

### 高等数学期中考试模拟试题(三)答案

#### 一. 单项选择题(共5道小题,每小题3分,共15分)



 $1.x \to 0$ 时,变量 $\frac{1}{r^2}\sin\frac{1}{r}$ 是(D).

A.无穷小 B. 无穷大 C. 有界但非无穷小 D. 无界但非无穷大

2. 设 $f(x) = \begin{cases} \sqrt{|x|} \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ ,则下列f(x)在x = 0处(C).

。 C 许绿

A. 极限不存在

C. 连续

B. 极限存在但不连续 D. 以上结论都不对

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3. 已知f(x)是奇函数且x < 0时单增,则当x > 0时,f(x)是(A). A.单增 B. 单减

C. 可能单增,可能单减 D. 既非单增,也非单减

4. 设f(x), g(x)都在x = a处取得极大值,则F(x) = f(x)g(x)



在
$$x = a$$
处( $\mathbb{D}$ ).

A. 必取得极大值

C. 不可能取得极值

B. 必取得极小值

D. 是否取得极值不能确定

# 5. 设f(x)在x = a的某邻域内有定义,则f(x)在x = a可导的一个充分条件是(D).

$$A. \lim_{h \to +\infty} h \left[ f\left(a + \frac{1}{h}\right) - f\left(a\right) \right]$$
存在

B. 
$$\lim_{h\to 0} \frac{f(a+2h)-f(a+h)}{h}$$
存在

$$C.\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{h}$$
存在

$$D.\lim_{h\to 0} \frac{f(a)-f(a-h)}{h}$$
存在

二. 填空题(共5道小题,每题4分,总计20分)



1.设f(x)的定义域为[0,1],则 $f(\ln x)$ 的定义域为[1,e].

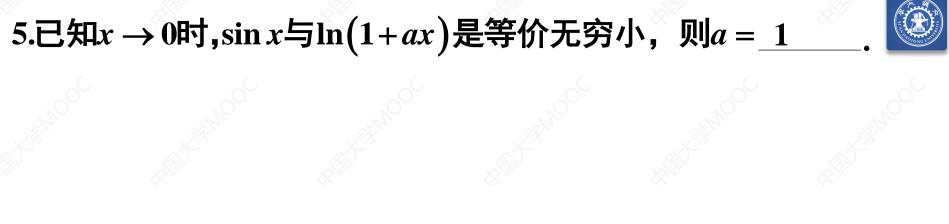
2.已知 
$$\lim_{x\to\infty} \left(\frac{x+a}{x-a}\right)^x = 9$$
,则  $a = \frac{\ln 3}{2}$ 

3.设
$$f(x) = \begin{cases} (1+2x)^{\frac{1}{x}}, x < 0 \\ b+1, & x=0 \end{cases}$$
, 当 $a = e^{2}$ ,  $b = e^{2} - 1$  时,  $f(x)$ 在  $\frac{\sin ax}{x}$ ,  $x > 0$ 

#### 4.函数 $f(x) = \frac{e^{\frac{1-x}{1-x}} \sin x}{|x|}$ 的第一类间断点x = 0,第二类间断点

$$|x| = |x|$$

## 5.已知 $x \to 0$ 时, $\sin x = \ln(1+ax)$ 是等价无穷小,则a = 1







#### 三. 计算题(共5道小题,每题7分,总计35分)



1.求极限
$$\lim_{x\to 0} \frac{e^x \sin x - x(x+1)}{1-\cos x}$$
.

#### 解 方法1

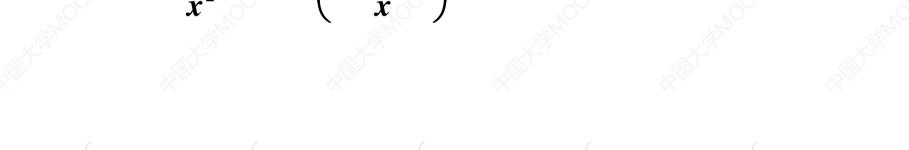
原式=
$$\lim_{x\to 0} \frac{e^x (\sin x + \cos x) - 2x - 1}{\sin x}$$
 =  $\lim_{x\to 0} \frac{2e^x \cos x - 2}{\cos x}$  = 0

#### 方法2

原式=
$$\lim_{x\to 0} \frac{(1+x+o(x))(x+o(x^2))-x^2-x}{\frac{1}{2}x^2} = 0$$

$$y' = 2\sin\left(\frac{1-\ln x}{x}\right)\cos\left(\frac{1-\ln x}{x}\right)\cdot\frac{-\frac{1}{x}x-(1-\ln x)}{x^2}$$

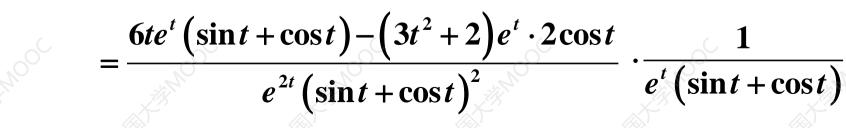
$$= \frac{\ln x - 2}{x^2} \sin 2\left(\frac{1 - \ln x}{x}\right)$$



3.设函数
$$y = y(x)$$
由
$$\begin{cases} x-e^t \sin t + 1 = 0 \\ y = t^3 + 2t \end{cases}$$
确定,求 $\frac{d^2y}{dx^2}$ .
$$\frac{dy}{dx} = \frac{3t^2 + 2}{e^t (\sin t + \cos t)}$$

$$\frac{dy}{dx} = \frac{3t^2 + 2}{e^t \left(\sin t + \cos t\right)}$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \cdot \frac{1}{x'(t)}$$



4.方程  $\sin(xy) - \ln \frac{x+1}{y} = 1$ 表示平面上一条曲线,试求该曲线在

$$x = 0$$
处的切线方程和法线方程.

解 方程两边对x求导,得: $\cos(xy)\cdot(y+xy')-\frac{1}{1+x}+\frac{1}{y}y'=0$ 

把x = 0, y = e代入,得y'(0) = e(1-e)

故切线方程为: 
$$y-e=e(1-e)x$$
 法线方程为:  $y-e=\frac{1}{e(e-1)}x$ 

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$$\frac{n \cdot n + 1 + 2 + \dots + n}{n^2 + n} \le \frac{n + 1}{n^2 + 1} + \frac{n + 2}{n^2 + 2} + \dots + \frac{n + n}{n^2 + n} \le \frac{n \cdot n + 1 + 2 + \dots + n}{n^2 + 1}$$

$$\lim_{n\to\infty} \frac{n \cdot n + 1 + 2 + \dots + n}{n^2 + n} = \lim_{n\to\infty} \frac{n^2 + \frac{n(n+1)}{2}}{n^2 + n} = \frac{3}{2}$$

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5.求极限 $\lim_{n\to\infty} \left(\frac{n+1}{n^2+1} + \frac{n+2}{n^2+2} + \dots + \frac{n+n}{n^2+n}\right)$ .

解

5.求极限
$$\lim_{n\to\infty} \left(\frac{n+1}{n^2+1} + \frac{n+2}{n^2+2} + \dots + \frac{n+n}{n^2+n}\right)$$
.



解

$$\lim_{n \to \infty} \frac{n \cdot n + 1 + 2 + \dots + n}{n^2 + n} = \lim_{n \to \infty} \frac{n^2 + \frac{n(n+1)}{2}}{n^2 + n} = \frac{3}{2}$$

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$$\therefore \lim_{n\to\infty} \left( \frac{n+1}{n^2+1} + \frac{n+2}{n^2+2} + \dots + \frac{n+n}{n^2+n} \right) = \frac{3}{2}.$$

四. (本题9分)设 $n \in N_+$ ,讨论函数 $f(x) = \begin{cases} x^n \sin \frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$ 

处的连续性与可导性及f'(x)在x = 0处的连续性.

解 
$$n = 1$$
时, $\lim_{x \to 0} x \sin \frac{1}{x} = 0 = f(0)$ 

n = 2FJ,  $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0$ 

$$n \ge 3$$
时, $f'(x) = \begin{cases} nx^{n-1} \sin \frac{1}{x} - x^{n-2} \cos \frac{1}{x}, x \ne 0 \\ 0, & x = 0 \end{cases}$ 

四. (本题9分)设 $n \in N_+$ , 讨论函数 $f(x) = \begin{cases} x^n \sin \frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$ 

处的连续性与可导性及f'(x)在x = 0处的连续性.

解 
$$n \ge 3$$
时, $f'(x) = \begin{cases} nx^{n-1} \sin \frac{1}{x} - x^{n-2} \cos \frac{1}{x}, x \ne 0 \\ 0, & x = 0 \end{cases}$ 

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left( nx^{n-1} \sin \frac{1}{x} - x^{n-2} \cos \frac{1}{x} \right) = 0 = f'(0)$$

所以, f'(x)在x = 0处连续.

#### 五.证明下列各题(共3道题,每小题7分,共21分)



1.设
$$x_1 < -1$$
,  $x_{n+1} + \sqrt{1-x_n} = 0$ , 证明 $\{x_n\}$ 收敛并求 $\lim_{n \to \infty} x_n$ .

证明 方法1(利用柯西收敛原理) 显然,  $x_n < 0$ 

$$\begin{aligned} x_{n+1} - x_n &= \sqrt{1 - x_{n-1}} - \sqrt{1 - x_n} = \frac{x_n - x_{n-1}}{\sqrt{1 - x_{n-1}} + \sqrt{1 - x_n}} \\ \left| x_{n+1} - x_n \right| &< \frac{1}{2} \left| x_n - x_{n-1} \right| \le \dots \le \frac{1}{2^{n-1}} \left| x_2 - x_1 \right| \end{aligned}$$

$$\begin{aligned} \left| x_{n+p} - x_n \right| &\leq \left| x_{n+p} - x_{n+p-1} \right| + \left| x_{n+p-1} - x_{n+p-2} \right| + \dots + \left| x_{n+1} - x_n \right| \\ &\leq \left( \frac{1}{2^{n+p-2}} + \dots + \frac{1}{2^{n-1}} \right) \left| x_2 - x_1 \right| \leq \frac{1}{2^{n-2}} \cdot \left| x_2 - x_1 \right| \end{aligned}$$

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由柯西收敛原理知,数列收敛,设 $\lim_{n\to\infty} x_n = A$ 

等式两端取极限,得: 
$$A = \frac{-1 - \sqrt{5}}{2}$$

1.设 $x_1 < -1$ ,  $x_{n+1} + \sqrt{1-x_n} = 0$ , 证明 $\{x_n\}$ 收敛并求 $\lim x_n$ . 证明 方法2 直接证明 $\lim_{n\to\infty} x_n = \frac{-1-\sqrt{5}}{2}$ 

五.证明下列各题(共3道题,每小题7分,共21分)

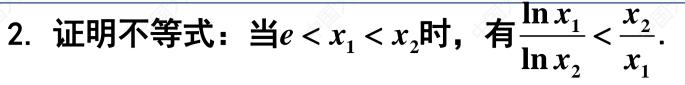
$$\begin{vmatrix} x_{n+1} - \frac{-1 - \sqrt{5}}{2} \end{vmatrix} = \begin{vmatrix} -\sqrt{1 - x_n} - \frac{-1 - \sqrt{5}}{2} \end{vmatrix} = \begin{vmatrix} \frac{1 + \sqrt{5}}{2} - \sqrt{1 - x_n} \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 + \sqrt{5} - 2\sqrt{1 - x_n} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} \frac{2 + 2\sqrt{5} + 4x_n}{1 + \sqrt{5} + 2\sqrt{1 - x_n}} \end{vmatrix} \le \frac{2}{5} \begin{vmatrix} x_n - \frac{-1 - \sqrt{5}}{2} \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 + \sqrt{5} - 2\sqrt{1 - x_n} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} \frac{2 + 2\sqrt{5} + 4x_n}{1 + \sqrt{5} + 2\sqrt{1 - x_n}} \end{vmatrix} \le \frac{2}{5} \begin{vmatrix} x_n - \frac{-1 - \sqrt{5}}{2} \end{vmatrix}$$

$$= \frac{1 - \sqrt{5}}{2} \begin{vmatrix} x_n - \frac{-1 - \sqrt{5}}{2} \end{vmatrix}$$

 $\cdots \leq \left(\frac{2}{5}\right)^n \left| x_1 - \frac{-1 - \sqrt{5}}{2} \right| \qquad \lim_{n \to \infty} \left(\frac{2}{5}\right)^n = 0, \quad \lim_{n \to \infty} x_{n+1} = \frac{-1 - \sqrt{5}}{2}.$ 





证明 
$$\diamondsuit f(x) = x \ln x$$

$$f'(x) = 1 + \ln x > 0(\because x > e)$$

∴ 当
$$e < x_1 < x_2$$
时,有 $x_1 \ln x_1 < x_2 \ln x_2$ .



即  $f'(\xi) + f'(\eta) = \xi^2 + \eta^2$ .

$$\left(\frac{1}{2}\right) - F\left(0\right)$$

$$\left(\frac{1}{2}\right) - F\left(0\right)$$

$$F\left(\frac{1}{2}\right) - F\left(0\right) =$$

$$\frac{F\left(\frac{1}{2}\right)-F\left(0\right)}{\frac{1}{2}-0}=F'\left(\xi\right)=f'\left(\xi\right)-\xi^{2}\quad\left(0<\xi<\frac{1}{2}\right)$$

$$\frac{\left(2\right)^{-1}\left(0\right)}{1}=F$$

$$\left(\frac{1}{2}\right) - F(0)$$

$$\theta:$$
 存任点 $\xi \in \left[0, \frac{\pi}{2}\right], \quad \eta \in \left[\frac{\pi}{2}\right]$ 

证明: 存在点
$$\xi \in \left(0, \frac{1}{2}\right)$$
,  $\eta \in \left(\frac{1}{2}, 1\right)$ , 使得 $f'(\xi) + f'(\eta) = \xi^2 + \eta^2$ .

月:存在点
$$\xi \in \left(0,\frac{1}{2}\right)$$
,  $\eta \in \left(\frac{1}{2}\right)$ 

3.设
$$f \in [0,1]$$
,  $f \in (0,1)$ 内可导,且 $f(0) = 0$ ,  $f(1) = \frac{1}{3}$ .

 $\frac{F(1)-F\left(\frac{1}{2}\right)}{1}=F'(\eta)=f'(\eta)-\eta^2\quad \left(\frac{1}{2}<\eta<1\right)$