期中考试模拟题（五）答案 2022.5.8

一、C　D　B　B　A

二、1. 1　2. $2e^2$　3. $-dx+dy$　4. $\int_0^{\sqrt{2}}dy\int_y^{\sqrt{4-y^2}}f(x,y)dx$　5. $\frac{8\pi}{5}$

三、1. $\frac{\partial z}{\partial x}=f_1\cdot 2+f_2\cdot y\cos x$

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}=2[f_{11}\cdot(-1)+f_{12}\cdot\sin x]+\cos x\cdot f_2+y\cos x[f_{21}\cdot(-1)+f_{22}\cdot\sin x]$$

$$=-2f_{11}+(2\sin x-y\cos x)f_{12}+y\sin x\cos xf_{22}+\cos xf_2$$

2. $\begin{cases}dx=-2udu+dv+dz\\dy=du+vdz+zdv\end{cases}\Rightarrow\begin{cases}2udu-dv=dz-dx\\du+zdv=-dy-vdz\end{cases}$

$$\Rightarrow\begin{cases}du=\frac{1}{1+2zu}[-zdx-dy+(z-v)dz]\\dv=\frac{1}{1+2zu}[dx-2udy+(1+2uv)dz]\end{cases}\Rightarrow\frac{\partial u}{\partial z}=\frac{z-v}{1+2zu},\frac{\partial v}{\partial x}=\frac{1}{1+2zu}$$

3. $\vec{n}=(2x+1-y\sin(xy),-x\sin(xy)+z,y)|_P=(1,-1,1)$,

切平面方程为 $x-y+z=-2$，法线方程为 $\frac{x}{1}=\frac{y-1}{-1}=\frac{z+1}{1}$

4. $\iint\limits_{(\sigma)}e^{\max(x^2,y^2)}d\sigma=\iint\limits_{(\sigma_1)}e^{x^2}d\sigma+\iint\limits_{(\sigma_2)}e^{y^2}d\sigma=\int_0^1dx\int_0^xe^{x^2}dy+\int_0^1dy\int_0^ye^{y^2}dx$

$$=\int_0^1xe^{x^2}dx+\int_0^1ye^{y^2}dy=2\int_0^1xe^{x^2}dx=e^{x^2}\Big|_0^1=e-1$$

5. 解法 1

$$I=\iiint\limits_{(V)}(x^2+y^2+z^2+2xy+2yz+2zx)dV=\iiint\limits_{(V)}(x^2+y^2+z^2)dV$$

$$=\iiint\limits_{(V)}r^2\cdot r^2\sin\varphi drd\theta d\varphi=\int_0^{2\pi}d\theta\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin\varphi d\varphi\int_0^{2\cos\varphi}r^4dr$$

$$=\frac{64\pi}{5}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin\varphi\cos^5\varphi d\varphi=-\frac{32\pi}{15}\cos^6\varphi\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\frac{4\pi}{15}$$

解法 2

$$I=\iiint\limits_{(V)}(x^2+y^2+z^2+2xy+2yz+2zx)dV=\iiint\limits_{(V)}(x^2+y^2+z^2)dV$$

$$=\int_0^1 dz\iint_{(\sigma_z)}(x^2+y^2+z^2)dxdy=\int_0^1 dz\int_0^{2\pi}d\theta\int_z^{\sqrt{2z-z^2}}\rho(\rho^2+z^2)d\rho$$

$$=2\pi\int_0^1(z^2-z^4)dz=\frac{4\pi}{15}$$

6. $\begin{cases} f_x=3x^2+2(x+y)=0 \\ f_y=3y^2+2(x+y)=0 \end{cases}$，解得驻点 $(0,0),(-\frac{4}{3},-\frac{4}{3})$

$f_{xx}(x,y)=6x+2, f_{xy}(x,y)=2, f_{yy}(x,y)=6y+2$

驻点 $(-\frac{4}{3},-\frac{4}{3})$ 处，$H=\begin{pmatrix}-6 & 2\\ 2 & -6\end{pmatrix}$ 负定，极大值 $f(-\frac{4}{3},-\frac{4}{3})=\frac{64}{27}$

对于 $(0,0)$，沿路径 $x+y=x^3$，$f(x,y)=x^3+(x^3-x)^3+x^6=x^5(3+x-3x^2+x^4)$

在 $x=0$ 附近 $f(x,y)$ 不定号，所以 $(0,0)$ 不是极值点。

7. $D\vec{f}(u,v)=\begin{pmatrix}2u & 2v\\ v & u\end{pmatrix}$，$D\vec{g}(x,y,z)=\begin{pmatrix}-\sin x & \cos y & \sec^2 z\\ e^{x+y+z} & e^{x+y+z} & e^{x+y+z}\end{pmatrix}$

$(x,y,z)=(0,0,0)$ 时，$(u,v)=(1,1)$，$D\vec{f}(1,1)=\begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}$，$D\vec{g}(0,0,0)=\begin{pmatrix}0 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$，

$$D\vec{w}(0,0,0)=Df(1,1)\cdot Dg(0,0,0)=\begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}=\begin{pmatrix}2 & 4 & 4\\ 1 & 2 & 2\end{pmatrix}$$

四、(1) 设 $L=x^{2022}+y^{2022}+\lambda(x^2+y^2-1)$，则 $\begin{cases}2022x^{2021}+2\lambda x=0\\ 2022y^{2021}+2\lambda y=0\\ x^2+y^2-1=0\end{cases}$，

解得驻点 $(0,\pm1),(\pm1,0),(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2}),(\pm\frac{\sqrt{2}}{2},\mp\frac{\sqrt{2}}{2})$，

最小值 $f(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2})=f(\pm\frac{\sqrt{2}}{2},\mp\frac{\sqrt{2}}{2})=\frac{1}{2^{1010}}$，最大值 $f(0,\pm1)=f(\pm1,0)=1$

证：(2) 设 $\varphi(x,y)=g(x,y)-x^2-y^2$，则 $\varphi(0,0)=g(0,0)\geq0$

当 $x^2+y^2=1$ 时，$\varphi(x,y)=f(x,y)-1\leq0$，所以 $\varphi(x,y)$ 一定在 D 的内部一点 (ξ,η) 处取得最大值，从而 $\varphi_x(\xi,\eta)=\varphi_y(\xi,\eta)=0$，即 $g_x(\xi,\eta)=2\xi, g_y(\xi,\eta)=2\eta$

故有 $[g_x(\xi,\eta)]^2+[g_y(\xi,\eta)]^2=4(\xi^2+\eta^2)<4$。