西安交通大学
XI'AN JIAOTONG UNIVERSITY

高等数学期中考试模拟试题（三）答案

一. 填空题（共5道小题，每题4分，总计20分）

1.设$z=e^{x^2y}$，则$\mathrm{d}z=$ $\underline{e^{x^2y}\left(2xy\mathrm{d}x+x^2\mathrm{d}y\right)}$.

2.函数$z=z(x,y)$由方程$xy=e^{xz}-z$确定，则$\dfrac{\partial z}{\partial x}=$ $\underline{\dfrac{y-ze^{xz}}{xe^{xz}-1}}$.

3.$u=2xy-z^2$在点$(2,-1,1)$处方向导数的最大值为 $\underline{\quad 2\sqrt{6} \quad}$.

4.空间曲线$\begin{cases} x^2+y^2=10 \\ x^2+z^2=10 \end{cases}$在点$M(3,1,1)$处的切线方程为

$\underline{\dfrac{x-3}{-1}=\dfrac{y-1}{3}=\dfrac{z-1}{3}}$.

5.平面曲线L为$y=-\sqrt{1-x^2}$,则$\int_L\left(x^2+y^2\right)ds=$ $\underline{\ \pi\ }$.

二. 计算题（共5道小题，每题8分，总计40分）

1.设$z=f\left(e^{x+y},\dfrac{x}{y}\right)$,其中$f$具有二阶连续的偏导数，求$\dfrac{\partial^2 z}{\partial x\partial y}$.

解

$$\frac{\partial z}{\partial x}=f_1 e^{x+y}+\frac{1}{y}f_2$$

$$\frac{\partial^2 z}{\partial x\partial y}=e^{x+y}f_1+e^{x+y}\left[f_{11}e^{x+y}-\frac{x}{y^2}f_{12}\right]-\frac{1}{y^2}f_2+\frac{1}{y}\left[f_{21}e^{x+y}-\frac{x}{y^2}f_{22}\right]$$

2.求曲面$\sqrt{x}+\sqrt{y}+\sqrt{z}=1$的一个切平面，使该切平面在三个坐标轴上的截距之积最大，并写出该切平面方程.

解 $\vec{n}=\left(\dfrac{1}{2\sqrt{x}},\dfrac{1}{2\sqrt{y}},\dfrac{1}{2\sqrt{z}}\right)$　　切平面方程:$\dfrac{X}{\sqrt{x}}+\dfrac{Y}{\sqrt{y}}\dfrac{Z}{\sqrt{z}}=1$

$$L=\sqrt{xyz}+\lambda\left(\sqrt{x}+\sqrt{y}+\sqrt{z}-1\right)$$

解联立方程:$\dfrac{\partial L}{\partial x}=0,\dfrac{\partial L}{\partial y}=0,\dfrac{\partial L}{\partial z}=0$,得: $x=y=z=\dfrac{1}{9}$

切平面方程:$3x+3y+3z=1$

3.方程组$\begin{cases} u+v+w=x \\ uv+vw+wu=y \\ uvw=z \end{cases}$可确定隐函数$\begin{cases} u=u(x,y,z) \\ v=v(x,y,z) \\ w=w(x,y,z) \end{cases}$，

求偏导数$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial z}$.

解 $\begin{cases} \mathrm{d}u+\mathrm{d}v+\mathrm{d}w=\mathrm{d}x \\ v\mathrm{d}u+u\mathrm{d}v+v\mathrm{d}w+w\mathrm{d}v+w\mathrm{d}u+u\mathrm{d}w=\mathrm{d}y \\ uv\mathrm{d}w+uw\mathrm{d}v+vw\mathrm{d}u=\mathrm{d}z \end{cases}$

$$\mathrm{d}u=\frac{\begin{vmatrix} \mathrm{d}x & 1 & 1 \\ \mathrm{d}y & u+w & u+v \\ \mathrm{d}z & uw & uv \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}}$$

$$\mathrm{d}u=\frac{u^2(v-w)\mathrm{d}x+u(w-v)\mathrm{d}y+(v-w)\mathrm{d}z}{(u-v)(u-w)(v-w)}$$

4.计算二重积分$\iint\limits_{D}\left(1-\sqrt{x^2+y^2}\right)\mathrm{d}x\mathrm{d}y$，其中$(D)$是由

$x^2+y^2=a^2$与$x^2+y^2=ax, x=0$围成的第一象限的区域.

解 $$\iint\limits_{D}\left(1-\sqrt{x^2+y^2}\right)\mathrm{d}x\mathrm{d}y=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_{a\cos\theta}^{a}(1-\rho)\rho\mathrm{d}\rho$$

$$=\frac{a^2}{2}\int_0^{\frac{\pi}{2}}\sin^2\theta\mathrm{d}\theta-\frac{a^2}{3}\int_0^{\frac{\pi}{2}}\left(1-\cos^3\theta\right)\mathrm{d}\theta$$

$$=\frac{\pi a^2}{2}-\frac{a^2}{3}\left(\frac{\pi}{2}-\frac{\pi}{3}\right)$$

5.计算$I=\iint\limits_{(\Sigma)}(x+y+z)\mathrm{d}S$，其中$(\Sigma)$为上半球面$z=\sqrt{a^2-x^2-y^2}$.

解 $$I=\iint\limits_{D}\left(x+y+\sqrt{a^2-x^2-y^2}\right)\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\,\mathrm{d}x\mathrm{d}y$$

$$=\iint\limits_{D}a\,\mathrm{d}x\mathrm{d}y=\pi a^3$$

三、(本题10分)计算三重积分$\iiint\limits_{\Omega} z\mathrm{d}V$,其中Ω由曲面

$z=\sqrt{a^2-x^2-y^2}$和$z=\sqrt{x^2+y^2}$围成.

解

$$\iiint\limits_{\Omega} z\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\mathrm{d}\varphi\int_0^{a} r^2\sin\varphi\cdot r\cos\varphi\mathrm{d}r$$

$$=2\pi\int_0^{\frac{\pi}{4}}\frac{1}{4}a^4\sin\varphi\cos\varphi\mathrm{d}\varphi$$

$$=\frac{1}{8}\pi a^4$$

四、(本题10分)计算$\int_L \left(x^2y+3xe^x\right)\mathrm{d}x+\frac{1}{3}\left(x^3-y\sin y\right)\mathrm{d}y$，其中$L$是摆线$x=t-\sin t, y=1-\cos t$上从到点$O(0,0)$到点$A(\pi,2)$的弧段.

解 设$B(\pi,0)$ $\because \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ $\therefore$积分与路径无关

$$\int_L=\int_{OB}+\int_{BA}$$

$$=\int_0^\pi 3xe^x\mathrm{d}x+\int_0^2\left(\frac{\pi^2}{3}-\frac{1}{3}y\sin y\right)\mathrm{d}y$$

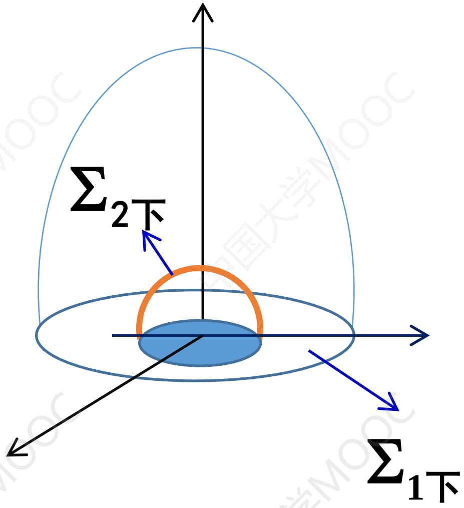
$$=3\left(\pi e^\pi-e^\pi+1\right)+\frac{2}{3}\pi^3+\frac{1}{3}(2\cos 2-\sin 2)$$

五、(本题10分)计算曲面积分$I=\iint\limits_{(\Sigma)}\dfrac{x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y}{\left(x^2+y^2+z^2\right)^{3/2}}$,

其中(Σ)为椭圆抛物面$z=4-\left(x^2+4y^2\right)$的上侧.

解 $I=\oiint\limits_{\Sigma+\Sigma_1+\Sigma_2}-\iint\limits_{\Sigma_1}-\iint\limits_{\Sigma_2}$

$\Sigma_2: x^2+y^2+z^2=\varepsilon^2$



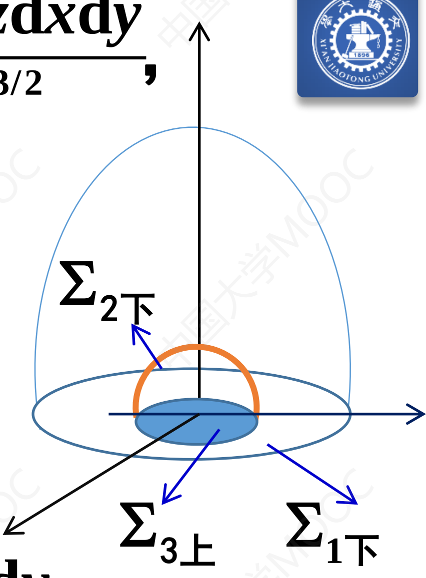
$$\frac{\partial P}{\partial x}=\frac{-2x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}\qquad \frac{\partial Q}{\partial y}=\frac{x^2-2y^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial R}{\partial z}=\frac{x^2+y^2-2z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\mathrm{div}\vec{A}=0$$

五、(本题10分)计算曲面积分$I=\iint\limits_{(\Sigma)}\dfrac{x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y}{\left(x^2+y^2+z^2\right)^{3/2}}$,

其中(Σ)为椭圆抛物面$z=4-\left(x^2+4y^2\right)$的上侧.



解 $I=\oint\!\!\!\oint\limits_{\Sigma+\Sigma_1+\Sigma_2}-\iint\limits_{\Sigma_1}-\iint\limits_{\Sigma_2}$ $\qquad \mathrm{div}\vec{A}=0$

$$=\iiint\limits_{(V)}\mathrm{div}\vec{A}\mathrm{d}V-\iint\limits_{\Sigma_2}=-\frac{1}{\varepsilon^3}\iint\limits_{\Sigma_2}x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y$$

$$=-\frac{1}{\varepsilon^3}\left[\oint\!\!\!\oint\limits_{\Sigma_2+\Sigma_3}-\iint\limits_{\Sigma_3}\right]=\frac{1}{\varepsilon^3}\iiint\limits_{(V)}3\mathrm{d}V+\frac{1}{\varepsilon^3}\iint\limits_{\Sigma_3}=2\pi$$

六、（本题10分） 设(Ω)为空间有界区域，其边界逐片光滑，$\vec{n}$是$(\partial\Omega)$的外单位法向量，函数$f(x,y,z)$在(Ω)内满足

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0,$$ 在$(\partial\Omega)$上有连续的偏导数.证明：

(1) $\displaystyle\oiint_{(\partial\Omega)}\frac{\partial f}{\partial \vec{n}}\mathrm{d}S=0;$

(2) $\displaystyle\oiint_{(\partial\Omega)} f(x,y,z)\frac{\partial f}{\partial \vec{n}}\mathrm{d}S=\iiint_{(\Omega)}\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2\right]\mathrm{d}V;$

(3) 若当$(x,y,z)\in(\partial\Omega)$时，$f(x,y,z)\equiv 0$，

证明：在(Ω)内，$f(x,y,z)\equiv 0.$

六、（本题10分） 设(Ω)为空间有界区域，其边界逐片光滑，$\vec{n}$是$(\partial\Omega)$的外单位法向量，函数$f(x,y,z)$在(Ω)内满足$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=0$，在$(\partial\Omega)$上有连续的偏导数.证明：

(1) $\displaystyle\oiint_{(\partial\Omega)}\frac{\partial f}{\partial\vec{n}}\mathrm{d}S=0$;

解 $$\oiint_{(\partial\Omega)}\frac{\partial f}{\partial\vec{n}}\mathrm{d}S=\oiint_{(\partial\Omega)}\mathbf{grad}f\cdot\vec{n}\mathrm{d}S=\iiint_{(V)}\mathrm{div}(\mathbf{grad}f)\mathrm{d}V=0.$$

六、（本题10分） 设(Ω)为空间有界区域，其边界逐片光滑，$\vec{n}$是$(\partial\Omega)$的外单位法向量，函数$f(x,y,z)$在(Ω)内满足

$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=0$，在$(\partial\Omega)$上有连续的偏导数.证明：

$$(2)\oint\!\!\!\oint_{(\partial\Omega)} f(x,y,z)\frac{\partial f}{\partial \vec{n}}\mathrm{d}S=\iiint_{(\Omega)}\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2\right]\mathrm{d}V;$$

解 左边 $=\displaystyle\iiint_{(\Omega)}\mathrm{div}[f\mathbf{grad}f]\mathrm{d}V$

$$=\iiint_{(\Omega)}\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2\right]\mathrm{d}V;$$

$$\oint\!\!\!\oint_{(\partial\Omega)}\frac{\partial f}{\partial \vec{n}}\mathrm{d}S=\iiint_{(V)}\mathrm{div}(\mathbf{grad}f)\mathrm{d}V$$

六、（本题10分）设为空间有界区域，其边界逐片光滑，$\vec{n}$是$(\partial\Omega)$的外单位法向量，函数$f(x,y,z)$在(Ω)内满足

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0,$$ 在$(\partial\Omega)$上有连续的偏导数.证明：

(3)若当$(x,y,z)\in(\partial\Omega)$时，$f(x,y,z)\equiv 0$，

证明：在(Ω)内，$f(x,y,z)\equiv 0$.

解 根据(2) $\displaystyle\oint\!\!\!\oint_{(\partial\Omega)} f(x,y,z)\frac{\partial f}{\partial \vec{n}}\mathrm{d}S=\iiint_{(\Omega)}\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2\right]\mathrm{d}V$

$\mathrm{grad}f=0\Rightarrow$ 在Ω内，f恒为常数. 又f连续，则在Ω内，$f\equiv 0$