# Transformations

Consider two orthogonal reference frames, $A$ and $B$, which are displaced by a vector $T_{BA}$ and related by a rotation $R_{BA}$. Additionally, frame $B$ is moving w.r.t. frame $A$, with linear velocity $V_{BA}$ and angular velocity (around its origin) $\Omega_{BA}$.

TODO picture

## Positions and Directions

Even though the two frames are not aligned, and some rotation $R_{BA}$ is needed to convert between the coordinates, we're not going to really worry about it in this document. Vectors are physical objects, and have coordinate-independent existences, and so we will write them without specifying a basis.

For positions, we'll specify them as vector offsets from the origin of the frame. So, given a point, we can measure both $r_A$, the displacement from the origin of $A$, and $r_B$, the displacement from the origin of $B$. They are related by $r_A = r_B + T_{BA}$.

Of course, in the code itself, we'll need to make use of $R_{BA}$ to convert between coordinates. Specifically, if $R_{BA}$ actively transforms the basis vectors of $A$ to those of $B$, it passively converts coordinates from $B$ to $A$. As an example, the above identity, when expressed in $A$'s coordinates, is $[r_A]_A = R_{BA}[r_B]_B + [T_{BA}]_A$, where $[-]_A$ is "this vector is expressed in $A$'s coordinates". But we don't have to worry about that except in code.

## Derivatives

These two frames move relative to one another, and so they will disagree on whether objects are moving or not. Let $D_A$ mean "derivative in frame $A$", and likewise for $D_B$. However, they will agree on scalar quantities, and so for those we'll just use the usual $d/dt$.

First, we'd like to determine how the bases move. Let the basis vectors in frame $A$ be $\hat{x}$, $\hat{y}$, and $\hat{z}$, and the ones in frame $B$ be $\hat{X}$, $\hat{Y}$, and $\hat{Z}$. We know from basic geometry that if a vector $u$ is fixed in frame $B$, then $D_A(u) = \Omega_{BA} \times u$. So in particular this is true of the basis vectors.

Now consider some vector, possibly time-dependent, $u(t)$. In frame $A$, it has coordinates $(x, y, z)_A$, where $v = x\hat{x} + y\hat{y} + z\hat{z}$, and in the $B$ frame, it has similar coordinates $(X, Y, Z)_B$.

Let's take the derivative in the $A$ frame. On one hand, it's simply $\frac{\mathrm{d}x}{\mathrm{d}t}\hat{x} +$

$\frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$. But if we expand it in $B$-coordinates, we get:

$$\begin{aligned}
D_A(u) &= D_A(X\hat{X} + Y\hat{Y} + Z\hat{Z}) \\
&= \frac{dX}{dt}\hat{X} + XD_A(\hat{X}) + \frac{dY}{dt}\hat{Y} + YD_A(\hat{X}) + \frac{dZ}{dt}\hat{Z} + ZD_A(\hat{Z}) \\
&= \left(\frac{dX}{dt}\hat{X} + \frac{dY}{dt}\hat{Y} + \frac{dZ}{dt}\hat{Z}\right) \\
&\qquad + \Big(X(\Omega_{BA} \times \hat{X}) + Y(\Omega_{BA} \times \hat{Y}) + Z(\Omega_{BA} \times \hat{Z})\Big) \\
&= D_B(u) + (\Omega_{BA} \times u)
\end{aligned}$$

### Velocities

Now that we know how derivatives work, we're equipped to talk about velocities. As mentioned above, if an object has position $r_A$ in frame $A$, and $r_B$ in frame $B$, then the two vectors are related by $r_A = r_B + T_{BA}$. Taking the derivative in the $A$ frame, we have:

$$D_A(r_A) = D_A(r_B) + D_A(T_{BA}) = D_B(r_B) + (\Omega_{BA} \times r_B) + D_A(T_{BA})$$

The $D_A(r_A)$ and $D_B(r_B)$ terms are the component-wise derivatives of position in their respective frames. By definition, these are the velocities $v_A$ and $v_B$. The last term is the velocity (as measured by $A$) of $B$'s origin, in other words, $V_{BA}$. (Note that there is an asymmetry here; $D_B(T_{BA})$ is *not* $V_{BA}$, because there's extra velocity that $A$'s origin acquires from $B$'s rotation!) So our final result is:

$$v_A = v_B + (\Omega_{BA} \times r_B) + V_{BA}$$

## Composition and Inverses

Let's now try to figure out the inverses by flipping some labels. Since $r_A = r_B + T_{BA}$, it should be the case that $r_B = r_A + T_{AB}$, which would imply that $T_{AB} = -T_{BA}$. Similarly, $D_A(u) = D_B(u) + \Omega_{BA} \times u$ implies that $\Omega_{AB} = \Omega_{BA}$.

The case for velocity is a little trickier, but nothing horrendous. Starting by flipping the labels, and isolating $V_{AB}$:

$$\begin{aligned}
v_B &= v_A + (\Omega_{AB} \times r_A) + V_{AB} \\
V_{AB} &= v_B - v_A - (\Omega_{AB} \times r_A) \\
V_{AB} &= -v_A + v_B + (\Omega_{BA} \times (r_B + T_{BA})) \\
V_{AB} &= (v_B + (\Omega_{BA} \times r_B)) - v_A + (\Omega_{BA} \times T_{BA}) \\
V_{AB} &= -V_{BA} + (\Omega_{BA} \times T_{BA})
\end{aligned}$$

Physically, what this means is that the velocity of $A$'s origin, w.r.t. $B$, is the negative of the velocity we saw before (as expected), but there's also an additional term that comes from the rotation $\Omega$.

For what it's worth, we could derive this a second way. We know that $D_A(T_{BA}) = V_{BA}$, and so we can use what we already know about derivatives:

$$V_{AB} = D_B(T_{AB}) = -D_B(T_{BA}) = -D_A(T_{BA}) - (\Omega_{AB} \times T_{BA})$$

$$= -V_{BA} + (\Omega_{BA} \times T_{BA})$$

Composition employs some similar tricks. Clearly $T_{CA} = T_{CB} + T_{BA}$ and with some thought, one can justify $\Omega_{CA} = \Omega_{CB} + \Omega BA$. As usual, it's the velocity that needs some work.

We can use the fact that $V_{CA} = D_A(T_{CA})$ to derive the correct velocity:

$$\begin{aligned} V_{CA} &= D_A(T_{CA}) \\ &= D_A(T_{CB}) + D_A(T_{BA}) \\ &= D_B(T_{CB}) + (\Omega_{BA} \times T_{CB}) + D_A(T_{BA}) \\ &= V_{CB} + V_{BA} + (\Omega_{BA} \times T_{CB}) \end{aligned}$$

Another way to read this result is as the conversion of the velocity of $C$'s origin from $B$'s frame to $A$'s frame. (Take the usual velocity transformation from $B$ to $A$, but with $v_B = V_{CB}$ and $r_B = T_{CB}$).