

# A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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ABSTRACT. We construct a quadratic double Dirichlet series  $Z(s, w)$  built from single variable quadratic Dirichlet  $L$ -functions  $L(s, \chi)$  over  $\mathbb{Q}$ . We prove that  $Z(s, w)$  admits meromorphic continuation to the  $(s, w)$ -plane and satisfies a group of functional equations.

## 1. PRELIMINARIES

We present an overview of quadratic Dirichlet  $L$ -functions over  $\mathbb{Q}$ . We begin with the Riemann zeta-function. The zeta function  $\zeta(s)$  is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m \geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for  $\operatorname{Re}(s) > 1$ . The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Now we recall characters on  $\mathbb{Z}$ . They are multiplicative functions  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ . The two flavors of characters of interest to use are:

- Dirichlet characters: multiplicative functions  $\chi_m : \mathbb{Z} \rightarrow \mathbb{C}$  modulo  $m \geq 1$  (in that they are  $m$ -periodic) and such that  $\chi_m(d) = 0$  if  $(d, m) > 1$ .
- Hilbert symbols: Dirichlet characters modulo 1.

In either case, the image always lands in the roots of unity. If  $\chi$  is a Dirichlet character then its conjugate  $\bar{\chi}$  is also a Dirichlet character. Moreover,  $\bar{\chi}$  is the multiplicative inverse to  $\chi$  and the Dirichlet characters modulo  $m$  form a group under multiplication. This group is always finite and its order is  $\phi(m) = |(\mathbb{Z}/m\mathbb{Z})^*|$ . Dirichlet characters also satisfy orthogonality relations:

**Theorem 1.1** (Orthogonality relations).

(i) For any two Dirichlet characters  $\chi$  and  $\psi$  modulo  $m$ ,

$$\frac{1}{\phi(m)} \sum'_{a \pmod{m}} \chi(a) \bar{\psi}(a) = \delta_{\chi, \psi}.$$

(ii) For any  $a, b \in (\mathbb{Z}/m\mathbb{Z})^*$ ,

$$\frac{1}{\phi(m)} \sum_{\chi \pmod{m}} \chi(a) \bar{\chi}(b) = \delta_{a, b}.$$

The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on  $\mathbb{F}_q[t]$ . First let us recall this symbol. For any irreducible  $p \in \mathbb{F}_q[t]$  and any  $m \in \mathbb{F}_q[t]$ , we define the

quadratic residue symbol  $\left(\frac{m}{p}\right)$  by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{|p|-1}{2}} \pmod{p} \in \{0, 1, -1\}.$$

In other words,  $\left(\frac{m}{p}\right) = 0, 1, -1$  according to whether  $p \mid m$ ,  $m$  is a square modulo  $p$ , or  $m$  is not a square modulo  $p$ . This symbol is multiplicative in  $m$ . Moreover, if  $b \in \mathbb{F}_q^*$ , we have

$$\left(\frac{b}{p}\right) = \text{sgn}(b)^{\deg(p)}.$$

where  $\text{sgn}(b) = \pm 1$  depending on if  $b \in (\mathbb{F}_q^\times)^2$  or not. For  $m \in \mathbb{F}_q[t]$  we define  $\text{sgn}(m) = \text{sgn}(b_n)$  if  $m(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_0$  (with  $b_n \neq 0$ ). We can extend the quadratic residue symbol multiplicatively in the denominator. If  $d = b p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  is the prime factorization of  $d$  (with  $b \in \mathbb{F}_q^*$ ), then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \leq i \leq k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any non-zero  $d \in \mathbb{F}_q[t]$ . The quadratic residue symbol also has the following reciprocity property:

**Theorem 1.2** (Quadratic reciprocity). *If  $d, m \in \mathbb{F}_q[t]$  are monic, relatively prime, and non-zero, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{q-1}{2} \deg(d) \deg(m)} \left(\frac{m}{d}\right).$$

Note that if  $q \equiv 1 \pmod{4}$ , the sign in the statement of quadratic reciprocity is always 1 so that reciprocity is perfect. We can now define the quadratic Dirichlet characters. For any non-zero  $d \in \mathbb{F}_q[t]$ , define the quadratic Dirichlet character  $\chi_d$  by the following quadratic residue symbol:

$$\chi_d(m) = \left(\frac{d}{m}\right) = m^{\frac{|d|-1}{2}} \pmod{d},$$

for any non-zero monic  $m \in \mathbb{F}_q[t]$ . Then  $\chi_d(m) \in \{\pm 1\}$  provided  $d$  and  $m$  are relatively prime and  $\chi_d(m) = 0$  if  $(m, d) > 1$ . Note that by quadratic reciprocity,  $\chi_d$  is a Dirichlet character modulo  $d$  if  $d$  is monic. Since the quadratic residue symbols are multiplicative,  $\chi_d$  is multiplicative in  $d$ . Therefore, factoring out a constant if necessary, we may always force  $d$  to be monic. Moreover, for  $b \in \mathbb{F}_q^\times$ , we see that  $\chi_b$  is a Hilbert symbol:

$$\chi_b(m) = \left(\frac{b}{m}\right) = \text{sgn}(b)^{\deg(m)},$$

where  $m \in \mathbb{F}_q[t]$  is a non-zero monic. The only Hilbert symbols we will need are those given by the quadratic residue symbol. There are only two of them: one nontrivial and one trivial. The nontrivial Hilbert symbol is  $\chi_\theta$  where  $\theta \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$ :

$$\chi_\theta(m) = (-1)^{\deg(m)},$$

where  $m \in \mathbb{F}_q[t]$  is a non-zero monic. Note that  $\overline{\chi_\theta} = \chi_\theta$ . The other Hilbert symbol is the trivial character  $\chi_\theta^2 = \chi_{\theta\theta} = \chi_1$ . In general, we denote a Hilbert symbol by  $\chi_a$  where  $a \in \{1, \theta\}$ .

With the Dirichlet characters and Hilbert symbols introduced, we are ready to discuss the  $L$ -functions associated to quadratic Dirichlet characters. We define the  $L$ -function  $L(s, \chi_d)$  attached to  $\chi_d$  by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \text{ monic}} \frac{\chi_d(m)}{|m|^s} = \prod_{P \text{ monic irr}} \left(1 - \frac{\chi_d(P)}{|P|^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character,  $L(s, \chi_d) \ll \zeta(s)$  for  $\operatorname{Re}(s) > 1$  so that  $L(s, \chi_d)$  is locally absolutely uniformly convergent in this region.  $L(s, \chi_d)$  also admits meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  if  $d$  is a perfect square and is analytic otherwise (see [?] for a proof). Moreover,  $L(s, \chi_d)$  is a polynomial in  $q^{-s}$  of degree at most  $\deg(d) - 1$ . The completed  $L$ -function is defined as follows:

$$L^*(s, \chi_d) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_d) & \text{if } \deg(d) \text{ is even,} \\ L(s, \chi_d) & \text{if } \deg(d) \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_d) = \begin{cases} q^{2s-1} |d|^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \deg(d) \text{ is even,} \\ q^{2s-1} (q|d|)^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \deg(d) \text{ is odd.} \end{cases}$$

Note that in the case  $\deg(d)$  is even, the conductor is  $|d|$  and in the case  $\deg(d)$  is odd, the conductor is  $q|d|$ . In other words, the gamma factors depend upon the degree of  $d$ . This will cause a small but important technical issue later when we want to derive functional equations for the quadratic double Dirichlet series.

## THE QUADRATIC DOUBLE DIRICHLET SERIES

### THE INTERCHANGE

### WEIGHTING TERMS

### FUNCTIONAL EQUATIONS

### MEROMORPHIC CONTINUATION

### POLES AND RESIDUES

### REFERENCES

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