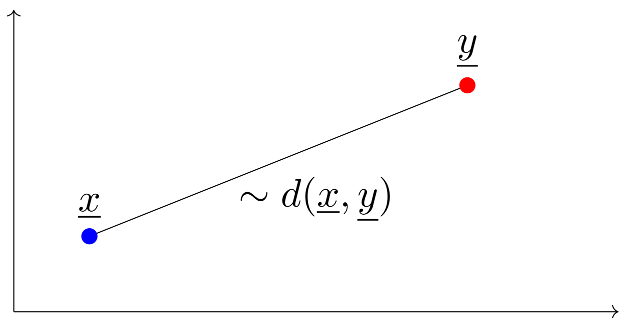
Start by using the model $\mathbb{R}^2$ wherein $(0,1) \subseteq \mathbb{R}$, we have that

$$\mathbb{R}^2 = \{(x,y)|x,y \in \mathbb{R}\}$$

Let's create a function called a "Metric" with a symbol d. Where

$$d(\underline{x},\underline{y}) = |\underline{x} - \underline{y}|, \quad \underline{x},\underline{y} \in \mathbb{R}^2$$

See that



where the essential properties of d on $\mathbb{R}^2$ are as follows:

1. $d(\underline{x},\underline{y}) \geq 0$,
2. $d(\underline{x},\underline{y}) = 0 \implies \underline{x} = \underline{y}$,
3. $d(\underline{x},\underline{y}) = d(\underline{y},\underline{x})$,
4. The distance is the shortest distance between the 2 points. So $d(\underline{x},\underline{y}) \leq d(\underline{x},\underline{z}) + d(\underline{z},\underline{y})$.

A metric d is a distance function between two points in a space. There are lots of different types of metrics for all different kinds of spaces. The standard Metric or "Euclidean Metric" for $\mathbb{R}$ is

$$d(x,y) = |x - y|.$$

2.2 Whats a Metric Space?

The abstract definition of a metric space. Let X be a non-empty set. A metric on X is a function $d : X \times X \to [0,\infty)$ satisfying:

M1) $d(x,y) \geq 0$
M2) $d(x,y) = 0 \implies x = y$
M3) $d(x,y) = d(y,x)$
M4) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)
$\forall x,y,z \in X$.

The pair (X,d) is the metric space of X with a metric d. Note that X is just a set, it needs an associated metric "d" in order to become a metric space.

2.2.1 Examples of Metric Spaces

1. The trivial metric on a set. Take $X \neq \emptyset$, define the function $\delta : X \times X \to [0,\infty)$ by

$$\delta(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

To show it is a metric on X we need only verify the axioms of a metric.

(a) Take arbitrary $x, y, z \in X$, then $\delta(x,y) \in \{0,1\}$ so **M1** holds.

(b) Suppose $x = y$, then by definition $\delta(x,y) = 0$. Furthermore if $x \neq y$ then $\delta(x,y) = 1 \neq 0$ so **M2** holds.

(c) Now take $x, y \in X$. If $x = y$ then $\delta(x,y) = 0 = \delta(y,x)$ and if $x \neq y$ then $\delta(x,y) = 1 = \delta(y,x)$ so $\delta(x,y) = \delta(y,x)$ so **M3** holds.

All that's left now is to verify the triangle inequality (often the hardest part).

(d) Take $x, y, z \in X$. Then $\delta(x,y) \leq 1$, $\delta(x,z) \leq 1$ and $\delta(z,y) \leq 1$. If $x = y$ then $\delta(x,y) = 0 \leq \delta(x,z) + \delta(z,y)$ as $\delta(x,z)$, $\delta(z,y) \geq 0$. If $x \neq y$ then $\delta(x,y) = 1$. Now z could equal x, but if it does then it cannot be equal to y and $\delta(z,y) = 1$. If $z = y$ then $z \neq x \implies \delta(x,z) = 1$. In both, $\delta(x,y) = 1 \leq 1 = \delta(x,z) + \delta(z,y)$. All that is left is if $z \neq x, y$. So $\delta(x,z) + \delta(z,y) = 2 > 1$. Therefore the triangle equality is conserved and **M4** holds.

Consider $\mathbb{R}$ equipped with the metric δ. What real numbers lie in the set $\{x \in \mathbb{R} \,|\, \delta(x,0) < 1\}$? In this event, 0 would be the only such element.

$$\{x \in \mathbb{R} \,|\, \delta(x,0) < 1\} = \{0\}.$$

What about the real numbers that lie in the set $\{x \in \mathbb{R} \,|\, \delta(x,0) \leq 1\}$? Every single real number satisfies this set!

$$\{x \in \mathbb{R} \,|\, \delta(x,0) \leq 1\} = \mathbb{R}$$

2. The standard metric on $\mathbb{R}$. The metric $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ where

$$d(x,y) = |x - y|.$$

Which follows all the axioms of a metric

(a) $d(x,y) \geq 0$ by definition **(M1)**

(b) If $x = y$ then $d(x,y) = |x - y| = 0$. Furthermore if $d(x,y) = 0$ then $|x - y| = 0 \implies x = y$. **(M2)**

(c) $d(x,y) = |x - y| = |y - x| = d(y,x)$ **(M3)**

(d) To show that $|x - y| = d(x,y) \leq d(x,z) + d(z,y) = |x - z| + |z - y|$. As we know that $d(x,y) \geq 0$, $d(x,z) \geq 0$ and $d(z,y) \geq 0$. If $x = y$ then $d(x,y) = 0 \leq d(x,z) + d(z,y)$. If $x \neq y$ then we can have $z = x$, then $|x - y| = |x - x| + |x - y| = |x - y|$, or $z = y$ and $|x - y| = |x - y| + |y - y|$. If $z \neq x, y$ then we have $|x - z| + |z - y| \geq |x - z + z - y| = |x - y|$ by the *triangle inequality*. So all cases are satisfied and the triangle inequality holds. **(M4)**

$\boxed{\text{Note: Unless otherwise stated, this is the metric used in a space. Hence it being “standard”.}}$

3. Case study in $\mathbb{R}^2$. *Used for the next few examples also!*

Three different metrics in $\mathbb{R}^2$:

(a) $d_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

(b) $d_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$

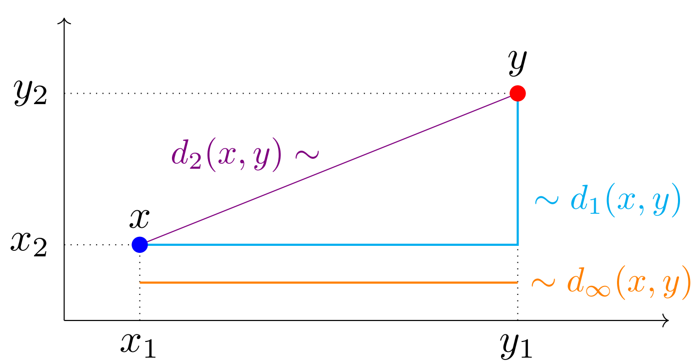
$$d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

This is the standard metric in $\mathbb{R}^2$

(c) $d_\infty : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$

$$d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

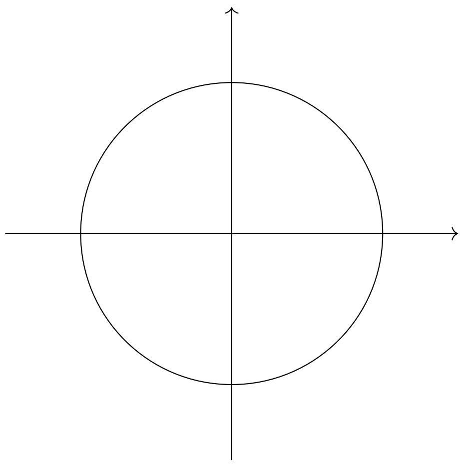
Throughout the above, we have that $x = (x_1, x_2)$ and $y = (y_1, y_2)$.



2.2.2 What is a Unit Circle?

Consider what the "Unit Circle" would look like in $(\mathbb{R}^2, d_1)$, $(\mathbb{R}^2, d_2)$ and $(\mathbb{R}^2, d_\infty)$. That is S_i^1 where $i = 1, 2, \infty$ or $S_i^1 = \{x \in \mathbb{R} \,|\, d_i(x, 0) = 1\}$:

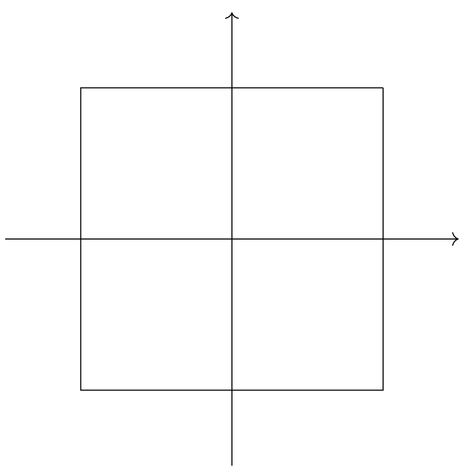
- S_2^1:



$$S_2^1 = \{x \in \mathbb{R} \,|\, d_2(x, 0) = 1\} = \{x \in \mathbb{R} : \sqrt{x_1^2 + x_2^2} = 1\}$$

The Unit circle in S_1^1 is exactly what you would expect a Unit circle to look like. Constant radius all around. This is because we're using the standard metric, which is one that we've been using for most of our academic career (Up till now of course). Note that this is called a circle, not a ball! This is important later.

- S_∞^1:



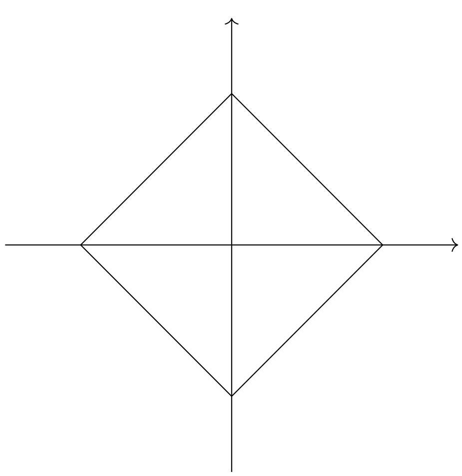
$$S_\infty^1 = \{x \in \mathbb{R} \,|\, d_\infty(x, 0) = 1\} = \{x \in \mathbb{R} : \max\{|x_1|, |x_2|\} = 1\}$$

Suppose $x = (x_1, 1)$, then

$$\begin{aligned} d_\infty(x, 0) &= \max\{|x_1 - 0|, \; |1 - 0|\} \\ &= \max\{|x_1|, 1\} \\ &= 1 \text{ if } -1 \leq x_1 \leq 1 \end{aligned}$$

This is also consistent for x_2 and is how we end up with the square shape.

- S_1^1:



$$S_1^1 = \{x \in \mathbb{R} \,|\, d_1(x, 0) = 1\} = \{x \in \mathbb{R} : |x_1| + |x_2| = 1\}$$

For $d_1(x, 0) = 1$ where $x = (x_1, x_2)$ we need $1 = x_1 + y$ which implies that $y = 1 - x_1$ where y is the absolute value.

A good way to think about how this shape occurs would be to see what values x_1 and x_2 need to take in order for the rules of the set S_1^1 to be satisfied. For example:

$$\begin{aligned} & d(x, 0) = 1 \\ \implies & |x_1| + |x_2| = 1 \end{aligned}$$

so if $x_1 = 1$ or -1, then $x_2 = 0$. And visa versa! This is where the term above arises. The equation $y = 1 - x_1$ is just another way of showing $|x_2| = 1 - |x_1|$

2.2.3 Real n-dimensional Metric

While still using the three example metrics from before, consider $(\mathbb{R}^n, d_1)$, $(\mathbb{R}^n, d_2)$ and $(\mathbb{R}^n, d_\infty)$ where $n \in \mathbb{N} \setminus \{1\}$.
So

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|,$$

$$(\star)\, d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$(\star)$ Standard or Euclidean metric on $\mathbb{R}^n$

2.2.4 Proof of a real n-dimensional metric space

For this we prove $(\mathbb{R}^n, d_2)$ is a metric space.

M1) Take $x, y \in \mathbb{R}^n$ then $d(x,y) \geq 0$ as $\sum_{i=1}^{n}(x_i - y_i)^2 \in [0, \infty) \therefore$ has a real square root.
So $\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \geq 0$ as $\sqrt{}$ is assumed to be positive.

M2) If $x = y$ then $x_i = y_i$ for $i = 1, 2, \ldots, n$. So $x_i - y_i = 0$ & $(x_i - y_i)^2 = 0$
It follows then that $\sum_{i=1}^{n}(x_i - y_i)^2 = 0$ & $\sqrt{0} = 0$.
If $d_2(x, y = 0)$ then $\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = 0$ so $\sum_{i=1}^{n}(x_i - y_i)^2 = 0$ & $(x_i - y_i)^2 \geq 0 \; \forall i \in \{1, 2, \ldots, n\}$
$\therefore (x_i - y_i)^2 = 0 \therefore x_i = y_i \; \forall i \in \{1, 2, \ldots, n\}$ Hence $x = y$.

M3) Symmetry
Take $x, y \in \mathbb{R}^2$ then $d_2(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n}(-1)^2(y_i - x_i)^2} = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = d_2(y,x)$

M4) The triangle inequality
Take $x, y \in \mathbb{R}^2$. Consider $d_2(x,z) + d_2(z,y) = \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} + \sqrt{\sum_{i=1}^{n}(z_i - y_i)^2}$
let $r_i = |x_i - z_i|$ and $s_i = |y_i - z_i|$
So $d_2(x,z) + d_2(z,y) = \sqrt{\sum_{i=1}^{n} r_i^2} + \sqrt{\sum_{i=1}^{n} s_i^2}$
We now apply a result due to *Hermann. Minkowski*, which allows us to deduce:
$d_2(x,z) + d_2(z,y) = \sqrt{\sum_{i=1}^{n} r_i^2} + \sqrt{\sum_{i=1}^{n} s_i^2} \geq \sqrt{\sum_{i=1}^{n}(r_i + s_i)^2}$
But $r_i + s_i = |x_i - z_i| + |z_i - y_i| \geq |x_i - z_i + z_i - y_i| = |x_i - y_i|$
And so $d_2(x,z) + d_2(z,y) \geq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = d_2(x,y)$.
So $d_2(x,y) \leq d_2(x,z) + d_2(z,y)$.

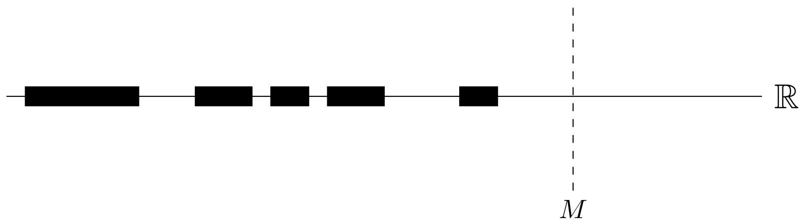
3 Functions and Analysis of Metric Spaces

Remember real-analysis?

3.1 The Real Analysis: Infima & Suprema

Consider $I = [0,1]$, $[0,1)$, $(0,1)$. Where $[0,1] = \{t : 0 \leq t \leq 1\}$, $[0,1) = \{t : 0 \leq t < 1\}$ and $(0,1) = \{t : 0 < t < 1\}$.

Take $S \subseteq \mathbb{R}$. Then S is bounded above $\iff \exists M \in \mathbb{R}$ such that $x \leq M \, \forall x \in S$



If S is bounded above by M, then any $M' > M$ is also an upper bound. The Supremum of S, $\sup S$, is the least (or smallest) upper bound for S. But what does "least" imply? If we take $\varepsilon > 0$ then $\sup(S) - \varepsilon$ is not an upper bound.[1] This means $\exists x \in S$ such that $\sup(S) - \varepsilon < x$.

Take $S = (0,1)$ then $\sup(S) = 1$ as there is an $x \in (0,1)$ between $\sup(S) - \varepsilon$ & 1. For example, take $1 - \frac{\varepsilon}{2}$.

Fact: There is no reason to assume $\sup(S) \in S$, if however $\sup(S) \in S$ then we say S has a maximal element, denoted $\max(S)$. If S has no upper bound $\exists x \in S$ for which $x > M$ for any choice of M then we define $\sup(S) = \infty$.

The set S is bounded below if $\exists m \in \mathbb{R}$ such that $m \leq x \, \forall x \in S$. The Infimum of S, $\inf(S)$ is the largest lower bound for S. Again, there is no reason for $\inf(S) \in S$, but if $\inf(S) \in S$ then S has a minimum element denoted by $\min(S)$. If no lower bound exists then $\inf(S) = -\infty$.

Take the empty set $\emptyset$; $\sup(\emptyset) = -\infty$ (The smallest upper bound) & $\inf(\emptyset) = \infty$ (The largest lower bound). The empty set is the only case wherein the supremum is less than the infimum.

But why should sup or inf even exist? The reason is that $\mathbb{R}$ has a property called completeness. $\mathbb{R}$ is complete and it satisfies the so-called completeness axiom, which states that any bounded above set in $\mathbb{R}$ which is non-empty has a supremum. This might be the case in other settings,

Take $x = \mathbb{Q}$ the rational numbers. We can construct a sequence of rationals $\frac{p_n}{q_n}$ such that $(\frac{p_n}{q_n})^2 \to 2$ as $n \to \infty$. Thus $\{\frac{p_n}{q_n} : n \in \mathbb{N}\}$ and 2 is an upper bound, but it has no supremum in $\mathbb{Q}$. This is because there is no rational number representation of $2^{1/2}$ or $\sqrt{2}$.

If you're confident with these ideas then you will find some of the later proofs easier to understand. If not, I would recommend you try a book by Alcock L. **How to Think about Analysis**[4]

3.2 Sets of sequences & Function Spaces

3.2.1 Sets of sequences

We already know that $\mathbb{R}^n$ is a finite-dimensional object, but what about $\mathbb{R}^{\mathbb{N}}$ (The set of all sequences of real numbers)? If $x \in \mathbb{R}^{\mathbb{N}}$ then $x = (x_1, x_2, x_3, \ldots, x_n, \ldots)$. We can't immediately use variants of d_1, d_2 or d_∞. Why? As an example of exactly why, let's try to apply $d_1(x, 0)$:

$$d_1(x,0) = d\Big((x_1, x_2, \ldots), (0, 0, \ldots)\Big) = \sum_{i=1}^{\infty} |x_i - 0| = \sum_{i=1}^{\infty} |x_i| = \infty?$$

There are many sequences where the evaluation of this metric is infinite, which doesn't work for us. We need the value to be a non-negative real number.

l_1 is the set of all sequences of real numbers that satisfy $\sum_{n=1}^{\infty} |x_i| < \infty \, (\star)$. Take $x = (1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots)$, then $x \in l_1$, but if $y = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots)$ then $\sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$. So l_1 is the set of all sequences satisfying $(\star)$ & $d_1(x,y) = \sum_{n=1}^{\infty} |x_n - y_n|$ is the metric.

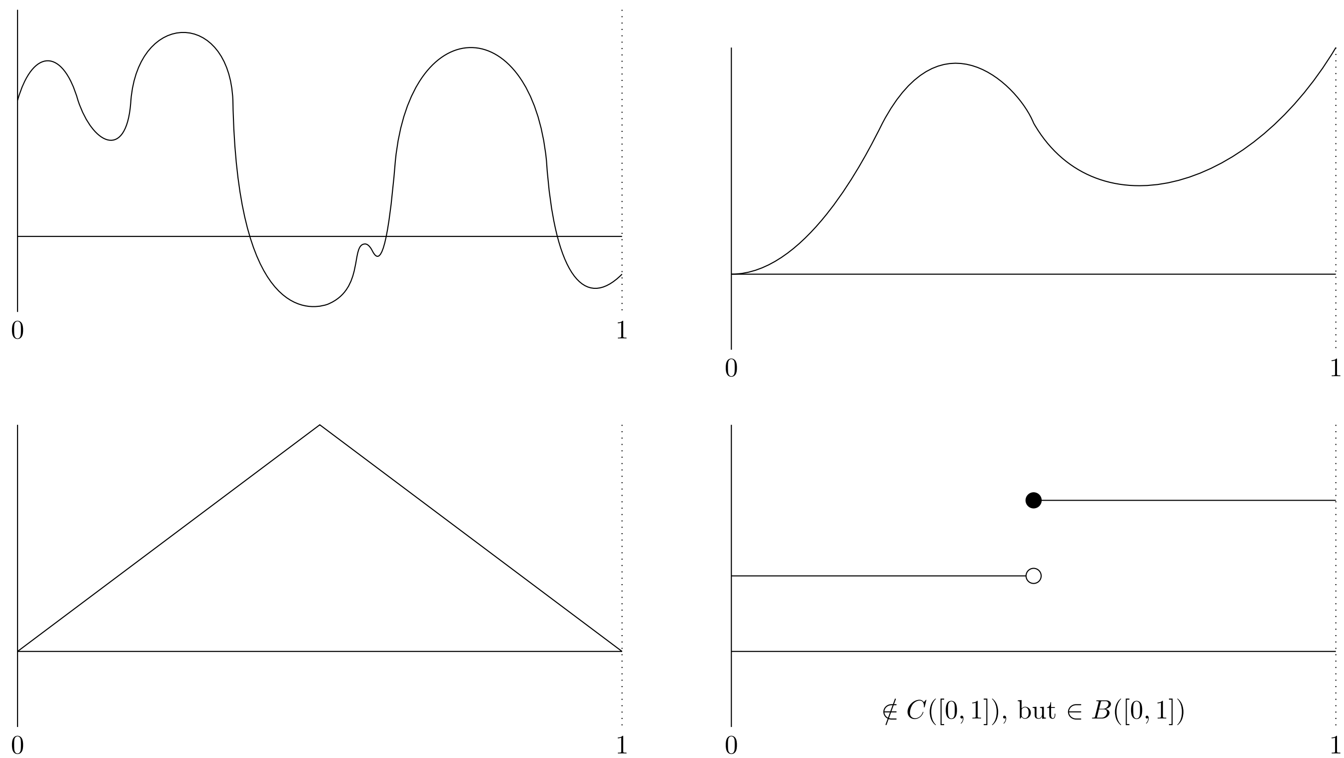
The first example of an infinite-dimensional vector space with a metric on it (A big space).

l_∞ is the set of all bounded sequences of real numbers; $x \in l_\infty \iff \exists M = M(x) > 0$ such that $|x_n| \leq M \forall n \in \mathbb{N}$, then $d_\infty(x,y) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$. Note that $x - y = (x_1 - y_1, x_2 - y_2, \ldots) \in l_\infty$.

[1] Note: $\varepsilon + \varepsilon = \varepsilon$. "Small plus small is small" - Jason

3.2.2 Function Spaces

Ideally, we would be able to apply all our theory thus far to $F(X,Y)$, the set of all functions from the set X to Y. Look at $C([0,1])$ & $B([0,1])$ where $C([0,1])$ is the set of all continuous functions $f:[0,1]\to\mathbb{R}$ & $B([0,1])$ is the set of all bounded[2] functions $f:[0,1]\to\mathbb{R}$. Let's try to draw a couple of functions:



$$f[0,1]\to\mathbb{R},\, t\to\begin{cases}1 & \text{if } t\in\mathbb{Q}\\ 0 & \text{if } t\notin\mathbb{Q}\end{cases}$$

In this case, with this particular function seen in the bottom right $f\notin C([0,1])$ however $f\in B([0,1])$. We want to be able to put a metric on these functions. We'll start with bounded functions and we'll try the sup metric across $d_\infty : B([0,1])\times B([0,1])\to[0,\infty)$ where $d_\infty(f,g)=\sup\{|f(t)-g(t)| : t\in[0,1]\}$.

For example: Take $f:[0,1]\to\mathbb{R}$ such that $f(t)=1$ & take $g:[0,1]\to\mathbb{R}$ such that $g(t)=t$,

$$\begin{aligned}d_\infty(f,g)&=\sup\{|f(t)-g(t)|\}\\&=\sup\{|1-g(t)|\}\\&=\sup\{1-g(t)\}\\&=1\end{aligned}$$

The pair $\Big(B([0,1]),d_\infty\Big)$ is a metric space. This is a function space. And it can be verified using the axioms of a metric space.

M1) f,g are bounded functions with domain $[0,1]$ and co-domain $\mathbb{R}$. So $f-g$ which is the function $f-g:[0,1]\to\mathbb{R};t\to f(t)-g(t)$ and is bounded. Here, $d_\infty(f,g)=\sup\{\underbrace{|f(t)-g(t)|}_{\geq 0}\}\leq M$ so $d_\infty(f,g)\geq 0$ $\forall f,g$.

M2) If $f=g$ then $f(t)=g(t)$ $\forall t\in[0,1]$, so $f(t)-g(t)=0$ $\forall t\in[0,1]$ thus $|f(t)-g(t)|=0$ $\forall t\in[0,1]$ and $\sup\{0\}=0$. If $d_\infty(f,g)=0 \implies \sup\{|f(t)-g(t)|\}=0$ so $|f(t)-g(t)|\leq 0$ $\forall t\in[0,1]$ But $|f(t)-g(t)|\geq 0$ so $|f(t)-g(t)|=0$. Thus $f=g$.

M3) Look at $d_\infty(f,g)=\sup\{|f(t)-g(t)| : t\in[0,1]\}=\sup\{|g(t)-f(t)| : t\in[0,1]\}=d_\infty(g,f)$

[2]Bounded $\implies \exists M>0$ such that $|f(x)|<M\forall x\in[0,1]$

M4) Take $f, g, h \in B([0,1])$, then

$$\begin{aligned}
d_\infty(f,g) &= \sup\{|f(t)-g(t)| : t \in [0,1]\} \\
&= \sup\{|f(t)-h(t)+h(t)-g(t)| : t \in [0,1]\} \\
\text{(triangle inequality)} \quad &\leq \sup\{|f(t)-h(t)| + |h(t)-g(t)| : t \in [0,1]\} \\
\text{(A property of sup)} &\leq \sup\{|f(t)-h(t)| : t \in [0,1]\} + \sup\{|h(t)-g(t)| : t \in [0,1]\} \\
&= d_\infty(f,h) + d_\infty(h,g).
\end{aligned}$$

4 Vector Spaces and the New Norm

In all the examples we have considered thus far, the metric d has involved a term of the form $|b-a|$ in some form. ie: Standard metric on $\mathbb{R}$:

$$d_2(x,y)=\sqrt{\sum_{i=1}^{n}\underbrace{(x_i-y_i)^2}_{|x_i-y_i|^2}}$$

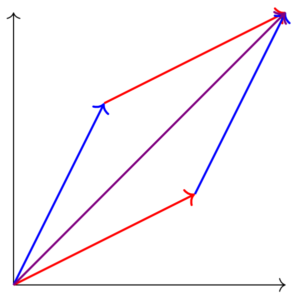
In general, a metric space need not support any algebraic structure. The definition of d on $X \neq \emptyset$ makes no mention of any such structure.

4.1 Hamming Distance

Take a finite set A, say $\{\omega_1,\omega_2,\dots,\omega_n\}$, called an alphabet & $A(N)$ is the set of all strings of length N. "string": $\omega^{(1)},\omega^{(2)},\omega^{(3)},\dots,\omega^{(n)}$. For example: $A=\{0,1\}$, $A(3)=\{000,001,010,011,100,101,110,111\}$. The Hamming distance $H : A(3)\to[0,\infty)$ is defined to be the number of places where string x differs from string y. $H(010,110)=1$.

4.2 Vector Spaces

Recall $\mathbb{R}^2$



So a vector space is basically a non-empty set V of elements that we call vectors which satisfy: $\mu\underline{u}+\lambda\underline{v}\in V$ if $\underbrace{\underline{u},\underline{v}}_{\text{Vectors}} \in V$ & $\underbrace{\mu,\lambda}_{\text{Scalars}} \in \underbrace{\mathbb{R}(\text{ or }\mathbb{C})}_{\text{Scalar Field}}$.

Abstracts $a-b$:

$$\underline{u}-\underline{v}=\underline{u}+(-1)\underline{v}$$

But we're missing the abstraction of the absolute value. The vector space version, $||\cdot||$ is called a norm.

4.3 The Norm

The definition of norm:

$$||\cdot|| : V\to[0,\infty)$$

N1) $||\underline{v}||\geq 0$ & $||\underline{v}||=0 \iff \underline{v}=0$

N2) $||\Lambda\underline{v}||=|\lambda|\cdot||\underline{v}||$ where $\lambda\in\mathbb{R}$

N3) $||\underline{u}+\underline{v}||\leq||\underline{u}||+||\underline{v}||$ where $\underline{u},\underline{v}\in V$

We call a vector space with a norm $||\cdot||$ a normed space. All normed spaces have a natural metric; called the metric induced by the norm & $d : V\times V\to[0,\infty); (\underline{u}, underlinev)\mapsto||\underline{u}-\underline{v}||=||\underline{u}+(-1)\underline{v}||$.

For example: The vector $\to$ normed spaces $\to$ metric. Chain.

Look at $C([0,1])$ the set of all continuous functions on the closed and bounded set $[0,1]$. $C([0,1])$ is a vector space. $f : [0,1]\to\mathbb{R}$ and $g : [0,1]\to\mathbb{R}$ are vectors because $f+g$ is continuous. $f+g : [0,1]\to\mathbb{R}; t\mapsto f(t)+g(t)$. Take $\lambda\in\mathbb{R}$, a scalar, then $\lambda f : [0,1]\to\mathbb{R}$ where $t\mapsto\lambda\cdot f(t)$ is continuous. But can we put norms on $C([0,1])$? Yes, yes we can.

$\mathbb{R}^2\sim d_1$, d_2, d_∞:

$$d_1(x,y)=\sum_{i=1}^{n}|x_i-y_i|$$

$$d_2(x,y)=\sqrt{\sum_{i=1}^{n}|x_i-y_i|^2}$$

$$d_\infty(x,y)=\max\{|x_i-y_i| : i=1,2\}$$

Construct 3 norms $||\cdot||_1$, $||\cdot||_2$ and $||\cdot||_\infty$ which will induce analogues of d_1, d_2 and d_∞.

$$||f||_1 := \int_0^1 |f(t)|\,dt$$
$$||f||_2 := \Big(\int_0^1 |f(t)|^2\Big)^{1/2}$$
$$||f||_\infty := \sup\{|f(t)| : t \in [0,1]\}$$

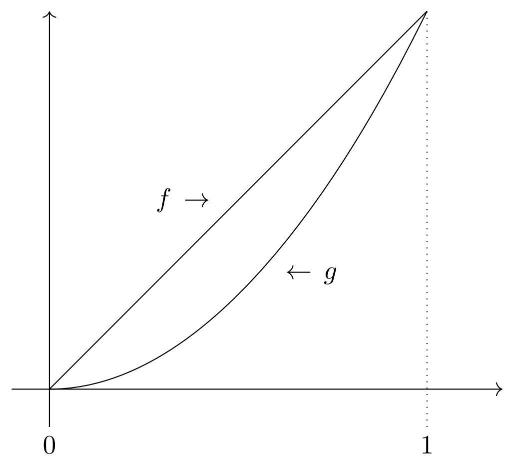
For example: Verify $||\cdot||_1$, $||\cdot||_2$ and $||\cdot||_\infty$ are norms.

Our corresponding induced metrics:

$$d_1(f,g) = ||f-g||_1 \qquad d_2 = ||f-g||_2$$

Example: Calculate the distance between $f(t) = t$ and $g(t) = t^2$ in d_2:

$$\begin{aligned} d_2^2(f,g) &= \int_0^1 |t^2 - t|^2\,dt \\ &= \int_0^1 (t-t^2)^2\,dt \\ &= \int_0^1 (t^2 - 2t^3 + t^4)\,dt \\ &= \left[\frac{1}{3}t^3 - \frac{1}{2}t^4 + \frac{1}{5}t^5\right]_0^1 \\ &= \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \\ &= \frac{1}{30} \end{aligned}$$



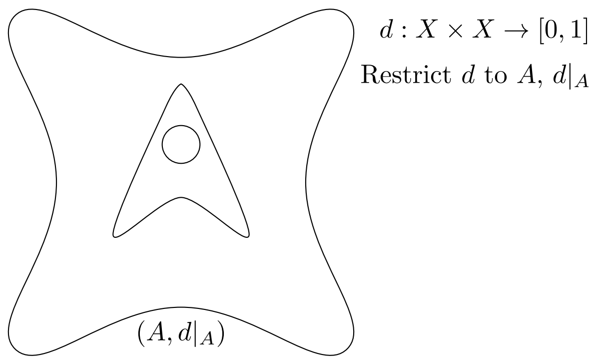
so $d_2(f,g) = \frac{1}{\sqrt{30}}$.

Just to go over some of what we've done, we've defined metrics over $\mathbb{R}^2$, namely d_1, d_2 and d_∞. We've defined metrics over $\mathbb{R}^n$. We've looked at the set of all sequences $\mathbb{R}^{\mathbb{N}} = \{(a_1, a_2, \ldots, a_n, \ldots) \mid n \in \mathbb{N}\}$ and how we could put on metrics (We'd need to impose extra restrictions!). We've got $C([0,1])$ along with our three metrics via our norm functions. If we wanted to, we could replace the closed interval $[0,1]$ with a closed interval $[a,b]$ as long as $-\infty < a < b < \infty$ and we can replace our $\mathbb{R} \in C([0,1])$ with $\mathbb{C}$, that is $f : [0,1] \to \mathbb{C}$ and abstract our definitions to cover complex functions!

5 Sub-spaces & Isometry

5.1 Sub-spaces

let $A \neq \emptyset$ be a subset of a metric space X. We can consider A to be a metric space by restricting d to A. What does this mean?

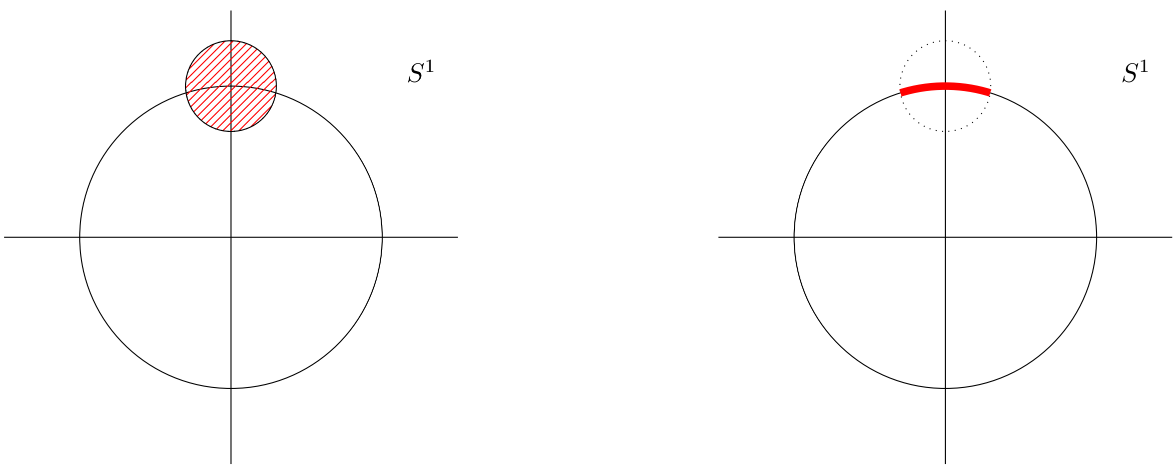


$d|_A$ is way of evaluating d only using points from A. It's what we refer to as a restriction.

Take $(\mathbb{R}, d)$ and take the subset $[0, 1]$. Consider $\Lambda = \{x \in \mathbb{R} : d(x, 1) < \frac{1}{2}\}$, consider the set of points x lying in the subspace $[0, 1]$ which are within $(1/2)$ of 1.

$$\left\{x \in A : d(x, 1) < \frac{1}{2}\right\} = \left(\frac{1}{2}, 1\right]$$

Now $(\mathbb{R}^2, d_2)$ & consider the subset $S^1 = \{(x, y) \in \mathbb{R} : (x^2 + y^2) = 1\}$:



If we restrict the area of the small circle to the big circle set or $d|_{\text{Big circle}}$, it becomes the arc shown in the second image.

5.2 & Isometry

Question: When is (X, d) the 'same' as $(X, \hat{d})$? It depends on our definition of 'same'. Perhaps when 'same' is an Isometry, it is most rigid.

> Let (X, d) be a metric space & let $A \subseteq Y$ where Y is a metric space $(Y, \hat{d})$. We say X is isometric to A if $\exists \psi : X \to A$ which is surjective & $(\star)$ $\hat{d}(\psi(a), \psi(b)) = d(a, b) \, \forall a, b \in X$ upshot $\psi(X)$ is a subspace of $(Y, \hat{d})$. We say that X and Y are isometric if $\psi(X) = Y$ and $(\star)$ holds.

The map ψ is an isometry. For example: $\psi : \mathbb{R} \to \mathbb{C}$; $t \mapsto t + 0i$. $\mathbb{C}$ is a metric space. The standard metric we can put on it is the Euclidean metric so $(\mathbb{C}, |\cdot|)$. $\psi^* : \mathbb{R} \to \mathbb{C}\,; t \mapsto t_2$ (maps onto the y-axis) is also an isometry. $\mathbb{R}^2$ is isometric to $\mathbb{C}$. Why? $\psi : \mathbb{R} \to \mathbb{C}$; $(x, y) \mapsto x + iy$. Verify the distance $|(x + iy) - (x' - iy')| = |(x - x') + i(y - y')| = \sqrt{(x - x')^2 + (y - y')^2} = |(x, y) - (x', y')|$. So as metric spaces, these two sets are the same as all the metric cares about is the distance.

6 The (Basic) Geometry of Metric Spaces

6.1 Interior, Exterior and Boundaries

Note: We can write the sets $(0,1) = \{x \in \mathbb{R} : |x - \frac{1}{2}| < \frac{1}{2}\}$ and $(a,b) = \{x \in \mathbb{R} : |x - \frac{a+b}{2}| < \frac{1}{2}\}$. Consider the open & closed interval in $\mathbb{R}$:

$$(0,1) = \{x \in \mathbb{R} : 0 < x < 1\} \qquad\qquad [0,1] = \{y \in \mathbb{R} : 0 \leq y \leq 1\}$$

Let (X,d) be a metric space. Then for any $x \in X$ and $r \in (a,b)$ the open ball centered at x of radius r which is denoted by $B(x,r)$ is the set $B(x,r) = \{y \in X : d(y,x) \leq r\}$. Analogue of $(a,b) \subset \mathbb{R}$:

$$\begin{aligned} &\text{Take } [a,b] = \{t \in \mathbb{R} : a \leq t \leq b\}, \\ &\text{then } [a,b] = \left\{ t \in \mathbb{R} : d\Big(t, \frac{a+b}{2}\Big) \leq \frac{b-a}{2} \right\}. \end{aligned}$$

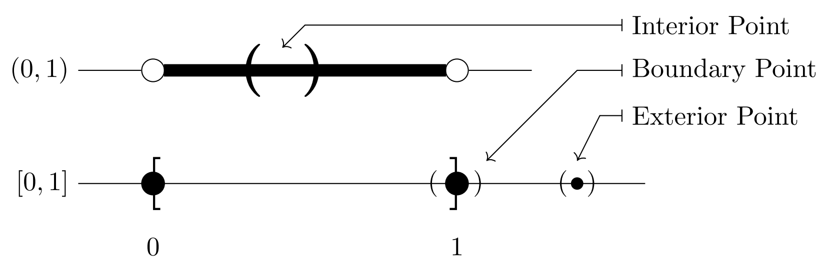
let (X,d) be a metric space & $x \in X$, $r \in (0,\infty)$. The closed ball centered x of radius r, $B(x,r)$ is the set:

$$\bar{B}(x,r) = \{y \in X : d(y,x) \leq r\}$$

But why do we care about $B(x,r)$ & $\bar{B}(y,s)$? Think about the role that (a,b) plays in analysis; Consider the continuous function f at x_0;

$f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R} \iff \forall \varepsilon > 0,\ \exists \delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.
$-\delta < x - x_0 < \delta \iff x_0 - \delta < x < x_0 + \delta \iff x \in B(x_0, \delta)$.

What is the difference geometrically?



After taking a small enough interval around x, where $x \in (0,1)$, it always stays inside the set. If we set $x = 1$ then for any interval of any length, it hits outside of the set. This is important when defining these points.

- **Interior Points**: let $A \subset X$. An interior point $y \in X$ of A is an element which $B(y,\varepsilon) \subset A$ for some $\varepsilon > 0$

  We call the set of interior points of A, the interior of A & we denote the set by A^o.

- **Boundary Points**: The element $y \in X$ is a boundary point of $A \iff$ for any $\varepsilon > 0$, $B(y,\varepsilon) \cap A \neq \emptyset$ and $B(y,\varepsilon) \cap (A^c) \neq \emptyset$. The boundary of A, denoted ∂A, is the set of all boundary points of A.

- **Exterior Points**: An element y of X is an exterior point of $A \iff \exists \varepsilon > 0$ for which $B(y,\varepsilon) \subset A^c$. The exterior of A is the set of all exterior points of A, denoted A^e.

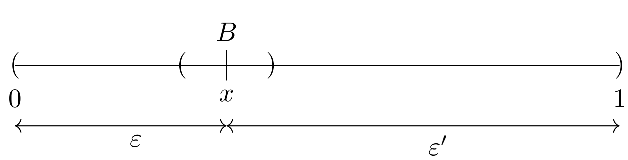
Example: Calculate A^2, ∂A and A^c
where $A = (0,1] \subset \mathbb{R}$:

$$\begin{aligned} A^e &= (-\infty, 0) \cup (1, \infty) \\ A^o &= (0,1) \\ \partial A &= \{0, 1\} \end{aligned}$$

Facts:

$$X = \underbrace{\partial A \sqcup A^e \sqcup A^o}_{\text{Disjoint Union}} \begin{cases} \partial A \cup A^e \cup A^o = X \\ \partial A \cap A^o = \emptyset \\ \partial A \cap A^e = \emptyset \\ A^o \cap A^e = \emptyset \end{cases}$$

$(\mathbb{R}, d)$ where $d(x,y) = |x - y|$. Take $x \in (0,1)$ so $0 < x < 1$.



Let $\varepsilon = x - 0$ and $\varepsilon' = 1 - x$, let $\varepsilon^* = \min\{\varepsilon, \varepsilon'\}$. Consider $B(x, \frac{\varepsilon^*}{2})$. Then $B(x, \frac{\varepsilon^*}{2}) \subset A$. **Why?** If $y \in B(x, \frac{\varepsilon^*}{2})$ then $|y - x| < \frac{\varepsilon^*}{2}$. So $-\frac{\varepsilon^*}{2} < y - x < \frac{\varepsilon^*}{2}$ and $0 < x - \varepsilon^* < x - \frac{\varepsilon^*}{2} < y < x + \frac{\varepsilon^*}{2}$. But **Why** is $0 < x - \frac{\varepsilon^*}{2}$? Because $\varepsilon = x - 0$ and $\varepsilon^* = \min\{\varepsilon, \varepsilon'\}$ so $\frac{\varepsilon^*}{2} < \varepsilon^* \leq \varepsilon$.

Thus far $(0,1) \subset (0,1]^o$. What about $x < 0$? Not possible. **Why?** See the definition of the interior point. Similarly, $x > 1$ cannot be an interior point. Any interior point is an element of the set. So $A^o \subseteq (0,1]$. What about $1 \in A^o$? No. Take $\varepsilon > 0$ and $B(1, \varepsilon)$ Then this contains a point $y > 1$. Then this contains a point $y > 1$ and $y \notin A$. Thus $A^o = (0,1)$.

$\partial A = \{0, 1\}$. Claim 0 and 1 are boundary points. **Why?** Take any $\varepsilon > 0$. Consider $B(0,\varepsilon) = B(-\varepsilon, \varepsilon)$ then $B(0,\varepsilon) \cap A \neq \emptyset$. **Why?** Take $\delta > 0$ such that $\delta > \varepsilon$ and $\delta > 1$. Further as $\exists y \in B(0,\varepsilon)$ for which $y < 0$ the intersection of $B(0,\varepsilon) \cap A^c \neq \emptyset$. Similar Argument for 1.

Fact: If $y \in \partial B$ then y might be an element of B, but it might not.

Are there any other boundary points? As $A^o \cap \partial A = \emptyset$ then no point in $(0,1)$ can be a boundary point. What about $y < 0$ or $y > 1$? No, let $y > 1$ then $\exists \varepsilon > 0$ such that $y = 1 + \varepsilon$. Take the ball $B(y, \frac{\varepsilon}{10})$ then $B(y, \frac{\varepsilon}{10}) \subset A^c$.

Fact: $\partial\emptyset = \emptyset$ and $\partial X = \emptyset$.

$(0,1)$: Open $[0,1]$: Closed $(0,1]$: Half-Closed

So let (X, d) be a metric space.

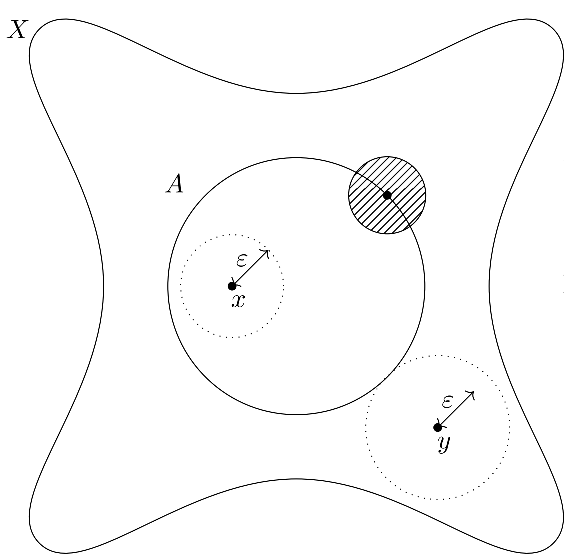
- A subset A of X is open $\iff A \cap \partial A = \emptyset$.
- A subset F of X is closed $\iff \partial F \subseteq F$.

X is Clopen. $\partial X = \emptyset$ so $\partial X \cap X = \emptyset \cap X = \emptyset$ (Open) and $\emptyset = \partial X \subseteq X$ (Closed).

Example: $(\mathbb{R}, d)$. $A = (0,1) \cup (1,2)$. As $\partial A = \{0,1,2\}$ we see that $A \cap \partial A = \emptyset$ so A is open. Further $\partial A \nsubseteq A$ so not closed. Now consider A as a subspace (A, d), $\partial A = \emptyset$, $A^o = A \implies A^e = \emptyset$. Now $(0,1) \subseteq (0,1) \cup (1,2)$ is clopen as $\partial(0,1) = \emptyset \implies \partial(0,1) \cap (0,1) = \emptyset$ (Open) and $\partial(0,1) = \emptyset \subseteq (0,1)$ thus, $(0,1)$ is closed.

6.2 Visualisation of Interior, Exterior and Boundary Points

Visualisation of A^o, A^e, ∂A in $\mathbb{R}^2$



$A^o = \{x \in X : \underbrace{x \text{ is an interior point of } A}_{\boxed{\exists}\, \varepsilon>0 \text{ such that } B(x,\varepsilon) \subseteq A}\}$

For any $\varepsilon > 0$ and any $x \in X$,

$B(x,\varepsilon) \supseteq B(x, \varepsilon')$ where $0 < \varepsilon' < \varepsilon$

$\partial A = \{y \in X : \underbrace{y \text{ is a boundary point}}_{\boxed{\forall}\, \varepsilon>0; B(y,\varepsilon)\cap A \neq \emptyset \text{ and } B(y,\varepsilon) \cap A^c \neq \emptyset}\}$

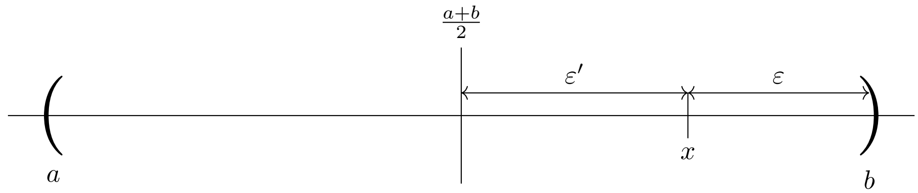
Open: $A \cap \partial A = \emptyset$, Closed: $\partial A \subseteq A$ and $\partial\emptyset = \emptyset$ and $\partial X = \emptyset$ **Fact:** If $A \subseteq X$ then $A^o \sqcup A^e \sqcup \partial A = X$. **Fact:** If $A \subseteq X$ is open $\iff A^c$ is closed. Proof:

Let's assume that $A \subseteq X$ is open. If $A = \emptyset$ then $A^c = X$ and X is closed. So we can assume that $A \neq \emptyset$. By definition, $\partial A \cap A = \emptyset$ (open), so $\partial A \subseteq A^c$. But $\partial A = \partial A^c$. As $\partial(A^c) = \{y \in X : y \text{ is a boundary point of } A^c\}$. y is such that $\forall \varepsilon > 0$, $B(y,\varepsilon) \cap A^c \neq \emptyset$ and $B(y,\varepsilon) \cap (A^c)^c \neq \emptyset$. Thus A^c is closed. Assume that A is closed. By definition, $\partial A \subseteq A$ so $\partial A \cap A^c = \emptyset$ and hence $\partial A^c \cap A^c = \emptyset$ and hence A^c is open. Thus; A is open $\iff A^c$ is closed.

6.3 Topology of a Metric Space

The topology of (X,d): So, the topology of a metric space, T_d is the collection of all open subsets of (X,d). Note: $T_d \subseteq \mathcal{P}(X) =$ (The power set of X) and $T_d \neq \emptyset$ as $\emptyset$, $X \in T_d$. (More on topology later).

Think about:



$\varepsilon + \varepsilon' = \frac{b-a}{2}$. One of ε and ε' are minimal, so without loss of generality assume $\varepsilon = \min\{\varepsilon, \varepsilon'\}$. The graph above is used to motivate $B(x,\varepsilon)$. So $B(x,\varepsilon^*) \subseteq (a,b)$ where $\varepsilon^* = \varepsilon/2$.

Fact: If $A \subseteq X$ is open, then every point of A is an interior point of A. Equivalently, For any $x \in A$, $\exists\varepsilon$ such that A (open) $\iff \varepsilon = \varepsilon(x) > 0$ such that $B(x,\varepsilon) \subseteq A$. A is open $\sim \partial A \cap A = \emptyset$ by definition.

1. $A = \emptyset$ as $\emptyset$ is open.
2. $A \neq \emptyset$. There must be at least one $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq A$ or $B(x,\varepsilon) \subseteq A^c$ for any $x \in A$. But $x \in A$ and $x \in B(x,\varepsilon)$, so $B(x,\varepsilon) \subseteq A^c$ is impossible! Therefore $B(x,\varepsilon) \subseteq A$.

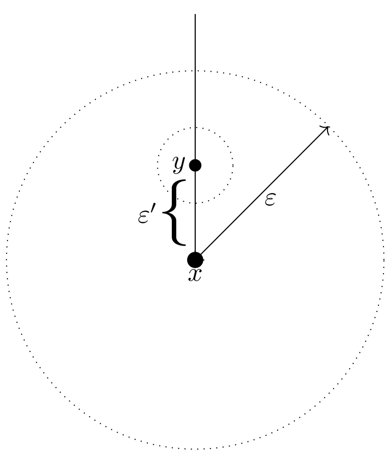
Assume that $\forall x \in A$, $\exists\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq A$. Thus $x \notin \partial A$. For $x \in \partial A$ we would need $B(x,\varepsilon) \cap A^c \neq \emptyset$ and this isn't possible.

- $A_{\text{open}} \iff A^c_{\text{closed}}$
- $A_{\text{open}} \iff \partial A \cap A = \emptyset$
- $A_{\text{open}} \iff \forall x \in A, \exists\varepsilon = \varepsilon(x) > 0$ such that $B(x\varepsilon) \subseteq A$.

T_d is the collection of all open subsets of (X,d); $\{\emptyset, X\} \subseteq T_d$.

Fact: For any $x \in X$ and any $\varepsilon > 0$, $B(x,\varepsilon) \in T_d$ (Open ball is open).

Proof:



Take $y \in B(x,\varepsilon)$. If $y = x$ we are done. Consider $B(y,\varepsilon) = B(x,\varepsilon) \subseteq B(x,\varepsilon)$.

Assume $y \neq x$. Then $d(x,y) = \Delta > 0$ and $0 < \Delta < \varepsilon$.

So we set $\varepsilon' = \min\{\Delta, \varepsilon - \Delta\}$ and $\varepsilon^* = \varepsilon'/2$. Claim: Ball $B(y,\varepsilon^*) \subseteq B(x,\varepsilon)$. We want to show that $d(x,z) < \varepsilon$ for any $z \in B(y,\varepsilon^*)$. It would follow that $z \in B(x,\varepsilon)$ and $B(y,\varepsilon^*) \subseteq B(x,\varepsilon)$. So

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ &\leq \Delta + \varepsilon^* \\ &\leq \Delta + \min\{\Delta, \varepsilon - \Delta\}/2 \\ &\leq \Delta + \frac{\varepsilon - \Delta}{2} \\ &\boxed{<} \Delta + \varepsilon - \Delta \\ &= \varepsilon. \end{aligned}$$

Thus $z \in B(x,\varepsilon) \forall z \in B(y,\varepsilon^*)$ and therefore $B(y,\varepsilon^*) \subseteq B(x,\varepsilon)$. Fact: If $A \in T_d$ & $A \neq \emptyset$ then A is a union of open balls

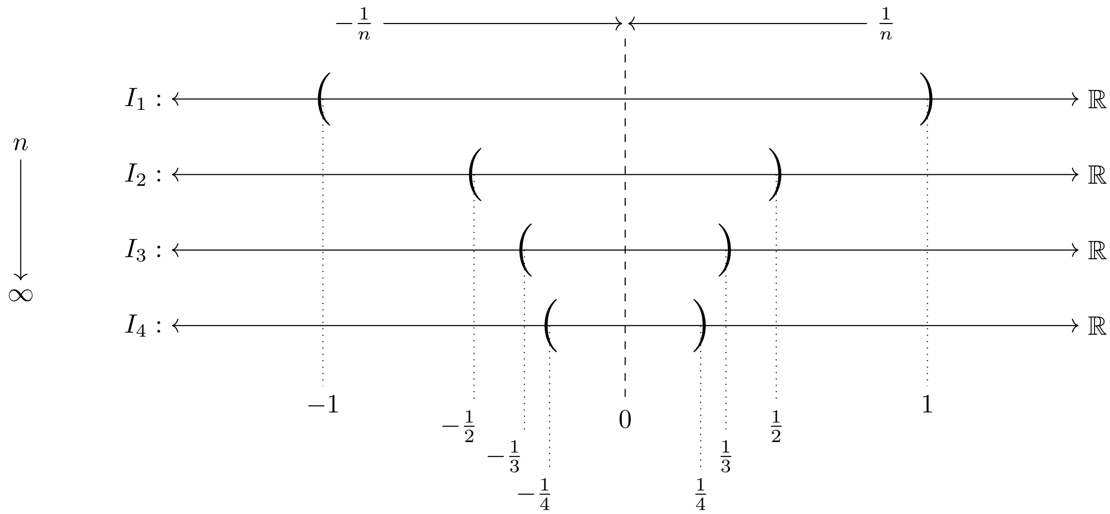
We are given $A \neq \emptyset$ and A_{open}.
Take an $x \in A$, we know $\exists\varepsilon = \varepsilon(x) > 0$ such that $B(x,\varepsilon(x)) \subseteq A$ then $A = \bigcup_{x\in A} B(x,\varepsilon(x))$.
Why? $A \subseteq \cup B(x,\varepsilon(x))$ as $x \in B(x,\varepsilon(x))$. Further $\cup B(x,\varepsilon(x)) \subseteq A$ as $B(x,\varepsilon(x)) \subseteq A$.

$\boxed{A = B\text{, can be proven showing } A \subseteq B \text{ and } B \subseteq A}$

Question: Can we now determine key properties of T_d? Yes!
Fact: Take any collection of open sets $\Lambda \in T_d$ then the union of the sets in Λ is open. For any $\Lambda \in T_d$, $\bigcup_{\Omega\in\Lambda} \Omega \in T_d$.
Take $x \in \bigcup_{\Omega\in\Lambda} \Omega$, we know $\exists\Omega(x) \ni x$ & $\Omega(x)$ is open $\implies \exists\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq \Omega(x) \implies B(x,\varepsilon) \subseteq \bigcup_{\Omega\in\Lambda} \Omega$.
Question: Unions work, but will intersections work? No! Finite intersections work, but infinite intersections do not.

Example: Take $(\mathbb{R}, d)$ and consider $I_n = (-\frac{1}{n}, \frac{1}{n})$. then $\bigcap_{n=1}^{\infty} I_n = \{0\}$ and $\{0\} \notin T_d$.

Visual Proof:



Fact: Take any finite collection of open sets $\Omega_1, \ldots, \Omega_n \in T_d$. Then $\bigcap_{i=1}^{N} \Omega_i$ is open. Why?
If $\bigcap_{i=1}^{N} \Omega_i = \emptyset$ then we're done as $\emptyset$ is open.
Assume $\bigcap_{i=1}^{N} \Omega_i \neq \emptyset$. Take any $x \in \bigcap_{i=1}^{N} \Omega_i$, then $\exists \varepsilon_i > 0$ for which $B(x, \varepsilon_i) \subseteq \Omega_i$ and take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ then $B(x, \varepsilon) \subseteq B(x, \varepsilon_1) \subseteq \Omega_i$ for $i = 1, 2, \ldots, N$ and thus $B(x, \varepsilon) \subseteq \bigcap_{i=1}^{N} \Omega_i$.

6.4 Rules of Topology

Quick recap:

- We've defined open & closed via open balls (Rather than via $\partial A \cap A = \emptyset$ (open) & $\partial A = A$ (closed))
- Derived the rules for working with the collection of all open sets, T_d: the set of all open subsets of X, the topology induced by the metric d.

$\Omega \subseteq X$, is an element of $T_d \iff \partial\Omega \cap \Omega = \emptyset \iff$ for any $x \in \Omega$, $\exists \varepsilon = \varepsilon(x) > 0$ such that $B(x, \varepsilon) \subseteq \Omega$.

Any open set is either empty or is the union of open balls. Let (X, d) be a metric space & T_d be the topology induced by d,

T1) $\emptyset, X \in T_d$

T2) T_d is closed under arbitrary unions

T3) T_d is closed under finite unions.[3]

Alternative:

T2) For any collection of open sets $\Lambda \subseteq T_d$, $\bigcup_{\Omega \in \Lambda} \Omega \in T_d$

T3) For any $\boxed{\text{finite}}$ collection of open sets $(\Omega_1, \Omega_2, \ldots, \Omega_N)$, $\bigcap_{i=1}^{N} \Omega_i \in T_d$.

Note: $\{a\} \subseteq X$ then $\{a\}$ is closed (closed: $\partial A \subseteq A$ or $A^c \in T_d$) if $X = \{a\}$ then $\{a\}$ is closed and open. So if $X = \{a\}$ then X has at least 2 elements inside. Well, $\partial\{a\} = \{a\}$ so $\partial A \subseteq A$. Why? Take any $\varepsilon > 0$, consider $B(a, \varepsilon)$ then $B(a, \varepsilon) \cap \{a\} \neq \emptyset$.

If $X = \{a, b\}$, $d : X \to [0, \infty)$ and $d(a, b) = \gamma > 0$. Then

$$B\left(a, \frac{\gamma}{2}\right) = \left\{x \in X \,\middle|\, d(a, x) < \frac{\gamma}{2}\right\} = \{a\} \qquad B\left(a, \frac{\gamma}{N}\right) = \left\{x \in X \,\middle|\, d(a, x) < \frac{\gamma}{N}\right\} = \{a\}$$

$\mathbb{Q}$-p-adic Metric where p is a prime number (this is the future):

$$d(q, q') \in \{p^{-n} \,|\, n \in \mathbb{Z}\}$$

[3]Subtle point on **(T3)**: It's possible that an infinite collection of open sets might have an open intersection. It just is not certain in general.

Question: What happens to T2) & T3) if we replace open with closed?

T2)? Take a collection of closed sets. Let $\mathcal{F}$ be an arbitrary collection of closed sets.

$$\begin{aligned} \forall F \in \mathcal{F},\ F^c \in T_d &\implies \bigcup (F^c) \in T_d \ \textbf{(T2)} \\ &\implies \bigcap (F) \text{ is closed.} \end{aligned}$$

De Morgan's Laws:

$$\left(\bigcap F\right)^c = \bigcup F^c \in T_d \implies \bigcap F \text{ is closed.}$$

Similarly:

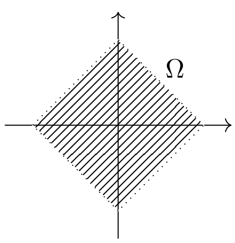
$$\left(\underbrace{\bigcap_{i=1}^{N} F_i}_{\text{closed}}\right)^c = \bigcap_{i=1}^{N} \underbrace{F^c}_{\text{open}} \in T_d$$

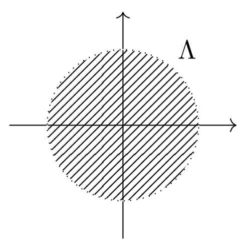
Thus:

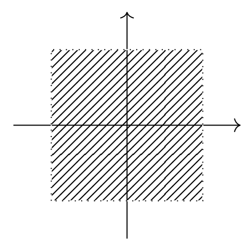
F2) The union of any finite collection of closed sets is closed.

F3) The intersection of any collection of closed sets is closed.

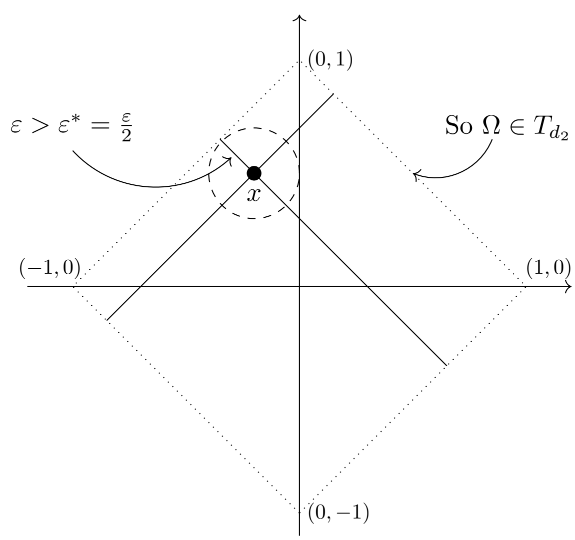
Back to $\mathbb{R}^2$:

d_1: 

d_2: 

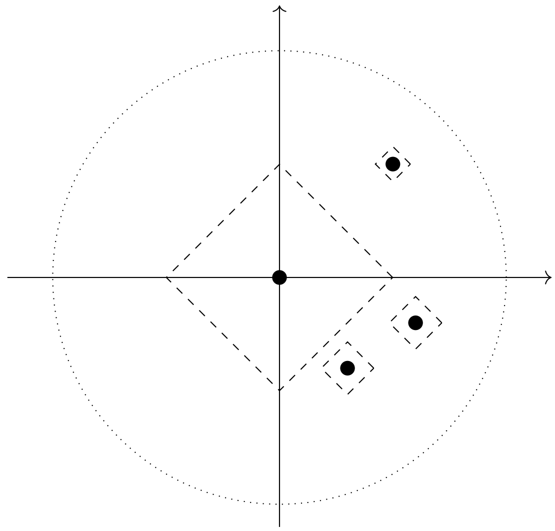
d_∞: 

What about T_1, T_{d_2} & T_{d_3}? They're all the same! Why?
Work in $(\mathbb{R}^2, d_2)$ & consider Ω. More of the 'same'?



No matter where the point x is placed, by choosing the shortest distance from the point to the boundary and setting that to ε, we can always construct an open ball $B(x,\varepsilon)$ that is within the set Ω. So $\Omega \in T_{d_2} \implies \Omega$ is open!

What about Λ in $(\mathbb{R}^2, d_1)$?



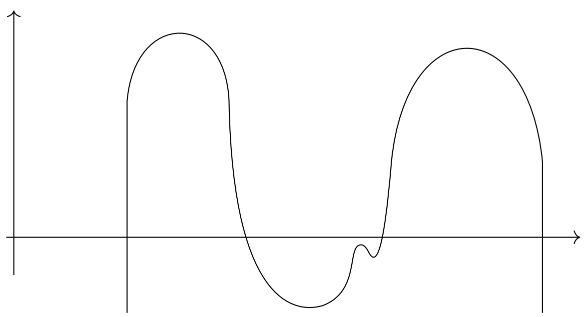
It's the same argument! $\{(x,y) \,|\, x^2+y^2 \leq 1\} = \Lambda$ is not an open ball in $(\mathbb{R}^2, d_1)$ but it is an open set.

Let d & d^* be metrics on X. Then (X,d) & (X,d^*) are equivalent $\iff T_d = T_{d^*}$.

Let X be a set & d, d^* be metrics on X such that $\exists\lambda > 0$ for which $\frac{1}{\lambda}d(x,y) \leq d^*(x,y) \leq \lambda d(x,y)$. Then $T_d = T_{d^*}$.

Goal: Take a set which is not closed and "make" it closed by adding as few points as possible.

Its maximum and its minimum on the interval $f : [a,b] \to \mathbb{R}$ where f is continuous.



Idea: Add the boundary[4] to the set.

Let (X,d) be a metric space & $A \subseteq X$. The closure of A, denoted by $\overline{A}$, is defined to be

$$\overline{A} = A \cup \partial A$$

"Fact"[5]: There is no superset of A, F such that $A \subseteq F \subset \overline{A}$ $(F \neq \overline{A})$.

Facts: $A \subset \overline{A} = A \cup \partial A$ ($\overline{A}$ is a superset of A)

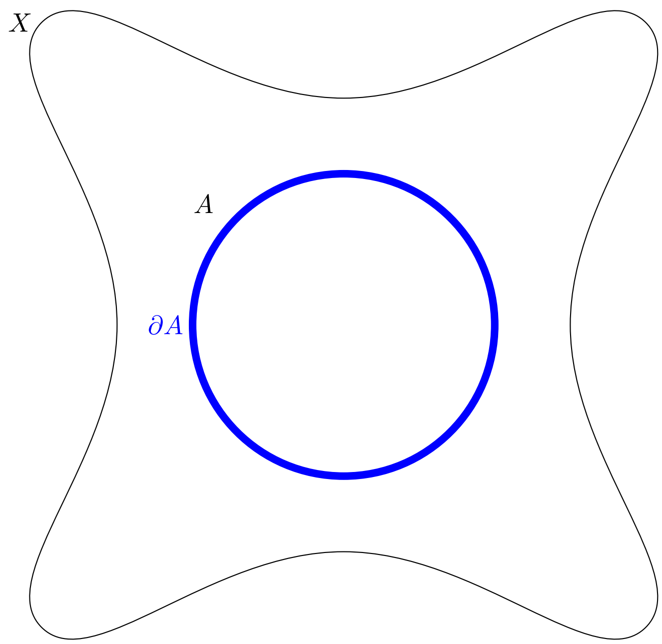
$\overline{A}$ is closed. $(\star)$

Proof of $(\star)$: Show that $(\overline{A})^c$ is open:

Consider $(\overline{A})^c = \emptyset$ then we're done as $\emptyset$ is open. Now assume $(\overline{A})^c \neq \emptyset$. To show $(\overline{A})^c$ is open we need to show that for any $x \in (\overline{A})^c$, $\exists B(x,\varepsilon) \subseteq (\overline{A})^c$. Take $x \in (\overline{A})^c$. Thus $x \notin A$ & $X \notin \partial A$ $(\overline{A} = A \cup \partial A)$. As $x \notin \partial A$ then $\exists \varepsilon > 0$ such that $B(x,\varepsilon) \subseteq A$ or $B(x,\varepsilon) \subseteq A^c$. So $B(x,\varepsilon) \subseteq A^c$

[4] Nothing less works and any more is too much!

[5] The proof for this so-called Fact comes later.



We also need to stay away from ∂A. Suppose $\exists y \in B(x,\varepsilon)$ such that $y \in \partial A$

Then $y \in \partial A$ and by definition, $\forall \delta > 0$,

$$B(y,\delta) \cap A \neq \emptyset \text{ and } B(y,\delta) \cap A^c \neq \emptyset$$

But $d(x,y) = \varepsilon^* < \varepsilon$. Consider then the ball $B(y, \frac{\hat{\varepsilon}}{2})$ where $\hat{\varepsilon} = \min\{\varepsilon^*, \varepsilon - \varepsilon^*\}$. Then the $B(y,\hat{\varepsilon}) \subseteq B(x,\varepsilon)$ but $B(y,\hat{\varepsilon}) \cap A \neq \emptyset$ which is a **contradiction** as $B(x,\varepsilon) \subseteq A^c$. We cannot simultaneously intersect with A and lie entirely within A^c.

Recall:

(X, d) & $A \subseteq X$ we associate with A, a set $\overline{A}$ known as the closure of A, where

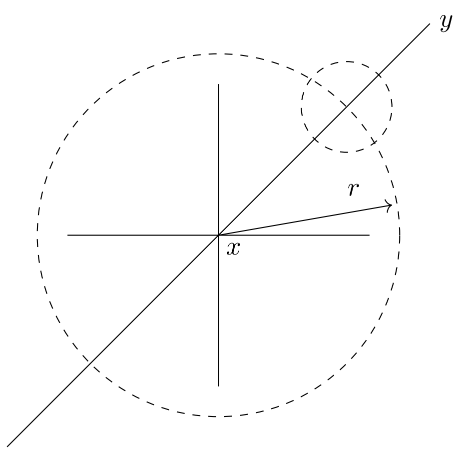
$\overline{A} := A \cup \partial A$ (∂A is the set of all boundary points of A)

$\overline{A}$ is closed. ($F \subseteq X$ is closed $\iff \partial F \subseteq F$.)

Example: In the metric space $(\mathbb{R}^n, d_2)$, where $\mathbb{R}^n$ is Euclidean n-space and $d_2(x,y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$,

$\overline{B(x,r)} = \overline{B}(x,r)$. In other words, the closure of the open ball centred at x of radius r is the closed ball.

Note: This is not true in general. $\exists (x,d)$ such that $\overline{B(x,r)} \neq \overline{B}(x,r)$



Observation: No point in $B(x,r) = \{y \in \mathbb{R}^n : d_2(x,y) < r\}$ is a boundary point; indeed they are all interior points.

Recall: ∂A, A^o & A^e are all pairwise disjoint.

Sufficient to show

$$\boxed{\partial B(x,r) = \{y \in \mathbb{R}^n : d_2(x,y) = r\} \subseteq \overline{B}(x,r) := C}$$

Straight forward to show that any $y \in \mathbb{R}^n$ for which $d_2(x,y) > r$ in in the exterior of $B(x,r)$.

We know that $\underline{r} \in Ly$ can be written in the form $\underline{r} = x + \lambda(y - x)$ where $\lambda \in \mathbb{R}$. If $\lambda \in (-1,1)$ then $\underline{r} \in B(x,r)$ & $\lambda \notin (-1,1)$ then $\underline{r} \notin B(x,r)$ & further $\underline{r}(1) = \underline{x} + 1(\underline{y} - \underline{x}) = \underline{y}$.

Let $\varepsilon > 0$ be given. Choose $\delta := \min\{\frac{\varepsilon}{2}, \frac{r}{2}\}$. Then $\underline{r}(\delta) = \underline{x} + \delta(\underline{y} - \underline{x}) \in B(x,r)$ & $\underline{r}(1+\delta) = \underline{x} + (1+\delta)(\underline{y} - \underline{x}) \in B(x,r)^c$. From this, we can deduce (by alternating δ slightly) that $B(y.\delta) \cap B(x,r) \neq \emptyset$ and $B(y,\delta) \cap B(x,r)^c \neq \emptyset$ which together define a boundary point.

Note: $\boxed{\overline{B}(x,r) = B(x,r) \cup C.}$

What we want to do is mimic closing $(0,1)$ in as 'efficient' a way as possible. (claim is to form the closure).

Fact: If $A \subseteq X$ is closed then $A = \overline{A}$ and if $A = \overline{A}$ then A is closed.

$(A \text{ is closed } \iff A = \overline{A})$

Proof: $A = \overline{A}$, $\overline{A}$ is closed (Lecture 6) $\implies$ A is closed.

A is closed $\implies \partial A \subseteq A \implies \overline{A} = A = A \cup \partial A$

What do we mean by 'efficient'? Suppose $A \subseteq F \subseteq \overline{A}$ and F are closed, then we want to show that $F = \overline{A}$.

Idea: The limiting properties of $\{0,1\}$ in relation to $(0,1)$ & $[0,1]$

$$\varepsilon_n = \frac{1}{n},\ n \in \mathbb{N}$$

Can construct a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \in (0,1)$ where $x_n \neq 0$ and $x_n \neq 1$ & $d(0, x_n) < \frac{1}{n} \to 0$ as $n \to \infty$.

Let (X,d) be a metric space & $A \subseteq X$. A **limit point**, $y \in X$, is an element of X for which $(B(y,\varepsilon) \setminus \{y\}) \cap A \neq \emptyset$ for any $\varepsilon > 0$.

The **derived set** of A, denoted by A', is the set of <u>all</u> limit points of A.

Note: $A' \cap \partial A \neq \emptyset$ is possible,

an element of A' is not always a boundary point and vice-versa

Eg: $(0,1)$: $\partial A = \{0,1\}$ but $A' = [0,1]$.

Take any $y \in (0,1)$ & any $\varepsilon > 0$, $(y-\varepsilon,\, y+\varepsilon) \supset \{y'\}$ where $y' \neq y$.

$\triangle$ Fact: $\overline{A} = A \cup A'$. $\square$ Fact: $F \subseteq X$ & F is closed. Then F contains all its limit points.

Proof of $\square$: Suppose F is closed and $\exists y$ such that $y \in F'$ but $y \notin F$

Claim: $y \in \partial F$ & this is a contradiction as $\partial F \subseteq F$ is closed by definition.

Take $\varepsilon > 0$ & consider $B(y,\varepsilon)$:

i) $\exists \underbrace{y' \in F}_{y' \neq y}$ such that $y' \in B(y,\varepsilon)$ (limit point)

ii) $B(y,\varepsilon) \ni y$ & $y \notin F$ we have $B(y,\varepsilon) \cap F^c \neq \emptyset$.

So i) & ii) $\implies$ y is a boundary point of F $\implies$ $y \in F$.

Proof of $\triangle$: If A is closed $\implies A = \overline{A} \implies A' \subseteq A = \overline{A}$ by $\square$.

Suppose A is not closed. We know $\overline{A} = A \cup \partial A$. If $A' = \emptyset$ then $A' \subseteq \overline{A}$.

Assume that $A' = \emptyset$ & further we can assume that $y \in A'$ is not in A wlog.

If $y \in A'$ & $y \in A$ then $y \in \overline{A}$ as $A \subseteq \overline{A}$. Thus $y \in \partial A$.

So for any $\varepsilon > 0$ $B(y,\varepsilon) \cap A \neq \emptyset$ & because $y \notin A \implies \exists y' \neq y$ which is in A & $B(y,\varepsilon)$.

$\therefore y$ is a limit point.

Thus $A' \subseteq \overline{A}$ & hence $A \cup A' \subseteq \overline{A}$. As A is not closed $\exists y \in A'$ such that $y \notin A$.
Take $\varepsilon > 0$ $B(y,\varepsilon) \cap A \neq \emptyset$ (limit point) & $B(y,\varepsilon) \cap A^c \ni y$ $\therefore y$ is a boundary point $\therefore \overline{A} \subseteq A \cup A'$.

Thus for

$$\begin{aligned}
A \text{ is closed} &\iff \partial A \subseteq A (\text{definition of closed}) \\
&\iff A^c \text{ is open} \\
&\iff A^c \cap \partial(A^c) = \emptyset \text{ (definition of open)} \\
&\iff A^c \cap \partial A = \emptyset \text{ as } \partial(A^c) = \partial A \\
&\iff \text{for any } x \in A^c \exists \varepsilon > 0 \text{ such that } B(x,\varepsilon) \subseteq A^c \\
&\iff A = \overline{A} \\
&\iff A \supseteq A'
\end{aligned}$$

$\overline{A} = A \cup \partial A = A \cup A'$ for any $A \subseteq X$.[6]

[6] no reason for $A' = \partial A$. They could be different.

Another version of closure; topological version: Let (X, d) be a metric space & $A \subseteq X$. Then

$$\overline{A} = \bigcap_{\mathcal{F}} F$$

where $\mathcal{F}$ is the collection of all closed supersets of A.

Let $A \subseteq X$. If $A \subseteq F \subset \overline{A}$ where F is closed, then F contains all its limit points but not all of A's limit points. Suppose that $y \in A'$ but $y \notin F$. By definition, for any $\varepsilon > 0$ $B(y, \varepsilon) \cap A \neq \emptyset$ (definition of a limit point) and contains $x \neq y$. But $x \in A \subseteq F \implies y \in F'$ but F is closed so $F' \subseteq F \implies y \in F$.

6.5 Density

Motivation: (rationals) $\mathbb{Q} \subseteq \mathbb{R}$ (reals)

Take any $x \in \mathbb{R}$ and any $\varepsilon > 0$, $\exists \frac{p}{q} \in \mathbb{Q}$ such that $|x - p/q| < \varepsilon$ ($\overline{\mathbb{Q}} = \mathbb{R}$)

Let (X, d) be a metric space & $A \subseteq X$. The A is said to be **dense** in $X \iff \overline{A} = X$. That is, for any $x \in X$ & any $\varepsilon > 0$, $\exists \alpha \in A$ such that $d(x, \alpha) < \varepsilon$.

6.6 Open & Closed in Sub-Spaces

Key Fact: open & closed do not behave 'well' in sub-spaces.

For example: $A \subseteq X$. Consider A is a sub-space, a set $\Omega \subseteq A$ could be open in A (as a sub-space) but not open in X.

We can characterise the open & closed sets in a sub-space;

- The open sets in A as a sub-space are all the sets of the form $A \cap \Omega$ where $\Omega \in T_d$ (the open sets in X).
- The closed sets in A as a sub-space are all sets of the form $A \cap F$ where $F^c \in T_d$.

7 Sequences & Convergence (in abstract metric spaces)

A **Sequence** in (X,d) is an element of $X^{\mathbb{N}}$, $x = \underbrace{(x_1, x_2, x_3, \dots, x_n, \dots)}_{\text{countable}}$ where $x_i \in X$. The notation we use for a sequence is $(x_n)_{n\in\mathbb{N}}$ or $(x_n)_{n=1}^{\infty}$. The order is important, it is not a set and nothing is stopping $x_i = x_j$ for some $i \neq j$.

Question: The main issue with sequences is that of convergence. Do we have similar convergence in (X,d)?

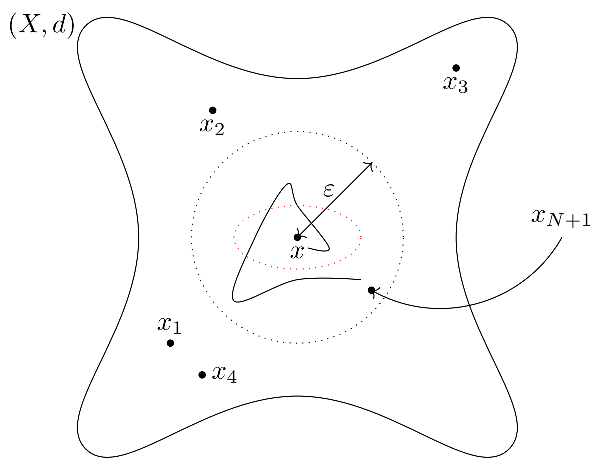
Real Analysis (Calculus): Take $(x_n)_{n=1}^{\infty}$ of real numbers. Then $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$.

for any $\varepsilon > 0$ (a tolerance) $\exists N = N(\varepsilon) > 0$ (make sure that any x_n where $n > N(\varepsilon)$ is close [within the desired tolerance to x]) such that $|x_n - x| < \varepsilon\ \forall n > N$.

Let $(x_n)_{n=1}^{\infty}$ in (X,d). Then (x_n) converges to $x \in X \iff \forall\varepsilon > 0 \exists N = N(\varepsilon) > 0$ such that $d(x_n, x) < \varepsilon \forall n > N$.
If this is the case, we write; $x_n \to x$ and we call x the limit of the sequence.
If no such x exists then we say that $(x_n)_{n=1}^{\infty}$ is divergent.

Note: The definition can be written readily in terms of open balls;

$x_n \to x$ as $n \to \infty \iff \forall\varepsilon > 0,\ \exists N > 0$ such that $x_n \in B(x,\varepsilon)\,\forall n > N$. [7]



A very useful equivalence: $x_n \to x$ as $n \to \infty \iff \underbrace{D(x_n, x) \to 0 \text{ as } n \to \infty}_{y_n = d(x_n,x)\in\mathbb{R}.(y_1,y_2,\dots)\in\mathbb{R}^{\mathbb{N}} \text{ can use calculus.}}$

Example: In $\mathbb{R}^k$ for $k \in \mathbb{N}$ with any of the metrics d_1, d_2 or d_∞ convergence is equivalent to simultaneous component wise convergence.

$$\underline{x}_n \in \mathbb{R}^k\ (n \in \mathbb{N})$$

$$\underline{x}_n \to x \text{ as } n \to \infty \iff x_j^{(n)} \to x_i \text{ as } n \to \infty \text{ for each } i \in \{1, 2, \dots, k\}.$$

where $\underline{x}_n = \left(x_1^{(n)}, x_2^{(n)}, \dots, x_k^{(n)}\right)$ and $\underline{x} = (x_1, x_2, \dots, x_k)$ where (n) is nothing but an index or label.

Prove in $(\mathbb{R}^k, d_\infty)$:

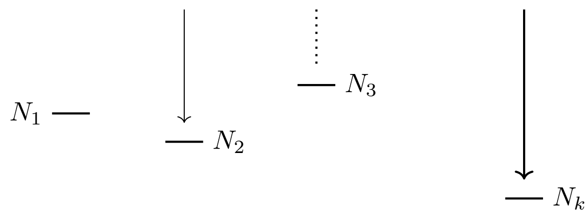
$$d_\infty(\underline{x}, \underline{y}) = \max\Big\{|x_i - y_i| : i \in \{1, 2, \dots, k\}\Big\}$$

i) Suppose that $\underline{x}_n \to \underline{x}$ as $n \to \infty$. Let $\varepsilon > 0$ be given. (Start with what you know.) Then $\exists N > 0$ such that $d_\infty(\underline{x}, \underline{y}) < \varepsilon\,\forall n > N$. That is $\max_{K i \leq k} |x_i^{(n)} - x_i| < \varepsilon\,\forall n > N$. & so $|x_i^{(n)} - x_i| < \varepsilon\,\forall n > N$ and any $i \in \{1, 2, \dots, k\}$. But this means the real sequence $\left(x_i^{(n)}\right)_{n=1}^{\infty}$ converges to x_i.

ii) For each $i \in \{1, 2, \dots, k\}$, the sequence $\left(x_i^{(n)}\right)_{n=1}^{\infty}$ converges to x_i as $n \to \infty$. We want to show that $\underline{x}_n \to \underline{x}$ as $n \to \infty$. Let $\varepsilon > 0$. Then for any $i \in \{1, 2, \dots, k\}$, $\exists N_i > 0$ such that $|x_i^{(n)} - x_i| < \varepsilon\,\forall n > N_i$. Let $N_i = \max\{N_1, N_2, \dots, N_k\}$. Then $|x_i^{(n)} - x_i| < \varepsilon\,\forall n > N$ and each $i \in \{1, 2, \dots, k\}$. At each such $n > N$ at least one of the terms $|x_i^{(n)} - x_i|$ is maximal, but this means that $d_\infty(\underline{x}_n, \underline{x}) < \varepsilon$ for each $n > N$.

[7] Note that in this we assume that x exists.

$$
\begin{array}{llllll}
\underline{x}_1 = x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \dots & x_k^{(1)} \\
\underline{x}_2 = x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & \dots & x_k^{(2)} \\
\underline{x}_3 = x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & \dots & x_k^{(3)}
\end{array}
$$



8 References

References

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