# Introduction to Applied Mathematics MAT00003C

## Part 1: Elements of Newtonian Mechanics

**Spring 2022–2023**

Sam Crawford,[1] Department of Mathematics, University of York

# Contents



[1] Based on notes lectured over several years by members of staff in the Department of Mathematics. Most recently Prof. Martin Bees

Please use the moodle discussion forum for questions on the lectures so that others can benefit.

# 1 Preliminaries

Newtonian mechanics (and more generally, Classical Mechanics) is arguably one of the biggest, if not the biggest, discoveries in science. It concerns in fundamental terms the motion of material bodies in space and time. Mechanics affects and impacts our everyday existence and has driven many ideas and branches in modern mathematics, including Differential Calculus, Integral Calculus, Calculus of Variations, Differential Equations, Differential Geometry, and much more. Therefore, it is essential that every mathematician is familiar with its concepts and tools. This is our focus in this first part of Introduction to Applied Mathematics.

Figure 1.1: Ski jump, Salt Lake City - view from the top. How far is it possible to jump?

Source: Reywas92; Creative Commons 3.0; https://commons.wikimedia.org/wiki/File:Ski_Jump_View_From_Top.JPG

**The syllabus includes the following topics:**

- Fundamental kinematical concepts: position, velocity, acceleration. (Examples include the inclined plane.)
- Newton's laws of motion.
- Conservation laws: energy and momentum. (Examples include skiflying.)
- Forces in one and two dimensions. (Examples include uniform gravity, Hooke's law, non-conservative systems - models of friction, and motion in a nonlinear potential.)
- Polar coordinates. (Examples include planetary motion.)

## 1.1 Dimensions and units

We start with the general notions of a **physical dimension**. (It is relevant not only to mechanics but also to any other branch of science and engineering.)

**Definition 1.1** **Dimensions**

Any physical quantity has a **physical dimension**. Notation: $[u]$ denotes the physical dimension of the physical quantity $u$. For example: [linear position] $= L$ (**length**), [mass] $= M$ (**mass**), [time] $= T$ (**time**), [velocity] $= L/T$ (**length divided by time**), [area] $= L^2$ (*length* *squared*). Note that **real numbers are dimensionless quantities**, i.e. the physical dimension of a real number is 1.

In mechanics, there are three fundamental dimensions: length, mass and time. These are denoted as $L$, $M$ and $T$, respectively. All other mechanical quantities can be expressed in terms of the fundamental dimensions, e.g. the physical dimension of the velocity, $v$, is length divided by time: $[v] = L/T$.

**Definition 1.2** Units Any physical quantity that can be measured needs units of measurement. For example, to measure a distance between two points in space, we need units of length such as meters, inches or miles. We shall use the International System of Units (SI). It uses *meters* (m) to measure length, *seconds* (s) to measure time and *kilograms* (kg) for mass.

It is important to understand the difference between physical dimensions and units: a physical quantity has one well-defined physical dimension, but it can be measured using many different units, e.g. distance (whose physical dimension is $L$) can be measured using meters, miles, light years, *etc.*

Physical dimensions obey certain rules (which come from common sense). We can say that a physical quantity $A$ is equal to another physical quantity $B$ only if the physical dimensions of $A$ and $B$ are the same. We cannot compare two quantities having different physical dimensions, e.g. the statement "$X\,kg = Y\,m$" for some numbers $X$ and $Y$ makes no sense as something measured in kilograms cannot be equal to anything measured in meters.

The above discussion can summarised as the following rules ("axioms").

1. Real numbers have a physical dimension of 1 (i.e. real numbers are dimensionless).
2. For any two physical quantities, $A$ and $B$,
$$A = B \quad \Rightarrow \quad [A] = [B].$$
3. For any three physical quantities, $A$, $B$ and $C$,
$$A \pm B = C \quad \Rightarrow \quad [A] = [B] = [C].$$
4. For any two physical quantities, $A$ and $B$,
$$[A \cdot B] = [A] \cdot [B] \quad \text{and} \quad [A/B] = [A]/[B].$$

Note that rules 1 and 4 imply that $[\lambda A] = [A]$ for any real number $\lambda$ and any physical quantity $A$. For example, $[2A] = [A]$.

**Example 1.1** Let $x(t)$ be the distance (at time $t$) of a moving car from a fixed point on the road. If $x(t)$ is given by

$$x(t) = \alpha\, t^2 + \beta\, t + \gamma\, e^{-\lambda t}$$

for come constants $\alpha$, $\beta$, $\gamma$ and $\lambda$. What are physical dimensions of these constants?

**Solution.** It follows from rule 3 that

$$[\alpha\, t^2] = [\beta\, t] = [\gamma\, e^{-\lambda t}] = [x] = L.$$

Using rule 4, we obtain

$$[\alpha] \cdot [t]^2 = L \quad \Rightarrow \quad [\alpha] \cdot T^2 = L \quad \Rightarrow \quad [\alpha] = \frac{L}{T^2}.$$

Similarly,

$$[\beta] = \frac{L}{T}.$$

The argument of exponential function must be dimensionless, i.e.

$$[\lambda t] = 1 \quad \Rightarrow \quad [\lambda] \cdot T = 1 \quad \Rightarrow \quad [\lambda] = \frac{1}{T}.$$

(The same is true for other functions such as sin, cos, tan, sinh, cosh, tan, etc. ) Also, $e^x$ for any real $x$ is a real number. Therefore,

$$[\gamma] \cdot [e^{-\lambda t}] = L \quad \Rightarrow \quad [\gamma] = L.$$

Physical dimensions will also be discussed in seminars and in the second part of the module.

## 1.2 Kinematics in 1D

### Coordinate systems

Consider a particle moving along a straight line (Fig. 1.2).

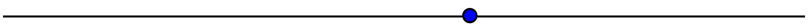


Figure 1.2: A particle on a straight line: a mathematical point, the simplest object.

Figure 1.3: Grass-covered tram line in Freiburg i. B., Germany

Source: CrazyD; CC BY-SA 3.0; https://commons.wikimedia.org/wiki/File:Combino_VAG_auf_Rasengleis.jpg

In this module part, we shall use the following simplification. Any material object whose motion is studied will be represented by a point (a particle). This is a reasonable assumption when we are interested in motion of the object as a whole over distances greater than its size. (But things get much more interesting when we allow rotation, *etc*! Gyroscopes and rattlebacks, for example.)

To describe its motion mathematically, we need a coordinate system on the line. First, we choose an arbitrary point $O$ on the line and call it the *origin*.

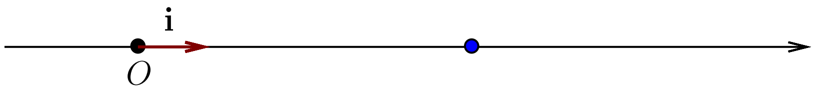



Figure 1.4: Coordinate system with an *origin* $O$, a *coordinate axis*, and a *unit vector* $\mathbf{i}$

Second, we introduce the coordinate axis along the line of motion and choose its positive direction (indicated by an arrow in Fig. 1.4. Finally, we define a unit vector $\mathbf{i}$, whose direction is the same as the positive direction of the coordinate axis. ($\mathbf{i}$ is a unit vector, which means that its magnitude equals 1: $|\mathbf{i}| = 1$).

**Definition 1.3** **Position vector**

We shall describe motion of the particle by a **position vector** $\mathbf{x}(t)$ shown in Fig. 1.5. If we know the position vector at all moments of time, we know everything about the motion of the particle. With the help of the unit vector $\mathbf{i}$, the position vector $\mathbf{x}(t)$ can presented in the form

$$\mathbf{x}(t) = x(t)\mathbf{i},$$

where $x(t)$ is a scalar function called the **coordinate** or the **position** or the **component** of the position vector of the particle.

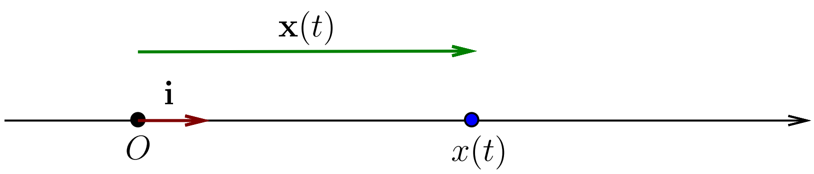



Figure 1.5: The position vector

Note that the coordinate of the particle, $x(t)$, can be **positive, negative or zero**. If for some fixed time $t$, $x(t) > 0$, then the direction of the position vector is the same as the positive direction of the coordinate axis. If for some fixed time $t$, $x(t) < 0$, then the direction of the position vector is opposite to the positive direction of the coordinate axis.

**Remark** It is important to distinguish **vector quantities** from **scalar ones**. We shall use the following convention: vectors quantities are presented as boldface type symbols in printed texts and as underlined symbols in hand-written texts, e.g. both $\mathbf{v}$ and $\underline{v}$ denote a vector quantity, while $v$ represents a scalar quantity. **You will lose marks in the exam if you get this wrong!**

**Definition 1.4** **Displacement vector**

Consider now two consecutive moments of time, $t$ and $t + \Delta t$. We define the **displacement**

**vector** on the interval $[t, t + \Delta t]$ as (see Fig. 1.6)

$$\Delta\mathbf{x} = \mathbf{x}(t + \Delta t) - \mathbf{x}(t).$$

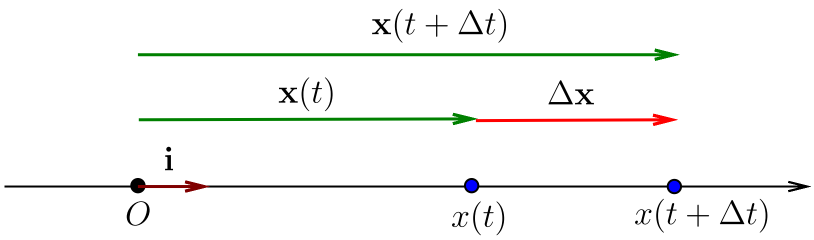



Figure 1.6: The displacement vector

Again, using the unit vector $\mathbf{i}$, we can write the displacement vector in the form

$$\Delta\mathbf{x} = \mathbf{x}(t + \Delta t) - \mathbf{x}(t) = (x(t + \Delta t) - x(t))\,\mathbf{i} = \Delta x\,\mathbf{i},$$

where $\Delta x$ is the **component** of the displacement vector. It is a scalar quantity and can be positive, negative or zero. If $\Delta x > 0$, then the particle has moved in the positive direction of the coordinate axis over the time interval $[t, t + \Delta t]$. If $\Delta x < 0$, then the particle has moved in the negative direction of the coordinate axis.

**Definition 1.5** **Average velocity**

Average velocity on the interval $[t, t + \Delta t]$ is defined as

$$\mathbf{v}_A = \frac{\Delta\mathbf{x}}{\Delta t} = \frac{\Delta x}{\Delta t}\,\mathbf{i} = v_A(t)\,\mathbf{i},$$

where $v_A(t)$ is the **component** of the average velocity vector (or simply the average velocity). Scalar $v_A(t)$ can be positive, negative or zero.

Geometric meaning of the average velocity: $v_A(t)$ is the slope of the line connecting two points of the graph of $x(t)$ corresponding to $t$ and $t + \Delta t$ (see Fig. 1.7).

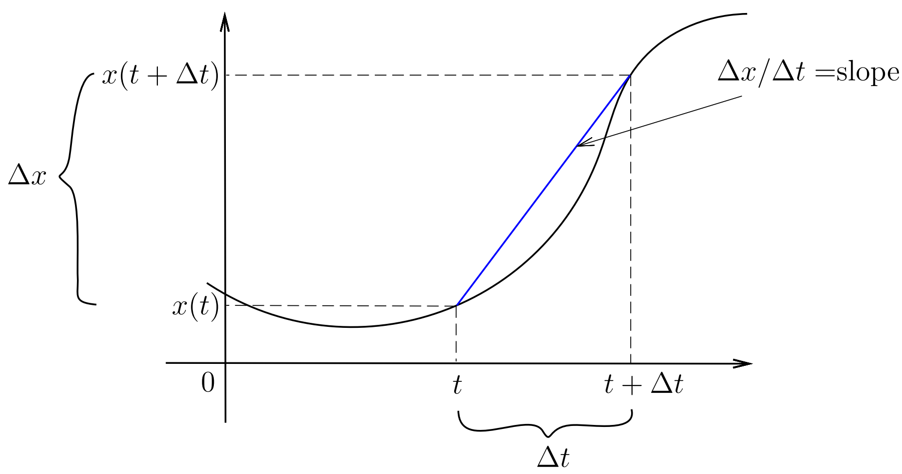



Figure 1.7: Average velocity

**Definition 1.6** **Instantaneous velocity**

We define the **instantaneous velocity** at time $t_1$ as

$$\mathbf{v}(t_1) = \lim_{\Delta t \to 0} \frac{\mathbf{x}(t_1 + \Delta t) - \mathbf{x}(t_1)}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t}\,\mathbf{i} = \frac{dx}{dt}(t_1)\,\mathbf{i} = v(t_1)\,\mathbf{i},$$

where $v(t_1) = dx/dt(t_1)$ is called the **component** of the instantaneous velocity at time $t_1$ or simply the velocity at time $t_1$ (which can be positive, negative or zero).

Geometric meaning of the instantaneous velocity: $v(t_1)$ is the **slope of the tangent line** to the graph of $x(t)$ at $t = t_1$ as shown in Fig. 1.8).

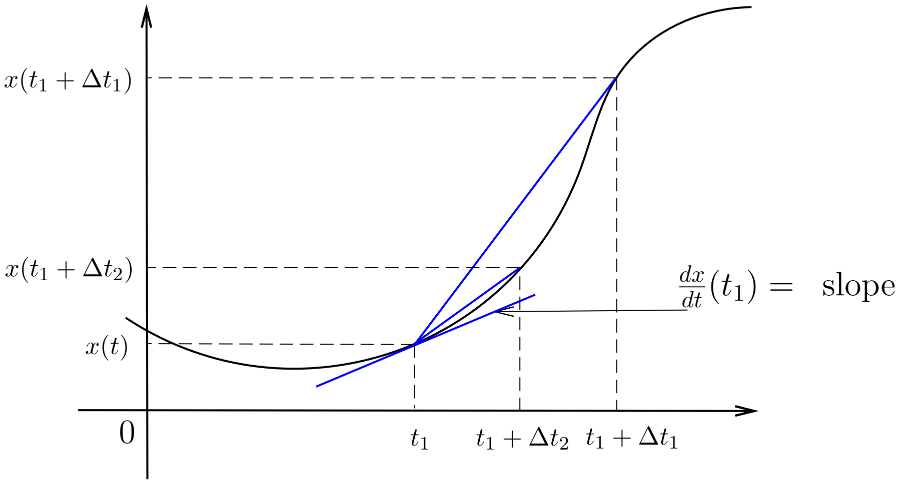



Figure 1.8: Instantaneous velocity

We have defined the velocity at time $t_1$. The same can be done for any moment in time. So, the instantaneous velocity at any time $t$ is

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt}(t) = \frac{dx}{dt}(t)\,\mathbf{i} = v(t)\,\mathbf{i}.$$

Physical meaning of the velocity: **the velocity of the particle is the rate of change of its position.**

**Definition 1.7** **Instantaneous acceleration**

The instantaneous **acceleration** $\mathbf{a}(t)$ can be defined in a similar manner as

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt}(t) = \frac{dv}{dt}(t)\,\mathbf{i} = a(t)\,\mathbf{i} \quad \text{or, equivalently,} \quad \mathbf{a}(t) = \frac{d^2\mathbf{x}}{dt^2}(t) = \frac{d^2x}{dt^2}(t)\,\mathbf{i} = a(t)\,\mathbf{i}.$$

Also, if $\mathbf{v} = v(x(t))\,\mathbf{i}$ then $\mathbf{a} = \dfrac{dv}{dx}\dfrac{dx}{dt}\,\mathbf{i} = \dfrac{dv}{dx}v\,\mathbf{i}$ (using the chain rule).

Physically, **the acceleration of the particle is the rate of change of its velocity.**

**Example 1.2** The position at time $t$ of particle moving on a straight line is given by

$$x(t) = \alpha\, t^2 + \beta\, t + \gamma\, e^{-\lambda t}$$

for some constants $\alpha$, $\beta$, $\gamma$ and $\lambda$. Find its velocity and acceleration. Check whether your answer is dimensionally correct.

**Solution.** The velocity and acceleration are

$$v(t) = \frac{dx}{dt}(t) = 2\alpha\, t + \beta - \lambda\gamma\, e^{-\lambda t}$$

and

$$a(t) = \frac{dv}{dt}(t) = 2\alpha + \lambda^2\gamma\, e^{-\lambda t}.$$

We shall only check that the answer for the acceleration is dimensionally correct. We need to show

$$[a(t)] = \left[2\alpha + \lambda^2\gamma\, e^{-\lambda t}\right].$$

On the left hand side, we have (using 'axioms' 4 and 3)

$$[a] = \left[\frac{dv}{dt}\right] = \frac{[v]}{[t]} = \frac{L/T}{T} = \frac{L}{T^2}.$$

On the right hand side, we obtain (using 'axioms' 1, 3 and 4)

$$[\alpha] = [\lambda]^2\,[\gamma] = [a] = \frac{L}{T^2}. \tag{1}$$

From Example 1, we know that

$$[\alpha] = L/T^2, \quad [\lambda] = 1/T \quad \text{and} \quad [\gamma] = L.$$

Therefore, Eq. (1) is satisfied, and our answer for the acceleration is dimensionally correct. The same analysis can done for the velocity.

## 1.3 Motion on an inclined plane

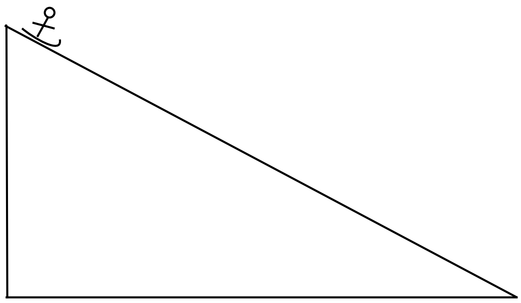


Figure 1.9: Downhill skiing - inclined planes.

LHS source: public domain image.

**Physical laws are revealed as a result of observations and experiments.** So, let us consider some 'experimental data' obtained from observing a body sliding down an inclined plane. (This is sometimes called Gallileo's inclined plane experiment, although there is no evidence that Gallileo Gallilei had actually performed the experiment.) We will see how the experimental data lead to a differential equation that describes motion on the inclined plane.

So what do we want? - a **quantitative** description of the motion. So we need measurements of the following quantities:

| Physical quantity | Physical dimension | Physical units (SI) |
|---|---|---|
| the mass $m$ of the athlete | [m]=M | kg |
| the distance travelled $d$ | [d]=L | m |
| the time elapsed $t$ | [t]=T | s |

First, we introduces the coordinate axis as shown in Fig. 1.10. Using this coordinate axis, we will be able to determine the position of a particle at a given moment of time.

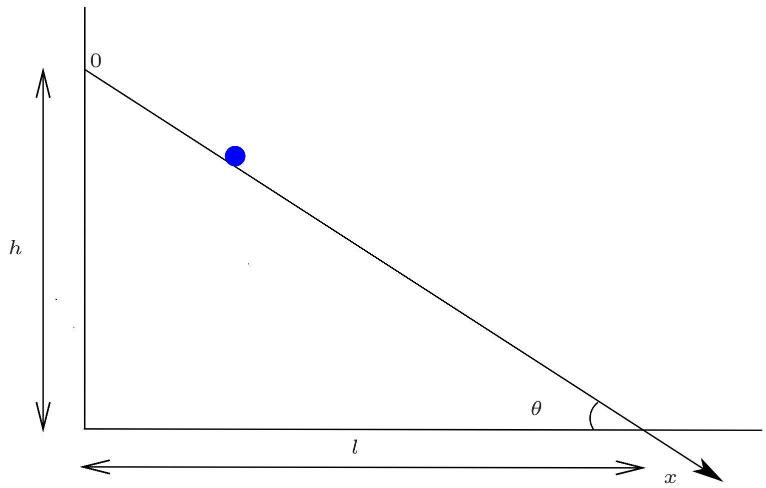



Figure 1.10: Coordinates on the inclined plane

Suppose that a body is moving down the inclined plane with the angle of inclination $\theta_0 = \pi/6\,(= 30°)$. We measure its position $x(t)$ at various times $t$. It is natural to expect that

$$x(t) = f(m, \theta_0, t).$$

We perform three experiments with bodies of different mass

$$m = 0.1\,kg, \quad m = 0.5\,kg, \quad m = 1\,kg.$$

The experimental data are shown is the table:

| $t$ (measured in $s$) | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | mass (kg) |
|---|---|---|---|---|---|---|---|
| $x$ (measured in $m$) | 0 | 0.6 | 2.5 | 5.6 | 10 | 15.6 | 0.1 |
| $x$ (measured in $m$) | 0 | 0.6 | 2.5 | 5.6 | 10 | 15.6 | 0.5 |
| $x$ (measured in $m$) | 0 | 0.6 | 2.5 | 5.6 | 10 | 15.6 | 1.0 |

The data show that the motion of a body on the inclined plane does not depend on its mass, i.e.

$$x(t) = f(\theta_0, t).$$

What else can be deduced from the data?

We can compute

- the average velocity (the rate of change of position): $\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x(t+\Delta t)-x(t)}{\Delta t}$ with dimension $[\bar{v}] = \left[\frac{\Delta x}{\Delta t}\right] = \frac{[x]}{[t]} = \frac{L}{T}$,
- the average acceleration (the rate of change of velocity): $\bar{a} = \frac{\Delta \bar{v}}{\Delta t} = \frac{\bar{v}(t+\Delta t)-\bar{v}(t)}{\Delta t}$ with dimension $[\bar{a}] = \left[\frac{\Delta \bar{v}}{\Delta t}\right] = \frac{[x/t]}{[t]} = \frac{L}{T^2}$.

The results are shown in the table:

| $t$ (measured in $s$) | 0 | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 | 2.25 | 2.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ (measured in $m$) | 0 | | 0.6 | | 2.5 | | 5.6 | | 10 | | 15.6 |
| $\bar{v}$ (measured in $m/s$) | | 1.2 | | 3.8 | | 6.2 | | 8.8 | | 11.2 | |
| $\bar{a}$ (measured in $m/s^2$) | | | 5.2 | | 4.8 | | 5.2 | | 4.8 | | |

The graphs of the experimental data are shown in Fig. 1.11

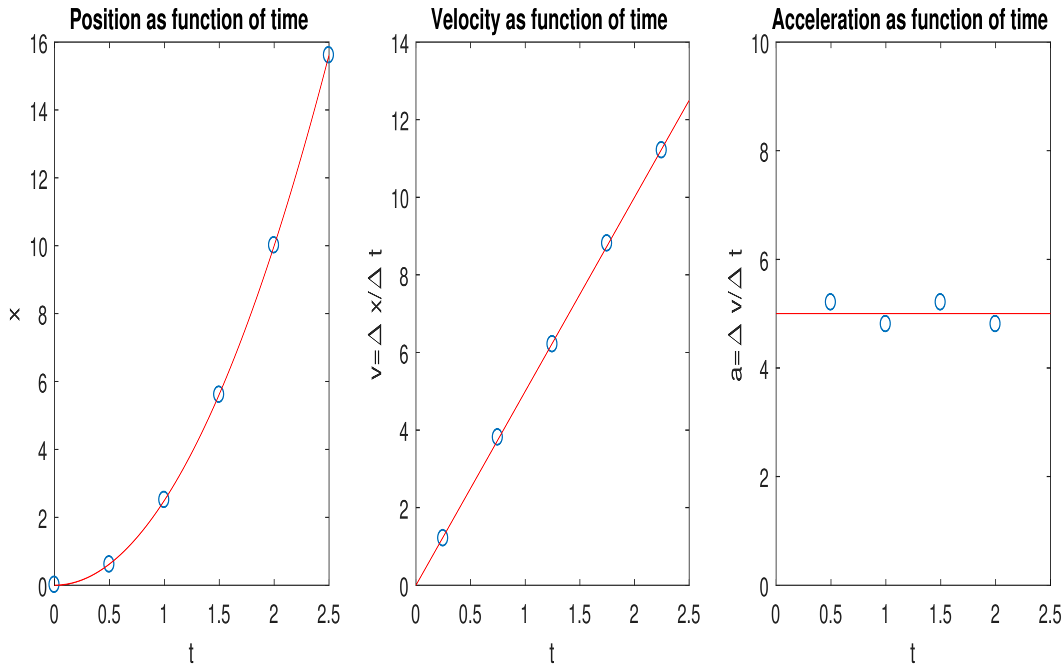



Figure 1.11: Experimental data plots.

Inspecting these graphs, we conclude that

- $x(t)$ looks like a parabola: $x(t) \sim t^2$.
- $v(t)$ looks like a straight line: $v(t) \sim t$.
- $a(t)$ looks like a horizontal line: $a(t) \sim \text{const} = 5\, m/s$.

[But note, in general, that acceleration can not be assumed constant.]

The experimental data suggest that

- $\frac{d^2x}{dt^2}(t) = g(\theta_0)$ with $g(\theta_0) = 5\, m/s^2$
- $\frac{dx}{dt}(t) = g(\theta_0)\, t$ since $\frac{dx}{dt}(t)|_{t=1s} \simeq 5\, m/s$
- $x(t) = \frac{1}{2}\, g(\theta_0)\, t^2$ since $x(t)|_{t=1s} \simeq 2.5\, m$

Note that $x(t)$ (position as a function of time) is a solution of the linear second-order differential equation (DE):

$$\frac{d^2x}{dt^2}(t) = g(\theta_0).$$

The above experiment was done for inclination angle $\theta = \pi/6$. If we repeat our experiments with $\theta = \pi/4$, $\theta = \pi/3$ and $\theta = \pi/2$, we'll find that in each experiment the acceleration $\Delta v/\Delta t$

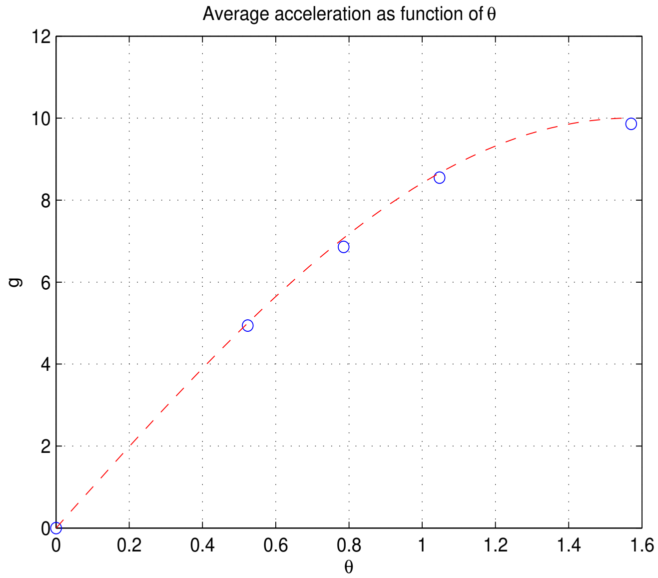



Figure 1.12: Experimental data plots.

is well approximated by a constant, but its magnitude depends on $\theta$. The experimental data are shown in Fig. 1.12. One can see that the maximum acceleration is approximately $10\,m/s^2$ and that the maximum is attained at $\theta = \pi/2$ (i.e. when we have a vertical plane).

Since the data are well approximated by the function $\phi(\theta) = 10\sin\theta$ (the red dashed curve in Fig. 1.12), we conclude that

$$\frac{d^2x}{dt^2}(t) \simeq g(\theta) = g_{max}\sin\theta \simeq 10\sin\theta\;\; m/s^2.$$

**Summary of our discussion of motion on an inclined plane**:

We have found that

- position as a function of time is $x(t) = \frac{1}{2}\,g\sin\theta\,t^2$,
- $x(t)$ satisfies the DE:

$$\frac{d^2x}{dt^2}(t) = g\sin\theta$$

where $g = 9.81\,m/s^2$ is the free fall acceleration.

## 1.4 Ordinary Differential Equations (ODEs): a reminder

In this section, we briefly summarize the terminology we shall use and some basic facts about 1st- and 2nd-order ODEs.

**Notation.** From now on, we shall use the following notation:

$$\dot{x}(t) = \frac{dx}{dt}(t), \quad \ddot{x}(t) = \frac{d^2x}{dt^2}(t), \quad \dot{\mathbf{x}}(t) = \frac{d\mathbf{x}}{dt}(t) = \dot{x}(t)\,\mathbf{i}, \quad \text{etc.}$$

and

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}}{dt}(t) = \dot{x}(t)\,\mathbf{i}, \quad \dot{\mathbf{v}}(t) = \frac{d\mathbf{v}}{dt}(t) = \dot{v}(t)\,\mathbf{i}, \quad \text{etc.}$$

**Definition 1.8** **1st-order ODEs**

The most general 1st-order ODE can be written as

$$F(x, \dot{x}, t) = 0$$

Here $x(t)$ is called the dependent variable and is an unknown function of the independent variable $t$.

**Example 1.3** the general solution of the ODE

$$\dot{x} + \lambda x = 0 \quad \text{(where } \lambda \text{ is a real constant)}$$

is $x(t) = C\, e^{-\lambda t}$ where $C$ is an arbitrary constant.

The general solution of any 1st-order ODE contains **one arbitrary constant**. In order to obtain a **unique solution**, we need to specify **one initial condition**, e.g.

$$x(0) = x_0 \quad \text{(for some given constant } x_0\text{).}$$

**Definition 1.9** **Linear**

If the function $F(x, \dot{x}, t)$ is linear in the unknown function $x(t)$ and its derivative $\dot{x}$, then the ODE is said to be **linear**.

The most general linear 1st-order ODE has the form

$$\dot{x} + A(t)\, x = f(t)$$

where $A(t)$ and $f(t)$ are given functions.

**Definition 1.10** **Homogeneous and inhomogeneous**

The above linear 1st-order ODE is **homogeneous** if $f(t) = 0$ for all $t$ and **inhomogeneous** if $f(t) \neq 0$.

We will discuss the strategy for solving inhomogeneous ODEs shortly.

**Definition 1.11** **2nd-order ODEs**

The most general 2nd-order ODE has the form

$$F(x, \dot{x}, \ddot{x}, t) = 0.$$

**Example:** the general solution of the 2nd-order ODE

$$\ddot{x} - \lambda^2 x = 0 \quad \text{(where } \lambda \text{ is a real constant)}$$

is $x(t) = C_1\, e^{\lambda t} + C_2\, e^{-\lambda t}$, where $C_1$ and $C_2$ are arbitrary constants.

The general solution of any 2nd-order ODE contains **two arbitrary constants**. To obtain a **unique solution**, we need **two initial conditions**, e.g.

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0 \text{ (for some given constants } x_0 \text{ and } v_0\text{).}$$

The most general **linear 2nd-order ODE** looks like this:

$$\ddot{x} + A(t)\,\dot{x} + B(t)\,x = f(t)$$

where $A(t)$, $B(t)$ and $f(t)$ are given functions.

**Definition** — **Homogeneous and inhomogeneous**

A linear, $2^{\text{th}}$-order ODE is **homogeneous** if $f(t) = 0$ for all $t$ and **inhomogeneous** if $f(t) \neq 0$.

**A very important and useful result:** The general solution of a linear inhomogeneous ODE is *the sum of any particular solution of the inhomogeneous equation and the general solution of the homogeneous equation*.

(This is the same result as that for linear 1st-order ODEs; in fact, this is true for linear ODEs of any order.)

Linear homogeneous 2nd-order equation **with constant coefficients**

$$\ddot{x} + A\,\dot{x} + B\,x = 0$$

can be solved by assuming that the solution has the form

$$x(t) = Ce^{\lambda t}.$$

Substitution yields the quadratic equation for $\lambda$

$$\lambda^2 + A\lambda + B = 0.$$

There are three possible cases:

- If it has two **distinct real roots**, $\lambda_1$ and $\lambda_2$, then the general solution of the ODE is

  $$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

- If it has two **complex conjugate roots**, $\alpha + i\beta$ and $\alpha - i\beta$, then the general solution is given by

  $$x(t) = C_1 e^{\alpha t}\sin(\beta t) + C_2 e^{\alpha t}\cos(\beta t).$$

- If it has **one double root**, $\lambda_0$, then the general solution has the form

  $$x(t) = C_1 e^{\lambda_0 t} + C_2\, t\, e^{\lambda_0 t}.$$

Finally, here is an example of a nonlinear ODE (which we will discuss in more detail later):

$$\ddot{x} + \mu\,\dot{x}^2 = C$$

where $\mu$ and $C$ are constants. This ODE is nonlinear in $\dot{x}$.

# 2 Equations of motion

## 2.1 Newton's laws (from "The mathematical principles of natural philosophy")

**Definition 2.1** **Newton's laws**

We can summarize Newton's laws as follows:

**Law I:** Every body continues in a state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

**Law II:** The change of motion is proportional to the motive force impressed; and is made in the direction of the straight line in which that force is impressed.

**Law III:** To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

## 2.2 Newton's second law

Newton's second law postulates a relation between the acceleration of a body and the forces acting on it. We can reformulate Newton's second law as follows:

The net force, $\mathbf{F}$, on a body of constant mass, $m$, causes the body to accelerate. The acceleration, $\ddot{\mathbf{x}}$, is in the direction of the force, proportional to the magnitude of the force and inversely proportional to the mass of the body, such that

$$\ddot{\mathbf{x}} = \frac{1}{m}\mathbf{F} \quad \text{or, equivalently,} \quad m\ddot{\mathbf{x}} = \mathbf{F}.$$

The above equation is known as the *equation of motion*.

Note that

- force has magnitude and direction (so that it is described by a vector),
- there may be a number of forces acting on the body but the acceleration is proportional to the net force,
- "inversely proportional to the mass" implies that the same force has a stronger impact on a smaller mass as it causes a higher acceleration.

## 2.3 Examples of forces

- Weight (or uniform gravity force): $\mathbf{F} = m\mathbf{g}$ (here $\mathbf{g}$ is the gravitational acceleration).
- Elastic force (coiled spring): $\mathbf{F} = -k\Delta x\,\mathbf{i}$ ($k$ - the coefficient of elasticity, $\Delta x$ is the change in the length of the spring relative to its natural length. This is usually called *Hooke's law*.
- Stokes friction: $\mathbf{F} = -\Gamma\mathbf{v} = -\Gamma\dot{\mathbf{x}}$ (here $\Gamma$ is a constant coefficient).

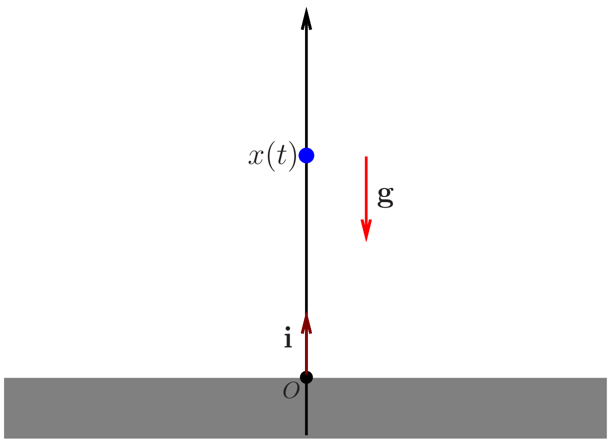



Figure 2.1: Coordinate system for vertical motion under uniform gravity force.

## 2.4 Examples of equations of motion

- **Vertical motion under uniform gravity with Stokes friction**:

  $$\mathbf{F} = m\mathbf{g} - \Gamma\mathbf{v} \quad \Rightarrow \quad m\ddot{\mathbf{x}} = m\mathbf{g} - \Gamma\dot{\mathbf{x}}.$$

  Let us choose a coordinate axis that is vertical and directed up with the origin at the ground level, as shown in Fig. 2.1. The gravity force is directed vertically down. This means that, relative to the chosen coordinate system, $\mathbf{g} = -g\mathbf{i}$ (where $g$ is the magnitude of the gravitational acceleration). Using this coordinate system, we can write

  $$m\ddot{x}\,\mathbf{i} = -mg\,\mathbf{i} - \Gamma\dot{x}\,\mathbf{i}$$

  which can be rewritten as the scalar equation

  $$m\ddot{x} = -mg - \Gamma\dot{x}.$$

- **Motion under an elastic force (Hooke's law)**: let the left end of a (weightless) coiled spring of natural length $L$ be fixed (attached to a wall) and a body of mass $m$ be attached to the right end of the spring and lie on a flat smooth surface (there is no friction between the body and the surface). Then we introduce the coordinate $x$ axis such that it is parallel to the surface with the origin at the wall (see Fig. 2.2) The force is given by

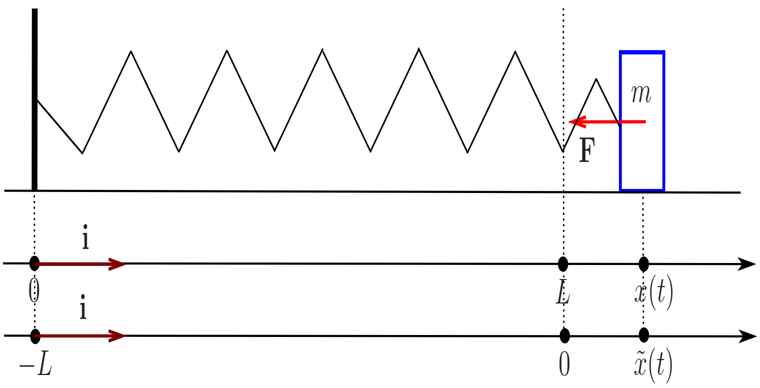



Figure 2.2: A body of mass $m$ attached to a coiled spring whose left end is fixed.

$$\mathbf{F} = -k\Delta\mathbf{x},$$

so that the equation of motion is

$$m\ddot{\mathbf{x}} = -k\Delta\mathbf{x}.$$

Here $\Delta\mathbf{x}(t)$ is the displacement of the body from the equilibrium position (where the length of the spring is equal to its natural length $L$) to its current position, i.e.

$$\Delta\mathbf{x}(t) = \mathbf{x}(t) - L\,\mathbf{i} \quad \text{or} \quad \Delta\mathbf{x}(t) = (x(t) - L)\,\mathbf{i}.$$

Hence the scalar equation for position $x(t)$ is

$$m\ddot{x} = -k(x - L) \quad \text{or, equivalently,} \quad m\ddot{x} + kx = kL.$$

Note that this is a linear inhomogeneous ODE. It can be made homogeneous by introducing new variable $\tilde{x} = x - L$. Indeed, in terms of $\tilde{x}$, the equation of motion has the form

$$m\ddot{\tilde{x}} = -k\tilde{x}.$$

Note that the new variable $\tilde{x}$ can interpreted as the position of the body relative to a new coordinate axis (the $\tilde{x}$ axis), which is parallel to the old one and whose origin is shifted by length $L$ to the right relative to the origin of the $x$ axis (see Fig. 2.2). From this example, we can conclude that sometimes the equation of motion can be simplified by an appropriate choice of the coordinate system.

## 2.5 A free particle: momentum and energy

When the net force is zero, the equation of motion reduces to

$$m\,\ddot{\mathbf{x}} = \mathbf{0} \quad \text{or} \quad \ddot{\mathbf{x}} = \mathbf{0}.$$

This vector equation is equivalent to the scalar ODE for the component $x(t)$ of the position vector:

$$\ddot{x}(t) = 0.$$

Integrating it twice, we obtain

$$x(t) = C_1 t + C_2$$

where $C_1$ and $C_2$ are arbitrary constants of integration. If $x_0$ and $v_0$ are the initial position and velocity (i.e. $x(0) = x_0$ and $\dot{x}(0) = v_0$), the solution can be re-written as

$$x(t) = x_0 + v_0 t.$$

This means that the particle moves with a constant velocity (if $v_0 \neq 0$) or is at rest at $x = x_0$ (if $v_0 = 0$). This agrees with Newton's first law.

**Definition 2.2** **Momentum**

The **momentum** of the particle is the quantity given by

$$p(t) = m\,\dot{x}(t).$$

In terms of the momentum, the equation of motion of a free particle becomes

$$\dot{p} = 0,$$

which means that $p$ does not change with time, i.e. $p(t) = p(0)$ for all $t > 0$.

**Constants of motion**:

Quantities which do not change with time are called **conserved quantities** or **constants of motion** in mechanics.

Thus, the momentum of a free particle is a constant of motion. This fact is also referred to as the *law of conservation of momentum*.

Multiplying the equation of motion of a free particle by $\dot{x}$, we obtain

$$m\,\ddot{x} = 0 \quad \Rightarrow \quad m\,\dot{x}\,\ddot{x} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{m\,\dot{x}^2}{2}\right) = 0.$$

**Definition 2.3** **Kinetic energy**

The kinetic energy of a particle with mass $m$ and velocity $\dot{x}$ is given by

$$T = \frac{m\,\dot{x}^2}{2}.$$

Therefore, $T$ is also a constant of motion for a free particle. This fact is a particular case of the *law of conservation of energy* in this system.

# 3 Solutions for motion in 1D

As discussed previously, here we shall consider the motion of a body as a whole ignoring any possible rotation. Thus it is appropriate to neglect its size and treat it as a **particle** (a material point). First we will discuss the motion of particles in one dimension, i.e. the motion along a straight line.

## 3.1 Motion with a constant force

Let the force be given by $\mathbf{F} = F\,\mathbf{i}$, where $F$ is a constant. The equation of motion is then

$$m\ddot{x} = F \quad \text{or} \quad \ddot{x} = \frac{F}{m} \tag{2}$$

where $F/m$ is also a constant.

This is a linear ODE. We supplement the ODE with two initial conditions (for the position and for the velocity):

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0 \tag{3}$$

where the initial position and velocity, $x_0$ and $v_0$, are given.

The ODE (2) can be solved by integrating it twice in variable $t$. This can be done in two equivalent ways: (i) using indefinite integrals and (ii) using definite integrals.

**Method 1**. We simply integrate (2). This yields

$$\int \frac{d^2x}{dt^2}(t)\,dt = \frac{F}{m}\int\,dt \quad \Rightarrow \quad \frac{dx}{dt}(t) = \frac{F}{m}\,t + C_1 \tag{4}$$

and

$$\int \frac{dx}{dt}(t)dt = \int \left(\frac{F}{m}\,t + C_1\right)dt \quad \Rightarrow \quad x(t) = \frac{F}{m}\frac{t^2}{2} + C_1 t + C_2. \tag{5}$$

In Eqs. (4) and (5), $C_1$ and $C_2$ are (arbitrary!) constants of integration. Equation (5) is the general solution of the equation of motion that depends on two constants $C_1$ and $C_2$. To determine these constants, we employ the initial conditions. Substituting (5) into the initial condition for $x$, we find that

$$\frac{F}{m}\frac{0^2}{2} + C_1 \cdot 0 + C_2 = x_0 \quad \Rightarrow \quad C_2 = x_0$$

Similarly, substitution of (4) into the initial condition for $\dot{x}$ yields

$$\frac{F}{m}\cdot 0 + C_1 = v_0 \quad \Rightarrow \quad C_1 = v_0.$$

Therefore, the solution of the equation of motion that satisfies the initial conditions (3) is given by

$$x(t) = x_0 + v_0 t + \frac{F}{m}\frac{t^2}{2}. \tag{6}$$

**Method 2**. Here we integrate the equation of motion (2) from 0 to $t$:

$$\int_0^t \ddot{x}(s)\,ds = \int_0^t \frac{F}{m}\,ds \quad \Rightarrow \quad \dot{x}(t) = \dot{x}(0) + \frac{F}{m}\,t \tag{7}$$

(note that the variable of integration has been re-named to distinguish it from the upper limit of integration $t$). Integrating (7) from 0 to $t$ again, we find that

$$\int_0^t \dot{x}(s)\,ds = \dot{x}(0)t + \frac{F}{m}\int_0^t s\,ds \quad \text{or} \quad x(t) = x(0) + \dot{x}(0)t + \frac{F}{m}\frac{t^2}{2}. \tag{8}$$

So, both methods lead to the same result (as expected). The second method is slightly shorter than the first one.

**Example 3.1 (uniform gravity force)**. Consider the motion of a body of mass $m$ under the action of the gravity force. Initially, the body is at height $x_0$ above the Earth's surface and is moving up with vertical velocity $v_0$.

(a) Find the maximum height the body will reach.

(b) Find the time when it will fall to the ground.

**Solution**. (a) Let $x(t)$ be the height of the body above the Earth's surface at time $t$. This means that the coordinate axis $Ox$ is vertical and directed upwards and that the origin is at the ground, as shown in Fig. 2.1. The only force applied to the body is the gravity force that is directed vertically downward and whose magnitude is $mg$ where $g$ is the free fall acceleration. The equation of motion is given by

$$m\ddot{\mathbf{x}} = m\mathbf{g} \quad \text{or} \quad \ddot{\mathbf{x}} = \mathbf{g}.$$

Since $\mathbf{g} = -g\mathbf{i}$, we obtain

$$\ddot{x} = -g.$$

This ODE should be solved subject to the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = v_0.$$

From Eqs. (7) and (8), we obtain

$$v(t) = v_0 - gt \quad \text{and} \quad x(t) = x_0 + v_0 t - g\frac{t^2}{2}. \tag{9}$$

Position $x$ as a function of $t$ is shown in Fig. 3.1. It is a part of a parabola. Let $t_1$ be the time when the body is at the highest point. This point is a maximum of the graph of $x(t)$, i.e. $\dot{x}(t_1) = v(t_1) = 0$. Therefore,

$$v_0 - gt_1 = 0 \quad \Rightarrow \quad t_1 = \frac{v_0}{g} \quad \Rightarrow \quad x(t_1) = x_0 + v_0 t_1 - g\frac{t_1^2}{2} = x_0 + \frac{v_0^2}{2g}.$$

Similarly, we can find the moment of time $t_2$ when the body will fall to the surface. At that moment of time, $x(t_2) = 0$, and we obtain

$$x_0 + v_0 t_2 - g\frac{t_2^2}{2} = 0.$$

Solving this quadratic equation for $t_2$, we find that

$$t_2 = \frac{v_0}{g} + \sqrt{\frac{v_0^2}{g^2} + \frac{2x_0}{g}}.$$

(The negative root of the quadratic equation has been excluded as not relevant for our problem.)

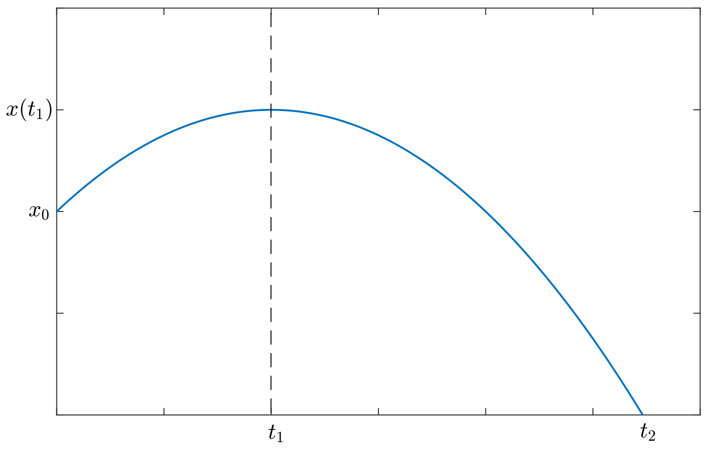



Figure 3.1: Position as a function of time.

## 3.2 Motion with a time-dependent force

If $\mathbf{F} = F(t)\,\mathbf{i}$, where $F(t)$ is a given function of time, then we have

$$m\ddot{x} = F(t).$$

Let's solve this ODE subject to the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = v_0.$$

Integrating the equation of motion over time from 0 to $t$, we obtain

$$\int_0^t \ddot{x}(s)ds = \frac{1}{m}\int_0^t F(s)ds \quad \Rightarrow \quad \dot{x}(t) - \dot{x}(0) = \frac{1}{m}\int_0^t F(s)ds \quad \Rightarrow \quad \dot{x}(t) = \dot{x}(0) + \frac{1}{m}\int_0^t F(s)ds.$$

Yet another integration yields

$$\int_0^t \dot{x}(\tau)d\tau = \int_0^t \dot{x}(0)d\tau + \frac{1}{m}\int_0^t \left(\int_0^\tau F(s)ds\right)d\tau \quad \Rightarrow$$

$$\Rightarrow \quad x(t) = x(0) + \dot{x}(0)t + \frac{1}{m}\int_0^t \left(\int_0^\tau F(s)ds\right)d\tau.$$

Taking account of the initial conditions, we obtain

$$v(t) = v_0 + \frac{1}{m}\int_0^t F(s)ds \quad \text{and} \quad x(t) = x_0 + v_0 t + \frac{1}{m}\int_0^t \left(\int_0^\tau F(s)ds\right)d\tau. \tag{10}$$

These equations give us the solution of the equation of motion in the case of a time-dependent force.

We note in passing that the repeated integral in the formula for $x(t)$ can be simplified by changing the order of integration

$$\int_0^t \left(\int 0^\tau F(s) ds\right) d\tau = \int_0^t \left(\int_s^t d\tau\right) F(s)\, ds = \int_0^t (t-s)\, F(s)\, ds.$$

(Prove it!) Therefore,

$$x(t) = x_0 + v_0 t + \frac{1}{m}\int_0^t (t-s)\, F(s)\, ds.$$

It is a good Calculus exercise to check (by differentiating this formula two times) that $x(t)$ satisfies the ODE and the initial conditions.

**Example** 3.2 **(oscillating force)**. Let $F = A\sin(\omega t)$ where $A$ and $\omega$ are constants and let $x(0) = x_0$ and $\dot{x}(0) = v(0) = v_0$. Find $x(t)$ and $v(t) = \dot{x}(t)$.

**Solution.** From (10), we obtain

$$v(t) = v_0 + \frac{1}{m}\int_0^t A\sin(\omega s)\, ds = v_0 + \frac{A}{m\omega}\,[1-\cos(\omega t)]\,.$$

Integrating this in $t$ from 0 to $t$, we find that

$$x(t) = x_0 + \left[v_0 + \frac{A}{m\omega}\right] t - \frac{A}{m\omega}\int_0^t \cos(\omega\tau)\, d\tau = x_0 + \left[v_0 + \frac{A}{m\omega}\right] t - \frac{A}{m\omega^2}\sin(\omega t).$$

It is interesting that although the force is an oscillating function of time with zero mean (over a period $T = 2\pi/\omega$), the solution is not necessarily periodic because it may contain a linearly growing (with time) term.

## 3.3 Motion with a force depending on velocity

When a body moves in a liquid or gaseous medium, the medium affects the motion. The interaction of the body and the medium results in a friction (or drag) force exerted on the body by the medium. The drag force (friction force) depends on the properties of the medium through which the body is moving, the shape of the body and its velocity.

**Stokes drag**:

It is known from experiments that when the velocity of the body is small (or the size of the body is small) the magnitude of the resistance force is proportional to the magnitude of the velocity, i.e.

$$\mathbf{F} = -\Gamma\mathbf{v} = -\Gamma\dot{\mathbf{x}} \tag{11}$$

where $\Gamma$ is the drag (resistance or Stokes friction) coefficient. Usually this force is called *Stokes drag* or *Stokes friction force* (named after George Gabriel Stokes who has computed the force exerted on a spherical body moving through a viscous fluid).

It is also known that for sufficiently large $v$ the drag force can be proportional to the square of

Figure 3.2: Skydiving: gravity *and* friction.
Source: Tony Danbury; Creative Commons 2.0; https://commons.wikimedia.org/wiki/File:AFF_Level_1_-_Skydive_Langar.jpg

the velocity, i.e.

$$\mathbf{F} = -\kappa|\mathbf{v}|\mathbf{v}, \tag{12}$$

where $\kappa$ is a constant coefficient.

**Example 3.3 (motion under Stokes drag force).** Consider a body of mass $m$ moving in air and assume that its position and velocity at time $t = 0$ are $x_0$ and $v_0$ respectively and that no external forces are acting on the body for $t > 0$, so that it is moving only under the action of the Stokes drag force.

(a) Find the motion of the body for $t > 0$, i.e. $x(t)$ for $t > 0$.

(b) Check whether your answer is dimensionally correct.

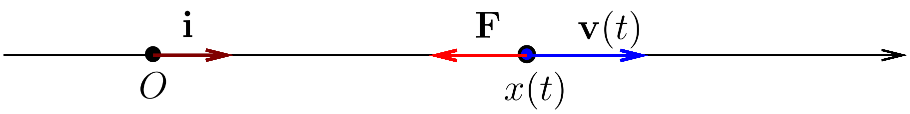



Figure 3.3: Motion with Stokes drag or friction

**Solution.** (a) We shall use the coordinate axis whose positive direction coincides with the direction of the initial velocity (see Fig. 3.3). The drag force is given by Eq. (11). So, the equation of motion is

$$m\ddot{\mathbf{x}} = -\Gamma\dot{\mathbf{x}}$$

or, equivalently,

$$m\ddot{x} = -\Gamma\dot{x}$$

It is convenient to rewrite this as the equation for velocity $v(t) = \dot{x}(t)$:

$$\dot{v} = -kv$$

where $k = \Gamma/m$. This ODE can be solved by assuming that the solution has the form $v(t) = Ce^{\lambda t}$ and substituting this into the ODE. This yields $\lambda = -k$, so that the general solution is $v(t) = Ce^{-kt}$. Then the initial condition, $v(0) = v_0$, implies that $C = v_0$. Hence,

$$v(t) = v_0 e^{-kt}. \tag{13}$$

This means that the velocity is an exponentially decreasing function of $t$ that goes to 0 as $t \to \infty$.

Furthermore, we have

$$\dot{x} = v_0 e^{-kt} \quad \Rightarrow \quad x(t) = x_0 + v_0 \int_0^t e^{-ks} ds = x_0 + \frac{v_0}{k}\left(1 - e^{-kt}\right).$$

Note that $x(t) \to x_0 + v_0/k$ as $t \to \infty$, so that the distance travelled by the body for the infinite time interval $(0, \infty)$ is finite and equal to

$$x(\infty) - x_0 = \frac{v_0}{k} = \frac{mv_0}{\Gamma}.$$

(b) First we determine the physical dimension of $\Gamma$. It follows from the equation of motion and the requirement that two physical quantities can be equal only if they have the same physical dimension that

$$[m][\ddot{x}] = [\Gamma][\dot{x}] \quad \Rightarrow \quad \frac{ML}{T^2} = [\Gamma]\,\frac{L}{T} \quad \Rightarrow \quad [\Gamma] = \frac{M}{T}.$$

Our answer for $x(t)$ makes sense only if the physical dimension of $v_0/k$ is length and $kt$ is a dimensionless quantity. We have

$$\left[\frac{v_0}{k}\right] = \left[\frac{mv_0}{\Gamma}\right] = \frac{ML/T}{M/T} = L$$

and

$$[kt] = \left[\frac{\Gamma}{m}\right]\,[t] = \frac{M/T}{M}\,T = 1.$$

Therefore, our formula for $x(t)$ is dimensionally correct.

**Example 3.4 (motion under a friction force which is quadratic in velocity).** Consider now the same problem as in Example 3.3 under the assumption that the friction force is given by Eq. (12).

**Solution.** (a) We shall use the same coordinate system as in Example 3.3 (see Fig. 3.3), so that the positive direction of the $x$ axis coincides with the direction of the initial velocity $\mathbf{v}(0) = v_0\,\mathbf{i}$ and, therefore, $v_0$ is positive.

The equation of motion in vector form is

$$m\ddot{\mathbf{x}} = -\kappa\,|\dot{\mathbf{x}}|\dot{\mathbf{x}}.$$

Hence, the equation for position $x(t)$ is given by

$$m\ddot{x} = -\kappa\,|\dot{x}|\dot{x}.$$

Initial conditions are

$$x(0) = x_0, \quad \dot{x}(0) = v_0 > 0.$$

Now we assume that the velocity, $v(t) = \dot{x}(t)$ does not change sign for $t > 0$ (this can can be verified once we know $v(t)$): $v(t) > 0$ for $t \in [0, \infty)$. With this assumption, $|\dot{x}| = \dot{x}$ and the equation of motion can be rewritten as

$$m\ddot{x} = -\kappa\,\dot{x}^2 \quad \Rightarrow \quad m\dot{v} = -\kappa v^2 \quad \Rightarrow \quad -\int \frac{dv}{v^2} = \frac{\kappa}{m}\int dt \quad \Rightarrow$$
$$\Rightarrow \quad \frac{1}{v} = qt + c \quad \Rightarrow \quad v(t) = \frac{1}{qt + c}.$$

where $q = \kappa/m$ and $c$ is an arbitrary constant. From the initial condition $v(0) = v_0$, we find that

$$v_0 = \frac{1}{c} \quad \Rightarrow \quad c = \frac{1}{v_0} \quad \Rightarrow \quad v(t) = \frac{v_0}{1 + v_0 qt}. \tag{14}$$

Note that the velocity is a decreasing function of $t$ for all $t > 0$ and $v(t) \to 0$ as $t \to \infty$. So, the body moves slower and slower with time and stops in the limit $t \to \infty$. Note also that $v(t) > 0$ for any finite $t > 0$, which means that our earlier assumption holds.

Further, we have

$$\dot{x} = \frac{v_0}{1 + v_0 qt} \quad \Rightarrow \quad x(t) = x_0 + v_0 \int_0^t \frac{ds}{1 + v_0 qs} = x_0 + \frac{1}{q}\ln|1 + v_0 qt|.$$

Here $x_0 = x(0)$ is the initial coordinate of the body. Note that $x(t) \to \infty$ as $t \to \infty$, so that the distance travelled by the body for the infinite time interval $(0, \infty)$ is infinite. Thus, we have obtained the paradoxical result: for large $v$ the magnitude of the Stokes friction force, given by (11), is smaller than the magnitude of the force, given by (12), but our result shows that in the former case the distance travelled by the body is finite, while in the latter case it is infinite. The explanation of the paradox is that the formula (12) is valid only for sufficiently large velocity. When the velocity becomes small (and it always does), the formula becomes incorrect. So the above formulas for $v(t)$ and $x(t)$ describe the motion of the body only on some finite time interval from 0 to $t_1$ for some $t_1$. This is illustrated in Fig. 3.4, where the velocity as a function of time is plotted for both forces (the velocities are given by Eqs. (13) and (14)). In Fig. 3.4: $m = 1\ kg$, $k = 1\ s^{-1}$, $q = 1\ m^{-1}$ and $v_0 = 3\ m/s$. One can see that initially, when the velocity of the body is high, the quadratic friction is more efficient in decelerating the body, but when the velocity becomes sufficiently small the linear friction is more efficient.

(b) As in Example 1, first we find the physical dimension of $\kappa$. From the equation of motion, we have

$$[m][\ddot{x}] = [\kappa][\dot{x}^2] \quad \Rightarrow \quad \frac{ML}{T^2} = [\kappa]\frac{L^2}{T^2} \quad \Rightarrow \quad [\kappa] = \frac{M}{L}.$$

The above formula for $x(t)$ makes sense only if $\left[\frac{1}{q}\right] = L$ and $v_0 qt$ is a dimensionless quantity. We have

$$\left[\frac{m}{\kappa}\right] = \frac{M}{M/L} = L$$

and

$$[v_0 qt] = \left[\frac{\kappa}{m}\right][v_0][t] = \frac{M/L}{M}\frac{L}{T}T = 1.$$

We conclude that the formula for $x(t)$ is dimensionally correct.

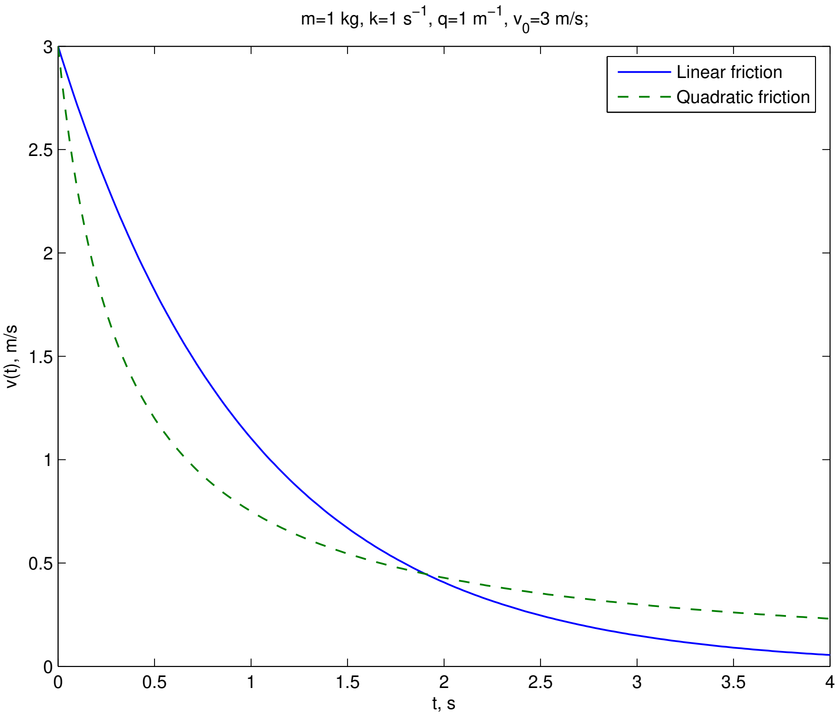



Figure 3.4: Velocity of the body, $v$ as a function of $t$ for linear (Stokes) friction, given by Eq. (13), and for quadratic friction, given by Eq. (14).

## 3.4 Motion under a force depending on position

Consider a particle of mass $m$ moving on a straight line under the action of a force $F$ which depends only on the coordinate $x$ of the particle. In this case, the equation of motion has the form

$$m\ddot{x} = F(x). \tag{15}$$

Before treating the general case, let's look at a particular problem.

**Example 3.5 (motion of a body attached to a spring).** Consider the system shown in Fig. 2.2: a body is attached to to the right end of a coiled spring of natural length $L$; the left end of the spring is fixed (attached to a fixed wall). The body lies on a flat smooth surface and is free to move on this surface without friction. If we choose the origin of the $x$ axis at the fixed wall, then the equation of motion is given by

$$m\ddot{x} = -k(x - L). \tag{16}$$

where $k$ is the elastic constant of the spring and $L$ is the natural length of the spring. Find the motion of the body for $t > 0$, given that $x(0) = x_0$ and $\dot{x}(0) = v_0$.

**Solution.** It is convenient to divide the equation of motion by $m$ and rewrite it as

$$\ddot{x} + \omega^2 x = \omega^2 L, \quad \omega = \sqrt{k/m}. \tag{17}$$

Equation (17) is a linear inhomogeneous second-order ODE with constant coefficients. Its general solution is the sum of a particular solution, $x_p(t)$, of the inhomogeneous equation and the general solution, $x_h(t)$, of the homogeneous equation: $x(t) = x_p(t) + x_h(t)$. To find $x_p(t)$, we assume that $x_p(t) = A$ where $A$ is a constant (this is a reasonable assumption as the right hand side of Eq. (17) is a constant). Substitution of this into Eq. (17) yields $A = L$ and therefore $x_p(t) = L$. To find the general solution of the homogeneous equation

$$\ddot{x}_h + \omega^2 x_h = 0,$$

we use the standard recipe: we assume that $x_h(t) = Ce^{\lambda t}$ and substitute is into the equation. This gives us the auxiliary equation $\lambda^2 + \omega^2 = 0$, which has two purely imaginary roots $\lambda = \pm i\,\omega$. Therefore, the general solution of the homogeneous equation is

$$x_h(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t),$$

where $c_1$, $c_2$ are arbitrary constants. Hence, the general solution of the inhomogeneous equation is given by

$$x(t) = L + c_1 \sin(\omega t) + c_2 \cos(\omega t), \tag{18}$$

Substituting this general solution into the initial conditions, we find that

$$\begin{aligned} x(0) &= L + c_1 \sin(\omega \cdot 0) + c_2 \cos(\omega \cdot 0) = L + c_2 = x_0 \quad \Rightarrow \quad c_2 = x_0 - L, \\ \dot{x}(0) &= \omega\, c_1 \cos(\omega \cdot 0) - \omega\, c_2 \sin(\omega \cdot 0) = \omega\, c_1 = v_0 \quad \Rightarrow \quad c_1 = \frac{v_0}{\omega}. \end{aligned}$$

Hence,

$$x(t) = L + \frac{v_0}{\omega} \sin(\omega t) + (x_0 - L) \cos(\omega t). \tag{19}$$

One can check that this formula is dimensionally correct.

**Discovering the energy.** Let

$$z(t) = x(t) - L, \quad z_0 = x_0 - L.$$

Then Eq. (19) can be written as

$$z(t) = \frac{v_0}{\omega} \sin(\omega t) + z_0 \cos(\omega t). \tag{20}$$

This is a solution of the equation

$$\ddot{z} + \omega^2 z = 0 \tag{21}$$

satisfying the initial conditions $z(0) = z_0$ and $\dot{z}(0) = v_0$. Equation (21) can be interpreted as the equation of a motion of a body attached to a spring relative to the coordinate axis whose origin is at the equilibrium position of the body (where the length of the spring is equal to its natural length).

Differentiating Eq. (20), we obtain

$$\dot{z}(t) = v_0 \cos(\omega t) - z_0 \omega \sin(\omega t). \tag{22}$$

<u>The idea:</u> let's eliminate the explicit dependence on $t$ from Eqs. (20) and (22) and see what relation between $z(t)$ and $\dot{z}(t)$ this will produce.

First, we divide (20) by $z_0$ and then square both sides of the resulting equation:

$$\frac{z^2(t)}{z_0^2} = \left(\frac{v_0}{z_0 \omega}\right)^2 \sin^2(\omega t) + 2\,\frac{v_0}{z_0 \omega} \sin(\omega t) \cos(\omega t) + \cos^2(\omega t).$$

Similarly, from (22), we obtain

$$\frac{\dot{z}^2(t)}{z_0^2\omega^2} = \left(\frac{v_0}{z_0\omega}\right)^2 \cos^2(\omega t) - 2\,\frac{v_0}{z_0\omega}\,\sin(\omega t)\cos(\omega t) + \sin^2(\omega t).$$

Adding these two equations, we find that

$$\frac{z^2(t)}{z_0^2} + \frac{\dot{z}^2(t)}{x_0^2\omega^2} = \frac{v_0^2}{z_0^2\omega^2} + 1$$

or, equivalently,

$$\dot{z}^2(t) + \omega^2 z^2(t) = v_0^2 + \omega^2 z_0^2.$$

If we multiply the last equation by $m/2$, we will get

$$\frac{m\dot{z}^2(t)}{2} + \frac{kz^2(t)}{2} = \frac{mv_0^2}{2} + \frac{kz_0^2}{2}. \tag{23}$$

This equation says that although the body is moving in accordance with (20), the quantity

$$E(t) = \frac{m\dot{z}^2(t)}{2} + \frac{kz^2(t)}{2} \tag{24}$$

remains the same as its initial value.

**Definition 3.1** **Energy**

Physical quantities that have this property are called **constants of motion** and $E$, given by (24) is called the **energy**. The fact that the energy does not change is known as the **law of conservation of energy**. The first term in (24), $m\dot{z}^2/2$, is called the **kinetic energy** (of a body of mass $m$ moving with velocity $\dot{z}$), while the second term, $kz^2/2$, is called the **potential energy**. In each particular physical problem, potential energy has its own physical nature.

In the present example, the potential energy is the elastic energy of the spring.

## 3.5 Motion in a potential

In preceding section, we have found a solution of the equation of motion, given by

$$m\ddot{x} = -kx, \tag{25}$$

that describes motion of a body of mass $m$ attached to a spring (with elastic constant $k$). Then we used this solution to show that the energy $E$ of this system,

$$E = \frac{m\dot{x}^2(t)}{2} + \frac{kx^2(t)}{2}, \tag{26}$$

does not depend on time, i.e.

$$E(t) = E(0).$$

Can we deduce that the energy is conserved directly from the equation of motion (without solving it)? - The answer is ‘yes’ and this can be done in the following way.

First, we multiply the equation of motion (25) by $\dot{x}$. This yields

$$m\,\dot{x}\ddot{x} = -k\,x\dot{x} \quad \text{or} \quad m\,\dot{x}\ddot{x} + k\,x\dot{x} = 0. \tag{27}$$

Then we observe that

$$\dot{x}\ddot{x} = \dot{x}\,\frac{d\dot{x}}{dt} = \frac{d}{dt}\left(\frac{\dot{x}^2}{2}\right) \quad \text{and} \quad x\dot{x} = x\,\frac{dx}{dt} = \frac{d}{dt}\left(\frac{x^2}{2}\right).$$

Therefore, Eq. (27) can be written as

$$\frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + \frac{kx^2}{2}\right) = 0 \quad \text{or} \quad \frac{dE}{dt} = 0 \tag{28}$$

with $E$ given by (26). So, we do not need to solve the equation of motion (25) to establish the fact that the energy is conserved.

**Question**:

*Is the law of conservation of energy generic for mechanical systems or is it just a random coincidence that the energy of 'a body + a spring' system is conserved?*

To answer this question, let's look at the general equation of motion of a particle of mass $m$ under a position dependent force:

$$m\ddot{x} = F(x). \tag{29}$$

Again, we multiply this equation by $\dot{x}$, This yields

$$m\dot{x}\ddot{x} = F(x)\dot{x} \quad \Rightarrow \quad \frac{d}{dt}\left(m\,\frac{\dot{x}^2}{2}\right) = F(x)\,\dot{x}.$$

Now let $V(x)$ be a function defined by the equation

$$F(x) = -\frac{dV}{dx}. \tag{30}$$

Equivalently, we can define $V(x)$ as

$$V(x) = -\int F(x)\,dx \quad \text{or} \quad V(x) = -\int_{x_0}^{x} F(s)\,ds$$

for arbitrarily chosen $x_0$. Then we have

$$F(x)\,\dot{x} = -\frac{dV}{dx}\,\dot{x} = -\frac{d}{dt}V(x),$$

Therefore, we obtain

$$\frac{d}{dt}\left(m\,\frac{\dot{x}^2}{2} + V(x)\right) = 0$$

or

$$\frac{dE}{dt} = 0 \tag{31}$$

where

$$E = m\,\frac{\dot{x}^2}{2} + V(x). \tag{32}$$

**Definition 3.2** **The potential and the conservation of energy**

The quantity $\frac{1}{2}m\dot{x}^2$ is the **kinetic energy** of the particle. $V(x)$ is, by definition, the **potential energy** of the particle or the **potential** of the force $F(x)$, defined by $dV/dx = -F(x)$. $E$ is called the **energy** (or the **total energy**). Equation (31) expresses the **conservation of the energy of the particle**.

Note that if $V(x)$ is a potential for $F(x)$, i.e. $V(x)$ satisfies Eq. (30), then $\tilde{V}(x) = V(x) + C$ for any constant $C$ is also a potential. This means that the potential energy $V(x)$ is defined up to an arbitrary constant.

**Examples of potentials:**

- Uniform gravity: if the $x$ axis is vertical and directed upward, then the gravity force is $F = -mg$. Therefore,

  $$V(x) = -\int F(x)\,dx = mg\int dx = mgx + C$$

  for an arbitrary constant $C$. It is convenient to choose $C = 0$. Then

  $$V(x) = mgx,$$

  so that the potential energy of a body (equivalently, the potential of the uniform gravity force) is zero when it is on the ground.

- A body + a spring: in this case, $F = -kx$. Therefore,

  $$V(x) = -\int F(x)\,dx = k\int x\,dx = k\,\frac{x^2}{2} + C$$

  for an arbitrary constant $C$. Again, it is convenient to choose $C = 0$. Then

  $$V(x) = k\,\frac{x^2}{2},$$

  and the elastic energy of the spring (equivalently, the potential energy of the body or the potential of the force $F = -kx$) is zero when the length of the spring is equal to its natural length.

- Newtonian gravitation: the Newtonian gravitational force of attraction between two bodies of masses $m$ and $M$ is $GmM/r^2$, where $G$ is the gravitational constant and $r$ is the distance between the centres of the bodies. Let $M$ be the mass of the Earth and $m$ be the mass of another spherically symmetric body. We introduce the $x$ axis that passes through the centre of the Earth and the centre of the body, with origin at the centre of the Earth, and is directed from the Earth to the body. Then the gravitational force acting on the body is given by

  $$F = -\frac{GmM}{x^2}.$$

  Hence,

  $$V(x) = -\int F(x)\,dx = -\frac{GmM}{x} + C.$$

If we choose $C = 0$, then

$$V(x) = -\frac{GmM}{x}.$$

This choice corresponds to zero potential energy of the body when it is far away (at infinity) from the Earth. Note also that $V(x)$ is negative for any finite $x > 0$. (There is nothing wrong with the potential energy being negative. The kinetic energy cannot be negative, but the potential energy, as well as the total energy, can be negative.)

**Example 3.6 (motion under uniform gravity).** To demonstrate that energy conservation can be useful, we consider a vertical motion of a particle of mass $m$ under the action of the uniform gravity force. Suppose that initially the particle is at height $x_0$ and has the velocity $v_0 > 0$ (the coordinate $x$-axis is vertical and directed upwards). We can use the energy conservation to find, for example, the highest point of the trajectory of the particle. Indeed, we have

$$E|_{t=0} = \frac{mv_0^2}{2} + mgx_0.$$

Since the energy is a constant of motion, $E(t) = E(0)$, i.e.

$$\frac{mv^{2(t)}}{2} + mgx(t) = \frac{mv_0^2}{2} + mgx_0.$$

At the highest point of the trajectory, at time $t = t^*$, the velocity of the particle is zero, i.e $v(t^*) = 0$. Therefore,

$$E(t^*) = E(0) \quad \Rightarrow \quad mgx(t^*) = \frac{mv_0^2}{2} + mgx_0 \quad \Rightarrow \quad x(t^*) = x_0 + \frac{v_0^2}{2g}.$$

## 3.6 Using energy conservation to solve equations of motion

Can conservation of energy help us in solving the equation of motion?

The energy, given by Eq. (32), is a constant, and its value can by found from initial conditions. Suppose that the value of $E$ is given (or computed using initial conditions). Then, it follows from (32) that

$$\dot{x} = \pm\sqrt{2(E - V(x))/m} \quad \Rightarrow \quad \frac{dx}{\sqrt{2(E - V(x))/m}} = \pm dt \quad \Rightarrow \quad \int \frac{dx}{\sqrt{2(E - V(x))/m}} = \pm t.$$

In principle, if we know $V(x)$, we can evaluate the integral. This introduces a constant of integration $C$ and the resulting equation can be inverted to give the position of the particle as a function of time, $x(t)$. Two constants of integration $E$ and $C$ will appear in this function which describes all possible motions of the particle consistent with a given force $F(x)$. In practice, this procedure leads to the answer expressed by an explicit formula only for a few simplest potential functions $V(x)$.

**Example 3.7 (a body attached to a spring).** Consider a particle of mass $m$ moving under the action of the force $F(x) = -kx$ where $k > 0$ (e.g. a body attached to a spring, discussed previously). The equation of motion is

$$m\ddot{x} = -kx. \tag{33}$$

We know that its general solution is given by

$$x(t) = c_1 \sin \omega t + c_2 \cos \omega t, \tag{34}$$

where $\omega = \sqrt{\frac{k}{m}}$, $c_1$ and $c_2$ are arbitrary constants. Nevertheless, it is instructive to obtain this general solution using the general method described above.

We have

$$V(x) = \frac{kx^2}{2} \quad \Rightarrow \quad E = \frac{m\dot{x}^2}{2} + \frac{kx^2}{2} \quad \Rightarrow \quad \dot{x} = \pm\sqrt{\frac{2E}{m} - \frac{k}{m}x^2} \quad \Rightarrow \quad t + C = \pm \int \frac{dx}{\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}}.$$

Let $y = x\sqrt{\frac{k}{2E}}$. Then $\sqrt{\frac{2E}{k}}dy = dx$, and we obtain

$$\int \frac{dx}{\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}} = \sqrt{\frac{m}{2E}} \int \frac{dx}{\sqrt{1 - \frac{k}{2E}x^2}} = \sqrt{\frac{m}{2E}}\sqrt{\frac{2E}{k}} \int \frac{dy}{\sqrt{1-y^2}} = \sqrt{\frac{m}{k}} \int \frac{dy}{\sqrt{1-y^2}}.$$

Making the change of variable $y = \sin\theta$, we find that

$$\sqrt{\frac{m}{k}} \int \frac{dy}{\sqrt{1-y^2}} = \sqrt{\frac{m}{k}} \int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} = \sqrt{\frac{m}{k}} \int \frac{\cos\theta d\theta}{\cos\theta} = \sqrt{\frac{m}{k}}\theta \; + \text{arb. const.}$$

Thus, we have

$$\pm\sqrt{\frac{m}{k}}\theta = t + C \quad \Rightarrow \quad y = \sin\theta = \sin\left(\pm\sqrt{\frac{k}{m}}(t+C)\right) = \pm\sin\left(\sqrt{\frac{k}{m}}(t+C)\right).$$

It follows that

$$x = \pm\sqrt{\frac{2E}{k}} \sin\left(\sqrt{\frac{k}{m}}(t+C)\right)$$

or

$$x = A\sin\left(\sqrt{\frac{k}{m}}t + \phi_0\right), \tag{35}$$

where $A$ and $\phi_0$ are arbitrary constants. Formula (35) represents the general solution of the equation of motion (32). It can be written in the form (34) with the help of the identity

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Of course, it is easier to directly solve the equation of motion in this example rather than to use the general method. Nevertheless, the example shows that the general method works.

## 3.7 Using energy to describe motion qualitatively

### 3.7.1 Qualitative analysis of the motion

Consider a particle of mass $m$ moving on a straight line under the action of a force $F$ with potential $V(x)$. It turns out that many features of the motion can be deduced even from the graph of the potential energy $V(x)$. Recall that $dV/dx = -F$, so that the force is minus the slope of the graph of $V(x)$.

**Observations**:

- if $V'(x_0) > 0$, then the force acting on the particle at the point $x_0$ is negative, i.e. its direction is opposite relative to the $x$ axis;
- if $V'(x_0) < 0$, then the force acting on the particle at the point $x_0$ is positive, i.e. its direction is the same as that of the $x$ axis.
- Since $E = \frac{m\dot{x}^2}{2} + V(x)$ is a constant of motion (whose value is determined by initial conditions for $x$ and $\dot{x}$), we conclude that $V(x) \leq E$. This means that whe particle whose total energy is $E$ cannot be located at points $x$ such that $V(x) > E$ (i.e. at points $x$ for which the graph of $y = V(x)$ is above the line $y = E$).

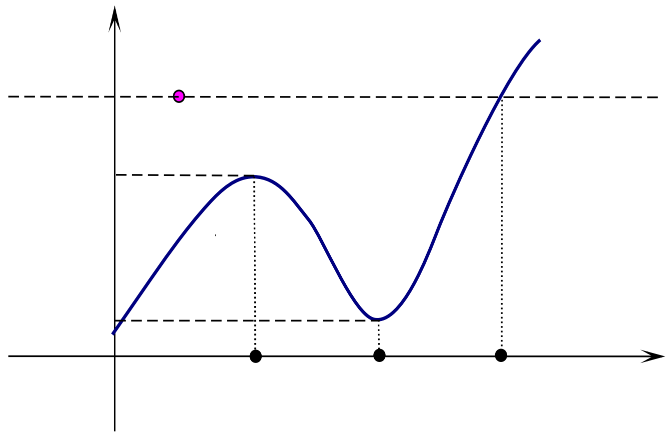


Figure 3.5: Motion in the potential $V(x)$ (Case I).

Suppose that the graph of the potential energy is as shown in Fig. 3.5. The potential $V(x)$ has a local maximum at $x = a$ and a local minimum at $x = b$. The total energy $E$ of the particle is represented by a dashed horizontal line. Let $x_0$ and $v_0$ be the initial position and the initial velocity of the particle. We consider the following three cases of possible motion.

- **Case I:** $E > V(a)$. According to Fig. 3.5 and our earlier observation, the particle cannot be at points $x > x^*$. So, let $x_0 < x^*$.

  (a) If $v_0 > 0$, the particle will continue to move in the positive direction of the $x$-axis until its velocity $v(t) = \dot{x}(t)$ becomes zero (it will be decelerating for $x < a$ and $b < x < x^*$ and accelerating for $a < x < b$, but its velocity will be nonzero for $x < x^*$ because $v = \sqrt{2(E - V(x))/m}$ and $E > V(x)$ for $x < x^*$). At $x = x^*$, the velocity of the particle is zero (because $v = \sqrt{2(E - V(x))/m}$ and $E = V(x)$ for $x = x^*$). The force $F(x^*) = -V'(x^*)$ at this point is nonzero and negative. Therefore, immediately after it is stopped at $x = x^*$ the particle will start moving in the negative direction of the $x$-axis and will keep moving to infinity (we will say that the particle *escapes to infinity*). Points at which the velocity changes direction to the opposite are called **turning points** of the motion (in our example, $x^*$ is a turning point).

  (b) If $v_0 < 0$, the particle will continue to move in the negative direction of the $x$-axis and eventually will escape to infinity.

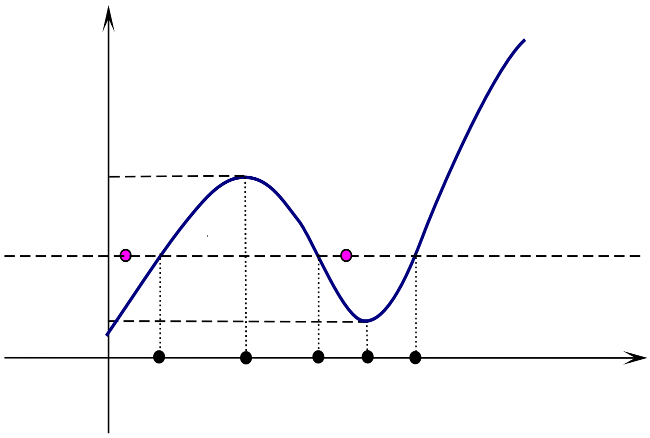


Figure 3.6: Motion in the potential $V(x)$ (Case II).

- **Case II:** $V(b) < E < V(a)$. According to Fig. 3.6, the particle cannot be at points $x > x_3$ or at $x \in (x_1, x_2)$. It can move either in the interval $(-\infty, x_1]$ or in the interval $[x_2, x_3]$.

  (a) If $x_0 < x_1$, then, depending on the direction of the initial velocity, the particle will either continue to move in the negative direction of the $x$-axis or move first towards the turning point $x_1$ and then back. In both cases, it will escape to infinity.

  (b) If $x_2 < x_0 < x_3$ and $v_0 > 0$, the particle will first move towards the turning point $x_3$, where the direction of the velocity is reversed, and then back towards the other turning point $x_2$. At $x_2$, the direction of the velocity changes to the opposite again and it starts moving towards $x_3$. And this motion will repeat itself indefinitely. Thus, in this case, the motion of the particle is periodic. This is an example of *finite motion* (i.e. for all $t > 0$, the motion occurs in a finite interval of the $x$-axis). If the initial velocity is negative ($v_0 < 0$), the motion is qualitatively the same.

  The period of motion, $T$, can determined as follows. First, we recall that, for a given value of $E$, the velocity can be found using the formula

  $$\dot{x} = \pm\sqrt{2(E - V(x))/m}.$$

  Then we consider the motion of the particle from point $x_2$ to point $x_3$. We assume that at time $t_2$ the particle is at point $x_2$ (i.e. $x_2 = x(t_2)$). Since $x_2$ is the turning point of motion, the particle's velocity is zero, i.e. $\dot{x}(t_2) = 0$. The force at $x_2$ is positive. So, the particle will start moving towards $x_3$ with the velocity given by

  $$\dot{x} = \sqrt{2(E - V(x))/m}. \tag{36}$$

  (The 'plus' sign was chosen because when the particle is moving from $x_2$ to $x_3$, the velocity is positive.) Suppose that it will arrive at point $x_3$ at some later time $t_3$, i.e. $x_3 = x(t_3)$. What will happen next? It will start moving back with the velocity that has exactly the same magnitude but the opposite direction and after a while will come back to $x_2$. Then the same motion will repeat again and again. Since the magnitude of the velocity is exactly the same (irrespective of whether the particle moves from $x_2$ to $x_3$ or from $x_3$ to $x_2$), it will take the same time for it to move from $x_2$ to $x_3$ and from $x_3$ to $x_2$. Hence, the period, $T$ is equal to twice the time needed to move from $x_2$ to $x_3$, i.e. $T = 2(t_3 - t_2)$. Separation of variables in Eq. (36) and integration yield the formula for $T$:

$$\int_{t_2}^{t_3} dt = \int_{x_2}^{x_3} \frac{dx}{\sqrt{2(E - V(x))/m}} \quad \Rightarrow \quad T = 2\int_{x_2}^{x_3} \frac{dx}{\sqrt{2(E - V(x))/m}}. \tag{37}$$

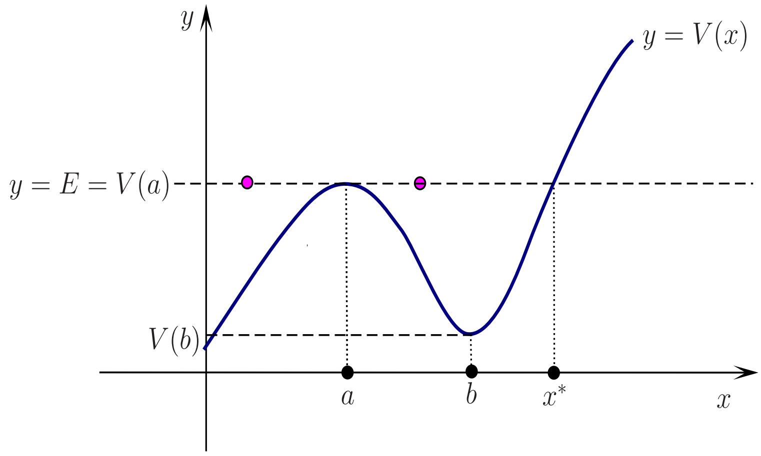



Figure 3.7: Motion in the potential $V(x)$ (Case III).

- **Case III:** $E = V(a)$ (Fig. 3.7).

    (a) If $a < x_0 < x^*$, then, depending on the direction of the initial velocity, the particle will either continue to move in the negative direction of the $x$-axis towards the point of local maximum of $V$, $x = a$ or move first towards the turning point $x^*$ and then back. In both cases, it will be approaching point $x = a$. Note that $V(x)$ has a local maximum at this point, i.e. $V'(a) = 0$, and the force is zero at $x = a$. If the particle was at this point, it would have zero velocity and acceleration and therefore it would remain there forever. It turns out that the particle will never reach this point (it will need infinite time for that).

    To show this, we consider motion of the particle near the point $x = a$. So, we assume that $v_0 < 0$ (motion towards $x = a$) and that initially it is close to $x = a$, i.e. $x_0 - a$ is small. For small $x - a$, we expand the potential energy $V(x)$ into Taylor's series[1] about the point $x = a$ and retain only the first three terms:

    $$V(x) \approx V(a) + (x - a)V'(a) + \frac{(x - a)^2}{2} V''(a).$$

    Since $x = a$ is the point of a local maximum of $V(x)$, its first derivative vanishes at this point and $V''(a) < 0$. Hence,

    $$V(x) \approx V(a) - k\frac{(x - a)^2}{2}$$

    where $k \equiv -V''(a) > 0$. Since the particle is moving in the negative $x$ direction, we have

    $$v = -\sqrt{2(E - V(x))/m} \quad \Rightarrow \quad v = -\sqrt{\frac{k}{m}(x - a)^2} \quad \Rightarrow \quad \dot{x} = -\sqrt{\frac{k}{m}}(x - a).$$

[1] Recall that if function $f(x)$ has $(n + 1)$ continuous derivatives in some interval containing point $x_0$, then $f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + R_n$, and the remainder $R_n$ is given by $R_n = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$ where $\xi$ is a number between $x$ and $x_0$.

Solving the last equation (which is a linear 1st-order inhomogeneous ODE with constant coefficients), we obtain

$$x(t) = a + (x_0 - a)e^{-\sqrt{\frac{k}{m}}t}. \tag{38}$$

Evidently, $x(t) > a$ for any finite $t > 0$ and $x(t) \to a$ as $t \to \infty$.

(b) If $x_0 < a$ and $v_0 < 0$, the particle will continue moving in the negative direction of the $x$-axis and will escape to infinity. If $x_0 < a$ and $v_0 > 0$, the particle will move towards $x = a$. When it is sufficiently close to $a$, its motion is approximately described by Eq. (38) [show this!], so that it will never reach point $x = a$.

### 3.7.2 Equilibrium points and their stability

The points $x = a$ and $x = b$ in Figs. 3.5–3.7 are critical (also called stationary) points of the potential energy $V(x)$ (i.e. $V'(x) = 0$ at these points) and therefore the force acting on the particle at each of these points is zero. Hence, if initially the particle is at rest at any of these points, it will remain there forever.

**Definition 3.3** **Equilibria**

If $x(0) = a$, $\dot{x}(0) = 0$ and $V'(a) = 0$, then the equation of motion,

$$m\ddot{x} = -V'(x),$$

has a constant solution $x(t) = a$ for all $t > 0$. Such points are called **equilibrium positions** (or, simply, **equilibria**) of the particle.

Now let $x^*$ be an equilibrium point of a particle in a potential $V(x)$. Suppose that the initial position and/or velocity of the particle is slightly perturbed from its state of rest at this point, i.e.

$$x(0) = x^* + \delta, \quad \dot{x}(0) = \epsilon$$

where $\delta$ and $\epsilon$ are small. The particle will start to move, and there are two possibilities:

**Definition 3.4** **Stability**

- The perturbed motion of the particle will remain close to the equilibrium for all $t > 0$. If this is so for all small perturbations (i.e. for all sufficiently small $\delta$ and $\epsilon$), the equilibrium is said to be **stable**.
- The perturbed particle will move away from the equilibrium. If there is at least one small perturbation (of the initial position and/or initial velocity) such that the particle moves away from the equilibrium, then this equilibrium is called **unstable**.

The potential shown in Figs. 3.5–3.7 has two equilibrium points at $x = a$ and $x = b$. If we slightly perturb the equilibrium of the particle at $x = b$, the particle will oscillate near this equilibrium, i.e. it will remain close to it. To see this, suppose that the perturbed particle is located at a point $x > b$. Since $x = b$ is a local minimum point of $V$, we observe that

$$V'(x) > 0 \quad \Rightarrow \quad F(x) < 0$$

so that the force is directed towards the equilibrium and the particle will move back towards it. Similarly, if initially the perturbed particle is at a point $x < b$, then

$$V'(x) < 0 \quad \Rightarrow \quad F(x) > 0,$$

so that, again, the force is directed towards the equilibrium and the particle will move back towards it. Thus, the equilibrium at $x = b$ is stable.

Consider now the equilibrium at the point $x = a$. If we perturb it, then the particle will move away from the equilibrium (it will either escape to $-\infty$ or move towards the point $x = b$), so that this equilibrium position of the particle is unstable. This happens because the force near $x = a$ is directed away from the equilibrium. We conclude therefore that the equilibrium at $x = a$ is unstable.

There is another type of an equilibrium corresponding to a critical point of $V$ which is neither a maximum nor a minimum but an inflection point. It can be shown that, in this case, the equilibrium is also unstable (again, this is because there are initial perturbations for which the force will be directed away from the equilibrium).

**Conclusion**:

An equilibrium is stable if it is a local minimum of $V(x)$, i.e. if $V''(x) > 0$ at the equilibrium, and unstable otherwise.

To remember this fact, it is helpful to imagine the following situation (that everyone can see in real life). Suppose that a solid ball is rolling on a curved surface under the uniform gravity force as in Figures 3.8 and 3.9. If a ball is at rest on the top of the hill as in Fig. 3.8, it is

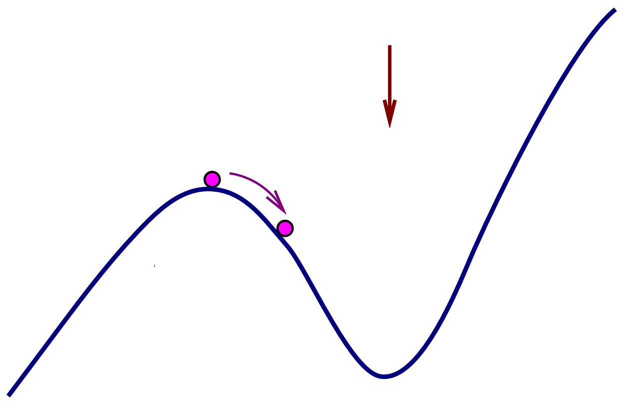


Figure 3.8: An example of an **unstable equilibrium**: a ball on the top of a hill.

at the unstable equilibrium position, because if it is slightly pushed away from this position, it will roll down the slope (as one would expect). However, if we push the ball away from the equilibrium at the bottom of a valley like in Fig. 3.9, it will move a little up the slope, but then the gravity will force it to move back towards the bottom, and it will remain not far from the equilibrium position for all $t > 0$. So, this equilibrium is stable. These examples give a nice visual illustration of the notion of stability and can help you to remember that stability is associated with a minimum of the potential, while instability - with a maximum. A precise mathematical treatment of the problem of a ball rolling on a curved surface is possible, but will not be considered in this course.

A rigorous proof of the above conclusion must be based on a more precise definition of stability.

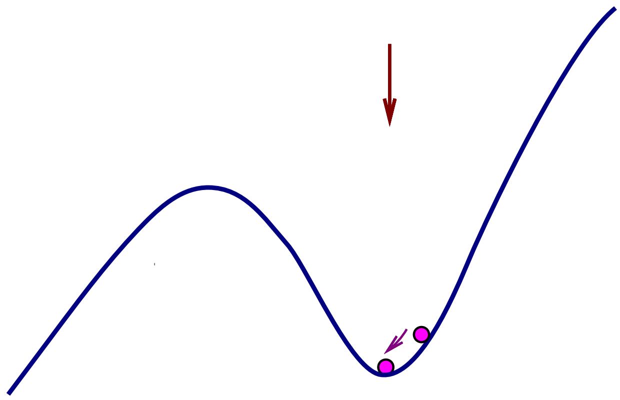


Figure 3.9: An example of a **stable equilibrium**: a ball at the bottom of a valley between two hills.

Although this is outside the scope of this course, it is instructive at least to have a brief look at the formal definition of stability.

**Formal definition of stability (NOT EXAMINABLE)**:

The notion of stability can be made precise in the following way. Let $x^*$ be an equilibrium point (i.e. $V'(x^*) = 0$). This means that if $x(0) = x^*$ and $\dot{x}(0) = 0$, then $x(t) = x^*$ for all $t > 0$. Let's perturb this equilibrium, that is, consider motion with slightly different initial conditions:

$$x(0) = x^* + \tilde{x}_0 \quad \text{and} \quad \dot{x}(0) = \tilde{v}_0 \tag{39}$$

for some small $\tilde{x}_0$ and $\tilde{v}_0$. Let $x(t)$ be the solution of the equation of motion (29) with these initial conditions. Then we say that the equilibrium at $x^*$ is **stable** if for $\forall \epsilon > 0$, there is a positive $\delta$ such that

$$|x(0) - x^*| < \delta \;\text{ and }\; |\dot{x}(0)| < \delta \quad \Rightarrow \quad |x(t) - x^*| < \epsilon \;\text{ for }\; \text{all } t > 0,$$

and **unstable** otherwise.

**Example 3.8 (motion in a potential).** Consider a particle of mass $m$ moving on a straight line under the action of a force with the potential

$$V(x) = \frac{kx}{x^2 + a^2},$$

where $k$ and $a$ are positive constants.

(a) Find the equilibria of the particle, sketch the graph of $V(x)$ and determine whether the equilibria are stable or not.

(b) The particle passes the origin $x = 0$, moving in the positive $x$ direction with the velocity $v_0 > 0$.

(i) Prove that the particle will subsequently pass the point $x = a$ if and only if

$$v_0^2 > \frac{k}{ma}.$$

(ii) Find a condition on $v_0$ such that the particle passes the point $x = -a$.

(c) Find all initial positions and initial velocities for which the particle's motion is periodic.

**Solution.** (a) Equilibria are critical points of $V(x)$. We have

$$V'(x) = \frac{k}{x^2 + a^2} - \frac{2kx^2}{(x^2 + a^2)^2} = \frac{k(x^2 + a^2) - 2kx^2}{(x^2 + a^2)^2} = \frac{k(a^2 - x^2)}{(x^2 + a^2)^2} = 0 \quad \Rightarrow \quad x = \pm a.$$

Also, we have

$$V(\pm a) = \pm\frac{k}{2a}, \quad V(0) = 0, \quad V(x) \to 0 \quad \text{as} \quad x \pm \infty.$$

Using the above information about $V(x)$, one can easily sketch the graph of $V(x)$. It is shown in Fig. 3.10 It is evident that $V(x)$ has a local maximum at $x = a$ and local minimum at $x = -a$,

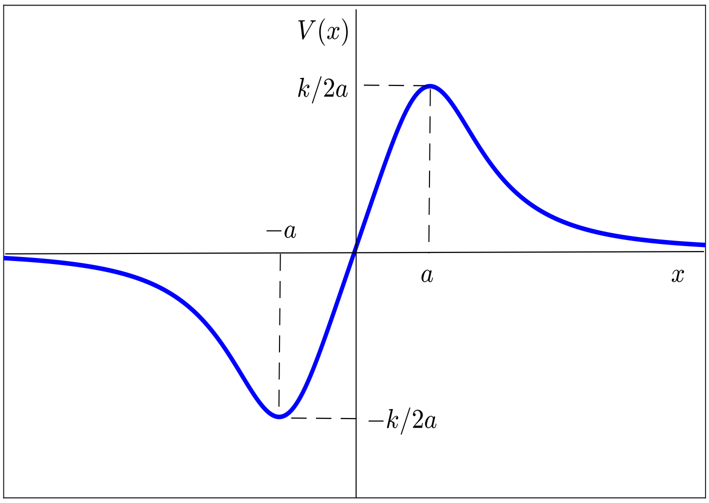



Figure 3.10: Sketch of the potential $V(x) = kx/(x^2 + a^2)$.

so that the equilibrium at $x = a$ is unstable and the equilibrium at $x = -a$ is stable.

(b)(i) The particle will pass the point $x = a$, if the total energy $E$ is greater than $V(a)$. Therefore,

$$\frac{mv_0^2}{2} + V(0) = \frac{mv_0^2}{2} > V(a) = \frac{k}{2a} \quad \Rightarrow \quad v_0^2 > \frac{k}{ma}.$$

(b)(ii) The particle will subsequently pass the point $x = -a$, if its direction of motion reverses, i.e. if the total energy $E$ is less than $V(a)$. Hence,

$$\frac{mv_0^2}{2} + V(0) = \frac{mv_0^2}{2} < V(a) = \frac{k}{2a} \quad \Rightarrow \quad v_0^2 < \frac{k}{ma}.$$

(c) Particle's motion will be periodic if $E < 0$, i.e. if

$$\frac{mv_0^2}{2} + V(x_0) = \frac{mv_0^2}{2} + \frac{kx_0}{x_0^2 + a^2} < 0.$$

This inequality can be satisfied only if $x_0 < 0$. Therefore, all possible values of the initial coordinate and velocity can be described as follows. First, we choose any $x_0 < 0$. Then all possible initial velocities are determined from the inequality

$$\frac{mv_0^2}{2} < -\frac{kx_0}{x_0^2 + a^2} \quad \text{or} \quad v_0^2 < -\frac{2kx_0}{m(x_0^2 + a^2)}.$$

Thus, the set of initial coordinates and velocities corresponding to periodic motion is

$$x_0 \in (-\infty, 0), \qquad v_0 \in \left( -\sqrt{\frac{-2kx_0}{m(x_0^2 + a^2)}}, \sqrt{\frac{-2kx_0}{m(x_0^2 + a^2)}} \right).$$

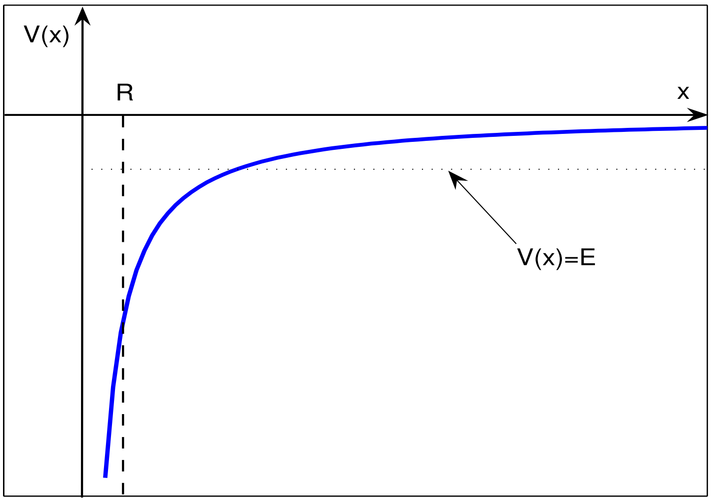



Figure 3.11: Sketch of the gravitational potential $V(x) = -GM/x$.

**Example 3.9 (escape velocity).** Let's find the minimum velocity that is needed for a body of mass $m$ to escape the Earth's gravity. Let $M$ be the mass of the Earth and $R$ its radius. We introduce the $x$ axis that passes through the centre of the Earth and the centre of the body, with origin at the centre of the Earth, and is directed from the Earth to the body. We know that the Newtonian gravitational force of attraction acting on the body and the corresponding gravitational potential are (see section 3.5 above) [2]

$$F = -\frac{GmM}{x^2} \quad \text{and} \quad V(x) = -\frac{GmM}{x}.$$

where $G$ is the gravitational constant. The sketch of $V(x)$ is shown in Fig. 3.11. It is clear that if the total energy is negative, $E < 0$ then the body that initially is at $x = R$ (i.e. at the surface of the Earth) with velocity $v_0 > 0$ will always reach a turning point ($V(x) = E$) where the direction of motion will be reversed, and it always falls back to the Earth. In order to escape

[2] The Earth is not a point mass. However, assuming that the mass distribution in the Earth is spherically symmetric, we can still replace the Earth by a point mass, using the following fundamental fact: the gravitational force exerted on a body by another body whose distribution of mass is spherically symmetric is the same as the force produced by the body of the same mass concentrated in the centre of the sphere.

the Earth's gravity, the total energy must be non-negative, $E \geq 0$. The minimum initial velocity corresponds to $E = 0$. Therefore,

$$E = m\,\frac{\dot{x}^2}{2} - \frac{GmM}{x} = m\,\frac{v_0^2}{2} - \frac{GmM}{R} = 0 \quad \Rightarrow \quad v_0 = \sqrt{\frac{2GM}{R}}.$$

To obtain the numerical value, we need only $R = 6400\,km$ and the free fall acceleration $g = 9.8\,m/s^2$. To see this, we recall the the gravitational force at the Earth's surface is $-mg$, so that

$$-\frac{GmM}{R^2} = -mg \quad \Rightarrow \frac{GM}{R} = gR.$$

Finally,

$$v_0 = \sqrt{2gR} \approx \sqrt{2 \cdot 6400 \cdot 10^3\text{m} \cdot 9.8\,\text{m/s}^2} = \sqrt{1.96}\cdot 8\cdot 10^3\,\text{m/s} = 1.4\cdot 8\cdot 10^3\,\text{m/s} = 11.2\,\text{km/s}.$$

Note that the escape velocity does not depend on the mass of the body, $m$.

## 3.8 Oscillations natural and forced

### 3.8.1 Motion near a stable equilibrium

We already know that if a particle in a stable equilibrium is disturbed slightly (say, by moving it to a nearby position), then it will oscillate about the position of equilibrium. Let us discuss this in more detail.

Consider a particle moving under the force with potential $V(x)$. Let $x = a$ be a stable equilibrium, i.e. $x = a$ is a point of local minimum of $V(x)$. For small $|x - a|$, we expand the potential energy $V(x)$ into Taylor's series about the point $x = a$ and retain only the first three terms:

$$V(x) \approx V(a) + (x - a)V'(a) + \frac{(x - a)^2}{2} V''(a).$$

Since $x = a$ is the point of a local minimum of $V(x)$, we have $V'(a) = 0$ and $V''(a) > 0$. Hence,

$$V(x) \approx V(a) + \frac{(x - a)^2}{2} V''(a) \quad \Rightarrow \quad V'(x) \approx V''(a)\,(x - a)$$

The equation of motion becomes

$$m\ddot{x} = -V''(a)(x - a) \quad \text{or} \quad \ddot{x} = -\omega^2(x - a)$$

where

$$\omega^2 \equiv V''(a)/m.$$

It is convenient to introduce a new variable, $z = x - a$ (this change of variable represents the shift of the origin to the point $x = a$).

**Definition 3.5** **Simple Harmonic Oscillator**

The equation of motion is

$$\ddot{z} + \omega^2 z = 0, \tag{40}$$

which is the equation of a **simple harmonic oscillator**. It undergoes **simple harmonic motion** with general solution

$$z(t) = C_1 \sin \omega t + C_2 \cos \omega t, \tag{41}$$

where $C_1$ and $C_2$ are arbitrary constants. These constants can be determined from suitable initial condition.

The general solution (41) can be written in several equivalent forms. For example, it can written as

$$z(t) = A\sin(\omega t + \delta), \tag{42}$$

where $A$ and $\delta$ are constants with $A \geq 0$ and $0 \leq \delta < 2\pi$. Equation (42) represents the same general solution as Eq. (41). Indeed, using the trigonometric identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, we can rewrite (42) as

$$z(t) = A\sin\omega t\cos\delta + A\cos\omega t\sin\delta.$$

Comparing this with Eq. (41), we conclude that these equations coincide provided that constants $A$, $\delta$ and $C_1$, $C_2$ are related by

$$C_1 = A\cos\delta \quad \text{and} \quad C_2 = A\sin\delta. \tag{43}$$

From these equations $A$ and $\delta$ can be found in terms of $C_1$ and $C_2$. Summing the squares of these equations yields

$$A^2 = C_1^2 + C_2^2 \quad \Rightarrow \quad A = \sqrt{C_1^2 + C_2^2}.$$

Then, if $A \neq 0$, the formulae

$$\cos\delta = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} \quad \text{and} \quad \sin\delta = \frac{C_2}{\sqrt{C_1^2 + C_2^2}}$$

give a unique value for $\delta \in [0, 2\pi)$. (Note that both equations are needed to obtain $\delta$ uniquely.)

It follows from Eq. (42), that the motion is periodic, i.e. there is a constant $T$ such that $z(t + T) = z(t)$ for all $t$. Indeed, we have

$$z(t + T) = A\sin(\omega(t + T) + \delta) = A\sin(\omega t + \delta).$$

Since $\sin(\theta) = \sin(\theta + 2\pi)$ for all $\theta$, this equation is satisfied if

$$\omega T = 2\pi \quad \Rightarrow \quad T = \frac{2\pi}{\omega}.$$

**Definition 3.6** **Period, frequency, angular frequency and phase**

$T = 2\pi/\omega$ is called the **period of oscillation**. The inverse of the period, $\omega/2\pi$ is the number of complete oscillations performed per unit time. This is called the **frequency**, and $\omega$ is called the **angular frequency**. Constant $\delta$ in Eq. (42) is called the **phase constant**.

The typical graph of $z(t)$ is shown in Fig. 3.12.

**Example** of simple harmonic motion (a body attached to a coiled spring) have been already discussed (see Section 3.4).

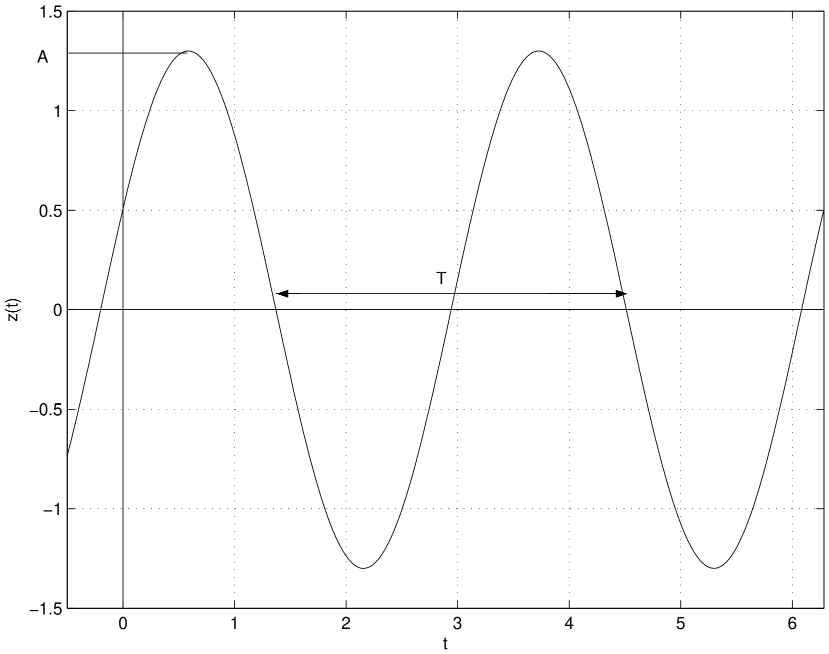



Figure 3.12: The graph of $z(t)$

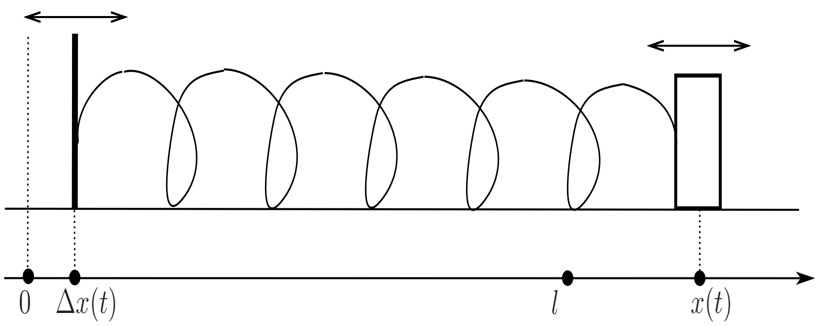



Figure 3.13: A body of mass $m$ attached to a coiled spring whose other end is attached to a vibrating wall.

### 3.8.2 Forced oscillations

Consider the following system shown in Fig. 3.13. A body of mass $m$ lies on a flat smooth surface (there is no friction between the body and the surface) and is attached to the right end of a (weightless) coiled spring of natural length $L$. The left end of the spring is attached to a wall which performs harmonic oscillations. Our aim is to describe the motion of the body.

First, we introduce the coordinate $x$ axis (see Fig. 3.13) such that it is parallel to the surface with the origin at the mean position of the wall. Let $b(t)$ be the position of the wall and $x(t)$ the position of the body at time $t$. Then the length of the spring at time $t$ is $x(t) - b(t)$. We know that the elastic force acting on the body is proportional to the difference between the current length of the spring and its natural length. Therefore, the force is given by $F(x) = -k[x(t) - b(t) - L]$, so that the equation of motion is

$$m\ddot{x} = -k[x(t) - b(t) - l].$$

where $k$ is the elastic constant of the spring. In fact, this equation is valid for arbitrary motion of the wall, but we will restrict our attention to the case of harmonic oscillations. So, we assume that

$$b(t) = \alpha \sin \Omega t, \quad \alpha > 0, \quad \Omega > 0.$$

The equation of motion becomes

$$m\ddot{x} = -k[x(t) - L] + k\alpha \sin \Omega t.$$

Note that the effect of vibration of the wall is described by an additional time-dependent force in the equation of motion. It is convenient to introduce a new variable $z(t) = x(t) - L$ and to divide the equation of motion by $m$. As a result, we obtain

$$\ddot{z} + \omega^2 z = \omega^2 \alpha \sin \Omega t, \tag{44}$$

where $\omega^2 = k/m$ is the natural angular frequency of small oscillations of a mass $m$ attached to one end of the spring whose other end is fixed.

Equation (44) is an example of a linear inhomogeneous second-order ODE. Its general solution is the sum of a particular solution $z_p(t)$ of the inhomogeneous equation (44) and the general solution $z_h(t)$ of the associated homogeneous equation

$$\ddot{z} + \omega^2 z = 0. \tag{45}$$

We already know that the general solution of the homogeneous equation (45) is

$$z_h = C_1 \sin \omega t + C_2 \cos \omega t \quad \text{or, equivalently,} \quad z_h = A \sin(\omega t + \delta).$$

To find a particular solution of Eq. (44), we make an educated guess and assume that it has the form

$$z_p(t) = C \sin \Omega t. \tag{46}$$

Substituting this into Eq. (44), we get

$$-\Omega^2 C \sin \Omega t + \omega^2 C \sin \Omega t = \omega^2 \alpha \sin \Omega t \quad \Rightarrow \quad C = \frac{\omega^2 \alpha}{\omega^2 - \Omega^2}.$$

Thus, the general solution of Eq. (44) is

$$z(t) = \frac{\omega^2 \alpha}{\omega^2 - \Omega^2} \sin \Omega t + A \sin(\omega t + \delta). \tag{47}$$

**Natural and forced oscillations**:

The second term in the solution (47) represents the general solution of the homogeneous equation, which corresponds to **free (natural) oscillations**. The first term is produced by the periodic force on the right side of Eq. (44) and describes the **forced oscillation**.

Choosing suitable initial conditions

$$z(0) = 0, \quad \dot{z}(0) = \Omega \frac{\omega^2 \alpha}{\omega^2 - \Omega^2},$$

we can get rid of the second term in Eq. (47). Indeed, substitution of the general solution into these initial conditions results in

$$z(0) = A\sin(\delta) = 0$$

and

$$\dot{z}(0) = \Omega\,\frac{\omega^2\alpha}{\omega^2-\Omega^2} + A\omega\cos(\delta) = \Omega\,\frac{\omega^2\alpha}{\omega^2-\Omega^2} \quad \Rightarrow \quad A\cos(\delta) = 0.$$

Since for any $\delta$, $\sin(\delta)$ and $\cos(\delta)$ cannot be zero simultaneously, we conclude that $A = 0$. So, the solution reduces to

$$z(t) = \frac{\omega^2\alpha}{\omega^2-\Omega^2}\sin\Omega t, \tag{48}$$

which represents forced oscillations. Let us discuss this solution in more detail. Evidently, the solution is valid only for $\Omega^2 \neq \omega^2$.

The solution given by (48) can also be written in the form (42), i.e.

$$z(t) = \tilde{A}\sin(\Omega t + \tilde{\delta}) \tag{49}$$

where $\tilde{A} \geq 0$ is the amplitude of the forced oscillation and $\tilde{\delta}$ is its phase ($\tilde{\delta} \in [0, 2\pi)$).

If $\Omega^2 < \omega^2$, then by comparing Eqs. (48) and (49) we deduce that the amplitude and the phase constant of the forced oscillations are

$$\tilde{A} = \frac{\omega^2\alpha}{\omega^2-\Omega^2} \quad \text{and} \quad \tilde{\delta} = 0.$$

So, the phase of the forcing term in Eq. (44) and the phase of the forced oscillations coincide, which means that the body ‘follows’ the wall (when the wall moves to the left (to the right), the body does the same).

If $\Omega^2 > \omega^2$, then we have

$$z(t) = -\frac{\omega^2\alpha}{\Omega^2-\omega^2}\sin\Omega t = \frac{\omega^2\alpha}{\Omega^2-\omega^2}\sin(\Omega t + \pi).$$

It follows that

$$\tilde{A} = \frac{\omega^2\alpha}{\Omega^2-\omega^2} \quad \text{and} \quad \tilde{\delta} = \pi,$$

i.e. the phase of the forcing term in Eq. (44) and the phase of the forced oscillations differ by $\pi$. This means that when the wall is moving to the left, the body is moving to the right and vice versa.

**Resonance**:

In both cases, the amplitude of forced oscillations increases without limit as $\Omega^2 \to \omega^2$. This phenomenon is called **resonance**.

So, the solution given by Eq. (48) is not valid if $\Omega^2 = \omega^2$. This case requires a separate treatment, and we seek a particular solution of Eq. (44) in the form

$$z_p = C_1 t\sin\Omega t + C_2 t\cos\Omega t.$$

Substitution of this into Eq. (44) yields

$$2\Omega(C_1\cos\Omega t - C_2\sin\Omega t) = \Omega^2\alpha\sin\Omega t \quad \Rightarrow \quad C_1 = 0 \quad \text{and} \quad 2\Omega C_2 = -\Omega^2\alpha.$$

Hence, $C_2 = -\Omega\alpha/2$, and we obtain

$$z_p = -\frac{\Omega\alpha}{2}\, t\, \cos\Omega t.$$

This can be written as

$$z_p = \tilde{A}(t)\, \sin\left(\Omega t + \tilde{\delta}\right), \quad \tilde{A}(t) \equiv \frac{\Omega\alpha}{2}\, t, \quad \tilde{\delta} \equiv \frac{3\pi}{2}.$$

Thus, in the case of resonance, the amplitude of the forced oscillation grows linearly with time.

## 3.9 Energy and friction

We know that if a particle is moving under a force $F(x) = -V'(x)$, then the energy is a constant of motion (i.e. it doesn't depend on time).

**Question:** will the energy be conserved in the presence of friction?

To answer this question, we consider a particle of mass $m$ moving under a conservative force with a potential $V(x)$ and the Stokes friction force. The equation of motion is

$$m\ddot{x} = -V'(x) - \Gamma\dot{x}. \tag{50}$$

Let's compute the derivative of the energy with respect to time:

$$\dot{E} = \frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + V(x)\right) = m\dot{x}\ddot{x} + V'(x)\dot{x} = \dot{x}\left(-V'(x) - \Gamma\dot{x}\right) + V'(x)\dot{x} = -\Gamma\dot{x}^2.$$

Thus,

$$\dot{E} = -\Gamma\dot{x}^2 \leq 0.$$

This means that the energy of a moving particle decreases with time, i.e. it is not a constant of motion.

**Definition 3.7** **Dissipation**

Forces which lead to a decrease in the energy (dissipate the energy) are said to be **dissipative forces**. And this phenomenon is known as **dissipation of energy** by friction forces.

**Conclusion**:

We conclude that **adding the Stokes friction to the dynamics of a particle leads to dissipation of the particle's energy**.

The natural question that arises in this context: does any force which depends on velocity lead to dissipation of energy? The answer is 'No', and we will see an example of a velocity dependent force that conserves the energy later.

The qualitative picture of motion in a potential is also different when a dissipative force is present. For example, consider a particle moving in potential $V(x) = kx^2/2$ when the Stokes friction is present. Without friction, the particle's energy would be conserved and it would move

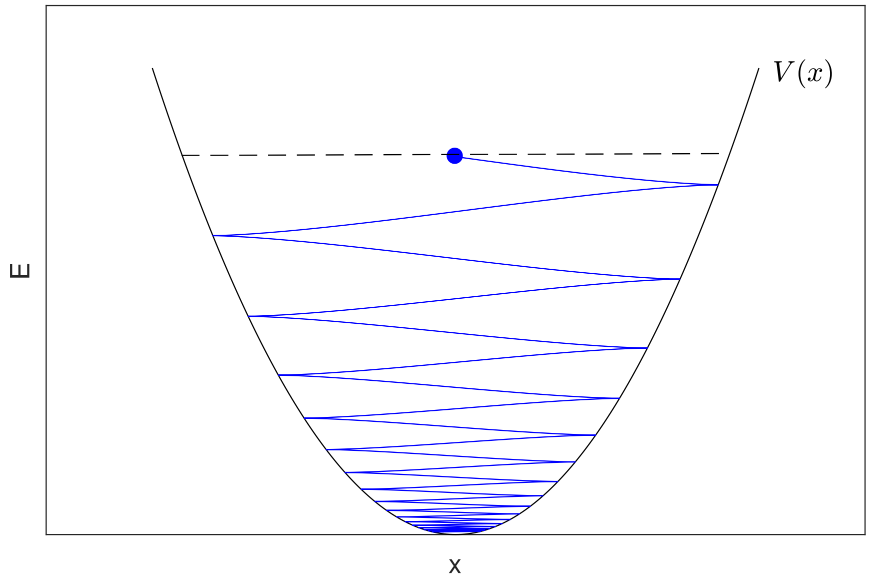



Figure 3.14: A particle moving in the potential $V(x) = kx^2/2$ when there is Stokes friction force.

periodically on the dashed horizontal line in Fig. 3.14. However, if there is even small friction, the energy will gradually decrease and the particle will end up at the bottom of the potential well (at the minimum point of $V(x)$) although this would take infinite time. This problem can be solved exactly if the friction is described by the Stokes friction force: in this case the equation of motion can be reduced to the equation of damped oscillations (see Problem sheet 4).

# 4 Solutions for motion in 2D

Figure 4.1: Motion in 2d - neglecting rotation, of course.



## 4.1 Equations of motion in 2D

Our aim is to generalise one-dimensional motion along a straight line to two-dimensional motion in a plane. Consider a particle of mass $m$ which is moving in a plane. To quantitatively describe the position of the particle, we introduce Cartesian coordinates $(x, y)$ in the plane (see Fig.

4.2). The position of the particle is described by its $x$ and $y$ coordinates or, equivalently, by its position vector

$$\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the direction of the $x$ and $y$ axes respectively (see Fig. 4.2).

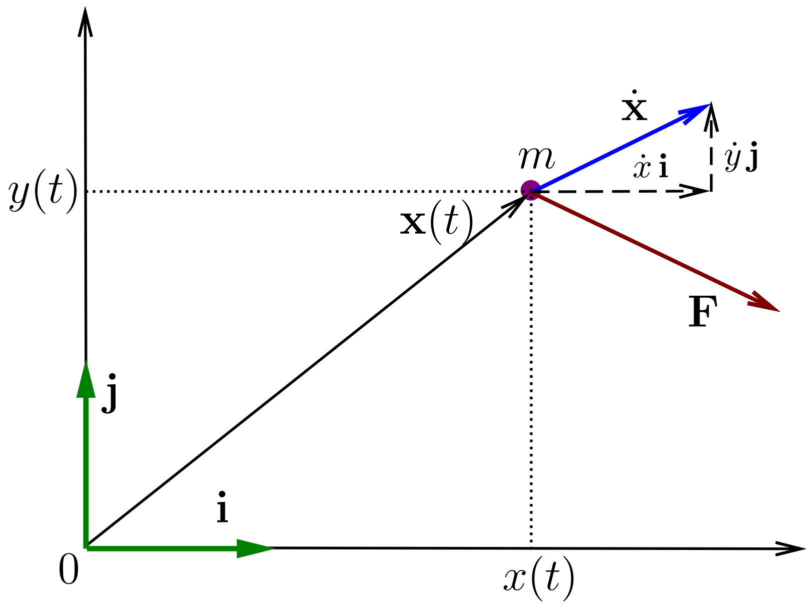



Figure 4.2: Position vector of a particle in the $xy$ plane.

A force acting on the particle is also a vector

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}.$$

Here $F_x$ and $F_y$ are the $x$ and $y$ components of the force $\mathbf{F}$.

When the particle is moving, its position vector is changing with time. This means that both coordinates (components of the position vector $\mathbf{x}$) are functions of time: $x = x(t)$ and $y = y(t)$. The position vector is also a vector-valued function of time:

$$\mathbf{x}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

The velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ are defined as the time derivatives of the position vector:

$$\mathbf{v}(t) = \dot{\mathbf{x}}(t) = \dot{x}(t)\,\mathbf{i} + \dot{y}(t)\,\mathbf{j} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}, \quad \mathbf{a}(t) = \ddot{\mathbf{x}}(t) = \ddot{x}(t)\,\mathbf{i} + \ddot{y}(t)\,\mathbf{j} = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix}.$$

Now we can write:

**Definition 4.1** **The equation of motion of a particle in 2D**

$$m\ddot{\mathbf{x}} = \mathbf{F} \quad \text{or} \quad m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}.$$

Note that this vector equation can be treated as a system of two ODEs, such that

$$\begin{cases} m\ddot{x} = F_x, \\ m\ddot{y} = F_y. \end{cases}$$

## 4.2 Motion under uniform gravity

Consider motion of a particle of mass $m$ under the uniform gravity force. First we introduce Cartesian coordinates $x, y$ such that the $x$ axis is horizontal and represents the surface of the Earth, and $y$ axis is vertical and directed upward. In this coordinate system, the free fall acceleration vector is given by $\mathbf{g} = -g\,\mathbf{j}$. The equation of motion of the particle is

$$m\ddot{\mathbf{x}} = m\mathbf{g} \quad \text{or} \quad \ddot{\mathbf{x}} = -g\mathbf{j} \quad \text{or} \quad \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix}.$$

So, we have two simultaneous ODEs:

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g.$$

The general solution of the first equation is $x(t) = A_1 + B_1 t$, and the general solution of the second is $y(t) = A_2 + B_2 t - gt^2/2$ (where $A_1, A_2, B_1, B_2$, are arbitrary constants). We can also write the general solution of these two equations in a vector form as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + t \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} - \begin{pmatrix} 0 \\ gt^2/2 \end{pmatrix}$$

or,

$$\mathbf{x}(t) = \mathbf{A} + \mathbf{B}\,t - \frac{gt^2}{2}\mathbf{j}, \quad \mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Note that the general solution contains 4 arbitrary constants $A_1, A_2, B_1, B_2$ or, equivalently, two arbitrary vectors $\mathbf{A}$, $\mathbf{B}$. To select particular values of these constants, we impose initial conditions

$$\mathbf{x}(0) = \mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad \text{and} \quad \dot{\mathbf{x}}(0) = \mathbf{v}_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

Substituting the general solution into these conditions, we obtain

$$\mathbf{A} = \mathbf{x}_0 \quad \text{and} \quad \mathbf{B} = \mathbf{v}_0.$$

So, the solution that satisfies the required initial conditions is

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_0\,t - \frac{gt^2}{2}\mathbf{j}. \tag{51}$$

**Example 4.1 (projectile motion)**. A projectile is launched with initial speed $|\mathbf{v}_0| = V_0$ at an angle $\theta$ to the horizontal. Find (i) the highest point of its trajectory and (ii) the range $D$ (the distance travelled in the horizontal direction).

**Solution**. We already have the formula for the solution (Eq. (51)) satisfying initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0$. In this particular problem,

$$\mathbf{x}_0 = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} V_0 \cos\theta \\ V_0 \sin\theta \end{pmatrix}.$$

So, the solution, given by (51), takes the form

$$\mathbf{x}(t) = \mathbf{v}_0\,t - \frac{gt^2}{2}\mathbf{j} \quad \text{or} \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} V_0 \cos\theta\, t \\ V_0 \sin\theta\, t - gt^2/2 \end{pmatrix}. \tag{52}$$

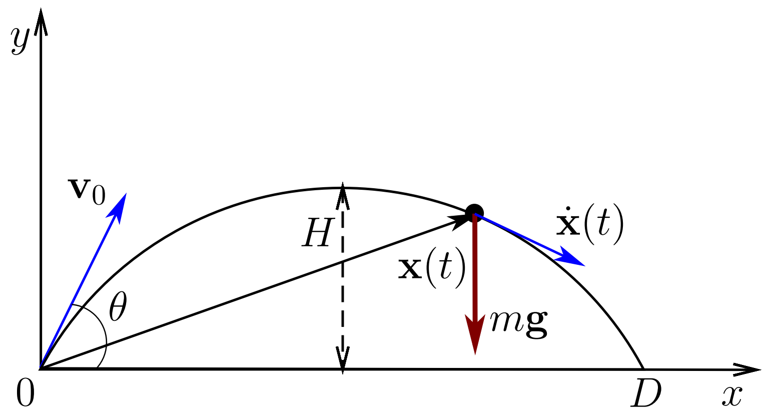



Figure 4.3: Motion of a projectile.

(i) To find the highest point of the trajectory, we need to know the time $t_1$ that corresponds to this point. Evidently, the vertical component of the velocity at this point is zero. Therefore, we obtain (using Eq. (52))

$$\dot{y}(t_1) = 0 \quad \Rightarrow \quad V_0 \sin\theta - g t_1 = 0 \quad \Rightarrow \quad t_1 = \frac{V_0 \sin\theta}{g}.$$

It follows that the highest point of the trajectory is

$$\mathbf{x}(t_1) = \begin{pmatrix} V_0 \cos\theta \, t_1 \\ V_0 \sin\theta \, t_1 - g t_1^2/2 \end{pmatrix} = \frac{V_0^2}{2g} \begin{pmatrix} \sin 2\theta \\ \sin^2\theta \end{pmatrix}.$$

So, now we know the coordinates of the highest point of the trajectory as functions of $\theta$. Evidently, if $\theta = \pi/2$ (i.e. the projectile is launched vertically), then $x(t_1) = 0$ and the maximum height is $y(t_1) = V_0^2/(2g)$ (which is the same as the answer we have obtained in the relevant one dimensional problem).

The time $t_2$ when the projectile hits the ground is a positive root of the quadratic equation $y(t) = V_0 \sin\theta \, t - g t^2/2 = 0$ and is given by $t_2 = 2V_0 \sin\theta/g$. Hence, the range is

$$D = x(t_2) = \frac{V_0^2}{g} \sin 2\theta.$$

(The same answer can also be obtained by observing that the trajectory (which is a part of parabola) is symmetric relative to its middle point, so that $t_2 = 2t_1$ and $x(t_2) = 2x(t_1)$.) It also follows from the formula for $D$ that the maximum range is attained at $\theta = \pi/4$.

## 4.3 Motion in a potential (under a conservative force)

Consider a particle of mass $m$ that is moving under a force depending on its position in two dimensions, i.e.

$$\mathbf{F} = \mathbf{F}(\mathbf{x}) = F_x(x,y)\mathbf{i} + F_y(x,y)\mathbf{j} = \begin{pmatrix} F_x(x,y) \\ F_y(x,y) \end{pmatrix}.$$

The equations of motion take the form

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \quad \text{or} \quad m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} F_x(x,y) \\ F_y(x,y) \end{pmatrix}.$$

Now we will focus on a particular (but very important) class of *conservative* forces.

### 4.3.1 Conservative forces

**Definition 4.2** **Conservative Force**

$\mathbf{F}(\mathbf{x})$ is a **conservative force** if there is a function $V(\mathbf{x})$ (called the **potential** of the force $\mathbf{F}$) such that

$$\mathbf{F} = -\nabla V = -\left(\frac{\partial V}{\partial x}(x,y)\mathbf{i} + \frac{\partial V}{\partial y}(x,y)\mathbf{j}\right) = -\begin{pmatrix} \partial V/\partial x \\ \partial V/\partial y \end{pmatrix}.$$

We note in passing that the potential $V(\mathbf{x})$, if it exists, is not unique as it is defined up to a constant (i.e. if $V(\mathbf{x})$ is a potential for a given $\mathbf{F}(\mathbf{x})$, then $V(\mathbf{x}) + C$ with arbitrary constant $C$ is also a potential for the same force).

In contrast with the one-dimensional case, not all forces are conservative in two (or three) dimensions. For example, let

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \lambda y \\ -\lambda x \end{pmatrix} \tag{53}$$

for a positive constant $\lambda$. If it was a conservative force, we would have the relations

$$\lambda y = -\frac{\partial V}{\partial x}(x,y) \quad \text{and} \quad -\lambda x = -\frac{\partial V}{\partial y}(x,y)$$

or

$$\frac{\partial V}{\partial x}(x,y) = -\lambda y, \tag{54}$$

$$\frac{\partial V}{\partial y}(x,y) = \lambda x. \tag{55}$$

Integrating (54) in variable $x$, we find that

$$V(x,y) = -\lambda xy + g(y)$$

where $g(y)$ is an arbitrary function of one variable. (Note that integration in one variable yields not a constant of integration but a function of the other variable.) Substituting this into (55), we obtain

$$-\lambda x + g'(y) = \lambda x \quad \Rightarrow \quad g'(y) = 2\lambda x.$$

It is impossible to satisfy the last equation for any choice of $g(y)$. So, we conclude that Eqs. (54) and (55) cannot be satisfied simultaneously and, therefore, the force is not conservative. (Note that we would arrive at the same conclusion if we first integrate Eq. (55) rather than Eq. (54). Verify this!)

**Test for non-existence of $V(\mathbf{x})$:**

There is a simple test to determine if the force is conservative or not. The test is based on the fact that if function $f(x, y)$ is sufficiently 'good' (more precisely, if all its second-order partial derivatives are continuous) then

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right). \tag{56}$$

Now let's assume that $\mathbf{F}(\mathbf{x}) = F_x(x, y)\mathbf{i} + F_y(x, y)\mathbf{j}$ is conservative. Then there is $V(x, y)$ such that

$$\frac{\partial V}{\partial x} = -F_x(x, y), \quad \frac{\partial V}{\partial y} = -F_y(x, y). \tag{57}$$

Let's differentiate the first of equations (57) with respect to $y$ and the second with respect to $x$:

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = -\frac{\partial F_x}{\partial y}, \quad \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) = -\frac{\partial F_y}{\partial x}.$$

Assuming that $V$ is sufficiently smooth, we employ property (56). This yields

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) \quad \Rightarrow \quad \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

or, equivalently,

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0. \tag{58}$$

So, if (58) is not satisfied, then the force is not conservative. The converse statement, saying that if (58) is satisfied, then the force is conservative, is also true under a certain additional restriction (which will be satisfied for all problems considered in this course). [a]

[a] Those who are interested can find more on this topic in any vector calculus textbook.

Let's apply this test on the above example. From (53), we have

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = -\lambda - \lambda = -2\lambda \neq 0.$$

So, the force is not conservative.

### 4.3.2 Examples of conservative forces

- Uniform gravity: if the $y$ axis is vertical and directed upward and the $x$ axis is parallel to the ground (see Fig. 4.4(a)), then

  $$\mathbf{F} = m\mathbf{g} = -mg\mathbf{j} \quad \text{and} \quad V(\mathbf{x}) = mgy.$$

  Let's verify that $V = mgy$ is the potential. We have

  $$\frac{\partial V}{\partial x} = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = mg \quad \Rightarrow \quad \mathbf{F} = \begin{pmatrix} 0 \\ -mg \end{pmatrix} = -mg\mathbf{j},$$

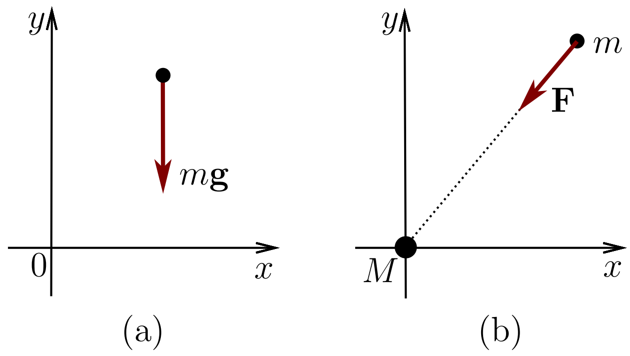



Figure 4.4: (a) uniform gravity force; (b) Newton's gravitational force.

as required.

- Newtonian gravitation: the Newtonian gravitational force of attraction between two bodies of masses $m$ and $M$ is $GmM/r^2$, where $G$ is the gravitational constant and $r$ is the distance between the centres of the bodies. Let $M$ be the mass of the Earth and $m$ be the mass of another spherically symmetric body and let the Cartesian $x$ and $y$ axes be such that (i) the origin is at the centre of the Earth (see Fig. 4.4(b)), (ii) the initial position of the body lies in the $xy$ plane and (iii) the initial velocity of the body is parallel to the $xy$ plane. The the motion of the body will be restricted to the $xy$ plane, and we have

$$\mathbf{F} = -\frac{GmM}{|\mathbf{x}|^2}\frac{\mathbf{x}}{|\mathbf{x}|} \quad \text{and} \quad V(\mathbf{x}) = -\frac{GmM}{|\mathbf{x}|}.$$

where $|\mathbf{x}| = \sqrt{x^2+y^2}$.

It is not difficult to show that $V = -GmM/|\mathbf{x}|$ is the potential. Indeed, we have

$$\frac{\partial V}{\partial x} = \frac{GmM}{|\mathbf{x}|^2}\frac{x}{|\mathbf{x}|} \quad \text{and} \quad \frac{\partial V}{\partial y} = \frac{GmM}{|\mathbf{x}|^2}\frac{y}{|\mathbf{x}|} \quad \Rightarrow \quad \nabla V = \frac{GmM}{|\mathbf{x}|^2}\frac{\mathbf{x}}{|\mathbf{x}|},$$

where we have used the formulae:

$$\frac{\partial|\mathbf{x}|}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2\right)^{1/2} = \frac{x}{|\mathbf{x}|}, \quad \frac{\partial|\mathbf{x}|}{\partial y} = \frac{\partial}{\partial x}\left(x^2+y^2\right)^{1/2} = \frac{y}{|\mathbf{x}|}.$$

Hence

$$\mathbf{F} = -\nabla V = -\frac{GmM}{|\mathbf{x}|^2}\frac{\mathbf{x}}{|\mathbf{x}|},$$

as required.

**Example 4.2** The force acting on a particle in the $xy$ plane is given by

$$\mathbf{F} = \begin{pmatrix} -Ax - Cy \\ -Cx - By \end{pmatrix}, \quad A > 0, \quad B > 0, \quad C > 0.$$

Is this force conservative? If yes, find its potential.

**Solution.** Let's check whether (58) holds:

$$\frac{\partial(-Cx - By)}{\partial x} - \frac{\partial(-Ax - Cy)}{\partial y} = -C - (-C) = 0.$$

So, the force is conservative. By definition of $V$,

$$\frac{\partial V}{\partial x}(x,y) = Ax + Cy \quad \text{and} \quad \frac{\partial V}{\partial y}(x,y) = Cx + By.$$

Integrating the first of these in $x$, we obtain

$$V(x,y) = A\frac{x^2}{2} + Cxy + g(y)$$

where $g(y)$ is an arbitrary function of one variable. Substituting this into the second equation, we find that

$$Cx + g'(y) = Cx + By \quad \Rightarrow \quad g'(y) = By \quad \Rightarrow \quad g(y) = B\frac{y^2}{2} + D$$

where $D$ is an arbitrary constant. Therefore, the potential for $\mathbf{F}$ is given by

$$V(x,y) = A\frac{x^2}{2} + Cxy + B\frac{y^2}{2} + D.$$

## 4.4 Conservation of energy

We know that motion of a particle of mass $m$ under a conservative force in one dimension conserves the energy of the particle:

$$E = \frac{m\dot{x}^2}{2} + V(x) = \text{const.}$$

**Question: what is the energy when the particle is moving under a conservative force in two dimensions?**

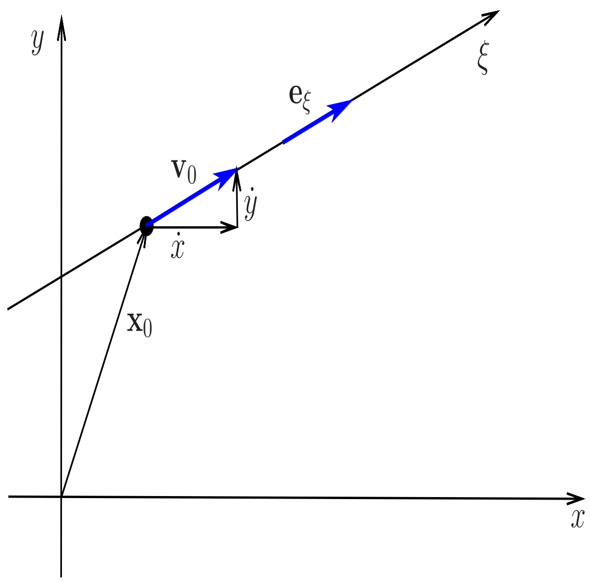



Figure 4.5

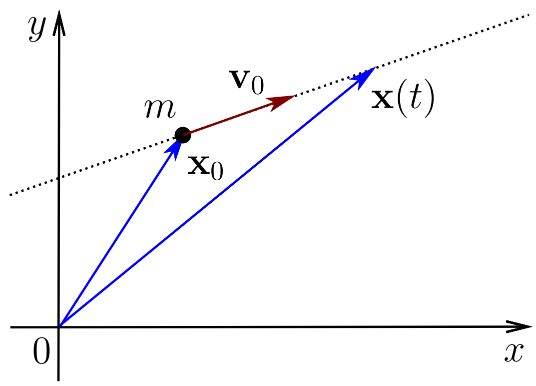



Figure 4.6: Motion of a free particle.

**Definition 4.3** **Energy in a potential in 2D**

By analogy with one-dimensional motion, we expect that the energy of a particle moving in a potential $V(x, y)$ in two dimensions is given by

$$E = m\frac{|\dot{\mathbf{x}}|^2}{2} + V(\mathbf{x}) \qquad \text{or, equivalently,} \quad E = m\frac{\dot{x}^2 + \dot{y}^2}{2} + V(x, y). \tag{59}$$

Here we use the notation $|\dot{\mathbf{x}}|^2 = \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} = \dot{x}^2 + \dot{y}^2$. In Eq. (59), $m(\dot{x}^2 + \dot{y}^2)/2 = m\,|\dot{\mathbf{x}}|^2/2$ is the **kinetic energy** of the particle, and $V(x, y)$ is the **potential energy** of the particle.

**Question: is the energy, given by (59), a constant of motion?**

For a particle moving in a potential $V(\mathbf{x})$ the equations of motion are

$$m\ddot{\mathbf{x}} = -\nabla V \quad \text{or} \quad m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -\begin{pmatrix} \partial V/\partial x \\ \partial V/\partial y \end{pmatrix}.$$

Let's compute the derivative of $E$ with respect to $t$. We have

$$\begin{aligned} \frac{dE}{dt} &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} \\ &= \dot{x}\left(-\frac{\partial V}{\partial x}\right) + \dot{y}\left(-\frac{\partial V}{\partial y}\right) + \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = 0. \end{aligned}$$

Here we have used the chain rule:

$$\frac{d}{dt}V(x(t), y(t)) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}.$$

So, the energy, given by (59), is indeed a constant of motion.

## 4.5 Solutions using energy conservation

**Example 4.3 (Sliding down)** Consider a body of mass $m$ that can slide along a smooth surface shown in Fig. 4.7. There is no friction. Find the velocity of the body $\mathbf{v}_1 = (v_1, 0)$ if its initial velocity is $\mathbf{v}_0 = (v_0, 0)$.

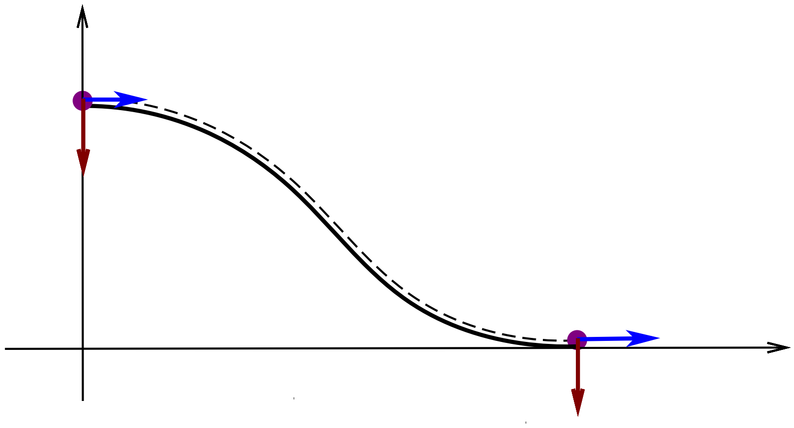


Figure 4.7

**Solution.** The energy of the body at $t = 0$ is

$$E(0) = \frac{mv_0^2}{2} + mgh.$$

The energy of the body at time $t_1$ when its velocity is $\mathbf{v}_1$ is given by

$$E(t_1) = \frac{mv_1^2}{2}.$$

Since the energy is a constant of motion, we obtain

$$E(0) = E(t_1) \quad \Rightarrow \quad \frac{mv_0^2}{2} + mgh = \frac{mv_1^2}{2} \quad \Rightarrow \quad v_1 = \sqrt{v_0^2 + 2gh}.$$

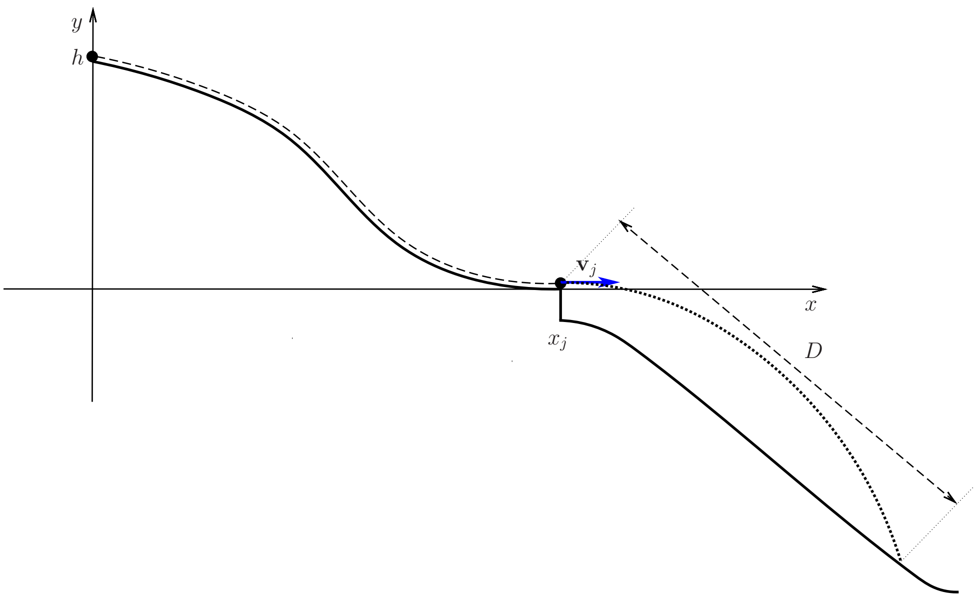



Figure 4.8: Sketch of a skiflying hill.

**Example 4.4** **(Skiflying)** Now let us try to estimate the length of the world record ski jump that can be seen here. The sketch of the hill is shown in Fig. 4.8.

First, we assume that there is no friction during both the initial descent and the flight. Second, we replace the skiflyer with a particle of mass $m$. Third, we assume that when the skiflyer takes off, his velocity is horizontal.

Our aim is to find the length of the jump, $D$ (see Fig. 4.8). To be able to do this, we need to know the velocity at the moment of take-off, $\mathbf{v}_j = (v_j, 0)$, and the angle $\beta$. The angle $\beta$ can be found at the scheme of the skiflying hill 'NOR VIKERSUND HS225' (available at ). We take $\beta = 35°$.

To find $v_j$, we need the difference in height between the starting point and the take-off point, $h$. From the same scheme it can be estimated as $h \approx 73m$. To determine $v_j$, we employ conservation of energy. At the start point, the velocity of the skiflyer is zero, and the energy is $E(0) = mgh$. At the time of take-off, $t_j$, the energy is $E(t_j) = mv_j^2/2$. Therefore,

$$mgh = \frac{mv_j^2}{2} \quad \Rightarrow \quad v_j = \sqrt{2gh} \approx \sqrt{2 \cdot 10 \cdot 73}\, m/s \approx 38\, m/s.$$

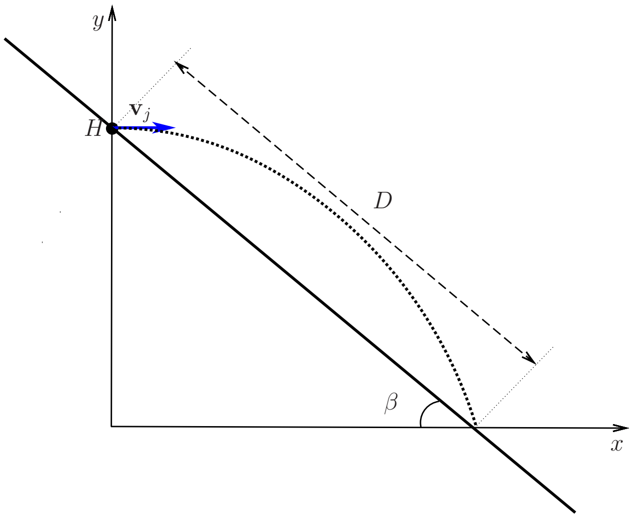



Figure 4.9: The stage of flight.

The stage of flight can be described as the motion of the projectile above the inclined plane as shown in Fig. 4.9. Let $H$ be the height of the point of take-off above the point of landing. Initially, at time $t = 0$ the velocity is horizontal, $\mathbf{v}_j = (v_j, 0)$, and the position is $\mathbf{x}_j = (0, H)$. The equations of motion are

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -mg \end{pmatrix}.$$

Solving these subject to the initial conditions

$$x(0) = 0, \quad y(0) = H, \quad \dot{x}(0) = v_j, \quad \dot{y}(0) = 0,$$

we obtain

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} v_j t \\ H - gt^2/2 \end{pmatrix}.$$

At $t = t_2$ (the moment of landing), $y(t_2) = 0$. Hence,

$$H - \frac{gt_2^2}{2} = 0 \quad \Rightarrow \quad t_2 = \sqrt{\frac{2H}{g}}.$$

It follows that the $x$ coordinate of the point of landing is

$$x(t_2) = v_j t_2 = v_j \sqrt{\frac{2H}{g}}.$$

On the other hand, Fig. 4.9 shows that $H/x(t_2) = \tan(\beta)$. Therefore,

$$\tan^2(\beta) = \frac{H^2}{(x(t_2))^2} = \frac{gH}{2v_j^2} \quad \text{or} \quad H = 2\,\frac{v_j^2}{g}\,\tan^2(\beta) \approx 144\,m.$$

Finally, we compute the length of the jump $D$:

$$D = H/\sin(\beta) \approx 253\,m.$$

So, our estimate matches the world record of 253.5 $m$ set by Stefan Kraft at Vikersund in 2017. It is amazing that a simple model can result in such a good estimate of a real life system. Of course, this might be coincidental as models are unlikely to be perfect; we may have missed some vital aspects of the physics that combine to give us the agreement seen. Does this mean that the world record cannot be beaten? - time will tell.

Figure 4.10: Fast enough to leave the road.

Source: Creative Commons 4.0; Yoann81; https://commons.wikimedia.org/wiki/File:Jump_off_road.jpg

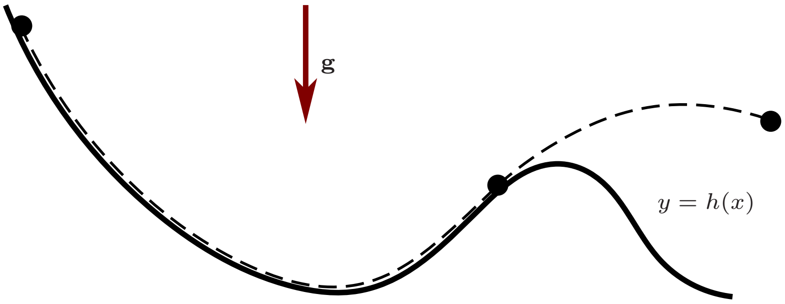
**Example 4.5 (Taking off)** Consider a particle of mass $m$ sliding along a given smooth surface $y = h(x)$ without friction (see Fig. 4.11). If its velocity is high enough it sometimes can take off from the surface and fly (those who drive a car fast over a hill can experience events like this). Can we predict conditions under which the particle will take off?

**Solution.** To answer this question, we first make an observation that when the particle is moving on the surface its velocity is tangent to it (i.e. parallel to the tangent line to the surface at the same point). Let at time $t = 0$ the particle be at point $\mathbf{x}(0) = \mathbf{x}_0 = (x_0, y_0)$ on the surface, i.e. $y_0 = h(x_0)$ where $y = h(x)$ is the equation of the surface. Its velocity $\mathbf{v}_0$ in parallel to the tangent line to the surface at this point, which means that the angle $\theta$ between $\mathbf{v}_0$ and the horizontal satisfies the relation

$$\tan\theta = h'(x_0). \tag{60}$$

If $\mathbf{x}_0$ is the point of take-off, then the particle will fly for $t > 0$, and its motion will be governed by the equations of motion of a projectile:

$$m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$$



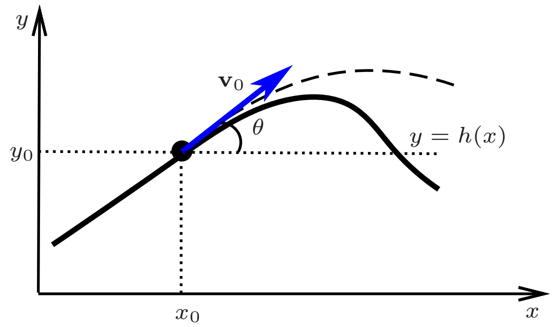




Figure 4.11: Will we take off?

Solving these subject to the initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad \dot{x}(0) = v_0 \cos\theta, \quad \dot{y}(0) = v_0 \sin\theta,$$

we obtain

$$x(t) = x_0 + v_0 \cos(\theta)\, t, \quad y(t) = y_0 + v_0 \sin(\theta)\, t - \frac{gt^2}{2}. \tag{61}$$

Functions $x(t)$ and $y(t)$ represent parametric equation of the particle's trajectory. It is convenient to rewrite it as $y = G(x)$ (i.e. as a Cartesian equation of the trajectory). To do this, we need to eliminate $t$ from Eqs. (61). From the first equation (61), we have

$$t = \frac{x - x_0}{v_0 \cos(\theta)}.$$

Substitution of this into the second equation (61) yields

$$y = G(x) = y_0 + \tan(\theta)(x - x_0) - \frac{g}{v_0^2 \cos^2(\theta)} \frac{(x - x_0)^2}{2}.$$

So, as expected, we have obtained the equation of a parabola. The particle's trajectory $y = G(x)$ will be above the surface $y = h(x)$ if

$$\Delta y(x) = G(x) - h(x) > 0 \quad \text{for} \quad x > x_0.$$

If we restrict our analysis to a small neighbourhood of $x_0$, we can expand $h(x)$ in Taylor's series about $x_0$, i.e.

$$h(x) = h(x_0) + (x - x_0)h'(x_0) + \frac{(x - x_0)^2}{2} h''(x_0) + \ldots$$

Substituting this into $\Delta y(x)$ and ignoring the higher order terms, we obtain

$$\Delta y(x) = y_0 - h(x_0) + [\tan(\theta) - h'(x_0)](x - x_0) + \left[-\frac{g}{v_0^2 \cos^2(\theta)} - h''(x_0)\right] \frac{(x - x_0)^2}{2}.$$

With the help of (60) and the fact that $y_0 = h(x_0)$, this simplifies to

$$\Delta y(x) = \left[-\frac{g}{v_0^2 \cos^2(\theta)} - h''(x_0)\right] \frac{(x - x_0)^2}{2}.$$

Now it is clear that $\Delta y(x) > 0$ provided that

$$-\frac{g}{v_0^2\cos^2(\theta)} - h''(x_0) > 0 \quad \Leftrightarrow \quad h''(x_0) < -\frac{g}{v_0^2\cos^2(\theta)}.$$

If we eliminate $\cos^2(\theta)$ with the help of the identity $1 + \tan^2\theta = 1/\cos^2\theta$, this inequality can be rewritten as

$$-\frac{h''(x_0)}{1 + (h'(x_0))^2} > \frac{g}{v_0^2}.$$

This inequality means that the surface must be concave (negative $h''(x_0)$). It also implies that $v_0$ cannot be too small (for any given $h(x)$ such that $h''(x_0) < 0$, the above inequality will not be satisfied for sufficiently small $v_0$).

**Example 4.6** **(A conservative force)** Consider a particle of mass $m$ moving under action of the conservative force

$$\mathbf{F} = -Ax\,\mathbf{i} - By\,\mathbf{j},$$

where $A$ and $B$ are positive constants.

(a) Find the potential $V(\mathbf{x})$.

(b) Write down the equations of motion and solve them subject to the initial conditions

$$\mathbf{x}(0) = x_0\,\mathbf{i} + y_0\,\mathbf{j}, \quad \dot{\mathbf{x}}(0) = u_0\,\mathbf{i} + v_0\,\mathbf{j}.$$

(c) Find a condition on constants $A$ and $B$ which must be satisfied for the trajectory of the particle to be a closed curve on the $(x, y)$-plane.

**Solution.** (a) If $V(\mathbf{x})$ is a potential, then

$$\begin{matrix} \frac{\partial V}{\partial x} = -F_x \\ \frac{\partial V}{\partial y} = -F_y \end{matrix} \quad \Rightarrow \quad \begin{matrix} \frac{\partial V}{\partial x} = Ax \\ \frac{\partial V}{\partial y} = By \end{matrix}.$$

Integrating the first equation in $x$, we find that

$$V(x, y) = \frac{Ax^2}{2} + g(y)$$

for an arbitrary function of one variable $g(y)$. Substituting this into the second equation and integrating in $y$, we obtain

$$g'(y) = By \quad \Rightarrow \quad g(y) = \frac{By^2}{2} + D$$

for an arbitrary constant $D$. Since the potential is defined up to addition of a constant, we can choose it as we wish. So, we choose $D = 0$. Then

$$V(x, y) = \frac{Ax^2 + By^2}{2}.$$

(b) The equations of motion:

$$m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -Ax \\ -By \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -\omega_1^2 x \\ -\omega_2^2 y \end{pmatrix}$$

where $\omega_1^2 = \frac{A}{m}$ and $\omega_2^2 = \frac{B}{m}$. Thus, we have the following system of two scalar equations

$$\begin{cases} \ddot{x} + \omega_1^2 x = 0, \\ \ddot{y} + \omega_2^2 y = 0. \end{cases}$$

These should be solved subject to the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = u_0, \quad y(0) = y_0, \quad \dot{y}(0) = v_0.$$

Each of the above equations coincides with the equation of the simple harmonic oscillator that has been solved earlier. The solutions are given by

$$\begin{aligned} x(t) &= x_0 \cos(\omega_1 t) + \frac{u_0}{\omega_1} \sin(\omega_1 t), \\ y(t) &= y_0 \cos(\omega_2 t) + \frac{v_0}{\omega_2} \sin(\omega_2 t). \end{aligned}$$

Typical trajectories of the particle for $x_0 = -0.2$, $y_0 = 0.5$, $u_0 = 0.6$, $v_0 = -0.7$, $\omega_1 = 1$ and different values of $\omega_1/\omega_2$, plotted with the help of MATLAB, are shown in Fig. 4.12. Clearly, the trajectories for $\omega_1/\omega_2 = 1, 1/2$ and $2/3$ are closed curves. The trajectory for $\omega_1/\omega_2 = (1+\sqrt{5})/2$ is not a closed curve.

(c) To explain the behaviour of the trajectories, it is convenient to present the solution in the form

$$x(t) = A_1 \sin(\omega_1 t + \delta_1), \tag{62}$$
$$y(t) = A_2 \sin(\omega_2 t + \delta_2), \tag{63}$$

where $A_1, A_2 > 0$ and $\delta_1, \delta_2 \in [0, 2\pi)$. (We know how this can be done from section 3.8.)

**Remark.** If function $f(t)$ is periodic with period $T$ (i.e. $f(t+T) = f(t)$ for all $t \in \mathbb{R}$), then it is also periodic with periods $2T$, $3T$, *etc*, i.e. with period $nT$ for $n \in \mathbb{N}$. (If you aren't convinced, try proving it!) For example, functions $\sin t$ and $\cos t$ are periodic with period $T = 2\pi n$ for $n \in \mathbb{N}$.

If we say that the trajectory of the particle is a closed curve in the $(x, y)$ plane, this means that there is $T > 0$ such that

$$\mathbf{x}(t+T) = \mathbf{x}(t) \quad \text{or, equivalently,} \quad \begin{cases} x(t+T) = x(t) \\ y(t+T) = y(t) \end{cases} \quad \text{for} \quad t \in \mathbb{R}.$$

It follows from Eqs. (62), (63) that the above conditions are equivalent to

$$\omega_1 T = 2\pi n_1 \quad \text{and} \quad \omega_2 T = 2\pi n_2$$

for some $n_1, n_2 \in \mathbb{N}$. These imply that

$$T = \frac{2\pi n_1}{\omega_1} = \frac{2\pi n_2}{\omega_2}.$$

The last equality in possible only if

$$\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} \quad \text{for some} \quad n_1, n_2 \in \mathbb{N}.$$

This means that the trajectory of the particle will be a closed curve only if the ratio of frequencies, $\omega_1/\omega_2$, is a rational number. This is consistent with the trajectories shown in Fig. 4.12.

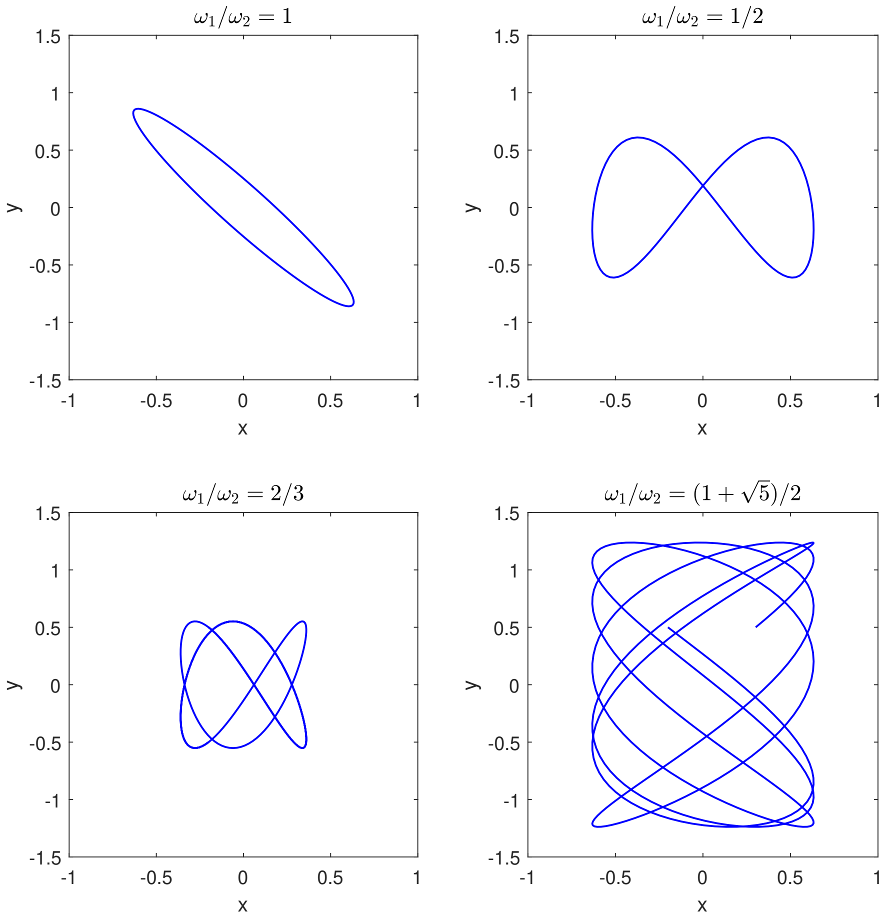



Figure 4.12: Typical trajectories of the particle for several values of $\omega_1/\omega_2$.

**Definition 4.4** **Commensurable/Incommensurable**

Two frequencies with rational ratio are called **commensurable**. If the ratio of two frequencies is an irrational number, then they are called **incommensurable**.

## 4.6 Motion with friction

Consider a particle of mass $m$ that is moving in a potential $V(\mathbf{x})$ in the presence of a friction force. We assume that the friction force $\mathbf{F}_f$ is proportional to the particle's velocity and is given by

$$\mathbf{F}_f = -\Gamma\dot{\mathbf{x}}$$

where $\Gamma > 0$ is a constant. The 'minus' sign reflects the fact that the direction of the force is opposite to the direction of the velocity.

The equations of motion take the form

$$m\ddot{\mathbf{x}} = -\nabla V - \Gamma\dot{\mathbf{x}} \quad \text{or} \quad m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -\partial V/\partial x - \Gamma\dot{x} \\ -\partial V/\partial y - \Gamma\dot{y}. \end{pmatrix}.$$

We know that if there were no friction force, the energy, given by

$$E = m\frac{|\dot{\mathbf{x}}|^2}{2} + V(\mathbf{x}), \tag{64}$$

would be a constant of motion. Let us show that the energy is a decreasing function of time if friction is present (i.e. that the situation is similar to what we had in the one-dimensional case).

Employing the equations of motion, we obtain

$$\begin{aligned}
\frac{dE}{dt} &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + \frac{\partial V}{\partial x}\,\dot{x} + \frac{\partial V}{\partial y}\,\dot{y} \\
&= \dot{x}\left(m\ddot{x} + \frac{\partial V}{\partial x}\right) + \dot{y}\left(m\ddot{y} + \frac{\partial V}{\partial y}\right) \\
&= \dot{x}\,(-\Gamma\,\dot{x}) + \dot{y}\,(-\Gamma\,\dot{y}) \\
&= -\Gamma\,\left(\dot{x}^2 + \dot{y}^2\right) = -\Gamma\,|\dot{\mathbf{x}}|^2.
\end{aligned}$$

Thus, $\dot{E} < 0$ unless $\dot{\mathbf{x}} = \mathbf{0}$, which means that the friction results in dissipation of energy.

# 5 Polar coordinates

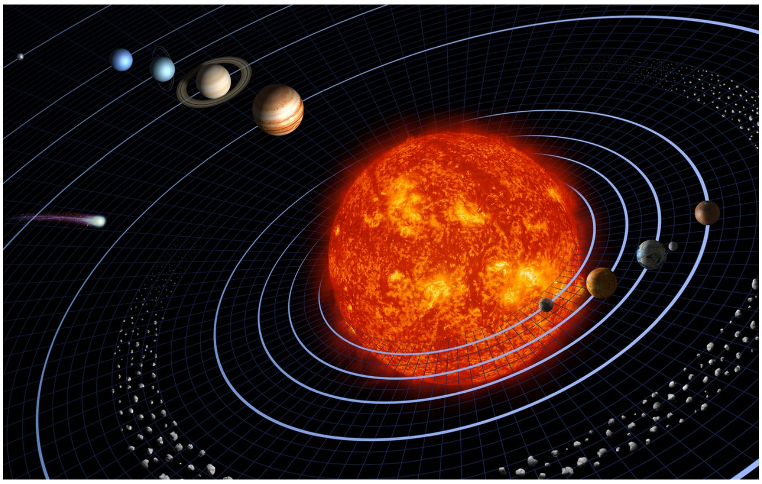


Figure 5.1: Planetary motion - not to scale!

Source: H. Smith & L Generosa, NASA, public domain image

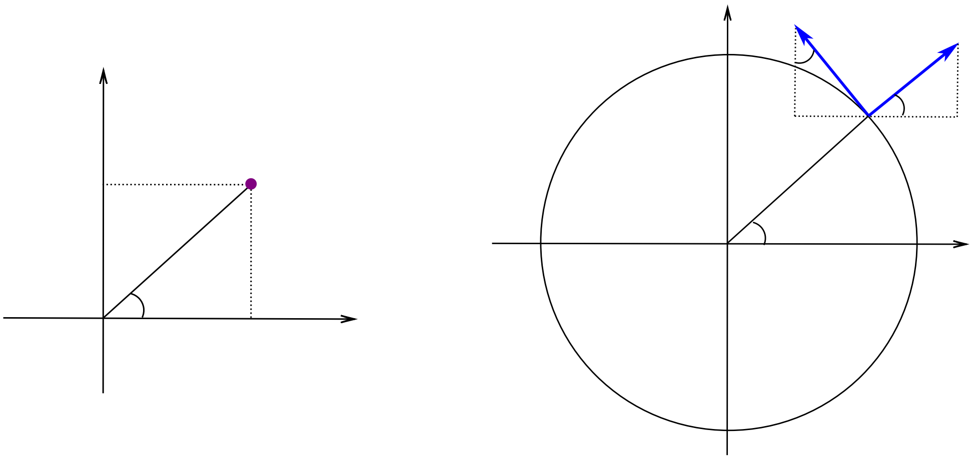


Figure 5.2: Left: Polar coordinates $(r, \theta)$ in the $xy$-plane. Right: Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

## 5.1 Equations of motion in polar coordinates

Sometimes it is more convenient to use polar coordinates $(r, \theta)$ rather that Cartesian coordinates $(x, y)$.

The relation between polar and Cartesian coordinates is given by (see Fig. 5.2)

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{65}$$

Our aim is to write the equation of motion of a particle in polar coordinates.

At any point $\mathbf{x}$ on the $xy$ plane, we can introduce two unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ (unit vectors in radial and azimuthal directions) as shown in Fig. 5.2. Any vector $\mathbf{b}$ associated with a point $\mathbf{x}$ (e.g. the velocity of a particle $\dot{\mathbf{x}}(t)$ whose position at time $t$ is $\mathbf{x}(t)$) can be presented in the form

$$\mathbf{b} = b_r\,\mathbf{e}_r + b_\theta\,\mathbf{e}_\theta. \tag{66}$$

The scalars $b_r$ and $b_\theta$ are called the **radial and azimuthal components** of vector $\mathbf{b}$.

The unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ can be expressed in terms of the Cartesian basis vectors $\mathbf{i}$ and $\mathbf{j}$. If follows from Fig. 5.2 that

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \tag{67}$$
$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}, \tag{68}$$

Assuming that the polar angle $\theta$ is a function of time $t$, let's compute the derivatives of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ with respect to $t$. We have

$$\dot{\mathbf{e}}_r = -\sin\theta\,\dot\theta\,\mathbf{i} + \cos\theta\,\dot\theta\,\mathbf{j},$$
$$\dot{\mathbf{e}}_\theta = -\cos\theta\,\dot\theta\,\mathbf{i} - \sin\theta\,\dot\theta\,\mathbf{j},$$

Comparing these with Eqs. (67) and (68), we deduce that

$$\dot{\mathbf{e}}_r = \dot\theta\,\mathbf{e}_\theta \quad \text{and} \quad \dot{\mathbf{e}}_\theta = -\dot\theta\,\mathbf{e}_r. \tag{69}$$

In polar coordinates, the position vector of a particle is simply

**Position (radial coordinates)**:

$$\mathbf{x} = r\,\mathbf{e}_r.$$

Let's compute the velocity, $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$, and the acceleration $\mathbf{a}(t) = \ddot{\mathbf{x}}(t)$. Differentiating the formula for $\mathbf{x}$ and employing (69), we get

**Velocity (radial coordinates)**:

$$\mathbf{v} = \dot{\mathbf{x}} = \dot r\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r = \dot r\,\mathbf{e}_r + r\,\dot\theta\,\mathbf{e}_\theta. \tag{70}$$

Thus, the radial component of the velocity is simply $\dot r$ while the azimuthal component is $r\,\dot\theta$. Differentiation of (70) yields

$$\begin{aligned}\mathbf{a} = \ddot{\mathbf{x}} &= \ddot r\,\mathbf{e}_r + \dot r\,\dot{\mathbf{e}}_r + \dot r\,\dot\theta\,\mathbf{e}_\theta + r\,\ddot\theta\,\mathbf{e}_\theta + r\,\dot\theta\,\dot{\mathbf{e}}_\theta \\ &= \ddot r\,\mathbf{e}_r + \dot r\,\dot\theta\,\mathbf{e}_\theta + \dot r\,\dot\theta\,\mathbf{e}_\theta + r\,\ddot\theta\,\mathbf{e}_\theta - r\,\dot\theta^2\,\mathbf{e}_r.\end{aligned}$$

Hence, we have

**Acceleration (radial coordinates)**:

$$\mathbf{a} = \left(\ddot r - r\,\dot\theta^2\right)\mathbf{e}_r + \left(r\,\ddot\theta + 2\dot r\,\dot\theta\right)\mathbf{e}_\theta. \tag{71}$$

Thus, the acceleration in polar coordinates has a quite complicated form: it contains not only intuitively natural terms $\ddot r$ and $r\,\ddot\theta$, but also two additional terms, $-r\,\dot\theta^2$ in the radial component and $2\dot r\,\dot\theta$ in the azimuthal component.

**Definition 5.1** **Centripetal acceleration**

The term $-r\,\dot{\theta}^2\,\mathbf{e}_r$ is what is called the **centripetal acceleration**. It is present, for instance, when a particle is moving along a circle.

The second additional term, $2\dot{r}\,\dot{\theta}\,\mathbf{e}_\theta$, is non-zero only if both $\dot{r}$ and $\dot{\theta}$ are non-zero. (Usually it is interpreted in terms of the so-called Coriolis force, but discussion will be postponed to later modules.)

Now we can write:

**Definition 5.2** **The equations of motion of a particle in polar coordinates**

These are

$$m\left(\ddot{r} - r\,\dot{\theta}^2\right) = F_r \quad \text{and} \quad m\left(r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right) = F_\theta \tag{72}$$

where $F_r$ and $F_\theta$ are the radial and azimuthal components of the force.

**Remark** Another (and easier) way to derive Eqs. (72) is as follows: Let $z(t) = x(t) + iy(t)$ and $f = F_x(x, y) + iF_y(x, y)$ where $x(t)$, $y(t)$ are Cartesian coordinates of the particle, $F_x$, $F_y$ are Cartesian components of the force and $i$ is the imaginary unit ($i^2 = -1$). Then the two scalar equations of motion in Cartesian coordinates can be written as one complex equation

$$\ddot{z} = f.$$

Substituting the polar form of $z$ ($z = re^{i\theta}$) into this equation, then multiplying the result by $e^{-i\theta}$ and separating real and imaginary parts of both sides, we can arrive at Eqs. (72) (check this!).

## 5.2 Planets and pendula

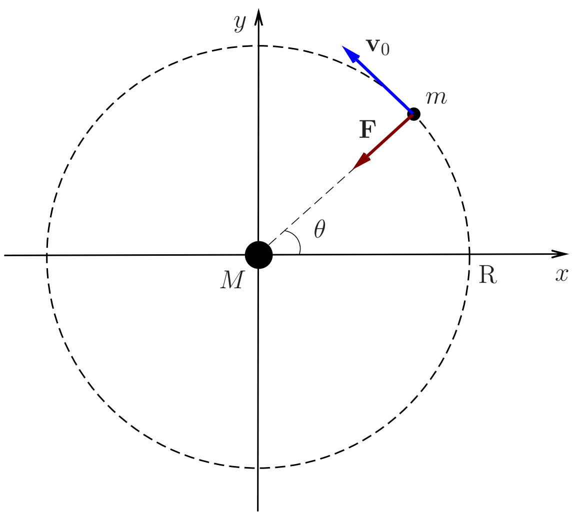



Figure 5.3: Circular motion.

**Example 5.1 (circular motion).** Consider a planet of mass $m$ which is moving with constant speed $v_0$ along a circular orbit around a star of mass $M$. Let the radius of the orbit be $R$. What is the azimuthal velocity $v_0$?

**Solution.** Let the centre of the star be the origin of polar coordinates $(r,\theta)$. The only force affecting the planet is gravitational force given by

$$\mathbf{F} = -\frac{GmM}{r^2}\,\mathbf{e}_r.$$

So, the equations of motion become

$$m\left(\ddot{r} - r\,\dot{\theta}^2\right) = -\frac{GmM}{r^2} \quad \text{and} \quad m\left(r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right) = 0. \tag{73}$$

Since the planet is moving with constant speed $v_0$ and along the circle of radius $R$, we have

$$r(t) = R = const \quad \text{and} \quad R\,\dot{\theta}(t) = v_0 = const$$

[The second equation can also be obtained by integration of the second equation in (73).] These results imply that the second equation is (73) is automatically satisfied, and the first equation in (73) yields the relation

$$-mR\,\dot{\theta}^2 = -\frac{GmM}{R^2}.$$

Therefore,

$$\dot{\theta} = \pm\sqrt{\frac{GM}{R^3}} \quad \Rightarrow \quad v_0 = R\dot{\theta} = \pm\sqrt{\frac{GM}{R}}.$$

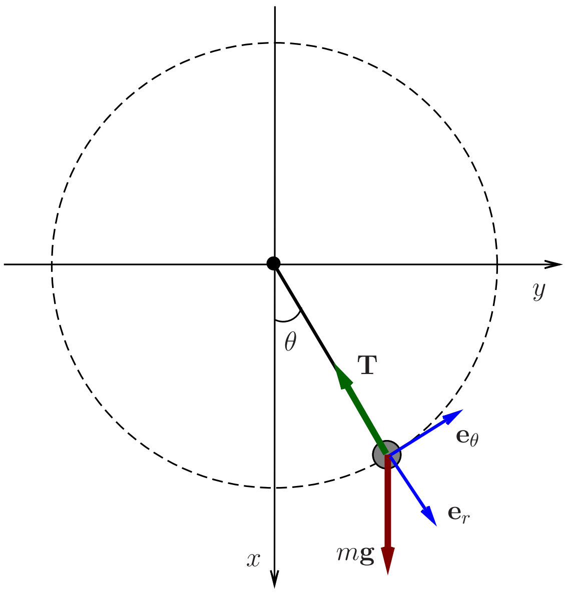



Figure 5.4: Simple pendulum.

**Example 5.2 (simple pendulum).** Consider the motion of an ideal simple pendulum shown in Fig. 5.4. An ideal pendulum is a weightless rigid rod of length $l$ with a point mass $m$ attached to its end, the other end being attached to a fixed point and able to turn around it without friction.

(a) Write down the equation of motion of the pendulum.

(b) Write down the equation describing small oscillations of the pendulum near the lower equilibrium position and find the period of oscillation.

**Solution.** (a) First we introduce the Cartesian coordinates as shown in Fig. 5.4. Then

$$\mathbf{g} = g\,\mathbf{i} = g\cos\theta\,\mathbf{e}_r - g\sin\theta\,\mathbf{e}_\theta.$$

The point mass moves under the uniform gravity force $m\mathbf{g}$ and the reaction force $\mathbf{T} = -T\mathbf{e}_r$ (due to the rod that keeps the mass on the circle of radius $l$). The equations of motion are

$$m\left[\left(\ddot{r} - r\,\dot{\theta}^2\right)\mathbf{e}_r + \left(r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right)\mathbf{e}_\theta\right] = \left(mg\cos\theta - T\right)\mathbf{e}_r + (-mg\sin\theta)\mathbf{e}_\theta$$

or

$$m\left(\ddot{r} - r\,\dot{\theta}^2\right) = mg\cos\theta - T \quad \text{and} \quad m\left(r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right) = -mg\sin\theta.$$

Since $r = l = const$, these equations simplify to

$$-ml\,\dot{\theta}^2 = mg\cos\theta - T \quad \text{and} \quad ml\,\ddot{\theta} = -mg\sin\theta.$$

The first of these allows us to determine $T$, when $\theta(t)$ is known. The second serves as an effective equation of motion in azimuthal coordinate $\theta$. It is convenient to rewrite it as

$$\ddot{\theta} = -\frac{g}{l}\sin\theta. \tag{74}$$

This is the exact differential equation for $\theta(t)$ (so far we didn't make any approximations).

**Remark** Note that Eq. (74) can be interpreted as one-dimensional motion of a particle of unit mass ($m = 1$) in the potential

$$V(\theta) = -\frac{g}{l}\cos\theta,$$

so that we can write down the 'energy'

$$\tilde{E} = \frac{\dot{\theta}^2}{2} + V(\theta)$$

and analyse motion of the 'particle' qualitatively (as we did in section 3.7).

(b) Evidently, $\theta = 0$ is a (constant) solution of Eq. (74). In other words, $\theta = 0$ is an equilibrium position of the pendulum. Let's look at the motion near the equilibrium. Assuming that $|\theta| \ll 1$, we replace $\sin\theta$ by the first nonzero term of its Taylor's expansion about the point $\theta = 0$:

$$\sin\theta \approx \theta.$$

As a result, we obtain the following (approximate) ODE:

$$\ddot{\theta} = -\frac{g}{l}\,\theta.$$

Up to the notation this is the same as the equation of a simple harmonic oscillator. It describes small oscillations of the pendulum with angular frequency $\omega = \sqrt{g/l}$ and the period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}.$$

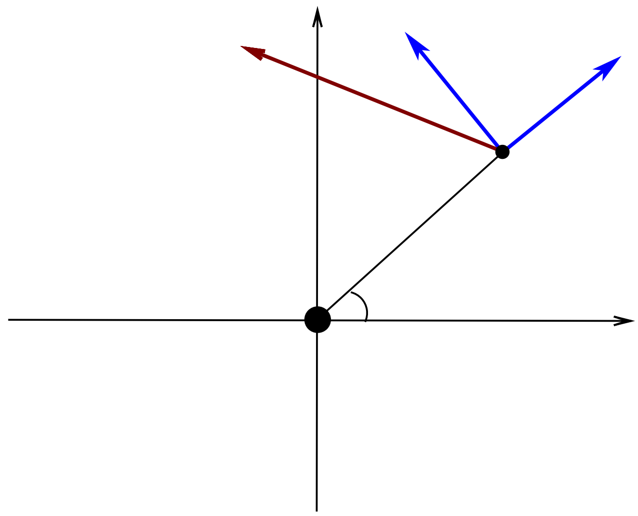


Figure 5.5: A planet of mass $m$ moving around a star of mass $M$.

**Example 5.3 (planetary motion).** Consider a planet of mass $m$ moving around a (fixed) star of mass $M$. Let the centre of the star be the origin of polar coordinates $(r, \theta)$ as shown in Fig. 5.5. The only force acting on the planet is Newton's gravitational force:

$$\mathbf{F} = -\frac{GmM}{r^2}\mathbf{e}_r.$$

So, the equations of motion become

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{GmM}{r^2} \quad \text{and} \quad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0. \tag{75}$$

Dividing the second equation by $m$ and then multiplying by $\dot{r}$, we find that

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0.$$

This means that

$$L = r^2\dot{\theta}$$

is a constant of motion, i.e. $L(t) = L(0)$.

**Definition 5.3** **Angular momentum**

$\mathcal{L} = mL = mr^2\dot{\theta}$ is called the **angular momentum**. It is conserved in the above example.

Can we use conservation of $L$ to simplify the problem? The answer is 'yes', and this can be done as follows. Since $L$ is a constant, we have

$$\dot{\theta}(t) = \frac{L}{r^2(t)} \tag{76}$$

Substituting this into the first equation (75), divided by $m$, yields

$$\ddot{r} = -\frac{\gamma}{r^2} + \frac{L^2}{r^3} \tag{77}$$

where $\gamma = GM$. This is called the equation of radial motion and describes one-dimensional motion in radial direction. It can be solved (subject to appropriate initial conditions), and this will give us $r(t)$. Then $r(t)$ is substituted into (76) and $\theta(t)$ is found by integration, such that

$$\theta(t) = \theta(0) + L \int_0^t \frac{ds}{r^2(s)}.$$

Thus, equations (76) and (77), supplemented with appropriate initial conditions, allow us to find the trajectory of the planet.

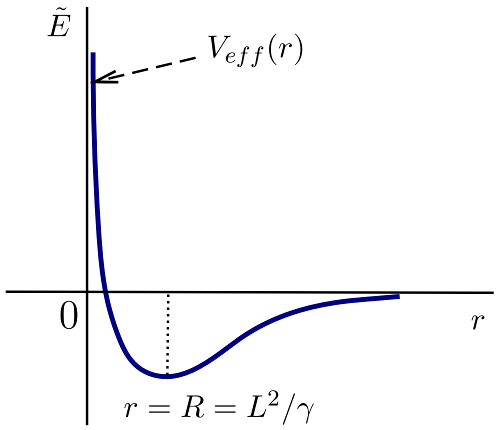



Figure 5.6: Sketch of the potential $V_{eff}(r)$.

Equation (77) can be interpreted as the equation of one-dimensional motion of a particle (of unit mass) in the effective potential

$$V_{eff}(r) = \frac{L^2}{2r^2} - \frac{\gamma}{r}.$$

Indeed, $V'_{eff}(r) = \frac{\gamma}{r^2} - \frac{L^2}{r^3}$, so that (77) can be written as

$$\ddot{r} = -V'_{eff}(r).$$

The energy of the particle moving in potential $V_{eff}(r)$ is

$$\tilde{E} = \frac{\dot{r}^2}{2} + V_{eff}(r).$$

Now we can use what we already know about motion in a potential in one dimension. The sketch of $V_{eff}(r)$ is shown in Fig. 5.6. The potential has a minimum point at

$$r = R = \frac{L^2}{\gamma} \quad \text{and} \quad V_{eff}(R) = \frac{L^2}{2R^2} - \frac{\gamma}{R} = -\frac{\gamma^2}{2L^2}.$$

Note that this 'equilibrium' point of radial motion is not a true equilibrium: it corresponds to a circular orbit of radius $R$, such that the azimuthal velocity is constant and equal to $R\dot{\theta} = L/R = \gamma/L$.

It follows from Fig. 5.6 that the motion of the particle is finite (i.e. takes place in a bounded region of $\mathbb{R}^2$) if $\tilde{E} < 0$ and that the particle will escape to infinity if $\tilde{E} \geq 0$. It can be shown that the planet's orbit will be an ellipse, if $\tilde{E} < 0$, a parabola, if $\tilde{E} = 0$, and a hyperbola, if $\tilde{E} > 0$. You will have a chance to learn more about planetary motion in the 2nd year module "Applied Mathematics".