- The real number system is a set of numbers with:
1) Two special numbers, 0 and 1 called the additive and multiplicative identities. 2) Two binary operations of addition and multiplication. 3) Two inverse "unary" operations - and $.^{-1}$. 4) An order relation "$<$"

- Axiom of completeness:
Suppose A and B are non-empty subsets of $\mathbb{R}$ with the property that if $a \in A$ and $b \in B$ then $a \leq b$. Then there exists $c \in \mathbb{R}$ such that for all $a \in A$ and $b \in B$, $a \leq c \leq b$.

- $L(f,P) = \sum_{n=1}^{N}(x_n - x_{n-1}) \inf_{x\in[x_{n-1},x_n]} f(x)$
$U(f,P) = \sum_{n=1}^{N}(x_n - x_{n-1}) \sup_{x\in[x_{n-1},x_n]} f(x)$

- Principle of Bounded Monotone Convergence:
If $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence which is bounded above, then as $n \to \infty$ it converges to $\sup\{a_n : n \in \mathbb{N}\}$.
If $(a_n)_{n\in\mathbb{N}}$ is an decreasing sequence which is bounded below, then as $n \to \infty$ it converges to $\inf\{a_n : n \in \mathbb{N}\}$

- Uniqueness of Limits:
Suppose $S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$, $x_0 \in \mathbb{R}$ and as $x \to x_0$, we have $f(x) \to L$ and $f(x) \to M$. We would expect to find that $L = M$: that limits are unique as they are for sequences.

- Thm 3.3: $\nexists x \in \mathbb{Q}$ s.t. $x^2 = 2$

- Thm 3.17: a non empty set I is an interval iff it has the intermediate value property: if $x \in I$ and $z \in I$ and $x < y < z$ then $y \in I$.

- Density of rationals: Suppose $a, b \in \mathbb{R}$ with $a < b$. Then $\exists p/q \in \mathbb{Q}$ with $a < p/q < b$

- Existence of Predecessors: If $n \in \mathbb{N}$ then either $n = 1$ or $n - 1 \in \mathbb{N}$

- Well ordering principle: Every non-empty subset of $\mathbb{N}$ has a minimal element.

- Thm 5.12: If there exists $N \in \mathbb{N}$ s.t. $a_n \geq b$ for $n > N$ then $a \geq b$
pf: For any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ s.t. if $n > N_\varepsilon$ then $|a_n - a| < \varepsilon$ or $a - \varepsilon < a_n < a + \varepsilon$. For $n > \max(N, N_\varepsilon)$ we also have $a_n \geq b$. It follows that $b < a + \varepsilon$ for all $\varepsilon > 0$ and so $b \leq a$.

- Triangle inequality: $|x + y| \leq |x| + |y|$
Reverse triangle inequality: $||x| - |y|| \leq |x - y|$

- Algebra of limis/ Combination Rules: If a sequence tends to something then a combination of sequences tends to a combination of limits.

- Hierarchy of limits: fast→slow as $n \to \infty$
n^n, $n!$, x^n $(x > 1)$, n^q $(q \in \mathbb{Q} > 0) \to \infty$
n^{-n}, $1/n!$, x^n $(|x| < 1)$, n^{-q} $(q \in \mathbb{Q} > 0) \to 0$

- Limit comparison test:
Suppose $(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ are strictly positive sequences and a_j/b_j converges to L as $j \to \infty$. Then:
1) If $\sum b_j$ converges then $\sum a_j$ converges
2) If $L > 0$ then $\sum a_j$ and $\sum b_j$ either both converge or both diverge.

- Cauchy Criterion, Cauchy's general Principle of Convergence: Every convergent sequence is a Cauchy sequence. Every Cauchy sequence converges, so the Cauchy property is equivalent to convergence,

- Leibniz alternating series test:
Suppose $(a_j)_{j\in\mathbb{N}}$ is a decreasing sequence tending to zero. Then $\sum_{j=1}^{\infty}(-1)^{j+1}a_j$ converges.

- Cauchy's Condensation Test:
Suppose $(a_j)_{j\in\mathbb{N}}$ is a decreasing sequence of non-negative terms. Then the following are equivalent:
1) $\sum_{j=1}^{\infty} a_j$ converges; 2) $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

- Absolute convergence implies convergence; the converse is not true.

- Comparison and limit comparison test for signed terms:
Comparison test: if $|a_j| \leq b_j$ for all $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} b_j$ converges, then $\sum_{j=1}^{\infty} a_j$ converges absolutely and $\left|\sum_{j=1}^{\infty} a_j\right| \leq \sum_{j=1}^{\infty} |a_j| \leq \sum_{j=1}^{\infty} b_j$
Limit comparison test: if $b_j > 0$, $|a_j/b_j| \to L \in \mathbb{R}$ as $j \to \infty$ and $\sum_{j=1}^{\infty} b_j$ converges, then $\sum_{j=1}^{\infty} a_j$ converges absolutely.

- Ratio test: suppose a sequence of non-zero terms such that $|a_{j+1}/a_j| \to r$ as $j \to \infty$. Then $\sum_{j=1}^{\infty} a_j$ either converges absolutely $(r < 1)$; diverges $(r > 1)$ or we don't know $(r = 1)$.

- Thm 19.6: $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is continuous and converges on $(x_0 - R, x_0 + R)$

- Bolzano's Thm: suppose $a < b$, $g : [a,b] \to \mathbb{R}$ is continuous and $g(a)$ and $g(b)$ have opposite signs. Then $\exists x_0 \in (a,b)$ s.t. $g(x_0) = 0$.

- Rolle's Thm: Suppose $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous and differentiable on (a,b), and s.t. $g(a) = g(b)$. Then $\exists x_0 \in (a,b)$ s.t. $g'(x_0) = 0$

- MVT: $\exists x_0 \in (a,b)$ s.t. $f'(x_0) = \frac{f(b)-f(a)}{b-a}$

- Taylor: $f(x_0 + h) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}h^n + \frac{f^{(N+1)}(c)}{(N+1)!}h^{N+1}$

- $\exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

- Thm 29.6: Any partitions P & Q of an interval have a common refinement. eg: make a new partition R that contains all the points of P & Q discarding duplicates.

- if P & Q are partitions of $[a,b]$ and P is a refinement of Q then: $L(f,P) \geq L(f,Q)$ and $U(f,P) \leq U(f,Q)$.

- Cauchy Criterion for Riemann Integrability: A function is integrable iff for any $h > 0$ $\exists P$ such that $U(f,P) - L(f,P) < h$

- FTC 1: $F(x) = \int_{p\in[a,b]}^{x} f$

- FTC 2: $\int_a^b = \int_a^b F' = F(b) - F(a)$

- Use the property of Archimedes to show that if: $S_1 = \left\{\frac{2n+1}{n+1} : n \in \mathbb{N}\right\}$, Then $\sup(S_1) = 2$
Soln: We have for $n \in \mathbb{N}$, $(2n+1)/(n+1) < (2n+2)/(n+1) = 2$, showing that 2 is an upper bound for S_1.
To show that 2 is the least upper bound, it is enough to show that any number $2 - h < 2$ is not an upper bound, i.e. that there exists $n \in \mathbb{N}$ s.t.
$\frac{2n+1}{n+1} > 2 - h \iff 2n+1 > 2n+2-(n+1)h \iff (n+1)h > 1 \iff n > \frac{1}{h} - 1$
Such n exists by Archimedes (otherwise $1/h - 1$ would be an upper bound for $\mathbb{N}$); all steps in the calculation being "if and only if", we conclude that $2 - h < 2$ is not an upper bound for S_1, and hence that 2 is the least upper bound for S_1, i.e. that $\sup(S_1) = 2$

- Use Archimedes to show if: $S_2 = \left\{1 - \frac{n}{2} : n \in \mathbb{N}\right\}$ that S_2 is not bounded below.
Suppose for contradiction that S_2 is bounded below. Then $\exists a \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $a \leq 1 - n/2$. Rearranging, $n \leq 2(1-a)$, $\forall n \in \mathbb{N}$ contradicting Archimedes.

- let $a_n = 1 - \frac{(-1)^n}{n}$ $(n \in \mathbb{N})$
Show directly from the definition of convergence of a sequence that as $n \to \infty$, $a_n \to 1$
Soln: To show that $1 - (-1)^n/n \to 1$ as $n \to \infty$, we need to lead up to the inequality $|1-(-1)^n/n-1| < \varepsilon$, or equivalently $1/n < \varepsilon$. So, given $\varepsilon > 0$ we choose $N_\varepsilon \in \mathbb{N}$ s.t. $N_\varepsilon \geq 1/\varepsilon$ (Archimedes). Now, if $n > N_\varepsilon$ then $1/n > \varepsilon$; as seen above, this is equivalent to $|a_n - 1| < \varepsilon$. We conclude that $a_n \to 1$ as $n \to \infty$

- let $b_n = 1 - (-1)^n$ $(n \in \mathbb{N})$ show directly from the definition of convergence that b_n has no limit as $n \to \infty$.
Soln: Note that, for even n, $b_n = 0$ and for odd, $b_n = 2$. If $b_n \to b$ as $n \to \infty$ then, because there are odd and even numbers greater than any N_ϵ, b would satisfy both $|b| < \varepsilon$ and $|b-2| < \varepsilon$. Equivalently, $-\varepsilon < b < \varepsilon$ and $2 - \varepsilon < b < 2 + \varepsilon$. But, if $\varepsilon = 1$ the first of these gives $b < 1$ and the second $b > 1$; this contradiction shows there is no limit b.

- Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{n!}{(2n)!}(x-1)^n$
$\left|\frac{((n+1)!)(x-1)^{n+1}}{(2n+2)!} \frac{(2n)!}{n!(x-1)^n}\right| = \frac{n+1}{(2n+2)(2n+1)}|x-1| = \frac{1}{2(2n+1)}|x-1| \to 0 < 1$ as $n \to \infty$. We thus have convergence $\forall x \in \mathbb{R}$ so $R = \infty$.

- Define $f : [-1,1] \to \mathbb{R}$; $f(x) = |x|$ (a) Consider the partition P of the interval given by $P = (-1, 0, 1/2, 1)$. Find the lower and upper sums $L(f,P)$ and $U(f,P)$.
Soln: Subintervals: $[-1,0]$, $[0,1/2]$, $[1/2,1]$ have widths 1, 1/2, 1/2 and infimums 0, 0, 1/2 and supremums 1, 1/2, 1 respectively. So the lower and upper sums are given by:
$L(f,P) = 1 \times 0 + (1/2) \times 0 + (1/2) \times (1/2) = 1/4$
$U(f,P) = 1 \times 1 + (1/2) \times (1/2) + (1/2) \times 1 = 7/4$

- Let P_N be the partition formed by dividing $[-1,1]$ into $2N$ subintervals of equal width. Given that $L(f,P_N) = 1 - 1/N$ and $U(f,P_N) = 1 + 1/N$, show directly from the definition of the integral, and the basic fact that $L_{-1}^{1} f \leq U_{-1}^{1} f$, that f is integrable on $[-1,1]$. Find $\int_{-1}^{1} f$.
Soln: We are given that $L(f,P_N) = 1 - 1/N \to 1$ as $N \to \infty$. Since this is an increasing sequence, it converges to $\sup\{L(f,P_N) : N \in \mathbb{N}\} \leq L_{-1}^{1} f$; combining these, we have $1 \leq L_{-1}^{1} f$. Similarly, we are given that $U(f,P_N) = 1 + 1/N \to 1$ as $N \to \infty$. Since this is a decreasing sequence, it converges to $\inf\{U(f,P_N) : N \in \mathbb{N}\} \geq U_{-1}^{1} f$; combining these we have, $1 \geq U_{-1}^{1} f$. Now using $L_{-1}^{1} f \leq U_{-1}^{1} f$, we have $1 \leq L_{-1}^{1} f \leq U_{-1}^{1} f \leq 1$ showing that the lower and upper integrals are both 1, so the function is integrable, with integral 1.

- Suppose A is a bounded, non-empty set of real numbers and let $B = \{x - y : x, y \in A\}$, Show that $\sup(B) = \sup(A) - \inf(A)$.
Soln: Suppose $z \in B$, so $z = x - y$ for some $x, y \in A$. Then $x \leq \sup(A)$ and $y \geq \inf(A)$, so $z = x - y \leq \sup(A) - \inf(A)$. This shows that $\sup(A) - \inf(A)$ is an upper bound for B. If $h > 0$ then $\sup(A) - h/2$ is not an upper bound for A, so $\exists x \in A$ with $x > \sup(A) - h/2$. Similarly, $\exists y \in A$ with $y < \inf(A) + h/2$. Subtracting, we find that $B \ni x - y > \sup(A) - \inf(A) - h$, showing that $\sup(A) - \inf(A) - h$ is not an upper bound for B. Combining these two, we see that $\sup(A) - \inf(A)$ is the least upper bound for B, and so $\sup(B) = \sup(A) - \inf(A)$

- let $f(x) = \frac{x^4+3}{4}$ let $x_1 = 2$ and (for $n \in \mathbb{N}$) let $x_{n+1} = f(x_n)$. Show that $(x_n)_{n\in\mathbb{N}}$ is a bounded monotonic sequence, and find it's limit.
Soln: Since $1 < x_1 < 3$, it follows that by (a) and induction that $1 < x_n < 3$ for all n, showing that $(x_n)_{n\in\mathbb{N}}$ is bounded. We also have from (a) that $f(x_n) - x_n < 0$, i.e. that $x_{n+1} < x_n$, showing that $(x_n)_{n\in\mathbb{N}}$ is strictly decreasing. Now, by the principle of bounded monotonic convergence, $(x_n)_{n\in\mathbb{N}}$ is convergent, say $x_n \to x$ as $n \to \infty$. Now on one hand, $x_{n+1} \to x$ as $n \to \infty$. But on the other hand, $x_{n+1} = (x_n^2 + 3)/4 \to (x^2+3)/4$ as $n \to \infty$. By uniqueness of limits, $x = (x^2+3)/4$. Finally we can solve this equation to give $x = 1$ or $x = 3$; but $x_1 = 2$ and $(x_n)_{n\in\mathbb{N}}$ is decreasing, so $x = 3$ is impossible and we must have $x = 1$.

- Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies: $|f(x)| \leq C|x|$ $\forall x \in \mathbb{R}$ and $C \geq 0$. use the "$\varepsilon - \delta$" definition of continuity to show that f is continuous at 0.
Soln: The inequality shows that $|f(0)| \leq 0$ so $f(0) = 0$. Thus, $|f(x) - f(0)| = |f(x)| \leq C|x|$. If $C \neq 0$ then, given $\varepsilon > 0$, we can let $\delta_\varepsilon = \varepsilon/C$; if $|x - 0| < \delta_\varepsilon$ then $|x| < \varepsilon/C$ so $|f(x) - f(0)| = |f(x)| \leq C\varepsilon/C = \varepsilon$ Which shows that f continuous at 0. If $C = 0$ then $f(x) = 0\,\forall x$ so f trivially continuous: any δ_ε works.