0.1 Todo: [The Kuznetsov Trace Formula]

The Kuznetsov trace formula is an analog of the Petersson trace formula for weight zero Maass forms. From ??, $\mathcal{L}(N,\chi)$ admits an orthonormal basis of Maass forms for the point spectrum (these forms are generally not Hecke-Maass eigenforms because they need not be Hecke normalized or even cuspidal in the case of the discrete spectrum). However, by ?? and ?? we make take this orthonormal basis to consist of Hecke-Maass eigenforms and the constant function. Denote this basis by $\{u_j\}_{j\geq 0}$ with $u_0(z)=1$ and let u_j be of type ν_j for $j\geq 1$. In particular, $\{u_j\}_{j\geq 1}$ is an orthonormal basis of Hecke-Maass eigenforms and each such form admits a Fourier series at the $\mathfrak{a}$ cusp given by

$$(u_j|\sigma_{\mathfrak{a}})(z)=\sum_{n\neq 0}a_{j,\mathfrak{a}}(n)\sqrt{y}K_{\nu_j}(2\pi ny)e^{2\pi inx}.$$

The Kuznetsov trace formula is an equation relating the Fourier coefficients $a_{j,\mathfrak{a}}(n)$ and $a_{j,\mathfrak{b}}(n)$ of the basis $\{u_j\}_{j\geq 1}$ for two cusps $\mathfrak{a}$ and $\mathfrak{b}$ of $\Gamma_0(N)\backslash\mathbb{H}$ to a sum of integral transforms involving test functions and Salié sums. Similar to the Petersson trace formula, we will compute the inner product of two Poincaré series $P_{n,\chi,\mathfrak{a}}(z,\psi)(z)$ and $P_{m,\chi,\mathfrak{b}}(z,\varphi)(z)$ in two different ways. The first will be geometric in nature while the second will be spectral. We first need to compute the Fourier series of such a Poincaré series. Although we will not need it explicitly, we will work over any congruence subgroup:

Proposition 0.1.1. *Let $m\geq 1$, χ be Dirichlet character with conductor dividing the level, $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$, and $\psi(y)$ be a smooth function such that $\psi(y)\ll_\varepsilon y^{1+\varepsilon}$ as $y\to 0$. The Fourier series of $P_{m,\chi,\mathfrak{a}}(z,\psi)$ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{b}$ cusp is given by*

$$(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z,\psi)=\sum_{t\in\mathbb{Z}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\mathrm{Im}(z))+\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,m,t,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c)\right)e^{2\pi itz},$$

where $\psi(y,m,t,c)$ is the integral transform given by

$$\psi(y,m,t,c)=\int_{\mathrm{Im}(z)=y}\psi\left(\frac{y}{|cz|^2}\right)e^{-\frac{2\pi im}{c^2z}-2\pi itz}\,dz.$$

Proof. From the cocycle condition and ??, we have

$$(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z,\psi)=\delta_{\mathfrak{a},\mathfrak{b}}\psi(\mathrm{Im}(z))e^{2\pi imz}+\sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}},d\in\mathbb{Z}\\ d\pmod{c}\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\overline{\chi}(d)\psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)e^{2\pi im\left(\frac{a}{c}-\frac{1}{c^2z+cd}\right)},$$

where a and b are chosen such that $\det\left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\right)=1$ and we have used the fact that

$$\frac{a}{c}-\frac{1}{c^2z+cd}=\frac{az+b}{cz+d}.$$

Summing over all pairs (c,d) with $c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $d\in\mathbb{Z}$, and $d\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is the same as summing over all triples (c,ℓ,r) with $c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $\ell\in\mathbb{Z}$, and r taken modulo c with $r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$. Indeed, this is seen by writing $d=c\ell+r$. Moreover, since $ad-bc=1$ we have $a(c\ell+r)-bc=1$ which further implies that

$ar \equiv 1 \pmod{c}$. So we may take a to be the inverse for r modulo c. Then

$$\begin{aligned}
\sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}, d \in \mathbb{Z} \\ d \pmod{c} \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(d)\psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + cd}\right)} &= \sum_{(c,\ell,r)} \overline{\chi}(c\ell + r)\psi\left(\frac{\mathrm{Im}(z)}{|cz + c\ell + r|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + c^2\ell + cr}\right)} \\
&= \sum_{(c,\ell,r)} \overline{\chi}(r)\psi\left(\frac{\mathrm{Im}(z)}{|cz + c\ell + r|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + c^2\ell + cr}\right)} \\
&= \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \sum_{\ell \in \mathbb{Z}} \overline{\chi}(r)\psi\left(\frac{\mathrm{Im}(z)}{|cz + c\ell + r|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + c^2\ell + cr}\right)} \\
&= \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(r) \sum_{\ell \in \mathbb{Z}} \psi\left(\frac{\mathrm{Im}(z)}{|cz + c\ell + r|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + c^2\ell + cr}\right)},
\end{aligned}$$

where on the right-hand side it is understood that we are summing over all triples (c, ℓ, r) with the prescribed properties and the second line holds since χ has conductor diving the level and $d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is determined modulo c. Now let

$$I_{c,r}(z, \psi) = \sum_{\ell \in \mathbb{Z}} \psi\left(\frac{\mathrm{Im}(z)}{|cz + c\ell + r|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + c^2\ell + cr}\right)}.$$

We apply the Poisson summation formula to $I_{c,r}(z, \psi)$. This is allowed since the summands are absolutely integrable by **??**, as they exhibit polynomial decay of order $\sigma > 1$ because $\psi(y) \ll_{\varepsilon} y^{1+\varepsilon}$ as $y \to 0$, and $I_{c,r}(z, \psi)$ is holomorphic because $(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z, \psi)$ is. By the identity theorem it suffices to apply the Poisson summation formula for $z = iy$ with $y > 0$. So let $f(x)$ be given by

$$f(x) = \psi\left(\frac{y}{|cx + r + icy|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 x + cr + ic^2 y}\right)}.$$

As we have just noted, $f(x)$ is absolutely integrable on $\mathbb{R}$. We compute the Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i t x}\, dx = \int_{-\infty}^{\infty} \psi\left(\frac{y}{|cx + r + icy|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 x + cr + ic^2 y}\right)} e^{-2\pi i t x}\, dx.$$

Complexify the integral to get

$$\int_{\mathrm{Im}(z)=0} \psi\left(\frac{y}{|cz + r + icy|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + cr + ic^2 y}\right)} e^{-2\pi i t z}\, dz.$$

Now make the change of variables $z \to z - \frac{r}{c} - iy$ to obtain

$$e^{2\pi i m \frac{a}{c} + 2\pi i t \frac{r}{c} - 2\pi t y} \int_{\mathrm{Im}(z)=y} \psi\left(\frac{y}{|cz|^2}\right) e^{-\frac{2\pi i m}{c^2 z} - 2\pi i t z}\, dz.$$

As the remaining integral is $\psi(y, m, t, c)$, it follows that

$$\hat{f}(t) = \psi(y, m, t, c)e^{2\pi i m \frac{a}{c} + 2\pi i t \frac{r}{c} - 2\pi t y}.$$

By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z, \psi) = \sum_{t \in \mathbb{Z}} (\psi(y, m, t, c)e^{2\pi i m \frac{a}{c} + 2\pi i t \frac{r}{c}})e^{2\pi i t z},$$

for all $z \in \mathbb{H}$. Substituting this back into the Eisenstein series gives a form of the Fourier series:

$$
\begin{aligned}
(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z,\psi) &= \delta_{\mathfrak{a},\mathfrak{b}}\psi(\mathrm{Im}(z))e^{2\pi imz} + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r)\sum_{t\in\mathbb{Z}}\psi(y,m,t,c)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}e^{2\pi itz} \\
&= \sum_{t\in\mathbb{Z}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\mathrm{Im}(z)) + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}}\overline{\chi}(r)\psi(y,m,t,c)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right)e^{2\pi itz} \\
&= \sum_{t\in\mathbb{Z}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\mathrm{Im}(z)) + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,m,t,c)\sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\overline{\chi}(r)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right)e^{2\pi itz}.
\end{aligned}
$$

We will simplify the innermost sum. Since a is the inverse for r modulo c, the innermost sum above becomes

$$
\sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\overline{\chi}(r)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}} = \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\overline{\chi}(\overline{a})e^{2\pi im\frac{a}{c}+2\pi it\frac{\overline{a}}{c}} = \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\chi(a)e^{\frac{2\pi i(am+\overline{a}t)}{c}} = S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c).
$$

So at last, we obtain our desired Fourier series:

$$
(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t\in\mathbb{Z}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\mathrm{Im}(z)) + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,m,t,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c)\right)e^{2\pi itz}.
$$

$\square$

We can now derive the first half of the Kuznetsov trace formula by computing the inner product between $P_{n,\chi,\mathfrak{a}}(z,\psi)$ and $P_{m,\chi,\mathfrak{b}}(z,\varphi)$:

$$
\begin{aligned}
\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), P_{m,\chi,\mathfrak{b}}(\cdot,\varphi)\rangle &= \frac{1}{V_{\Gamma_0(N)}}\int_{\mathcal{F}_{\Gamma_0(N)}} P_{n,\chi,\mathfrak{a}}(z,\psi)\overline{P_{m,\chi,\mathfrak{b}}(z,\varphi)}\,d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}}\int_{\mathcal{F}_{\Gamma_0(N)}}\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma_0(N)}\chi(\gamma)P_{n,\chi,\mathfrak{a}}(z,\psi)\overline{\varphi(\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}\gamma z))}e^{-2\pi im\overline{\sigma_{\mathfrak{b}}^{-1}\gamma z}}\,d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}}\int_{\mathcal{F}_{\Gamma_0(N)}}\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma_0(N)}P_{n,\chi,\mathfrak{a}}(\gamma z,\psi)\overline{\varphi(\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}\gamma z))}e^{-2\pi im\overline{\sigma_{\mathfrak{b}}^{-1}\gamma z}}\,d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}}\int_{\mathcal{F}_{\sigma_{\mathfrak{b}}^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}}}\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma_0(N)}P_{n,\chi,\mathfrak{a}}(\gamma\sigma_{\mathfrak{b}}z,\psi)\overline{\varphi(\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}\gamma\sigma_{\mathfrak{b}}z))}e^{-2\pi im\overline{\sigma_{\mathfrak{b}}^{-1}\gamma\sigma_{\mathfrak{b}}z}}\,d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}}\int_{\mathcal{F}_{\sigma_{\mathfrak{b}}^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}}}\sum_{\gamma\in\Gamma_\infty\backslash\sigma_{\mathfrak{b}}^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}^{-1}}P_{n,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}}\gamma z,\psi)\overline{\varphi(\mathrm{Im}(\gamma z))}e^{-2\pi im\overline{\gamma z}}\,d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}}\int_{\Gamma_\infty\backslash\mathbb{H}}(P_{n,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z,\psi)\overline{\varphi(\mathrm{Im}(z))}e^{-2\pi im\overline{z}}\,d\mu,
\end{aligned}
$$

where in the third line we have used the automorphy of $P_{n,\chi,\mathfrak{a}}(z,\psi)$, in the forth and fifth lines we have made the change of variables $z \to \sigma_{\mathfrak{b}}z$ and $\gamma \to \sigma_{\mathfrak{b}}\gamma\sigma_{\mathfrak{b}}^{-1}$ respectively, and in the sixth line we have unfolded. Now substitute in the Fourier series of $P_{n,\chi,\mathfrak{a}}(z,\psi)$ at the $\mathfrak{b}$ cusp to obtain

$$
\frac{1}{V_{\Gamma_0(N)}}\int_{\Gamma_\infty\backslash\mathbb{H}}\sum_{t\in\mathbb{Z}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,t}\psi(\mathrm{Im}(z)) + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,n,t,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(n,t,c)\right)\overline{\varphi(\mathrm{Im}(z))}e^{2\pi itz-2\pi im\overline{z}}\,d\mu,
$$

which is equivalent to

$$\frac{1}{V_{\Gamma_0(N)}}\int_0^\infty\int_0^1\sum_{t\geq 1}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,t}\psi(y)+\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,n,t,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(n,t,c)\right)\overline{\varphi(y)}e^{2\pi i(t-m)x}e^{-2\pi(t+m)y}\,\frac{dx\,dy}{y^2}.$$

By the dominated convergence theorem, we can interchange the sum and the two integrals. Then **??** implies that the inner integral cuts off all of the terms except the diagonal $t=m$. This leaves

$$\frac{1}{V_{\Gamma_0(N)}}\int_0^\infty\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m}\psi(y)+\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,n,m,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c)\right)\overline{\varphi(y)}e^{-4\pi my}\,\frac{dy}{y^2}.$$

Interchanging the integral and the remaining sum by the dominated convergence theorem again, we arrive at

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),P_{m,\chi,\mathfrak{b}}(\cdot,\varphi)\rangle=\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m}(\psi,\varphi)_{n,m}+\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c)V(n,m,c,\psi,\varphi),$$

where we have set

$$(\psi,\varphi)_{n,m}=\frac{1}{V_{\Gamma_0(N)}}\int_0^\infty\psi(y)\overline{\varphi(y)}e^{-2\pi(n+m)y}\,\frac{dy}{y^2},$$

and

$$V(n,m,c;\psi,\varphi)=\frac{1}{V_{\Gamma_0(N)}}\int_0^\infty\int_{\mathrm{Im}(z)=y}\psi\left(\frac{y}{|cz|^2}\right)\overline{\varphi(y)}e^{-\frac{2\pi im}{c^2z}-2\pi inz-4\pi my}\,\frac{dz\,dy}{y^2}.$$

This is the first half of the Kuznetsov trace formula. For the second half, **??** gives

$$P_{n,\chi,\mathfrak{a}}(\cdot,\psi)=\sum_{j\geq 0}\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),u_j\rangle u_j(z)+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^\infty\left\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right)\right\rangle E_{\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)\,dr,$$

and

$$P_{m,\chi,\mathfrak{a}}(\cdot,\varphi)=\sum_{j\geq 0}\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi),u_j\rangle u_j(z)+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^\infty\left\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi),E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right)\right\rangle E_{\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)\,dr.$$

By orthonormality, it follows that

$$\begin{aligned}\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),P_{m,\chi,\mathfrak{a}}(\cdot,\varphi)\rangle&=\sum_{j}\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),u_j\rangle\overline{\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi),u_j\rangle}\\&\quad+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^\infty\left\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right)\right\rangle\overline{\left\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi),E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right)\right\rangle}\,dr.\end{aligned}$$

Now we must simplify the remaining inner products. Let $f\in\mathcal{L}(N,\chi)$ with Fourier series

$$f(z)=a^+(0)y^{\frac{1}{2}+\nu}+a^-(0)y^{\frac{1}{2}-\nu}+\sum_{n\neq 0}a(n)\sqrt{y}K_\nu(2\pi|n|y)e^{2\pi inx}.$$

By unfolding the integral in the Petersson inner product and cutting off everything except the diagonal using **??** exactly as in the case for $\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),P_{m,\chi,\mathfrak{a}}(\cdot,\varphi)\rangle$, we see that

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),f\rangle=\frac{1}{V_\Gamma}\int_0^\infty\overline{a(n)\sqrt{y}K_\nu(2\pi ny)}\psi(y)e^{-4\pi my}\frac{dy}{y^2}.$$

Now set

$$\omega_\nu(n,\psi) = \frac{1}{V_\Gamma}\int_0^\infty \sqrt{y}K_\nu(2\pi|n|y)\overline{\psi(y)}e^{-4\pi m y}\frac{dy}{y^2}.$$

Then it follows from the Fourier series of cusp forms and Eisenstein series that

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), u_j\rangle = \overline{a_j(n)\omega_{\nu_j}(n,\psi)},$$

for $j \geq 1$ and

$$\left\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right)\right\rangle = \overline{\tau_{\mathfrak{a}}\left(n,\frac{1}{2}+ir\right)\omega_{ir}(n,\psi)}.$$

In particular, $\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), u_0\rangle = 0$. So we obtain

$$\begin{aligned}\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), P_{m,\chi,\mathfrak{a}}(\cdot,\varphi)\rangle &= \sum_{j\geq 1}\overline{a_j(n)}a_j(m)\overline{\omega(n,\psi)}\omega(m,\varphi)\\ &\quad+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\overline{\tau_{\mathfrak{a}}\left(n,\frac{1}{2}+ir\right)}\tau_{\mathfrak{a}}\left(m,\frac{1}{2}+ir\right)\overline{\omega(n,\psi)}\omega(m,\varphi)\,dr.\end{aligned}$$

This is the second half of the Kuznetsov trace formula. Equating the first and second halves we get the **Kuznetsov trace formula**:

$$\begin{aligned}\delta_{n,m}(\psi,\varphi)+\sum_{\substack{c\geq 1\\ c\equiv 0 \pmod N}}\frac{1}{c}S_\chi(n,m,c)V(n,m,c,\psi,\varphi) &= \sum_{j\geq 1}\overline{a_j(n)}a_j(m)\overline{\omega(n,\psi)}\omega(m,\varphi)\\ &\quad+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\overline{\tau_{\mathfrak{a}}\left(n,\frac{1}{2}+ir\right)}\tau_{\mathfrak{a}}\left(m,\frac{1}{2}+ir\right)\overline{\omega(n,\psi)}\omega(m,\varphi)\,dr.\end{aligned}$$

The left-hand side is called the **geometric side** and the right-hand side is called the **spectral side**. We collect our work as a theorem:

Theorem 0.1.1 (Kuznetsov trace formula). *Let $\{u_j\}_{j\geq 1}$ be an orthonormal basis of Hecke-Maass eigenforms for $\mathcal{L}(N,\chi)$ of types ν_j with Fourier coefficients $a_j(n)$. Then for any positive integers $n, m \geq 1$, we have*

$$\begin{aligned}\delta_{n,m}(\psi,\varphi)+\sum_{\substack{c\geq 1\\ c\equiv 0 \pmod N}}\frac{1}{c}S_\chi(n,m,c)V(n,m,c,\psi,\varphi) &= \sum_{j\geq 1}\overline{a_j(n)}a_j(m)\overline{\omega(n,\psi)}\omega(m,\varphi)\\ &\quad+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\overline{\tau_{\mathfrak{a}}\left(n,\frac{1}{2}+ir\right)}\tau_{\mathfrak{a}}\left(m,\frac{1}{2}+ir\right)\overline{\omega(n,\psi)}\omega(m,\varphi)\,dr.\end{aligned}$$

0.2 Absolute, Relative & Galois Extensions

The concept of ramification can be extended to more general extensions of number fields. Let K and L be number fields with $K \subseteq L$. We say that $K/\mathbb{Q}$ is an **absolute extension** and L/K is a **relative extension**. In other words, an absolute extension is simply when the base field is the number field $\mathbb{Q}$. For a relative extension L/K, we have rings of integers $\mathcal{O}_K \subseteq \mathcal{O}_L$. Let $\mathfrak{P}$ be a prime integral ideal of $\mathcal{O}_L$. Then $\mathfrak{P}\cap\mathcal{O}_K$ is a prime integral ideal of K. It is clear that $\mathfrak{P}\cap\mathcal{O}_K$ is an integral ideal of $\mathcal{O}_K$ and it is proper because $1 \notin \mathfrak{P}\cap\mathbb{Z} \subseteq \mathfrak{P}\cap\mathcal{O}_K$ as $\mathfrak{P}$ does not contain units. Moreover, it is nonzero since any integral ideal contains its norm (as we have noted) and so $\mathrm{N}(\mathfrak{P}) \in \mathfrak{P} \cap \mathbb{Z} \subseteq \mathfrak{P} \cap \mathcal{O}_K$. To see that it is prime, suppose $\alpha, \beta \in \mathcal{O}_K$ are such that $\alpha\beta \in \mathfrak{P} \cap \mathcal{O}_K$. Then $\alpha\beta \in \mathfrak{P}$ and since $\mathfrak{P}$ is prime either $\alpha \in \mathfrak{P}$ or $\beta \in \mathfrak{P}$. But then $\alpha \in \mathfrak{P} \cap \mathcal{O}_K$ or $\beta \in \mathfrak{P} \cap \mathcal{O}_K$ as desired. We have shown that $\mathfrak{P} \cap \mathcal{O}_K$ is a prime integral ideal of $\mathcal{O}_K$ and so

$$\mathfrak{P} \cap \mathcal{O}_K = \mathfrak{p},$$

for some prime integral ideal of $\mathcal{O}_K$. We say that $\mathfrak{P}$ is **above** $\mathfrak{p}$, or equivalently, $\mathfrak{p}$ is **below** $\mathfrak{P}$. If $\mathfrak{P}$ is above $\mathfrak{p}$, then $\mathfrak{P}$ is a prime factor of $\mathfrak{p}\mathcal{O}_L$. Indeed, since $\mathfrak{p} \subseteq \mathfrak{P}$ we have $\mathfrak{p}\mathcal{O}_L \subseteq \mathfrak{P}$ and hence some prime factor of $\mathfrak{p}\mathcal{O}_L$ is contained in $\mathfrak{P}$ since $\mathfrak{P}$ is prime. Since prime integral ideals are maximal by **??**, this prime factor must be $\mathfrak{P}$ itself. This situation directly generalizes a prime integral ideal $\mathfrak{P}$ lying over the prime p for the absolute extension $L/\mathbb{Q}$ to the relative extension L/K by replacing p with a prime integral ideal $\mathfrak{p}$. When $K = \mathbb{Q}$, the situations are equivalent since $\mathbb{Z}$ is a principal ideal domain. The situation can be illustrated via the extension

$$\begin{array}{c} \mathfrak{P} \subset \mathcal{O}_L \subset L \\ | \\ \mathfrak{p} \subset \mathcal{O}_K \subset K \\ | \\ p\mathbb{Z} \subset \mathbb{Z} \subset \mathbb{Q}. \end{array}$$

As $\mathfrak{P}$ and $\mathfrak{p}$ are maximal in $\mathcal{O}_L$ and $\mathcal{O}_K$ respectively by **??**, we have the residue fields $\mathcal{O}_L/\mathfrak{P}$ and $\mathcal{O}_K/\mathfrak{p}$. As we might expect, $\mathcal{O}_L/\mathfrak{P}$ is a finite dimensional vector space over $\mathcal{O}_K/\mathfrak{p}$ which is itself a finite dimensional vector space over $\mathbb{F}_p$. This latter fact we have already established since $\mathfrak{p}$ lies over a prime p. As for the former, consider the homomorphism

$$\phi : \mathcal{O}_K \to \mathcal{O}_L/\mathfrak{P} \qquad \alpha \mapsto \alpha \pmod{\mathfrak{P}}.$$

We have $\ker \phi = \mathfrak{P} \cap \mathcal{O}_K$ and so $\ker \phi = \mathfrak{p}$ since $\mathfrak{P}$ is above $\mathfrak{p}$. By the first isomorphism theorem, ϕ induces an injection $\phi : \mathcal{O}_K/\mathfrak{p} \to \mathcal{O}_L/\mathfrak{P}$. By **??**, $\mathcal{O}_L/\mathfrak{P}$ is finite and thus a finite field containing $\mathcal{O}_K/\mathfrak{p}$. Therefore $\mathcal{O}_L/\mathfrak{P}$ is a finite dimensional vector space over $\mathcal{O}_K/\mathfrak{p}$ and hence $\mathbb{F}_p$ as well. Accordingly, we define the **inertia degree** $f_{\mathfrak{p}}(\mathfrak{P})$ of $\mathfrak{P}$ by

$$f_{\mathfrak{p}}(\mathfrak{P}) = [\mathcal{O}_L/\mathfrak{P} : \mathcal{O}_K/\mathfrak{p}].$$

That is, $f_{\mathfrak{p}}(\mathfrak{P})$ is the dimension of the residue field $\mathcal{O}_L/\mathfrak{P}$ as a vector space over $\mathcal{O}_K/\mathfrak{p}$. It follows that

$$f_p(\mathfrak{P}) = f_{\mathfrak{p}}(\mathfrak{P}) f_p(\mathfrak{p}).$$