A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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Abstract. We construct a quadratic double Dirichlet series $Z(s, w)$ built from single variable quadratic Dirichlet L-functions $L(s, \chi)$ over $\mathbb{Q}$. We prove that $Z(s, w)$ admits meromorphic continuation to the (s, w)-plane and satisfies a group of functional equations.

1. Preliminaries

We present an overview of quadratic Dirichlet L-functions over $\mathbb{Q}$. We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m \geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Now we recall characters on $\mathbb{Z}$. They are multiplicative functions $\chi : \mathbb{Z} \to \mathbb{C}$. They form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions $\chi_d : \mathbb{Z} \to \mathbb{C}$ modulo $d \geq 1$ (in that they are d-periodic) and such that $\chi_d(m) = 0$ if $(m, d) > 1$.
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. If χ is a Dirichlet character then its conjugate $\overline{\chi}$ is also a Dirichlet character. Moreover, $\overline{\chi}$ is the multiplicative inverse to χ and the Dirichlet characters modulo m form a group under multiplication. This group is always finite and its order is $\phi(d) = |(\mathbb{Z}/d\mathbb{Z})^*|$. Dirichlet characters also satisfy orthogonality relations:

Theorem 1.1 (Orthogonality relations)**.**

(i) For any two Dirichlet characters χ and ψ modulo d,

$$\frac{1}{\phi(d)} \sideset{}{'}\sum_{a \pmod{d}} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

(ii) For any $a, b \in (\mathbb{Z}/d\mathbb{Z})^*$,

$$\frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

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The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{Z}$. First let us recall this symbol. For any odd prime p and any $m \geq 1$, we define the quadratic residue symbol $\left(\frac{m}{p}\right)$ by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod{p} = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod{p} \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod{p} \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod{p}. \end{cases}$$

This symbol only depends upon m modulo p and is multiplicative in m. We can extend the quadratic residue symbol multiplicatively in the denominator. If $d = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of d, then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \leq i \leq k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any odd $d \geq 1$. We can extend this symbol further and allow $d \geq 1$ to be even. To this end, we define

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases}$$

and extend $\left(\frac{m}{d}\right)$ multiplatively in d when d is even. Now the quadratic residue symbol makes sense for any $m, d \geq 1$. Moreover, it is multiplicative in both m and d but no longer depends upon only m modulo d (it also depends upon m modulo 8). In particular,

$$\left(\frac{-1}{d}\right) = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases}$$

and if $d \not\equiv 0 \pmod{2}$, we can compactly write

$$\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}} = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = (-1)^{\frac{d^2-1}{8}} = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}. \end{cases}$$

The quadratic residue symbol also admits the following reciprocity law:

Theorem 1.2 (Quadratic reciprocity)**.** *If $d, m \geq 1$, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2}\frac{m^{(2)}-1}{2}} \left(\frac{m}{d}\right),$$

where $d^{(2)}$ and $m^{(2)}$ are the parts of d and m relatively prime to 2 respectively.

We can now define the quadratic Dirichlet characters. For any odd square-free $d \in \mathbb{Z}$, define the quadratic Dirichlet character χ_d by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod{4}, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod{4}. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension $\mathbb{Q}(\sqrt{d})$. We extend χ_d multiplicatively in the denominator so that χ_d makes sense for any odd d. In particular, $\chi_d(m) = \pm 1$ provided d and m are relatively prime and $\chi_d(m) = 0$ if $(m, d) > 1$. Quadratic reciprocity implies that χ_d is a Dirichlet character modulo d if $d \equiv 1 \pmod{4}$ and is a Dirichlet character modulo $4d$ if $d \equiv 2, 3 \pmod{4}$. Indeed, if $d \equiv 1 \pmod{4}$ then $d^{(2)} = d$ and the sign is always 1. If $d \equiv 3 \pmod{4}$, then $d^{(2)} = d$ and the sign is

$\left(\frac{-1}{m}\right)$ which is a character modulo 4. If $d \equiv 2 \pmod 4$, then $d^{(2)} \equiv 1, 3 \pmod 4$ and we are reduced to one of the previous two cases. We will also set

$$q(d) = \begin{cases} d & \text{if } d \equiv 1 \pmod 4, \\ 4d & \text{if } d \equiv 2, 3 \pmod 4, \end{cases} \qquad \text{and} \quad \varepsilon_\chi = \frac{\tau(\chi_d)}{\sqrt{q(d)}},$$

where $\tau(\chi_d)$ is the Gauss sum attached to χ_d. We will also require an associated character. For each χ_m (here we are purposely interchanging the roles of d and m to keep consistency with the latter notation when discussing the double Dirichlet series), we define $\widetilde{\chi}_m$ by

$$\widetilde{\chi}_m(d) = (-1)^{\frac{m^{(2)}-1}{2}\frac{d^{(2)}-1}{2}} \chi_m(d).$$

Equivalently, $\widetilde{\chi}_m(d)$ can be expressed as

$$\widetilde{\chi}_m(d) = \begin{cases} \chi_m(d) & \text{if } m \equiv 1, 2 \pmod 4, \\ \chi_{-1}(d)\chi_m(d) & \text{if } m \equiv 3 \pmod 4, \end{cases}$$

and it follows that $\widetilde{\chi}_m(d)$ is a quadratic Dirichlet character of the same modulus as χ_m. We now discuss the Hilbert characters. We will only need four of them: the quadratic Dirichlet characters modulo 8. They are given as follows:

$$\chi_1(m) = \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod 2, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases} \qquad \chi_{-1}(m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4, \\ -1 & \text{if } m \equiv 3 \pmod 4, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

$$\chi_2(m) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod 8, \\ -1 & \text{if } m \equiv 3, 5 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases} \qquad \chi_{-2}(m) = \begin{cases} 1 & \text{if } m \equiv 1, 3 \pmod 8, \\ -1 & \text{if } m \equiv 5, 7 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

In general, we will denote a Hilbert character by χ_a with $a \in \{\pm 1, \pm 2\}$. Morevoer, observe that $\chi_a(m) = \widetilde{\chi}_a(m)$ for all $a \in \{\pm 1, \pm 2\}$ and any $m \geq 1$. Also, can write

$$\chi_{-1}(m) = \left(\frac{-1}{m}\right) \quad \text{and} \quad \chi_2(m) = \left(\frac{m}{2}\right),$$

and have the relations

$$\chi_{-2}(m) = \chi_{-1}(m)\chi_2(m), \quad \chi_1(m) = \chi_{-1}(m)\chi_{-1}(m), \quad \text{and} \quad \chi_{-1}(m) = \chi_2(m)\chi_{-2}(m).$$

Suppose d is square-free. If $d \equiv 1, 2, 5 \pmod 8$, then $d^{(2)} \equiv 1 \pmod 4$ so that the sign in the statement of quadratic recipricty is 1. If $d \equiv 3, 6, 7 \pmod 8$, then $d^{(2)} \equiv 3 \pmod 4$ and the sign is $(-1)^{\frac{m^{(2)}-1}{2}}$. This fact together with the relations for the quadratic characters modulo 8 imply

$$\chi_d(m) = \begin{cases} \chi_m(d) & \text{if } d \equiv 1 \pmod 4, \\ \chi_{-1}(m)\chi_m(d) & \text{if } d \equiv 3 \pmod 4, \\ \chi_2(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 2 \pmod 8, \\ \chi_{-2}(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 6 \pmod 8. \end{cases}$$

With the Dirichlet and Hilbert characters introduced, we are ready to discuss the L-functions associated to quadratic Dirichlet characters. We define the L-function $L(s, \chi_d)$ attached to χ_d for square-free d, by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \geq 1} \frac{\chi_d(m)}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character, $L(s,\chi_d) \ll \zeta(s)$ for $\mathrm{Re}(s) > 1$ so that $L(s,\chi_d)$ is locally absolutely uniformly convergent in this region. $L(s,\chi_d)$ also admits analytic continuation to $\mathbb{C}$. The completed L-function $L^*(s,\chi_d)$ is defined as

$$L^*(s,\chi_d) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi_d) & \text{if } \chi_d \text{ is even,} \\ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi_d) & \text{if } \chi_d \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s,\chi_d) = \begin{cases} \varepsilon_\chi q(d)^{\frac{1}{2}-s}L^*(1-s,\chi_d) & \text{if } \chi_d \text{ is even,} \\ -\varepsilon_\chi q(d)^{\frac{1}{2}-s}L^*(1-s,\chi_d) & \text{if } \chi_d \text{ is odd.} \end{cases}$$

Note that the gamma factors depend upon the partiy of χ_d. This the root cause of an important technical issue later when deriving functional equations for the quadratic double Dirichlet series. Analogously, the Dirichlet L-function $L(w,\widetilde{\chi}_m)$ attached to $\widetilde{\chi}_m$ for square-free m is defined by a Dirichlet series or Euler product:

$$L(w,\widetilde{\chi}_m) = \sum_{d\geq 1}\frac{\widetilde{\chi}_m(d)}{d^w} = \prod_{p \text{ prime}}\left(1-\frac{\widetilde{\chi}_m(p)}{p^w}\right)^{-1}.$$

As for $L(s,\chi_d)$, $L(w,\widetilde{\chi}_m) \ll \zeta(w)$ for $\mathrm{Re}(w) > 1$ so that $L(w,\widetilde{\chi}_m)$ is locally absolutely uniformly convergent in this region. Moreover, $L(w,\widetilde{\chi}_m)$ admits analytic continuation to $\mathbb{C}$ and the completed L-function $L^*(w,\widetilde{\chi}_m)$ is defined as

$$L^*(w,\widetilde{\chi}_m) = \begin{cases} \pi^{-\frac{w}{2}}\Gamma\left(\frac{w}{2}\right)L(w,\widetilde{\chi}_m) & \text{if } \widetilde{\chi}_m \text{ is even,} \\ \pi^{-\frac{w}{2}}\Gamma\left(\frac{w+1}{2}\right)L(w,\widetilde{\chi}_m) & \text{if } \widetilde{\chi}_m \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(w,\widetilde{\chi}_m) = \begin{cases} \varepsilon_{\widetilde{\chi}} q(m)^{\frac{1}{2}-w}L^*(1-w,\widetilde{\chi}_m) & \text{if } \widetilde{\chi}_m \text{ is even,} \\ -\varepsilon_{\widetilde{\chi}} q(m)^{\frac{1}{2}-w}L^*(1-w,\widetilde{\chi}_m) & \text{if } \widetilde{\chi}_m \text{ is odd.} \end{cases}$$

Remark 1.1. *The definitions for $L(s,\chi_d)$, $L^*(s,\chi_d)$, $L(w,\widetilde{\chi}_m)$, and $L^*(w,\widetilde{\chi}_m)$ work perfectly well even when d and m are not square-free (however the functional equations do not hold). We purposely do not define these L-functions, yet, for d and m not necessarily square-free.*

The Quadratic Double Dirichlet Series

We will now define the quadratic double Dirichlet series $Z(s,w)$. For any integer $d \geq 1$, write $d = d_0d_1^2$ where d_0 is square-free. Equivalently, d_0 is the square-free part of d and $\frac{d}{d_0}$ is a perfect square. The **quadratic double Dirichlet series** $Z(s,w)$ is defined as

$$Z(s,w) = \sum_{d \text{ odd}}\frac{L^{(2)}(s,\chi_{d_0})Q_{d_0d_1^2}(s)}{d^w},$$

where the superscript (2) indicates that the local factor at 2 has been removed, $Q_{d_0d_1^2}(s)$ is the **correction polynomial** defined by

$$Q_{d_0d_1^2}(s) = \sum_{e_1e_2 | d_1}\mu(e_1)\chi_{d_0}(e_1)e_1^{-s}e_2^{1-s} = \sum_{e_1e_2e_3 = d_1}\mu(e_1)\chi_{d_0}(e_1)e_1^{-s}e_2^{1-s},$$

and μ is the usual Möbius function. For $\mathrm{Re}(s) > 1$, there is the trivial estimate

$$Q_{d_0d_1^2}(s) \ll \sum_{e_1e_2|d_1} 1 \ll \sigma_0(d_1)^2 \ll_\varepsilon d_1^{2\varepsilon} \ll_\varepsilon d^\varepsilon,$$

for any $\varepsilon > 0$. As $L(s, \chi_{d_0}) \ll 1$ for $\mathrm{Re}(s) > 1$, $Z(s, w)$ is locally absolutely uniformly convergent in the region $\Lambda = \{(s, w) \in \mathbb{C}^2 : \mathrm{Re}(s) > 1, \mathrm{Re}(w) > 1\}$. It will also be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters χ_{a_1} and χ_{a_2}. The **quadratic double Dirichlet series** $Z_{a_1,a_2}(s, w)$ twisted by χ_{a_1} and χ_{a_2} is defined as

$$Z_{a_1,a_2}(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w},$$

where $Q_{d_0 d_1^2}(s, \chi_{a_1})$ is the **correction polynomial** twisted by χ_{a_1} defined by

$$Q_{d_0 d_1^2}(s, \chi_{a_1}) = \sum_{e_1 e_2 | d_1} \mu(e_1)\chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s},$$

and μ is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound $Q_{d_0 d_1^2}(s, \chi_{a_1}) \ll d_\varepsilon$ so that $Z_{a_1,a_2}(s, w)$ converges locally absolutely uniformly in the same region as $Z(s, w)$ does. In particular, $Z(s, w) = Z_{1,1}(s, w)$. As a final comment, we will also need the correction polynomials $Q_{m_0 m_1^2}(w)$ and $Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})$. They are defined by

$$Q_{m_0 m_1^2}(w) = \sum_{e_1 e_2 | m_1} \mu(e_1)\chi_{m_0}(e_1) e_1^{-w} e_2^{1-w} = \sum_{e_1 e_2 e_3 = m_1} \mu(e_1)\chi_{m_0}(e_1) e_1^{-w} e_2^{1-w},$$

and

$$Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2}) = \sum_{e_1 e_2 | m_1} \mu(e_1)\widetilde{\chi}_{a_2 m_0}(e_1) e_1^{-w} e_2^{1-2w} = \sum_{e_1 e_2 e_3 = m_1} \mu(e_1)\widetilde{\chi}_{a_2 m_0}(e_1) e_1^{-w} e_2^{1-2w}.$$

Clearly they satisfy analogous estimates.

The Interchange

As defined, $Z_{a_1,a_2}(s, w)$ is a sum of L-functions, and hence Euler products, in s. We will prove an interchange formula for $Z_{a_1,a_2}(s, w)$ which will show that it can be expressed as a sum of L-functions in w. That is, we want the variables s and w to change places. Precisely:

Theorem 1.3 (Interchange)**.** *Wherever $Z_{a_1,a_2}(s, w)$ converges locally absolutely uniformly,*

$$Z_{a_1,a_2}(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w} = \sum_{m \text{ odd}} \frac{L^{(2)}(w, \widetilde{\chi}_{a_2 m_0})\widetilde{\chi}_{a_1}(m) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})}{m^s}.$$

Proof. Only the second equality needs to be proved. To do this, first expand the L-function $L^{(2)}(s, \chi_{a_1 d_0})$ and polynomial $Q_{d_0 d_1^2}(s, \chi_{a_1})$ to get

$$\begin{aligned} Z(s, w) &= \sum_{d \text{ odd}} \frac{L(s, \chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w} \\ &= \sum_{d \text{ odd}} \left(\sum_{m \text{ odd}} \chi_{a_1 d_0}(m) m^{-s} \right) \left(\sum_{e_1 e_2 | d_1} \mu(e_1)\chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} \right) \chi_{a_2}(d) d^{-w} \\ &= \sum_{m, d \text{ odd}} \sum_{e_1 e_2 | d_1} \mu(e_1)\chi_{a_2}(d)\chi_{a_1 d_0}(m e_1) e_1^{-s} e_2^{1-2s} m^{-s} d^{-w}. \end{aligned}$$

Now $\chi_{a_1 d_0}(m e_1) = 0$ unless $(d_0, m e_1) = 1$. We make this restriction on the sum giving

$$\sum_{m, d \text{ odd}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, m e_1) = 1}} \mu(e_1)\chi_{a_2}(d)\chi_{a_1 d_0}(m e_1) e_1^{-s} e_2^{1-2s} m^{-s} d^{-w}.$$

Making the change of variables $me_1 \to m$ yields

$$\sum_{d \text{ odd}} \sum_{\substack{m \text{ odd} \\ e_1 | m}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, m)=1}} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m) e_2^{1-2s} m^{-s} d^{-w}.$$

For fixed $d = d_0 d_1^2$ and e_2, the subsum over m and e_1 is

$$\sum_{\substack{m \text{ odd} \\ e_1 | m}} \sum_{\substack{e_1 | \frac{d_1}{e_2} \\ (d_0, m)=1}} \mu(e_1) \chi_{a_1 d_0}(m) m^{-s} = \sum_{\substack{m \text{ odd} \\ (d_0, m)=1}} \chi_{a_1 d_0}(m) m^{-s} \left(\sum_{e_1 | \left(\frac{d_1}{e_2}, m\right)} \mu(e_1) \right).$$

The inner sum over e_1 of the Möbius function vanishes unless $\left(\frac{d_1}{e_2}, m\right) = 1$ in which case it is 1. Therefore the triple sum above becomes

$$\sum_{m, d \text{ odd}} \sum_{\substack{e_2 | d_1 \\ \left(\frac{d_0 d_1}{e_2}, m\right)=1}} \chi_{a_2}(d) \chi_{a_1 d_0}(m) e_2^{1-2s} m^{-s} d^{-w}.$$

Making the change of variables $d \to de_2^2$, the condition $\left(\frac{d_0 d_1}{e_2}, m\right) = 1$ becomes $(d_0 d_1, m) = 1$ which is equivalent to $(d, m) = 1$. Moreover, $\chi_{a_2}(de_2^2) = \chi_{a_2}(d)$. Altogether, we obtain

$$\sum_{\substack{m, d \text{ odd} \\ (d,m)=1}} \sum_{e_2} \chi_{a_2}(d) \chi_{a_1 d_0}(m) e_2^{1-2s-2w} m^{-s} d^{-w}.$$

Writing $m = m_0 m_1^2$ analogously as for d, quadratic reciprocity implies $\chi_{d_0}(m) = \widetilde{\chi}_m(d_0) = \widetilde{\chi}_{m_0}(d)$ where the last equality holds because $(d, m) = 1$ and both d_0 and m_0 differ from d and m respectively by perfect squares. As $\chi_{a_1}(m) = \widetilde{\chi}_{a_1}(m)$ and $\chi_{a_2}(d) = \widetilde{\chi}_{a_2}(d)$, the previous fact implies $\chi_{a_2}(d) \chi_{a_1 d_0}(m) = \widetilde{\chi}_{a_1}(m) \widetilde{\chi}_{a_2 m_0}(d)$ and so our expression becomes

$$\sum_{\substack{d, m \geq 1 \\ (d,m)=1}} \sum_{e_2} \widetilde{\chi}_{a_1}(m) \widetilde{\chi}_{a_2 m_0}(d) e_2^{1-2s-2w} m^{-s} d^{-w}.$$

But now we can reverse the argument with the roles of d, m, χ_{a_1}, and χ_{a_2} interchanged respectively, but with $\widetilde{\chi}_{a_1}$ and $\widetilde{\chi}_{a_2}$, to obtain

$$Z(s, w) = \sum_{m \geq 1} \frac{L(w, \widetilde{\chi}_{a_2 m_0}) \widetilde{\chi}_{a_1}(m) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})}{m^s}.$$

□

Note that the interchange is not completely symmetric because of the characters $\widetilde{\chi}_{a_2 m_0}$, $\widetilde{\chi}_{a_1}$, and $\widetilde{\chi}_{a_2}$ in the second expression for $Z_{a_1, a_2}(s, w)$. This is due to the fact that recipricty is not perfect. In even more general settings the correction polynomials in w need not be equal to those in s.

Weighting Terms

We will now study the coefficients of $Z_{a_1, a_2}(s, w)$ expanded in s and w. By expanding $L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1})$ in the numerator of $Z_{a_1, a_2}(s, w)$, we can write

$$Z_{a_1, a_2}(s, w) = \sum_{d \geq 1} \frac{L(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w} = \sum_{d, m \geq 1} \frac{\chi_{a_1 d_0}(\widehat{m}) \chi_{a_2}(d) a(m, d)}{m^s d^w},$$

where $\widehat{m}$ is the part of m relatively prime to d_0 and the **weighting coefficient** $a(m,d)$ is given by

$$a(m,d) = \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2.$$

To see this, the coefficient of $m^{-s} d^{-w}$ in the definition of $Z_{a_1,a_2}(s,w)$ is

$$\begin{aligned} \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1}} \mu(e_1) \chi_{a_1 d_0}(e_1 e_3) e_2 &= \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) \chi_{a_1 d_0}(e_1 e_3) e_2 \\ &= \chi_{a_1 d_0}(\widehat{m}) \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2 \\ &= \chi_{a_1 d_0}(\widehat{m}) \chi_{a_2}(d) a(m,d), \end{aligned}$$

where the first equality holds because $\chi_{d_0}(e_1 e_3) = 0$ unless $(d_0, e_1 e_3) = 1$ and the second equality holds because if $(d_0, e_1 e_3) = 1$, $\widehat{m}$ differs from $e_1 e_3$ by a perfect square (the divisors of which belong to (d_0, e_2)) and so $\chi_{d_0}(e_1 e_3) = \chi_{d_0}(\widehat{m})$.

Remark 1.2. *Also, $a(m,d) = 0$ unless $m = e_1 e_2^2 e_3$ with $(d_0, e_1 e_3) = 1$ and $e_1 e_2^2 \mid d_1$.*

We will define $L(s, \chi_{a_1 d})$ to be the Dirichlet series given by

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1}) = \sum_{m \geq 1} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m,d)}{m^s}.$$

In particular, $L(s, \chi_d)$ now makes sense for d not necessarily square-free and this definition agrees with the former when d is square-free. Moreover, we have the representation

$$Z_{a_1,a_2}(s,w) = \sum_{d \geq 1} \frac{\chi_{a_2}(d) L(s, \chi_{a_1 d})}{d^w}.$$

If we perform the same procedure to the interchange, then

$$Z_{a_1,a_2}(s,w) = \sum_{m \geq 1} \frac{L(w, \widetilde{\chi}_{a_2 m_0}) \widetilde{\chi}_{a_1}(m) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})}{m^s} = \sum_{d,m \geq 1} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d}) \widetilde{\chi}_{a_1}(m) a(d,m)}{m^s d^w},$$

where $\widehat{d}$ is the part of d relatively prime to m_0. Analogously, we define $L(w, \widetilde{\chi}_{a_2 m})$ to be the Dirichlet series given by

$$L(w, \widetilde{\chi}_{a_2 m}) = L(w, \widetilde{\chi}_{a_2 m_0}) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2}) = \sum_{d \geq 1} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d}) a(d,m)}{d^w},$$

so that

$$Z_{a_1,a_2}(s,w) = \sum_{m \geq 1} \frac{\widetilde{\chi}_{a_1}(m) L(w, \widetilde{\chi}_{a_2 m})}{m^s}.$$

We now investigate the structure of the weighting coefficients $a(m,d)$. Their structure controls the majority of the information about both the double Dirichlet series and the correction polynomials. We first show that the weighting coefficients posess a multiplicativity property:

Proposition 1.1. *We have* $a(m,1) = a(1,d) = 1$ *and*

$$a(m,d) = \prod_{\substack{p^\alpha || m \\ p^\beta || d}} a(p^\alpha, p^\beta).$$

Proof. From the definition of the weighting coefficients, $a(m,1) = a(1,d) = 1$. We will prove multiplicativity in m and then in d. Letting $m = m'p^\alpha$, we must show

$$a(m,d) = a(m',d)a(p^\alpha,d).$$

To acomplish this, for $e_1e_2^2e_3 = m$, let $e_1 = c_1d_1$, $e_2 = c_2d_2$, and $e_3 = c_3d_3$ with $c_1, c_2, c_3 \mid m'$ and $d_1, d_2, d_3 \mid p^\alpha$. Because $(m', p^\alpha) = 1$, as $e_1e_2^2e_3$ runs over decompositions of m, $c_1c_2^2c_3$ and $d_1d_2^2d_3$ run over decompositions of m' and p^α respectively. Moreover, as e_1e_2 runs over the divisors of d_1 so does $c_1d_1c_2d_2$. These facts combined with multiplicativity of the Möbius function gives

$$\begin{aligned}
a(m,d) &= \sum_{\substack{e_1e_2^2e_3 = m \\ e_1e_2 | d_1 \\ (d_0, e_1e_3) = 1}} \mu(e_1)e_2 \\
&= \sum_{\substack{c_1c_2^2c_3 = m' \\ d_1d_2^2d_3 = p^\beta \\ c_1d_1c_2d_2 | d_1 \\ (d_0, c_1d_1c_3d_3) = 1}} \mu(c_1)(d_1)|c_2|d_2 \\
&= \left(\sum_{\substack{c_1c_2^2c_3 = m' \\ c_1c_2 | d_1 \\ (d_0, c_1c_3) = 1}} \mu(c_1)|c_2| \right) \left(\sum_{\substack{d_1d_2^2d_3 = p^\alpha \\ d_1d_2 | d_1 \\ (d_0, d_1d_3) = 1}} \mu(d_1)d_2 \right) \\
&= a(m',d)a(p^\alpha,d),
\end{aligned}$$

as desired. Now we prove multiplicativity in d. Since we have already proven multiplicativity in m, we may assume $m = p^\alpha$. Letting $d = d'p^\beta$, we must show

$$a(p^\alpha, d) = a(p^\alpha, p^\beta).$$

As $e_1e_2^2e_3 = p^\alpha$, the e_i are powers of p for $1 \le i \le 3$. It follows that $e_1e_2 \mid d_1$ is equivalent to $e_1e_2 \mid p^\beta$. Moreover, $(d_0, e_1e_2) = 1$ is equivalent to $(1, e_1e_2) = 1$ or $(p, e_1e_2) = 1$ depending on of β is even or odd. These facts imply the desired identity. ☐

The correction polynomials $Q_{d_0d_1^2}(s, \chi_{a_1})$ are tightly connected to the weighting coefficients $a(m,d)$. In particular, $Q_{d_0d_1^2}(s, \chi_{a_1})$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients. We first prove this relationship when d is an odd prime power:

Lemma 1.1. *For any prime* p *and* $\alpha \ge 1$, *we have*

$$Q_{p^{2\alpha+1}}(s) = \sum_{k \le 2\alpha} \frac{a(p^k, p^{2\alpha+1})}{p^{ks}}.$$

Moreover, the same holds for $Q_{p^{2\alpha+1}}(w)$.

Proof. Expanding the correction polynomial in p^{-s} yields

$$Q_{p^{2\alpha+1}}(s) = \sum_{e_1e_2 | p^\alpha} \mu(e_1)\chi_p(e_1)e_1^{-s}e_2^{1-2s} = \sum_{k \le 2\alpha} \frac{b(p^k, p^{2\alpha+1})}{p^{ks}}.$$

where
$$b(p^k, p^{2\alpha+1}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1)\chi_p(e_1)e_2.$$

The proof will be finished if we can show $b(p^k, p^{2\alpha+1}) = a(p^k, p^{2\alpha+1})$. To see this, first observe $\mu(e_1)\chi_p(e_1) = 0$ unless $e_1 = 1$ in which case it is 1. So $b(p^k, p^{2\alpha+1}) = 0$ if k is odd and $p^{\frac{k}{2}}$ if k is even. Compactly stated,
$$b(p^k, p^{2\alpha+1}) = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

On the other hand, $k \leq \alpha$ so that
$$a(p^k, p^{2\alpha+1}) = \sum_{\substack{e_1 e_2^2 e_3 = p^k \\ e_1 e_2 | p^\alpha \\ (p, e_1 e_3) = 1}} \mu(e_1)e_2 = \sum_{\substack{e_1 e_2^2 | p^k \\ (p, e_1 e_3) = 1}} \mu(e_1)e_2 = \sum_{e_2^2 = p^k} e_2 = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

This finishes the proof. Clearly the same holds for $Q_{p^{2\alpha+1}}(w)$. ☐

There is an analogous statement when d is an even prime power up to a square-free factor and relatively prime factor:

Lemma 1.2. *For any square-free integer $d_0 \geq 1$, $a_1 \in \{\pm 1, \pm 2\}$, prime p not dividing d_0, and $\beta \geq 1$, we have*
$$Q_{d_0 p^{2\beta}}(s, \chi_{a_1}) = (1 - \chi_{a_1 d_0}(p)p^{-s}) \sum_{k \leq 2\beta} \frac{\chi_{a_1 d_0}(p^k) a(p^k, p^{2\beta})}{p^{ks}}.$$

Moreover, the same holds for $Q_{m_0 p^{2\beta}}(w, \widetilde{\chi}_{a_2})$.

Proof. Expand the correction polynomial in p^{-s} to get
$$Q_{d_0 p^{2\beta}}(s, \chi_{a_1}) = \sum_{e_1 e_2 | p^\alpha} \mu(e_1)\chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} = \sum_{k \leq 2\beta} \frac{b(p^k, p^{2\beta})}{p^{ks}}.$$

where
$$b(p^k, p^{2\beta}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1)\chi_{a_1 d_0}(e_1)e_2.$$

It suffices to show $b(p^k, p^{2\beta}) = \chi_{a_1 d_0}(p^k)\left(a(p^k, p^{2\beta}) - a(p^{k-1}, p^{2\beta})\right)$. On the one hand, $\mu(e_1) = 0$ unless $e_1 = 1, p$ in which case $\mu(e_1) = \pm 1$ accordingly. So
$$b(p^k, p^{2\beta}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1)\chi_{a_1 d_0}(e_1)e_2 = \begin{cases} \chi_{a_1 d_0}(p^k) p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(p^k) p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

where we have used the identity $\chi_{a_1 d_0}(e_1) = \chi_{a_1 d_0}(p^k)$ which holds because this quadratic Dirichlet character only depends upon the parity of k. On the other hand, as in the proof of Lemma 1.1
$$a(p^k, p^{2\beta}) = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

But then
$$\chi_{a_1 d_0}(p^k)\left(a(p^k, p^{2\beta}) - a(p^{k-1}, p^{2\beta})\right) = \begin{cases} \chi_{a_1 d_0}(p^k) p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(p^k) p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

which completes the proof. Clearly the same holds for $Q_{m_0 p^{2\beta}}(w, \widetilde{\chi}_{a_2})$. ☐

Lemmas 1.1 and 1.2 together show that $Q_{d_0d_1^2}(s,\chi_{a_1})$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients $a(m,d)$ when d is an prime power. The proof of these lemmas also give the value of $a(p^k,p^l)$ and we collect this as a corollary:

Corollary 1.1. *For any prime p,*

$$a(p^k,p^l)=\begin{cases}\min\left(p^{\frac{k}{2}},p^{\frac{l}{2}}\right) & \text{if } \min(k,l) \text{ is even,}\\ 0 & \text{otherwise.}\end{cases}$$

If we combine Proposition 1.1 and Corollary 1.1 we can compute $a(m,d)$ in general:

Corollary 1.2. *For any integers $d,m\geq 1$,*

$$a(m,d)=\begin{cases}(m,d)^{\frac{1}{2}} & \text{if } (m,d) \text{ is a perfect square,}\\ 0 & \text{otherwise.}\end{cases}$$

As an immediate consquence of Corollary 1.2, $a(m,d)$ is symmetric in m and d. As the weighting coefficients are multiplicative, $Q_{d_0d_1^2}(s,\chi_{a_1})$ will posess an Euler product. To state the Euler product explicitely, we write $d=d_0d_1^2d_2^2$ with d_0 square-free and, d_2 relatively prime to d_0d_1, and such that every prime divisor of d_1 divides d_0. In other words, d_0 is the square-free part of d, d_1 is the square part of d whose prime factors divide d to odd power, and d_2 is the square part of d whose prime factors divide d to even power. We have the following Euler product:

Theorem 1.4. *Let $d=d_0d_1^2d_2^2$ be the square decomposition of d stratified by even and odd powers. Then for any $a_1\in\{\pm1,\pm2\}$,*

$$Q_{d_0d_1^2d_2^2}(s,\chi_{a_1})=\prod_{p^\alpha||d_1}Q_{p^{2\alpha+1}}(s)\cdot\prod_{p^\beta||d_2}Q_{d_0p^{2\beta}}(s,\chi_{a_1}).$$

Moreover, the same holds for $Q_{m_0m_1^2m_2^2}(w,\widetilde{\chi}_{a_2})$.

Proof. Recall that

$$L(s,\chi_{a_1d})=L(s,\chi_{a_1d_0})Q_{d_0d_1^2d_2^2}(s,\chi_{a_1})=\sum_{m\geq1}\frac{\chi_{a_1d_0}(\widehat{m})a(m,d)}{m^s}.$$

We will now derive an alterate expression for $L(s,\chi_{a_1d})$. By Proposition 1.1, the coefficients of $L(s,\chi_{a_1d})$ are multiplicative. Therefore $L(s,\chi_{a_1d})$ admits the Euler product

$$L(s,\chi_{a_1d})=\prod_{p\text{ prime}}\left(\sum_{k\geq0}\frac{\chi_{a_1d_0}(\widehat{p^k})a(p^k,d)}{p^{ks}}\right).$$

Decomposing the product according to primes dividing $d = d_0 d_1^2 d_2^2$, we get

$$
\begin{aligned}
&L(s, \chi_{a_1 d}) \\
&= \prod_{p \text{ prime}} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, d)}{p^{ks}} \right) \\
&= \prod_{p \nmid d} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, 1)}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^\beta)}{p^{ks}} \right) \\
&= \prod_{p \nmid d} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^\beta)}{p^{ks}} \right) \\
&= \prod_{p \nmid d} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left(\sum_{k \geq 0} \frac{a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(p^k) a(p^k, p^\beta)}{p^{ks}} \right).
\end{aligned}
$$

Including the factors corresponding to primes $p \mid d_2$ into the first product, we must multiply the last factor by the inverse of $\sum_{k \geq 0} \chi_{a_1 d_0}(p) p^{-ks} = (1 - \chi_{a_1 d_0}(p) p^{-s})^{-1}$ obtaining

$$
\prod_{p \nmid d_0} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left(\sum_{k \geq 0} \frac{a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left((1 - \chi_{a_1 d_0}(p) p^{-s}) \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(p^k) a(p^k, p^\beta)}{p^{ks}} \right),
$$

as every prime divisor of d_1 divdes d_0. The first product is $L(s, \chi_{a_1 d_0})$. For the second and third products, Remark 1.2 implies that the sums run up to $k \leq 2\alpha$ and $k \leq 2\beta$ respectively. Therefore they are $Q_{p^{2\alpha+1}}(s)$ and $Q_{d_0 p^{2\beta}}(s, \chi_{a_1})$ respectively. It follows that

$$
L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) \cdot \prod_{p^\alpha || d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta || d_2} Q_{d_0 p^{2\beta}}(s, \chi_{a_1}).
$$

This is our alternate expression for $L(s, \chi_{a_1 d})$ and equating the two results in

$$
L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = L(s, \chi_{a_1 d_0}) \cdot \prod_{p^\alpha || d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta || d_2} Q_{d_0 p^{2\beta}}(s, \chi_{a_1}),
$$

from which the proof is complete since $L(s, \chi_{a_1 d_0}) \neq 0$ for $\mathrm{Re}(s) > 1$ (so that we may divide by $L(s, \chi_{a_1 d_0})$). Clearly the same holds for $Q_{m_0 m_1^2 m_2^2}(w, \widetilde{\chi}_{a_2})$. $\square$

Observe that for $d = d_0 d_1^2 d_2^2$, the prime factors that divide $d_1 d_2$ are exactly those factors that divide d to power larger than 1. Thus, from Theorem 1.4 the Euler product for $Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1})$ is supported on exactly the primes dividing d to order larger than 1 and also depends upon the character $\chi_{a_1 d_0}$.

Functional Equations

We can now derive functional equations for $Z_{a_1, a_2}(s, w)$. These functional equations will be induced from the functional equations for $L(s, \chi_{a_1 d})$ and $L(s, \widetilde{\chi}_{a_2 m})$. To prove these latter functional equations, we require a functional equation for the correction polynomials:

Theorem 1.5. *$Q_{d_0 d_1^2}(s, \chi_{a_1})$ admits the functional equation.*

$$
Q_{d_0 d_1^2}(s, \chi_{a_1}) = d_1^{1-2s} Q_{d_0 d_1^2}(1 - s, \chi_{a_1}).
$$

Moreover, the same holds for $Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})$.

Proof. The stragety is to interchange e_2 and e_3 in the sum defining $Q_{d_0 d_1^2}(s, \chi_{a_1})$:

$$\begin{aligned} d_1^{1-2s} Q_{d_0 d_1^2}(1-s) &= d_1^{1-2s} \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{s-1} e_2^{2s-1} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{s-1} \left(\frac{d_1}{e_2} \right)^{1-2s} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{s-1} (e_1 e_3)^{1-2s} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_3^{1-2s} \\ &= Q_{d_0 d_1^2}(s, \chi_{a_1}). \end{aligned}$$

Clearly the same holds for $Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})$. □

We will define the completed L-function $L^*(s, \chi_{a_1 d})$ by

$$L^*(s, \chi_{a_1 d}) = L^*(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1}).$$

In particular, $L^*(s, \chi_d)$ makes sense even when d is not square-free and agrees with the previous definition when d is square-free. Combining Theorem 1.5, the functional equation for $L^*(s, \chi_{a_1 d_0})$, that $\deg(d) \equiv \deg(d_0) \pmod 2$, and that $d \equiv d_0 \pmod 4$, we obtain a functional equation for $L^*(s, \chi_{a_1 d})$:

$$L^*(s, \chi_{a_1 d}) = \begin{cases} q(d)^{\frac{1}{2}-s} L^*(1-s, \chi_{a_1 d}) & \text{if } \deg(d) \text{ is even,} \\ q(d)^{\frac{1}{2}-s} L^*(1-s, \chi_{a_1 d}) & \text{if } \deg(d) \text{ is odd.} \end{cases}$$

Analogously, define the completed L-function $L^*(w, \widetilde{\chi}_{a_2 m})$ by

$$L^*(w, \widetilde{\chi}_{a_2 m}) = L^*(w, \widetilde{\chi}_{a_2 m_0}) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2}).$$

Then, as before, we have a functional equation for $L^*(w, \widetilde{\chi}_{a_2 m})$:

$$L^*(w, \widetilde{\chi}_{a_2 m}) = \begin{cases} q(m)^{\frac{1}{2}-w} L^*(1-w, \widetilde{\chi}_{a_2 m}) & \text{if } \deg(m) \text{ is even,} \\ q(m)^{\frac{1}{2}-w} L^*(1-w, \widetilde{\chi}_{a_2 m}) & \text{if } \deg(m) \text{ is odd.} \end{cases}$$

The functional equations for $L^*(s, \chi_{a_1 d})$ and $L^*(w, \widetilde{\chi}_{a_2 m})$ will induce functional equations for $Z_{a_1, a_2}(s, w)$. However, there is an obstruction caused by the gamma factors. Indeed, the gamma factor for $L^*(s, \chi_{a_1 d})$ and $L^*(w, \widetilde{\chi}_{a_2 m})$ depend upon the degrees of d and m respectively and to induce functional equations we need these gamma factors to be constant. Using the orthogonality for Dirichlet characters, in particular the Hilbert characters, will allow us to get past this issue.

Meromorphic Continuation

Poles and Residues

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