SUBCONVEXITY FOR GL_2 L-FUNCTIONS VIA MULTIPLE DIRICHLET SERIES

HENRY TWISS

Abstract. We use the amplification method and the analytic properties of shifted multiple Dirichlet series to obtain a subconvexity reuslt for twisted GL_2 holomorphic cusp forms. This proves upon the subconvexity bound established in [1].

Subconvexity

Let f be a weight k holomorphic cusp form on $\Gamma_1(q)\backslash\mathbb{H}$, for a prime q, and with trivial character. Let χ be a primitive Dirichlet character of conductor p. Suppose $(p,q)=1$. Then it is a result of [1] that

$$L\left(\frac{1}{2}, f\times\chi\right) \ll_{k,\varepsilon} (qp^2)^{\frac{1}{4}+\varepsilon}(p^{-\frac{1}{4}}+q^{\frac{\theta}{4}-\frac{1}{8}}),$$

where θ is the current best bound towards the Ramanujan-Petersson conjecture. We will improve upon this bound in the level 1 case using Weyl group multiple Dirichlet series.

1. Background Setup

Let $q\geq 1$ and let ψ be a Dirichlet character modulo q. For $m\in\mathbb{Z}$, let

$$c_q(m) = \sum_{a \pmod q} e^{\frac{2\pi i m a}{q}} \quad \text{and} \quad c_\psi(m) = \sum_{a \pmod q} \psi(a) e^{\frac{2\pi i m a}{q}},$$

be the Ramanujan and Gauss sums respectively. In particular, for ℓ such that $(\ell,q)=1$, we have

$$c_\psi(\ell m) = \overline{\psi(\ell)} c_\psi(m),$$

and moreover

$$c_\psi(m) = \overline{\psi(m)} c_\psi(1),$$

provided ψ is primitive. Throughout we will let

$$f(z) = \sum_{m\geq 1} a(m)e^{2\pi i m z} = \sum_{m\geq 1} A(m) m^{\frac{k-1}{2}} e^{2\pi i m z} \quad \text{and} \quad g(z) = \sum_{m\geq 1} b(m)e^{2\pi i m z} = \sum_{m\geq 1} B(m) m^{\frac{k-1}{2}} e^{2\pi i m z},$$

be the Fourier expansions of two weight k and level 1 Hecke eigenforms f and g. We define the L-function $L(s, f\times c_\psi)$ by

$$L(s, f\times c_\psi) = \sum_{m\geq 1} \frac{A(m) c_\psi(m)}{m^s}.$$

When ψ is primitive this is related to the L-function $L(s, f\times\psi)$ by

$$L(s, f\times c_\psi) = \sqrt{q} L(s, f\times\psi).$$

We will now define our primary object of interest:

$$S_{f,g}(s_1,s_2;q) = \frac{1}{\varphi(q)} \sum_{\psi \pmod q} L(s_1, f\times c_\psi) L(s_2, g\times \overline{c_\psi}).$$

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There are multiple Dirichlet series that are connected to these sums. Throughout we let $\{\mu_j\}$ represent an orthonormal basis of Maass forms with spectral parameter t_j for μ_j. Moreover, let ℓ_1 and ℓ_2 be fixed primes. Also set

$$V_{f,g} = V_{f,g}(z;\ell_1,\ell_2) = \overline{f(\ell_1 z)}g(\ell_2 z)\mathrm{Im}(z)^k \quad \text{and} \quad V_{f,v} = V_f(z;\ell_1) = \overline{f(\ell_1 z)}E(z,s;k)\mathrm{Im}(z)^{\frac{k}{2}}.$$

The Dirichlet Series $D_{f,g}(s;h,\ell_1,\ell_2)$. Let $h \geq 1$. Our first multiple Dirichlet series $D_{f,g}(s;h,\ell_1,\ell_2)$ is given by

$$D_{f,g}(s;h,\ell_1,\ell_2) = \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m)b(n)}{n^{s+k-1}}.$$

This series is absolutely convergent for $\mathrm{Re}(s) > 1$ and admits meromorphic continuation to $\frac{1-k}{2} - C_1 < \mathrm{Re}(s)$, for any $C_1 > 0$, and in these two regions it satisfies the bounds

$$D_{f,g}(s;h,\ell_1,\ell_2) \ll_{\text{Todo:}[\ell_1,\ell_2]} h^{\frac{k-1}{2}+\varepsilon} \quad \text{and} \quad D_{f,g}(s;h,\ell_1,\ell_2) \ll_{\text{Todo:}[\ell_1,\ell_2]} h^{k+2C_1+\varepsilon},$$

respectively. In the region $\frac{1-k}{2} - C_1 < \mathrm{Re}(s)$, the meromorphic continuation is given by the absolutely convergent spectral expansion

$$D_{f,g}(s;h,\ell_1,\ell_2) = \frac{\Gamma(1-s)}{\Gamma(s+k-1)} \sum_j \overline{\rho_{t_j}(-h)} h^{\frac{1}{2}-s} \frac{\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)} \overline{\langle V_{f,g},\mu_j\rangle}.$$

These two representations for $D_{f,g}(s;h,\ell_1,\ell_2)$ give meromorphic continuation to $\mathbb{C}$ but we do not have a repsentation in the strip $\frac{1-k}{2} \leq \mathrm{Re}(s) \leq 1$.

The Dirichlet Series $D_{f,v}(w;n,\ell_1,\ell_2)$. Let $n \geq 1$. Our second multiple Dirichlet series $D_{f,v}(w;n,\ell_1,\ell_2)$ is given by

$$D_{f,v}(w;n,\ell_1,\ell_2) = \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m)\sigma_{1-2v}(h)h^{v-\frac{1}{2}}}{h^{w+\frac{k-1}{2}}}.$$

Let $c > 0$ be such that if v satisfies $\zeta(2v) \neq 0$, then $\mathrm{Re}(v) \geq \frac{1}{2} - \frac{8c}{\log(2+\mathrm{Im}(v))}$. For such a c, we set

$$\delta(s,v,u) = \frac{c}{\log(3+|\mathrm{Im}(s+u)|+|\mathrm{Im}(v)|)} \quad \text{and} \quad \delta_v = \delta(0,0,v).$$

The former series converges absolutely for $\mathrm{Re}(s) > 1$ while the latter does for $\mathrm{Re}(w) > \mathrm{Re}(v) + \frac{1}{2}$ and $\mathrm{Re}(v) > \frac{1}{2}$. In these regions, the series satisfy the estimates

$$D_{f,g}(s;h,\ell_1,\ell_2) \ll h^{\frac{k-1}{2}+\varepsilon} \quad \text{and} \quad D_{f,v}(w;n,\ell_1,\ell_2) \ll n^{\frac{k-1}{2}+\varepsilon}.$$

We define the associted multiple Dirichlet series

$$Z_{f,g}(s,u,v;\ell_1,\ell_2) = \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m)b(n)\sigma_{1-2v}(h)}{n^{s+k-1}h^u}.$$

This converges absolutely for $\mathrm{Re}(s) > 1$, $\mathrm{Re}(u) > \frac{k+1}{2}$, and $\mathrm{Re}(v) > \frac{1}{2}$. Moreover, in this region $Z_{f,g}$ can be expressed in terms of $D_{f,g}$ and $D_{f,v}$ as

$$Z_{f,g}(s,u,v;\ell_1,\ell_2) = \sum_{h \geq 1} \frac{D_{f,g}(s;h,\ell_1,\ell_2)\sigma_{1-2v}(h)}{h^u} = \sum_{n \geq 1} \frac{D_{f,v}\left(u+v-\frac{k}{2};n,\ell_1,\ell_2\right)b(n)}{n^{s+k-1}},$$

with both representations converging absolutely. The series $D_{f,g}$ and $D_{f,v}$ also admit spectral expansions. To state them, set

$$V_{f,g} = V_{f,g}(z;\ell_1,\ell_2) = \overline{f(\ell_1 z)}g(\ell_2 z)\mathrm{Im}(z)^k \quad \text{and} \quad V_{f,v} = V_f(z;\ell_1) = \overline{f(\ell_1 z)}E(z,s;k)\mathrm{Im}(z)^{\frac{k}{2}}.$$

From [2], $D_{f,g}$ admits the spectral expansion (modulo the continuous spectrum and up to constants)

$$D_{f,g}(s;h,\ell_1,\ell_2)=\frac{\Gamma(1-s)}{\Gamma(s+k-1)}\sum_j\overline{\rho_{t_j}(-h)}h^{\frac{1}{2}-s}\frac{\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)}\overline{\langle V_{f,g},\mu_j\rangle},$$

which converges absolutely for $\mathrm{Re}(s)<\frac{1-k}{2}$ and at least ε away from the poles. By analytic continuation, $D_{f,g}$ is meromorphic on $\mathbb{C}$. This induces a spectral expansion for $Z_{f,g}$ given by

$$Z_{f,g}(s,u,v;\ell_1,\ell_2)=\frac{\Gamma(1-s)}{\Gamma(s+k-1)}\sum_j\overline{\rho_{t_j}(-1)}h^{\frac{1}{2}-s}\frac{\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)}\overline{\langle V_{f,g},\mu_j\rangle}$$
$$\cdot\frac{L\left(s+u-\frac{1}{2},\mu_j\right)L\left(s+u+2v-\frac{3}{2},\mu_j\right)}{\zeta(2s+2u+2v-2)},$$

which converges absolutely for $\mathrm{Re}(s)<\frac{1-k}{2}$, $\mathrm{Re}(u)>\frac{k+1}{2}$, and $\mathrm{Re}(v)>\frac{1}{2}$. Similarily, $D_{f,v}$ admits the spectral expansion (modulo the continuous spectrum and up to constants)

$$D_{f,v}\left(w;n,\ell_1,\ell_2\right)=\frac{\Gamma(1-w)\Gamma(w)}{\Gamma\left(w+v+\frac{k}{2}-1\right)\Gamma\left(w-v+\frac{k}{2}\right)}\sum_j\overline{\rho_{t_j}(-\ell_2 n)}(\ell_2 n)^{\frac{1}{2}-w}\frac{\Gamma\left(w-\frac{1}{2}+it_j\right)\Gamma\left(w-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)}$$
$$\cdot\overline{\langle V_{f,v},\mu_j\rangle},$$

which converges absolutely for $\mathrm{Re}(w)<\frac{1-k}{2}$. By analytic continuation, $D_{f,v}$ is meromorphic on $\mathbb{C}$. This induces another spectral expansion for $Z_{f,g}$ given by

$$Z_{f,g}(s,u,v;\ell_1,\ell_2)=\frac{\Gamma\left(\frac{k}{2}+1-u-v\right)\Gamma\left(u+v-\frac{k}{2}\right)}{\Gamma(u+2v-1)\Gamma(u)}\sum_j\overline{\rho_{t_j}(-1)}\frac{\Gamma\left(u+v-\frac{k+1}{2}+it_j\right)\Gamma\left(u+v-\frac{k+1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)}$$
$$\cdot\overline{\langle V_{f,v},\mu_j\rangle}\ell_2^{\frac{k+1}{2}-u-v}\frac{L^{(\ell_2)}(s+u+v-1,g\times\mu_j)}{\zeta^{(\ell_2)}(2s+2u+2v-2)}\sum_{\alpha\geq 0}\frac{b(\ell_2^\alpha)\lambda_f(\ell_2^{\alpha+1})}{(\ell_2^\alpha)^{s+u+v-1+\frac{k-1}{2}}},$$

which converges absolutely for $\mathrm{Re}(s)>1$, $\mathrm{Re}(u)<1-2\mathrm{Re}(v)$ Todo: [subtly from Jeff], and $\mathrm{Re}(v)>\frac{1}{2}$ Todo: [update subtly from Jeff].

2. An Amplified Series

Let ℓ be prime, fix a primitive Dirichlet character χ modulo $Q\gg 1$, and set

$$S_f(s;q,L)=\frac{1}{\varphi(q)}\sum_{\psi\pmod q}|L(s,f\times c_\psi)|^2\left|\sum_{\ell\sim L}\chi(\ell)\overline{\psi(\ell)}\right|^2,$$

where $\ell\sim L$ means $\ell\in[L,2L]$. We will primarily be interested in an average of $S_f(s;q,L)$ over q in a short interval around Q. Accordingly, for any $\varepsilon>0$ define

$$S_f(s;Q,L)=\sum_{|q-Q|\ll Q^\varepsilon}S_f(s;q,L).$$

Our desired result will follow from upper and lower bounds for this sum. The presence of the sum over $\ell\sim L$ in each $S_f(s;q,L)$ is to amplify the term attached to the character χ (note that this only happens when $q=Q$). For the lower bound, consider $S_f(s;Q,L)$. When $\psi=\chi$ the prime number theorem gives

$$\left|\sum_{\ell\sim L}\chi(\ell)\overline{\chi}(\ell)\right|^2\sim\frac{L^2}{\log(L)^2}.$$

So the contribution coming from the term corresponding to χ is

$$\frac{L^2}{\varphi(Q)\log(L)^2}|L(s, f\times c_\chi)|^2 = \frac{QL^2}{\varphi(Q)\log(L)^2}|L(s, f\times\chi)|^2.$$

Since every term in $S_f(s;Q,L)$ is nonnegative, we can discard them and obtain a lower bound of the form

$$\frac{QL^2}{\varphi(Q)\log(L)^2}|L(s, f\times\chi)|^2 \ll S_f(s;q,L).$$

Recalling that $\varphi(Q)\sim Q$ and discarding the other $S_f(s;q,L)$, we arrive at the associated lower bound

$$\frac{L^2}{\log(L)^2}|L(s, f\times\chi)|^2 \ll S_f(s;Q,L).$$

The upper bound requires much more delicate treatment. We first expand all of the sums in $S_f(s;q,L)$:

$$\begin{aligned} S_f(s;q,L) &= \frac{1}{\varphi(q)}\sum_{\psi \pmod q}\sum_{m,n\geq 1}\frac{A(m)c_\psi(m)A(n)\overline{c_\psi(n)}}{(mn)^s}\sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\psi}(\ell_1)\overline{\chi}(\ell_2)\psi(\ell_2) \\ &= \frac{1}{\varphi(q)}\sum_{\psi \pmod q}\sum_{m,n\geq 1}\frac{A(m)A(n)}{(mn)^s}\sum_{\ell_1,\ell_2\sim L} c_\psi(\ell_1 m)\overline{c_\psi}(\ell_2 n)\chi(\ell_1)\overline{\chi}(\ell_2) \\ &= \frac{1}{\varphi(q)}\sum_{m,n\geq 1}\sum_{\ell_1,\ell_2\sim L}\frac{A(m)A(n)}{(mn)^s}\sum_{\psi \pmod q} c_\psi(\ell_1 m)\overline{c_\psi}(\ell_2 n)\chi(\ell_1)\overline{\chi}(\ell_2). \end{aligned}$$

Now

$$\frac{1}{\varphi(q)}\sum_{\psi \pmod q} c_\psi(\ell_1 m)\overline{c_\psi}(\ell_2 n) = c_q(\ell_1 m - \ell_2 n),$$

so that

$$\begin{aligned} S_f(s;q,L) &= \sum_{\ell_1,\ell_2\sim L}\sum_{m,n\geq 1}\frac{A(m)A(n)}{(mn)^s}c_q(\ell_1 m-\ell_2 n)\chi(\ell_1)\overline{\chi}(\ell_2) \\ &= \sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)\sum_{m,n\geq 1}\frac{A(m)A(n)}{(mn)^s}c_q(\ell_1 m-\ell_2 n) \\ &= \sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)S_f(s;q,\ell_1,\ell_2), \end{aligned}$$

where we have set

$$S_f(s;q,\ell_1,\ell_2) = \sum_{m,n\geq 1}\frac{A(m)A(n)}{(mn)^s}c_q(\ell_1 m-\ell_2 n).$$

Therefore

$$S_f(s;Q,L) = \sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)\sum_{|q-Q|\ll Q^\varepsilon} S_f(s;q,\ell_1,\ell_2).$$

In order to carefully estimate the sum over q via Perron-type formulas, we need to understand the analytic properties of the Dirichlet series with coefficients $S_f(s;q,\ell_1,\ell_2)$. Thus, we define

$$S_f(s,v;\ell_1,\ell_2) = \sum_{q\geq 1}\frac{S_f(s;q,\ell_1,\ell_2)}{q^{2v}} = \sum_{m,n\geq 1}\frac{A(m)A(n)}{(mn)^s}\sum_{q\geq 1}\frac{c_q(\ell_1 m-\ell_2 n)}{q^{2v}}.$$

The inner sum can be expressed as

$$\sum_{q\geq 1}\frac{c_q(\ell_1 m-\ell_2 n)}{q^{2v}} = \begin{cases} \frac{\zeta(2v-1)}{\zeta(2v)} & \text{if } \ell_1 m = \ell_2 n, \\ \frac{\sigma_{1-2v}(\ell_1 m-\ell_2 n)}{\zeta(2v)} & \text{if } \ell_1 m \neq \ell_2 n. \end{cases}$$

So if we write $\ell_1 m = \ell_2 n + h$ with $h \geq 1$, then $S_f(s, v; \ell_1, \ell_2)$ can be expressed as the sum of a diagonal and an off-diagonal contribution:

$$S_f(s, v; \ell_1, \ell_2) = \frac{\zeta(2v-1)}{\zeta(2v)} \sum_{\ell_1 m = \ell_2 n} \frac{A(m)A(n)}{(mn)^s} + \frac{2}{\zeta(2v)} \sum_{\ell_1 m = \ell_2 n + h} \frac{A(m)A(n)\sigma_{1-2v}(h)}{(mn)^s}.$$

3. Perron-type Estimates

We can now apply Perron-type formulas to upper bound

$$S_f(s; Q, L) = \sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1)\overline{\chi}(\ell_2) \sum_{|q-Q| \ll Q^\varepsilon} S_f(s; q, \ell_1, \ell_2).$$

Recall the inverse Mellin transform

$$\frac{1}{2\pi i} \int_{(2)} \frac{e^{\frac{\pi v^2}{Q^2}} x^{2v}}{Q} \, dv = e^{-\frac{(y \log(x))^2}{\pi}}.$$

An application of smoothed Perron's formula with this transform when $x = Q$, yields

$$\begin{aligned}
\frac{1}{2\pi i} \int_{(2)} S_f(s, v; \ell_1, \ell_2) \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} \, dv &= \frac{1}{2\pi i} \int_{(2)} \sum_{q \geq 1} \frac{S_f(s; q, \ell_1, \ell_2)}{q^{2v}} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} \, dv \\
&= \sum_{q \geq 1} S_f(s; q, \ell_1, \ell_2) \frac{1}{2\pi i} \int_{(2)} \frac{e^{\frac{\pi v^2}{Q^2}} \left(\frac{Q}{q}\right)^{2v}}{Q} \, dv \\
&= \sum_{|q-Q| \ll Q^\varepsilon} S_f(s; q, \ell_1, \ell_2) + O_s(Q^{-B}),
\end{aligned}$$

with $B \gg 1$. We will now compute the Mellin transform in another way. We can decompose the integral into a diagonal and an off-diagonal term:

$$\begin{aligned}
\frac{1}{2\pi i} \int_{(2)} S_f(s, v; \ell_1, \ell_2) \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} \, dv &= \frac{1}{2\pi i} \int_{(2)} \frac{\zeta(2v-1)}{\zeta(2v)} \sum_{\ell_1 m = \ell_2 n} \frac{A(m)A(n)}{(mn)^s} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} \, dv \\
&\quad + \frac{1}{2\pi i} \int_{(2)} \frac{2}{\zeta(2)} \sum_{\ell_1 m = \ell_2 n + h} \frac{A(m)A(n)\sigma_{1-2v}(h)}{(mn)^s} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} \, dv.
\end{aligned}$$

For the diagonal term, write

$$\frac{\zeta(2v-1)}{\zeta(2v)} = \sum_{q \geq 1} \frac{\varphi(q)}{q^{2v}}.$$

Then we compute

$$\begin{aligned}
\frac{1}{2\pi i} \int_{(2)} \frac{\zeta(2v-1)}{\zeta(2v)} \sum_{\ell_1 m = \ell_2 n} \frac{A(m)A(n)}{(mn)^s} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} \, dv &= \frac{1}{2\pi i} \int_{(2)} \sum_{q \geq 1} \frac{\varphi(q)}{q^{2v}} \sum_{\ell_1 m = \ell_2 n} \frac{A(m)A(n)}{(mn)^s} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} \, dv \\
&= \sum_{q \geq 1} \varphi(q) \sum_{\ell_1 m = \ell_2 n} \frac{A(m)A(n)}{(mn)^s} \frac{1}{2\pi i} \int_{(2)} \frac{e^{\frac{\pi v^2}{Q^2}} \left(\frac{Q}{q}\right)^{2v}}{Q} \, dv \\
&= \sum_{|q-Q| \ll Q^\varepsilon} \varphi(q) \sum_{\ell_1 m = \ell_2 n} \frac{A(m)A(n)}{(mn)^s} + O_s(Q^{-B}).
\end{aligned}$$

Therefore the diagonal contribution has size

$$\frac{1}{2\pi i}\int_{(2)}\frac{\zeta(2v-1)}{\zeta(2v)}\sum_{\ell_1 m=\ell_2 n}\frac{A(m)A(n)}{(mn)^s}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv=\sum_{|q-Q|\ll Q^{\varepsilon}}\varphi(q)\sum_{\ell_1 m=\ell_2 n}\frac{A(m)A(n)}{(mn)^s}+O_s(Q^{-B}).$$

To estimate the sum, writing $\ell_1 m=\ell_2 n=\ell_1\ell_2 d$ and noting that $A(m),A(n)\ll 1$ gives

$$\sum_{\ell_1 m=\ell_2 n}\frac{A(m)A(n)}{(mn)^s}\ll\sum_{d\geq 1}\frac{1}{(\ell_1\ell_2)^s d^{2s}}=\frac{1}{(\ell_1\ell_2)^s}\zeta(2s).$$

Specializing $s=\frac{1}{2}+\varepsilon$, we find that the diagonal contribution is

$$\ll_{\varepsilon}\frac{Q^{1+\varepsilon}}{L^{1+2\varepsilon}},$$

where we have again used that $\varphi(q)\sim q$. For the off-diagonal term, first make the following computation:

$$\begin{aligned}\sum_{\ell_1 m=\ell_2 n+h}\frac{A(m)A(n)\sigma_{1-2v}(h)}{(mn)^s}&=\sum_{\ell_1 m=\ell_2 n+h}\frac{A(m)A(n)\sigma_{1-2v}(h)(\ell_1\ell_2)^{s+\frac{k-1}{2}}}{(mn)^s(\ell_1\ell_2)^{s+\frac{k-1}{2}}}\\&=(\ell_1\ell_2)^{s+\frac{k-1}{2}}\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{(\ell_1 m)^{s+\frac{k-1}{2}}(\ell_2 n)^{s+\frac{k-1}{2}}}\\&=(\ell_1\ell_2)^{s+\frac{k-1}{2}}\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{(\ell_2 n+h)^{s+\frac{k-1}{2}}(\ell_2 n)^{s+\frac{k-1}{2}}}\\&=(\ell_1\ell_2)^{s+\frac{k-1}{2}}\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{\left(1+\frac{h}{\ell_2 n}\right)^{s+\frac{k-1}{2}}(\ell_2 n)^{2s+k-1}}.\end{aligned}$$

Recall the identity

$$\frac{1}{(1+t)^{\beta}}=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\beta-u)\Gamma(u)}{\Gamma(\beta)}t^{-u}\,du,$$

for any $0<c<\mathrm{Re}(\beta)$. Applying this identity to our sum with $t=\frac{h}{\ell_2 n}$ and $\beta=s+\frac{k-1}{2}$ yields

$$\frac{1}{2\pi i}\int_{(c)}(\ell_1\ell_2)^{s+\frac{k-1}{2}}\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{(\ell_2 n)^{2s-u+k-1}h^u}\frac{\Gamma\left(s-u+\frac{k-1}{2}\right)\Gamma(u)}{\Gamma\left(s+\frac{k-1}{2}\right)}\,du.$$

This integral can be expressed as

$$\frac{1}{2\pi i}\int_{(c)}\ell_1^{s+\frac{k-1}{2}}\ell_2^{u-s-\frac{k-1}{2}}Z_f(2s-u,u,v;\ell_1,\ell_2)\frac{\Gamma\left(s-u+\frac{k-1}{2}\right)\Gamma(u)}{\Gamma\left(s+\frac{k-1}{2}\right)}\,du.$$

So the off-diagonal contribution is

$$\frac{1}{(2\pi i)^2}\int_{(2)}\int_{(c)}\frac{2}{\zeta(2v)}\ell_1^{s+\frac{k-1}{2}}\ell_2^{u-s-\frac{k-1}{2}}Z_f(2s-u,u,v;\ell_1,\ell_2)\frac{\Gamma\left(s-u+\frac{k-1}{2}\right)\Gamma(u)}{\Gamma\left(s+\frac{k-1}{2}\right)}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,du\,dv.$$

Shifting the line in v at (2) to $\left(\frac{1}{2}+\varepsilon\right)$, we are still within the region of absolute convergence for Z_f and do not pass over any poles of the integrand. This gives

$$\frac{1}{(2\pi i)^2}\int_{\left(\frac{1}{2}+\varepsilon\right)}\int_{(c)}\frac{2}{\zeta(2v)}\ell_1^{s+\frac{k-1}{2}}\ell_2^{u-s-\frac{k-1}{2}}Z_f(2s-u,u,v;\ell_1,\ell_2)\frac{\Gamma\left(s-u+\frac{k-1}{2}\right)\Gamma(u)}{\Gamma\left(s+\frac{k-1}{2}\right)}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,du\,dv.$$

Now we shift the line in u at (c) to $\left(\varepsilon - \frac{1}{2}\right)$. Here we pass over polar lines of Z_f. The off-diagonal contribution then becomes

$$\mathrm{Res}(s) + \frac{1}{(2\pi i)^2} \int_{\left(\frac{1}{2}+\varepsilon\right)} \int_{\left(\varepsilon-\frac{1}{2}\right)} \frac{2}{\zeta(2v)} \ell_1^{s+\frac{k-1}{2}} \ell_2^{u-s-\frac{k-1}{2}} Z_f(2s-u,u,v;\ell_1,\ell_2) \frac{\Gamma\left(s-u+\frac{k-1}{2}\right)\Gamma(u)}{\Gamma\left(s+\frac{k-1}{2}\right)} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q}\, du\, dv,$$

where

$$\mathrm{Res}(s) = \textsf{Todo : [xxx]}$$

Ignoring the residue term for the moment, the second spectral expansion of $Z_f(s,u,v;\ell_1,\ell_2)$ along with an analgous result to one in [2], we have

$$Z_f(s,u,v;\ell_1,\ell_2) \ll \ell_1^{-\frac{k-1}{2}} \ell_2^{\frac{k+1}{2}-u-v} \textsf{Todo : [polynomial in } s\textsf{]}.$$

Therefore the integral is

$$\ll \frac{1}{(2\pi i)^2} \int_{\left(\frac{1}{2}+\varepsilon\right)} \int_{\left(\varepsilon-\frac{1}{2}\right)} \frac{2}{\zeta(2v)} \ell_1^{s} \ell_2^{\frac{1}{2}-u-v} \textsf{Todo : [polynomial in } s\textsf{]} \frac{\Gamma\left(s-u+\frac{k-1}{2}\right)\Gamma(u)}{\Gamma\left(s+\frac{k-1}{2}\right)} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q}\, du\, dv,$$

and from this we deduce that the off-diagonal contribution at $s = \frac{1}{2} + \varepsilon$ is

$$\ll_\varepsilon Q^\varepsilon L^{\frac{1}{2}+3\varepsilon}.$$

Combining the diagonal and off-diagonal estimates, we have

$$\sum_{|q-Q| \ll Q^\varepsilon} S_f\left(\frac{1}{2}; q, \ell_1, \ell_2\right) \ll_\varepsilon \frac{Q^{1+\varepsilon}}{L^{1+2\varepsilon}} + Q^\varepsilon L^{\frac{1}{2}+3\varepsilon}$$

and summing over ℓ_1 and ℓ_2 yields

$$S_f\left(\frac{1}{2}, Q, L\right) \ll_\varepsilon \sum_{\ell_1,\ell_2 \sim L} \chi(\ell_1)\overline{\chi}(\ell_2) \left(\frac{Q^{1+\varepsilon}}{L^{1+2\varepsilon}} + Q^\varepsilon L^{\frac{1}{2}+3\varepsilon}\right) \ll Q^{1+\varepsilon} L^{1-2\varepsilon} + L^{\frac{5}{2}+3\varepsilon} Q^\varepsilon.$$

Ignoring the ε factors, setting $L = Q^{\frac{2}{3}}$ balances the error terms so that

$$S_f\left(\frac{1}{2}, Q, L\right) \ll_\varepsilon Q^{\frac{5}{3}}.$$

This the desired upper bound. Combining with the lower bound and our choice of L results in

$$\frac{Q^{\frac{4}{3}}}{\log(Q^{\frac{2}{3}})^2} |L(s, f \times \chi)|^2 \ll S_f\left(\frac{1}{2}, Q, Q^{\frac{2}{3}}\right) \ll_\varepsilon Q^{\frac{5}{3}},$$

which, ignoring logarithmic factors, implies

$$|L(s, f \times \chi)| \ll_\varepsilon Q^{\frac{1}{6}}.$$

A Bound Analgous to Resnikov

We will prove the bound

$$\sum_{t_j \sim T} |\langle V_{f,v}, \mu_j \rangle|^2 e^{\frac{\pi}{2}|t_j|} \ll \ell_1^{-k} T^{2k+\varepsilon} \log(T).$$

To this end, we first estimate the inner product:

$$\begin{aligned}\langle V_{f,v}, \mu_j\rangle &= \frac{1}{\mathcal{V}(\ell_1)} \int_{\mathcal{F}(\ell_1)} \overline{f(\ell_1 z)} E(z,s;k) \overline{\mu_j(z)} \mathrm{Im}(z)^{\frac{k}{2}}\, d\mu \\ &= \frac{1}{\mathcal{V}(\ell_1)} \int_{\mathcal{F}(\ell_1)} \overline{f(\ell_1 z)\mu_j(z)} \mathrm{Im}(z)^{\frac{k}{2}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(1)} \left(\frac{j(\gamma,z)}{|j(\gamma,z)|}\right)^{-k} \mathrm{Im}(\gamma z)^s\, d\mu \\ &= \frac{1}{\mathcal{V}(\ell_1)} \int_{\mathcal{F}(\ell_1)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(1)} \overline{f(\gamma\ell_1 z)\mu_j(\gamma z)} \left(\frac{\overline{j(\gamma,\ell_1 z)} j(\gamma,z)}{|j(\gamma,z)|^2}\right)^{-k} \mathrm{Im}(\gamma z)^{s+\frac{k}{2}}\, d\mu.\end{aligned}$$

Writing $\mathcal{F}(\ell_1) = \bigcup_{\eta\in\Gamma_0(\ell_1)\backslash\Gamma_0(1)} \eta\mathcal{F}$, we can express the integral as

$$\frac{1}{\mathcal{V}(\ell_1)} \sum_{\eta\in\Gamma_0(\ell_1)\backslash\Gamma_0(1)} \int_{\eta\mathcal{F}(\ell_1)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(1)} \overline{f(\gamma\ell_1 z)\mu_j(\gamma z)} \left(\frac{\overline{j(\gamma,\ell_1 z)} j(\gamma,z)}{|j(\gamma,z)|^2}\right)^{-k} \mathrm{Im}(\gamma z)^{s+\frac{k}{2}}\, d\mu,$$

which is equivalent to

$$\frac{1}{\mathcal{V}(\ell_1)} \int_{\mathcal{F}} \sum_{\eta\in\Gamma_0(\ell_1)\backslash\Gamma_0(1)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(1)} \overline{f(\gamma\eta\ell_1 z)\mu_j(\gamma\eta z)} \left(\frac{\overline{j(\gamma,\eta\ell_1 z)} j(\gamma,\eta z)}{|j(\gamma,\eta z)|^2}\right)^{-k} \mathrm{Im}(\gamma\eta z)^{s+\frac{k}{2}}\, d\mu$$

References

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