Here's the math behind this simulator. (heavily abbreviated).
TODO all the other math

# 1 Universal Variable Formulation

## 1.1 Regularizing the Equation of Motion

The equation of motion for our satellite is

$$\ddot{\boldsymbol{x}} + \frac{\mu}{r^3}\boldsymbol{x} = 0 \tag{1.1.1}$$

We introduce a new variable $s$ such that $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{r}$. Since dots are being used to represent derivatives with respect to $t$, we'll use primes to represent derivatives with respect to $s$. From the chain rule, we can derive the following relations for any state function $f$, which will be used repeatedly:

$$\dot{f} = \frac{1}{r}f' \qquad \ddot{f} = \frac{rf'' - f'r'}{r^3} \tag{1.1.2}$$

Using this, we rewrite (1.1.1) in terms of $s$:

$$\boldsymbol{x}'' - \frac{r'}{r}\boldsymbol{x}' + \frac{\mu}{r}\boldsymbol{x} = 0 \tag{1.1.3}$$

This can be further simplified by introducing some conserved physical quantities. In particular, consider the eccentricity vector and the energy:

$$e = \left(\frac{v^2}{\mu} - \frac{1}{r}\right)\boldsymbol{x} - \frac{\boldsymbol{x}\cdot\boldsymbol{v}}{\mu}\boldsymbol{v} \qquad E = \frac{v^2}{2} - \frac{\mu}{r} \tag{1.1.4}$$

It's conventional to use $\beta = -2E$ instead of the energy itself, for reasons that will be clear soon. Substituting $\beta$ into the first term, and using the identity $\boldsymbol{x}\cdot\dot{\boldsymbol{x}} = r\dot{r}$, we get the following expression:

$$\mu\boldsymbol{e} = \left(\frac{\mu}{r} - \beta\right)\boldsymbol{x} - r\dot{r}\boldsymbol{v} = \left(\frac{\mu}{r} - \beta\right)\boldsymbol{x} - \frac{r'}{r}\boldsymbol{x}' \tag{1.1.5}$$

This contains an $(r'/r)\boldsymbol{x}'$ term, which, when substituted it back into (1.1.3), gives a particularly simple differential equation:

$$\boldsymbol{x}'' + \beta\boldsymbol{x} = -\mu\boldsymbol{e} \tag{1.1.6}$$

## 1.2 $f$ and $g$ Functions

Since $\boldsymbol{x}$ is a vector, we want to pick a basis to represent it in. Since the initial position (we'll denote it $\boldsymbol{x}_0$) and the initial velocity ($\boldsymbol{v}_0$) determine the orbital plane, we'll use those two vectors. There are then some functions $f(s)$ and $g(s)$ such that

$$\boldsymbol{x}(s) = f(s)\boldsymbol{x}_0 + g(s)\boldsymbol{v}_0 \tag{1.2.1}$$

Conveniently, we also know the coefficients of $\mu \boldsymbol{e}$ in this basis, from (1.1.5). Applying this basis to (1.1.6)

$$
\begin{aligned}
\boldsymbol{x}'' + \beta \boldsymbol{x} &= -\mu \boldsymbol{e} \\
(f''\boldsymbol{x}_0 + g''\boldsymbol{v}_0) + (\beta f \boldsymbol{x}_0 + \beta g \boldsymbol{v}_0) &= -\left[\left(\frac{\mu}{r_0} - \beta\right)\boldsymbol{x}_0 - r_0\dot{r}_0\boldsymbol{v}_0\right] \\
(f'' + \beta f)\boldsymbol{x}_0 + (g'' + \beta g)\boldsymbol{v}_0 &= \left(\beta - \frac{\mu}{r_0}\right)\boldsymbol{x}_0 + r_0\dot{r}_0\boldsymbol{v}_0
\end{aligned}
$$

Since $\boldsymbol{x}_0$ and $\boldsymbol{v}_0$ are independent (in the general case), we can separately equate the coefficients on each basis vector:

$$
f'' + \beta f = \beta - \frac{\mu}{r_0} \qquad g'' + \beta g = r_0\dot{r}_0 \tag{1.2.2}
$$

We now have two uncoupled, second-order, linear differential equations. Solving these would give us the solution for $\boldsymbol{x}$, parameterized by $s$.

## 1.3 Stumpff Functions

To solve the inhomogeneous diffeqs in (1.2.2), we first have to solve the associated homogeneous equations. Fortunately, this equation has well-known solutions. Unfortunately, we have to separate out cases depending on the sign of $\beta$:

$$
f(s) = g(s) = \begin{cases} A\cos(\sqrt{\beta}s) + B\sin(\sqrt{\beta}s) & \beta > 0 \\ A\cosh(\sqrt{-\beta}s) + B\sinh(\sqrt{-\beta}s) & \beta < 0 \\ A + Bs & \beta = 0 \end{cases} \tag{1.3.1}
$$

We'd like to unify these cases, so that our universal variables are truly universal. We can do so with the help of *Stumpff functions*. The $k$th Stumpff function is defined by the following series, which converges absolutely for all $x$:

$$
c_k(x) = \frac{1}{k!} - \frac{x}{(k+2)!} + \frac{x^2}{(k+4)!} - \frac{x^3}{(k+6)!} + \cdots = \sum_{i=0}^{\infty} \frac{(-x)^i}{(k+2i)!} \tag{1.3.2}
$$

With some thought, the series for $c_0$ and $c_1$ can be recognized, and are expressible in a nice-ish form. Furthermore, there's a recurrence relation between $c_k$ and $c_{k+2}$:

$$
c_0(x) = \begin{cases} \cos(\sqrt{x}) & x > 0 \\ \cosh(\sqrt{-x}) & x < 0 \\ 1 & x = 0 \end{cases} \tag{1.3.3}
$$

$$
c_1(x) = \begin{cases} \sin(\sqrt{x})/\sqrt{x} & x > 0 \\ \sinh(\sqrt{-x})/\sqrt{-x} & x < 0 \\ 1 & x = 0 \end{cases} \tag{1.3.4}
$$

$$xc_{k+2}(x) = \frac{1}{k!} - c_k(x) \tag{1.3.5}$$

Note that $c_0$ and $c_1$ are are quite similar to the solutions to our diffeq. In order to get the actual solutions, we introduce the two-variable functions $G_k(\beta, s)$:

$$G_k(\beta, s) = s^k c_k(\beta s^2) \tag{1.3.6}$$

Like the $c_k$, these are smooth functions, defined everywhere. Expanding the definition of $G_k$, we see that $G_0$ and $G_1$ form a basis for the solution space of $f'' + \beta f = 0$ (compare with (1.3.1)).

$$G_0(\beta, s) = c_0(\beta s^2) = \begin{cases} \cos(\sqrt{\beta} s) & \beta > 0 \\ \cosh(\sqrt{-\beta} s) & \beta < 0 \\ 1 & \beta = 0 \end{cases} \tag{1.3.7}$$

$$G_1(\beta, s) = s c_1(\beta s^2) = \begin{cases} \sin(\sqrt{\beta} s)/\sqrt{\beta} & \beta > 0 \\ \sinh(\sqrt{-\beta} s)/\sqrt{-\beta} & \beta < 0 \\ s & \beta = 0 \end{cases} \tag{1.3.8}$$

The $G_k$ have two more nice properties, which will be useful later.

$$\frac{\mathrm{d}}{\mathrm{d}s} G_{k+1} = G_k \qquad G_k(\beta, 0) = 0 \text{ for } k > 0 \tag{1.3.9}$$

*Aside: This may seem like sweeping the casework under the rug, but it isn't. The reason that this is acceptable is that $G_k(\beta, s)$ is an analytic function in both variables, and varying $\beta$ lets us smoothly transition between the cases. The casework here is only about how we* ***choose to express it*** *in elementary functions.*

## 1.4 Solving for Initial Conditions

Now that we have the solution set for the associated homogeneous diffeq of (1.2.2), we need to find any solution to the inhomogeneous equation. Conveniently, constants will work here:

$$f(s) = 1 - \frac{\mu}{\beta r_0} \qquad g(s) = \frac{r_0 \dot{r}_0}{\beta} \tag{1.4.1}$$

So our solutions for $f$ and $g$ are of the form:

$$f(s) = 1 - \frac{\mu}{\beta r_0} + A G_0(\beta, s) + B G_1(\beta, s) \tag{1.4.2}$$

$$g(s) = \frac{r_0 \dot{r}_0}{\beta} + C G_0(\beta, s) + D G_1(\beta, s) \tag{1.4.3}$$

for some constants $A$, $B$, $C$, and $D$, determined by the initial conditions.

Our initial conditions are in terms of $\boldsymbol{x}$, but they can be converted into initial conditions on $f$ and $g$ without much fuss. By evaluating $\boldsymbol{x} = f\boldsymbol{x}_0 + g\boldsymbol{v}_0$

at $s = 0$, we immediately see that $f(0) = 1$ and $g(0) = 0$. Likewise, evaluating $\dot{\boldsymbol{x}} = \dot{f}\boldsymbol{x}_0 + \dot{g}\boldsymbol{v}_0$ at zero gives $\dot{f}(0) = 0$ and $\dot{g}(0) = 1$. Converting to $s$-derivatives, we get our initial conditions:

$$\begin{aligned} f(0) &= 1 & g(0) &= 0 \\ f'(0) &= r_0\dot{f}(0) = 0 & g'(0) &= r_0\dot{g}(0) = r_0 \end{aligned} \tag{1.4.4}$$

Evaluating these equations at $s = 0$, we obtain the values of $A$, $C$, $B$, and $D$, respectively.

$$A = \frac{\mu}{\beta r_0} \qquad B = 0 \qquad C = -\frac{r_0\dot{r}_0}{\beta} \qquad D = r_0 \tag{1.4.5}$$

Plugging back into (1.4.2) and (1.4.3) gives us our complete solutions for $f$ and $g$.

$$f(s) = \left(1 - \frac{\mu}{\beta r_0}\right) + \frac{\mu}{\beta r_0} G_0(\beta, s) \tag{1.4.6}$$

$$g(s) = \frac{r_0\dot{r}_0}{\beta} - \frac{r_0\dot{r}_0}{\beta} G_0(\beta, s) + r_0 G_1(\beta, s) \tag{1.4.7}$$

These solutions have a $\beta$ in some denominators, and that could prove troublesome when $\beta = 0$. Fortunately, substituting $G_0 = 1 - \beta G_2$ lets us eliminate these denominators:

$$f(s) = 1 - \frac{\mu}{r_0} G_2(\beta, s) \qquad g(s) = r_0 G_1(\beta, s) + r_0\dot{r}_0 G_2(\beta, s) \tag{1.4.8}$$

Together the equations in (1.4.8) give us the position for any value of $s$. To get the velocity, we'll also have to take the time derivative (not the $s$-derivative!). Since $\frac{\mathrm{d}}{\mathrm{d}s} G_{k+1} = G_k$, we have $\frac{\mathrm{d}}{\mathrm{d}t} G_{k+1} = \frac{1}{r} G_k$, giving:

$$\dot{f}(s) = -\frac{\mu}{r r_0} G_1(\beta, s) \qquad \dot{g}(s) = \frac{r_0}{r} G_0(\beta, s) + \frac{r_0}{r}\dot{r}_0 G_1(\beta, s) \tag{1.4.9}$$

At this point, we can now compute $\boldsymbol{x}$ and $\boldsymbol{v}$ at any given $s$. It remains to connect $s$ to $t$.

## 1.5 Connecting Time and Anomaly

Given a value of $t$, how do we convert it to a value of $s$? Recall the defining relation $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{r}$. Rearranging, we get that $\mathrm{d}t = r\ \mathrm{d}s$, and if we know $r$ in terms of $s$, we can integrate.

We can express $r(s)$ in terms of our $G$-functions. Start with the equation $r' = r\dot{r} = \boldsymbol{x} \cdot \dot{\boldsymbol{x}}$, and take the time derivative of both sides. Then, apply the equations of motion (1.1.1), and the definition of energy (1.1.4):

$$\frac{\mathrm{d}}{\mathrm{d}t} r' = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x} \cdot \dot{\boldsymbol{x}}) = \boldsymbol{x} \cdot \ddot{\boldsymbol{x}} + \dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}$$

$$\frac{r''}{r} = -\frac{\mu}{r^3}\boldsymbol{x}^2 + 2\left(E + \frac{\mu}{r}\right) = \frac{\mu}{r} - \beta$$

$$r'' + \beta r = \mu \tag{1.5.1}$$

We've already seen how to solve diffeqs of this form; we know that the general solution for $r(s)$ looks like $r(s) = A + BG_1(\beta, s) + CG_2(\beta, s)$ (note: we could use $G_0$ and $G_1$ here instead, but if we did, we'd have to clear $\beta$ from the denominator, just like before).

Evaluating this and its $s$-derivatives at $s = 0$, it's straightforward to find these constants:

$$A = r(0) = r_0 \qquad B = r'(0) = r_0\dot{r}_0 \qquad C = r''(0) = \mu - \beta r_0 \tag{1.5.2}$$

This gives us a clean equation for $r(s)$:

$$r(s) = r_0 G_0(\beta, s) + r_0\dot{r}_0 G_1(\beta, s) + \mu G_2(\beta, s) \tag{1.5.3}$$

Integrating this with respect to $s$ bumps up the subscript on the $G_i$, giving us the desired link between $s$ and $t$:

$$t = \int r(s)\ \mathrm{d}s = r_0 G_1(\beta, s) + r_0\dot{r}_0 G_2(\beta, s) + \mu G_3(\beta, s) + C \tag{1.5.4}$$

Since we only know $ds/dt$, we have total freedom to pick the initial value of $s$. It's convenient to pick $s = 0$ at $t = 0$, giving $C = 0$. So if we want to advance the system by some $s$, we can update position and velocity using (1.4.8) and (1.4.9), and update time using (1.5.4).

As a side note, we can use this to simplify the $\dot{g}$ expression in (1.4.9):

$$\dot{g}(s) = 1 - \frac{\mu}{r} G_2(\beta, s) \tag{1.5.5}$$

(TODO: can i do something similar for $g$?)

## 1.6 Connecting Universal, Eccentric, and True Anomalies

What exactly is $s$? It's some kind of time-like parameter that describes how far along its orbit a satellite is. It turns out that it bears a close relation to the eccentric anomaly, and could be thought of as a "universal" kind of anomaly.

### Elliptical Orbits

For elliptical orbits, the *true anomaly* is the actual angle that a satellite makes with its focal axis, and is denoted by $\theta$. The *mean anomaly* $M$ is an averaged measure of motion; if $t$ is the time elapsed since periapsis, and $T$ is the period of the orbit, $M = 2\pi t/T$.

The *eccentric anomaly* is somewhat of a hybrid of these two quantities. The important things for us to know are Kepler's equation, which relates it to the mean anomaly, and a relation to the true-anomaly:

$$M = E - e \sin E \tag{1.6.1}$$

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} \tag{1.6.2}$$

To connect $s$ to $E$, we will integrate $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{r}$, and miraculously, the $r$ and $t$ quantities will coincide into a single expression in $E$.

First, we expand $r$ using the polar form of an ellipse:

$$s = \int \frac{\mathrm{d}t}{r} = \int \frac{1+e\cos\theta}{a(1-e^2)}\ \mathrm{d}t \tag{1.6.3}$$

Taking the derivative of (1.6.1), we can switch our integration variable to $E$:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{2\pi/T}{1-e\cos E} \tag{1.6.4}$$

Plugging it into (1.6.3), we get an integrand without time:

$$s = \frac{T/2\pi}{a(1-e^2)}\int (1+e\cos\theta)(1-e\cos E)\ \mathrm{d}E \tag{1.6.5}$$

Lastly, we need to express $\cos\theta$ in terms of $E$. By applying several different half-angle formulas to (1.6.2), and grinding away, we can get

$$\cos\theta = \frac{\cos E - e}{1-e\cos E} \tag{1.6.6}$$

which, when plugged back into the previous integral, gives

$$s = \frac{T/2\pi}{a(1-e^2)}\int (1-e^2)\ \mathrm{d}E = \frac{T}{2\pi a}E = \frac{E}{\sqrt{\beta}} \tag{1.6.7}$$

where the last equality comes from Kepler's third law.

### Non-Elliptical Orbits

Similarly, for hyperbolic orbits, $s = H/\sqrt{-\beta}$, where $H$ is the hyperbolic eccentric anomaly, related to the true anomaly by:

$$\tanh\frac{H}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2} \tag{1.6.8}$$

Lastly, we look at the parabola. The polar form of a parabola is $r = \frac{h^2/\mu}{1+\cos\theta}$, and instead of Kepler's equation, we use Barker's equation:

$$t = \frac{h^3}{2\mu^2}(D + \frac{1}{3}D^3) \tag{1.6.9}$$

where $D = \tan\frac{\theta}{2}$ and $h$ is the angular momentum.

Taking the derivative of (1.6.9), we can change our integral to be over $D$:

$$s = \frac{h}{2\mu}\int (1+\cos\theta)(1+D^2)\ \mathrm{d}D \tag{1.6.10}$$

Through double-angle identities, we rephrase $\cos\theta$ to be $(1-D^2)/(1+D^2)$, allowing us to simplify the integrand to a constant 2, and so

$$s = \frac{h}{\mu} D = \frac{h}{\mu} \tan\frac{\theta}{2} \tag{1.6.11}$$

In summary, when converting from true to universal anomaly, we first convert to eccentric/hyperbolic anomaly, using (1.6.2) and (1.6.8). Then we divide through by $\sqrt{\pm\beta}$ to get $s$. Parabolas are a special case, and are simply handled by (1.6.11).