You will do a series of exercises on Mellin transforms, the Riemann zeta function, and Poisson summation. The exercises build on each other so try to do them in order.

If $f(x)$ is a continuous function on $\mathbb{R}_+$ then the **Mellin transform** $(\mathcal{M}f)(s)$ of $f(x)$ is defined by

$$(\mathcal{M}f)(s) = \int_{(0,\infty)} f(x)x^s \, \frac{dx}{x},$$

for $s \in \mathbb{C}$. However, this integral is not guaranteed to converge unless specific conditions upon $f(x)$ are imposed. For example, if $f(x)$ exhibits rapid decay and remains bounded as $x \to 0$ then the integral is locally absolutely uniformly convergent for $\sigma > 0$. The **gamma function** $\Gamma(s)$ is defined to be the Mellin transform of $e^{-x}$.

(i) Write down the definition of the gamma function. Show $\Gamma(1) = 1$ and that $\Gamma(s+1) = s\Gamma(s)$. Use these facts to prove $\Gamma(n) = (n-1)!$.

(ii) Consider the function

$$\omega(z) = \sum_{n \geq 1} e^{\pi i n^2 z},$$

which is defined for $z \in \mathbb{H}$. Use the Weierstrass $M$-test to show that $\omega(z)$ is locally absolutely uniformly convergent for $z \in \mathbb{H}$.

(iii) Compute the following Mellin transform:

$$\int_0^\infty \omega(iy) y^{\frac{s}{2}} \, \frac{dy}{y},$$

using the fact that you may freely interchange sums and integrals since $\omega(z)$ is locally absolutely uniformly convergent (Fubini-Tonelli theorem). Deduce an integral representation for $\zeta(s)$.

(iv) Using the integral representation derived in part (iii) and the identity

$$\omega\left(\frac{i}{y}\right) = \sqrt{y}\,\omega(iy) + \frac{\sqrt{y}}{2} - \frac{1}{2}, \tag{1}$$

derive the following integral representation:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left[ -\frac{1}{s(1-s)} + \int_1^\infty \omega(iy) y^{\frac{1-s}{2}} \, \frac{dy}{y} + \int_1^\infty \omega(iy) y^{\frac{s}{2}} \, \frac{dy}{y} \right].$$

Deduce that $\zeta(s)$ admits analytic continuation to $\mathbb{C}$.

(v) Using part (iv), derive the following functional equation:

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}}\zeta(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\pi^{\frac{1-s}{2}}}\zeta(1-s).$$

Compute $\operatorname{Res}_{s=1} \zeta(s)$ using the functional equation and the fact that

$$\operatorname*{Res}_{s=1} \zeta(s) = \lim_{s \to 1} (1-s)\zeta(s).$$

We now introduce Fourier transforms and Fourier coefficients. Suppose $f(x)$ is absolutely integrable on $\mathbb{R}$. The **Fourier transform** $(\mathcal{F}f)(\xi)$ of $f(x)$ is defined by

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\,dx,$$

for $\xi \in \mathbb{R}$. This integral is absolutely convergent precisely because $f(x)$ is absolutely integrable on $\mathbb{R}$. The Fourier transform is intimately related to periodic functions. If $f(x)$ is 1-periodic and integrable on $[0,1]$ then we define the $n$-th **Fourier coefficient** $\hat{f}(n)$ of $f(x)$ by

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\,dx.$$

The **Fourier series** of $f(x)$ is defined by the series

$$\sum_{n\in\mathbb{Z}^n} \hat{f}(n)e^{2\pi inx}.$$

There is the question of whether the Fourier series of $f(x)$ converges at all and if so does it even converge to $f(x)$ itself. Under reasonable conditions this is possible:

**Proposition 0.1.** *If $f(x)$ is smooth and 1-periodic then it converges uniformly to its Fourier series.*

The link between the Fourier transform and Fourier series is given by the **Poisson summation formula**:

**Theorem (Poisson summation formula).** *Suppose $f(x)$ is absolutely integrable on $\mathbb{R}$, and the function*

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n),$$

*is locally absolutely uniformly convergent and smooth. Then*

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{t\in\mathbb{Z}} (\mathcal{F}f)(t)e^{2\pi itx},$$

*and*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{t\in\mathbb{Z}} (\mathcal{F}f)(t).$$

(i) Prove the Poisson summation formula. (*hint*: can you compute the Fourier coefficients of $F(x)$? What is it's Fourier series?)

(ii) There are two ways of building a periodic function from an absolutely integrable function on $\mathbb{R}$. What does the Poisson summation formula say about these periodic functions?

Now we develop some basic properties of Fourier transforms. In practical settings, we need a class of functions $f(x)$ for which the assumptions of the Poisson summation formula hold. We say that $f(x)$ is of **Schwarz class** if $f \in C^\infty(\mathbb{R})$ and $f(x)$ along with all of its partial derivatives have **rapid decay**. This means $f(x) = o(|x|^{-n})$ for all $n \geq 0$. If $f(x)$ is of Schwarz class, the rapid decay implies that $f(x)$ and all of its derivatives are absolutely integrable over $\mathbb{R}$. Moreover, this also implies that $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ and all of its derivatives are locally absolutely uniformly convergent by the Weierstrass $M$-test. The uniform limit theorem then implies $F(x)$ is smooth and thus the conditions of the Poisson summation formula are satisfied. We will now derive some properties of the Fourier transform including a case specific to Schwarz class functions:

Let $f(x)$ and $g(x)$ be absolutely integrable on $\mathbb{R}$. Prove the following:

(i) For any $\alpha, \beta \in \mathbb{R}$, we have

$$(\mathcal{F}(\alpha f + \beta g))(\xi) = \alpha(\mathcal{F}f)(\xi) + \beta(\mathcal{F}g)(\xi).$$

(ii) If $g(x) = f(x+\alpha)$ for any $\alpha \in \mathbb{R}$ then

$$(\mathcal{F}g)(\xi) = e^{2\pi i \alpha \xi}(\mathcal{F}f)(\xi).$$

(iii) If $g(x) = f(\alpha x)$ for any $\alpha \in \mathbb{R}^*$ then

$$(\mathcal{F}g)(\xi) = \frac{1}{|\alpha|}(\mathcal{F}f)\left(\frac{\xi}{\alpha}\right).$$

(iv) If $f(x)$ is of Schwarz class and $g(x) = \frac{\partial}{\partial x}^k f(x)$ for some $k \geq 0$ then

$$(\mathcal{F}g)(\xi) = (2\pi i \xi)^k (\mathcal{F}f).$$

(v) Let $f(x) = e^{-2\pi\alpha x^2}$. Show that

$$(\mathcal{F}f)(\zeta) = \frac{e^{-\frac{\pi\zeta^2}{2\alpha}}}{\sqrt{2\alpha}}.$$

In particular, what function is its own Fourier transform?

You will now derive Equation (1) using the theory of Fourier transforms and Poisson summation. Consider the function

$$\vartheta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z},$$

defined for $\zeta \in \mathbb{H}$. We will use this function to prove Equation (1):

(i) Apply the Poisson summation formula to $\vartheta(z)$ to prove

$$\vartheta(z) = \frac{1}{\sqrt{-2iz}}\vartheta\left(-\frac{1}{4z}\right).$$

You may use the identity theorem to assume $z = iy$ for $y > 0$.

(ii) Show that

$$\omega(z) = \frac{\vartheta\left(\frac{z}{2}\right) - 1}{2}.$$

(iii) Deduce Equation (1).