A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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Abstract. We construct a quadratic double Dirichlet series $Z(s,w)$ built from single variable quadratic Dirichlet L-functions $L(s,\chi)$ over $\mathbb{Q}$. We prove that $Z(s,w)$ admits meromorphic continuation to the (s,w)-plane and satisfies a group of functional equations.

1. Preliminaries

We present an overview of quadratic Dirichlet L-functions over $\mathbb{Q}$. We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m\geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Now we recall characters on $\mathbb{Z}$. They are multiplicative functions $\chi : \mathbb{Z} \to \mathbb{C}$. They form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions $\chi_d : \mathbb{Z} \to \mathbb{C}$ modulo $d \geq 1$ (in that they are d-periodic) and such that $\chi_d(m) = 0$ if $(m,d) > 1$.
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. If χ is a Dirichlet character then its conjugate $\overline{\chi}$ is also a Dirichlet character. Moreover, $\overline{\chi}$ is the multiplicative inverse to χ and the Dirichlet characters modulo m form a group under multiplication. This group is always finite and its order is $\phi(d) = |(\mathbb{Z}/d\mathbb{Z})^*|$. Dirichlet characters also satisfy orthogonality relations:

Theorem 1.1 (Orthogonality relations)**.**

(i) For any two Dirichlet characters χ and ψ modulo d,

$$\frac{1}{\phi(d)} \sideset{}{'}\sum_{a \pmod{d}} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

(ii) For any $a, b \in (\mathbb{Z}/d\mathbb{Z})^*$,

$$\frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

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The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{Z}$. First let us recall this symbol. For any odd prime p and any $m \geq 1$, we define the quadratic residue symbol $\left(\frac{m}{p}\right)$ by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod p = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod p \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod p \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod p. \end{cases}$$

This symbol only depends upon m modulo p and is multiplicative in m. We can extend the quadratic residue symbol multiplicatively in the denominator. If $d = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of d, then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \leq i \leq k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any odd $d \geq 1$. We can extend this symbol further and allow $d \geq 1$ to be even. To this end, we define

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod 8, \\ -1 & \text{if } m \equiv 3, 5 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

and extend $\left(\frac{m}{d}\right)$ multiplatively in d when d is even. Now the quadratic residue symbol makes sense for any $m, d \geq 1$. Moreover, it is multiplicative in both m and d but no longer depends upon only m modulo d (it also depends upon m modulo 8). In particular,

$$\left(\frac{-1}{d}\right) = \begin{cases} 1 & d \equiv 1 \pmod 4, \\ -1 & d \equiv 3 \pmod 4, \\ 0 & d \equiv 0 \pmod 2, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = \begin{cases} 1 & d \equiv 1, 7 \pmod 8, \\ -1 & d \equiv 3, 5 \pmod 8, \\ 0 & d \equiv 0 \pmod 2, \end{cases}$$

and if $d \not\equiv 0 \pmod 2$, we can compactly write

$$\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}} = \begin{cases} 1 & d \equiv 1 \pmod 4, \\ -1 & d \equiv 3 \pmod 4, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = (-1)^{\frac{d^2-1}{8}} = \begin{cases} 1 & d \equiv 1, 7 \pmod 8, \\ -1 & d \equiv 3, 5 \pmod 8. \end{cases}$$

The quadratic residue symbol also admits the following reciprocity law:

Theorem 1.2 (Quadratic reciprocity)**.** *If $d, m \geq 1$, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2} \frac{m^{(2)}-1}{2}} \left(\frac{m}{d}\right),$$

where $d^{(2)}$ and $m^{(2)}$ are the parts of d and m relatively prime to 2 respectively.

We can now define the quadratic Dirichlet characters. For any odd square-free $d \in \mathbb{Z}$, define the quadratic Dirichlet character χ_d by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod 4, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension $\mathbb{Q}(\sqrt{d})$. We extend χ_d multiplicatively in the denominator so that χ_d makes sense for any odd d. In particular, $\chi_d(m) = \pm 1$ provided d and m are relatively prime and $\chi_d(m) = 0$ if $(m, d) > 1$. Quadratic reciprocity implies that χ_d is a Dirichlet character modulo d if $d \equiv 1 \pmod 4$ and is a Dirichlet character modulo $4d$ if $d \equiv 2, 3 \pmod 4$. Indeed, if $d \equiv 1 \pmod 4$ then $d^{(2)} = d$ and the sign is always 1. If $d \equiv 3 \pmod 4$, then $d^{(2)} = d$ and the sign is

$\left(\frac{-1}{m}\right)$ which is a character modulo 4. If $d \equiv 2 \pmod 4$, then $d^{(2)} \equiv 1, 3 \pmod 4$ and we are reduced to one of the previous two cases. We will also require an associated character. For each χ_d, we define $\widetilde{\chi}_d$ by

$$\widetilde{\chi}_d(m) = (-1)^{\frac{d^{(2)}-1}{2}\frac{m^{(2)}-1}{2}}\chi_d(m).$$

Equivalently, $\widetilde{\chi}_d(m)$ can be expressed as

$$\widetilde{\chi}_d(m) = \begin{cases} \chi_d(m) & \text{if } d \equiv 1, 2 \pmod 4, \\ \chi_{-1}(m)\chi_d(m) & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

and it follows that $\widetilde{\chi}_d(m)$ is a quadratic Dirichlet character of the same modulus as χ_d. We now discuss the Hilbert characters. We will only need four of them: the quadratic Dirichlet characters modulo 8. They are given as follows:

$$\chi_1(m) = \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod 2, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases} \quad \chi_{-1}(m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4, \\ -1 & \text{if } m \equiv 3 \pmod 4, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

$$\chi_2(m) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod 8, \\ -1 & \text{if } m \equiv 3, 5 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases} \quad \chi_{-2}(m) = \begin{cases} 1 & \text{if } m \equiv 1, 3 \pmod 8, \\ -1 & \text{if } m \equiv 5, 7 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

In general, we will denote a Hilbert character by χ_a with $a \in \{\pm 1, \pm 2\}$. Note that

$$\chi_{-1}(m) = \left(\frac{-1}{m}\right) \quad \text{and} \quad \chi_2(m) = \left(\frac{m}{2}\right).$$

Moreover, we have the relations

$$\chi_{-2}(m) = \chi_{-1}(m)\chi_2(m), \quad \chi_1(m) = \chi_{-1}(m)\chi_{-1}(m), \quad \text{and} \quad \chi_{-1}(m) = \chi_2(m)\chi_{-2}(m).$$

Suppose d is square-free. If $d \equiv 1, 2, 5 \pmod 8$, then $d^{(2)} \equiv 1 \pmod 4$ so that the sign in the statement of quadratic recipricty is 1. If $d \equiv 3, 6, 7 \pmod 8$, then $d^{(2)} \equiv 3 \pmod 4$ and the sign is $(-1)^{\frac{m^{(2)}-1}{2}}$. This fact together with the relations for the quadratic characters modulo 8 imply

$$\chi_d(m) = \begin{cases} \chi_m(d) & \text{if } d \equiv 1 \pmod 4, \\ \chi_{-1}(m)\chi_m(d) & \text{if } d \equiv 3 \pmod 4, \\ \chi_2(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 2 \pmod 8, \\ \chi_{-2}(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 6 \pmod 8. \end{cases}$$

With the Dirichlet and Hilbert characters introduced, we are ready to discuss the L-functions associated to quadratic Dirichlet characters. We define the L-function $L(s, \chi_d)$ attached to χ_d for square-free d, by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \geq 1} \frac{\chi_d(m)}{|m|^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi_d(P)}{|P|^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character, $L(s, \chi_d) \ll \zeta(s)$ for $\mathrm{Re}(s) > 1$ so that $L(s, \chi_d)$ is locally absolutely uniformly convergent in this region. $L(s, \chi_d)$ also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ if d is a perfect square. For square-free d, the completed L-function $L^*(s, \chi_d)$ is defined as

$$L^*(s, \chi_d) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) L(s, \chi_d) & \text{if } \chi_d \text{ is even}, \\ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right) L(s, \chi_d) & \text{if } \chi_d \text{ is odd}, \end{cases}$$

and satisfies the functional equation

$$L^*(s,\chi_d) = \begin{cases} \varepsilon_\chi q^{\frac{1}{2}-s} L^*(1-s,\chi_d) & \text{if } \chi_d \text{ is even,} \\ -\varepsilon_\chi q^{\frac{1}{2}-s} L^*(1-s,\chi_d) & \text{if } \chi_d \text{ is odd.} \end{cases}$$

Note that the gamma factors depend upon the partiy of χ_d. This the root cause of an important technical issue later when deriving functional equations for the quadratic double Dirichlet series.

Remark 1.1. *The definitions for $L(s,\chi_d)$ and $L^*(s,\chi_d)$ work perfectly well even when d is not square-free (however the functional equations do not hold). We purposely do not define these L-functions, yet, for d not necessarily square-free.*

The Quadratic Double Dirichlet Series

We will now define the quadratic double Dirichlet series $Z(s,w)$. For any integer $d \geq 1$, write $d = d_0 d_1^2$ where d_0 is square-free. Equivalently, d_0 is the square-free part of d and $\frac{d}{d_0}$ is a perfect square. The **quadratic double Dirichlet series** $Z(s,w)$ is defined as

$$Z(s,w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s,\chi_{d_0}) Q_{d_0 d_1^2}(s)}{d^w},$$

where the superscript (2) indicates that the local factor at 2 has been removed, $Q_{d_0 d_1^2}(s)$ is the **correction polynomial** defined by

$$Q_{d_0 d_1^2}(s) = \sum_{e_1 e_2 | d_1} \mu(e_1)\chi_{d_0}(e_1) e_1^{-s} e_2^{1-s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{d_0}(e_1) e_1^{-s} e_2^{1-s},$$

and μ is the usual Möbius function. For $\mathrm{Re}(s) > 1$, there is the trivial estimate

$$Q_{d_0 d_1^2}(s) \ll \sum_{e_1 e_2 | d_1} 1 \ll \sigma_0(d_1)^2 \ll_\varepsilon |d_1^2|^\varepsilon \ll_\varepsilon |d|^\varepsilon,$$

for any $\varepsilon > 0$. As $L(s,\chi_{d_0}) \ll 1$ for $\mathrm{Re}(s) > 1$, $Z(s,w)$ is locally absolutely uniformly convergent in the region $\Lambda = \{(s,w) \in \mathbb{C}^2 : \mathrm{Re}(s) > 1, \mathrm{Re}(w) > 1\}$. It will also be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters χ_{a_1} and χ_{a_2}. The **quadratic double Dirichlet series** $Z_{a_1,a_2}(s,w)$ twisted by χ_{a_1} and χ_{a_2} is defined as

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s,\chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s,\chi_{a_1})}{|d|^w},$$

where $Q_{d_0 d_1^2}(s,\chi_{a_1})$ is the **correction polynomial** twisted by χ_{a_1} defined by

$$Q_{d_0 d_1^2}(s,\chi_{a_1}) = \sum_{e_1 e_2 | d_1} \mu(e_1)\chi_{a_1 d_0}(e_1)|e_1|^{-s}|e_2|^{1-2s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{a_1 d_0}(e_1)|e_1|^{-s}|e_2|^{1-2s},$$

and μ is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound $Q_{d_0 d_1^2}(s,\chi_{a_1}) \ll |d|_\varepsilon$ so that $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly in the same region as $Z(s,w)$ does. In particular, $Z(s,w) = Z_{1,1}(s,w)$.

The Interchange

As defined, $Z_{a_1,a_2}(s,w)$ is a sum of L-functions, and hence Euler products, in s. We will prove an interchange formula for $Z_{a_1,a_2}(s,w)$ which will show that it can be expressed as a sum of L-functions in w. That is, we want the variables s and w to change places. Precisely:

Theorem 1.3 (Interchange)**.** *Wherever $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly,*

$$Z_{a_1,a_2}(s,w)=\sum_{d\ odd}\frac{L^{(2)}(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{|d|^w}=\sum_{m\ odd}\frac{L^{(2)}(w,\widetilde{\chi}_{a_2m_0})\chi_{a_1}(m)Q_{m_0m_1^2}(w,\chi_{a_2})}{|m|^s}.$$

Proof. Only the second equality needs to be proved. To do this, first expand the L-function $L^{(2)}(s,\chi_{a_1d_0})$ and polynomial $Q_{d_0d_1^2}(s,\chi_{a_1})$ to get

$$\begin{aligned}Z(s,w)&=\sum_{d\text{ odd}}\frac{L(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{|d|^w}\\&=\sum_{d\text{ odd}}\left(\sum_{m\text{ odd}}\chi_{a_1d_0}(m)|m|^{-s}\right)\left(\sum_{e_1e_2|d_1}\mu(e_1)\chi_{a_1d_0}(e_1)|e_1|^{-s}|e_2|^{1-2s}\right)\chi_{a_2}(d)|d|^{-w}\\&=\sum_{m,d\text{ odd}}\sum_{e_1e_2|d_1}\mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(me_1)|e_1|^{-s}|e_2|^{1-2s}|m|^{-s}|d|^{-w}.\end{aligned}$$

Now $\chi_{a_1d_0}(me_1)=0$ unless $(d_0,me_1)=1$. We make this restriction on the sum giving

$$\sum_{m,d\text{ odd}}\sum_{\substack{e_1e_2|d_1\\(d_0,me_1)=1}}\mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(me_1)|e_1|^{-s}|e_2|^{1-2s}|m|^{-s}|d|^{-w}.$$

Making the change of variables $me_1\to m$ yields

$$\sum_{d\text{ odd}}\sum_{\substack{m\text{ odd}\\e_1|m}}\sum_{\substack{e_1e_2|d_1\\(d_0,m)=1}}\mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(m)|e_2|^{1-2s}|m|^{-s}|d|^{-w}.$$

For fixed $d=d_0d_1^2$ and e_2, the subsum over m and e_1 is

$$\sum_{\substack{m\text{ odd}\\e_1|m}}\sum_{\substack{e_1|\frac{d_1}{e_2}\\(d_0,m)=1}}\mu(e_1)\chi_{a_1d_0}(m)|m|^{-s}=\sum_{\substack{m\text{ odd}\\(d_0,m)=1}}\chi_{a_1d_0}(m)|m|^{-s}\left(\sum_{e_1|\left(\frac{d_1}{e_2},m\right)}\mu(e_1)\right).$$

The inner sum over e_1 of the Möbius function vanishes unless $\left(\frac{d_1}{e_2},m\right)=1$ in which case it is 1. Therefore the triple sum above becomes

$$\sum_{m,d\text{ odd}}\sum_{\substack{e_2|d_1\\\left(\frac{d_0d_1}{e_2},m\right)=1}}\chi_{a_2}(d)\chi_{a_1d_0}(m)|e_2|^{1-2s}|m|^{-s}|d|^{-w}.$$

Making the change of variables $d\to de_2^2$, the condition $\left(\frac{d_0d_1}{e_2},m\right)=1$ becomes $(d_0d_1,m)=1$ which is equivalent to $(d,m)=1$. Moreover, $\chi_{a_2}(de_2^2)=\chi_{a_2}(d)$. Altogether, we obtain

$$\sum_{\substack{m,d\text{ odd}\\(d,m)=1}}\sum_{e_2}\chi_{a_2}(d)\chi_{a_1d_0}(m)|e_2|^{1-2s-2w}|m|^{-s}|d|^{-w}.$$

Writing $m=m_0m_1^2$ analogously as for d, quadratic reciprocity implies $\chi_{d_0}(m)=\widetilde{\chi}_m(d_0)=\widetilde{\chi}_{m_0}(d)$ where the last equality holds because $(d,m)=1$ and both d_0 and m_0 differ from d and m respectively by perfect squares. This implies $\chi_{a_2}(d)\chi_{a_1d_0}(m)=\chi_{a_1}(m)\widetilde{\chi}_{a_2m_0}(d)$ and so our expression becomes

$$\sum_{\substack{m,d\text{ monic}\\(d,m)=1}}\sum_{e_2}\chi_{a_1}(m)\widetilde{\chi}_{a_2m_0}(d)|e_2|^{1-2s-2w}|m|^{-s}|d|^{-w}.$$

But now we can reverse the argument with the roles of d, m, χ_{a_1}, and $\widetilde{\chi}_{a_2}$ interchanged respectively to obtain

$$Z(s,w) = \sum_{m \text{ monic}} \frac{L(w, \widetilde{\chi}_{a_2 m_0})\chi_{a_1}(m) Q_{m_0 m_1^2}(w, \chi_{a_2})}{|m|^s}.$$

□

Note that the interchange is not completely symmetric because of the character $\widetilde{\chi}_{a_2 m_0}$ in the second expression for $Z_{a_1,a_2}(s,w)$. This is due to the fact that reciprocty is not perfect. In even more general settings the correction polynomials in w need not be equal to those in s.

Weighting Terms

We will now study the coefficients of $Z_{a_1,a_2}(s,w)$ expanded in s and w. By expanding $L(s,\chi_{a_1 d_0})Q_{d_0 d_1^2}(s,\chi_{a_1})$ in the numerator of $Z_{a_1,a_2}(s,w)$, we can write

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ monic}} \frac{L(s,\chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s,\chi_{a_1})}{|d|^w} = \sum_{m,d \text{ monic}} \frac{\chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) a(m,d)}{|m|^s |d|^w},$$

where $\widehat{m}$ is the part of m relatively prime to d_0 and the **weighting coefficient** $a(m,d)$ is given by

$$a(m,d) = \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1)|e_2|.$$

To see this, the coefficient of $|m|^{-s}|d|^{-w}$ in the definition of $Z_{a_1,a_2}(s,w)$ is

$$\begin{aligned}
\chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1}} \mu(e_1)\chi_{a_1 d_0}(e_1 e_3)|e_2| &= \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1)\chi_{a_1 d_0}(e_1 e_3)|e_2| \\
&= \chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1)|e_2| \\
&= \chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) a(m,d),
\end{aligned}$$

where the first equality holds because $\chi_{d_0}(e_1 e_3) = 0$ unless $(d_0, e_1 e_3) = 1$ and the second equality holds because if $(d_0, e_1 e_3) = 1$, $\widehat{m}$ differs from $e_1 e_3$ by a perfect square (the divisors of which belong to (d_0, e_2)) and so $\chi_{d_0}(e_1 e_3) = \chi_{d_0}(\widehat{m})$.

Remark 1.2. *Also, $a(m,d) = 0$ unless $m = e_1 e_2^2 e_3$ with $(d_0, e_1 e_3) = 1$ and $e_1 e_2^2 \mid d_1$.*

Todo: [resume here]

We will let $L(s,\chi_{a_1 d})$ be the Dirichlet series defined by

$$L(s,\chi_{a_1 d}) = L(s,\chi_{a_1 d_0}) Q_{d_0 d_1^2}(s,\chi_{a_1}) = \sum_{m \text{ monic}} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m,d)}{|m|^s},$$

so that

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ monic}} \frac{\chi_{a_2}(d) L(s,\chi_{a_1 d})}{|d|^w}.$$

If we perform the same procedure to the interchange, then

$$Z_{a_1,a_2}(s,w) = \sum_{m \text{ monic}} \frac{L(w,\widetilde{\chi}_{a_2 m_0})\chi_{a_1}(m) Q_{m_0 m_1^2}(w,\chi_{a_2})}{|m|^s} = \sum_{m,d \text{ monic}} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d})\chi_{a_1}(m) a(d,m)}{|m|^s |d|^w},$$

where $\widehat{d}$ is the part of d relatively prime to m_0. Analogously, we define $L(w, \widetilde{\chi}_{a_2 m})$ by

$$L(w, \widetilde{\chi}_{a_2 m}) = L(w, \widetilde{\chi}_{a_2 m_0}) Q_{m_0 m_1^2}(w, \chi_{a_2}) = \sum_{d \text{ monic}} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d}) a(d, m)}{|d|^w},$$

so that

$$Z_{a_1, a_2}(s, w) = \sum_{m \text{ monic}} \frac{\chi_{a_1}(m) L(w, \widehat{\chi}_{a_2 m})}{|m|^s}.$$

We can now derive an important relationship for the weighting coefficients. Equating the coefficients of the two expansions for $Z_{a_1,a_2}(s, w)$ gives

$$\chi_{a_1 d_0}(\widehat{m}) \chi_{a_2}(d) a(m, d) = \chi_{a_2 m_0}(\widehat{d}) \chi_{a_1}(m) a(d, m)$$

By applying quadratic reciprocity twice we have

$$\chi_{d_0}(\widehat{m}) = \chi_{\widehat{m}}(d_0) = \chi_{\widehat{m_0}}(d_0) = \chi_{m_0}(\widehat{d_0}) = \chi_{m_0}(\widehat{d}),$$

so upon setting $a_1 = a_2 = 1$ we have

$$a(m, d) = a(d, m).$$

In other words, the weighting coefficients are symmetric. The weighting coefficients also posess a multiplicativity property:

Proposition 1.1. *We have* $a(m, 1) = a(1, d) = 1$ *and*

$$a(m, d) = \prod_{\substack{P^\alpha || m \\ P^\beta || d}} a(P^\alpha, P^\beta).$$

Functional Equations

Meromorphic Continuation

Poles and Residues

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