VECTOR CALCULUS 2023/24

LECTURER: Dr FRANCESCO COSENTINO

francesco.cosentino@york.ac.uk
Office: G/N/005

Lecture notes by Dr Eli Hawkins
(With minor modifications by Francesco Cosentino)

Acknowlegments

These lecture notes were written by Dr Eli Hawkins for the "Vector Calculus" module taught by him from 2017/18 to 2022/23. I have made some minor changes and added some content. I would like to express my deep gratitude to Eli for consenting to the use of these lecture notes for the "Vector Calculus" part of "Vector and Complex Calculus 2023-24".

Francesco Cosentino

1. Derivatives

1.1. **Vectors.**

Definition 1.1.1. An *n-dimensional vector* is an element of

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}.$$

The numbers x_a for $a = 1, \dots, n$ are the *coordinates* or *components* of the vector. It is extremely convenient to use abbreviated "vector notation":

$$\underline{x} = (x_1, \dots, x_n),$$

wherever possible.

The coordinates will often be given more individual names, as in (x, y, z) or (r, θ).
In the context of vectors, numbers are often called *scalars*.
It will also be useful to think of an n-dimensional vector as an $n \times 1$-matrix:

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

This is not quite the same as a $1 \times n$-matrix, $\begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}$, which is written without commas.

Recall that vectors can be added, the sum being defined as

$$\underline{x} + \underline{y} = (x_1 + y_1, \dots, x_n + y_n),$$

and that a scalar can be multiplied with a vector

$$\lambda \underline{x} = (\lambda x_1, \dots, \lambda x_n).$$

There is also a zero vector, $\underline{0} = (0, \dots, 0)$, such that $\underline{x} + \underline{0} = \underline{x}$. Multiplication of vectors is another story, which we will get to later.

Remark. $\mathbb{R}^n$ is a commutative group, with group operation the sum of vectors, and unit element the zero vector.

1.2. **Topology.**

Definition 1.2.1. A subset $U \subseteq \mathbb{R}^n$ is *open* if for any $\underline{x} \in U$, there is a number $\delta > 0$ such that any point at distance less than δ from $\underline{x}$ is also in U.

Example 1.2.2. Some open sets in $\mathbb{R}^n$:

- $\mathbb{R}^n$ itself
- The empty set, $\emptyset$
- The open cube, $(0, 1)^n = \{\underline{x} \in \mathbb{R}^n \mid 0 < x_a < 1, \forall a = 1, \dots, n\}$

$\lhd$

Definition 1.2.3. The *interior* $\operatorname{int} E$ of $E \subset \mathbb{R}^n$ is the union of all open subsets of E.

Example 1.2.4. The interior of the closed interval $[0, 1] \subset \mathbb{R}$ is the open interval $(0, 1)$. $\lhd$

Definition 1.2.5. For $U \subseteq \mathbb{R}^n$ open, a function $\underline{f} : U \to \mathbb{R}^m$ is *continuous* if for any open $V \subseteq \mathbb{R}^m$, the subset

$$\underline{f}^{-1}V := \{\underline{x} \in U \mid \underline{f}(\underline{x}) \in V\} \subseteq \mathbb{R}^n$$

is open. A continuous function is also called a *map*. The set of maps $\underline{f} : U \to \mathbb{R}^m$ is denoted $\mathcal{C}(U, \mathbb{R}^m)$.

As with functions of a single variable, the idea of continuity is that if $\underline{x}$ and $\underline{y}$ are sufficiently close together, then $\underline{f}(\underline{x})$ and $\underline{f}(\underline{y})$ will be sufficiently close together. There are no sudden jumps in the values of $\underline{f}$.

Remark. $\underline{f}$ is continuous if and only if the components $f_1, \ldots, f_m$ are continuous.

Remark. To avoid clutter, I will make most statements and definitions for functions defined on $\mathbb{R}^n$, but unless I say otherwise, these will extend in an obvious way to functions defined on any open subset.

Remark. In a more general setting, a *topoloy* is a mathematical concept described as follows: let X be a set and let τ be a family of subsets of X. Then τ is called a *topoloy* on X if (i), (ii), and (iii) apply:

(i) both the empty set and X are elements of τ;
(ii) the union of any family of elements of τ is an element of τ;
(iii) the intersection of finitely many elements of τ is an element of τ.

The pair (X, τ) is called a topological space. Topological spaces are fundamental in many mathematical fields but our elementary treatment of topological concepts, essentially based only on the definition of open sets, without resorting to the general concept of topological space, is adequate to the material treated in these notes.

1.3. **Derivatives.** Recall from single-variable calculus that the derivative of a function $g : \mathbb{R} \to \mathbb{R}$ describes how the value of g changes when the argument (input) is changed slightly:

$$x \approx a \implies g(x) - g(a) \approx g'(a)(x - a).$$

Definition 1.3.1. The *(total) derivative* (or *Jacobian matrix*) of $\underline{f} : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$-matrix made up of the partial derivatives of the components:

$$D\underline{f}(\underline{x}) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

$D_a\underline{f}(\underline{x}) := \frac{\partial \underline{f}}{\partial x_a}$ will denote the a'th column of $D\underline{f}(\underline{x})$.

Remark. This might not exist. However, when I talk about the derivative of a function, I am usually implicitly assuming that it exists.

If it exists, this is a function $D\underline{f} : \mathbb{R}^n \to \mathbb{R}^{mn}$.

This also describes how $\underline{f}$ changes when its argument is changed slightly:

$$\underline{x} \approx \underline{a} \implies \underline{f}(\underline{x}) - \underline{f}(\underline{a}) \approx D\underline{f}(\underline{a})(\underline{x} - \underline{a}),$$

where we are using matrix multiplication.

We can take derivatives of the derivative. For example, $D^2\underline{f}$ contains all second order partial derivatives of $\underline{f}$.

Definition 1.3.2. The set of r-times continuously differentiable functions, denoted $\mathcal{C}^r(\mathbb{R}^n, \mathbb{R}^m)$, is the set of maps $\underline{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that all derivatives up to order r (i.e., $\underline{f}$, $D\underline{f}$,... $D^r\underline{f}$) exist and are continuous. Likewise, for an open set $U \subseteq \mathbb{R}^n$, $\mathcal{C}^r(U, \mathbb{R}^m)$ denotes the set of r-times continuously differentiable functions $\underline{f} : U \to \mathbb{R}^m$. More generally, for an arbitrary subset $E \subset \mathbb{R}^n$, a function $\underline{f} : E \to \mathbb{R}^m$ is $\underline{f} \in \mathcal{C}^r(E, \mathbb{R}^m)$ if for some open $U \supseteq E$, $\underline{f}$ is the restriction of some function in $\mathcal{C}^r(U, \mathbb{R}^m)$.

I will often use $\mathcal{C}^r$ as an adjective, as in "a $\mathcal{C}^r$ map".

1.4. **Properties of the derivative.** The derivative has the obvious properties under addition and multiplication by a constant scalar: For $\underline{f}, \underline{g} : \mathbb{R}^n \to \mathbb{R}^m$,

$$D(\underline{f} + \underline{g})(\underline{x}) = D\underline{f}(\underline{x}) + D\underline{g}(\underline{x})$$

and for $\lambda \in \mathbb{R}$,

$$D(\lambda\underline{f})(\underline{x}) = \lambda\, D\underline{f}(\underline{x}).$$

For a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ and a vector function $\underline{g} : \mathbb{R}^n \to \mathbb{R}^m$, there is a product rule:

$$D(f\underline{g})(\underline{x}) = f(\underline{x})D\underline{g}(\underline{x}) + \underline{g}(\underline{x})Df(\underline{x}).$$

A more sophisticated property is the *chain rule*. If $\underline{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\underline{g} : \mathbb{R}^m \to \mathbb{R}^p$, then $\underline{g} \circ \underline{f} : \mathbb{R}^n \to \mathbb{R}^p$ (defined by $(\underline{g} \circ \underline{f})(\underline{x}) := \underline{g}[\underline{f}(\underline{x})]$) has derivative

$$D(\underline{g} \circ \underline{f})(\underline{x}) = D\underline{g}(\underline{f}[\underline{x}])D\underline{f}(\underline{x})$$

(using matrix multiplication again).

Example 1.4.1. *For*

$$\underline{f}(r, \theta) = (r\cos\theta, r\sin\theta)$$

and

$$g(x, y) = x^2 + y^2$$

compute $D(g \circ \underline{f})$ by using the chain rule and directly.

Solution.

$$D\underline{f}(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$Dg(x, y) = \begin{pmatrix} 2x & 2y \end{pmatrix}$$

$$Dg\left(\underline{f}[r, \theta]\right) = \begin{pmatrix} 2r\cos\theta & 2r\sin\theta \end{pmatrix}$$

$$Dg\left(\underline{f}[r, \theta]\right) D\underline{f}(r, \theta) = \begin{pmatrix} 2r\cos\theta & 2r\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 2r & 0 \end{pmatrix}$$

On the other hand, the composition is

$$(g \circ \underline{f})(r, \theta) = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

so

$$D(g \circ \underline{f})(r, \theta) = \begin{pmatrix} 2r & 0 \end{pmatrix}. \qquad \lhd$$

Example 1.4.2. *Let $\underline{h} : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function, defined by $\underline{h}(\underline{x}) = \underline{x}$. Compute the derivative .*

Solution.

$$D\underline{h}(\underline{x}) = \mathbb{I} := \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}$$

is the identity matrix. ◁

Example 1.4.3. *Let $\underline{g} : \mathbb{R}^n \to \mathbb{R}^n$ be the inverse map of $\underline{f} : \mathbb{R}^n \to \mathbb{R}^n$. That is, $\underline{f} \circ \underline{g}$ and $\underline{g} \circ \underline{f}$ both equal the identity function. Compute $D\underline{g}$.*

Solution. By the previous example and the chain rule, the derivative of $f \circ g$ is the identity matrix

$$\mathbb{I} = D\underline{f}(\underline{g}[\underline{x}])Dg(\underline{x}).$$

Therefore, $Dg(\underline{x}) = \left[D\underline{f}(\underline{g}[\underline{x}])\right]^{-1}$ is the inverse matrix. ◁

2. Line Integrals

2.1. **The dot product and gradient.**

Definition 2.1.1. Given two vectors $\underline{a}, \underline{b} \in \mathbb{R}^n$, their *dot product* (also known as the *scalar product* or *inner product*) is

$$\underline{a} \cdot \underline{b} := a_1 b_1 + \cdots + a_n b_n.$$

The *norm* of $\underline{a}$ is

$$\|\underline{a}\| := \sqrt{\underline{a} \cdot \underline{a}}\,.$$

Nota bene. This bold dot $\cdot$ is not optional. It is an essential part of the notation. (I will occasionally use a light dot $\cdot$ for multiplication of numbers.)

The dot product is symmetric, $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$, and is compatible with scalar multiplication

$$(\lambda \underline{a}) \cdot \underline{b} = \lambda(\underline{a} \cdot \underline{b})$$

and addition

$$(\underline{a} + \underline{b}) \cdot \underline{c} = \underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{c}.$$

It also has a geometric meaning. If θ is the angle between $\underline{a}$ and $\underline{b}$, then

$$\underline{a} \cdot \underline{b} = \|\underline{a}\|\,\|\underline{b}\| \cos\theta.$$

Definition 2.1.2. A *scalar field* is a map $f : \mathbb{R}^n \to \mathbb{R}$. A *vector field* is a map $\underline{g} : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 2.1.3. The *gradient* of a scalar field $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ is the vector field $\nabla f = (D_1 f, \ldots, D_n f)$ (denoted $\mathbf{grad}\, f$ in some books).

This is essentially the same as Df, but rearranged ("transposed") from a $1 \times n$ matrix to a $n \times 1$ matrix.

2.2. **Curves.**

Definition 2.2.1. An *orientation* of a curve $C \subset \mathbb{R}^n$ is a choice of direction along C.

There are always 2 possible orientations of a curve. Usually, this means a choice of which end is the beginning and which is the end.

Orientation is so important in this module that when I talk about a curve, it will always have a chosen orientation.

Definition 2.2.2. A *parametric curve* (or *path*) in $\mathbb{R}^n$ is a continuously differentiable map $\underline{p} \in \mathcal{C}^1([a, b], \mathbb{R}^n)$, for some numbers $a < b \in \mathbb{R}$.

Notation 2.2.3. I will denote the derivative of such a map with a dot: $\dot{\underline{p}} = D\underline{p}$. (I'm saving the prime symbol for other uses.)

Definition 2.2.4. Another parametric curve $\underline{q} : [a', b'] \to \mathbb{R}^n$ is a *reparametrization* of $\underline{p} : [a, b] \to \mathbb{R}^n$, if $\underline{q} = \underline{p} \circ h$ (i.e., $\underline{q}(\tau) = \underline{p}[h(\tau)]$) for some $\mathcal{C}^1$ function $h : [a', b'] \to [a, b]$ such that $\dot{h} \geq 0$, $h(a') = a$, and $h(b') = b$.

Definition 2.2.5. An (oriented) *curve* C is an equivalence class of parametric curves, where two parametric curves are equivalent if they are related by reparametrization. If $\underline{p}$ is in this equivalence class, then it is a *parametrization* of C ("$\underline{p}$ parametrizes C").

Remark. Less precisely, you can think of a curve as a subset $C \subset \mathbb{R}^n$. A parametric curve $\underline{p} : [a, b] \to \mathbb{R}^n$ is a parametrization of C if it is the image of $\underline{p}$, i.e.,

$$C = \{\underline{p}(t) \mid a \leq t \leq b\}.$$

and as t increases, $\underline{p}(t)$ moves in the direction of the orientation of C.

Definition 2.2.6. If $\underline{p} : [a, b] \to \mathbb{R}^n$ parametrizes C, then the *endpoints* of C are $\underline{p}(a)$ (the beginning) and $\underline{p}(b)$ (the end).

Definition 2.2.7. If C is parametrized by $\underline{p} : [a, c] \to \mathbb{R}^n$ and $a < b < c$, then the restriction to $[a, b]$ parametrizes a curve C_1 and the restriction to $[b, c]$ parametrizes a curve C_2. We call C the concatenation of C_1 and C_2 and denote $C = C_1 \cup C_2$.

Definition 2.2.8. A curve C is *smooth* if there exists a parametrization $\underline{p}$ such that $\dot{\underline{p}}(t) \neq \underline{0}$ for all t.

When a curve is not smooth, it is usually more convenient to express it as a concatenation of smooth curves, rather than to construct a single parametrization.

2.3. **The vector line integral.** The line integral of a vector field $\underline{g} : \mathbb{R}^n \to \mathbb{R}^n$ along a curve C in $\mathbb{R}^n$ is a number, which I will denote

$$\int_C \underline{g}(\underline{x}) \cdot d\underline{x}.$$

Remark. "Line integral" is not a good name for this, because C is almost certainly *not* a line. However, this is standard terminology.

This has some key properties:

- If $C = C_1 \cup C_2$, then
$$\int_C \underline{g}(\underline{x}) \cdot d\underline{x} = \int_{C_1} \underline{g}(\underline{x}) \cdot d\underline{x} + \int_{C_2} \underline{g}(\underline{x}) \cdot d\underline{x}. \tag{2.1}$$
- If $\underline{f} \approx \underline{g}$, then
$$\int_C \underline{f}(\underline{x}) \cdot d\underline{x} \approx \int_C \underline{g}(\underline{x}) \cdot d\underline{x}. \tag{2.2}$$
(This can be made precise, but I won't do so here.)
- If C is a curve from $\underline{a}$ to $\underline{b}$ and $\underline{v} \in \mathbb{R}^n$ is constant, then
$$\int_C \underline{v} \cdot d\underline{x} = \underline{v} \cdot (\underline{b} - \underline{a}). \tag{2.3}$$

Example 2.3.1. Suppose that an object moves through space along a curve C, while acted on by a force $\underline{F}$ that depends on position, but not time. The amount of energy transferred to the object by the force is called *work*. This is the line integral

$$W = \int_C \underline{F}(\underline{x}) \cdot d\underline{x}.$$

$\triangleleft$

To try to compute the line integral of $\underline{g}$ along C, we can try breaking C into small pieces. Let $\underline{x}_0, \ldots, \underline{x}_N$ be points on C, ordered with respect to the orientation, with $\underline{x}_0$ the beginning and $\underline{x}_N$ the end of C. Let C_k be the piece of C from $\underline{x}_{k-1}$ to $\underline{x}_k$, so that

$$C = C_1 \cup \cdots \cup C_N.$$

If these pieces are small enough, and $\underline{g} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then $\underline{g}$ is approximately constant over C_k, therefore

$$\begin{aligned}
\int_C \underline{g}(\underline{x}) \cdot d\underline{x} &= \sum_{k=1}^{N} \int_{C_k} \underline{g}(\underline{x}) \cdot d\underline{x} && \text{by (2.1)} \\
&\approx \sum_{k=1}^{N} \int_{C_k} \underline{g}(\underline{x}_k) \cdot d\underline{x} && \text{by (2.2)} \\
&= \sum_{k=1}^{N} \underline{g}(\underline{x}_k) \cdot (\underline{x}_k - \underline{x}_{k-1}) && \text{by (2.3).}
\end{aligned}$$

This is the line integral version of a Riemann sum. The line integral can be defined as a limit of such sums as C is broken into smaller and smaller pieces.

A parametrization is now useful. Suppose that $\underline{p} : [a, b] \to \mathbb{R}^n$ is a parametrization of C, and $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ are closely spaced numbers. We can let $\underline{x}_k = \underline{p}(t_k)$. This gives

$$\begin{aligned}
\int_C \underline{g}(\underline{x}) \cdot d\underline{x} &\approx \sum_{k=1}^{N} \underline{g}(\underline{p}[t_k]) \cdot (\underline{p}[t_k] - \underline{p}[t_{k-1}]) \\
&\approx \sum_{k=1}^{N} \underline{g}(\underline{p}[t_k]) \cdot \dot{\underline{p}}(t_k)(t_k - t_{k-1}) \\
&\approx \int_a^b \underline{g}(\underline{p}[t]) \cdot \dot{\underline{p}}(t)\, dt,
\end{aligned}$$

since the last sum is just a Riemann sum for an ordinary integral.

This becomes an equality in the limit of breaking $[a, b]$ up into arbitrarily small pieces, and leads to a practical definition for the line integral.

Definition 2.3.2. The *line integral* of a vector field $\underline{g}$ along a curve C in $\mathbb{R}^n$ is

$$\int_C \underline{g}(\underline{x}) \cdot d\underline{x} = \int_a^b \underline{g}(\underline{p}[t]) \cdot \dot{\underline{p}}(t) dt$$

for any parametrization $\underline{p} : [a, b] \to \mathbb{R}^n$ of C.

This formula can be seen as evaluating the line integral by the substitution $\underline{x} = \underline{p}(t)$.

Example 2.3.3. *Let $\underline{g}(x, y) = (-y, x)$ and let C be the part of the parabola defined by $y = x^2$ going from $(0, 0)$ to $(1, 1)$. Compute the line integral*

$$\int_C \underline{g}(\underline{x}) \cdot d\underline{x}.$$

Solution. Because y is easily computed from x along this curve, it is easiest to parametrize C by the x coordinate. Define $\underline{p} : [0,1] \to \mathbb{R}^2$ by $\underline{p}(t) = (t, t^2)$. This has derivative $\dot{\underline{p}}(t) = (1, 2t)$ and composition $\underline{g}(\underline{p}[t]) = (-t^2, t)$. This gives the line integral,

$$\int_C \underline{g}(\underline{x}) \cdot d\underline{x} = \int_0^1 \underline{g}(\underline{p}[t]) \cdot \dot{\underline{p}}(t)\, dt = \int_0^1 (-t^2, t) \cdot (1, 2t)\, dt = \int_0^1 t^2 dt = \left[\tfrac{1}{3}t^3\right]_{t=0}^{t=1} = \frac{1}{3}.$$

We can also write this computation as

$$\begin{aligned}\int_C \underline{g}(\underline{x}) \cdot d\underline{x} &= \int_C (-y\, dx + x\, dy) = \int_C (-x^2 dx + x \cdot 2x\, dx) \\ &= \int_C x^2 dx = \int_C d\left(\tfrac{1}{3}x^3\right) = \frac{1}{3}\end{aligned}$$

by using the fact that $y = x^2$ along C. ◁

Example 2.3.4. *What happens if we use a reparametrization, as in Definition 2.2.4, to compute the line integral?*

Solution. Making the substitution $t = h(\tau)$ in the integral gives

$$\int_a^b \underline{g}(\underline{p}[t]) \cdot \dot{\underline{p}}(t) dt = \int_{a'}^{b'} \underline{g}(\underline{p}[h(\tau)]) \cdot \dot{\underline{p}}(h[\tau])\, \dot{h}(\tau) d\tau = \int_{a'}^{b'} \underline{g}(\underline{q}[\tau]) \cdot \dot{\underline{q}}(\tau) d\tau.$$

So, integrating with $\underline{p}$ or $\underline{q}$ gives the same result. This shows that the integral depends on a curve, rather than a particular parametrization. ◁

The restriction $\dot{h} \geq 0$ in Definition 2.2.4 is important.

Example 2.3.5. *Compute the line integral of $\underline{g}(x, y) = (x, x)$ along the line segment from $(0, 0)$ to $(1, 2)$ using $\underline{p} : [0, 1] \to \mathbb{R}^2$ defined by $\underline{p}(t) = (t, 2t)$. Then try to use $\underline{q}(\tau) = \underline{p}(1 - \tau)$.*

Solution. $\dot{\underline{p}}(t) = (1, 2)$ and $\underline{g}(\underline{p}[t]) = (t, t)$, so

$$\int_C \underline{g}(\underline{x}) \cdot d\underline{x} = \int_0^1 \underline{g}(\underline{p}[t]) \cdot \dot{\underline{p}}(t)\, dt = \int_0^1 (t, t) \cdot (1, 2)\, dt = \int_0^1 3t\, dt = \frac{3}{2}.$$

On the other hand, with $\underline{q}(\tau) = (1 - \tau, 2 - 2\tau)$, $\dot{\underline{q}}(\tau) = (-1, -2)$ and $\underline{g}(\underline{q}[\tau]) = (1 - \tau, 1 - \tau)$, giving

$$\int_0^1 \underline{g}(\underline{q}[\tau]) \cdot \dot{\underline{q}}(\tau)\, d\tau = \int_0^1 (-1, -2) \cdot (1 - \tau, 1 - \tau)\, d\tau = \int_0^1 (-3 + 3\tau)\, d\tau = \frac{-3}{2}. \quad \triangleleft$$

This $\underline{q}$ is a parametrization of C with the *opposite* orientation.

Definition 2.3.6. $\bar{C}$ denotes the curve C with the orientation reversed.

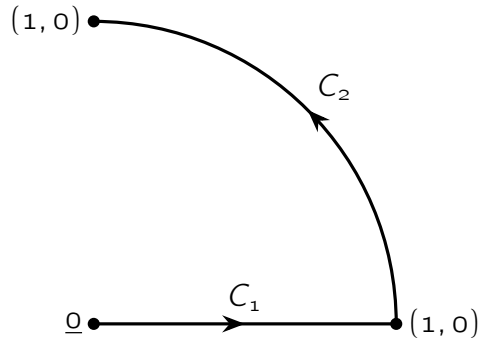
The line integral along $\bar{C}$ is

$$\int_{\bar{C}} \underline{g}(\underline{x}) \cdot d\underline{x} = -\int_C \underline{g}(\underline{x}) \cdot d\underline{x}.$$

Example 2.3.7. *Compute the line integral of $\underline{g}(x, y) = (x + y, 0)$ along C, where C consists of the straight line segment from $(0, 0)$ to $(1, 0)$ and then the part of the unit circle anticlockwise from $(1, 0)$ to $(0, 1)$.*

Solution. This C is not a smooth curve, so it is easiest to express this as a concatenation of smooth curves and use property (2.1) to evaluate the integral.

Let C_1 be the straight segment and C_2 the circular arc, so that $C = C_1 \cup C_2$.



C_1 is easily parametrized by the x coordinate,

$$\int_{C_1} \underline{g}(\underline{x}) \cdot d\underline{x} = \int_{C_1} x\, dx = \frac{1}{2}.$$

C_2 is parametrized by $\underline{p} : [0, \frac{\pi}{2}] \to \mathbb{R}^2$, $\underline{p}(t) = (\cos t, \sin t)$,

$$\begin{aligned}\int_{C_2} \underline{g}(\underline{x}) \cdot d\underline{x} &= \int_0^{\frac{\pi}{2}} (\cos t + \sin t, 0) \cdot (-\sin t, \cos t)\, dt \\ &= \int_0^{\frac{\pi}{2}} (-\cos t\, \sin t - \sin^2 t)\, dt = -\frac{1}{2} - \frac{\pi}{4}.\end{aligned}$$

Therefore,

$$\int_C \underline{g}(\underline{x}) \cdot d\underline{x} = \frac{-\pi}{4}. \qquad \triangleleft$$

2.4. **The fundamental theorem.** One version of the Fundamental Theorem of Calculus states that

$$\int_a^b f'(x)dx = f(b) - f(a), \tag{2.4}$$

i.e., integration undoes differentiation.

Similarly, the line integral undoes the gradient.

Theorem 2.4.1 (Fundamental Theorem of Line Integrals)**.** *If C is a curve in $\mathbb{R}^n$, going from $\underline{a}$ to $\underline{b}$, and $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ is a scalar field, then*

$$\int_C \nabla f(\underline{x}) \cdot d\underline{x} = f(\underline{b}) - f(\underline{a}).$$

Proof. Let $\underline{p} \in \mathcal{C}^1([a, b], \mathbb{R}^n)$ be a parametrization of C. The chain rule for $f \circ \underline{p}$ can be written as

$$\frac{d}{dt} f(\underline{p}[t]) = Df(\underline{p}[t])\, D\underline{p}(t) = \nabla f(\underline{p}[t]) \cdot \dot{\underline{p}}(t).$$

By eq. (2.4), this gives

$$\int_C \nabla f(\underline{x}) \cdot d\underline{x} = \int_a^b \nabla f(\underline{p}[t]) \cdot \dot{\underline{p}}(t)\, dt = \int_a^b \frac{d}{dt} f(\underline{p}[t])\, dt = f(\underline{b}) - f(\underline{a}). \qquad \square$$

Remark. This is not quite as useful as the Fundamental Theorem of Calculus. Because any nice function of a single variable is the derivative of another function, the FTC can be used to compute any reasonable integral. However, not every vector field is a gradient; thus, the Fundamental Theorem of Line Integrals cannot be directly applied to compute most line integrals.

Example 2.4.2. *The vector field $\underline{g}(x, y) = (y, x + y)$ happens to be a gradient. Find a scalar field $f \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ such that $\nabla f = \underline{g}$ and $f(\underline{0}) = 0$.*

Solution. We solved this type of problem in Calculus by treating it as a pair of differential equations:

$$\frac{\partial f}{\partial x} = y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x + y.$$

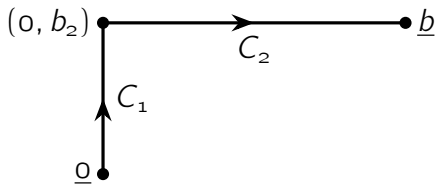
Integrating the first equation shows that $f(x, y) = xy + h(y)$, for some function h, because this h term is always killed by the x derivative. Inserting this into the second equation shows that

$$\frac{\partial f}{\partial y} = x + h'(y) \stackrel{!}{=} x + y,$$

so $h'(y) = y$. With the assumption that $f(\underline{0}) = 0$, this gives $h(y) = \frac{1}{2}y^2$ and hence

$$f(x, y) = xy + \tfrac{1}{2}y^2. \tag{2.5}$$

This solution is equivalent to computing a line integral. Let C_1 be the line segment from $\underline{0}$ to $(0, b_2)$ and C_2 the line segment from $(0, b_2)$ to $\underline{b}$.



With the assumption that $f(\underline{0}) = 0$,

$$\begin{aligned} f(\underline{b}) &= \int_{C_1 \cup C_2} \underline{g}(\underline{x}) \cdot d\underline{x} = \int_{C_1} \underline{g}(\underline{x}) \cdot d\underline{x} + \int_{C_2} \underline{g}(\underline{x}) \cdot d\underline{x} = \int_{C_1} y \, dy + \int_{C_2} b_2 dx \\ &= \tfrac{1}{2}b_2^2 + b_1 b_2. \end{aligned}$$

After changing the variable names, this is eq. (2.5).

We can also compute this in one step with a simpler curve. Let C be the straight line from $\underline{0}$ to $\underline{b}$, and define $\underline{p} : [0, 1] \to \mathbb{R}^2$ by $\underline{p}(t) = t\,\underline{b}$.

$$\begin{aligned} f(\underline{b}) &= \int_C \underline{g}(\underline{x}) \cdot d\underline{x} = \int_0^1 (tb_2, tb_1 + tb_2) \cdot (b_1, b_2)\, dt = \int_0^1 t(2b_1 b_2 + b_2^2) dt \\ &= b_1 b_2 + \tfrac{1}{2}b_2^2. \end{aligned}$$

◁

2.5. **Arc length.** How long is a curve C?

Again, cut C into small segments $C_1, \dots, C_N$, where C_k goes from $\underline{x}_{k-1}$ to $\underline{x}_k$. If C_k is smooth and small, then it is approximately straight, so

$$\operatorname{Length} C = \sum_{k=1}^{N} \operatorname{Length} C_k \approx \sum_{k=1}^{N} \left\| \underline{x}_k - \underline{x}_{k-1} \right\|.$$

If, again, $\underline{p} : [a, b] \to \mathbb{R}^n$ is a parametrization of C and $\underline{x}_k = \underline{p}(t_k)$, then

$$\operatorname{Length} C \approx \sum_{k=1}^{N} \left\| \underline{p}(t_k) - \underline{p}(t_{k-1}) \right\| \approx \sum_{k=1}^{N} \left\| \dot{\underline{p}}(t_k) \right\| (t_k - t_{k-1}) \approx \int_a^b \left\| \dot{\underline{p}}(t) \right\| dt.$$

In the limit of small spacing, this shows that

$$\operatorname{Length} C = \int_a^b \left\| \dot{\underline{p}}(t) \right\| dt. \tag{2.6}$$

Example 2.5.1. *Compute the length of the curve C, parametrized by $\underline{p} : [0, 1] \to \mathbb{R}^3$, $\underline{p}(t) = (t, t^2, \frac{2}{3}t^3)$.*

Solution. The derivative of $\underline{p}$ is $\dot{\underline{p}}(t) = (1, 2t, 2t^2)$, so

$$\left\| \dot{\underline{p}}(t) \right\|^2 = 1 + 4t^2 + 4t^4 = (1 + 2t^2)^2.$$

This gives the arc length

$$\operatorname{Length} C = \int_0^1 \left\| \dot{\underline{p}}(t) \right\| dt = \int_0^1 (1 + 2t^2) dt = \left[t + \tfrac{2}{3}t^3 \right]_{t=0}^{t=1} = \frac{5}{3}. \qquad \lhd$$

The computation of arc length generalizes to another type of integral:

Definition 2.5.2. The integral of a scalar field $f \in \mathcal{C}(\mathbb{R}^n, \mathbb{R})$ with respect to arc length along a curve C is

$$\int_C f \, ds := \int_a^b f(\underline{p}[t]) \left\| \dot{\underline{p}}(t) \right\| dt$$

for any parametrization $\underline{p} : [a, b] \to \mathbb{R}^n$ of C.

This integral is independent of the parametrization *and* the orientation of C. It is less natural and less well-behaved than the vector line integral, because it requires computing a square root.

Heuristically, $ds = \|d\underline{x}\|$.

With this notation,

$$\operatorname{Length} C = \int_C ds.$$

Example 2.5.3. *Let C be the curve parametrized by $\underline{p} : [0, 1] \to \mathbb{R}^3$,*

$$\underline{p}(t) = (\cosh t, \sinh t, t).$$

Suppose that we have a wire in the shape of C and the temperature of the air is $T(x, y, z) = y$. At any point, the wire absorbs heat from the air at a rate proportional to the temperature there, so the total rate of heat absorption is proportional to the integral of T along C. What is that integral?

Solution. In this case, $\dot{\underline{p}}(t) = (\sinh t, \cosh t, 1)$,

$$\left\|\dot{\underline{p}}(t)\right\|^2 = \sinh^2 t + \cosh^2 t + 1 = 2\cosh^2 t.$$

This gives $ds = \sqrt{2}\cosh t\, dt$. The rate of heat absorption is proportional to

$$\int_C T\, ds = \int_0^1 \sqrt{2}\sinh t\, \cosh t\, dt = \frac{1}{\sqrt{2}}\left[\sinh^2 t\right]_{t=0}^{t=1} = \frac{\sinh^2 1}{\sqrt{2}}.$$

◁

Definition 2.5.4. For a curve C parametrized by $\underline{p}$, if $\dot{\underline{p}}(t) \neq \underline{0}$, then the *unit tangent vector* to C at $\underline{p}(t)$ is

$$\underline{T}(t) := \frac{\dot{\underline{p}}(t)}{\left\|\dot{\underline{p}}(t)\right\|}.$$

This allows us to relate these two types of integral:

$$\int_C \underline{g}(\underline{x}) \cdot d\underline{x} = \int_a^b \underline{g}(\underline{p}[t]) \cdot \dot{\underline{p}}(t) dt = \int_a^b \underline{g}(\underline{p}[t]) \cdot \frac{\dot{\underline{p}}(t)}{\left\|\dot{\underline{p}}(t)\right\|} \left\|\dot{\underline{p}}(t)\right\| dt = \int_C \underline{g} \cdot \underline{T}\, ds.$$

Conversely,

$$\int_C f\, ds = \int_C f\, \underline{T} \cdot d\underline{x}.$$

In particular, the most natural (but not usually best) way to parametrize a curve is by arc length. At any $\underline{x} \in C$, define s to be the arc length along C from the beginning to $\underline{x}$. If we parametrize C by s, then $\underline{T}(s) = \dot{\underline{p}}(s)$.

Example 2.5.5. *Let C be the curve parametrized by $\underline{p} : [0, 1] \to \mathbb{R}^2$,*

$$\underline{p}(t) = \left(\tfrac{1}{1+t^2}, \tfrac{t}{1+t^2}\right).$$

Reparametrize C such that the new parameter is the arc length from the beginning of C.

Solution. First, differentiate,

$$\dot{\underline{p}}(t) = \left(\tfrac{-2t}{[1+t^2]^2}, \tfrac{1-t^2}{[1+t^2]^2}\right) = \frac{(-2t, 1-t^2)}{[1+t^2]^2}.$$

Taking the norm and squaring this gives,

$$\left\|\dot{\underline{p}}(t)\right\|^2 = \frac{(-2t)^2 + (1-t^2)^2}{(1+t^2)^4} = \frac{(1+t^2)^2}{(1+t^2)^4} = \frac{1}{(1+t^2)^2}.$$

So,

$$\frac{ds}{dt} = \left\|\dot{\underline{p}}(t)\right\| = \frac{1}{1+t^2}.$$

This gives the arc length

$$s = \int \frac{dt}{1+t^2} = \tan^{-1} t$$

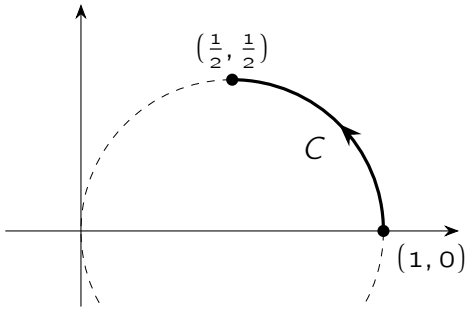
(with the constant so that $s = 0$ when $t = 0$). This is easily inverted to $t = \tan s$. Note that

$$t = 0 \implies s = 0, \qquad t = 1 \implies s = \tfrac{\pi}{4}.$$

The parametrization of C by arc length is $\underline{q} : [0, \frac{\pi}{4}] \to \mathbb{R}^2$,

$$\underline{q}(s) = \underline{p}(\tan s) = \left(\frac{1}{1+\tan^2 s}, \frac{\tan s}{1+\tan^2 s} \right) = \left(\frac{1}{\sec^2 s}, \frac{\tan s}{\sec^2 s} \right)$$
$$= (\cos^2 s, \sin s \cos s) = \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2s, \tfrac{1}{2}\sin 2s \right).$$

This C is just a quarter of the circle of radius $\frac{1}{2}$, centered at $(\frac{1}{2}, 0)$.



$\triangleleft$

3. Green's Theorem

For this section, we specialize to 2 dimensions.

3.1. **Double integration.** The integral of a scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ over a region $R \subset \mathbb{R}^2$ is denoted

$$\iint_R f \, dA.$$

Nota bene. Here, A is *not* a variable name. It stands for "area" and refers to the property that integrating 1 over R gives the area of R.

The most important property of this is *additivity*: If $R, S \subset \mathbb{R}^2$ are non-overlapping regions ($\operatorname{int} R \cap \operatorname{int} S = \emptyset$) then

$$\iint_{R \cup S} f \, dA = \iint_R f \, dA + \iint_S f \, dA.$$

Recall from Calculus:

Theorem 3.1.1 (Fubini's theorem)**.** *Let $f \in \mathcal{C}(R, \mathbb{R})$.*

- *If $R = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$ ("type I") where $\varphi_1, \varphi_2 : [a, b] \to \mathbb{R}$ are both continuous, then*

$$\iint_R f \, dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) \, dy \, dx \tag{3.1a}$$

- *If $R = \{(x, y) \in \mathbb{R}^2 \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$ ("type II") where $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ are both continuous, then*

$$\iint_R f \, dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) \, dx \, dy \tag{3.1b}$$

With this in mind, I will also denote the double integral more explicitly as

$$\iint_R f \, dA = \iint_R f(x, y) \, dx \, dy.$$

In this way, we can evaluate a double integral over any region that we can express as a union of non-overlapping regions of type I and II. Another method is to use another coordinate system.

3.2. **Substitution.**

Definition 3.2.1. For a map $\underline{P} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, the *Jacobian* determinant $J_{\underline{P}} \in \mathcal{C}(\mathbb{R}^n, \mathbb{R})$ is the determinant of the total derivative,

$$J_{\underline{P}}(\underline{x}) := \det D\underline{P}(\underline{x}).$$

Example 3.2.2. *Let $\underline{P}(r, \theta) = (r \cos\theta, r \sin\theta)$. What is the Jacobian determinant of this map?*

Solution.

$$J_{\underline{P}}(r, \theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r. \qquad \lhd$$

Definition 3.2.3. For $S \subseteq \mathbb{R}^n$, a map $\underline{P} \in \mathcal{C}^1(S, \mathbb{R}^n)$ is *orientation preserving* if its Jacobian determinant, $J_{\underline{P}}$, is positive over the interior, $\operatorname{int} S$. It is *orientation reversing* if its Jacobian is negative over $\operatorname{int} S$.

Example 3.2.4. The map $\underline{P}$ in the previous example, restricted to $r \geq 0$ is orientation preserving. ◁

Example 3.2.5. The map $\underline{P} : \mathbb{R}^2 \to \mathbb{R}^2$, $\underline{P}(x, y) = (y, x)$ has $J_{\underline{P}}(\underline{x}) = -1$, so it is orientation reversing. ◁

Example 3.2.6. Orientation is important in integration. Consider a 1-dimensional integral such as

$$\int_0^1 f(x)\, dx.$$

If we make the simple substitution $x = -u$, then the limits are $u = -1$ and $u = 0$, and $\frac{dx}{du} = -1$. We might naively write the integral as

$$\int_{-1}^{0} f(-u)\frac{dx}{du}du = -\int_{-1}^{0} f(-u)\, du,$$

but this is equal to $-\int_0^1 f(x)\, dx$. What went wrong?

This substitution is given by the function $g : \mathbb{R} \to \mathbb{R}$, $g(u) = -u$. The Jacobian determinant of g is just its derivative, which is $J_g(u) = -1$, so g is orientation reversing. It flips the domain of integration over. In integrating by substitution, we should either put the limits of integration in reverse order or correct with an overall negative sign to account for the reversal of orientation. ◁

Theorem 3.2.7. *Consider regions $R, S \subset \mathbb{R}^2$ and a map $\underline{P} : S \to R$ that is $\mathcal{C}^1$, onto, one-to-one over $\operatorname{int} S$, and orientation preserving. If $f \in \mathcal{C}(R, \mathbb{R})$, then*

$$\iint_R f\, dA = \iint_S f(\underline{P}[u, v])\, J_{\underline{P}}(u, v)\, du\, dv = \iint_S (f \circ \underline{P})\, J_{\underline{P}}\, dA.$$

This can be proven by cutting S into arbitrarily small pieces, which gives a decomposition of R into small pieces. Over a small piece of S, $\underline{P}$ is approximately linear, and $\underline{P}$ transforms the area by (approximately) a factor of $J_{\underline{P}}$ evaluated there.

Example 3.2.8. *Let $R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the unit disk, and*

$$S = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}.$$

Let $\underline{P} : S \to R$, $\underline{P}(r, \theta) = (r\cos\theta, r\sin\theta)$. Express the double integral of $f \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ over R using $\underline{P}$ as a substitution.

Solution. As we have already seen, $J_{\underline{P}}(r, \theta) = r$, so

$$\iint_R f\, dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta. \qquad ◁$$

Notation 3.2.9. If $(x, y) = \underline{P}(u, v)$, then the Jacobian determinant is also denoted as

$$\frac{\partial(x, y)}{\partial(u, v)} = J_{\underline{P}}(u, v).$$

If $\underline{G}$ is the inverse map, so that $(u, v) = \underline{G}(x, y)$, then by Example 1.4.3, $D\underline{G}(x, y) = [D\underline{P}(u, v)]^{-1}$. The multiplicative property of determinants ($\det AB = \det A \cdot \det B$) shows that $J_{\underline{G}}(x, y) = \left[J_{\underline{P}}(u, v)\right]^{-1}$, i.e.,

$$\frac{\partial(u, v)}{\partial(x, y)} = \left(\frac{\partial(x, y)}{\partial(u, v)}\right)^{-1}.$$

Example 3.2.10. *Compute the double integral*

$$\iint_R e^{(y-x+3)/(x+y+1)}\,dx\,dy,$$

where R is the triangle bounded by $x + y = 1$, $x = 1$, and $y = -2$.

Solution. The integrand is quite unpleasant, since it involves division within an exponent. We can clean things up by using the numerator and denominator as new variables. So, try setting $u = y - x + 3$ and $v = x + y + 1$. This gives the Jacobian (of the *inverse* of the substitution)

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2,$$

which tells us that

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}.$$

This is negative, so the substitution is orientation reversing, and we introduce another factor of -1 to cancel this negative sign.

To describe the integration region in the u-v-plane, look what happens to the bounding lines,

$$\begin{aligned} x + y = 1 &\implies v = 2, \\ x = 1 &\implies v = y + 2 = u, \end{aligned}$$

and

$$y = -2 \implies v = x - 1 = -u.$$

Letting S denote the triangle in the u-v-plane bounded by these lines, we have

$$\begin{aligned} \iint_R e^{(y-x+3)/(x+y+1)}\,dx\,dy &= -\iint_S e^{u/v}\left(\tfrac{-1}{2}\right)du\,dv = \frac{1}{2}\int_0^2\int_{-v}^{v} e^{u/v}\,du\,dv \\ &= \frac{1}{2}\int_0^2 \left[v\,e^{u/v}\right]_{u=-v}^{u=v} dv = \frac{1}{2}(e - e^{-1})\int_0^2 v\,dv \\ &= e - e^{-1}. \end{aligned}$$

Note that we never actually needed to solve for x and y in terms of u and v. ◁

3.3. **Closed curves.**

Definition 3.3.1. A *closed curve* is a curve that ends where it begins. Integration around a closed curve is indicated with $\oint$.

Remark. The $\oint$ notation is not really necessary. It is just a reminder that we are dealing with a closed curve.

For a closed curve, we don't really care where the end/beginning is.

Suppose that C is a closed curve, beginning and ending at $\underline{a}$. If $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, then the line integral of ∇f is just

$$\oint_C \nabla f(\underline{x}) \cdot d\underline{x} = f(\underline{a}) - f(\underline{a}) = 0,$$

by the Fundamental Theorem of Line Integrals. This suggests that the line integral of a (more general) vector field around a closed curve might be special. We will first consider this in $\mathbb{R}^2$.

Definition 3.3.2. For a region $R \subset \mathbb{R}^2$, ∂R denotes the boundary of R. This is a closed curve or union of closed curves, oriented with R on the left.

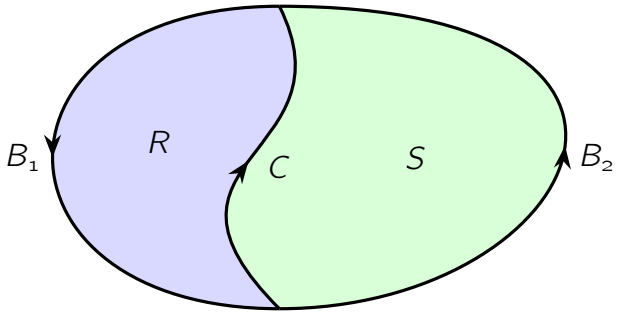
Remark. Imagine cutting R out of paper using right-handed scissors. The orientation of ∂R is the direction that you would cut.

Nota bene. The symbol ∂ is used to denote both partial derivatives and boundaries, but these are not the same thing.

3.4. **Green's theorem.** For a region $R \subset \mathbb{R}^2$ and a vector field $\underline{g} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$, I would like to consider the line integral

$$\oint_{\partial R} \underline{g}(\underline{x}) \cdot d\underline{x}.$$

Consider two regions $R, S \subset \mathbb{R}^2$ that touch along a curve C, oriented anticlockwise around R. Let B_1 be the rest of the boundary of R, so that $\partial R = C \cup B_1$. Note that C goes *clockwise* around S. Let B_2 be the rest of the boundary of S, so that $\partial S = \bar{C} \cup B_2$ and $\partial(R \cup S) = B_1 \cup B_2$.



With this notation,

$$\oint_{\partial R} \underline{g}(\underline{x}) \cdot d\underline{x} + \oint_{\partial S} \underline{g}(\underline{x}) \cdot d\underline{x} =$$
$$\int_C \underline{g}(\underline{x}) \cdot d\underline{x} + \int_{B_1} \underline{g}(\underline{x}) \cdot d\underline{x} - \int_C \underline{g}(\underline{x}) \cdot d\underline{x} + \int_{B_2} \underline{g}(\underline{x}) \cdot d\underline{x}$$
$$= \oint_{B_1 \cup B_2} \underline{g}(\underline{x}) \cdot d\underline{x} = \oint_{\partial(R \cup S)} \underline{g}(\underline{x}) \cdot d\underline{x}.$$

This means that this integral around the boundary has the same additive property as the double integral. If a region R is divided into small pieces, then the integral around ∂R is the sum of the integrals around the boundaries of the small pieces. Perhaps it *is* a double integral — but of what?

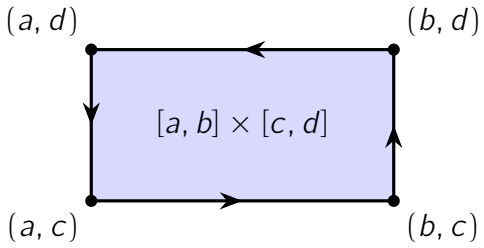
Notation 3.4.1. The symbol $\times$ denotes the Cartesian product of sets (Mathematical Skills I). It will have another meaning later.

Example 3.4.2. *Consider a rectangle,*

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

and a vector field $\underline{g} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$. *What is the line integral of* $\underline{g}$ *around* ∂R*?*

Solution.



$$\oint_{\partial R} \underline{g}(\underline{x}) \cdot d\underline{x} = \int_a^b g_1(x, c)dx + \int_c^d g_2(b, y)dy - \int_a^b g_1(x, d)dx - \int_c^d g_2(a, y)dy$$
$$= \int_c^d \big[g_2(b, y) - g_2(a, y)\big]dy - \int_a^b \big[g_1(x, d) - g_1(x, c)\big]dy$$
$$= \int_c^d \int_a^b \frac{\partial g_2}{\partial x} dx\, dy - \int_a^b \int_c^d \frac{\partial g_1}{\partial y} dy\, dx = \iint_R \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) dA. \qquad \triangleleft$$

Definition 3.4.3. For $\underline{g} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$, the *planar curl*, $\operatorname{curl} \underline{g} \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$, is

$$\operatorname{curl} \underline{g} = D_1 g_2 - D_2 g_1,$$

i.e.,

$$\operatorname{curl} \underline{g}(x, y) = \frac{\partial g_2(x, y)}{\partial x} - \frac{\partial g_1(x, y)}{\partial y}.$$

Nota bene. This is not standard notation. There is no common notation for the planar curl.

Theorem 3.4.4 (Green's Theorem)**.** *For a region $R \subset \mathbb{R}^2$ and $\underline{g} \in \mathcal{C}^1(R, \mathbb{R}^2)$,*

$$\oint_{\partial R} \underline{g}(\underline{x}) \cdot d\underline{x} = \iint_R \operatorname{curl} \underline{g}\, dA. \tag{3.2}$$

This is also often written as, for $P, Q \in \mathcal{C}^1(R, \mathbb{R})$,

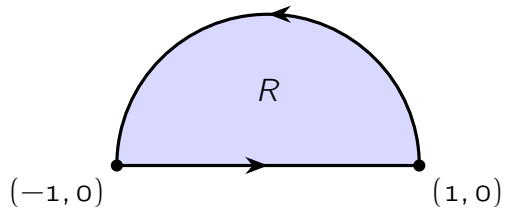
$$\oint_{\partial R} \big(P\, dx + Q\, dy\big) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy,$$

but the variables could have any names.

Example 3.4.5. *Compute the line integral of $\underline{g} : \mathbb{R}^2 \to \mathbb{R}^2$, $\underline{g}(x, y) = (x^2 + x^2y^3, xy + x^3y^2)$ around the curve that consists of the line segment from $(-1, 0)$ to $(1, 0)$ and the unit semicircle anticlockwise back to $(-1, 0)$.*

Solution. This curve is the boundary of the half disk,

$$R = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0,\ x^2 + y^2 \leq 1\}.$$



For this vector field, the planar curl is

$$\begin{aligned} \operatorname{curl} \underline{g}(x, y) &= \frac{\partial(xy + x^3y^2)}{\partial x} - \frac{\partial(x^2 + x^2y^3)}{\partial y} \\ &= y + 3x^2y^2 - 3x^2y^2 = y. \end{aligned}$$

By Green's theorem,

$$\begin{aligned} \oint_{\partial R} \underline{g}(\underline{x}) \cdot d\underline{x} = \iint_R y\, dA &= \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} y\, dy\, dx = \int_{-1}^{1} \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\sqrt{1-x^2}} dx \\ &= \int_{-1}^{1} \tfrac{1}{2}(1 - x^2) dx = \left[\tfrac{1}{2}x - \tfrac{1}{6}x^3\right]_{x=-1}^{x=1} = 2\left(\tfrac{1}{2} - \tfrac{1}{6}\right) = \frac{2}{3}. \end{aligned}$$ ◁

One application of Green's theorem is to compute area. If $\operatorname{curl} \underline{g} = 1$, then Green's theorem computes the area of a region by a line integral around its boundary. For example, $\underline{g}(x, y) = (-\frac{1}{2}y, \frac{1}{2}x)$ is often a good choice for this purpose.

Example 3.4.6. *$\underline{p} : [0, 1] \to \mathbb{R}^2$, $\underline{p}(t) = (t[1 - t]^2, t^2[1 - t])$ parametrizes a closed curve, which is the boundary of a region, R, in the first quadrant. What is the area of R?*

Solution. The boundary curve is given by $xy = (x + y)^3$. It is messy to solve for y in terms of x from this, so although R is a type I region, it is not convenient to express it in that way, and the methods of elementary Calculus do not work well to compute its area.

On the other hand, this is easy to compute by a line integral. If we set $\underline{g}(x,y) = (-\frac{1}{2}y, \frac{1}{2}x)$, then $\mathbf{curl}\,\underline{g} = 1$, so

$$\begin{aligned}\mathbf{Area}(R) &= \iint_R 1\,dA = \iint_R \mathbf{curl}\,\underline{g}\,dA = \int_{\partial R} \underline{g}(\underline{x})\cdot d\underline{x}\\ &= \int_0^1 \left(-\tfrac{1}{2}t^2[1-t], \tfrac{1}{2}t[1-t]^2\right)\cdot([1-3t][1-t], 2t-3t^2)\,dt\\ &= \int_0^1 \tfrac{1}{2}t^2(1-t)^2\big[-(1-3t)+(2-3t)\big]\,dt\\ &= \int_0^1 \tfrac{1}{2}t^2(1-t)^2 dt = \int_0^1 \left(\tfrac{1}{2}t^2 - t^3 + \tfrac{1}{2}t^4\right)dt = \frac{1}{6}-\frac{1}{4}+\frac{1}{10} = \frac{1}{60}. \quad \lhd\end{aligned}$$

Example 3.4.7. In the simplest mathematical model of a steam engine, the state of the system is described by 4 variables: p (pressure), V (volume), T (temperature), and S (entropy). These all depend on t (time). The purpose of a steam engine is to convert heat energy (from burning coal) into mechanical work (which is also energy). Normally, the state of the system runs through a repeating cycle.

The work done by the steam pushing on the piston is the integral of $p(t)\dot{V}(t)dt$, so the work done in one cycle, C, can be expressed as a line integral,

$$W = \oint_C p\,dV.$$

If C encloses a region R in the p-V-plane, then Green's theorem tells us that

$$W = \iint_R dp\,dV = \mathbf{Area}(R).$$

So, the work done by a steam engine in one cycle is the area enclosed by that cycle in the p-V-plane! $\lhd$

Example 3.4.8. Imagine a flowing stream and let $\underline{v}(x,y)$ be the velocity at the point (x,y) on the surface of the stream. A small object floating on the surface will be pushed along by the flow of the water, but it may also rotate because the velocity of water on its left and right or front and back may be different. The planar curl describes this rotational effect. If $\mathbf{curl}\,\underline{v} = 0$ at the location of the object then it will not rotate. If $\mathbf{curl}\,\underline{v} > 0$, then it will rotate anticlockwise, and if $\mathbf{curl}\,\underline{v} < 0$, then it will rotate clockwise. $\lhd$

3.5. **Gradients.** The gradient of a scalar field is a vector field. The planar curl of a vector field is a scalar field. So, what is the planar curl of a gradient? For $f \in \mathcal{C}^2(\mathbb{R}^2,\mathbb{R})$,

$$\mathbf{curl}\,\nabla f = \mathbf{curl}(D_1 f, D_2 f) = D_1D_2 f - D_2D_1 f = 0,$$

because of Clairault's Theorem,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

This means that if $\underline{g} = \nabla f$, then $0 = \mathbf{curl}\,\underline{g}$, which is just the necessary condition for $\underline{g}$ to be a gradient that we saw in Calculus. The planar curl, $\mathbf{curl}\,\underline{g}$, measures the failure of $\underline{g}$ to be a gradient.

This is a *necessary* condition for $\underline{g}$ to be a gradient, but when is it sufficient?

Let $U \subseteq \mathbb{R}^n$ be an open subset. If we suspect that $\underline{g} \in \mathcal{C}^1(U, \mathbb{R}^n)$ is the gradient of some $f \in \mathcal{C}^2(U, \mathbb{R})$, then we can try to reconstruct f (up to a constant) by line integration. For some curve C_1 from $\underline{a}$ to $\underline{b}$,

$$f(\underline{b}) - f(\underline{a}) = \int_{C_1} \underline{g}(\underline{x}) \cdot d\underline{x}.$$

It turns out that the only thing that can go wrong with this plan is if a different curve, C_2, from $\underline{a}$ to $\underline{b}$ gives a different answer, so a sufficient condition for $\underline{g}$ to be a gradient is that its line integral is the same for any 2 curves from $\underline{a}$ to $\underline{b}$.

Note that $\bar{C}_2$ goes from $\underline{b}$ to $\underline{a}$, so $C_1 \cup \bar{C}_2$ is a closed curve and

$$\oint_{C_1 \cup \bar{C}_2} \underline{g}(\underline{x}) \cdot d\underline{x} = \int_{C_1} \underline{g}(\underline{x}) \cdot d\underline{x} - \int_{C_2} \underline{g}(\underline{x}) \cdot d\underline{x} = 0.$$

In fact, $C_1 \cup \bar{C}_2$ is *any* closed curve.

Theorem 3.5.1. *For $U \subseteq \mathbb{R}^n$ open and $\underline{g} \in \mathcal{C}^1(U, \mathbb{R}^n)$, there exists $f \in \mathcal{C}^2(U, \mathbb{R})$ such that $\underline{g} = \nabla f$ if an only if for every closed curve C in U,*

$$0 = \oint_C \underline{g}(\underline{x}) \cdot d\underline{x}. \tag{3.3}$$

Unfortunately, it may not be convenient to check this for every single closed curve.

Definition 3.5.2. A *simple* curve is one that does not intersect itself (other than at endpoints).

Example 3.5.3. *Suppose $U = \mathbb{R}^2$. If $\underline{g} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ satisfies $\operatorname{curl} \underline{g} = 0$, is it a gradient?*

Solution. It is actually sufficient to check eq. (3.3) for simple closed curves. Any simple closed curve in $\mathbb{R}^2$ (if oriented anticlockwise) is the boundary of a region. For any region $R \subset \mathbb{R}^2$, Green's theorem tells us that

$$\oint_{\partial R} \underline{g}(\underline{x}) \cdot d\underline{x} = \iint_R \operatorname{curl} \underline{g} \, dA = 0,$$

therefore $\underline{g}$ is the gradient of some scalar field. ◁

Example 3.5.4. *Let $U = \mathbb{R}^2 \setminus \{\underline{0}\}$ (the plane, except for the origin). Is $\underline{g} : U \to \mathbb{R}^2$,*

$$\underline{g}(x, y) = \left(\tfrac{-y}{x^2+y^2}, \tfrac{x}{x^2+y^2} \right)$$

a gradient?

Solution. Firstly,

$$D_1 g_2(x, y) = \frac{1}{x^2 + y^2} - \frac{2x \cdot x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and

$$D_2 g_1(x, y) = \frac{-1}{x^2 + y^2} - \frac{2y \cdot (-y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

so $\operatorname{curl} \underline{g} = 0$.

This looks promising. For the boundary of any region, eq. (3.3) will be satisfied, and this extends to any closed curve that doesn't go around the origin.

The danger is a curve that *does* go around the origin. Let C be the circle of radius 1 around the origin, oriented anticlockwise. This is parametrized by $\underline{p} : [0, 2\pi] \to \mathbb{R}^2$, $\underline{p}(t) = (\cos t, \sin t)$. Note that $\underline{g}(\underline{p}[t]) = (-\sin t, \cos t) = \dot{\underline{p}}(t)$, so

$$\oint_C \underline{g}(\underline{x}) \cdot d\underline{x} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} dt = 2\pi.$$

(Any closed curve that goes once anticlockwise around the origin will give the same number.) Therefore, $\underline{g}$ is not a gradient! ◁

In general, $\operatorname{curl} \underline{g} = 0$ implies that $\underline{g}$ is a gradient if the domain of $\underline{g}$ is *simply connected*, which means that any closed curve can be shrunken continuously down to a point. This means that there are no holes to get caught around. If there are holes, then it is sufficient to check the line integrals of $\underline{g}$ around curves encircling each of the holes.

4. Vector Algebra

4.1. **Index notation.** Any statement about some vectors is really a statement about all components of those vectors. For example, consider two vectors $\underline{v} = (v_1, \ldots, v_n)$ and $\underline{w} = (w_1, \ldots, w_n)$. These are equal, $\underline{v} = \underline{w}$, if and only if $v_1 = w_1$, $v_2 = w_2$, ... and $v_n = w_n$. More succinctly, we can just write that $v_a = w_a$, where "for all values of a from 1 to n" is left understood.

In this context, a is called an *index*. Its name doesn't have to be a, so for example, we can write $v_b = w_b$.

Nota bene. Do not confuse $\underline{v}$ with v_a. These are different ways of expressing the same structure, but they are not equal.

We can go back and forth between vector notation and this index notation for the components using the standard basis vectors:

Definition 4.1.1. The a'th basis vector is

$$\underline{e}_a = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$$

with 1 for the a'th component, and all other components 0.

With this notation, $x_a = \underline{e}_a \cdot \underline{x}$ and

$$\begin{aligned} \underline{x} &= x_1\underline{e}_1 + x_2\underline{e}_2 + \cdots + x_n\underline{e}_n \\ &= \sum_{a=1}^{n} x_a\underline{e}_a. \end{aligned} \tag{4.1}$$

However, expressions like this are cumbersome to write repeatedly. Instead, from now on, we will use the *summation convention*:

Notation 4.1.2. When an index name is repeated twice in the same term, this denotes summation over all possible values of the index.

Example 4.1.3. We can write eq. (4.1) more succinctly as

$$\underline{x} = x_a\underline{e}_a.$$

◁

Example 4.1.4. In index notation, the dot product is

$$\underline{u} \cdot \underline{v} = u_a v_a.$$

◁

Definition 4.1.5. Such a repeated index is called a *dummy index*. A non-repeated index is called a *free index*.

We can always rename a dummy index. It can have any name that's not being used for a different purpose in the same equation or expression, so we can equally well write $\underline{x} = x_b\underline{e}_b$ for eq. (4.1).

Example 4.1.6.

$$(\underline{u} \cdot \underline{v})(\underline{w} \cdot \underline{x}) = u_a v_a w_b x_b = u_a w_b v_a x_b$$

but you **cannot** write this as $u_a v_a w_a x_a$. ◁

Notation 4.1.7. For any vector expression, $(\dots)_a$ will denote the a'th component. We can use any available name for this index.

Example 4.1.8.

$$([\underline{u} \cdot \underline{v}]\underline{w})_a = u_b v_b w_a.$$

◁

Index notation can be used for matrix algebra. Let A_{ab} denote the component of a square matrix A in the a'th row and the b'th column. The product of a matrix and a column vector is computed by

$$(A\underline{x})_a = A_{ab} x_b.$$

The product of two matrices is computed by

$$(AB)_{ab} = A_{ac} B_{cb}.$$

Definition 4.1.9. The *Kronecker delta symbol* is

$$\delta_{ab} = \begin{cases} 1 & a = b \\ 0 & a \neq b. \end{cases}$$

This is the identity matrix, expressed in index notation. Any expression with a dummy index on a Kronecker delta can be simplified.

Example 4.1.10.

$$\delta_{ab} v_b = v_a. \tag{4.2}$$

◁

With index notation, we can express more complicated structures called *tensors*, which aren't limited to 1 or 2 indices. It is also useful for many calculations that would otherwise be quite confusing.

4.2. **The laws of index notation.** There are two laws of index notation. Do not break them.

(1) No index should be repeated more than twice in one term.
(2) The free indices must be the same in every term of an equation (unless the term is just 0).

The reason is that everything we do should be geometrically meaningful. In particular, we might rotate the basis vectors to get a new basis, $\{\underline{e}'_{a'}\}_{a'=1}^{n}$, and computations using these different bases should always be equivalent.

Example 4.2.1. In $\mathbb{R}^2$, we can rotate the standard basis vectors

$$\underline{e}_1 = (1, 0) \quad \text{and} \quad \underline{e}_2 = (0, 1)$$

by $\frac{\pi}{4}$ to get

$$\underline{e}'_1 = (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) \quad \text{and} \quad \underline{e}'_2 = (\tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}).$$

◁

Example 4.2.2. Consider the vector $\underline{u} = (1, 1)$. If we were to repeat an index 3 times, we would get

$$u_a u_a u_a = (u_1)^3 + (u_2)^3 = 2.$$

Using the alternate basis from Example 4.2.1, $\underline{u} = \sqrt{2}\,\underline{e}'_1 + 0\,\underline{e}'_2$, so computing in this alternate basis would give

$$(\sqrt{2})^3 + 0^3 = 2\sqrt{2},$$

which is not 2. So, these numbers depend on the basis and are not meaningful. ◁

Example 4.2.3. Suppose that we tried to write $u_a = v_b$ in 2 dimensions. The only plausible interpretation is that this is true for all values of a and b, so $u_1 = v_1$ and $u_2 = v_1$, therefore $u_1 = u_2$.

Again, using the basis from Example 4.2.1, this implies that $\underline{u}$ is a multiple of $\underline{e}'_1$, so the coefficients of $\underline{e}'_1$ and $\underline{e}'_2$ are *not* equal (unless $\underline{u} = \underline{0}$). ◁

4.3. **The cross product.** The cross product of vectors is only defined in 3 dimensions, so this section is only about $\mathbb{R}^3$. You have previously known the basis vectors as $\underline{e}_1 = \underline{i}$, $\underline{e}_2 = \underline{j}$, and $\underline{e}_3 = \underline{k}$.

Recall that the *cross product* of $\underline{u}, \underline{v} \in \mathbb{R}^3$ is a vector $\underline{u} \times \underline{v}$ that is orthogonal to $\underline{u}$ and $\underline{v}$, points in the direction given by the right hand rule, and has magnitude

$$\|\underline{u} \times \underline{v}\| = \|\underline{u}\|\,\|\underline{v}\| \sin\theta$$

where θ is the angle between $\underline{u}$ and $\underline{v}$. Geometrically, this magnitude is the area of the parallelogram with vertices $\underline{0}$, $\underline{u}$, $\underline{v}$, and $\underline{u} + \underline{v}$.

Nota bene. The symbol $\times$ is used for both the Cartesian product of sets and the cross product of (3-dimensional) vectors. Do not confuse this with multiplication of numbers.

This has the properties:

- $\underline{v} \times \underline{u} = -\underline{u} \times \underline{v}$
- $\underline{u} \times (\lambda \underline{v}) = \lambda(\underline{u} \times \underline{v})$
- $\underline{u} \times (\underline{v} + \underline{w}) = \underline{u} \times \underline{v} + \underline{u} \times \underline{w}$.

By these properties, any cross product can be computed from the cross products of basis vectors. Using $\underline{u} = u_a \underline{e}_a$ and $\underline{v} = v_a \underline{e}_a$,

$$\underline{u} \times \underline{v} = (u_b \underline{e}_b) \times \underline{v} = u_b(\underline{e}_b \times \underline{v}) = u_b(\underline{e}_b \times [v_c \underline{e}_c]) = u_b v_c (\underline{e}_b \times \underline{e}_c).$$

For example, $\underline{e}_1$ and $\underline{e}_2$ are orthogonal unit vectors, and the right hand rule gives $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$. In general, the cross product of 2 basis vectors is $\pm$ the other basis vector.

Definition 4.3.1. The *alternating tensor* ε_{abc} is defined by $\varepsilon_{123} = 1$ and total anti-symmetry:

$$\varepsilon_{abc} = -\varepsilon_{bac} = -\varepsilon_{acb} = -\varepsilon_{cba}.$$

Equivalently,

$$\varepsilon_{abc} = \begin{cases} \operatorname{sgn}\left(\begin{smallmatrix} 1 & 2 & 3 \\ a & b & c \end{smallmatrix}\right) & \text{if } abc \text{ is a permutation of } 123. \\ 0 & \text{otherwise.} \end{cases}$$

That's permutation notation, not a matrix. The sign is $+1$ for an even permutation and -1 for an odd permutation.

With this notation, the a'th component of the cross product is given explicitly by

$$(\underline{u} \times \underline{v})_a = \varepsilon_{abc} u_b v_c.$$

Because cyclic permutations of 3 objects are even, the alternating tensor is invariant under cyclic permutations of the indices:

$$\varepsilon_{abc} = \varepsilon_{bca} = \varepsilon_{cab}.$$

Definition 4.3.2. The *triple product* of $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^3$ is

$$\underline{u} \cdot (\underline{v} \times \underline{w}).$$

This has symmetries that are not obvious until we use index notation.

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = u_a (\underline{v} \times \underline{w})_a = \varepsilon_{abc} u_a v_b w_c.$$

Now it is apparent that it has the same symmetries as the alternating tensor:

$$\begin{aligned} \underline{u} \cdot (\underline{v} \times \underline{w}) &= \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v}) \\ &= -\underline{v} \cdot (\underline{u} \times \underline{w}). \end{aligned}$$

4.4. **An identity for the alternating tensor.** Nice things happen if we multiply alternating tensors together. First, note that

$$\varepsilon_{abc}\varepsilon_{def} = \operatorname{sgn}\left(\begin{smallmatrix} 1 & 2 & 3 \\ a & b & c \end{smallmatrix}\right) \cdot \operatorname{sgn}\left(\begin{smallmatrix} 1 & 2 & 3 \\ d & e & f \end{smallmatrix}\right) = \operatorname{sgn}\left(\begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix}\right)$$

if abc and def are permutations of $1\,2\,3$, and 0 otherwise. Now, for the index values such that $\varepsilon_{abe}\varepsilon_{cde} \neq 0$, there is only one term in the sum over values of e. This is the term where e is not equal to a or b. This means that

$$\varepsilon_{abe}\varepsilon_{cde} = \operatorname{sgn}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$

or 0 if $a\,b$ is not a permutation of $c\,d$. This can be written as

$$\begin{aligned} \varepsilon_{abe}\varepsilon_{cde} &= \begin{cases} 1 & a = c \neq b = d \\ -1 & a = d \neq b = c \\ 0 & \text{otherwise} \end{cases} \\ &= \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}, \end{aligned} \tag{4.3}$$

which is an extremely useful identity.

Example 4.4.1. *Compute* $\underline{u} \times (\underline{v} \times \underline{w})$.

Solution. This is a vector, so we compute it by computing its components, using eqs. (4.3) and (4.2):

$$\begin{aligned}[\underline{u} \times (\underline{v} \times \underline{w})]_a &= \varepsilon_{abc} u_b (\underline{v} \times \underline{w})_c = \varepsilon_{abc} u_b \varepsilon_{cde} v_d w_e \\ &= \varepsilon_{abc}\varepsilon_{dec} u_b v_d w_e = (\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}) u_b v_d w_e \\ &= u_b v_a w_b - u_b v_b w_a = (\underline{u} \cdot \underline{w}) v_a - (\underline{u} \cdot \underline{v}) w_a,\end{aligned}$$

therefore

$$\underline{u} \times (\underline{v} \times \underline{w}) = (\underline{u} \cdot \underline{w})\, \underline{v} - (\underline{u} \cdot \underline{v})\, \underline{w}. \qquad \lhd$$

4.5. **Determinant formulae.** For explicit computations, it is also useful to express the cross product as a determinant

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

although $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ are *not* numbers.

Likewise, the triple product can be written as

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

5. Vector Differentiation

5.1. **The operations.** For scalar and vector fields in 3 dimensions, we have 3 differential operations that appear in the integral theorems of vector calculus. We have already seen the first.

- The *gradient* of a scalar field $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ is the vector field ∇f, defined by
$$(\nabla f)_a = D_a f.$$
- The *curl* of a vector field $\underline{g} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$ is the vector field $\nabla \times \underline{g}$ defined by
$$(\nabla \times \underline{g})_a = \varepsilon_{abc} D_b g_c.$$
- The *divergence* of a vector field $\underline{g} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ is the scalar field $\nabla \cdot \underline{g}$ defined by
$$\nabla \cdot \underline{g} = D_a g_a.$$

Remark. Note that the gradient and divergence are defined in any dimension, but we will focus on dimension 3. The curl in 3 dimensions is different from the planar curl that we used before.

Like the cross product, the curl is sometimes expressed formally as a determinant,

$$\nabla \times \underline{g} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ D_1 & D_2 & D_3 \\ g_1 & g_2 & g_3 \end{vmatrix}.$$

Nota bene. As the notation suggests, these operations are structured like the scalar multiplication, cross product, and dot product. However, ∇ is a symbol for differentiation, not a vector, and these are not really products.

Example 5.1.1. *Compute the curl of* $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\underline{F}(x, y, z) = (x^2yz, xy^2z, xyz^2).$$

Solution.

$$(\nabla \times \underline{F})(x, y, z) = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} = xz^2\underline{e}_1 + x^2y\underline{e}_2 + y^2z\underline{e}_3 - xy^2\underline{e}_1 - yz^2\underline{e}_2 - x^2z\underline{e}_3$$
$$= (x[z^2 - y^2], y[x^2 - z^2], z[y^2 - x^2]). \qquad \triangleleft$$

5.2. **Identities.** There are many identities involving these operations, along with the algebraic operations with vectors. These identities can all be derived easily using index notation.

Theorem 5.2.1. *For any scalar field* $f \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R})$,

$$\nabla \times \nabla f = \underline{0}. \tag{5.1}$$

Proof. Since the LHS is a vector field, we should compute its a'th component:

$$\begin{aligned}(\nabla \times \nabla f)_a &= \varepsilon_{abc} D_b (\nabla f)_c = \varepsilon_{abc} D_b D_c f && \text{by definition}\\ &= \varepsilon_{acb} D_c D_b f && \text{renaming } b \leftrightarrow c\\ &= \varepsilon_{acb} D_b D_c f && \text{reordering derivatives}\\ &= -\varepsilon_{abc} D_b D_c f && \text{antisymmetry}\\ &= 0 && \text{equals minus itself.}\end{aligned}$$ □

Theorem 5.2.2. *For any vector field, $\underline{g} \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R}^3)$,*

$$\nabla \cdot (\nabla \times \underline{g}) = 0. \tag{5.2}$$

Proof. The LHS is a scalar field, so compute it directly. Since ε_{abc} is constant,

$$\nabla \cdot (\nabla \times \underline{g}) = D_a (\nabla \times \underline{g})_a = D_a (\varepsilon_{abc} D_b g_c) = \varepsilon_{abc} D_a D_b g_c.$$

Again, $D_a D_b = D_b D_a$, so

$$\varepsilon_{abc} D_a D_b g_c = \varepsilon_{bac} D_b D_a g_c = \varepsilon_{bac} D_a D_b g_c = -\varepsilon_{abc} D_a D_b g_c = 0.$$ □

Example 5.2.3. *For a scalar field $f \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$ and a vector field $\underline{g} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$, compute the divergence*

$$\nabla \cdot (f\, \underline{g}).$$

Solution. This is a scalar field. By the product rule,

$$\begin{aligned}\nabla \cdot (f\, \underline{g}) &= D_a (f\, g_a) = (D_a f) g_a + f\, (D_a g_a)\\ &= (\nabla f) \cdot \underline{g} + f\, (\nabla \cdot \underline{g}).\end{aligned}$$ ◁

Example 5.2.4. *For vector fields $\underline{F}, \underline{G} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$, compute the divergence*

$$\nabla \cdot (\underline{F} \times \underline{G}).$$

Solution. This is a scalar field.

$$\begin{aligned}\nabla \cdot (\underline{F} \times \underline{G}) &= \varepsilon_{abc} D_a (F_b G_c) = \varepsilon_{abc} [(D_a F_b) G_c + F_b (D_a G_c)]\\ &= \varepsilon_{cab} (D_a F_b) G_c - \varepsilon_{bac} F_b (D_a G_c)\\ &= (\nabla \times \underline{F}) \cdot \underline{G} - \underline{F} \cdot (\nabla \times \underline{G}).\end{aligned}$$ ◁

Definition 5.2.5. The *Laplacian* operator is $\nabla^2 = D_a D_a$.

That is, for a scalar field $f \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R})$, $\nabla^2 f$ is the divergence of the gradient,

$$\nabla^2 f = \nabla \cdot (\nabla f) = D_a D_a f.$$

Example 5.2.6. *For a vector field $\underline{g} \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R}^3)$, what is the curl of the curl?*

Solution. This is a vector field, so we compute its components.

$$\begin{aligned}\big[\nabla \times (\nabla \times \underline{g})\big]_a &= \varepsilon_{abc} D_b (\nabla \times \underline{g})_c = \varepsilon_{abc} \varepsilon_{cde} D_b D_d g_e\\ &= (\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}) D_d D_b g_e = D_a D_b g_b - D_b D_b g_a,\end{aligned}$$

so

$$\nabla \times (\nabla \times \underline{g}) = \nabla (\nabla \cdot \underline{g}) - \nabla^2 \underline{g}.$$ ◁

Recall that the derivative $D\underline{F}$ is a matrix-valued function. Given a vector $\underline{v}$, the product $(D\underline{F})\underline{v}$ can be expressed in index notation as

$$([D\underline{F}]\underline{v})_a = (D_b F_a)v_b = v_b D_b F_a.$$

If we want to avoid matrix notation or index notation, we can stretch vector notation a bit to write this as

$$(D\underline{F})\underline{v} = (\underline{v}\cdot\nabla)\underline{F}.$$

(You will need this for one of the homework exercises.)

Example 5.2.7. *For $\underline{F},\underline{G}\in\mathcal{C}^1(\mathbb{R}^3,\mathbb{R}^3)$, reexpress*

$$\nabla(\underline{F}\cdot\underline{G}).$$

Solution. This is a vector field. Its a'th component is

$$[\nabla(\underline{F}\cdot\underline{G})]_a = D_a(F_bG_b) = F_bD_aG_b + G_bD_aF_b.$$

One of these terms occurs if we expand $\underline{F}\times(\nabla\times\underline{G})$:

$$\begin{aligned}[\underline{F}\times(\nabla\times\underline{G})]_a &= \varepsilon_{abc}\varepsilon_{cde}F_bD_dG_e = (\delta_{ad}\delta_{be}-\delta_{ae}\delta_{bd})F_bD_dG_e\\ &= F_bD_aG_b - F_bD_bG_a.\end{aligned}$$

Switching $\underline{F}$ and $\underline{G}$ shows that

$$[\underline{G}\times(\nabla\times\underline{F})]_a = G_bD_aF_b - G_bD_bF_a.$$

Adding these together gives

$$\begin{aligned}[\underline{F}\times(\nabla\times\underline{G})+\underline{G}\times(\nabla\times\underline{F})]_a &= D_a(F_bG_b) - F_bD_bG_a - G_bD_bF_a\\ &= [\nabla(\underline{F}\cdot\underline{G}) - (\underline{F}\cdot\nabla)\underline{G} - (\underline{G}\cdot\nabla)\underline{F}]_a.\end{aligned}$$

Finally, rearranging this gives

$$\nabla(\underline{F}\cdot\underline{G}) = \underline{F}\times(\nabla\times\underline{G}) + \underline{G}\times(\nabla\times\underline{F}) + (\underline{F}\cdot\nabla)\underline{G} + (\underline{G}\cdot\nabla)\underline{F}. \qquad \lhd$$

6. Triple Integration

6.1. **The integral.**

Notation 6.1.1. The integral of a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ over a solid region $E \subset \mathbb{R}^3$ is denoted

$$\iiint_E f \, dV.$$

Nota bene. This V is not a variable. It stands for "volume".

This has the same properties as integration over 2 dimensional regions, except that

$$\iiint_E 1 \, dV = \mathrm{Volume}(E).$$

Example 6.1.2. Consider an object consisting of matter that is unevenly distributed over $E \subset \mathbb{R}^3$. Its mass density is a scalar field $\rho : \mathbb{R}^3 \to \mathbb{R}$. That is, the mass of a bit of it around a point $\underline{x}$ is approximately $\rho(\underline{x})$ times the volume of that bit. The total mass of this is

$$M = \iiint_E \rho \, dV. \tag{6.1}$$

◁

The same idea holds for densities of other things such as charge.

6.2. **Integration in Cartesian coordinates.**

An integral over a suitable solid region can be computed as an iterated integral by another version of Fubini's Theorem.

Theorem 6.2.1. *If $R \subset \mathbb{R}^2$ is a planar region, $\psi_1 \leq \psi_2 \in \mathcal{C}(R, \mathbb{R})$,*

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in R, \ \psi_1(x, y) \leq z \leq \psi_2(x, y)\},$$

and $f \in \mathcal{C}(E, \mathbb{R})$, then

$$\iiint_E f \, dV = \iint_R \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) \, dz \right) dA.$$

In particular, if

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid a \leq x \leq b, \ \varphi_1(x) \leq y \leq \varphi_2(x), \ \psi_1(x, y) \leq z \leq \psi_2(x, y)\},$$

then

$$\iiint_E f \, dV = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) \, dz \, dy \, dx.$$

Similar statements are true for the other 5 possible orders of integration.

Definition 6.2.2. The triple integral of a vector field is the vector whose components are the integrals of the components of the vector field. For an object occupying a solid region $E \subset \mathbb{R}^3$, with mass density $\rho : E \to \mathbb{R}$, the *center of mass* is the point given by

$$\underline{x}_{\mathrm{cm}} := \frac{1}{M} \iiint_E \underline{x} \, \rho(\underline{x}) \, dV, \tag{6.2}$$

where M is the total mass, given by eq. (6.1).

Example 6.2.3. *Let E be the tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$. If E is filled with matter with constant mass density ρ, then where is its center of mass?*

Solution. We can compute by integrating in z, y, x order. First, we need to express E in the appropriate form.

- Over all of E, $0 \le x \le 1$.
- For a given value of x, the smallest value of y is 0 and the largest occurs at the intersection of the x-y-plane and the tilted plane (where $x+y+z=1$).
- For given values of x and y, the smallest value of z is 0 and the largest occurs when $x+y+z=1$.

Therefore,

$$E = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}.$$

Because ρ is constant, $M = \rho\,\mathrm{Volume}(E)$ and

$$\begin{aligned}\mathrm{Volume}(E) &= \iiint_E dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx \\ &= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx = \int_0^1 \Big[y - xy - \tfrac{1}{2}y^2\Big]_{y=0}^{y=1-x} dx \\ &= \int_0^1 \tfrac{1}{2}(1-x)^2 dx = \Big[-\tfrac{1}{6}(1-x)^3\Big]_{x=0}^{x=1} = \frac{1}{6}.\end{aligned}$$

So, $M = \rho/6$. The x-coordinate of the center of mass is

$$x_{\mathrm{cm}} = \frac{1}{M}\iiint_E x\,\rho\,dV = 6\iiint_E x\,dV = 6\int_0^1 x \int_0^{1-x}\int_0^{1-x-y} dz\,dy\,dx.$$

We have just seen in the previous step that

$$\int_0^{1-x}\int_0^{1-x-y} dz\,dy = \tfrac{1}{2}(1-x)^2,$$

so

$$\begin{aligned}x_{\mathrm{cm}} &= 3\int_0^1 x(1-x)^2 dx = \int_0^1 (3x - 6x^2 + 3x^3) = \Big[\tfrac{3}{2}x^2 - 2x^3 + \tfrac{3}{4}x^4\Big]_{x=0}^{x=1} \\ &= \frac{3}{2} - 2 + \frac{3}{4} = \frac{1}{4}.\end{aligned}$$

The tetrahedron E is symmetrical with respect to permuting the coordinates, therefore the other coordinates of the center of mass are the same. The center of mass is at

$$\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right). \qquad \lhd$$

6.3. **Substitution.** Now, suppose that we are trying to compute the integral

$$\iiint_E f\,dV$$

by a substitution $\underline{P} : W \to E$, where $W \subset \mathbb{R}^3$ is another solid region. Let u, v, and w be the coordinates on W and imagine breaking W (and thence E) into small pieces. A small Δu by Δv by Δw piece of W at (u,v,w) will be approximately transformed linearly by the matrix $D\underline{P}(u,v,w)$, so its volume will be (approximately) transformed

by the Jacobian determinant $J_{\underline{P}} = \det D\underline{P}$ and the volume of the image in E is approximately

$$J_{\underline{P}}(u, v, w)\, \Delta u\, \Delta v\, \Delta w.$$

(Provided that $\underline{P}$ is orientation preserving, i.e., $J_{\underline{P}} \geq 0$.)

Theorem 6.3.1. *Consider solid regions $E, W \subset \mathbb{R}^3$ and a map $\underline{P} : W \to E$ that is $\mathcal{C}^1$, onto, 1-1 over $\operatorname{int} W$, and orientation preserving. If $f \in \mathcal{C}(E, \mathbb{R})$, then*

$$\iiint_E f\, dV = \iiint_W f(\underline{P}[u, v, w]) J_{\underline{P}}(u, v, w)\, du\, dv\, dw = \iiint_W (f \circ \underline{P})\, J_{\underline{P}}\, dV.$$

Notation 6.3.2. If $(x, y, z) = \underline{P}(u, v, w)$, then the Jacobian determinant is also denoted as

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = J_{\underline{P}}(u, v, w).$$

Just as with 2 variables, this has the property that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \left(\frac{\partial(x, y, z)}{\partial(u, v, w)}\right)^{-1}.$$

Example 6.3.3. *Given that the volume inside a sphere of radius r is $\frac{4}{3}\pi r^3$, compute the volume inside the ellipsoid with radii a, b, and c:*

$$E = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; \left(\tfrac{x}{a}\right)^2 + \left(\tfrac{y}{b}\right)^2 + \left(\tfrac{z}{c}\right)^2 \leq 1\right\}.$$

Solution. If we let $u = x/a$, $v = y/b$, and $w = z/c$, then this substitution is given by

$$\underline{P}(u, v, w) = (au, bv, cw)$$

and W is the ball of radius 1 (bounded by the sphere). This substitution has Jacobian,

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

The volume of the ellipsoid is thus

$$\operatorname{Volume}(E) = \iiint_E dV = \iiint_W abc\, dV = abc \cdot \operatorname{Volume}(W) = \frac{4\pi}{3} abc. \qquad \lhd$$

6.4. **Cylindrical coordinates.** There are 2 generalizations of polar coordinates to 3 dimensions. The first is particularly suited to situations where there is symmetry with respect to rotation around the z-axis.

Definition 6.4.1. The *cylindrical coordinates* (r, θ, z) are related to Cartesian coordinates by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

These are usually restricted to $r \geq 0$ and $0 \leq \theta \leq 2\pi$; in this way, they parametrize all of $\mathbb{R}^3$.

Geometrically, r is the distance from the z-axis to $\underline{x}$, and θ is the angle between the x-z-plane and the line from the z-axis to $\underline{x}$.

This substitution has Jacobian

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Example 6.4.2. *Let $E \subset \mathbb{R}^3$ be the half-ball above the x-y-plane and inside the sphere of radius 1 centered at $\underline{0}$. Let $f : \mathbb{R}^3 \to \mathbb{R}$,*

$$f(x,y,z) = z\sqrt{x^2+y^2}.$$

Compute the integral of f over E.

Solution. Both f and E are invariant under rotations around the z-axis, so cylindrical coordinates are suitable.

- "Above the x-y-plane" means that $z \geq 0$.
- "Inside the sphere" means that $z \leq \sqrt{1-x^2-y^2} = \sqrt{1-r^2}$, which also implies $r \leq 1$.
- To avoid repeating points, we require $r \geq 0$ and $0 \leq \theta \leq 2\pi$.

In terms of cylindrical coordinates,

$$f(r\cos\theta, r\sin\theta, z) = rz.$$

Together, these give

$$\begin{aligned} \iiint_E f\,dV &= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-r^2}} rz \cdot r\,dz\,d\theta\,dr = \int_0^1 \int_0^{2\pi} r^2 \left[\tfrac{1}{2}z^2\right]_{z=0}^{z=\sqrt{1-r^2}} d\theta\,dr \\ &= \frac{1}{2}\int_0^1 \int_0^{2\pi} r^2(1-r^2)\,d\theta\,dr = \pi\int_0^1 (r^2-r^4)dr = \pi\left[\tfrac{1}{3}r^3 - \tfrac{1}{5}r^5\right]_{r=0}^{r=1} = \frac{2\pi}{15}. \end{aligned}$$

◁

6.5. **Spherical coordinates.** These coordinates are particularly suited to situations where there is complete rotational symmetry around the origin.

Definition 6.5.1. The *spherical coordinates* (r,θ,ϕ) are related to the Cartesian coordinates by

$$x = r\sin\theta\,\cos\phi, \qquad y = r\sin\theta\,\sin\phi, \qquad z = r\cos\theta.$$

These are usually restricted to $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$.

Geometrically, $r = \|\underline{x}\|$ is the distance from $\underline{0}$ to $\underline{x}$; θ is the angle between the positive z-axis and the line from $\underline{0}$ to $\underline{x}$; ϕ is the angle between the x-z-plane and the line from the z-axis to $\underline{x}$.

Nota bene. The coordinates r and θ are quite different from those in cylindrical coordinates. In fact, the cylindrical θ is equal to the spherical ϕ.

This can be expressed as a map, $\underline{P} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\underline{P}(r,\theta,\phi) = (r\sin\theta\,\cos\phi, r\sin\theta\,\sin\phi, r\cos\theta).$$

This has Jacobian,

$$\begin{aligned} J_{\underline{P}}(r,\theta,\phi) &= \begin{vmatrix} \sin\theta\,\cos\phi & r\cos\theta\,\cos\phi & -r\sin\theta\,\sin\phi \\ \sin\theta\,\sin\phi & r\cos\theta\,\sin\phi & r\sin\theta\,\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \\ &= r^2\sin\theta \begin{vmatrix} \sin\theta\,\cos\phi & \cos\theta\,\cos\phi & -\sin\phi \\ \sin\theta\,\sin\phi & \cos\theta\,\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} \\ &= r^2\sin\theta \end{aligned}$$

Definition 6.5.2. For an object occupying a solid region $E \subset \mathbb{R}^3$, with mass density $\rho : E \to \mathbb{R}$, the *moment of inertia* about the z-axis is

$$I_z := \iiint_E (x^2+y^2)\,\rho(\underline{x})\,dV.$$

(In general, the moment of inertia about a line is the integral of the mass density times the square of the distance to the line.)

Just as the mass of an object characterizes the effort needed to move it, the moment of inertia characterizes the effort needed to rotate it.

Example 6.5.3. *Let*

$$E := \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| \le a\}$$

be the ball of radius a. If E has constant mass density and total mass M, then what is its moment of inertia?

Solution. Firstly, if the constant mass density is ρ, then the mass is

$$M = \rho\,\mathrm{Volume}(E) = \frac{4\pi\rho a^3}{3},$$

so $\rho = \frac{3M}{4\pi a^3}$.

In terms of spherical coordinates, $x^2+y^2 = r^2\sin^2\theta$. The equivalent region in spherical coordinates is

$$W = \{(r,\theta,\phi) \in \mathbb{R}^3 \mid 0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi\}.$$

The moment of inertia is thus

$$\begin{aligned} I_z &= \iiint_E (x^2+y^2)\rho\,dx\,dy\,dz = \rho\iiint_W r^2\sin^2\theta\cdot r^2\sin\theta\,d\phi\,d\theta\,dr \\ &= \frac{3M}{4\pi a^3}\int_0^a\int_0^\pi\int_0^{2\pi} r^4\sin^3\theta\,d\phi\,d\theta\,dr = \frac{3M}{2a^3}\int_0^a\int_0^\pi r^4\sin^3\theta\,d\theta\,dr. \end{aligned}$$

We need to compute

$$\begin{aligned} \int_0^\pi \sin^3\theta\,d\theta &= \int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta \qquad \text{Let } u = -\cos\theta. \\ &= \int_{-1}^1 (1-u^2)\,du = \Big[u - \tfrac{1}{3}u^3\Big]_{u=-1}^{u=1} = \frac{4}{3}. \end{aligned}$$

This gives

$$I_z = \frac{2M}{a^3}\int_0^a r^4 dr = \frac{2M}{a^3}\cdot\frac{a^5}{5} = \frac{2Ma^2}{5}.$$

By symmetry, the moment of inertia about any other line through the origin is the same. ◁

7. Surface Integration

7.1. **Orientation.** Consider a smooth 2-dimensional surface in 3-dimensional space, $\Sigma \subset \mathbb{R}^3$. For any point $\underline{x} \in \Sigma$, there exists a plane tangent to Σ at $\underline{x}$, and we say that a vector $\underline{N}$ is *normal* to Σ at $\underline{x}$ if it is perpendicular to that tangent plane. There are only 2 possible normal directions at $\underline{x}$, and if $\underline{N}$ points on one normal direction, then $-\underline{N}$ points in the other.

Definition 7.1.1. An *orientation* of a surface $\Sigma \subset \mathbb{R}^3$ is a continuous choice of a normal direction at every point of Σ.

In contrast to curves, not all surfaces are orientable, but if a surface is orientable, then there are exactly 2 possible orientations.

Example 7.1.2. For a sphere, the possible orientations are outward and inward. ◁

Example 7.1.3. For the graph of a function of 2 variables, the possible orientations are upward and downward. ◁

Example 7.1.4. The Möbius strip is unorientable. ◁

Orientation is so important here that when I talk about a surface I will always mean a surface with a chosen orientation.

7.2. **Parametric surfaces.** It is often easiest to describe a surface implicitly as a level set such as $\Sigma = \{\underline{x} \in \mathbb{R}^3 \mid f(\underline{x}) = 0\}$, where $f : \mathbb{R}^3 \to \mathbb{R}$. However, this is not so convenient for integration. Instead, we need to parametrize Σ.

Definition 7.2.1. A *parametrization* of a surface $\Sigma \subset \mathbb{R}^3$ consists of a region $R \subset \mathbb{R}^2$ and a map $\underline{p} \in \mathcal{C}^1(R, \mathbb{R}^3)$ such that

- Σ is the image (range) of $\underline{p}$,
- the restriction of $\underline{p}$ to the interior $\operatorname{int} R$ is one-to-one, and
- $D_1\underline{p} \times D_2\underline{p}$ points in the direction of the orientation.

For any $\underline{u} \in R \subset \mathbb{R}^2$, $D_1\underline{p}(\underline{u})$ and $D_2\underline{p}(\underline{u})$ are tangent to Σ at $\underline{p}(\underline{u})$, therefore $D_1\underline{p}(\underline{u}) \times D_2\underline{p}(\underline{u})$ is normal to Σ at $\underline{p}(\underline{u})$. This is why the last condition makes sense.

Example 7.2.2. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a region $R \subset \mathbb{R}^2$, the piece of the graph of f over R,

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in R,\ z = f(x, y)\}$$

with the upward orientation can be parametrized by $\underline{p} : R \to \mathbb{R}^3$,

$$\underline{p}(u, v) = (u, v, f[u, v]).$$

◁

Example 7.2.3. For the unit sphere,

$$\Sigma = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| = 1\}$$

with the outward orientation, spherical coordinates with $r = 1$ provide a parametrization $\underline{p} : R \to \mathbb{R}^3$,

$$\underline{p}(\theta, \phi) = (\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta),$$

where $R = [0, \pi] \times [0, 2\pi] = \{(\theta, \phi) \in \mathbb{R}^2 \mid 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi\}$. ◁

7.3. **Flux integrals.**

Example 7.3.1. *First, let's consider some stuff with constant density ρ that is flowing through space with constant velocity $\underline{v} \in \mathbb{R}^3$. Let P be the parallelogram determined by two vectors $\underline{a}, \underline{b} \in \mathbb{R}^3$ (with vertices $\underline{0}$, $\underline{a}$, $\underline{b}$, and $\underline{a} + \underline{b}$). At what rate is stuff flowing across P?*

Solution. The vector $\underline{a} \times \underline{b}$ is normal to P, so this determines an orientation of P. We can say that the rate that stuff is crossing P is positive if it is crossing in this direction and negative if it is crossing in the other direction. Let's first suppose that this is positive.

Because of the constant velocity $\underline{v}$, in time Δt, a bit of stuff moves from $\underline{x}$ to $\underline{x} + \underline{v}\,\Delta t$. The stuff that has crossed P in that time fills the parallelepiped defined by $\underline{a}$, $\underline{b}$, and $\Delta t\,\underline{v}$. The volume of this is the triple product

$$\underline{a} \cdot (\underline{b} \times [\Delta t\,\underline{v}]) = \Delta t\,\underline{v} \cdot (\underline{a} \times \underline{b}).$$

The total amount of stuff that has crossed is $\rho\,\Delta t\,\underline{v} \cdot (\underline{a} \times \underline{b})$, and the rate is

$$\rho \underline{v} \cdot (\underline{a} \times \underline{b}).$$

If the stuff is crossing in the other direction, then the triple product is minus the volume, and the same expression gives the rate (which is now negative).

In this case the *current* of stuff is $\underline{J} = \rho \underline{v}$, and the rate of stuff flowing across P is

$$\underline{J} \cdot (\underline{a} \times \underline{b}), \tag{7.1}$$

which is called the *flux* of $\underline{J}$ across P. ("Flux" means "flow".) The flux of a constant current is given by (7.1). ◁

More generally, the motion of stuff in space is described by a current, which is a vector field $\underline{J} : \mathbb{R}^3 \to \mathbb{R}^3$. The flux of $\underline{J}$ across a surface Σ is denoted

$$\iint_\Sigma \underline{J} \cdot d\underline{S}\ .$$

Nota bene. The bold dot and underscore are part of the notation. They are not optional. $\underline{S}$ is not a variable name.

Let $\underline{p} : R \to \mathbb{R}^3$ be a parametrization of a smooth (and oriented) surface $\Sigma \subset \mathbb{R}^3$. If R is cut into small rectangles, then this cuts Σ into small pieces, which are approximately parallelograms. The image of a $\Delta u \times \Delta v$ rectangle at $\underline{u} = (u, v)$ is approximately a parallelogram with edges $D_1\underline{p}(\underline{u})\Delta u$ and $D_2\underline{p}(\underline{u})\Delta v$ at $\underline{p}(\underline{u})$. The current is approximately constant over this piece, so the flux is approximately

$$\underline{J}(\underline{p}[\underline{u}]) \cdot (D_1\underline{p}[\underline{u}] \times D_2\underline{p}[\underline{u}])\,\Delta u\,\Delta v.$$

The flux across Σ is the sum of the fluxes across all these pieces.

This suggests a precise definition:

Definition 7.3.2. If a surface Σ is parametrized by $\underline{p} : R \to \mathbb{R}^3$, and $\underline{J} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, then the *flux* of $\underline{J}$ across Σ is

$$\begin{aligned}\iint_\Sigma \underline{J} \cdot d\underline{S} &= \iint_R (\underline{J} \circ \underline{p}) \cdot (D_1\underline{p} \times D_2\underline{p})\, dA \\ &= \iint_R \underline{J}(\underline{p}[\underline{u}]) \cdot (D_1\underline{p}[\underline{u}] \times D_2\underline{p}[\underline{u}])\, du\, dv.\end{aligned}$$

More generally, if a surface Σ is not smooth, but is a union of finitely many smooth pieces, then the flux of $\underline{J}$ across Σ is the sum of the fluxes across the pieces.

Example 7.3.3. *Compute the flux of the vector field $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$,*

$$\underline{F}(x, y, z) = (0, x, z)$$

across the surface

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2 \leq 1\}$$

with the upward orientation.

Solution. This is a piece of a graph. Its projection to the x-y-plane is the unit disc

$$R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

Σ is parametrized by $\underline{p} : R \to \mathbb{R}^3$,

$$\underline{p}(u, v) = (u, v, u^2 + v^2).$$

From this, we compute

$$D_1\underline{p}(u, v) = (1, 0, 2u)$$

and

$$D_2\underline{p}(u, v) = (0, 1, 2v).$$

Note that

$$(1, 0, 2u) \times (0, 1, 2v) = (-2u, -2v, 1)$$

points upward, so this parametrization is compatible with the orientation.

We also need,

$$\underline{F}(\underline{p}[u, v]) = (0, u, u^2 + v^2)$$

and

$$(0, u, u^2 + v^2) \cdot (-2u, -2v, 1) = u^2 + v^2 - 2uv = (u - v)^2.$$

This gives the flux as

$$\iint_\Sigma \underline{F} \cdot d\underline{S} = \iint_R (u - v)^2 du\, dv.$$

Using the polar coordinate substitution $u = r\cos\theta$, $v = r\sin\theta$, this is

$$\begin{aligned}\iint_\Sigma \underline{F} \cdot d\underline{S} &= \int_0^{2\pi} \int_0^1 (r\cos\theta - r\sin\theta)^2 r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 r^3 (1 - \sin 2\theta)\, dr\, d\theta \\ &= \int_0^{2\pi} \left[\tfrac{1}{4} r^4\right]_{r=0}^{r=1} (1 - \sin 2\theta)\, d\theta = \frac{1}{4} \int_0^{2\pi} (1 - \sin 2\theta)\, d\theta = \frac{\pi}{2}.\end{aligned}$$

◁

7.4. **Surface area.** What is the area of a surface $\Sigma \subset \mathbb{R}^3$?

Suppose that $\underline{p} : R \to \mathbb{R}^3$ is a parametrization of Σ and again imagine cutting R into small rectangles. The image of a small $\Delta u \times \Delta v$ rectangle at $\underline{u} = (u, v)$ is approximately a parallelogram with area

$$\left\| D_1\underline{p}[\underline{u}] \times D_2\underline{p}[\underline{u}] \right\| \Delta u \, \Delta v.$$

The area of Σ is the sum of the areas of these pieces, so it can be computed as a double integral

$$\operatorname{Area}(\Sigma) = \iint_R \|D_1\underline{p} \times D_2\underline{p}\| dA = \iint_R \left\| D_1\underline{p}[\underline{u}] \times D_2\underline{p}[\underline{u}] \right\| du \, dv.$$

This generalizes to another type of integral:

Definition 7.4.1. If Σ is a surface parametrized by $\underline{p} : R \to \mathbb{R}^3$ and $f \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$ a scalar field, then the integral of f over Σ with respect to surface area is

$$\begin{aligned} \iint_\Sigma f \, dS &= \iint_R (f \circ \underline{p}) \|D_1\underline{p} \times D_2\underline{p}\| \, dA \\ &= \iint_R f(\underline{p}[\underline{u}]) \left\| D_1\underline{p}[\underline{u}] \times D_2\underline{p}[\underline{u}] \right\| du \, dv. \end{aligned}$$

Nota bene. This is similar to integration with respect to arc length (Sec. 2.5) but note the capital S in the notation.

With this notation,

$$\operatorname{Area}(\Sigma) = \iint_\Sigma dS.$$

Note that this is similar to how we computed a flux integral, but there is a vector norm in this case. Formally, $dS = \|d\underline{S}\|$. The square root makes this integral less natural than the flux integral.

Example 7.4.2. *Compute the surface area of a sphere of radius $a > 0$,*

$$\Sigma = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| = a\}.$$

Solution. This sphere is parametrized by $\underline{p} : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$,

$$\underline{p}(\theta, \phi) = (a \sin\theta \, \cos\phi, a \sin\theta \, \sin\phi, a \cos\theta).$$

From this, compute

$$D_1\underline{p}(\theta, \phi) = a(\cos\theta \, \cos\phi, \cos\theta \, \sin\phi, -\sin\theta)$$

and

$$\begin{aligned} D_2\underline{p}(\theta, \phi) &= a(-\sin\theta \, \sin\phi, \sin\theta \, \cos\phi, 0) \\ &= a \sin\theta \, (-\sin\phi, \cos\phi, 0). \end{aligned}$$

Note that

$$\left\|(\cos\theta \, \cos\phi, \cos\theta \, \sin\phi, -\sin\theta)\right\| = \left\|(-\sin\phi, \cos\phi, 0)\right\| = 1$$

and

$$(\cos\theta \, \cos\phi, \cos\theta \, \sin\phi, -\sin\theta) \cdot (-\sin\phi, \cos\phi, 0) = 0,$$

so those are orthogonal unit vectors, therefore their cross product is a unit vector and

$$\left\| D_1\underline{p}(\theta,\phi) \times D_2\underline{p}(\theta,\phi) \right\| = a^2 \sin\theta$$

for $0 \leq \theta \leq \pi$.

This means that the area is

$$\mathrm{Area}(\Sigma) = a^2 \int_0^{\pi} \int_0^{2\pi} \sin\theta \, d\phi \, d\theta = 2\pi a^2 \int_0^{\pi} \sin\theta \, d\theta = 2\pi a^2 \Big[-\cos\theta \Big]_{\theta=0}^{\theta=\pi} = 4\pi a^2.$$

◁

7.5. **Relationship between the surface integrals.** This scalar surface integral is related to the flux integral. Let $\underline{N}$ be the unit normal vector field to Σ in the direction of the orientation. At $\underline{p}(\underline{u})$, this is the unit vector in the direction of $D_1\underline{p}[\underline{u}] \times D_2\underline{p}[\underline{u}]$, so we can write $d\underline{S} = \underline{N}\, dS$ and

$$\iint_{\Sigma} \underline{J} \cdot d\underline{S} = \iint_{\Sigma} (\underline{J} \cdot \underline{N}) dS.$$

This is useful for computing flux integrals in some simple cases.

8. Gauss' Theorem

8.1. **The theorem.**

Notation 8.1.1. For a solid region, $E \subset \mathbb{R}^3$, ∂E denotes the boundary of E with the outward orientation, i.e., the unit normal vector field to ∂E points away from E.

Notation 8.1.2. For any (oriented) surface $\Sigma \subset \mathbb{R}^3$, $\overline{\Sigma}$ will denote Σ with the opposite orientation.

Theorem 8.1.3 (Gauss' Theorem [The Divergence Theorem])**.** *If $E \subset \mathbb{R}^3$ is a solid region and $\underline{F} \in \mathcal{C}^1(E, \mathbb{R}^3)$ a vector field, then*

$$\iint_{\partial E} \underline{F} \cdot d\underline{S} = \iiint_E (\nabla \cdot \underline{F})\, dV.$$

Example 8.1.4. *Let Σ be the surface consisting of the hemisphere*

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\}$$

with the upward orientation and the disk

$$\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$$

with the downward orientation. Let $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\underline{F}(x, y, z) = (-xz, 0, e^x + z^2).$$

Compute the flux of $\underline{F}$ across Σ.

Solution. This Σ is the boundary of the half ball (Ex. 6.4.2)

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1,\ z \geq 0\}.$$

In cylindrical coordinates, E is characterized by $0 \leq z \leq 1$, $0 \leq r \leq \sqrt{1 - z^2}$, and $0 \leq \theta \leq 2\pi$.

The divergence of $\underline{F}$ is

$$\nabla \cdot \underline{F}(x, y, z) = -z + 2z = z.$$

Gauss' Theorem gives the flux,

$$\iint_\Sigma \underline{F} \cdot d\underline{S} = \iiint_E z\, dV = \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{1-z^2}} z \cdot r\, dr\, d\theta\, dz = \int_0^1 \int_0^{2\pi} \left[\tfrac{1}{2} r^2 z\right]_{r=0}^{r=\sqrt{1-z^2}} d\theta\, dz$$
$$= \int_0^1 \int_0^{2\pi} \tfrac{1}{2}(1 - z^2) z\, d\theta\, dz = \pi \int_0^1 (z - z^3) dz = \pi \left[\tfrac{1}{2} z^2 - \tfrac{1}{4} z^4\right]_{z=0}^{z=1} = \frac{\pi}{4}. \lhd$$

Example 8.1.5. *Let*

$$B = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| \leq a\}$$

be the ball (filled sphere) of radius $a > 0$, centered at $\underline{0}$. Given that the area of the sphere ∂B is $4\pi a^2$ (Ex. 7.4.2) compute the volume of B.

Solution. Let $\underline{r} : \mathbb{R}^3 \to \mathbb{R}^3$, $\underline{r}(\underline{x}) = \underline{x}$.

Note that ∂B is the sphere of radius a. At any point $\underline{x} \in \partial B$ of the sphere, $\underline{r}(\underline{x}) = \underline{x}$ points directly away from the origin and has norm a; the unit normal vector $\underline{N}(\underline{x})$ points directly away from the origin and has norm 1. Therefore, $a\underline{N} = \underline{r}$ and $\underline{r} \cdot \underline{N} = a$.

The flux of this vector field is thus

$$\iint_{\partial B} \underline{r} \cdot d\underline{S} = \iint_{\partial B} \underline{r} \cdot \underline{N}\, dS = \iint_{\partial B} a\, dS = a\,\mathrm{Area}(\partial B) = a \cdot 4\pi a^2 = 4\pi a^3.$$

The divergence of this vector field is

$$\nabla \cdot \underline{r}(\underline{x}) = \frac{\partial x_b}{\partial x_b} = \delta_{bb} = 3.$$

Gauss' Theorem gives

$$4\pi a^3 = \iiint_B \nabla \cdot \underline{r}\, dV = 3\,\mathrm{Volume}\, B.$$

Therefore

$$\mathrm{Volume}\, B = \frac{4\pi a^3}{3}. \qquad \lhd$$

Example 8.1.6. In an *incompressible* fluid, the density ρ is constant. The velocity of the fluid is a vector field $\underline{v} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$, i.e., the velocity of the bit of fluid at $\underline{x}$ is $\underline{v}(\underline{x})$. The current is simply $\rho\underline{v}$.

Because the density is constant, the amount of stuff in some solid region E does not change, therefore the flux of $\rho\underline{v}$ across ∂E must be 0:

$$0 = \iint_{\partial E} \rho\underline{v} \cdot d\underline{S} = \rho \iiint_E (\nabla \cdot \underline{v})\, dV.$$

Since this is true for *any* region E, we must have $\nabla \cdot \underline{v} = 0$. $\lhd$

8.2. **Conservation.** In general, some stuff distributed and moving around space is described by its density ρ (a scalar field) and current $\underline{J}$ (a vector field). These may depend on time, t, as well as position.

The amount of stuff in a solid region E is

$$\iiint_E \rho\, dV.$$

The time derivative of this is

$$\frac{d}{dt} \iiint_E \rho\, dV = \iiint_E \frac{\partial \rho}{\partial t} dV.$$

The rate of stuff leaving E is the flux of $\underline{J}$ across ∂E:

$$\iint_{\partial E} \underline{J} \cdot d\underline{S} = \iiint_E \nabla \cdot \underline{J}\, dV.$$

If stuff is neither created nor destroyed, then the amount of stuff in E decreases at the rate that it leaves E, therefore

$$0 = \frac{d}{dt} \iiint_E \rho\, dV + \iint_{\partial E} \underline{J} \cdot d\underline{S} = \iiint_E \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} \right) dV.$$

Since this is true for *any* region E, we must have,

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J}\,, \tag{8.1}$$

which is called the *continuity equation* or *conservation equation*. It expresses as a differential equation the condition that stuff is neither created nor destroyed.

This applies to many things. For example charge, mass, heat, or atoms.

8.3. **Gauss' Law.** Newton's law of gravity describes the gravitational force between two objects, but it is not clear that this is consistent. If we assume it to be true for point masses, then is it actually true for finite objects?

Newton addressed this with his "shell theorem" which shows that his law of gravity is true for rotationally symmetric objects. He proved this with geometry, but it is simpler to prove by using Gauss' Theorem.

Definition 8.3.1. The *gravitational field* is a vector field $\underline{g}$ such that in the absence of other forces, when a point particle is at position $\underline{x}$, it will move with acceleration $\ddot{\underline{x}} = \underline{g}(\underline{x})$.

For a point mass M at $\underline{0}$, Newton's law implies that the gravitational field $\underline{g} : \mathbb{R}^3 \smallsetminus \{\underline{0}\} \to \mathbb{R}^3$ has magnitude $\|\underline{g}(\underline{x})\| = GM\|\underline{x}\|^{-2}$ and is directed toward $\underline{0}$. Here, G is *Newton's constant*. The unit vector pointing from $\underline{x}$ toward $\underline{0}$ is $-\|\underline{x}\|^{-1}\underline{x}$, so

$$\underline{g}(\underline{x}) = -\frac{GM}{\|\underline{x}\|^3}\underline{x}. \tag{8.2}$$

Let S_a be the sphere of radius $a > 0$ around $\underline{0}$. We can compute the flux of $\underline{g}$ across S_a. At any point $\underline{x} \in S_a$, $\|\underline{x}\| = a$ and $\underline{N}(\underline{x})$ is the unit vector pointing away from $\underline{0}$, so $\underline{g} \cdot \underline{N} = -GMa^{-2}$. This gives the flux,

$$\iint_{S_a} \underline{g} \cdot d\underline{S} = \iint_{S_a} \underline{g} \cdot \underline{N}\, dS = -GMa^{-2} \cdot \operatorname{Area}(S_a) = -GMa^{-2} \cdot 4\pi a^2 = -4\pi GM.$$

Note that this is independent of a. For any $b > a > 0$,

$$0 = \iint_{S_b} \underline{g} \cdot d\underline{S} - \iint_{S_a} \underline{g} \cdot d\underline{S} = \iint_{S_b \cup \overline{S_a}} \underline{g} \cdot d\underline{S} = \iiint_{a \le \|\underline{x}\| \le b} \nabla \cdot \underline{g}\, dV.$$

(Again, $\overline{S_a}$ denotes the reversed orientation.) By rotational symmetry, $\nabla \cdot \underline{g}(\underline{x})$ only depends on $\|\underline{x}\|$, so this shows that $0 = \nabla \cdot \underline{g}$. (This can also be computed directly from eq. (8.2).)

We can use this to compute the flux across an arbitrary closed surface. Let $\Sigma = \partial E$ be the boundary of some solid region, $E \subset \mathbb{R}^3$. If $\underline{0} \notin E$, then

$$\iint_{\Sigma} \underline{g} \cdot d\underline{S} = \iiint_{E} \nabla \cdot \underline{g}\, dV = 0.$$

If $\underline{0} \in E$, then we cannot apply Gauss' Theorem directly, because $\underline{g}(\underline{0})$ is undefined. Instead, we can choose $a > 0$ small enough that $S_a \subset E$. Let E' be the region between S_a and Σ, so that $\partial E' = \Sigma \cup \overline{S_a}$. By Gauss' Theorem,

$$0 = \iiint_{E'} \nabla \cdot \underline{g}\, dV = \iint_{\Sigma \cup \overline{S_a}} \underline{g} \cdot d\underline{S} = \iint_{\Sigma} \underline{g} \cdot d\underline{S} - \iint_{S_a} \underline{g} \cdot d\underline{S},$$

therefore

$$\iint_{\Sigma} \underline{g} \cdot d\underline{S} = \iint_{S_a} \underline{g} \cdot d\underline{S} = -4\pi GM.$$

In summary, for the gravitational field of a point mass, M, and any closed surface Σ,

$$\iint_\Sigma \underline{g} \cdot d\underline{S} = \begin{cases} -4\pi GM & \Sigma \text{ encloses } M \\ 0 & \text{otherwise.} \end{cases}$$

To get the gravitational field of multiple point masses, just add. This shows that

$$\iint_\Sigma \underline{g} \cdot d\underline{S} = -4\pi G \cdot (\text{mass enclosed by } \Sigma).$$

This is called *Gauss' Law*.

If, instead of point masses, we have a continuous distribution of matter with mass density $\rho \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, then the mass in a region E is the integral of ρ. Gauss' Law and Theorem give,

$$-4\pi G \iiint_E \rho \, dV = \iint_{\partial E} \underline{g} \cdot d\underline{S} = \iiint_E \nabla \cdot \underline{g} \, dV.$$

Because this holds for any region E, the integrands must be equal:

$$\nabla \cdot \underline{g} = -4\pi G\rho.$$

This is the differential form of Gauss' Law.

Now consider mass M distributed in a rotationally symmetric way over the ball where $\|\underline{x}\| \leq a$. Rotational symmetry implies that $\underline{g}(\underline{x})$ points directly toward $\underline{0}$, and $\|\underline{g}(\underline{x})\|$ only depends on $\|\underline{x}\|$. For the sphere S_b of any radius $b > a$, Gauss' Law tells us

$$-4\pi GM = \iint_{S_b} \underline{g} \cdot d\underline{S} = \iint_{S_b} \underline{g} \cdot \underline{N} \, dS = \iint_{S_b} -\|\underline{g}\| \, dS = -4\pi b^2 \|\underline{g}(\underline{x})\|$$

for any point $\underline{x} \in S_b$; hence, $\|\underline{g}(\underline{x})\| = GMb^{-2} = GM\|\underline{x}\|^{-2}$.

This shows that the gravitational field outside the sphere is the same as the gravitational field of a point mass.

9. Stokes' Theorem

9.1. **Boundaries.**

Notation 9.1.1. If $\Sigma \subset \mathbb{R}^3$ is a surface, then $\partial\Sigma$ denotes the *boundary* of Σ.

You can think of the boundary as the edge of Σ. This is a closed curve or union of closed curves. The orientation is a generalization of the orientation of the boundary of a planar region. Roughly, it looks anticlockwise with the preferred normal direction to Σ pointing toward you.

At any point of $\partial\Sigma$, if $\underline{T}$ is the preferred unit tangent vector, $\underline{n}$ is the outward unit normal vector to $\partial\Sigma$ (pointing away from Σ), and $\underline{N}$ is the preferred unit normal vector to Σ, then $\underline{N} = \underline{n} \times \underline{T}$.

Definition 9.1.2. If $\partial\Sigma = \emptyset$ (i.e., Σ has no boundary), then Σ is a *closed surface*.

Example 9.1.3. For any solid region, E, the boundary ∂E is a closed surface. $\lhd$

If $\underline{p} : R \to \mathbb{R}^3$ is a parametrization of Σ and $\underline{\ell} : [a, b] \to \mathbb{R}^2$ is a parametrization of ∂R, then $\partial\Sigma$ is generally the curve parametrized by

$$\underline{p} \circ \underline{\ell} : [a, b] \to \mathbb{R}^3.$$

However, this can be a little more subtle if ∂R is a concatenation of smooth curves. If a segment just gives a point in $\mathbb{R}^3$, then it can be ignored. It 2 segments give the same curve in opposite directions, then these cancel in line integrals and should be ignored.

Example 9.1.4. *Let Σ be the upper unit hemisphere,*

$$\Sigma = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| = 1,\ x_3 \geq 0\}$$

with the outward orientation. What is $\partial\Sigma$?

Solution. The boundary $\partial\Sigma$ is the circle of radius 1 where the sphere is cut off along the x-y-plane.

This hemisphere is parametrized by restricting the standard parametrization of the sphere to $R = [0, \frac{\pi}{2}] \times [0, 2\pi]$, giving $\underline{p} : R \to \mathbb{R}^3$,

$$\underline{p}(\theta, \phi) = (\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta).$$

Now, ∂R has 4 sides. The side where $\theta = 0$ just gives the point $(0, 0, 1)$. The sides where $\phi = 0$ and $\phi = 2\pi$ both give the same quarter-circle in the x-z-plane, so these cancel and can be ignored.

This leaves the side where $\theta = \frac{\pi}{2}$. This is parametrized by $\underline{\ell} : [0, 2\pi] \to \mathbb{R}^2$, $\underline{\ell}(\phi) = (\frac{\pi}{2}, \phi)$. This parametrizes $\partial\Sigma$ by $\underline{p} \circ \underline{\ell} : [0, 2\pi] \to \mathbb{R}^3$,

$$\underline{p} \circ \underline{\ell}(\phi) = \underline{p}(\tfrac{\pi}{2}, \phi) = (\cos\phi, \sin\phi, 0).$$

This gives the expected circle. $\lhd$

9.2. **The theorem.**

Theorem 9.2.1 (Stokes' Theorem)**.** *If $\Sigma \subset \mathbb{R}^3$ is a surface and $\underline{F} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$, then*

$$\oint_{\partial\Sigma} \underline{F}(\underline{x}) \cdot d\underline{x} = \iint_{\Sigma} (\nabla \times \underline{F}) \cdot d\underline{S}. \tag{9.1}$$

This follows from Green's theorem and some calculation.

Example 9.2.2. *Let $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$,*

$$\underline{F}(x, y, z) = (-y, yz, y^2 - \tfrac{1}{2}x^2).$$

Let $C \subset \mathbb{R}^3$ be the closed curve parametrized by $\underline{q} : [0, 2\pi] \to \mathbb{R}^3$,

$$\underline{q}(t) = (\cos t, \sin t, \cos 2t).$$

Compute $\oint_C \underline{F}(\underline{x}) \cdot d\underline{x}$.

Solution. Note that $\cos 2t = \cos^2 t - \sin^2 t$, so any point $\underline{x} \in C$ satisfies $z = x^2 - y^2$. This suggests defining Σ as the piece of this graph bounded by C. So let

$$R = \left\{ \underline{x} \in \mathbb{R}^2 \;\middle|\; \|\underline{x}\| \leq 1 \right\}$$

be the unit disc and $\underline{p} : R \to \mathbb{R}^3$,

$$\underline{p}(x, y) = (x, y, x^2 - y^2).$$

This parametrizes Σ, and $\partial\Sigma = C$. Note that this parametrization gives Σ the usual upward orientation, and this orients $\partial\Sigma$ anticlockwise, which is consistent with our parametrization $\underline{q}$ of C.

The curl of $\underline{F}$ is

$$(\nabla \times \underline{F})(x, y, z) = (y, x, 1),$$

So, $(\nabla \times \underline{F})(\underline{p}[x, y]) = (y, x, 1)$.

This parametrization of Σ gives

$$D_1\underline{p}(x, y) = (1, 0, 2x), \qquad D_2\underline{p}(x, y) = (0, 1, -2y),$$
$$D_1\underline{p}(x, y) \times D_2\underline{p}(x, y) = (-2x, 2y, 1).$$

By Stokes' Theorem, the integral we want is

$$\oint_C \underline{F}(\underline{x}) \cdot d\underline{x} = \iint_{\Sigma} (\nabla \times \underline{F}) \cdot d\underline{S} = \iint_R (y, x, 1) \cdot (-2x, 2y, 1)\, dx\, dy$$
$$= \iint_R dA = \mathrm{Area}(R) = \pi.$$

We can also compute this directly. We need

$$\dot{\underline{q}}(t) = (-\sin t, \cos t, -2\sin 2t),$$

$$\underline{F}(\underline{q}[t]) = (-\sin t, \sin t\, \cos 2t, \sin^2 t - \tfrac{1}{2}\cos^2 t)$$
$$= (-\sin t, \sin t\, \cos 2t, \tfrac{1}{4} - \tfrac{3}{4}\cos 2t),$$

and

$$\begin{aligned}\underline{F}(\underline{q}[t]) \cdot \dot{\underline{q}}(t) &= \sin^2 t + \sin t \, \cos t \, \cos 2t - \tfrac{1}{2}\sin 2t + \tfrac{3}{2}\sin 2t \cos 2t \\ &= \tfrac{1}{2} - \tfrac{1}{2}\cos 2t + 2\sin 2t \, \cos 2t - \tfrac{1}{2}\sin 2t \\ &= \tfrac{1}{2} - \tfrac{1}{2}\cos 2t + \sin 4t - \tfrac{1}{2}\sin 2t.\end{aligned}$$

This gives the line integral

$$\begin{aligned}\oint_C \underline{F}(\underline{x}) \cdot d\underline{x} &= \int_0^{2\pi} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2t + \sin 4t - \tfrac{1}{2}\sin 2t\right) dt \\ &= \left[\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t - \tfrac{1}{4}\cos 4t + \tfrac{1}{4}\cos 2t\right]_{t=0}^{t=2\pi} = \pi\end{aligned}$$

which agrees with Stokes' Theorem. ◁

9.3. **Applications.** The line integral of a vector field around a closed curve is often called *circulation*.

Example 9.3.1. If a fluid flows without turbulence, then the circulation of its velocity, $\underline{v}$, around any closed curve is 0. For any surface $\Sigma \subset \mathbb{R}^3$,

$$0 = \oint_{\partial\Sigma} \underline{v}(\underline{x}) \cdot d\underline{x} = \iint_\Sigma (\nabla \times \underline{v}) \cdot d\underline{S}.$$

Since this is true for any surface, it implies that $0 = \nabla \times \underline{v}$.

In general, the vector field $\nabla \times \underline{v}$ is called the *vorticity*. This is usually $\underline{0}$ outside of thin tubes. Stokes' Theorem tells us that the circulation of $\underline{v}$ around a closed curve C is equal to the flux of vorticity across a surface spanning C.

Since this is true for *any* surface spanning C, this implies that a tube of vorticity cannot end within the fluid. It must either continue to the boundary of the fluid or come back to form a loop. ◁

Example 9.3.2. In electromagnetism, *Ampère's law* relates the magnetic field $\underline{H}$ and electric charge current $\underline{J}$ (constant in time) by

$$\oint_{\partial\Sigma} \underline{H}(\underline{x}) \cdot d\underline{x} = \iint_\Sigma \underline{J} \cdot d\underline{S},$$

for any surface, Σ. By Stokes' Theorem,

$$\iint_\Sigma (\nabla \times \underline{H}) \cdot d\underline{S} = \iint_\Sigma \underline{J} \cdot d\underline{S},$$

and since this is true for *any* surface,

$$\nabla \times \underline{H} = \underline{J}.$$

In Theorem 5.2.2, we proved eq. (5.2) that $\underline{0} = \nabla \cdot (\nabla \times \underline{g})$. In this case

$$\nabla \cdot \underline{J} = \nabla \cdot (\nabla \times \underline{H}) = 0.$$

However, the conservation equation shows that this is $-\frac{\partial \rho}{\partial t}$, so Ampère's law cannot hold if the electrical charge density changes with time. This observation led Maxwell to discover the complete equations of electromagnetism. ◁

9.4. **Gradients.** According the Theorem 5.2.1, for a scalar field, f,

$$\underline{0} = \nabla \times (\nabla f).$$

In other words, the curl of a gradient is $\underline{0}$. This shows that

$$\underline{0} = \nabla \times \underline{g} \tag{9.2}$$

is a necessary condition for $\underline{g}$ to be the gradient of some scalar field.

Conversely, suppose that $\underline{g} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$ with $\underline{0} = \nabla \times \underline{g}$. Any closed curve $C \subset \mathbb{R}^3$ is the boundary of some surface, Σ, so

$$\oint_C \underline{g}(\underline{x}) \cdot d\underline{x} = \iint_\Sigma (\nabla \times \underline{g}) \cdot d\underline{S} = 0.$$

According to Theorem 3.5.1, this implies that $\underline{g}$ is the gradient of some scalar field.

More generally, if $U \subset \mathbb{R}^3$ is an open set and $\underline{g} \in \mathcal{C}^1(U, \mathbb{R}^3)$ with $\nabla \times \underline{g} = \underline{0}$, then $\underline{g}$ is a gradient if any closed curve $C \subset U$ is the boundary of some surface $\Sigma \subset U$.

Example 9.4.1. Let $U = \mathbb{R}^3 \setminus \{\underline{0}\}$. Any vector field on U satisfying eq. (9.2) is a gradient, because we can span a curve with a surface that avoids $\underline{0}$. ◁

Example 9.4.2. Let $U = \mathbb{R}^3 \setminus \{(0, 0, z) \in \mathbb{R}^3\}$. The vector field $\underline{g} : U \to \mathbb{R}^3$,

$$\underline{g}(x, y, z) = \left(\tfrac{-y}{x^2+y^2}, \tfrac{x}{x^2+y^2}, 0\right)$$

satisfies eq. (9.2) but is not a gradient. (See Example 3.5.4.) ◁

9.5. **The Big Picture.** We now have an assortment of integral theorems involving integrating scalar or vector fields over various domains (curves, surfaces, etc.). Most of these take the following form: An integral of some field around the boundary of some domain is equal to an integral of some derivative of that field over the domain.

The exception is the Fundamental Theorem of Line Integrals (FTLI), but that can also be expressed in this form.

Suppose that C is a curve from $\underline{a}$ to $\underline{b}$. The boundary is the set of endpoints $\{\underline{a}, \underline{b}\}$. How do we integrate over a point?

The simple answer is that the integral of a scalar field f over the set of one point $\{\underline{a}\}$ is the value,

$$\int_{\{\underline{a}\}} f = f(\underline{a}). \tag{9.3}$$

However, even points have orientation. For the obvious orientation, eq. (9.3) works, but there is also the opposite orientation; denote that as $\overline{\{\underline{a}\}}$. The opposite orientation should change the sign of the integral, so

$$\int_{\overline{\{\underline{a}\}}} f = -f(\underline{a}).$$

In this sense, the boundary of C is $\partial C = \overline{\{\underline{a}\}} \cup \{\underline{b}\}$ and

$$\int_{\partial C} f = \int_{\{\underline{b}\}} f - \int_{\{\underline{a}\}} f = f(\underline{b}) - f(\underline{a}).$$

Finally, the FTLI can be stated as

$$\int_{\partial C} f = \int_C \nabla f(\underline{x}) \cdot d\underline{x}.$$

In $\mathbb{R}^3$, the natural ways of integrating and the integral theorems fit together like this:

Dimension	Domain	Integrand	Integral	Theorem
0	Points	Scalar field	Evaluation	
	$\uparrow \partial$	$\downarrow \nabla$		FTLI
1	Curve	Vector field	Line integral	
	$\uparrow \partial$	$\downarrow \nabla\times$		Stokes'
2	Surface	Vector field	Flux integral	
	$\uparrow \partial$	$\downarrow \nabla\cdot$		Gauss'
3	Solid region	Scalar field	Triple integral	

For any scalar field $f \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R})$,

$$\nabla \times (\nabla f) = \mathrm{o}.$$

The boundary $\partial\Sigma$ of a surface is a closed curve, so its boundary is empty. By the FTLI and Stokes' Theorem,

$$\mathrm{o} = \oint_{\partial\Sigma} \nabla f(\underline{x}) \cdot d\underline{x} = \iint_{\Sigma} (\nabla \times \nabla f) \cdot d\underline{S} = \mathrm{o}.$$

For any vector field $\underline{F} \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R}^3)$,

$$\nabla \cdot (\nabla \times \underline{F}) = \mathrm{o}.$$

The boundary ∂E of a solid region is a closed surface, so its boundary is empty. By Stokes' Theorem and Gauss' Theorem,

$$\mathrm{o} = \iint_{\partial E} (\nabla \times \underline{F}) \cdot d\underline{S} = \iiint_E \nabla \cdot (\nabla \times \underline{F})\, dV = \mathrm{o}.$$

In $\mathbb{R}^2$, there is a similar pattern:

Dimension	Domain	Integrand	Integral	Theorem
0	Points	Scalar field	Evaluation	
	$\uparrow \partial$	$\downarrow \nabla$		FTLI
1	Curve	Vector field	Line integral	
	$\uparrow \partial$	$\downarrow \mathbf{curl}$		Green's
2	Region	Scalar field	Double integral	

Even the Fundamental Theorem of Calculus (in $\mathbb{R}$) can be viewed in this way:

Dimension	Domain	Integrand	Integral
0	Points	Function	Evaluation
	$\uparrow \partial$	$\downarrow \frac{d}{dx}$	
1	Interval	Function	Definite integral

Summary of Notation

$a \in \mathbb{R}$, $\underline{v} \in \mathbb{R}^n$	$a\underline{v} \in \mathbb{R}^n$	Product of a scalar and vector is a vector.
$\underline{v}, \underline{w} \in \mathbb{R}^n$	$\underline{v} \cdot \underline{w} \in \mathbb{R}$	Dot product of 2 vectors is a scalar.
$\underline{v}, \underline{w} \in \mathbb{R}^3$	$\underline{v} \times \underline{w} \in \mathbb{R}^3$	Cross product of 3-dimensional vectors is a vector.
$\underline{v} \in \mathbb{R}^n$	v_a	The a'th component of $\underline{v}$.
$\underline{v}, \underline{w} \in \mathbb{R}^n$	$\underline{v} \cdot \underline{w} = v_a w_a$	Always sum over a repeated index. Never repeat an index more that 2 times. The unrepeated indices must be the same in every term.
$\underline{f} : \mathbb{R}^n \to \mathbb{R}^m$	$D\underline{f}(\underline{x})$	m by n matrix of partial derivatives of $\underline{f}$ at $\underline{x} \in \mathbb{R}^n$.
	$\mathcal{C}^r(\mathbb{R}^n, \mathbb{R}^m)$	The set of r-times continuously differentiable functions $\underline{f} : \mathbb{R}^n \to \mathbb{R}^m$.
$f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$	∇f	The gradient of a scalar field is a vector field.
$\underline{F} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$	$\nabla \cdot \underline{F}$	The divergence of a vector field is a scalar field.
$\underline{F} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$	$\operatorname{curl} \underline{F}$	The planar curl of a vector field is a scalar.
$\underline{F} \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^3)$	$\nabla \times \underline{F}$	The curl of a vector field is a vector field.
$C \subset \mathbb{R}^n$	$\int_C \underline{F}(\underline{x}) \cdot d\underline{x}$	Line integral of a vector field along a curve.
	$\int_C f \, ds$	Integral of a scalar field along a curve with respect to arc length.
$R \subset \mathbb{R}^2$	$\iint_R f \, dA$	Integral of a scalar over a region with respect to area.
$\Sigma \subset \mathbb{R}^3$	$\iint_\Sigma \underline{F} \cdot d\underline{S}$	Flux integral of a vector field across a surface.
	$\iint_\Sigma f \, dS$	Integral of a scalar over a surface with respect to area.
$E \subset \mathbb{R}^3$	$\iiint_E f \, dV$	Integral of a scalar over a solid with respect to volume.