Classical Dynamics

Dynamical Systems Notes[1]

Lecturer: Konstantin Ilin[2]

Recommended texts

1. S. H. Strogatz, Nonlinear Dynamics and Chaos, Westview Press (Perseus), 1994 (electronic copy available from the library, you will need chapters 2, 3, 5 and 6).
2. V. I. Arnol'd, Geometric methods in the theory of ordinary differential equations, Springer, 1983.
3. D. K. Arrowsmith and C. M. Place, An introduction to dynamical systems, Cambridge University Press, 1990.
4. P. Glendinning, Stability, instability and chaos: an introduction to the theory of nonlinear differential equations, Cambridge Texts in Applied Mathematics, 1994.
5. M.H. Holmes, J. Guckenheimer and P. Holmes, Nonlinear oscillations, dynamical systems and bifurcations of vector fields, Springer Applied Math. Sciences 42, 1983.

---

[1]Based on the notes originally written by George Constable, A. Jamie Wood and Ian McIntosh.

[2]Room G/142, email: konstantin.ilin@york.ac.uk

Contents

1 Systems in a single dimension **3**

1.1 Flows on a line . . . 3
1.2 Flows for arbitrary 1D systems . . . 4
1.3 Linear stability analysis . . . 5
1.4 Nondimensionalisation . . . 6
1.5 Logistic equation . . . 7
1.6 Bifurcations . . . 8
1.7 A model of spruce budworm outbreak . . . 12

2 Planar Linear Systems **18**

2.1 The general solution of planar linear systems . . . 18
2.2 The classification of planar linear systems . . . 21

3 Planar Nonlinear Systems **23**

3.1 Fixed points and their stability . . . 24
3.2 Linearisation about a fixed point . . . 25
3.3 Linear stability analysis . . . 26
3.4 Structural Stability . . . 29
3.5 Features of interest . . . 29
3.6 Drawing a phase portrait . . . 30

1 Systems in a single dimension

In general we want to consider a way to extract the interesting behaviour from systems of n ordinary differential equations of the form

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1,\ldots,x_n) \\ f_2(x_1,\ldots,x_n) \\ \vdots \\ f_n(x_1,\ldots,x_n) \end{pmatrix} \quad \text{or} \quad \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{1.1}$$

where $\boldsymbol{x}, \boldsymbol{f} \in \mathbb{R}^n$ and the dot stands for the derivative with respect to time t. Systems like this one arise in many different application areas.

The space where solutions of system (1.1) live, $\mathbb{R}^n$, is called the *phase space*, and the solutions are curves in the phase space, called *solution curves* or *trajectories* or *orbits*.

Two important points to note.

(a) We consider (1.1) to be a system, regardless of the value of n.

(b) Note that the right hand sides in (1.1) do not explicitly depend on time.

As such, we refer to them as *autonomous systems*. This is important as the techniques we use will not be of use for non-autonomous (time-dependent) systems.

For simplicity we begin by looking at systems with $n = 1$ (i.e. dynamical systems whose phase space in the real line $\mathbb{R}$).

$$\dot{x} = f(x) \tag{1.2}$$

We will introduce many of the ideas in this section that we will extend to $n = 2$.

1.1 Flows on a line

Consider the differential equation

$$\dot{x} = \cos x \tag{1.3}$$

which has the exact solution, satisfying $x(0) = x_0$,

$$\ln\frac{|\tan x + \sec x|}{|\tan x_0 + \sec x_0|} = t \tag{1.4}$$

Can we use this to provide qualitative answers about the interesting behaviour of the system for different values of x_0 in the long time limit? Difficult.

Instead we can think about plotting a vector field straight from (1.3). For a 1D equation this is particularly simple, we just plot x and $\dot{x} = f(x)$, Fig. 1.

We can see immediately that places where the flow is positive, moving to the right, and negative, moving to the left. Annotating these onto the plot we then see there are two types of point. Those where the flows move towards on both sides, and those where is moves away. It is natural to think about these as stable and unstable respectively.

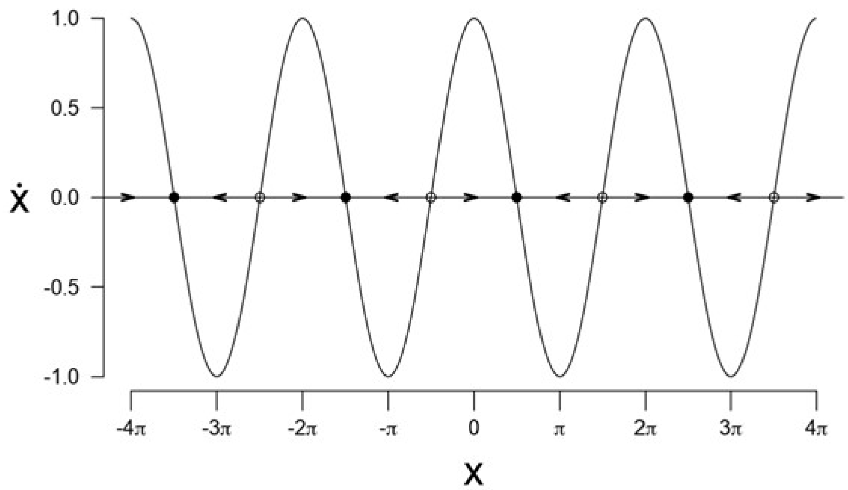

Figure 1 Plot of $\cos x$. Giving a visualisation of the flow for (1.3.)

Can we find this out directly? Yes we can, we are interested in the local gradient near the points where the time derivative is zero. This can be found by taking the derivative with respect to x of the right hand side of equation as long as the function is well behaved around this point. Why? The derivative with respect to x is telling us whether $\dot{x} = f(x)$ is going from positive to negative as x increases through a point with $\dot{x} = 0$ (i.e. stable) and therefore has a negative derivative or alternatively $\dot{x}$ is going from negative to positive as x increases through a point with $\dot{x} = 0$ (i.e. unstable) and therefore has a positive derivative.

$$f(x) = \cos x \quad \Rightarrow \quad f'(x) = -\sin x \tag{1.5}$$

Therefore where $x = \pi/2$, $f'(\pi/2) = -1 < 0$ and where $x = -\pi/2$, $f'(-\pi/2) = 1 > 0$ assigning these points as stable and unstable respectively, and so on for all the periodic points.

It will be useful for later to think of computing $f'(x)$ for a one-dimensional system as computing an eigenvalue of a 1D matrix, despite the apparent banality of this statement.

1.2 Flows for arbitrary 1D systems

So, for the general system

$$\dot{x} = f(x) \tag{1.6}$$

as long as $f'(x)$ is well defined and sufficiently smooth (see below) for all x and $f'(x_i) \neq 0$ for all x_i such that $f(x_i) = 0$, we can determine the behaviour for long times for any $f(x)$ for any x_0 just by drawing the picture! In particular we can construct a phase line of just the directions which tells us what will happen in the system. In 2D we will call this a phase portrait.

Some interesting points about phase lines[3]:

(a) Strictly we need to worry about existence and uniqueness. In practise this is a bit of a Pandora's box, so for the purposes of this course we consider $f(x)$ and it's derivatives to be

[3] See Strogatz, pages 26-30, for discussion of these points.

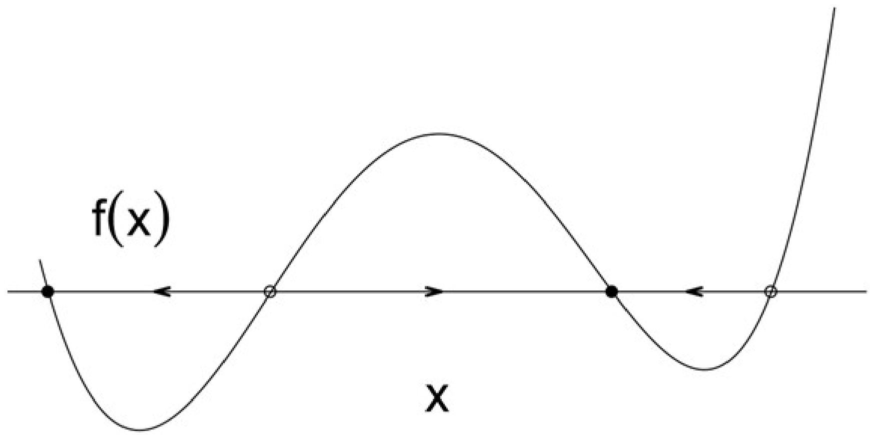

Figure 2 Plot of an arbitrary function. This is all that is necessary to construct a qualitatively accurate visualisation of a phase line.

sufficiently smooth and well behaved that a solution exists and is unique. If there is an $f(x)$ which contradicts this assumption, it will be heavily labelled to that effect and guidance will be given.

(b) No oscillations are possible: the trajectories can only be on the line, and the flows can only go away from unstable points and towards stable points or to infinity. This means: **No Oscillations** for $x(t)$ that are solutions of (1.6).

(c) As long as the fixed points are simple (i.e. $f'(x^*) \neq 0$), the fixed points have to alternate between unstable and stable along the phase line.

1.3 Linear stability analysis

Let us now be a bit more precise about the analysis above.

Definition 1.1. For the systems dx $\dot{x} = f(x)$ all the points x_i^* such that $f(x_i^*) = 0$ are called the **fixed points** of the system.

Consider a fixed point x^* and a small perturbation, η away from x^*, i.e. $x = x^* + \eta$ near x^*. The dynamics of this will be given by

$$\begin{aligned}\dot{\eta} &= \frac{d}{dt}(x - x^*)\\ &= \dot{x}\\ &= f(x^* + \eta)\\ &= f(x^*) + \eta f'(x^*) + O(\eta^2)\end{aligned}$$

Discarding higher order term, we obtain

$$\dot{\eta} = \eta f'(x^*), \tag{1.7}$$

which means that

- if $f'(x^*) < 0$, the perturbation will exponentially decay back to x^*
- $f'(x^*) > 0$, the perturbation will exponentially increase away from x^*
- $f'(x^*) = 0$, it is unclear, we can determine graphically by inspection or by examining the $O(\eta^2)$ term, or worse.

Example 1.1. Classify the fixed points of $\dot{x} = x^2 - 1$.

Example 1.2. Classify the fixed points of $\dot{x} = -x^3$.

Example 1.3. Classify the fixed points of $\dot{x} = x^3$.

Example 1.4. Classify the fixed points of $\dot{x} = x^2$.

Example 1.5. Classify the fixed points of $\dot{x} = 0$.

Example 1.1 can be linearised without difficulty. Examples 1.2 and 1.3 cannot be linearised but the behaviour can easily be deduced from the graphical solution. Example 1.4 gives rise to an unknown answer where the trajectories appear to be passing straight through the fixed point (but they do not!). Strogatz's defines points like this as *half stable*. Example 1.5 gives rise to a line of *neutrally stable* fixed points that neither attract nor repel trajectories. Examples 1.4 and 1.5 turn out to be useful as marginal cases that arise transiently during *bifurcations* (we will encounter these in section 1.6).

1.4 Nondimensionalisation

As you have seen in Introduction to Applied Mathematics, differential equations that arise in physics, chemistry, biology, etc. are usually expressed in term of dimensional variables representing physical quantities. Any physical quantity that can be measured needs units of measurement, e.g. to measure a distance between two point in space, we need units of length such as meters, inches or miles. In mathematics, we work with numbers which have no physical dimension (i.e. dimensionless quantities). Therefore, in order to obtain results that are independent of a particular set of units used to formulate a differential equation, we need to *nondimensionalise* the equation. This is achieved by rewriting the differential equation in terms of the dimensionless variables, obtained by dividing dimensional variables by characteristic scales for these variables which we can choose as we wish as long as these characteristic scales have the same physical dimensions as the corresponding dimensional variables. (This is equivalent to assigning the characteristic scales as the units of measurement.) We shall see a few examples of this procedure later on.

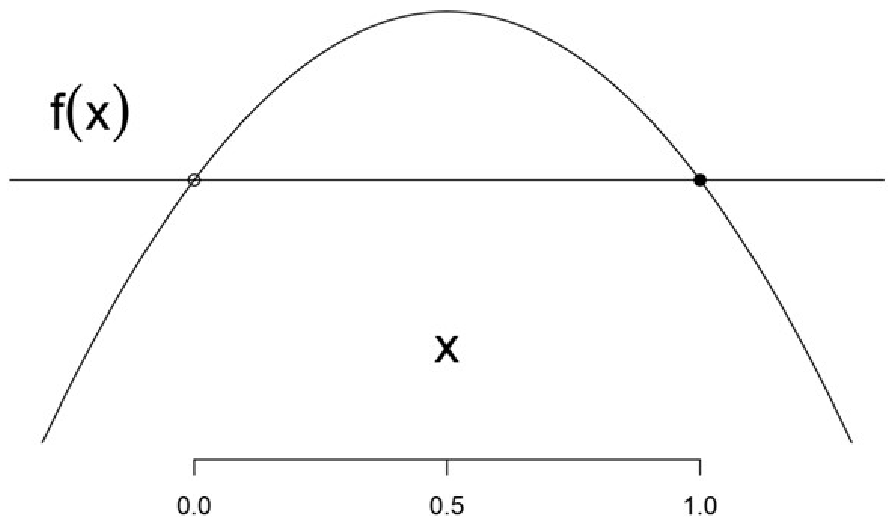

Figure 3 A plot of $\dot{x}$ against x for the logistic equation.

1.5 Logistic equation

We can pull all this together to look in more detail at the logistic equation (an important equation in mathematical biology and epidemiology):

$$\frac{d}{dt}N = rN\left(1 - \frac{N}{K}\right), \tag{1.8}$$

where $N(t)$ is the population size, r is the growth rate and K is the carrying capacity (i.e. the maximum population size).

Let us use the carrying capacity and the inverse growth rate, r^{-1}, as characteristic scales for the population N and time t. This means that we introduce the dimensionless population x and time τ as

$$x = \frac{N}{K}, \quad \tau = t\,r. \tag{1.9}$$

Since $d/dt = r\,d/d\tau$, Eq. (1.8), written in the new dimensionless variables, simplifies to

$$\frac{dx}{d\tau} \equiv \dot{x} = x(1 - x). \tag{1.10}$$

Solving $f(x^*) = 0$ gives two fixed points, $x^* = 0$ and $x^* = 1$, and we can determine $f'(x) = 1-2x$. For $x^* = 0$, $f'(0) = 1 > 0$, so this is unstable. For $x^* = 1$, $f'(1) = -1 < 0$, so this is stable.

This equation is very important in mathematical biology where it is also known as the Pearl-Verhulst equation. We will leave the discussion on it here for the moment. Further commentary can be found in Strogatz.

1.6 Bifurcations

We have outlined methodology whereby we can produce qualitatively a accurate descriptions of any dynamical system in one dimension. We are now interested in situations where the qualitative description changes as the result of a parameter changing its value. This parameter is known as a *bifurcation parameter*, the change is known as a *bifurcation*, and the study of them is called *bifurcation theory*. This is an area of mathematics which can get extremely complex. For the purposes of this course, we consider a bifurcation to be a qualitative change in the phase portrait caused by a change in a parameter value. We will consider three different types of bifurcation in detail in this course. Here we introduce *transcritical*, *fold* (also called *saddle-node*) and *pitchfork* bifurcations.

Definition 1.2. A bifurcation diagram shows how the fixed points, x^*, change their behaviour as the bifurcation parameter, μ, changes value. Stable fixed points are shown as a solid curve. Unstable fixed points are shown as dashed curves.

Definition 1.3. A transcritical bifurcation occurs when two fixed points collide as a bifurcation parameter ν changes in value and the two fixed points exchange stability.

Transcritical bifurcations are quite common in the systems we will study. Note that exactly at the bifurcation point (i.e. when $\mu = 0$ in the prototypical example (1.7) below) the system is *half-stable* (using Strogatz's definition).

Example 1.6. Find and categorise the fixed points and draw the bifurcation diagram for the system

$$\dot{x} = \mu x - x^2. \tag{1.11}$$

Fixed points:

$$f(x^*) = 0 \quad \Rightarrow \quad x^*(\mu - x^*) = 0 \quad \Rightarrow \quad x^* = 0 \;\text{ or }\; x^* = \mu$$

So, the fixed points are $x^* = 0$ and $x^* = \mu$. They merge if $\mu = 0$.
Stability:

$$f'(x) = \mu - 2x \quad \Rightarrow \quad f'(0) = \mu \quad \text{and} \quad f'(\mu) = -\mu$$

Hence, $x^* = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$, and $x^* = \mu$ is unstable for $\mu < 0$ and stable for $\mu > 0$. If $\mu = 0$, then the only fixed point $x^* = 0$ is half-stable. Alternatively, the stability of the fixed points follows from Fig. 4.

Since two fixed points collide and exchange stability, this is a transcritical bifurcation. The bifurcation diagram is shown in Fig. 5.

Definition 1.4. A Fold bifurcation occurs either when two fixed points collide and vanish as a bifurcation parameter μ changes in value or when two fixed points are simultaneously created as a bifurcation parameter μ changes in value.

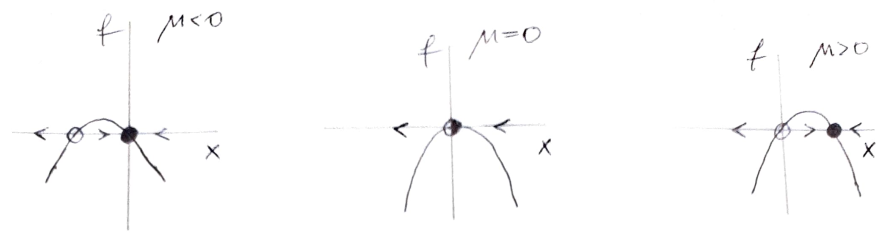

Figure 4 Plots of $\dot{x}$ against x for example 1.6.

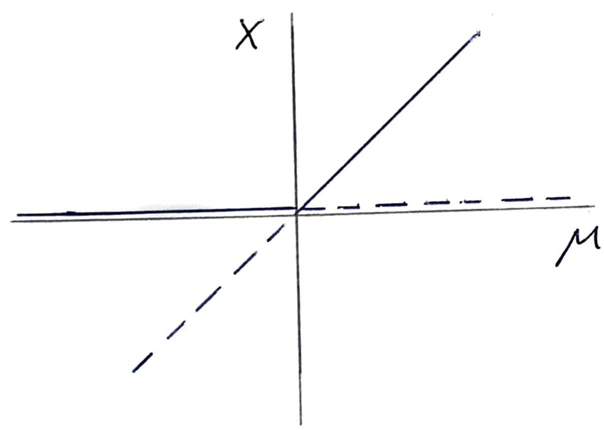

Figure 5 The bifurcation diagram for example 1.6

Example 1.7. Find and categorise the fixed points and draw the bifurcation diagram for the system

$$\dot{x} = \mu - x^2. \tag{1.12}$$

Fixed points:

$$f(x^*) = 0 \quad \Rightarrow \quad \mu - x^{*2} = 0 \quad \Rightarrow \quad x^* = \pm\sqrt{\mu}$$

So, there are no fixed points for $\mu < 0$, one fixed point, $x^* = 0$, for $\mu = 0$ and two fixed points, $x^* = \pm\sqrt{\mu}$, for $\mu > 0$.
Stability:

$$f'(x) = -2x \quad \Rightarrow \quad f'(\sqrt{\mu}) = -2\sqrt{\mu} < 0 \quad \text{and} \quad f'(\sqrt{\mu}) = 2\sqrt{\mu} > 0$$

Hence, $x^* = \sqrt{\mu}$ is stable, and $x^* = -\sqrt{\mu}$ is unstable. If $\mu = 0$, the only fixed point $x^* = 0$ is half-stable. Alternatively, the stability of the fixed points follows from Fig. 6.

Since two fixed points are simultaneously created, this is a fold (saddle-node) bifurcation. The bifurcation diagram is shown in Fig. 7.

Fold bifurcations are fundamental in the systems we will study. Again, note that exactly at the bifurcation point (i.e. when $\mu = 0$ in the prototypical example above) the system is *half-stable*,

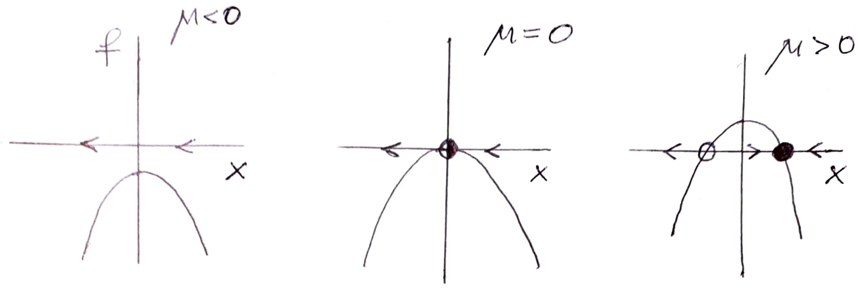

Figure 6 Plots of $\dot{x}$ against x for example 1.7.

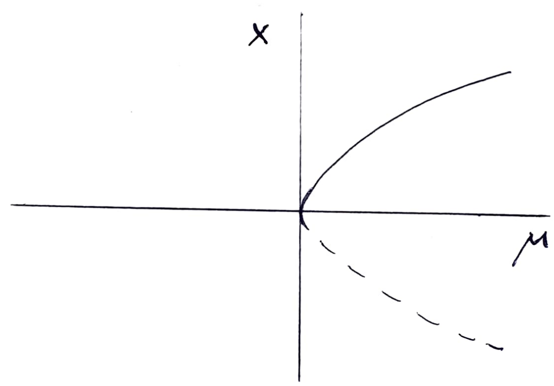

Figure 7 The bifurcation diagram for example 1.7

to use Strogatz's definition. The theory of them can get quite technical as the change induced is equivalent to a discontinuity.

Remark 1.1. A fold bifurcation must give rise to an unstable and a stable pair of fixed points in order to preserve the stability order of fixed points.

Remark 1.2. An alternative way to see the spontaneous emergence of two fixed points is to note that the fixed points are given by $x^* = \pm\sqrt{\mu}$, so no fixed points exist for $\mu < 0$. A half-stable point exists for $\mu = 0$ and two exist for $\mu > 0$.

Remark 1.3. Fold bifurcations are sometimes called saddle-node bifurcations, including Strogatz's book. Nonlinear Dynamics and Chaos [textbook]. To avoid confusion with saddles later we will call this type of bifurcation a fold in this course.

Remark 1.4. In 2D (and higher-dimensional) systems, there is also *Hopf* bifurcation, but it will not considered in this course.

Definition 1.5. A Pitchfork bifurcation occurs either when there are three fixed points two of which vanish and the third one changes stability as a bifurcation parameter μ changes in value or when there is one fixed point which changes stability and simultaneously two new fixed points are created as a bifurcation parameter μ changes in value. A Pitchfork bifurcation is called *supercritical* when two fixed points that vanish or are created are stable and *subcritical* when these two fixed points are unstable.

Example 1.8. Find and classify the fixed points and draw the bifurcation diagram for the system

$$\dot{x} = \mu x - x^3. \tag{1.13}$$

Fixed points:

$$f(x^*) = 0 \quad \Rightarrow \quad x^*(\mu - x^{*2}) = 0 \quad \Rightarrow \quad x^* = 0 \text{ or } x^* = \pm\sqrt{\mu}$$

So, there is one fixed point, $x^* = 0$, for $\mu \leq 0$, and there are three fixed points, $x^* = 0$ and $x^* = \pm\sqrt{\mu}$, for $\mu > 0$.
Stability:

$$f'(x) = \mu - 3x^2 \quad \Rightarrow \quad f'(0) = \mu, \;\; f'(\sqrt{\mu}) = -2\mu \quad \text{and} \quad f'(-\sqrt{\mu}) = -2\mu$$

Hence, $x^* = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$, $x^* = \pm\sqrt{\mu}$ are both stable for $\mu > 0$. If $\mu = 0$, the only fixed point $x^* = 0$ is stable. Alternatively, the stability of the fixed points follows from Fig. 8.

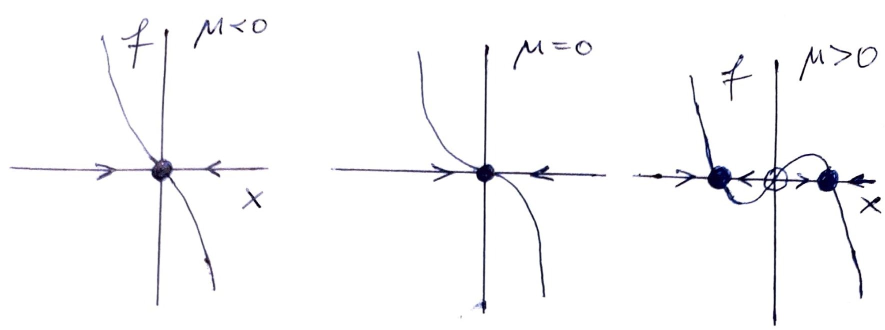

Figure 8 Plots of $\dot{x}$ against x for example 1.8.

Since one fixed point changes stability and simultaneously two new fixed points are created, this is a supercritical pitchfork bifurcation. The bifurcation diagram is shown in Fig. 9.

Example 1.9. Find and classify the fixed points and draw the bifurcation diagram for the system

$$\dot{x} = \mu x + x^3.$$

Fixed points:

$$f(x^*) = 0 \quad \Rightarrow \quad x^*(\mu + x^{*2}) = 0 \quad \Rightarrow \quad x^* = 0 \text{ or } x^* = \pm\sqrt{-\mu}$$

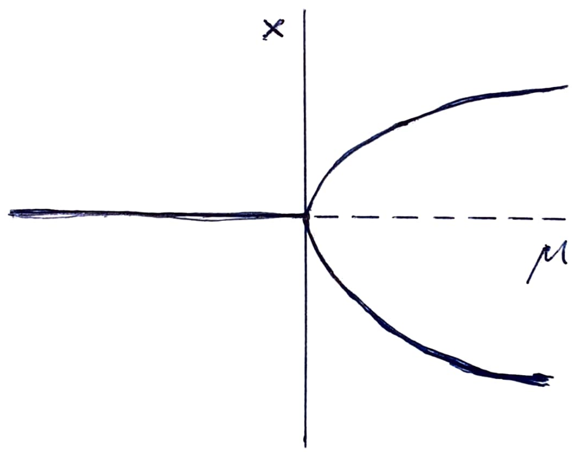

Figure 9 The bifurcation diagram for example 1.8

So, there are three fixed points, $x^* = 0$ and $x^* = \pm\sqrt{-\mu}$, for $\mu < 0$, and there is one fixed point, $x^* = 0$, for $\mu \geq 0$.
Stability:

$$f'(x) = \mu + 3x^2 \quad \Rightarrow \quad f'(0) = \mu, \;\; f'(\pm\sqrt{-\mu}) = -2\mu$$

Hence, $x^* = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$, $x^* = \pm\sqrt{-\mu}$ are both unstable for $\mu < 0$. If $\mu = 0$, the only fixed point $x^* = 0$ is unstable. Alternatively, the stability of the fixed points follows from Fig. 10.

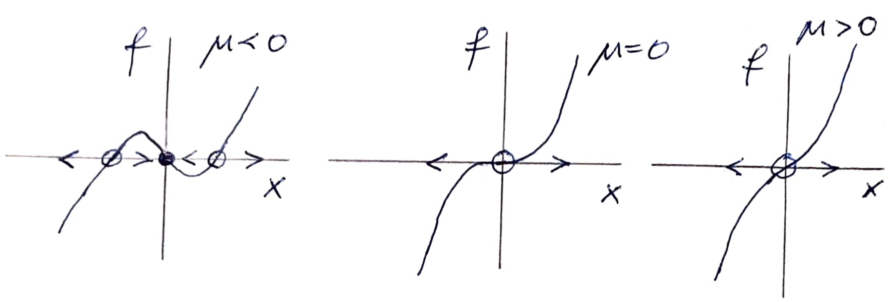

Figure 10 Plots of $\dot{x}$ against x for example 1.9.

Since one fixed point changes stability and simultaneously two fixed points annihilate, this is a subcritical pitchfork bifurcation. The bifurcation diagram is shown in Fig. 11.

1.7 A model of spruce budworm outbreak

Consider the dynamical system

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2} \tag{1.14}$$

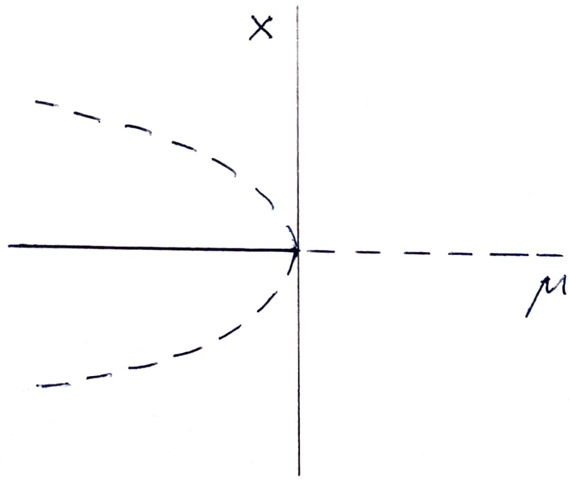

Figure 11 The bifurcation diagram for example 1.9

where N is the population of insects, R is the growth rate, K is the carrying capacity, A and B are constant parameters modelling the decrease in population due predation (e.g. by birds). For more details see, e.g., Strogatz, pp. 73-74.

Dimensionless form of (1.14). Let dimensionless variables, x and τ, and parameters, r and k be defined as

$$x = \frac{N}{A}, \quad \tau = \frac{Bt}{A}, \quad r = \frac{RA}{B}, \quad k = \frac{K}{A}.$$

Rewriting (1.14) in terms of variables x and τ, we obtain

$$\frac{dx}{d\tau} = \dot{x} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}. \tag{1.15}$$

Note that parameters k and r must be positive.

Fixed points. We have

$$x\left(r - \frac{r}{k}\,x - \frac{x}{1+x^2}\right) = 0.$$

So, there is an obvious fixed point at the origin: $x^* = 0$. The other fixed points are solutions of the equation

$$f_1(x) = f_2(x) \quad \text{where} \quad f_1(x) = r - \frac{r}{k}\,x, \quad f_2(x) = \frac{x}{1+x^2}. \tag{1.16}$$

The easiest way to see that this equation may have one, two or three positive solutions is to use the diagram in Fig. 12. Figure 12 shows the graph of $f_2(x)$ (blue curve) and four straight lines representing graphs of $f_1(x)$ for different pairs of values of parameters r and k. Solutions of equation (1.16) are the intersection points of these lines with the graph of $f_2(x)$. It is evident from Fig. 12 that there may be one positive fixed point (the line corresponding to parameter values (k_1, r_1)), two positive fixed points (the lines (k_2, r_2) and (k_4, r_4)) or three fixed points (the line (k_3, r_3)).

Dynamics on the phase line. We can use the derivative test to determine the stability of the fixed points. But it is enough to do this only for fixed point $x^* = 0$ that exists for all values

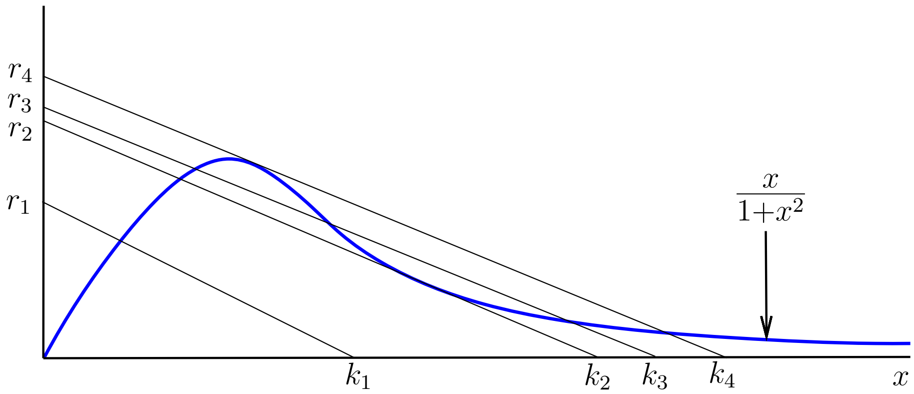

Figure 12 The graph of $f_1(x)$ and graphs of $f_2(x)$ for several values of parameters k and r.

of parameters k and r, because, as we observed earlier, stable and unstable fixed points must alternate when we move along the phase line (sometimes with half-stable points between stable and unstable ones).

We have

$$f'(x) = r - \frac{x}{k} - \frac{x}{1+x^2} + x\,\frac{d}{dx}\left(r - \frac{x}{k} - \frac{x}{1+x^2}\right) \quad \Rightarrow \quad f'(0) = r > 0.$$

Hence, $x^* = 0$ is unstable for all k and r.

Consider now the values of parameters $(k, r) = (k_1, r_1)$ as in Fig. 12. Since the fixed point at the origin is unstable, the only positive fixed point must be stable, so that the dynamics on the phase line will look like that shown in Fig. 13(a). If the values of parameters $(k, r) = (k_2, r_2)$ as in Fig. 12, so that there are two positive fixed points, with the phase portrait shown in Fig. 13(b). The phase portraits for $(k, r) = (k_3, r_3)$ and $(k, r) = (k_4, r_4)$ are shown in 13(c) and (d) respectively.

The stability diagram on the (k, r) plane. Let's find a region in the parameter space (i.e. on the (k, r) plane) where the system has three positive fixed points. First, we note that there is a critical value k_* of parameter k such that for $0 < k \le k_*$ and any $r > 0$, there is only one positive fixed point. One can see in Fig. 14 that this critical value corresponds to the situation where the graph of $f_1(x)$ coincides with the tangent line to the graph of $f_2(x)$ at the inflection point where $f_2''(x) = 0$ (the red line in Fig. 14).

Now let parameter k have a fixed value $k > k_*$. There are exactly two lines $f_1(x) = r_1 - r_1 x/k$ and $f_1(x) = r_2 - r_2 x/k$ that intersect with the graph of $f_2(x)$ and tangent to it at the intersection points. It follows from Fig. 14) that there is only one intersection point for all $0 < r < r_1(k)$ and for $r > r_2(k)$. This means that for $0 < r < r_1(k)$ and $r > r_2(k)$ there is only one positive fixed point and it is stable (as shown in Fig. 13(a)). For $r_1(k) < r < r_2(k)$, there are three intersection points, which implies there are three positive fixed points, say $x^* = x_1$, $x^* = x_2$ and $x^* = x_3$ with $0 < x_1 < x_2 < x_3$, and x_1 and x_3 are stable while x_2 is unstable, as shown in Fig. 13(c). Thus,

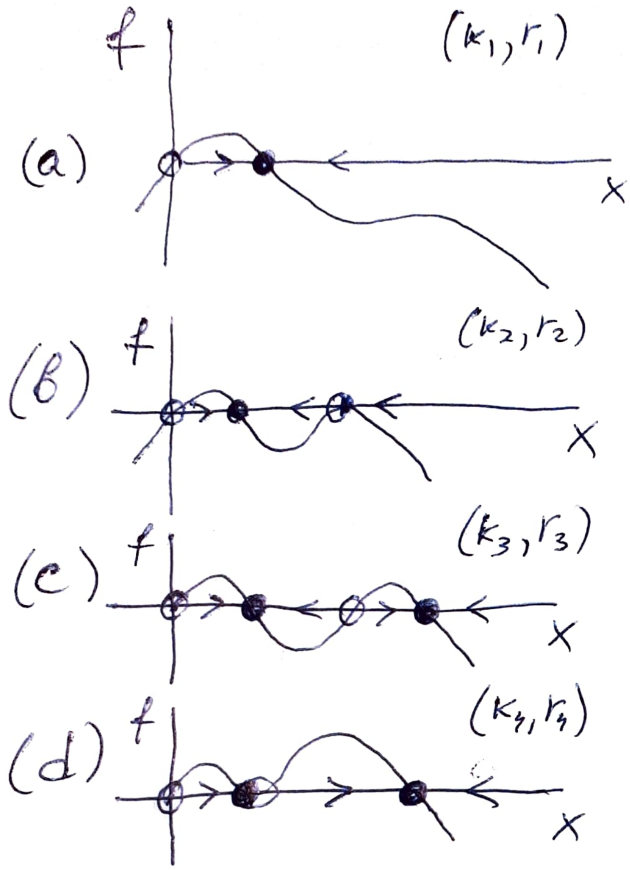

Figure 13 Phase portraits in four cases (k_n, r_n) corresponding to Fig. 12.

we can conclude that for a fixed $k > k_*$, fold bifurcations occur when r passes through the values $r_1(k)$ and $r_2(k)$.

There are two boundary cases when $r = r_1(k)$ and $r = r_2(k)$ and there are two fixed points, say $x^* = x_1$ and $x^* = x_2$ with $0 < x_1 < x_2$. In the 1st case, x_1 is stable and x_2 is half-stable, see Fig. 13(b). In the 2nd case, x_1 is half-stable and x_2 is stable, see Fig. 13(d).

This picture remain qualitatively the same for all $k > k_*$. It also follows from Fig. 14) that as $k \to k_*$ from above, both $r_1(k)$ and $r_2(k)$ tend to r_* and that as $k \to \infty$, $r_1(k) \to 0$ and $r_2(k) \to \max_{x>0} f_2(x) = f_2(1) = 1/2$.

The graphical analysis above gives us a qualitatively clear picture of the region where there are three positive fixed point (sometimes called the region of *bi-stability* as there are two stable fixed points there). Nevertheless some analytical arguments are also useful to, at the very least, confirm our qualitative graphical analysis. So, let's find a parametric equation of the boundary of the bistability region in the form $(k(x), r(x)$ where x the half-stable fixed point. On the boundary of this region, we have the following two equations

$$f_1(x) = f_2(x) \quad \Leftrightarrow \quad r - \frac{r}{k}\,x = \frac{x}{1 + x^2} \tag{1.17}$$

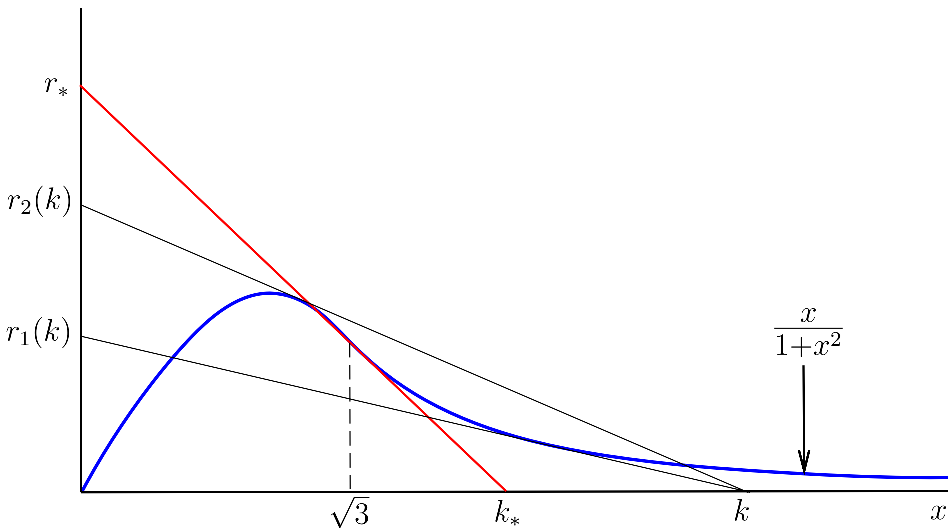

Figure 14 Graph of $f_2(x)$ with graphs of $f_1(x)$ for a fixed k and two critical values $r_1(k)$ and $r_2(k)$ of parameters r. Red line corresponds to the critical value k_* of k and the corresponding value $r = r_*$.

(because it is an intersection point of the two graphs) and

$$f_1'(x) = f_2'(x) \quad \Leftrightarrow \quad -\frac{r}{k} = \frac{1-x^2}{(1+x^2)^2} \tag{1.18}$$

(because the line $f_1(x)$ must be tangent to the graph of $f_2(x)$). Following Strogatz (p. 77), we substitute r/k from (1.18) into (1.17). This yields the formula for $r(x)$:

$$r(x) = \frac{2x^3}{(1+x^2)^2}. \tag{1.19}$$

Then we substitute (1.19) into (1.18). As a result, we obtain

$$k(x) = \frac{2x^3}{x^2-1}. \tag{1.20}$$

Since $k(x)$ must be positive, Eq. (1.20) implies that $x > 1$ (the same conclusion can also be reached by analysing Fig. 14)). Equations (1.19) and (1.20) for $x > 1$ give us the parametric equation of a curve in the (k, r) plane, which is the boundary of the bistability region. For each $x > 1$, we can find $k(x)$ and $r(x)$ and thus plot this curve.

The location of the point (k_*, r_*) can also be found analytically. The only thing we need to do is to find a value of x where $f_2(x)$ has the inflection point:

$$f_2''(x) = \frac{2x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad f_2''(x) = 0 \text{ at } x = \sqrt{3}.$$

Substituting this value into (1.18) and (1.19), we find

$$k_* = 3\sqrt{3} \approx 5.17, \quad r_* = \frac{3\sqrt{3}}{8} \approx 0.65$$

Other analytical results, which can be easily obtained, are the limits of $k(x)$ and $r(x)$ as $x \to 1$ and as $x \to infty$:

$$\lim_{x\to 1} r(x) = \frac{1}{2} \quad \text{and} \quad r(x) \sim \frac{2}{x} \quad \text{as} \quad x \to \infty$$

and

$$k(x) \to \infty \quad \text{as} \quad x \to 1 \quad \text{and} \quad k(x) \sim 2x \quad \text{as} \quad x \to \infty.$$

The asymptotic behaviour of $k(x)$ and $r(x)$ as $x \to \infty$ implies that

$$r \sim \frac{4}{k} \quad \text{as} \quad k \to \infty.$$

Note that this confirms our earlier observation that $r_1(k) \to 0$ as $k \to \infty$.

The information we have obtained is enough to produce a sketch of the bistability region. It is shown in Fig. 15.

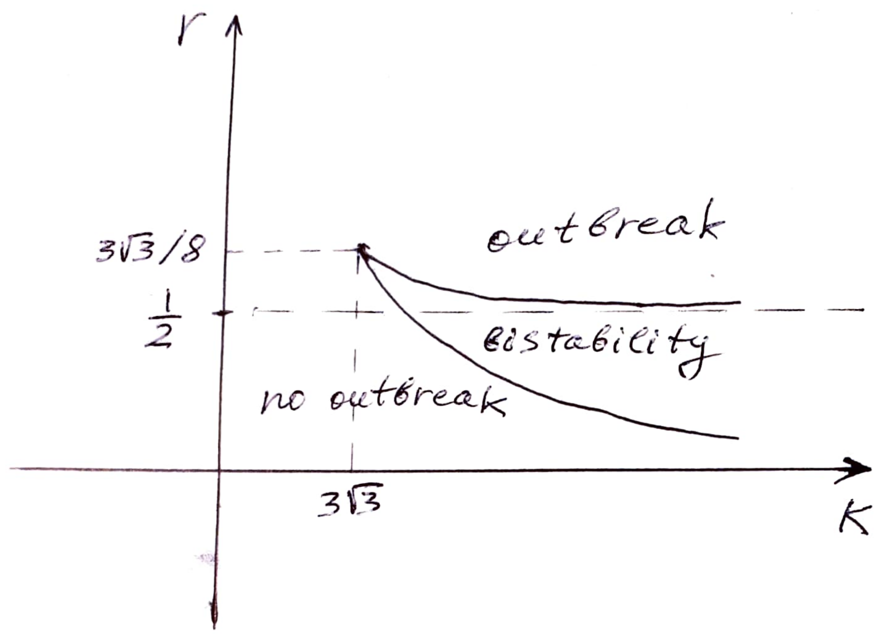

Figure 15 The sketch of bistability region on the (k, r) plane.

Suppose that the value $k > 3\sqrt{3}$ doesn't change, but r can slowly change with time. We assume that initially r has a value below the bistability region in Fig. 15. Then whatever initial population of insects we have, it will quickly move towards the only fixed point where the population is low (no outbreak), and the population will stay near this fixed point forever. When the value of the slowly varying parameter r crosses the lower boundary of the bistability region, nothing drastic happens because the population is still near the same stable fixed point and the appearance of new stable and unstable fixed points will not affect the population. However, when the value of parameter r crosses the upper boundary of the bistability region, the stable fixed point (corresponding to a low population level) simply vanishes and the only stable fixed point that is left is at a much higher population level. So, the system will quickly move towards this fixed point, which means that it will rapidly grow until it reached an equilibrium at a high population level. This is when we have an outbreak. More about this model including its connection with the *catastrophe theory* can be found in Strogatz (see pp. 78-79 for this example and section 3.6 and references there for the *catastrophe theory*).

2 Planar Linear Systems

In this chapter we will study and completely classify the behaviour of solutions to first order linear ODE in two variables. Geometrically these represent vector fields in the plane. Their solutions are curves in the plane (see Fig. 16)). Such ODEs are all of the form

$$
\begin{aligned}
\dot{x} &= ax + by \\
\dot{y} &= cx + dy
\end{aligned}
\tag{2.1}
$$

where a, b, c, d are real constants.

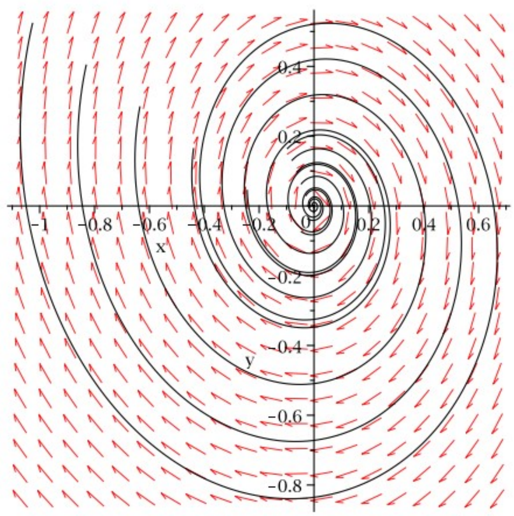

Figure 16 An example of a vector field with some solution curves.

Such systems are also called *autonomous*, meaning that there is no explicit dependence on the dependent variable t. Here we have chosen $n = 2$ and assumed that both f_1 and f_2 are simple linear functions of both variables with no constant term. Any constant terms can be simply removed in a linear system by a translation.

2.1 The general solution of planar linear systems

We may rewrite (2.1) as an ODE for the vector $\boldsymbol{x} = (x, y)$.

$$
\dot{\boldsymbol{x}} = A\boldsymbol{x}, \tag{2.2}
$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.3}$$

For any initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0 = (x_0, y_0)$ there is a unique solution $\boldsymbol{x}(t) = (x(t), y(t))$ to (2.2) for which $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

Since Eq. (2.2) is linear, if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are solutions, then so is any linear combination $C_1\boldsymbol{x}_1 + C_2\boldsymbol{x}_2$. Note that $\dot{\boldsymbol{x}} = 0$ when $\boldsymbol{x} = \mathbf{0}$, so $\mathbf{x}^* = \mathbf{0}$ is always a fixed point (for any matrix A) .

How to find a solution? Let's consider an example first.

Example 2.1. Find a solution of

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

We have two uncoupled ODEs

$$\begin{aligned} \dot{x} &= -2x \\ \dot{y} &= 4y \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 e^{-2t} \\ y_0 e^{4t} \end{pmatrix}$$

or, equivalently,

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} = x_0 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y_0 e^{4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Phase portrait of the system is shown is Fig. 17. This is an example of a saddle fixed point where

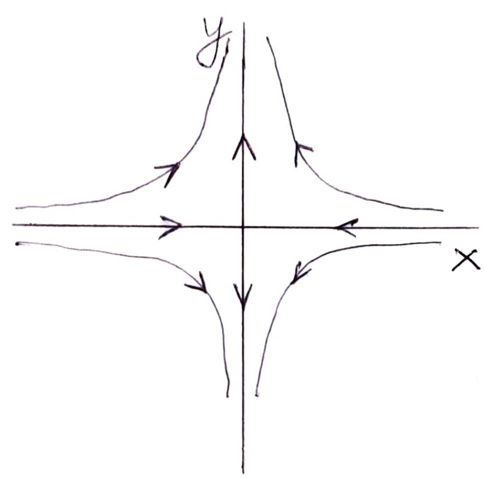

Figure 17 Phase portrait for example 2.1.

we have stable and unstable directions (the former is the x-axis and the latter - the y-axis). This example also demonstrates an important general property: *in a phase portrait no two trajectories cross.* This is because each initial condition uniquely determines a solution.

Consider now the general case. We seek solutions of (2.2) in the form

$$\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{v}, \tag{2.4}$$

where $\boldsymbol{v}$ is a fixed nonzero vector to be determined, and λ is a fixed constant(which may be complex), also to be determined. If such solutions exist, they correspond to motion along the line spanned by the vector $\boldsymbol{v}$.

Substituting (2.4) in (2.2), we find

$$A\boldsymbol{v} = \lambda\boldsymbol{v}, \tag{2.5}$$

which means that $\boldsymbol{v}$ is an eigenvector of A with corresponding eigenvalue λ.

Let's recall how to find eigenvalues and eigenvectors. The eigenvalues of a matrix A are solutions of the characteristic equation $\det(A-\lambda I) = 0$, where I is the identity matrix. For a 2 matrix A, given by (2.3), we have

$$\det(A-\lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+b)\lambda + ad - bc = 0$$

Let

$$\tau = \operatorname{tr} A \quad \text{and} \quad \Delta = \det A,$$

then the characteristic equations takes the form

$$\lambda^2 - \tau\,\lambda + \Delta = 0,$$

and the eigenvalues are given by

$$\lambda = \lambda^{\pm} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \tag{2.6}$$

If the eigenvalues are distinct, i.e. $\lambda^+ \neq \lambda^-$, then the corresponding eigenvectors $\boldsymbol{v}^+$, and $\boldsymbol{v}^-$ are linearly independent, and hence any initial condition $\boldsymbol{x}_0$ can be written as a linear combination of eigenvectors, say $\boldsymbol{x}_0 = C_1\boldsymbol{v}^+ + C_2\boldsymbol{v}^-$. Therefore the solution satisfying $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is given by

$$\boldsymbol{x}(t) = C_1 e^{\lambda^+ t}\boldsymbol{v}^+ + C_2 e^{\lambda^- t}\boldsymbol{v}^-. \tag{2.7}$$

If $\lambda^+ = \lambda^- = \lambda$, then there are two possible cases to consider.

(i) There are two linearly independent eigenvectors that span the whole plane, so that any $2D$ vector is an eigenvector. Then for any $\boldsymbol{x}_0$, the solution satisfying $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is given by

$$\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{x}_0. \tag{2.8}$$

(ii) There is only one eigenvector, $\boldsymbol{v}$. In this case, we can define vector $\boldsymbol{w}$ (called a *generalised eigenvector of rank* 2) as a solution of the equation

$$(A - \lambda I)\boldsymbol{w} = \boldsymbol{v}$$

(note that $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent). Then any initial condition $\boldsymbol{x}_0$ can be written as a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$: $\boldsymbol{x}_0 = C_1\boldsymbol{v} + C_2\boldsymbol{w}$. The solution, satisfying $\boldsymbol{x}(0) = \boldsymbol{x}_0$, is given by

$$\boldsymbol{x}(t) = e^{\lambda t}\left((C_1 + C_2 t)\boldsymbol{v} + C_2\boldsymbol{w}\right). \tag{2.9}$$

[Verify by direct substitution that (2.9) satisfies $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ for any $C_1, C_2 \in \mathbb{R}$.]

2.2 The classification of planar linear systems

A. If $\tau^2 - 4\Delta > 0$, there are two distinct eigenvalues, λ^+ and λ^-, $\lambda^+ \neq \lambda^-$. Let $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$ be the corresponding eigenvectors. Then there are 4 possible cases:

(i) If $\lambda^- < \lambda^+ < 0$ (i.e. $\tau < 0$, $\Delta > 0$), then the fixed point at the origin is a *stable node*, with the phase portrait shown in Fig. 18(a).

Figure of stable node

(ii) If $0 < \lambda^- < \lambda^+$ (i.e. $\tau < 0$, $\Delta > 0$), then the fixed point at the origin is an *unstable node*, with the phase portrait shown in Fig. 18(b).

(iii) If $\lambda^- < 0 < \lambda^+$ (i.e. $\Delta < 0$), then the origin is a *saddle point*, with the phase portrait shown in Fig. 18(c).

(iv) If $\lambda^- < \lambda^+ = 0$ or $0 = \lambda^- < \lambda^+$ (i.e. $\tau \neq 0$, $\Delta = 0$), then the origin is not an isolated fixed point, but a point on a line of fixed points. [If $\tau = 0$ and $\Delta = 0$, then $\lambda^- = \lambda^+ = 0$ and all points on the plane is a fixed point.]

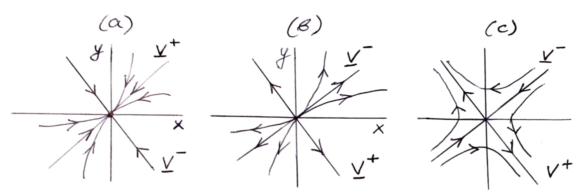

Figure 18 Phase portraits for cases A(i)-(iii): (a) a stable node; (b) an unstable node; (c) a saddle point.

B. If $\tau^2 - 4\Delta < 0$, there are two complex conjugate eigenvalues, λ and $\overline{\lambda}$. Let $\lambda = \mu + i\nu$ (where $\mu, \nu \in \mathbb{R}$). Let $\boldsymbol{v}$ and $\overline{\boldsymbol{v}}$ be the corresponding (complex conjugate) eigenvectors. If

$$\boldsymbol{x}_0 = C\,\boldsymbol{v} + \overline{C}\,\overline{\boldsymbol{v}}$$

where C is a complex constant and $\overline{C}$ is its complex conjugate (the 2nd constant must be $\overline{C}$ because $\boldsymbol{x}_0$ is a real vector), then

$$\boldsymbol{x}(t) = C\,e^{(\mu+i\nu)t}\boldsymbol{v} + \overline{C}\,e^{(\mu-i\nu)t}\overline{\boldsymbol{v}}.$$

This can be rewritten in real form as follows. Let

$$\boldsymbol{v} = \boldsymbol{u} + i\boldsymbol{w} \quad \text{and} \quad C = \xi + i\eta$$

where $\boldsymbol{u}$, $\boldsymbol{w}$ are real vectors and ξ, η are real constants. Then

$$\begin{aligned} \boldsymbol{x}(t) \;&=\; (\xi + i\eta)\,e^{(\mu+i\nu)t}(\boldsymbol{u} + i\boldsymbol{w}) + (\xi - i\eta)\,e^{(\mu-i\nu)t}(\boldsymbol{u} - i\boldsymbol{w}) \\ &= e^{\mu t}\;[2(\xi\boldsymbol{u} - \eta\boldsymbol{w})\cos(\nu t) - 2(\xi\boldsymbol{w} + \eta\boldsymbol{u})\sin(\nu t)] \\ &= e^{\mu t}\;[(C_1\boldsymbol{u} - C_2\boldsymbol{w})\cos(\nu t) - (C_1\boldsymbol{w} + C_2\boldsymbol{u})\sin(\nu t)] \end{aligned}$$

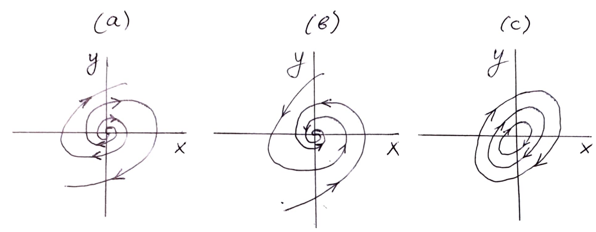

Figure 19 Phase portraits for cases B(i)-(iii)(a) an unstable focus ($\mu > 0$); (b) a stable focus ($\mu < 0$); (c) a centre ($\mu = 0$).

where $C_1 = 2\xi$ and $C_2 = 2\eta$ are real constants.

There are 3 possible cases, depending on the value of $\mu = Re(\lambda)$:

(i) If $\mu > 0$, then the fixed point at the origin is an *unstable focus*, with the phase portrait shown in Fig. 19(a).

(ii) If $\mu < 0$, then the fixed point at the origin is a *stable focus*, with the phase portrait shown in Fig. 19(b).

(iii) If $\mu = 0$, then the fixed point at the origin is a *centre*, with the phase portrait shown in Fig. 19(c).

C. If $\tau^2 - 4\Delta = 0$, there is one repeated eigenvalue, λ. There are only two possibilities:

(i) There are two independent eigenvectors, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Then any $2D$ vector is an eigenvector, and the fixed point at the origin is a *stable star* if $\lambda < 0$ and an *unstable star* if $\lambda > 0$. The phase portrait of a stable star is shown in Fig. 20(a). (The phase portrait of an unstable star can be obtained from Fig. 20(a) by reversing the direction of the flow.)

(ii) There is only one eigenvector, $\boldsymbol{v}$, then the fixed point at the origin is a *stable degenerate node*, if $\lambda < 0$ or an *unstable degenerate node*. The phase portrait of a stable degenerate node shown in Fig. 20(b). (The phase portrait of an unstable degenerate node can be obtained from Fig. 20(b) by reversing the direction of the flow.)

We can summarise the different types of behaviour in the following classification diagram (Fig. 21). The important point of this diagram is that to understand the qualitative nature of the dynamics of most planar linear systems it is enough to know the trace τ and determinant Δ of the matrix AA appearing in Eq. (2.2). We should be impressed by this: instead of having to solve differential equations we need only calculate two numbers which are easily read from the equation itself.

In practice it is more practical to think about this as being determined by two different numbers, the eigenvalues, even though they are potentially complex. This is because the eigenvalues have a more easily understandable interpretation in terms of the trajectories around the fixed point. For more complex systems the eigenvalues are a better way of understanding behaviour.

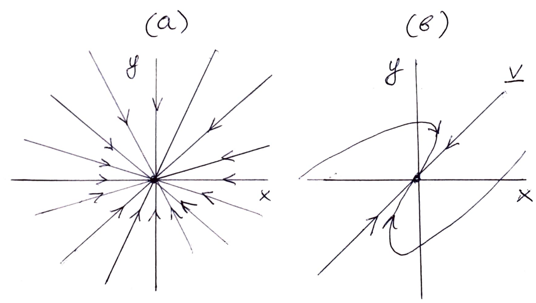

Figure 20 Phase portraits for cases C(i)-C(ii): (a) a stable star ($\lambda < 0$); (b) a stable degenerate node ($\lambda < 0$).

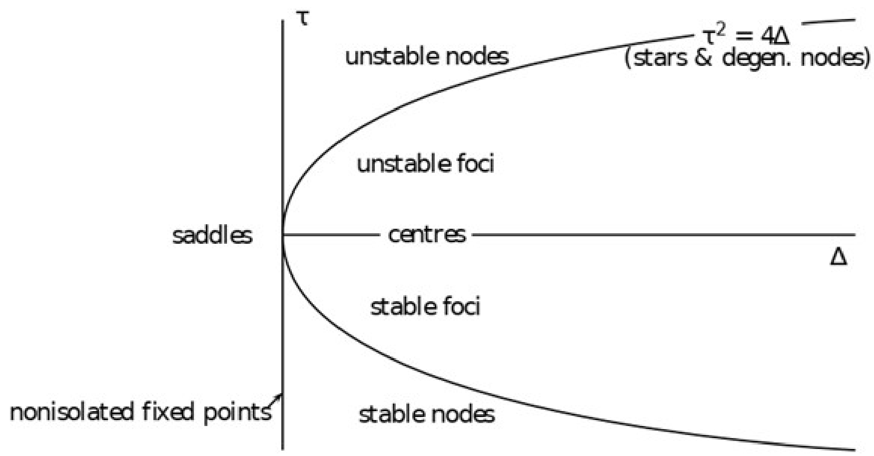

Figure 21 Classification diagram for planar linear systems.

3 Planar Nonlinear Systems

In this chapter we consider systems of the form

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{3.1}$$

where $f(x, y)$ and $g(x, y)$ are not necessarily linear.

We can rewrite (3.1) in vector form:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \quad \text{or} \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}. \tag{3.2}$$

Such systems are called *first order planar autonomous systems* of which the linear systems in the previous chapter are a special case. To ensure that we still have existence and uniqueness of a solution for every initial condition we will insist that f, g are differentiable with continuous partial derivatives in both variables and throughout $\mathbb{R}^2$. Under these conditions we have the following theorem, which we will take for granted.

Theorem 3.1. Under the conditions above, for every $\boldsymbol{x} = (x_0, y_0) \in \mathbb{R}^2$ there exists a unique solution $\boldsymbol{x}(t) = (x(t), y(t))$ for $t \in (-\epsilon, \epsilon)$, for some $\epsilon > 0$, for which $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Consequently, distinct trajectories do not intersect.

We will not be concerned with the proof of this theorem: the interested reader can find a discussion of it in Strogatz.

3.1 Fixed points and their stability

A point $\mathbf{x}^* \in \mathbb{R}^2$ is called a fixed point (or point of equilibrium) for (3.1) if $\boldsymbol{f}(\mathbf{x}^*) = \mathbf{0}$. For in that case $\dot{\boldsymbol{x}} = \mathbf{0}$ at the point $\mathbf{x}^*$ so there is no motion away from $\mathbf{x}^*$: the solution with initial condition $\mathbf{x}^*$ is the constant solution $\boldsymbol{x}(t) = \mathbf{x}^*$ for all t.

Nearby a fixed point we classify the types of possible behaviour as follows.

(a) If there exists a $\delta > 0$, for which

$$\|\boldsymbol{x}(0) - \mathbf{x}^*\| < \delta \quad \Rightarrow \quad \lim_{t\to\infty} \boldsymbol{x}(t) = \mathbf{x}^*,$$

we say that $\mathbf{x}^*$ is an *attracting fixed point*. In this case $\boldsymbol{x}(t)$ can leave the neighbourhood (open disc) $\|\boldsymbol{x} - \mathbf{x}^*\| < \delta$ but must eventually return to it, since it must return arbitrarily close to $\mathbf{x}^*$. Notice that $\boldsymbol{x}(t)$ can never actually reach $\mathbf{x}^*$ in finite time, since $\mathbf{x}^*$ is a distinct solution. We say $\boldsymbol{x}(t)$ is *asymptotic* to $\mathbf{x}^*$.

(b) If for each $\epsilon > 0$ there exists $\delta > 0$ for which

$$\|\boldsymbol{x}(0) - \mathbf{x}^*\| < \delta \quad \Rightarrow \quad \|\boldsymbol{x}(t) - \mathbf{x}^*\| < \epsilon, \quad \forall t > 0$$

we say $\mathbf{x}^*$ is a *Lyapunov stable* fixed point. In this case $\boldsymbol{x}(t)$ must stay "close"(within a disc of radius ϵ) for all time if it starts sufficiently close to $\mathbf{x}^*$. But solutions starting in this disc of radius δ do not necessarily have to be asymptotic to $\mathbf{x}^*$.

(c) If $\mathbf{x}^*$ is both attracting and Lyapunov stable, we say it is an *asymptotically stable fixed point*.

(d) If $\mathbf{x}^*$ is neither attracting nor Lyapunov stable we say it is *unstable*. An equivalent way of saying this is: $\mathbf{x}^*$ is an *unstable fixed point* if for each open disc D about $\mathbf{x}^*$ there is some initial condition $\boldsymbol{x}_0 \in D$ for which $\boldsymbol{x}(t)$ leaves D and never returns to D.

Example 3.1. Since every linear system has a fixed point at the origin, we can try to classify the stability type of this fixed point. We will do this just by inspection: we shall not try to prove what we claim (although it is a useful exercise to think about how you might do this).

(a) For all stable foci and stable nodes, the origin is both attracting and Liapunov stable, hence they are *asymptotically stable*.

(b) Centres have a Liapunov stable origin, but it is not attracting.

(c) There are no linear systems for which the origin is attracting without being Liapunov stable.

(d) For all saddles, unstable nodes and unstable foci, the origin is unstable.

(e) Systems of type A(iv) which have one zero eigenvalue (say $\lambda_1 = 0$) have Liapunov stable origin if $\lambda_2 < 0$ and unstable origin if $\lambda_2 > 0$. Systems of type C(ii) have asymptotically stable origin if $\lambda < 0$ and unstable origin if $\lambda > 0$.

We can summarise most of this is a neat way, which will have great importance for us later. A linear system has stable origin if its matrix A has trace $\tau < 0$ and determinant $\Delta > 0$. It has unstable origin if A has $\tau > 0$ or $\Delta < 0$.

3.2 Linearisation about a fixed point

. Suppose $\mathbf{x}^*$ is a fixed point of (3.2). The *linearisation* of this equation about $\mathbf{x}^* = (x^*, y^*)$ is the linear system (also called the *linearised system*)

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{3.3}$$

The matrix in (3.3) is the Jacobi matrix at (x^*, y^*) of the function $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$, given by $\boldsymbol{f} = (f, g)$. We denote it by $J(\boldsymbol{x})$, i.e. we set

$$J(\boldsymbol{x}) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

Example 3.2. All linear systems (2.1) have a fixed point at the origin $(0, 0)$. For these equations $\boldsymbol{f} = (ax + by, cx + dy)$ and therefore

$$J(\boldsymbol{x}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Hence, the linearisation about the origin is the same as the linear system itself.

Example 3.3. Consider the nonlinear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x^3 - y \\ x - y \end{pmatrix}.$$

The fixed point is the intersection of the graphs of $y = -x^3$ and $y = x$, which is the origin $\mathbf{x}^* = (0, 0)$. The Jacobi matrix is

$$J(\boldsymbol{x}) = \begin{pmatrix} -2x & -1 \\ 1 & -1 \end{pmatrix} \quad \Rightarrow \quad J(\mathbf{x}^*) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

So, the linearised system is

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

3.3 Linear stability analysis

For linear systems the linearisation about the origin completely describes the behaviour of the system (since they are the same). It is natural to ask whether a similar thing is true for general nonlinear systems. The answer is given by the Linearisation Theorem, which says that qualitatively the linearisation is locally the same provided the fixed point has the following crucial property.

Definition 3.1. A fixed point $\mathbf{x}^*$ of (3.1) is *hyperbolic* when the eigenvalues of $J(\boldsymbol{x})$ both have non-zero real parts.

Example 3.4. A linear system (2.2) has a hyperbolic fixed point at the origin precisely when the eigenvalues of the matrix A have non-zero real parts. Since the eigenvalues of A are given by (2.5), this happens if and only if $\Delta \neq 0$ with $\tau \neq 0$ when $\Delta > 0$. Therefore the linear systems which have hyperbolic fixed point correspond to the points on the classification diagram (Figure 2.4) lying in the open half-plane $\Delta < 0$ or one of the open quadrants $\Delta > 0, \tau > 0$ or $\Delta > 0, \tau < 0$.

Thus all nodes, foci and saddles have hyperbolic fixed points at the origin. Centres have a non-hyperbolic fixed point at the origin.

Linearisation Theorem (Hartman-Grobman). The dynamics of (3.1) near a hyperbolic fixed point $\mathbf{x}^*$ is topologically equivalent near $\mathbf{x}^*$ to that of its linearisation of (3.1) at $\mathbf{x}^*$.

The expression topologically equivalent near $\mathbf{x}^*$ means, roughly speaking, that there is an open disc centred at $\mathbf{x}^*$ in which the dynamics looks like the dy- namics of the linearisation near its fixed point.

Topological equivalence separates the hyperbolic fixed points of the linear sys- tems into three groups: attractors (stable nodes and foci), repellors (unstable nodes and foci) and saddles. Because the dynamics of a nonlinear system near a hyperbolic fixed point is locally equivalent to the linearisation, we can likewise label every hyperbolic fixed point as either an attractor, repellor or saddle. We summarise this in the following corollary to the Linearisation Theorem.

Corollary (Linear Stability Analysis). Let $\mathbf{x}^*$ be a hyperbolic fixed point of (3.1). The stability of $\mathbf{x}^*$ is the same as the stability of the linearisation about $\mathbf{x}^*$, and is determined by the trace τ and determinant Δ of the Jacobi matrix $J(\mathbf{x}^*)$. More precisely:

(a) $\tau < 0$ and $\Delta > 0$ implies $\mathbf{x}^*$ is an attractor, hence stable,

(b) $\tau > 0$ and $\Delta > 0$ implies $\mathbf{x}^*$ is a repellor, hence unstable,

(c) $\Delta < 0$ implies $\mathbf{x}^*$ is a saddle, hence unstable.

Note that this does more than just tell us whether the fixed point is stable or unstable. In particular, it distinguishes saddles from repellors, even though both are unstable. Saddles have an important characteristic feature. A saddle fixed point lies at the intersection of two curves, called the *stable manifold* and the *unstable manifold*.

Example 3.5. Let us analyse the example

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + x^3 \\ -2y \end{pmatrix}.$$

using linear stability analysis wherever it applies.

The fixed points occur at the intersection of $-x+x^3=0$ and $-2y=0$, hence there are three: $(0,0)$, $(1,0)$ and $(-1,0)$. To find out which of these are hyperbolic we first calculate the Jacobi matrix

$$J(\boldsymbol{x}) = \begin{pmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{pmatrix}$$

Now we evaluate this at each fixed point.

(a) At $\mathbf{x}^* = (0,0)$,

$$J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

Since, $\tau = -3 < 0$ and $\Delta = 2 > 0$, $(0,0)$ is a hyperbolic fixed point of attractor type.

(b) At $\mathbf{x}^* = (1,0)$,

$$J(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Since, $\tau = 0$ and $\Delta = -4 < 0$, $(1,0)$ is a hyperbolic fixed point of saddle type.

(c) Since $J(-1,0) = J(1,0)$, the fixed point $\mathbf{x}^* = (-1,0)$ is the same as the one in (b): a hyperbolic fixed point of saddle type.

This gives us the local picture about each fixed point. Because this system is uncoupled we can actually deduce the global phase portrait using some extra observations. When $y=0$ the velocity is $\dot{\boldsymbol{x}} = (-x+x^3, 0)$. Since this is tangent to $y=0$, the line $y=0$ is an invariant straight line: the flow stays along this line. When $-x+x^3=0$ we have $\dot{\boldsymbol{x}} = (0,-2y)$, so the three lines $x=0$, $x=\pm 1$ are also invariant straight lines. Together with the dynamics near the fixed points, we deduce the phase portrait in Fig. 22.

Note: neither the linearisation theorem nor its corollary tell us anything about what happens near non-hyperbolic fixed points. The next example demonstrates that linear stability analysis can fail for non- hyperbolic fixed points.

Example 3.6. Consider the one parameter family of systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y+ax(x^2+y^2) \\ x+ay(x^2+y^2) \end{pmatrix}. \tag{3.4}$$

where a is a constant (i.e. a parameter independent of x, y).It has one fixed point, at $\mathbf{x}^* = (0,0)$. The Jacobi matrix is

$$J(\boldsymbol{x}) = \begin{pmatrix} 3ax^2+ay^2) & -1+2axy \\ 1+2axy & 3ay^2+ax^2 \end{pmatrix} \quad \Rightarrow \quad J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This has $\tau = 0$ and $\Delta = 1$, so $(0,0)$ is NOT a hyperbolic fixed point. In fact, this matrix corresponds to a linear system of centre type. When $a=0$ the system is linear, so we do indeed get a centre, but when $a \neq 0$ we can show that this is no longer true, and the behaviour is not predicted by the linearisation.

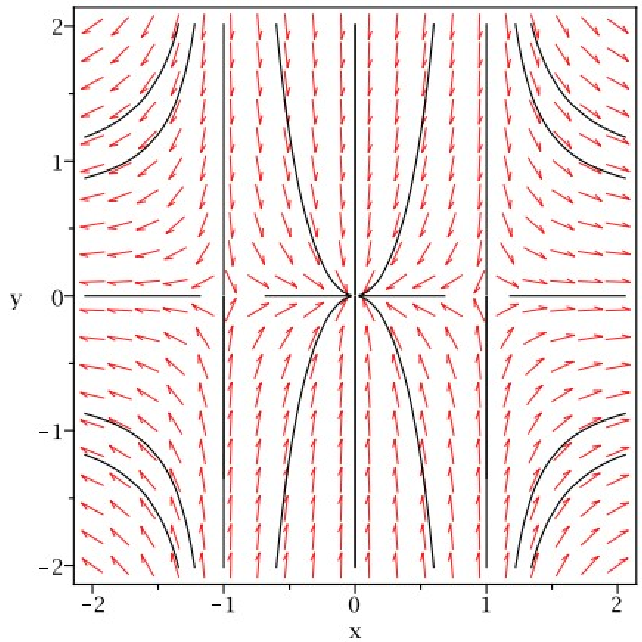

Figure 22 The phase portrait for example 3.5.

To analyse this system, it is convenient to use polar coordinates (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$, so that $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$. It is easy to show that

$$r\dot{r} = x\dot{x} + y\dot{y}, \quad r^2\dot{\theta} = x\dot{y} - y\dot{x}.$$

Therefore in polar coordinates Eq. (3.4) becomes

$$\dot{r} = ar^3, \quad \dot{\theta} = 1.$$

Hence for $a < 0$ we obtain an attractor at the origin, while $a > 0$ yields a repellor. See Figure 23.

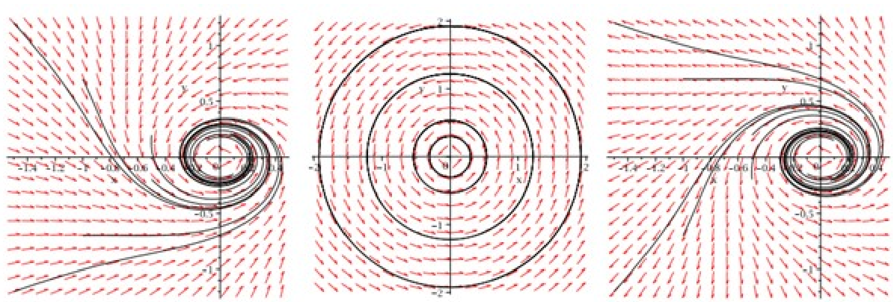

Figure 23 The three cases $a < 0$, $a = 0$ and $a > 0$ for system (3.4)

.

Note carefully the full lesson of this example. It does not say the behaviour near a non-hyperbolic fixed point is not the same as the linearisation (that is clearly false for linear systems): it says that for a non-hyperbolic fixed point the linearisation theorem cannot be applied.

3.4 Structural Stability

The example 3.6 above also helps to illustrate an- other important difference between hyperbolic and non-hyperbolic fixed points. In that example the dynamics of the system for $a = 0$ is utterly different than the dynamics for any non-zero value near $a = 0$: at $a = 0$ we have a Lyapunov stable fixed point, while $a \neq 0$ gives either a stable or unstable fixed point. In general non-hyperbolic fixed points have the property that a small change in parameters of the system can change the stability of the fixed point. This is called structural instability. By contrast, hyperbolic fixed points are structurally stable: their local dynamics is not qualitatively changed by suitably small changes of parameters of the system (and they remain hyperbolic).

More generally, we say a whole dynamical system is structurally stable if all sufficiently small perturbations of its parameters do not change the shape of the phase portrait: otherwise we say it is structurally unstable. In particular, a necessary condition for structural stability is that all the fixed points are hyperbolic[4].

3.5 Features of interest

As well as the fixed points and their linearisation there are other features of interest that we will define here.

- *Nullclines.* Nullclines are the curves given by $\dot{x} = 0$ or $\dot{y} = 0$ and hence $f(x, y) = 0$ or $g(x, y) = 0$ respectively. Nullclines are related to fixed points as follows.
- *Fixed points.* Fixed points are points given by $\dot{x} = 0$ and $\dot{y} = 0$ and hence $f(x, y) = 0$ and $g(x, y) = 0$. This is equivalent to finding the intersections of the two nullclines.
- It is important to remember that even where the linearisation theorem fails, and not all fixed points are asymptotically stable, we may still be able to construct a qualitatively accurate phase portrait. Don't stop!
- *Principal directions.* The principle directions of a saddle are directions where trajectories move purely away or purely towards the fixed point. They are given by the eigenvectors of the Jacobi matrix.
- *Separatrix.* A Separatrix is a division between two different areas of phase space that have different behaviours and/or different end points for trajectories. Note that whilst a separatrix often includes a principal direction of a saddle it is not usually analytically calculable away from the fixed point.
- *Heteroclinic connection.* A heteroclinic connection is a trajectory connecting two fixed points.
- *Homoclinic connection.* A homoclinic connection is a loop in phase space that links a fixed point to itself.

[4]This is not always sufficient, because other features (e.g. closed orbits) can change even when the fixed points remain hyperbolic.

3.6 Drawing a phase portrait

Here is the checklist for drawing a phase portrait:

- Classify all the fixed points. This is a common cause of errors. Make sure you have all of them. Two more complex errors can occur - fixed points not on the physical plane can still significantly affect the system. Secondly, where nullcline limits are more complicated it is easy to over count fixed points, take care!
- Draw the nullclines. As the fixed points are on the intersections, this gives you valuable extra information to make sure your plot is accurate and you have found all the fixed points.
- Mark on the known directions of the trajectories around the fixed points. Where more detail is required also compute the principal directions of the relevant fixed points. The linearisation theorem completely classifies the behaviour in a small region around the hyperbolic fixed points, but not beyond.
- Mark on the direction of trajectories across the nullclines. It is useful to compute asymptotic behaviour of the original system to do this. Remember - the sign of the direction of trajectories along nullclines only changes at fxed points, analogous to 1D.
- Mark on key other features: e.g. separatrices and heteroclinic connections , where these exist and can be computed.
- Finally, add arrows, consistent with all the above features, that indicate the direction of flow. At a minimum in each section of the plot, as divided up by the nullclines, there should be an indication of the direction of flow.

Remember, **draw big, and draw clearly**. It often helps to draw the sim- ple, key features in pen (axes, fixed points, nullclines if brave) and then add trajectories in pencil. You must include all relevant information on the plot.

Example 3.7. Let's draw the phase portrait of the following system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x - \gamma xy \\ \gamma xy - \delta y \end{pmatrix}. \tag{3.5}$$

This is a particular case of the Lotka-Volterra predator-prey model. In (3.5), $x(t)$ is the population of prey, $y(t)$ is the population of predators, γ and δ are positive parameters. This model make sense only for $x, y \geq 0$. So, we restrict our analysis to the positive quadrant of the (x, y) plane.

- First, we find and sketch nullclines of the system:

$$\begin{aligned} &x-\text{nullclines: } x - \gamma xy = 0 \quad \Rightarrow \quad x = 0 \ \text{ or } \ y = 1/\gamma \ \ \text{(green curves in Fig. 24)} \\ &y-\text{nullclines: } x - \gamma xy = 0 \quad \Rightarrow \quad y = 0 \ \text{ or } \ x = \delta/\gamma \ \ \text{(red curves in Fig. 24)} \end{aligned}$$

Also, we can determine the direction of the flow on the nullclines:

$$
\begin{aligned}
&\text{if } x = 0,\ \dot{y} = -\delta y < 0 \text{ for } y > 0;\\
&\text{if } y = 1/\gamma,\ \dot{y} = x - \delta/\gamma \ \Rightarrow\ \dot{y} > 0 \text{ for } 0 < x < \delta/\gamma,\ \dot{(y)} < 0 \text{ for } x > \delta/\gamma\\
&\text{if } y = 0,\ \dot{x} = x > 0 \text{ for } x > 0;\\
&\text{if } x = \delta/\gamma,\ \dot{x} = (1-\gamma y)\delta/\gamma \ \Rightarrow\ \dot{x} > 0 \text{ for } 0 < y < 1/\gamma,\ \dot{(y)} < 0 \text{ for } y > 1/\gamma
\end{aligned}
$$

The direction of the flow in the nullclines is shown by green and red arrows in Fig. 24.

- Second, we find the fixed points. Fixed points are intersection points of the x- and y-nullclines (see Fig. 24). There are two fixed points: $\mathbf{x}^* = (0,0)$ and $\mathbf{x}^* = (\delta/\gamma, 1/\gamma)$.

- Third, we use linear stability analysis to determine the type of each fixed points. The Jacobi matrix for this system is

$$
J(\boldsymbol{x}) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 1-\gamma y & -\gamma x \\ \gamma y & \gamma x - \delta \end{pmatrix}
$$

For $\mathbf{x}^* = (0,0)$, we obtain

$$
J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -\delta \end{pmatrix} \quad \Rightarrow \quad \lambda^+ = 1,\ \lambda^- = -\delta \ \text{ and } \ \boldsymbol{v}^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \boldsymbol{v}^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

Hence, $(0,0)$ is a saddle point with the unstable direction along the x-axis and the stable direction along the y-axis (we already know that from the direction of flow on the corresponding nullclines).

For $\mathbf{x}^* = (\delta/\gamma, 1/\gamma)$, we obtain

$$
J(\delta/\gamma, 1/\gamma) = \begin{pmatrix} 0 & -\delta \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \lambda^{\pm} = \pm i\sqrt{\delta} \quad \Rightarrow \quad \mathbf{x}^* = (\delta/\gamma, 1/\gamma) \text{ is a centre}
$$

- Fourth, we draw the phase portrait using the information obtained above. The result is shown in Fig. 24

Example 3.8. Consider the following system:

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(3-x-2y) \\ y(2-x-y) \end{pmatrix}. \tag{3.6}
$$

[This example is taken from Strogatz. It is a competition model for two species. For more details, see Section 6.4 in Strogatz.] Again, we restrict our analysis to the positive quadrant of the (x,y) plane.

- First, we find and sketch nullclines of the system:

$$
\begin{aligned}
&x-\text{nullclines: } x(3-x-2y) = 0 \quad \Rightarrow \quad x = 0 \ \text{ or } \ x = 3-2y \ \text{ (green curves in Fig. 25)}\\
&y-\text{nullclines: } x(2-x-y) = 0 \quad \Rightarrow \quad y = 0 \ \text{ or } \ y = 2-x \ \text{ (red curves in Fig. 25)}
\end{aligned}
$$

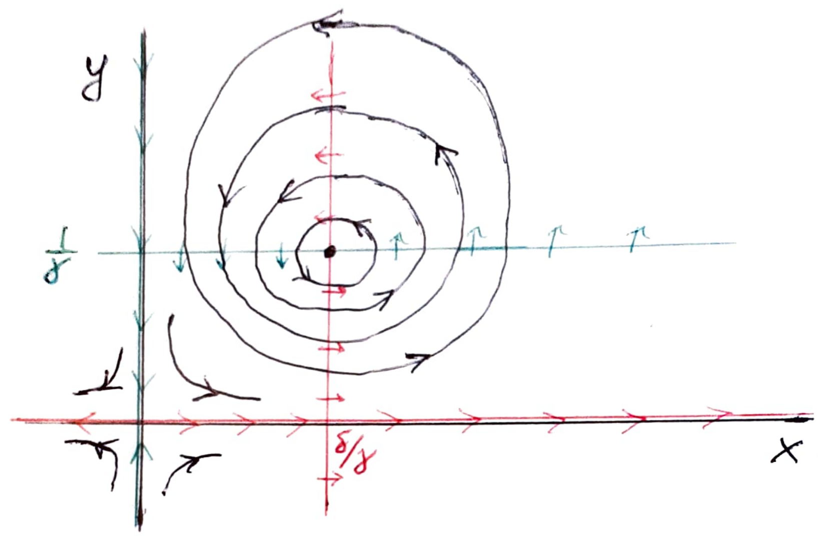

Figure 24 Phase portrait of the Lotka-Volterra model.

Also, we can determine the direction of the flow on the nullclines:

$$
\begin{aligned}
&\text{if } x = 0,\ \dot{y} = y(2-y) > 0 \text{ for } 0 < y < 2 \text{ and } \dot{y} < 0 \text{ for } y > 2;\\
&\text{if } x = 3-2y,\ \dot{y} = y(y-1) > 0 \text{ for } y > 1 \text{ and } \dot{y} < 0 \text{ for } 0 < y < 1;\\
&\text{if } y = 0,\ \dot{x} = x(3-x) > 0 \text{ for } 0 < x < 3 \text{ and } \dot{x} < 0 \text{ for } x > 3;\\
&\text{if } y = 2-x,\ \dot{x} = x(x-1) > 0 \text{ for } x > 1 \text{ and } \dot{x} < 0 \text{ for } 0 < x < 1.
\end{aligned}
$$

The direction of the flow in the nullclines is shown by green and red arrows in Fig. 25.

- Second, we find the fixed points. Fixed points are intersection points of the x- and y-nullclines (see Fig. 25). There are two fixed points: $\mathbf{x}^* = (0,0)$, $\mathbf{x}^* = (0,2)$, $\mathbf{x}^* = (1,1)$ and $\mathbf{x}^* = (3,0)$.

- Third, we use linear stability analysis to determine the type of each fixed points. The Jacobi matrix for this system is

$$
J(\boldsymbol{x}) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 3-2y-2x & -2x \\ -y & \gamma 2 - x - 2y \end{pmatrix}
$$

For $\mathbf{x}^* = (0,0)$, we obtain

$$
J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \quad \Rightarrow \quad \lambda^+ = 3,\ \lambda^- = 2 \ \text{ and } \ \boldsymbol{v}^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \boldsymbol{v}^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

Hence, $(0,0)$ is an unstable node the unstable direct along the y-axis (which is the slow direction of the node).

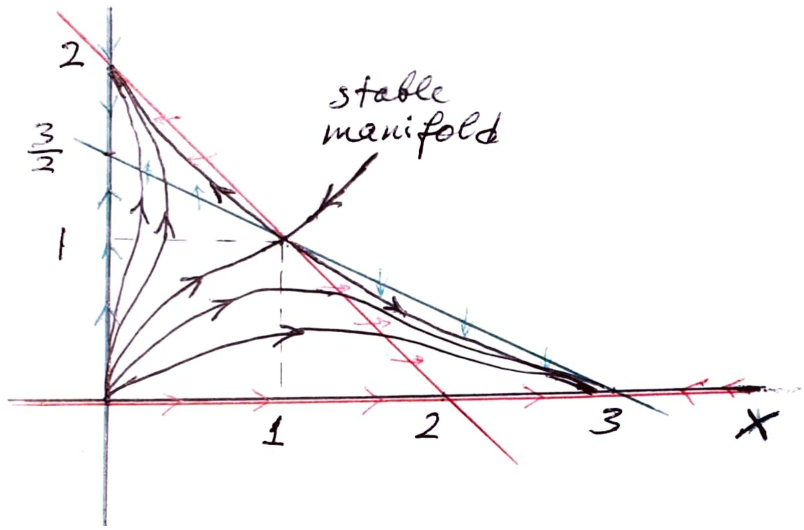

Figure 25 Phase portrait of the system in example 3.8.

For $\mathbf{x}^* = (0, 2)$, we obtain

$$J(0,0) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \quad \Rightarrow \quad \lambda^+ = -1,\ \lambda^- = -2 \ \text{ and } \ \boldsymbol{v}^+ = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Hence, $(0, 2)$ is a stable node with the slow direction parallel to $\boldsymbol{v}^+ = (1, -2)$.

For $\mathbf{x}^* = (1, 1)$, we obtain

$$J(1,1) = \begin{pmatrix} -\sqrt{2} & -2 \\ -1 & -\sqrt{2} \end{pmatrix} \quad \Rightarrow \quad \lambda^\pm = -1 \pm \sqrt{2} \quad \Rightarrow \quad \mathbf{x}^* = (1,1) \text{ is a saddle point}$$

To produce a correct sketch, we need to determine stable and unstable directions of the saddle. So, let's find eigenvectors. For λ^+, we have

$$(J(1,1) - \lambda^+ I)\boldsymbol{v}^+ = \begin{pmatrix} -\sqrt{2} & -2 \\ -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1^+ \\ v_2^+ \end{pmatrix} \quad \Rightarrow \quad -v_1^+ - \sqrt{2} v_2^+ = 0 \quad \Rightarrow \quad \begin{pmatrix} v_1^+ \\ v_2^+ \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

Similarly, for λ^-, we obtain

$$\begin{pmatrix} \sqrt{2} & -2 \\ -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} v_1^- \\ v_2^- \end{pmatrix} \quad \Rightarrow \quad -v_1^- + \sqrt{2} v_2^- = 0 \quad \Rightarrow \quad \begin{pmatrix} v_1^- \\ v_2^- \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}.$$

Finally, for $\mathbf{x}^* = (3, 0)$, we have

$$J(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1\delta \end{pmatrix} \quad \Rightarrow \quad \lambda^+ = -1,\ \lambda^- = -3 \ \text{ and } \ \boldsymbol{v}^+ = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Hence, $(3, 0)$ is a stable node with the slow direction parallel to $\boldsymbol{v}^+ = (3, -1)$.

- Fourth, we draw the phase portrait using the information obtained above. The result is shown in Fig. 25 (for a nicer picture, see Strogatz, Figure 6.4.6 on p. 158).