A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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Abstract. We construct a quadratic double Dirichlet series $Z(s,w)$ built from single variable quadratic Dirichlet L-functions $L(s,\chi)$ over $\mathbb{Q}$. We prove that $Z(s,w)$ admits meromorphic continuation to the (s,w)-plane and satisfies a group of functional equations.

1. Preliminaries

We present an overview of quadratic Dirichlet L-functions over $\mathbb{Q}$. We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m\geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Now we recall characters on $\mathbb{Z}$. They are multiplicative functions $\chi : \mathbb{Z} \to \mathbb{C}$ and form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions $\chi_d : \mathbb{Z} \to \mathbb{C}$ modulo $d \geq 1$ (in that they are d-periodic) and such that $\chi_d(m) = 0$ if $(m,d) > 1$.
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. Moreover, $\overline{\chi}$ is the multiplicative inverse to χ and the Dirichlet characters modulo d form a subgroup under multiplication. This group is always finite and its order is $\phi(d) = |(\mathbb{F}_q[t]/d\mathbb{F}_q[t])^\times|$. The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{Z}$. First let us recall this symbol. For any odd prime p and any $d \in \mathbb{Z}$, we define the quadratic residue symbol $\left(\frac{d}{p}\right)$ by

$$\left(\frac{d}{p}\right) \equiv d^{\frac{p-1}{2}} \pmod{p} = \begin{cases} 1 & \text{if } x^2 \equiv d \pmod{p} \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv d \pmod{p} \text{ is not solvable,} \\ 0 & \text{if } d \equiv 0 \pmod{p}. \end{cases}$$

This symbol only depends upon d modulo p and is multiplicative in d. We can extend the quadratic residue symbol multiplicatively in the denominator. First we define

$$\left(\frac{d}{-1}\right) = \begin{cases} 1 & \text{if } d \geq 0, \\ -1 & \text{if } d < 0, \end{cases} \quad \text{and} \quad \left(\frac{d}{2}\right) = \begin{cases} 1 & \text{if } d \equiv 1,7 \pmod{8}, \\ -1 & \text{if } d \equiv 3,5 \pmod{8}, \\ 0 & \text{if } d \equiv 0 \pmod{2}. \end{cases}$$

Date: 2024.

If $m = up_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ is the prime factorization of m (with $u = \pm 1$), then we define

$$\left(\frac{d}{m}\right) = \left(\frac{d}{u}\right)\prod_{1\le i\le k}\left(\frac{d}{p_i}\right)^{e_i}.$$

The quadratic residue symbol now makes sense for any $m \in \mathbb{Z}$ and is multiplicative in both d and m. The quadratic residue symbol also admits the following reciprocity law:

Theorem 1.1 (Quadratic reciprocity)**.** *If $d, m \in \mathbb{Z}$, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2}\frac{m^{(2)}-1}{2}}\left(\frac{m}{|d|}\right),$$

where $d^{(2)}$ and $m^{(2)}$ are the parts of d and m relatively prime to 2 respectively.

Moreover, we have the additional relations

$$\left(\frac{-1}{m}\right) = (-1)^{\frac{m^{(2)}-1}{2}} \quad \text{and} \quad \left(\frac{2}{m}\right) = (-1)^{\frac{m^2-1}{8}},$$

and if $m \not\equiv 0 \pmod 2$, we can write

$$\left(\frac{-1}{m}\right) = (-1)^{\frac{m-1}{2}} = \begin{cases} 1 & m \equiv 1 \pmod 4, \\ -1 & m \equiv 3 \pmod 4, \end{cases} \quad \text{and} \quad \left(\frac{2}{m}\right) = (-1)^{\frac{m^2-1}{8}} = \begin{cases} 1 & m \equiv 1,7 \pmod 8, \\ -1 & m \equiv 3,5 \pmod 8. \end{cases}$$

We can now define the quadratic Dirichlet characters. For any square-free $d \in \mathbb{Z}$, define the quadratic Dirichlet character χ_d by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod 4, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2,3 \pmod 4. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension $\mathbb{Q}(\sqrt{d})$. We extend χ_d multiplicatively in the denominator so that χ_d makes sense for any odd d. In particular, $\chi_d(m) = \pm 1$ provided d and m are relatively prime and $\chi_d(m) = 0$ if $(m, d) > 1$. Quadratic reciprocity implies that χ_d is a Dirichlet character modulo $|d|$ if $d \equiv 1 \pmod 4$ and is a Dirichlet character modulo $|4d|$ if $d \equiv 2,3 \pmod 4$. Indeed, if $d \equiv 1 \pmod 4$ then $d^{(2)} = d$ and the sign is always 1. If $d \equiv 3 \pmod 4$, then $d^{(2)} = d$ and the sign is $\left(\frac{-1}{m}\right)$ which is a character modulo 4. If $d \equiv 2 \pmod 4$, then $d^{(2)} \equiv 1,3 \pmod 4$ and we are reduced to one of the previous two cases. We will also set

$$q(d) = \begin{cases} |d| & \text{if } d \equiv 1 \pmod 4, \\ |4d| & \text{if } d \equiv 2,3 \pmod 4, \end{cases} \quad \text{and} \quad \varepsilon_{\chi_d} = \frac{\tau(\chi_d)}{\sqrt{q(d)}} = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ 1+i & \text{if } d \equiv 2,3 \pmod 4, \end{cases}$$

where $\tau(\chi_d)$ is the Gauss sum attached to χ_d. We will also require an associated character. For each χ_m (here we are purposely interchanging the roles of d and m to keep consistency with the notation when discussing the quadratic double Dirichlet series later), we define $\widetilde{\chi}_m$ by

$$\widetilde{\chi}_m(d) = (-1)^{\frac{m^{(2)}-1}{2}\frac{d^{(2)}-1}{2}}\chi_m(|d|).$$

By quadratic reciprocity, $\widetilde{\chi}_m$ is a quadratic Dirichlet character of the same modulus as χ_m and is multiplicative in m. Moreover, we have the identity $\widetilde{\widetilde{\chi}}_m(d) = \chi_m(|d|)$. Analogously, we set

$$q(m) = \begin{cases} |m| & \text{if } m \equiv 1 \pmod 4, \\ |4m| & \text{if } m \equiv 2,3 \pmod 4, \end{cases} \quad \text{and} \quad \varepsilon_{\widetilde{\chi}_m} = \frac{\tau(\widetilde{\chi}_m)}{\sqrt{q(m)}} = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4, \\ 1+i & \text{if } m \equiv 2,3 \pmod 4, \end{cases}$$

where $\tau(\widetilde{\chi}_m)$ is the Gauss sum attached to $\widetilde{\chi}_m$. We now discuss the Hilbert characters. We will only need four of them: the quadratic Dirichlet characters modulo 8. They are given as follows:

$$\chi_1(m) = \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod 2, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases} \quad \chi_{-1}(m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4, \\ -1 & \text{if } m \equiv 3 \pmod 4, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

$$\chi_2(m) = \begin{cases} 1 & \text{if } m \equiv 1,7 \pmod 8, \\ -1 & \text{if } m \equiv 3,5 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases} \quad \chi_{-2}(m) = \begin{cases} 1 & \text{if } m \equiv 1,3 \pmod 8, \\ -1 & \text{if } m \equiv 5,7 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

In general, we will denote a Hilbert character by χ_a with $a \in \{\pm 1, \pm 2\}$. The Hilbert characters also satisfy an important orthogonality property:

Theorem 1.2 (Orthogonality of Hilbert characters)**.** *If $d, m \in \mathbb{Z}$ are odd, then*

$$\frac{1}{4} \sum_{a \in \{\pm 1, \pm 2\}} \chi_a(dm) = \begin{cases} 1 & \textit{if } d \equiv m \pmod 8, \\ 0 & \textit{if } d \not\equiv m \pmod 8. \end{cases}$$

Also, we have the identities

$$\widetilde{\chi}_a(m) = \chi_a(|m|), \quad \chi_{-1}(m) = \left(\frac{-1}{m}\right), \quad \text{and} \quad \chi_2(m) = \left(\frac{2}{m}\right),$$

and the relations

$$\chi_{-2}(m) = \chi_{-1}(m)\chi_2(m), \quad \chi_1(m) = \chi_{-1}(m)\chi_{-1}(m), \quad \text{and} \quad \chi_{-1}(m) = \chi_2(m)\chi_{-2}(m).$$

With the Dirichlet and Hilbert characters introduced, we are ready to discuss the L-functions associated to quadratic Dirichlet characters. We define the L-function $L(s, \chi_d)$ attached to χ_d for square-free d, by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \geq 1} \frac{\chi_d(m)}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character, $L(s, \chi_d) \ll \zeta(s)$ for $\mathrm{Re}(s) > 1$ so that $L(s, \chi_d)$ is locally absolutely uniformly convergent in this region. $L(s, \chi_d)$ also admits analytic continuation to $\mathbb{C}$. The completed L-function $L^*(s, \chi_d)$ is defined as

$$L^*(s, \chi_d) = \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi_d) & \text{if } d > 0, \\ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi_d) & \text{if } d < 0. \end{cases}$$

We have the functional equation

$$L^*(s, \chi_d) = \varepsilon_{\chi_d} q(d)^{\frac{1}{2} - s} L^*(1 - s, \chi_d),$$

which can be equivalently expressed as

$$L^*(s, \chi_d) = \begin{cases} |d|^{\frac{1}{2} - s} L^*(1 - s, \chi_d) & \text{if } d \equiv 1,5 \pmod 8, \\ (1+i)|4d|^{\frac{1}{2} - s} L^*(1 - s, \chi_d) & \text{if } d \equiv 2,3,6,7 \pmod 8. \end{cases}$$

Note that the gamma factor depends upon d modulo 8. This the root cause of an important technical issue later when deriving functional equations for the quadratic double Dirichlet series. Analogously, the Dirichlet L-function $L(w, \widetilde{\chi}_m)$ attached to $\widetilde{\chi}_m$ for square-free m is defined by a Dirichlet series or Euler product:

$$L(w, \widetilde{\chi}_m) = \sum_{d \geq 1} \frac{\widetilde{\chi}_m(d)}{d^w} = \prod_{p \text{ prime}} \left(1 - \frac{\widetilde{\chi}_m(p)}{p^w}\right)^{-1}.$$

As for $L(s,\chi_d)$, $L(w,\widetilde{\chi}_m) \ll \zeta(w)$ for $\mathrm{Re}(w) > 1$ so that $L(w,\widetilde{\chi}_m)$ is locally absolutely uniformly convergent in this region. Moreover, $L(w,\widetilde{\chi}_m)$ admits analytic continuation to $\mathbb{C}$ and the completed L-function $L^*(w,\widetilde{\chi}_m)$ is defined as

$$L^*(w,\widetilde{\chi}_m) = \begin{cases} \pi^{-\frac{w}{2}}\Gamma\left(\frac{w}{2}\right) L(w,\widetilde{\chi}_m) & \text{if } m \equiv 1,2,5 \pmod 8, \\ \pi^{-\frac{w}{2}}\Gamma\left(\frac{w+1}{2}\right) L(w,\widetilde{\chi}_m) & \text{if } m \equiv 3,6,7 \pmod 8). \end{cases}$$

We have the functional equation

$$L^*(w,\widetilde{\chi}_m) = \varepsilon_{\widetilde{\chi}_m} q(m)^{\frac{1}{2}-w} L^*(1-w,\widetilde{\chi}_m),$$

which can be equivalently expressed as

$$L^*(w,\widetilde{\chi}_m) = \begin{cases} |m|^{\frac{1}{2}-w} L^*(1-w,\widetilde{\chi}_m) & \text{if } m \equiv 1,5 \pmod 8, \\ (1+i)|4m|^{\frac{1}{2}-w} L^*(1-w,\widetilde{\chi}_m) & \text{if } m \equiv 2,3,6,7 \pmod 8). \end{cases}$$

Analogously, note that the gamma factor depends upon m modulo 8.

Remark 1.1. *The definitions for $L(s,\chi_d)$, $L^*(s,\chi_d)$, $L(w,\widetilde{\chi}_m)$, and $L^*(w,\widetilde{\chi}_m)$ work perfectly well even when d and m are not square-free (however the functional equations do not hold). We purposely do not define these L-functions, yet, for d and m not necessarily square-free.*

2. The Quadratic Double Dirichlet Series

We will now define the quadratic double Dirichlet series $Z(s,w)$. For any integer $d \geq 1$, write $d = d_0 d_1^2$ where d_0 is square-free. Equivalently, d_0 is the square-free part of d and $\frac{d}{d_0}$ is a perfect square. The **quadratic double Dirichlet series** $Z(s,w)$ is defined as

$$Z(s,w) = \sum_{\substack{d \geq 1 \\ (d,2)=1}} \frac{L^{(2)}(s,\chi_{d_0}) Q_{d_0 d_1^2}(s)}{d^w},$$

where $Q_{d_0 d_1^2}(s)$ is the **correction polynomial** defined by

$$Q_{d_0 d_1^2}(s) = \sum_{e_1 e_2 | d_1} \mu(e_1)\chi_{d_0}(e_1) e_1^{-s} e_2^{1-s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{d_0}(e_1) e_1^{-s} e_2^{1-s},$$

and μ is the usual Möbius function. For $\mathrm{Re}(s) > 1$, there is the trivial estimate

$$Q_{d_0 d_1^2}(s) \ll \sum_{e_1 e_2 | d_1} 1 \ll \sigma_0(d_1)^2 \ll_\varepsilon d_1^{2\varepsilon} \ll_\varepsilon d^\varepsilon,$$

for any $\varepsilon > 0$. As $L(s,\chi_{d_0}) \ll 1$ for $\mathrm{Re}(s) > 1$, $Z(s,w)$ is locally absolutely uniformly convergent in the region $\Lambda = \{(s,w) \in \mathbb{C}^2 : \mathrm{Re}(s) > 1, \mathrm{Re}(w) > 1\}$. It will also be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters χ_{a_1} and χ_{a_2}. The **quadratic double Dirichlet series** $Z_{a_1,a_2}(s,w)$ twisted by χ_{a_1} and χ_{a_2} is defined as

$$Z_{a_1,a_2}(s,w) = \sum_{\substack{d \geq 1 \\ (d,2)=1}} \frac{L^{(2)}(s,\chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s,\chi_{a_1})}{d^w},$$

where $Q_{d_0 d_1^2}(s,\chi_{a_1})$ is the **correction polynomial** twisted by χ_{a_1} defined by

$$Q_{d_0 d_1^2}(s,\chi_{a_1}) = \sum_{e_1 e_2 | d_1} \mu(e_1)\chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s},$$

and μ is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound $Q_{d_0 d_1^2}(s,\chi_{a_1}) \ll d_\varepsilon$ so that $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly in the same region as

$Z(s,w)$ does. In particular, $Z(s,w) = Z_{1,1}(s,w)$. We will also require quadratic double Dirichlet series corresponding to the characters $\widetilde{\chi}_m$. Analogously writing $m = m_0 m_1^2$, the **quadratic double Dirichlet series** $\widetilde{Z}(s,w)$ is defined as

$$\widetilde{Z}(w,s) = \sum_{\substack{m \geq 1 \\ (m,2)=1}} \frac{L^{(2)}(w, \widetilde{\chi}_{m_0}) Q_{m_0 m_1^2}(w)}{m^s},$$

where $Q_{d_0 d_1^2}(w)$ is the **correction polynomial** defined by

$$Q_{m_0 m_1^2}(w) = \sum_{e_1 e_2 | m_1} \mu(e_1) \chi_{m_0}(e_1) e_1^{-w} e_2^{1-w} = \sum_{e_1 e_2 e_3 = m_1} \mu(e_1) \chi_{m_0}(e_1) e_1^{-w} e_2^{1-w},$$

and μ is the usual Möbius function. We have the analogous estimate $Q_{m_0 m_1^2}(w) \ll_\varepsilon m^\varepsilon$ and as $L(w, \widetilde{\chi}_{m_0}) \ll 1$ for $\mathrm{Re}(w) > 1$, $\widetilde{Z}(w,s)$ is locally absolutely uniformly convergent in the same region as $Z(s,w)$. We also need to consider twists by a pair of Hilbert characters $\widetilde{\chi}_{a_2}$ and $\widetilde{\chi}_{a_1}$. The **quadratic double Dirichlet series** $\widetilde{Z}_{a_2,a_1}(w,s)$ twisted by $\widetilde{\chi}_{a_2}$ and $\widetilde{\chi}_{a_1}$ is defined as

$$\widetilde{Z}_{a_2,a_1}(w,s) = \sum_{\substack{m \geq 1 \\ (m,2)=1}} \frac{L^{(2)}(w, \widetilde{\chi}_{a_2 m_0}) \widetilde{\chi}_{a_1}(m) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})}{m^s},$$

where $Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})$ is the **correction polynomial** twisted by $\widetilde{\chi}_{a_2}$ defined by

$$Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2}) = \sum_{e_1 e_2 | m_1} \mu(e_1) \widetilde{\chi}_{a_2 m_0}(e_1) e_1^{-w} e_2^{1-2w} = \sum_{e_1 e_2 e_3 = m_1} \mu(e_1) \widetilde{\chi}_{a_2 m_0}(e_1) e_1^{-w} e_2^{1-2w}.$$

and μ is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound $Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2}) \ll_\varepsilon m^\varepsilon$ so that $\widetilde{Z}_{a_2,a_1}(w,s)$ converges locally absolutely uniformly in the same region as $\widetilde{Z}(w,s)$ does. In particular, $\widetilde{Z}(w,s) = \widetilde{Z}_{1,1}(w,s)$.

3. The Interchange

As defined, $Z_{a_1,a_2}(s,w)$ is a sum of L-functions, and hence Euler products, in s. We will prove an interchange formula for $Z_{a_1,a_2}(s,w)$ which will show that it can be expressed as a sum of L-functions in w. That is, we want the variables s and w to change places. Precisely:

Theorem 3.1 (Interchange)**.** *Wherever $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly,*

$$Z_{a_1,a_2}(s,w) = \sum_{\substack{d \geq 1 \\ (d,2)=1}} \frac{L^{(2)}(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w} = \sum_{\substack{m \geq 1 \\ (m,2)=1}} \frac{L^{(2)}(w, \widetilde{\chi}_{a_2 m_0}) \widetilde{\chi}_{a_1}(m) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})}{m^s}.$$

Moreover, the same holds for $\widetilde{Z}_{a_2,a_1}(w,s)$.

Proof. Only the second equality needs to be proved. To do this, first expand the L-function $L^{(2)}(s,\chi_{a_1d_0})$ and polynomial $Q_{d_0d_1^2}(s,\chi_{a_1})$ to get

$$\begin{aligned} Z(s,w) &= \sum_{\substack{d\geq 1\\(d,2)=1}} \frac{L^{(2)}(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{d^w} \\ &= \sum_{\substack{d\geq 1\\(d,2)=1}} \left(\sum_{\substack{m\geq 1\\(m,2)=1}} \chi_{a_1d_0}(m)m^{-s}\right)\left(\sum_{e_1e_2|d_1} \mu(e_1)\chi_{a_1d_0}(e_1)e_1^{-s}e_2^{1-2s}\right)\chi_{a_2}(d)d^{-w} \\ &= \sum_{\substack{d,m\geq 1\\(dm,2)=1}} \sum_{e_1e_2|d_1} \mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(me_1)e_1^{-s}e_2^{1-2s}m^{-s}d^{-w}. \end{aligned}$$

Now $\chi_{a_1d_0}(me_1)=0$ unless $(d_0,me_1)=1$. We make this restriction on the sum giving

$$\sum_{\substack{d,m\geq 1\\(dm,2)=1}} \sum_{\substack{e_1e_2|d_1\\(d_0,me_1)=1}} \mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(me_1)e_1^{-s}e_2^{1-2s}m^{-s}d^{-w}.$$

Making the change of variables $me_1\to m$ yields

$$\sum_{\substack{d\geq 1\\(d,2)=1}} \sum_{\substack{m\geq 1\\(m,2)=1\\e_1|m}} \sum_{\substack{e_1e_2|d_1\\(d_0,m)=1}} \mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(m)e_2^{1-2s}m^{-s}d^{-w}.$$

For fixed $d=d_0d_1^2$ and e_2, the subsum over m and e_1 is

$$\sum_{\substack{m\geq 1\\(m,2)=1\\e_1|m}} \sum_{\substack{e_1|\frac{d_1}{e_2}\\(d_0,m)=1}} \mu(e_1)\chi_{a_1d_0}(m)m^{-s} = \sum_{\substack{m\geq 1\\(m,2)=1\\(d_0,m)=1}} \chi_{a_1d_0}(m)m^{-s}\left(\sum_{e_1|\left(\frac{d_1}{e_2},m\right)} \mu(e_1)\right).$$

The inner sum over e_1 of the Möbius function vanishes unless $\left(\frac{d_1}{e_2},m\right)=1$ in which case it is 1. Therefore the triple sum above becomes

$$\sum_{\substack{d,m\geq 1\\(dm,2)=1}} \sum_{\substack{e_2|d_1\\\left(\frac{d_0d_1}{e_2},m\right)=1}} \chi_{a_2}(d)\chi_{a_1d_0}(m)e_2^{1-2s}m^{-s}d^{-w}.$$

Making the change of variables $d\to de_2^2$, the condition $\left(\frac{d_0d_1}{e_2},m\right)=1$ becomes $(d_0d_1,m)=1$ which is equivalent to $(d,m)=1$. Moreover, $\chi_{a_2}(de_2^2)=\chi_{a_2}(d)$. Altogether, we obtain

$$\sum_{\substack{d,m\geq 1\\(dm,2)=1\\(d,m)=1}} \sum_{e_2} \chi_{a_2}(d)\chi_{a_1d_0}(m)e_2^{1-2s-2w}m^{-s}d^{-w}.$$

Writing $m=m_0m_1^2$ analogously as for d, quadratic reciprocity and positivity of m and d together imply $\chi_{d_0}(m)=\widetilde{\chi}_m(d_0)=\widetilde{\chi}_{m_0}(d)$ where the last equality holds because $(d,m)=1$ and both d_0 and m_0 differ from d and m respectively by perfect squares. As $\chi_{a_1}(m)=\widetilde{\chi}_{a_1}(m)$ and $\chi_{a_2}(d)=\widetilde{\chi}_{a_2}(d)$ (again we use the positivity of m and d), the previous fact implies $\chi_{a_2}(d)\chi_{a_1d_0}(m) = \widetilde{\chi}_{a_1}(m)\widetilde{\chi}_{a_2m_0}(d)$ and so our expression becomes

$$\sum_{\substack{d,m\geq 1\\(dm,2)=1\\(d,m)=1}}\sum_{e_2}\widetilde{\chi}_{a_1}(m)\widetilde{\chi}_{a_2m_0}(d)e_2^{1-2s-2w}m^{-s}d^{-w}.$$

But now we can reverse the argument with the roles of d, m, χ_{a_1}, and χ_{a_2} interchanged respectively, but with $\widetilde{\chi}_{a_1}$ and $\widetilde{\chi}_{a_2}$, to obtain

$$Z(s,w) = \sum_{\substack{m\geq 1\\(m,2)=1}}\frac{L^{(2)}(w,\widetilde{\chi}_{a_2m_0})\widetilde{\chi}_{a_1}(m)Q_{m_0m_1^2}(w,\widetilde{\chi}_{a_2})}{m^s}.$$

Clearly the same holds for $\widetilde{Z}_{a_2,a_1}(w,s)$. □

Note that the interchange is not completely symmetric because of the characters $\widetilde{\chi}_{a_2m_0}$, $\widetilde{\chi}_{a_1}$, and $\widetilde{\chi}_{a_2}$ in the second expression for $Z_{a_1,a_2}(s,w)$. This is due to the fact that reciprocity is not perfect. In even more general settings the correction polynomials in w need not be equal to those in s.

Remark 3.1. *When $a_1 = a_2 = 1$, the interchange implies*

$$Z(s,w) = \widetilde{Z}(w,s).$$

More generally, the interchange implies the following relations for twisted quadratic double Dirichlet series:

$$Z_{a_1,a_2}(s,w) = \widetilde{Z}_{a_2,a_1}(w,s),$$

for $a_1, a_2 \in \{\pm 1, \pm 2\}$.

4. Weighting Terms

We will now study the coefficients of $Z_{a_1,a_2}(s,w)$ expanded in s and w. Expanding $L^{(2)}(s,\chi_{a_1d_0})Q_{d_0d_1^2}(s,\chi_{a_1})$ in the numerator of $Z_{a_1,a_2}(s,w)$, we can write

$$Z_{a_1,a_2}(s,w) = \sum_{\substack{d\geq 1\\(d,2)=1}}\frac{L^{(2)}(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{d^w} = \sum_{\substack{d,m\geq 1\\(dm,2)=1}}\frac{\chi_{a_1d_0}(\widehat{m})\chi_{a_2}(d)H(m,d)}{m^sd^w},$$

where $\widehat{m}$ is the part of m relatively prime to d_0 and the **weighting coefficient** $H(m,d)$ is given by

$$H(m,d) = \sum_{\substack{e_1e_2^2e_3=m\\e_1e_2|d_1\\(d_0,e_1e_3)=1}}\mu(e_1)e_2.$$

To see this, the coefficient of $m^{-s}d^{-w}$ in the definition of $Z_{a_1,a_2}(s,w)$ is

$$\begin{aligned}\chi_{a_2}(d)\sum_{\substack{e_1e_2^2e_3=m\\e_1e_2|d_1}}\mu(e_1)\chi_{a_1d_0}(e_1e_3)e_2 &= \chi_{a_2}(d)\sum_{\substack{e_1e_2^2e_3=m\\e_1e_2|d_1\\(d_0,e_1e_3)=1}}\mu(e_1)\chi_{a_1d_0}(e_1e_3)e_2\\ &= \chi_{a_1d_0}(\widehat{m})\chi_{a_2}(d)\sum_{\substack{e_1e_2^2e_3=m\\e_1e_2|d_1\\(d_0,e_1e_3)=1}}\mu(e_1)e_2\\ &= \chi_{a_1d_0}(\widehat{m})\chi_{a_2}(d)H(m,d),\end{aligned}$$

where the first equality holds because $\chi_{d_0}(e_1e_3) = 0$ unless $(d_0,e_1e_3) = 1$ and the second equality holds because if $(d_0,e_1e_3) = 1$, $\widehat{m}$ differs from e_1e_3 by a perfect square (the divisors of which belong to (d_0,e_2))

and so $\chi_{d_0}(e_1e_3) = \chi_{d_0}(\widehat{m})$. For completeness, we extend the definition of $H(m,d)$ to all $d, m \geq 1$. In particular, $H(m,d)$ makes sense when m or d may be even.

Remark 4.1. *Also,* $H(m,d) = 0$ *unless* $m = e_1e_2^2e_3$ *with* $(d_0, e_1e_3) = 1$ *and* $e_1e_2^2 \mid d_1$.

We will define $L(s, \chi_{a_1d})$ to be the Dirichlet series given by

$$L(s, \chi_{a_1d}) = L(s, \chi_{a_1d_0})Q_{d_0d_1^2}(s, \chi_{a_1}) = \sum_{m\geq 1} \frac{\chi_{a_1d_0}(\widehat{m})H(m,d)}{m^s}.$$

Clearly $L(s, \chi_{a_1d})$ is locally absolutely uniformly convergent for $\mathrm{Re}(s) > 1$. In particular, $L(s, \chi_d)$ now makes sense for d not necessarily square-free and this definition agrees with the former when d is square-free. Moreover, we have the representation

$$Z_{a_1,a_2}(s,w) = \sum_{\substack{d\geq 1\\(d,2)=1}} \frac{\chi_{a_2}(d)L^{(2)}(s, \chi_{a_1d})}{d^w}.$$

If we perform the same procedure but with the interchange, we get

$$\widetilde{Z}_{a_2,a_1}(w,s) = \sum_{\substack{m\geq 1\\(m,2)=1}} \frac{L^{(2)}(w, \widetilde{\chi}_{a_2m_0})\widetilde{\chi}_{a_1}(m)Q_{m_0m_1^2}(w, \widetilde{\chi}_{a_2})}{m^s} = \sum_{\substack{d,m\geq 1\\(dm,2)=1}} \frac{\widetilde{\chi}_{a_2m_0}(\widehat{d})\widetilde{\chi}_{a_1}(m)H(d,m)}{m^sd^w},$$

where $\widehat{d}$ is the part of d relatively prime to m_0. Analogously, we define $L(w, \widetilde{\chi}_{a_2m})$ to be the Dirichlet series given by

$$L(w, \widetilde{\chi}_{a_2m}) = L(w, \widetilde{\chi}_{a_2m_0})Q_{m_0m_1^2}(w, \widetilde{\chi}_{a_2}) = \sum_{d\geq 1} \frac{\widetilde{\chi}_{a_2m_0}(\widehat{d})H(d,m)}{d^w}.$$

Again, $L(w, \widetilde{\chi}_{a_2m})$ is locally absolutely uniformly convergent for $\mathrm{Re}(w) > 1$. We also have

$$\widetilde{Z}_{a_2,a_1}(w,s) = \sum_{\substack{m\geq 1\\(m,2)=1}} \frac{\widetilde{\chi}_{a_1}(m)L^{(2)}(w, \widetilde{\chi}_{a_2m})}{m^s}.$$

We now investigate the structure of the weighting coefficients $H(m,d)$. Their structure controls the majority of the information about both the quadratic double Dirichlet series and the correction polynomials. We first show that the weighting coefficients possess a multiplicativity property:

Proposition 4.1. *We have* $H(m,1) = H(1,d) = 1$ *and*

$$H(m,d) = \prod_{\substack{p^\alpha || m\\p^\beta || d}} H(p^\alpha, p^\beta).$$

Proof. From the definition of the weighting coefficients, $H(m,1) = H(1,d) = 1$. We will prove multiplicativity in m and then in d. Letting $m = m'p^\alpha$, we must show

$$H(m,d) = H(m',d)H(p^\alpha, d).$$

To accomplish this, for $e_1e_2^2e_3 = m$, let $e_1 = c_1d_1$, $e_2 = c_2d_2$, and $e_3 = c_3d_3$ with $c_1, c_2, c_3 \mid m'$ and $d_1, d_2, d_3 \mid p^\alpha$. Because $(m', p^\alpha) = 1$, as $e_1e_2^2e_3$ runs over decompositions of m, $c_1c_2^2c_3$ and $d_1d_2^2d_3$ run over decompositions of m' and p^α respectively. Moreover, as e_1e_2 runs over the divisors of d_1 so does $c_1d_1c_2d_2$.

These facts combined with multiplicativity of the Möbius function gives

$$\begin{aligned}
H(m,d) &= \sum_{\substack{e_1e_2^2e_3=m\\ e_1e_2|d_1\\ (d_0,e_1e_3)=1}} \mu(e_1)e_2\\
&= \sum_{\substack{c_1c_2^2c_3=m'\\ d_1d_2^2d_3=p^\beta\\ c_1d_1c_2d_2|d_1\\ (d_0,c_1d_1c_3d_3)=1}} \mu(c_1)(d_1)|c_2|d_2\\
&= \left(\sum_{\substack{c_1c_2^2c_3=m'\\ c_1c_2|d_1\\ (d_0,c_1c_3)=1}} \mu(c_1)|c_2|\right)\left(\sum_{\substack{d_1d_2^2d_3=p^\alpha\\ d_1d_2|d_1\\ (d_0,d_1d_3)=1}} \mu(d_1)d_2\right)\\
&= H(m',d)H(p^\alpha,d),
\end{aligned}$$

as desired. Now we prove multiplicativity in d. Since we have already proven multiplicativity in m, we may assume $m=p^\alpha$. Letting $d=d'p^\beta$, we must show

$$H(p^\alpha,d)=H(p^\alpha,p^\beta).$$

As $e_1e_2^2e_3=p^\alpha$, the e_i are powers of p for $1\le i\le 3$. It follows that $e_1e_2\mid d_1$ is equivalent to $e_1e_2\mid p^\beta$. Moreover, $(d_0,e_1e_2)=1$ is equivalent to $(1,e_1e_2)=1$ or $(p,e_1e_2)=1$ depending on of β is even or odd. These facts imply the desired identity. □

The correction polynomials $Q_{d_0d_1^2}(s,\chi_{a_1})$ are tightly connected to the weighting coefficients $H(m,d)$. In particular, $Q_{d_0d_1^2}(s,\chi_{a_1})$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients. We first prove this relationship when d is an odd prime power:

Lemma 4.1. *For any prime p and $\alpha\ge 1$, we have*

$$Q_{p^{2\alpha+1}}(s)=\sum_{k\le 2\alpha}\frac{H(p^k,p^{2\alpha+1})}{p^{ks}}.$$

Moreover, the same holds for $Q_{p^{2\alpha+1}}(w)$.

Proof. Expanding the correction polynomial in p^{-s} yields

$$Q_{p^{2\alpha+1}}(s)=\sum_{e_1e_2|p^\alpha}\mu(e_1)\chi_p(e_1)e_1^{-s}e_2^{1-2s}=\sum_{k\le 2\alpha}\frac{H'(p^k,p^{2\alpha+1})}{p^{ks}}.$$

where

$$H'(p^k,p^{2\alpha+1})=\sum_{e_1e_2^2=p^k}\mu(e_1)\chi_p(e_1)e_2.$$

The proof will be finished if we can show $H'(p^k,p^{2\alpha+1})=H(p^k,p^{2\alpha+1})$. To see this, first observe $\mu(e_1)\chi_p(e_1)=0$ unless $e_1=1$ in which case it is 1. So $H'(p^k,p^{2\alpha+1})=0$ if k is odd and $p^{\frac{k}{2}}$ if k is even. Compactly stated,

$$H'(p^k,p^{2\alpha+1})=\begin{cases}p^{\frac{k}{2}} & \text{if } k \text{ is even},\\ 0 & \text{if } k \text{ is odd.}\end{cases}$$

On the other hand, $k \le \alpha$ so that

$$H(p^k, p^{2\alpha+1}) = \sum_{\substack{e_1e_2^2e_3=p^k \\ e_1e_2|p^\alpha \\ (p,e_1e_3)=1}} \mu(e_1)e_2 = \sum_{\substack{e_1e_2^2|p^k \\ (p,e_1e_3)=1}} \mu(e_1)e_2 = \sum_{e_2^2=p^k} e_2 = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

This finishes the proof. Clearly the same holds for $Q_{p^{2\alpha+1}}(w)$. □

There is an analogous statement when d is an even prime power up to a square-free factor and relatively prime factor:

Lemma 4.2. *For any square-free integer $d_0 \ge 1$, $a_1 \in \{\pm 1, \pm 2\}$, prime p not dividing d_0, and $\beta \ge 1$, we have*

$$Q_{d_0p^{2\beta}}(s, \chi_{a_1}) = (1 - \chi_{a_1d_0}(p)p^{-s}) \sum_{k \le 2\beta} \frac{\chi_{a_1d_0}(p^k)H(p^k, p^{2\beta})}{p^{ks}}.$$

Moreover, the same holds for $Q_{m_0p^{2\beta}}(w, \widetilde{\chi}_{a_2})$.

Proof. Expand the correction polynomial in p^{-s} to get

$$Q_{d_0p^{2\beta}}(s, \chi_{a_1}) = \sum_{e_1e_2|p^\alpha} \mu(e_1)\chi_{a_1d_0}(e_1)e_1^{-s}e_2^{1-2s} = \sum_{k \le 2\beta} \frac{H'(p^k, p^{2\beta})}{p^{ks}}.$$

where

$$H'(p^k, p^{2\beta}) = \sum_{e_1e_2^2=p^k} \mu(e_1)\chi_{a_1d_0}(e_1)e_2.$$

It suffices to show $H'(p^k, p^{2\beta}) = \chi_{a_1d_0}(p^k)\left(H(p^k, p^{2\beta}) - H(p^{k-1}, p^{2\beta})\right)$. On the one hand, $\mu(e_1) = 0$ unless $e_1 = 1, p$ in which case $\mu(e_1) = \pm 1$ accordingly. So

$$H'(p^k, p^{2\beta}) = \sum_{e_1e_2^2=p^k} \mu(e_1)\chi_{a_1d_0}(e_1)e_2 = \begin{cases} \chi_{a_1d_0}(p^k)p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1d_0}(p^k)p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

where we have used the identity $\chi_{a_1d_0}(e_1) = \chi_{a_1d_0}(p^k)$ which holds because this quadratic Dirichlet character only depends upon the parity of k. On the other hand, as in the proof of Lemma 4.1

$$H(p^k, p^{2\beta}) = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

But then

$$\chi_{a_1d_0}(p^k)\left(H(p^k, p^{2\beta}) - H(p^{k-1}, p^{2\beta})\right) = \begin{cases} \chi_{a_1d_0}(p^k)p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1d_0}(p^k)p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

which completes the proof. Clearly the same holds for $Q_{m_0p^{2\beta}}(w, \widetilde{\chi}_{a_2})$. □

Lemmas 4.1 and 4.2 together show that $Q_{d_0d_1^2}(s, \chi_{a_1})$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients $H(m, d)$ when d is an prime power. The proof of these lemmas also give the value of $H(p^k, p^l)$ and we collect this as a corollary:

Corollary 4.1. *For any prime p,*

$$H(p^k, p^l) = \begin{cases} \min\left(p^{\frac{k}{2}}, p^{\frac{l}{2}}\right) & \textit{if } \min(k, l) \textit{ is even,} \\ 0 & \textit{otherwise.} \end{cases}$$

If we combine Proposition 4.1 and Corollary 4.1 we can compute $H(m, d)$ in general:

Corollary 4.2. *For any integers $d, m \geq 1$,*

$$H(m,d) = \begin{cases} (m,d)^{\frac{1}{2}} & \text{if } (m,d) \text{ is a perfect square,} \\ 0 & \text{otherwise.} \end{cases}$$

As an immediate consequence of Corollary 4.2, $H(m,d)$ is symmetric in m and d. As the weighting coefficients are multiplicative, $Q_{d_0 d_1^2}(s, \chi_{a_1})$ will possess an Euler product. To state the Euler product explicitly, we write $d = d_0 d_1^2 d_2^2$ with d_0 square-free and, d_2 relatively prime to $d_0 d_1$, and such that every prime divisor of d_1 divides d_0. In other words, d_0 is the square-free part of d, d_1 is the square part of d whose prime factors divide d to odd power, and d_2 is the square part of d whose prime factors divide d to even power. We have the following Euler product:

Theorem 4.1. *Let $d = d_0 d_1^2 d_2^2$ be the square decomposition of d stratified by even and odd powers. Then for any $a_1 \in \{\pm 1, \pm 2\}$,*

$$Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = \prod_{p^\alpha || d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta || d_2} Q_{d_0 p^{2\beta}}(s, \chi_{a_1}).$$

Moreover, the same holds for $Q_{m_0 m_1^2 m_2^2}(w, \widetilde{\chi}_{a_2})$.

Proof. Recall that

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = \sum_{m \geq 1} \frac{\chi_{a_1 d_0}(\widehat{m}) H(m,d)}{m^s}.$$

We will now derive an alternate expression for $L(s, \chi_{a_1 d})$. By Proposition 4.1, the coefficients of $L(s, \chi_{a_1 d})$ are multiplicative. Therefore $L(s, \chi_{a_1 d})$ admits the Euler product

$$L(s, \chi_{a_1 d}) = \prod_{p \text{ prime}} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) H(p^k, d)}{p^{ks}} \right).$$

Decomposing the product according to primes dividing $d = d_0 d_1^2 d_2^2$, we get

$$\begin{aligned}
&L(s, \chi_{a_1 d}) \\
&= \prod_{p \text{ prime}} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) H(p^k, d)}{p^{ks}} \right) \\
&= \prod_{p \nmid d} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) H(p^k, 1)}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) H(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) H(p^k, p^\beta)}{p^{ks}} \right) \\
&= \prod_{p \nmid d} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) H(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) H(p^k, p^\beta)}{p^{ks}} \right) \\
&= \prod_{p \nmid d} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left(\sum_{k \geq 0} \frac{H(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(p^k) H(p^k, p^\beta)}{p^{ks}} \right).
\end{aligned}$$

Including the factors corresponding to primes $p \mid d_2$ into the first product, we must multiply the last factor by the inverse of $\sum_{k \geq 0} \chi_{a_1 d_0}(p) p^{-ks} = (1 - \chi_{a_1 d_0}(p) p^{-s})^{-1}$ obtaining

$$\prod_{p \nmid d_0} \left(\sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left(\sum_{k \geq 0} \frac{H(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left((1 - \chi_{a_1 d_0}(p) p^{-s}) \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(p^k) H(p^k, p^\beta)}{p^{ks}} \right),$$

as every prime divisor of d_1 divides d_0. The first product is $L(s,\chi_{a_1 d_0})$. For the second and third products, Remark 4.1 implies that the sums run up to $k \le 2\alpha$ and $k \le 2\beta$ respectively. Therefore they are $Q_{p^{2\alpha+1}}(s)$ and $Q_{d_0 p^{2\beta}}(s,\chi_{a_1})$ respectively. It follows that

$$L(s,\chi_{a_1 d}) = L(s,\chi_{a_1 d_0}) \cdot \prod_{p^\alpha || d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta || d_2} Q_{d_0 p^{2\beta}}(s,\chi_{a_1}).$$

This is our alternate expression for $L(s,\chi_{a_1 d})$ and equating the two results in

$$L(s,\chi_{a_1 d_0}) Q_{d_0 d_1^2 d_2^2}(s,\chi_{a_1}) = L(s,\chi_{a_1 d_0}) \cdot \prod_{p^\alpha || d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta || d_2} Q_{d_0 p^{2\beta}}(s,\chi_{a_1}),$$

from which the proof is complete since $L(s,\chi_{a_1 d_0}) \neq 0$ for $\mathrm{Re}(s) > 1$ (so that we may divide by $L(s,\chi_{a_1 d_0})$). Clearly the same holds for $Q_{m_0 m_1^2 m_2^2}(w,\widetilde{\chi}_{a_2})$. $\square$

Observe that for $d = d_0 d_1^2 d_2^2$, the prime factors that divide $d_1 d_2$ are exactly those factors that divide d to power larger than 1. Thus, from Theorem 4.1 the Euler product for $Q_{d_0 d_1^2 d_2^2}(s,\chi_{a_1})$ is supported on exactly the primes dividing d to order larger than 1 and also depends upon the character $\chi_{a_1 d_0}$.

5. Functional Equations

We can now derive functional equations for $Z_{a_1,a_2}(s,w)$. These functional equations will be induced from the functional equations for $L(s,\chi_{a_1 d})$ and $L(s,\widetilde{\chi}_{a_2 m})$. To prove these latter functional equations, we require a functional equation for the correction polynomials:

Theorem 5.1. *$Q_{d_0 d_1^2}(s,\chi_{a_1})$ admits the functional equation.*

$$Q_{d_0 d_1^2}(s,\chi_{a_1}) = d_1^{1-2s} Q_{d_0 d_1^2}(1-s,\chi_{a_1}).$$

Moreover, the same holds for $Q_{m_0 m_1^2}(w,\widetilde{\chi}_{a_2})$.

Proof. The strategy is to interchange e_2 and e_3 in the sum defining $Q_{d_0 d_1^2}(s,\chi_{a_1})$:

$$\begin{aligned} d_1^{1-2s} Q_{d_0 d_1^2}(1-s) &= d_1^{1-2s} \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{a_1 d_0}(e_1) e_1^{s-1} e_2^{2s-1} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{a_1 d_0}(e_1) e_1^{s-1} \left(\frac{d_1}{e_2}\right)^{1-2s} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{a_1 d_0}(e_1) e_1^{s-1} (e_1 e_3)^{1-2s} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{a_1 d_0}(e_1) e_1^{-s} e_3^{1-2s} \\ &= Q_{d_0 d_1^2}(s,\chi_{a_1}). \end{aligned}$$

Clearly the same holds for $Q_{m_0 m_1^2}(w,\widetilde{\chi}_{a_2})$. $\square$

We will define the completed L-function $L^*(s,\chi_{a_1 d})$ by

$$L^*(s,\chi_{a_1 d}) = L^*(s,\chi_{a_1 d_0}) Q_{d_0 d_1^2}(s,\chi_{a_1}).$$

In particular, $L^*(s,\chi_d)$ makes sense even when d is not square-free and agrees with the previous definition when d is square-free. Combining Theorem 5.1, the functional equation for $L^*(s,\chi_{a_1 d_0})$, and that $d \equiv d_0 \pmod 4$, we obtain a functional equation for $L^*(s,\chi_{a_1 d})$:

$$L^*(s,\chi_{a_1 d}) = \begin{cases} |d|^{\frac{1}{2}-s} L^*(1-s,\chi_{a_1 d}) & \text{if } a_1 d \equiv 1,5 \pmod 8, \\ (1+i)|8d|^{\frac{1}{2}-s} L^*(1-s,\chi_{a_1 d}) & \text{if } a_1 d \equiv 2,3,6,7 \pmod 8. \end{cases}$$

Analogously, define the completed L-function $L^*(w, \widetilde{\chi}_{a_2m})$ by

$$L^*(w, \widetilde{\chi}_{a_2m}) = L^*(w, \widetilde{\chi}_{a_2m_0})Q_{m_0m_1^2}(w, \widetilde{\chi}_{a_2}).$$

Then, as before, we have a functional equation for $L^*(w, \widetilde{\chi}_{a_2m})$:

$$L^*(w, \widetilde{\chi}_{a_2m}) = \begin{cases} |m|^{\frac{1}{2}-w}L^*(1-w, \widetilde{\chi}_{a_2m}) & \text{if } a_2m \equiv 1,5 \pmod 8, \\ (1+i)|8m|^{\frac{1}{2}-w}L^*(1-w, \widetilde{\chi}_{a_2m}) & \text{if } a_2m \equiv 2,3,6,7 \pmod 8). \end{cases}$$

The functional equations for $L^*(s, \chi_{a_1d})$ and $L^*(w, \widetilde{\chi}_{a_2m})$ will induce functional equations for $Z_{a_1,a_2}(s,w)$ and $\widetilde{Z}_{a_2,a_1}(w,s)$. However, there is an obstruction caused by the gamma factors. Indeed, the gamma factors for $L^*(s, \chi_{a_1d})$ and $L^*(w, \widetilde{\chi}_{a_2m})$ depend a_1d and a_2m modulo 8 respectively. To induce functional equations we need the gamma factors to be constant. Orthogonality of the Hilbert characters will allow us to get past this issue. For $b \in \{1,3,5,7\}$, define $Z^b_{a_1,a_2}(s,w)$ and $\widetilde{Z}^b_{a_2,a_1}(w,s)$ by

$$Z^b_{a_1,a_2}(s,w) = \frac{1}{4}\sum_{a\in\{\pm1,\pm2\}} \chi_a(b)Z_{a_1,aa_2}(s,w) \quad \text{and} \quad \widetilde{Z}^b_{a_2,a_1}(w,s) = \frac{1}{4}\sum_{a\in\{\pm1,\pm2\}} \widetilde{\chi}_a(b)\widetilde{Z}_{a_2,aa_1}(w,s).$$

In terms of the representations

$$Z_{a_1,a_2}(s,w) = \sum_{\substack{d\geq 1\\(d,2)=1}} \frac{\chi_{a_2}(d)L^{(2)}(s,\chi_{a_1d})}{d^w} \quad \text{and} \quad \widetilde{Z}_{a_2,a_1}(w,s) = \sum_{\substack{m\geq 1\\(m,2)=1}} \frac{\widetilde{\chi}_{a_1}(m)L^{(2)}(w,\widetilde{\chi}_{a_2m})}{m^s},$$

and orthogonality of the Hilbert characters, $Z^b_{a_1,a_2}(s,w)$ and $\widetilde{Z}^b_{a_2,a_1}(w,s)$ are the subseries containing only those d and m equivalent to b modulo 8 respectively. Then $Z^b_{a_1,a_2}(s,w)$ and $\widetilde{Z}^b_{a_2,a_1}(w,s)$ are sums of L-functions with a fixed gamma factor and so we can obtain functional equations. The fact that $Z_{a_1,a_2}(s,w)$ and $\widetilde{Z}_{a_2,a_1}(w,s)$ are linear combinations of these series will induce function equations. Precisely, we have the following statement:

Theorem 5.2. *$Z_{a_1,a_2}(s,w)$ admits the functional equations*

$$\begin{aligned}
Z_{a_1,a_2}(s,w) = {} & \frac{1}{4}\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \sum_{\substack{a_1b>0\\ a_1b\equiv 1,5 \ (mod\ 8)\\ a\in\{\pm1,\pm2\}}} \chi_a(b)\frac{(1-\chi_{a_1b}(2)2^{s-1})^{-1}}{(1-\chi_{a_1b}(2)2^{-s})^{-1}} Z_{a_1,aa_2}\left(1-s, s+w-\frac{1}{2}\right) \\
& + \frac{1}{4}\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \sum_{\substack{a_1b<0\\ a_1b\equiv 1,5 \ (mod\ 8)\\ a\in\{\pm1,\pm2\}}} \chi_a(b)\frac{(1-\chi_{a_1b}(2)2^{s-1})^{-1}}{(1-\chi_{a_1b}(2)2^{-s})^{-1}} Z_{a_1,aa_2}\left(1-s, s+w-\frac{1}{2}\right) \\
& + \frac{1+i}{8^s}\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \sum_{\substack{a_1b>0\\ a_1b\equiv 2,3,6,7 \ (mod\ 8)\\ a\in\{\pm1,\pm2\}}} \chi_a(b)\frac{(1-\chi_{a_1b}(2)2^{s-1})^{-1}}{(1-\chi_{a_1b}(2)2^{-s})^{-1}} Z_{a_1,aa_2}\left(1-s, s+w-\frac{1}{2}\right) \\
& + \frac{1+i}{8^s}\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \sum_{\substack{a_1b<0\\ a_1b\equiv 2,3,6,7 \ (mod\ 8)\\ a\in\{\pm1,\pm2\}}} \chi_a(b)\frac{(1-\chi_{a_1b}(2)2^{s-1})^{-1}}{(1-\chi_{a_1b}(2)2^{-s})^{-1}} Z_{a_1,aa_2}\left(1-s, s+w-\frac{1}{2}\right).
\end{aligned}$$

and

$$Z_{a_1,a_2}(s,w)=\frac{1}{4}\pi^{w-\frac{1}{2}}\frac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)}\sum_{\substack{a_1b\equiv 1,5 \ (mod\ 8)\\ a\in\{\pm1,\pm2\}}}\chi_a(b)\frac{(1-\chi_{a_2b}(2)2^{w-1})^{-1}}{(1-\chi_{a_2b}(2)2^{-w})^{-1}}Z_{aa_1,a_2}\left(s+w-\frac{1}{2},1-w\right)$$
$$+\frac{1+i}{8^w}\pi^{w-\frac{1}{2}}\frac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)}\sum_{\substack{a_1b\equiv 2 \ (mod\ 8)\\ a\in\{\pm1,\pm2\}}}\chi_a(b)\frac{(1-\chi_{a_2b}(2)2^{w-1})^{-1}}{(1-\chi_{a_2b}(2)2^{-w})^{-1}}Z_{aa_1,a_2}\left(s+w-\frac{1}{2},1-w\right)$$
$$+\frac{1+i}{8^w}\pi^{w-\frac{1}{2}}\frac{\Gamma\left(\frac{-w}{2}\right)}{\Gamma\left(\frac{w+1}{2}\right)}\sum_{\substack{a_1b\equiv 3,6,7 \ (mod\ 8)\\ a\in\{\pm1,\pm2\}}}\chi_a(b)\frac{(1-\chi_{a_2b}(2)2^{w-1})^{-1}}{(1-\chi_{a_2b}(2)2^{-w})^{-1}}Z_{aa_1,a_2}\left(1-w,s+w-\frac{1}{2}\right).$$

Proof. Set

$$L_{a_1b}(s)=\frac{(1-\chi_{a_1b}(2)2^{s-1})^{-1}}{(1-\chi_{a_1b}(2)2^{-s})^{-1}}.$$

For the fist functional equation, the functional equation for L-functions attached to quadratic Dirichlet characters implies the functional equation

$$Z^b_{a_1,a_2}(s,w)=\begin{cases}\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}L_{a_1b}(s)Z^b_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right) & \text{if } a_1b>0,\ a_1b\equiv 1,5 \pmod 8,\\ \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}L_{a_1b}(s)Z^b_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right) & \text{if } a_1b<0,\ a_1b\equiv 1,5 \pmod 8,\\ \frac{1+i}{8^{s-\frac{1}{2}}}\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}L_{a_1b}(s)Z^b_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right) & \text{if } a_1b>0,\ a_1b\equiv 2,3,6,7 \pmod 8,\\ \frac{1+i}{8^{s-\frac{1}{2}}}\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}L_{a_1b}(s)Z^b_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right) & \text{if } a_1b<0,\ a_1b\equiv 2,3,6,7 \pmod 8.\end{cases}$$

But as

$$Z_{a_1,a_2}(s,w)=\sum_{b\in\{1,3,5,7\}}Z^b_{a_1,a_2}(s,w),$$

the functional equation above gives

$$Z_{a_1,a_2}(s,w)=\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\sum_{\substack{a_1b>0\\ a_1b\equiv 1,5 \pmod 8}}L_{a_1b}(s)Z^b_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right)$$
$$+\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\sum_{\substack{a_1b<0\\ a_1b\equiv 1,5 \pmod 8}}L_{a_1b}(s)Z^b_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right)$$
$$+\frac{1+i}{8^{s-\frac{1}{2}}}\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\sum_{\substack{a_1b>0\\ a_1b\equiv 2,3,6,7 \pmod 8}}L_{a_1b}(s)Z^b_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right)$$
$$+\frac{1+i}{8^{s-\frac{1}{2}}}\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\sum_{\substack{a_1b<0\\ a_1b\equiv 2,3,6,7 \pmod 8}}L_{a_1b}(s)Z^b_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right).$$

The first functional equation for $Z_{a_1,a_2}(s,w)$ follows by writing $Z^b_{a_1,a_2}(s,w)$ in terms of $Z_{a_1,aa_2}(s,w)$ for $a \in \{\pm 1, \pm 2\}$. For the second functional equation, first set

$$L_{a_2b}(w) = \frac{(1-\widetilde{\chi}_{a_2b}(2)2^{w-1})^{-1}}{(1-\widetilde{\chi}_{a_2b}(2)2^{-w})^{-1}}.$$

Now argue as before but for $\widetilde{Z}_{a_2,a_1}(w,s)$ using the functional equation

$$\widetilde{Z}^b_{a_2,a_1}(w,s) = \begin{cases} \pi^{w-\frac{1}{2}} \dfrac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} L_{a_2b}(w) \widetilde{Z}^b_{a_2,a_1}\left(1-w, s+w-\dfrac{1}{2}\right) & \text{if } a_2b \equiv 1,5 \pmod 8, \\ \dfrac{1+i}{8^{w-\frac{1}{2}}} \pi^{w-\frac{1}{2}} \dfrac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} L_{a_2b}(w) L_{a_2b}(w) \widetilde{Z}^b_{a_2,a_1}\left(1-w, s+w-\dfrac{1}{2}\right) & \text{if } a_2b \equiv 2 \pmod 8, \\ \dfrac{1+i}{8^{w-\frac{1}{2}}} \pi^{w-\frac{1}{2}} \dfrac{\Gamma\left(\frac{-w}{2}\right)}{\Gamma\left(\frac{w+1}{2}\right)} L_{a_2b}(w) \widetilde{Z}^b_{a_2,a_1}\left(1-w, s+w-\dfrac{1}{2}\right) & \text{if } a_2b \equiv 3,6,7 \pmod 8, \end{cases}$$

induced from the corresponding L-function. We then apply the interchange in the form $\widetilde{Z}_{a_2,a_1}(w,s) = Z_{a_1,a_2}(s,w)$ and the fact that

$$L_{a_2b}(w) = \frac{(1-\chi_{a_2b}(2)2^{w-1})^{-1}}{(1-\chi_{a_2b}(2)2^{-w})^{-1}}.$$

to obtain the second functional equation for $Z_{a_1,a_2}(s,w)$. Clearly analogous functional equations hold for $\widetilde{Z}_{a_2,a_1}(w,s)$. □

These functional equations are quite unruly and it is often far more simple to compactify them in terms of vectors. For simplicity we do this only for $Z_{a_1,a_2}(s,w)$. Define $\mathbf{Z}(s,w)$ by

$$\mathbf{Z}(s,w) = (Z_{a_1,a_2}(s,w))_{a_1,a_2 \in \{\pm 1, \pm 2\}},$$

with the lexicographical ordering determined by $1 > -1 > 2 > -2$. Also, for $i \in \{1,2,3\}$, set

$$\gamma^{\pm,i}_{a_1,a,b}(s) = \begin{cases} \dfrac{1}{4} \pi^{s-\frac{1}{2}} \dfrac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \chi_a(b) \dfrac{(1-\chi_{a_1b}(2)2^{s-1})^{-1}}{(1-\chi_{a_1b}(2)2^{-s})^{-1}} & \text{if } + \text{ and } i=1, \\ \dfrac{1}{4} \pi^{s-\frac{1}{2}} \dfrac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \chi_a(b) \dfrac{(1-\chi_{a_1b}(2)2^{s-1})^{-1}}{(1-\chi_{a_1b}(2)2^{-s})^{-1}} & \text{if } - \text{ and } i=1, \\ \dfrac{1+i}{8^s} \pi^{s-\frac{1}{2}} \dfrac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \chi_a(b) \dfrac{(1-\chi_{a_1b}(2)2^{s-1})^{-1}}{(1-\chi_{a_1b}(2)2^{-s})^{-1}} & \text{if } + \text{ and } i=2,3, \\ \dfrac{1+i}{8^s} \pi^{s-\frac{1}{2}} \dfrac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \chi_a(b) \dfrac{(1-\chi_{a_1b}(2)2^{s-1})^{-1}}{(1-\chi_{a_1b}(2)2^{-s})^{-1}} & \text{if } - \text{ and } i=2,3, \end{cases}$$

and

$$\gamma^{i}_{a_2,a,b}(w) = \begin{cases} \dfrac{1}{4} \pi^{w-\frac{1}{2}} \dfrac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} \chi_a(b) \dfrac{(1-\chi_{a_2b}(2)2^{w-1})^{-1}}{(1-\chi_{a_2b}(2)2^{-w})^{-1}} & \text{if } i=1, \\ \dfrac{1+i}{8^w} \pi^{w-\frac{1}{2}} \dfrac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} \chi_a(b) \dfrac{(1-\chi_{a_2b}(2)2^{w-1})^{-1}}{(1-\chi_{a_2b}(2)2^{-w})^{-1}} & \text{if } i=2, \\ \dfrac{1+i}{8^w} \pi^{w-\frac{1}{2}} \dfrac{\Gamma\left(\frac{-w}{2}\right)}{\Gamma\left(\frac{w+1}{2}\right)} \chi_a(b) \dfrac{(1-\chi_{a_2b}(2)2^{w-1})^{-1}}{(1-\chi_{a_2b}(2)2^{-w})^{-1}} & \text{if } i=3. \end{cases}$$

These are the gamma factors appearing in the functional equations. By Theorem 5.2, there exist 16×16 matrices

$$\Phi(s) \quad \text{and} \quad \Psi(w),$$

whose coefficients are functions of $\gamma^{\pm,i}_{a_1,a,b}(s)$ and $\gamma^{i}_{a_2,a,b}(w)$ respectively, and satisfy functional equations

$$\mathbf{Z}(s,w) = \Phi(s)\mathbf{Z}\left(1-s, s+w-\frac{1}{2}\right) \quad \text{and} \quad \mathbf{Z}(s,w) = \Psi(s)\mathbf{Z}\left(s+w-\frac{1}{2}, 1-w\right),$$

which are equivalent to those for $Z_{a_1,a_2}(s,w)$ given in Theorem 5.2. So we have two functional equations of shapes

$$\sigma_1 : (s,w) \to \left(1-s, s+w-\frac{1}{2}\right) \quad \text{and} \quad \sigma_2 : (s,w) \to \left(s+w-\frac{1}{2}, 1-w\right).$$

These transformations also act on the (s,w)-plane and satisfy the relations

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 : (s,w) \to (1-w, 1-s) \quad \text{or equivalently} \quad (\sigma_1\sigma_2)^3 = 1 : (s,w) \to (s,w).$$

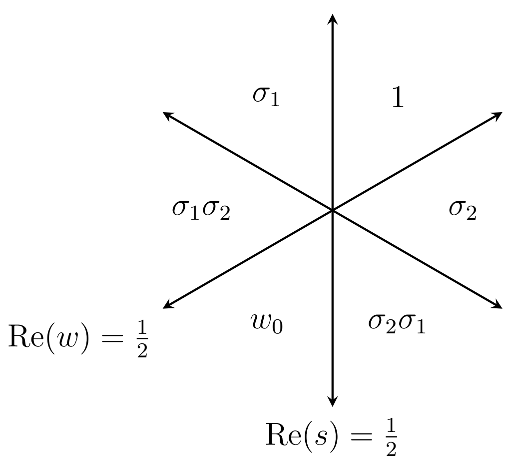
As $\sigma_1^2 = \sigma_2^2 = 1$, σ_1 and σ_2 generate the group

$$W = \langle \sigma_1, \sigma_2 : \sigma_1^2 = \sigma_2^2 = (\sigma_1\sigma_2)^3 = 1 \rangle \cong D_6 \cong S_3.$$

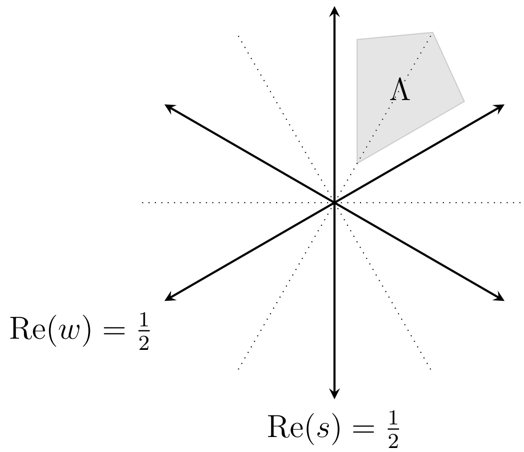
For convenience we set $w_0 = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. It follows that $Z_{a_1,a_2}(s,w)$ possess a group of 6 functional equations. These functional equations can be used to meromorphically continue $Z_{a_1,a_2}(s,w)$ to the entire (s,w)-plane. Of course, all the same can be achieved for $\widetilde{Z}_{a_2,a_1}(w,s)$ as well.

6. Meromorphic Continuation

We will now show how to meromorphically continue $Z(s,w)$ to the entire (s,w)-plane. We could do this for each twisted quadratic double Dirichlet series $Z_{a_1,a_2}(s,w)$ and $\widetilde{Z}_{a_2,a_1}(w,s)$, but we will not be concerned with this level of generality here or further on. We will start by describing the action of W on the (s,w)-plane. It is clear from the definition of the actions σ_1 and σ_2 that there is a unique W-invariant point $\left(\frac{1}{2},\frac{1}{2}\right)$. Representing the point (s,w) by $(\mathrm{Re}(s), \mathrm{Re}(w))$ we represent the action of W on the (s,w)-plane as follows:



In this diagram we have transformed the (s,w)-plane so that the origin lies at $\left(\frac{1}{2},\frac{1}{2}\right)$ and the (s,w)-axes intersect at $\frac{\pi}{3}$ angles. We have done this so that σ_1 and σ_2 act by rigid motions sending the region enclosing 1 (corresponding to the identity) to either of the adjacent triangles. The other regions are obtained by acting by the corresponding element of W. The initial region Λ that $Z(s,w)$ is defined on is displayed in the figure below:



To meromorphically continue $Z(s,w)$ to all of the (s,w)-plane, we first need to show that the quadratic double Dirichlet series $Z_{a_1,a_2}(s,w)$ are locally absolutely uniformly convergent on a slightly larger region than Λ. This will be achieved by the Phragmén-Lindelöf convexity principal. Fix some small $\varepsilon > 0$. The functional equations for $L^*(s,\chi_{a_1d})$ and $L^*(w,\widetilde{\chi}_{a_2m})$ and Stirling's formula together imply the estimates

$$L(-\varepsilon,\chi_{a_1d}) \ll (a_1d)^{\frac{1}{2}+\varepsilon} \quad \text{and} \quad L(-\varepsilon,\widetilde{\chi}_{a_2m}) \ll (a_2m)^{\frac{1}{2}+\varepsilon},$$

because $L(1+\varepsilon,\chi_{a_1d}) \ll 1$ and $L(1+\varepsilon,\widetilde{\chi}_{a_2m}) \ll 1$. As both of these L-functions have at most a simple pole at $s=1$ and $w=1$ respectively, the Phragmén-Lindelöf convexity principal gives the weak estimates

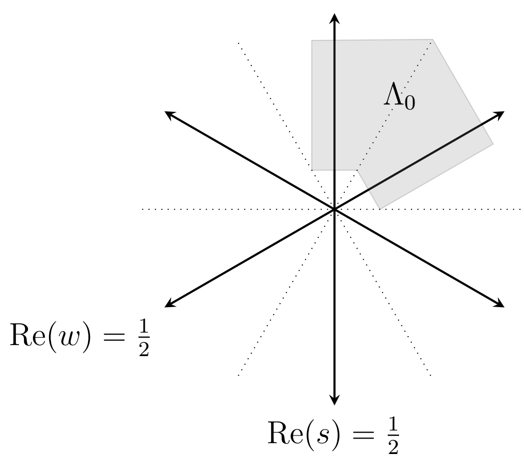
$$(s-1)L(s,\chi_{a_1d}) \ll (a_1d)^{\frac{1}{2}+\varepsilon} \quad \text{and} \quad (w-1)L(w,\widetilde{\chi}_{a_2m}) \ll (a_2m)^{\frac{1}{2}+\varepsilon},$$

and hence

$$(s-1)L^{(2)}(s,\chi_{a_1d}) \ll (a_1d)^{\frac{1}{2}+\varepsilon} \quad \text{and} \quad (w-1)L^{(2)}(w,\widetilde{\chi}_{a_2m}) \ll (a_2m)^{\frac{1}{2}+\varepsilon},$$

for $\mathrm{Re}(s) > -\varepsilon$ and $\mathrm{Re}(w) > -\varepsilon$. It follows from the interchange that $(s-1)(w-1)Z_{a_1,a_2}(s,w)$ is locally absolutely uniformly convergent on the region

$$\Lambda_0 = \Lambda \cup \left\{(s,w) \in \mathbb{C}^2 : \mathrm{Re}(s) > 0, \mathrm{Re}(w) > \frac{3}{2}\right\} \cup \left\{(s,w) \in \mathbb{C}^2 : \mathrm{Re}(s) > \frac{3}{2}, \mathrm{Re}(w) > 0\right\}.$$



Therefore $Z_{a_1,a_2}(s,w)$ is meromorphic on this region with at most polar lines at $s=1$ and $w=1$. The key difference between Λ and Λ_0 is that Λ_0 intersects the hyperplanes $s=\frac{1}{2}$ and $w=\frac{1}{2}$ so that the union of the reflections $w\Lambda_0$ for $w \in W$ is connected. This guarantees that the functional equations imply meromorphic continuation since adjacent reflections of Λ_0 overlap on open sets. We now reflect Λ_0 via the functional equations and then apply a theorem of Bochner which we now state. First, we say that a domain $\Omega \subset \mathbb{C}^n$ is a **tube domain** if there is an open set $\omega \subset \mathbb{R}^n$ such that

$$\Omega = \{(s_1,\ldots,s_n) \in \mathbb{C}^n : \mathrm{Re}((s_1,\ldots,s_n)) \in \omega\}.$$

Tube domains are generalizations of vertical strips in the complex plane. Now we can state the theorem of Bochner (see [1] for a proof):

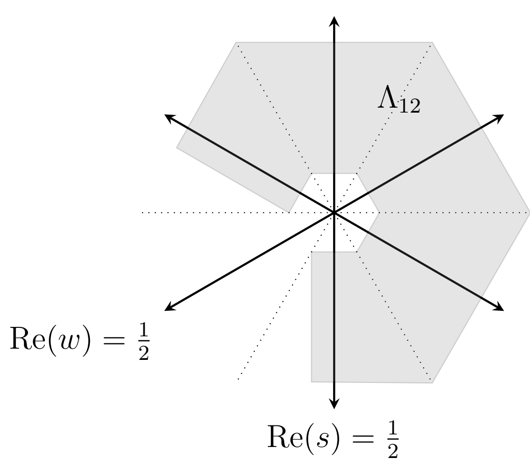
Theorem 6.1 (Bochner's continuation theorem)**.** *If Ω is a connected tube domain, then any holomorphic function on Ω can be extended to a holomorphic function on the convex hull $\widehat{\Omega}$.*

Clearing polar divisors if necessary, Bochner's continuation theorem implies that any meromorphic function on a connected tube domain possessing a finite amount of hyperplane polar divisors can be extended to a meromorphic function on the convex hull. This is exactly the situation for $Z(s,w)$. Clearly Λ_0 is a tube domain and on Λ_0 there are a most polar lines at $s=1$ and $w=1$. Reflecting these hyperplanes via W we obtain the finite set of possible polar divisors:

$$\left\{s=1, w=1, s=0, w=0, s+w=\frac{1}{2}, s+w=\frac{3}{2}\right\}.$$

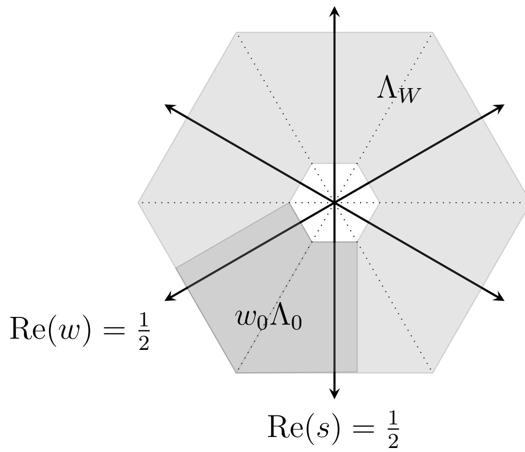
So by the previous argument, we are reduced to extending $Z(s,w)$ meromorphically. By applying the functional equations corresponding to σ_1, σ_2, and $\sigma_1\sigma_2$, $Z(s,w)$ admits meromorphic continuation to the region

$$\Lambda_{12} = \Lambda_0 \cup \sigma_1\Lambda_0 \cup \sigma_2\Lambda_0 \cup \sigma_1\sigma_2\Lambda_0.$$



Now Λ_{12} is a connected tube domain whose convex hull is $\mathbb{C}^2$. So by applying Bochner's continuation theorem (or rather our comment for meromorphic functions) we see that $Z(s,w)$ admits meromorphic continuation to the (s,w)-plane with at most a finite set of polar divisors. This argument is more elegant than repeatedly applying the functional equations corresponding to every $w \in W$. Indeed, if we did we would obtain meromorphic continuation to the region

$$\Lambda_W = \bigcup_{w \in W} w\Lambda_0.$$



There are two issues here. The first is that $Z(s,w)$ has two meromorphic continuations to the region $w_0\Lambda_0$ given by the functional equations corresponding to $w_0 = \sigma_1\sigma_2\sigma_1$ and $w_0 = \sigma_2\sigma_1\sigma_2$ and we would need to show that these agree. The second is that we have not obtained meromorphic continuation to $\mathbb{C}^2 - \Lambda_W$ which is a compact hexagon about the origin. By using Bochner's theorem after meromorphically continuing to Λ_{12}, we have avoided these issues and as a consequence shown that the meromorphic continuations given by $w_0 = \sigma_1\sigma_2\sigma_1$ and $w_0 = \sigma_2\sigma_1\sigma_2$ agree.

7. Poles & Residues

We can now determine the poles and residues of $Z(s,w)$. Recall that the set of possible polar divisors is

$$\left\{ s = 1, w = 1, s = 0, w = 0, s + w = \frac{1}{2}, s + w = \frac{3}{2} \right\}.$$

The poles of $Z(s,w)$ is actually smaller than this set as there are no poles on the hyperplanes $s = 0$, $w = 0$, and $s + w = \frac{3}{2}$. To see this, first observe that by our earlier application of the Phragmén-Lindelöf convexity principal we actually obtained continuation to an open set containing Λ_0 (because our estimates held for $\mathrm{Re}(s) > -\varepsilon$ and $\mathrm{Re}(w) > -\varepsilon$). We did not need this slightly larger region for the meromorphic continuation but we do require it to study the poles. Now consider the possible polar divisor $s = 0$. Therefore $(s-1)(w-1)Z_{a_1,a_2}(s,w)$ is holomorphic on an open set containing Λ_0 which contains half of the hyperplane defined by $s = 0$ outside of the hexagon $\mathbb{C}^2 - \Lambda_W$. Since $(s-1)(w-1)$ is holomorphic on this region, $Z_{a_1,a_2}(s,w)$ can not have a polar divisor at $s = 0$ on an open set containing Λ_0. Moreover, an open set containing $\sigma_1\sigma_2\Lambda_0$ contains the other half of the hyperplane defined by $s = 0$ outside of the hexagon $\mathbb{C}^2 - \Lambda_W$. Upon applying the functional equation corresponding to $\sigma_1\sigma_2$, Theorem 5.2 implies that the gamma factors in the corresponding functional equation have a simple pole when $s + w = \frac{3}{2}$ (the gamma factors in the functional equation for σ_1 have a simple pole at $s = 1$ and $s - 1 \to s + w - \frac{3}{2}$ under σ_2). Therefore $Z_{a_1,a_2}(s,w)$ does not have polar divisors at $s = 0$ on an open set containing $\sigma_1\sigma_2\Lambda_0$ away from $s + w = \frac{3}{2}$. In particular, $Z(s,w)$ does not have a polar divisor at $s = 0$ on Λ_W and away from the other polar divisors. By Bochner's continuation theorem (after clearing all of the other possible polar divisors), $Z(s,w)$ does not have a polar divisors at $s = 0$ on all of $\mathbb{C}^2$ and away from the other polar divisors. An identical argument holds for the case $w = 0$ using the regions Λ_0 and $\sigma_2\sigma_1\Lambda_0$. For the polar divisor $s + w = \frac{1}{2}$, we argue in the same way with the regions $\sigma_2\sigma_1\Lambda_0$, $\sigma_1\sigma_2\Lambda_0$, and $w_0\Lambda_0$, but there is one difference. For these regions, the gamma factors in the corresponding functional equations are different. For the first two regions $\sigma_2\sigma_1\Lambda_0$ and $\sigma_1\sigma_2\Lambda_0$, the gamma factors have a simple pole when $s + w = \frac{3}{2}$. For the third region $w_0\Lambda_0$, the gamma factors have simple poles at $s = 1$ and $w = 1$ which is seen by using both representations $w_0 = \sigma_1\sigma_2\sigma_1$ and $w_0 = \sigma_2\sigma_1\sigma_2$. Nevertheless, there are no poles on the hyperplanes $s = 0$, $w = 0$, and $s + w = \frac{1}{2}$ and away from the other polar divisors. As for the hyperplanes $s = 1$, $w = 1$, and $s + w = \frac{3}{2}$, there are clearly genuine poles for $s = 1$ and $w = 1$ coming from $L(s, \chi_{d_0})$ and $L(w, \chi_{m_0})$

when d and m are perfect squares (so that $d_0 = m_0 = 1$). For $s + w = \frac{3}{2}$, we have a pole coming from the gamma factors corresponding to the functional equations for $\sigma_2\sigma_1$ and $\sigma_1\sigma_2$. We collect all of our work as a theorem:

Theorem 7.1. *$Z(s,w)$ admits meromorphic continuation to $\mathbb{C}^2$ with polar divisors $s = 1$, $w = 1$, and $s + w = \frac{3}{2}$.*

We can also study the residues of $Z(s,w)$ at these poles. Since all of the poles are obtained from each other by applying the functional equations of $Z(s,w)$, we begin by looking at the pole when $w = 1$. To compute the residue we use the representation

$$Z(s,w) = \sum_{\substack{m \geq 1 \\ (m,2)=1}} \frac{L^{(2)}(w, \widetilde{\chi}_{m_0}) Q_{m_0 m_1^2}(w)}{m^s},$$

coming from the interchange. The numerator $L(w, \widetilde{\chi}_{m_0}) Q_{m_0 m_1^2}(w)$ in the summand corresponding to m has a pole at $w = 1$ if and only if m_0 is a perfect square, that is $m_0 = 1$, or equivalently $m = m_1^2$ itself is a perfect square. In this case, $L(w, \widetilde{\chi}_{m_0}) = \zeta(w)$ and

$$\operatorname*{Res}_{w=1} L^{(2)}(w, \chi_{m_0}) Q_{m_0 m_1^2}(w) = \frac{1}{2} Q_{m_1^2}(1).$$

But from Lemma 4.2 and Theorem 4.1 we see that $Q_{m_1^2}(1) = 1$, and hence

$$\operatorname*{Res}_{w=1} Z(s,w) = \frac{1}{2} \sum_{\substack{m \text{ perfect square} \\ (m,2)=1}} \frac{Q_{m_1^2}(1)}{m^s} = \frac{1}{2} \sum_{\substack{m \geq 1 \\ (m,2)=1}} \frac{1}{m^{2s}} = \frac{1}{2} \zeta^{(2)}(2s).$$

Notice that this expression has a simple pole at $s = \frac{1}{2}$ which is exactly when the polar lines $w = 1$ and $s + w = \frac{3}{2}$ intersect. The residue of $Z(s,w)$ at $s = 1$ is computed in an analogous way. Indeed, by applying the interchange, the exact same argument holds with the roles of s and w interchanged so that

$$\operatorname*{Res}_{s=1} Z(s,w) = \frac{1}{2} \zeta^{(2)}(2w).$$

Again, this expression has a simple pole at $w = \frac{1}{2}$ which is when the polar lines $s = 1$ and $s + w = \frac{3}{2}$ intersect. The other residues at the simple poles can be computed by applying the functional equations for $Z(s,w)$ and using the residues at $s = 1$ and $w = 1$. Now consider the point where the polar lines $w = 1$ and $s + w = \frac{3}{2}$ intersect. Setting $s = \frac{1}{2}$, we see that $Z\left(\frac{1}{2}, w\right)$ has a pole at $w = 1$ and we would like to study this pole more. To achieve this, the Mittag-Leffler theorem applied to $Z(s,w)$ (in w) implies

$$Z(s,w) = \frac{R_1(s)}{w - 1} + \frac{R_2(s)}{s + w - \frac{3}{2}} + V(s,w),$$

in some neighborhood of $\left(\frac{1}{2}, 1\right)$, where $V(s,w)$ is holomorphic, and we have set

$$R_1(s) = \operatorname*{Res}_{w=1} Z(s,w) \quad \text{and} \quad R_2(s) = \operatorname*{Res}_{w=\frac{3}{2}-s} Z(s,w).$$

From our residue computations above, $R_1(s) = \frac{1}{2}\zeta^{(2)}(2s)$ which implies that it has a simple pole at $s = \frac{1}{2}$. The residue is given by $A = \frac{1}{8}$. On the other hand, $Z\left(\frac{1}{2}, w\right)$ is holomorphic for $\mathrm{Re}(w) > 1$. These two facts together imply that $R_2(s)$ must have a simple pole at $s = \frac{1}{2}$ which cancels the simple pole coming from $R_1(s)$. So by Mittag-Leffler again, we may write

$$R_1(s) = \frac{A}{s - \frac{1}{2}} + R_3(s) \quad \text{and} \quad R_2(s) = -\frac{A}{s - \frac{1}{2}} + R_4(s),$$

in a neighborhood of $s = \frac{1}{2}$ and where $R_3(s)$ and $R_4(s)$ are holomorphic. Then

$$\begin{aligned} Z(s,w) &= \frac{R_1(s)}{w-1} + \frac{R_2(s)}{s+w-\frac{3}{2}} + V(s,w) \\ &= \frac{A}{(w-1)\left(s-\frac{1}{2}\right)} + \frac{R_3(s)}{w-1} - \frac{A}{\left(s+w-\frac{3}{2}\right)\left(s-\frac{1}{2}\right)} + \frac{R_4(s)}{s+w-\frac{3}{2}} + V(s,w) \\ &= \frac{A}{(w-1)\left(s+w-\frac{3}{2}\right)} + \frac{R_3(s)}{w-1} + \frac{R_4(s)}{s+w-\frac{3}{2}} + V(s,w). \end{aligned}$$

We can now set $s = \frac{1}{2}$ and let $B = R_3\left(\frac{1}{2}\right) + R_4\left(\frac{1}{2}\right)$ so that

$$Z\left(\frac{1}{2}, w\right) = \frac{A}{(w-1)^2} + \frac{B}{w-1} + O(1).$$

It follows that $Z\left(\frac{1}{2}, w\right)$ has a double pole at $w = 1$. This can be thought of as follows: the polar lines $w = 1$ and $s + w = \frac{3}{2}$ correspond to simple poles of $Z(s,w)$ except in the case when they intersect and the order of the poles combine to give $Z\left(\frac{1}{2}, w\right)$ a double pole at $w = 1$. Applying the interchange, the exact same argument holds to show that $Z\left(s, \frac{1}{2}\right)$ has a double pole at $s = 1$. We collect this work as a theorem:

Theorem 7.2. *$Z\left(\frac{1}{2}, w\right)$ and $Z\left(s, \frac{1}{2}\right)$ have double poles at $w = 1$ and $s = 1$ respectively. In particular, in neighborhoods of $w = 1$ and $s = 1$ respectively, we have*

$$Z\left(\frac{1}{2}, w\right) = \frac{A}{(w-1)^2} + \frac{B}{w-1} + O(1) \quad \textit{and} \quad Z\left(s, \frac{1}{2}\right) = \frac{A}{(s-1)^2} + \frac{B}{s-1} + O(1),$$

for some constants A and B with $A = \frac{1}{8}$.

References

[1] Lars Hörmander. *An introduction to complex analysis in several variables*. Number 7 in North-Holland mathematical library. North-Holland, 3., rev. ed., 3. impr edition, 2000.