A MULTIPLE DIRICHLET SERIES APPROACH TO SHIFTED CONVOLUTION SUMS

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Abstract. A generalization of the classical additive divisor problem is that of shifted convolution sums for holomorphic cusp forms: find asymptotics for sums of the form

$$S(X,h)=\sum_{1<m\le X}A(m)\overline{B(m+h)},$$

where $A(m)$ and $B(m)$ are the Hecke normalized Fourier coefficients of weight k holomorphic cusp forms f and g on say $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. One method of obtaining good estimates for $S(X,h)$ when $h\ge 1$ involves a detailed study of the spectral expansion of the shifted Dirichlet series

$$D(s,h)=\sum_{m\ge 1}\frac{a(m)b(m+h)}{m^{s+k-1}},$$

where $a(m)$ and $b(m)$ are the unnormalized Fourier coefficients. In the following, we discuss the analytic properties of the multiple Dirichlet series

$$Z(s,w)=\sum_{m,h\ge 1}\frac{a(m)b(m+h)c(h)}{m^{s+k-1}h^{w+k-1}},$$

where $c(h)$ are the Fourier coefficients of another weight k holomorphic cusp form l. We exploit the analytic properties of this multiple Dirichlet series to sketch the proof of estimates for the following triple shifted convolution sums:

$$T(X,Y)=\sum_{\substack{\frac{X}{2}<m\le X\\ \frac{Y}{2}<h\le Y}}A(m)B(m+h)C(h),$$

where $C(h)$ are the Hecke normalized Fourier coefficients of l.

1. Motivating Shifted Convolution Sums

The prototypical example of shifted convolution sums is when $a(m)=b(m)=\tau_2(n)$ is the usual divisor function. Obtaining estimates for the sum

$$D_2(X,h)=\sum_{m\le X}\tau_2(m)\tau_2(m+h),$$

is the well-known **binary additive divisor problem**. More generally, sums of the form

$$D_{k,\ell}(X,h)=\sum_{m\le X}\tau_k(m)\tau_\ell(m+h),$$

where τ_k is the k-th divisor function, are called **additive divisor sums**. These objects are of interest because $D_k(X,h)=D_{k,k}(X,h)$ is attached to the $2k$-th moment of the Riemann zeta function, defined by

$$I_k(T)=\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt.$$

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The first appearance of this phenomena occured in 1926 when Ingham (see [3]) showed that $D_2(X,h)$ was related to the 4-th moment by using estimate for $D_2(X,h)$ to establish the asymptotitc

$$I_2(T) \sim \frac{T}{2\pi^2}(\log T)^4.$$

The essential ingredient in Ingham's proof was an approximation function equation for $|\zeta\left(\frac{1}{2}+it\right)|^4$ involving the sums $D_2(X,h)$. Since Ingham's proof, many others have used results about additive divisor sums to establish asymptotics for moments (for example, see [5] and [6]). Unfortunately, not much is known about $D_k(X,h)$ when $k>2$.

On the other hand, when $k=2$, additive divisor sums are also attached to the spectral theory of automorphic forms. This is because $\tau_2(m)$ appears as the m-th Fourier coefficeint of the Eisenstein series $E(z,s)$ on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ when $s=\frac{1}{2}$. When the $a(m)$ and $b(m)$ are Fourier coefficients of automorphic forms, the associated sum

$$S(X,h)=\sum_{m\leq X}A(m)B(m+h),$$

with Hecke normalized Fourier coefficients, is called a **shifted convolution sum**. If the automorphic forms are non-cuspdial and the Fourier coefficients are Hecke normalized, we have

$$S(X,h)=XP(\log(X))+O_{h,\varepsilon}(X^{\frac{2}{3}+\varepsilon}),$$

where P is some polynomial. Note that the error term here has cube root cancellation. Conjecturally, we should have square root cancellation in the error term (with an additional h^{ε} factor). When the automorphic forms are cuspidal, there is no main term, but the error term is of the same size as in the non-cuspdial case. Again, the correct order of magnitude should have square root cancellation. The specific triple shifted convolution sums we will discuss below are when $a(m)$, $b(m)$, and $c(m)$, are Fourier coefficients of weight k holomorphic cuspforms

$$f(z)=\sum_{m\geq 1}a(m)e^{2\pi imz},\quad g(z)=\sum_{m\geq 1}b(m)e^{2\pi imz},\quad \text{and}\quad l(z)=\sum_{m\geq 1}c(m)e^{2\pi imz},$$

on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. We will denote the Hecke normalized coefficients by $A(m)$, $B(m)$, and $C(m)$, respectively.

2. Spectral Exapansions for Shifted Dirichlet Series

Let $h\geq 1$. We will estimate our shifted convolution sums using the shifted Dirichlet series

$$D(s,h)=\sum_{m\geq 1}\frac{a(m)b(m+h)}{m^{s+k-1}}.$$

The Ramanujan conjecture implies $a(m)\ll m^{\frac{k-1}{2}}$ and $b(m+h)\ll (m+h)^{\frac{k-1}{2}}\ll_h m^{\frac{k-1}{2}}$ and so $D(s,h)$ converges locally absolutely uniformly for $\mathrm{Re}(s)>1$. It will be necessairy to analytically continue $D(s,h)$ to $\mathbb{C}$ and this is provided by its spectral exapansion. To obtain this spectral expansion, we consider a Poincaré series $P_{h,Y}(z,s;\delta)$ which can be thought of as a δ-deformed version of the more classical Poincaré $P_h(z,s)$ (studied by Selberg) and truncated outside of $Y^{-1}\leq \mathrm{Im}(\gamma z)\leq Y$. We compute $I_{Y,\delta}(s,h)$ given by

$$I_{Y,\delta}(s,h)=\left\langle P_{h,Y}(z,s;\delta),f(z)\overline{g(z)}\mathrm{Im}(z)^k\right\rangle,$$

in two ways. After unfolding and computing the spectral expansion, one analytically continutes $I_{Y,\delta}(s,h)$. Upon taking the limit $Y\to\infty$, we arrive at

$$I_{\delta}(s,h)=\lim_{Y\to\infty}I_{Y,\delta}(s,h)=\frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}}D(s,h;\delta),$$

where $D(s,h;\delta)$ is a δ-deformed version of $D(s,h)$ and $I_{\delta}(s,h)$ admits a spectral exapansion. The analytic continuation of $I_{\delta}(s,h)$ gives the analytic continuation of $D(s,h;\delta)$ to all of $\mathbb{C}$. Taking the limit as $\delta\to 0$

results in the analytic contunation of $D(s,h)$ to the region $\mathrm{Re}(s) < \frac{1-k}{2}$. A detailed derivation of the spectral expansion can be found in [2]. Explicitly, the spectral expansion modulo constants and the continuous spectrum is

$$D(s,h) = \sum_{t_j} h^{\frac{1}{2}-s}\overline{\rho_j(-h)}\frac{\Gamma\left(1-s\right)\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma(s+k-1)}\left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k\right\rangle, \tag{1}$$

which is valid when $\mathrm{Re}(s) < \frac{1-k}{2}$. Since $D(s,h) \ll 1$ for $\mathrm{Re}(s) > 1$, we see that $D(s,h)$ admits meromorphic continuation to $\mathbb{C}$ but we do not have an explicit expression for $D(s,h)$ in the strip $\frac{1-k}{2} \leq \mathrm{Re}(s) \leq 1$. As for residues, the residue of $D(s,h)$ at $s = \frac{1}{2} - \ell + it_j$ for $0 \leq \ell \leq \frac{k}{2}$ is

$$\operatorname*{Res}_{s=\frac{1}{2}-\ell+it_j} D(s,h) = \frac{(-1)^\ell}{\ell!}\frac{\Gamma(-\ell+2it_j)\Gamma\left(\frac{1}{2}+\ell-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma\left(k-\frac{1}{2}-\ell+it_j\right)}\frac{\overline{\rho_j(-h)}}{h^{it_j-\ell}}\cdot\left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k\right\rangle. \tag{2}$$

We also have an estimate in vertical strips. For $\frac{1-k}{2} \leq \mathrm{Re}(s) \leq 1$, there exists $N \geq 1$ such that

$$D(s,h) \ll_\varepsilon (1+|s|)^N h^{\frac{1}{2}+\theta-\mathrm{Re}(s)+\varepsilon}, \tag{3}$$

where θ is the best bound toward the Ramanujan-Petersson conjecture, provided s is at least distance ε away from the closest pole of $D(s,h)$.

A Double Dirichlet Series for Shifted Convolution Sums

Let

$$f(z) = \sum_{m\geq 1} a(m)e^{2\pi imz} \quad \text{and} \quad g(z) = \sum_{m\geq 1} b(m)e^{2\pi imz} \quad \text{and} \quad l(z) = \sum_{m\geq 1} c(m)e^{2\pi imz},$$

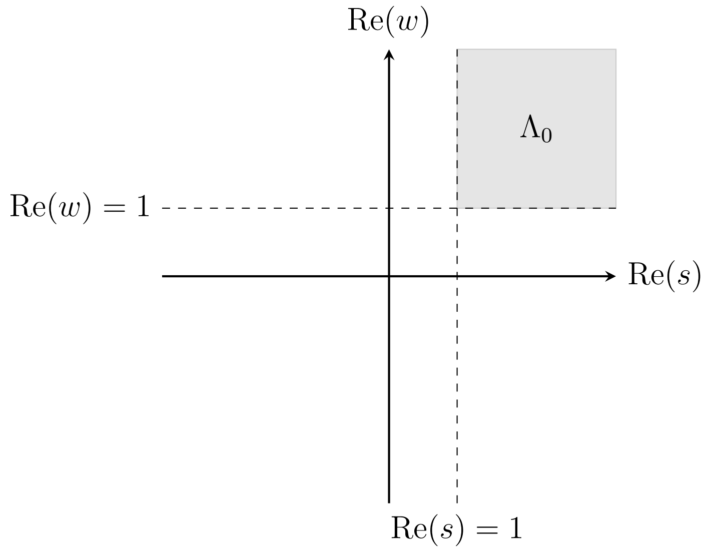
be weight k holomorphic cusp forms on $\Gamma = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. Define the double Dirichlet series

$$Z(s,w) = \sum_{m,h\geq 1}\frac{a(m)b(m+h)c(h)}{m^{s+k-1}h^{w+k-1}} = \sum_{h\geq 1}\frac{D(s,h)c(h)}{h^{w+k-1}} = \sum_{m\geq 1}\frac{a(m)D(w,m)}{m^{s+k-1}}.$$

As either of the latter two expressions converge absolutely uniformly on compacta provided $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(w) > 1$, we see that $Z(s,w)$ is converges locally absolutely uniformly convergence in the region $\Lambda_0 = \{(s,w) \in \mathbb{C}^2 : \mathrm{Re}(s) > 1, \mathrm{Re}(w) > 1\}$ and all of the equalities are justified. The last equality deserves as special name and is known as **the interchange** for $Z(s,w)$:

$$\sum_{h\geq 1}\frac{D(s,h)c(h)}{h^{w+k-1}} = \sum_{m\geq 1}\frac{a(m)D(w,m)}{m^{s+k-1}}.$$

Representing the point (s,w) by $(\mathrm{Re}(s),\mathrm{Re}(w))$, graphically we have continuation to the region



To obtain meromorphic continuation to a larger region in s, suppose $\mathrm{Re}(s) < \frac{1-k}{2}$ and replace $D(s,h)$ with its spectral expansion (modulo constants and the continuous spectrum) given in Equation (1) to obtain

$$Z(s,w) = \sum_{h\geq 1}\sum_{t_j} h^{\frac{1}{2}-s}\overline{\rho_j(-h)}\frac{\Gamma\left(1-s\right)\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma(s+k-1)}\left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k\right\rangle\frac{c(h)}{h^{w+k-1}},$$

Interchanging sums and collecting terms yields

$$Z(s,w) = \sum_{t_j}\frac{\Gamma\left(1-s\right)\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma(s+k-1)}\left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k\right\rangle\sum_{h\geq 1}\frac{\overline{\rho_j(-h)}c(h)}{h^{s+w+k-\frac{3}{2}}}.$$

Now the sum over h is the Rankin-Selberg convolution of l and u_j up to a zeta factor. Indeed,

$$L(s,u_j\otimes l) = \zeta(2s)\sum_{h\geq 1}\frac{\overline{\rho_j(-h)}C(h)}{h^s} = \zeta(2s)\sum_{h\geq 1}\frac{\overline{\rho_j(-h)}c(h)}{h^{s+\frac{k-1}{2}}},$$

so that

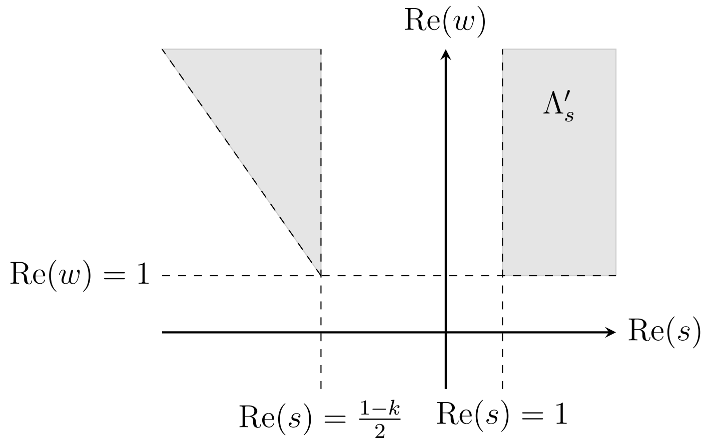
$$L\left(s+w+\frac{k}{2}-1, u_j\otimes l\right) = \zeta(2s+2w+k-2)\sum_{h\geq 1}\frac{\overline{\rho_j(-h)}c(h)}{h^{s+w+k-\frac{3}{2}}}.$$

This L-function admits analytic continuation to $\mathbb{C}$. Moreover, as the zeta factor has no zeros provided $\mathrm{Re}(2s+2w+k-2) > 1$, it follows that the sum over h is holomorphic for $\mathrm{Re}\left(s+w+\frac{k}{2}-1\right) > \frac{1}{2}$ or equivalently $\mathrm{Re}(s+w) > \frac{3-k}{2}$. Therefore $Z(s,w)$ admits meromorphic continuation to the region

$$\Lambda_s' = \Lambda_0\cup\left\{(s,w)\in\mathbb{C}^2 : \mathrm{Re}\left(s+w\right) > \frac{3-k}{2}, \mathrm{Re}(s) < \frac{1-k}{2}\right\},$$

where in the region $\Lambda_s' - \Lambda_0$, $Z(s,w)$ can be expressed as

$$Z(s,w) = \sum_{t_j}\frac{\Gamma\left(1-s\right)\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma(s+k-1)}\left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k\right\rangle\frac{L\left(s+w+\frac{k}{2}-1, u_j\otimes l\right)}{\zeta(2s+2w+k-2)}.$$

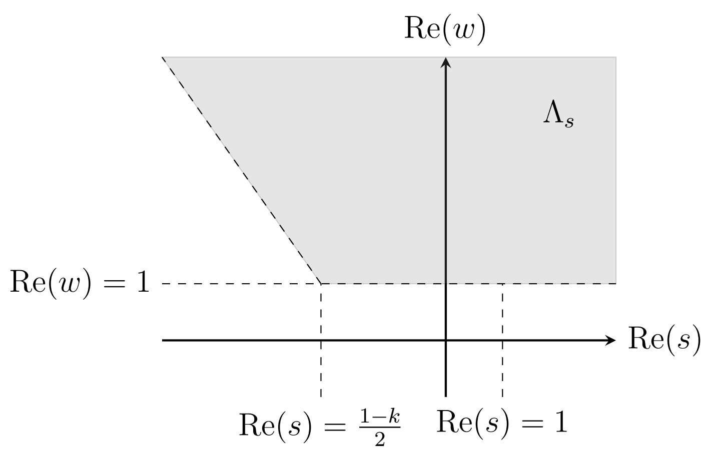


We can actually obtain continuation to a slightly large region at the expense of an explicit formula. To see this, using Equation (3) we have

$$Z(s,w) \ll_\varepsilon \sum_{h\geq 1} \frac{(1+|s|)^N c(h)}{h^{\mathrm{Re}(s+w)+k-\theta-\varepsilon-\frac{3}{2}}},$$

for $\frac{1-k}{2} < \mathrm{Re}(s) < 1$ and $\mathrm{Re}(w) > 1$ provided s is at least distance ε away from the closest pole. The latter expression converges in this region if $\mathrm{Re}(s+w) > \frac{5}{2} - k + \theta + \varepsilon$, or equivalently, $k > 2 + \theta + \varepsilon$ wich holds because $k \geq 4$ and $\theta < 1$. Since $D(s,h)$ admits meromorphic continuation to $\mathbb{C}$, the above bound implies that we obtain continuation to the region

$$\Lambda_s = \Lambda_0 \cup \left\{ (s,w) \in \mathbb{C}^2 : \mathrm{Re}\,(s+w) > \frac{3}{2}, \mathrm{Re}(w) > 1 \right\}.$$



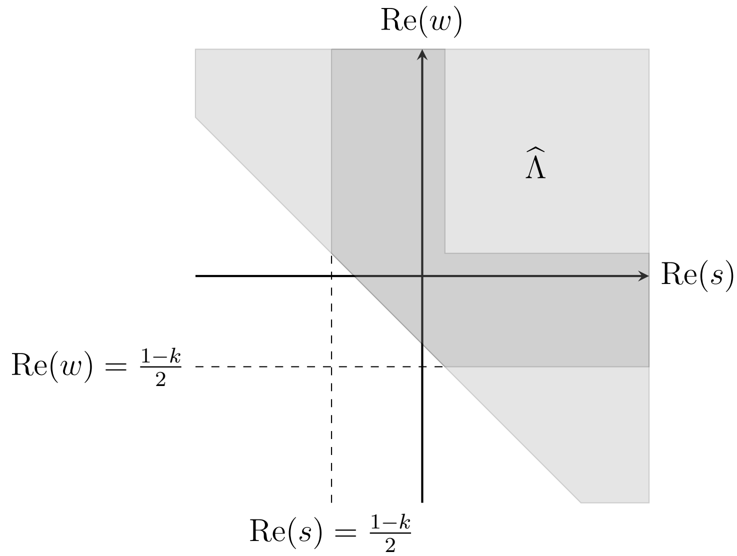
Using the interchange, we can perform the same procedure with the roles of s and w flipped to obtain meromorphic continuation to the region

$$\Lambda'_w = \Lambda_0 \cup \left\{ (s,w) \in \mathbb{C}^2 : \mathrm{Re}\,(s+w) > \frac{3-k}{2}, \mathrm{Re}(w) < \frac{1-k}{2} \right\},$$

and hence to the larger region

$$\Lambda_w = \Lambda_0 \cup \left\{ (s,w) \in \mathbb{C}^2 : \mathrm{Re}\,(s+w) > \frac{3-k}{2}, \mathrm{Re}(s) > 1 \right\}.$$

Since the region $\Lambda = \Lambda_s \cup \Lambda_w$ is a connected tube domain, we can appeal to Bochner's continuation theorem to at last obtain meromorphic continuation to the convex hull $\widehat{\Lambda}$.



The primary benifit of meromorphic continuation to $\widehat{\Lambda}$ is that we obtain continuation in a region containing the poles at $s = \frac{1}{2} + it_j$ (or $w = \frac{1}{2} + it_j$) provided w (or s) has large enough real part. In particular, the representations

$$Z(s,w) = \sum_{m,h \geq 1} \frac{a(m)b(m+h)c(h)}{m^{s+k-1}h^{s+k-1}},$$

$$Z(s,w) = \sum_{t_j} \frac{\Gamma\left(1-s\right)\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma(s+k-1)} \left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k \right\rangle \frac{L\left(s+w+\frac{k}{2}-1, u_j \otimes l\right)}{\zeta(2s+2w+k-2)},$$

$$Z(s,w) = \sum_{t_j} \frac{\Gamma\left(1-w\right)\Gamma\left(w-\frac{1}{2}+it_j\right)\Gamma\left(w-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma(w+k-1)} \left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k \right\rangle \frac{L\left(s+w+\frac{k}{2}-1, u_j \otimes l\right)}{\zeta(2s+2w+k-2)},$$

are valid on the corresponding regions

$$\Lambda_0 = \{(s,w) \in \mathbb{C}^2 : \mathrm{Re}(s) > 1, \mathrm{Re}(w) > 1\},$$
$$\Lambda'_s - \Lambda_0 = \cup \left\{(s,w) \in \mathbb{C}^2 : \mathrm{Re}\left(s+w\right) > \frac{3-k}{2}, \mathrm{Re}(s) < \frac{1-k}{2}\right\},$$
$$\Lambda'_w - \Lambda_0 = \cup \left\{(s,w) \in \mathbb{C}^2 : \mathrm{Re}\left(s+w\right) > \frac{3-k}{2}, \mathrm{Re}(s) < \frac{1-k}{2}\right\},$$

and in general, $Z(s,w)$ admits meromorphic continuation to the region

$$\widehat{\Lambda} = \left\{(s,w) \in \mathbb{C}^2 : \mathrm{Re}(s+w) > \frac{3-k}{2}\right\}.$$

Accordingly, the darker shaded region in the figure above is exactly where we have meromorphic continuation of $Z(s,w)$ but do not have an expression for $Z(s,w)$. This region is

$$\widehat{\Lambda} - \Lambda'_s - \Lambda'_w,$$

and it includes some of the poles of $Z(s,w)$. In general, the poles of the shifted Dirichlet series $D(s,h)$ and $D(w,m)$ induce poles of $Z(s,w)$. No additional poles arise from the Rankin-Selberg convolutions appearing in the meromorphic continuation. Indeed, these convolutions are of a Maass cusp form and a modular form which are orthogonal with respect to the Petersson inner product. Let us compute the residue of $Z(s,w)$ at $s=\frac{1}{2}-\ell+it_j$ where $0\le \ell \le \frac{k}{2}$ provided $\mathrm{Re}(w)>1$. We use the representation

$$Z(s,w)=\sum_{h\ge 1}\frac{D(s,h)c(h)}{h^{w+k-1}}.$$

As each term in the numerator has a pole at $s=\frac{1}{2}-\ell+it_j$ via the spectral expansion, using Equation (2) we obtain

$$\operatorname*{Res}_{s=\frac{1}{2}-\ell+it_j} Z(s,w)=\sum_{h\ge 1}\frac{(-1)^\ell}{\ell!}\frac{\Gamma(-\ell+2it_j)\Gamma\left(\frac{1}{2}+\ell-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma\left(k-\frac{1}{2}-\ell+it_j\right)}\frac{\overline{\rho_j(-h)}c(h)}{h^{w+k-1-\ell+it_j}} \cdot\left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k\right\rangle, \tag{4}$$

Collecting the sum over h gives the simplified expression

$$\operatorname*{Res}_{s=\frac{1}{2}-\ell+it_j} Z(s,w)=\frac{(-1)^\ell}{\ell!}\frac{\Gamma(-\ell+2it_j)\Gamma\left(\frac{1}{2}+\ell-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma\left(k-\frac{1}{2}-\ell+it_j\right)}\frac{L\left(\frac{1}{2}-\ell+it_j+w+\frac{k}{2}-1,u_j\otimes l\right)}{\zeta(1-2\ell+2it_j+2w+k-2)} \cdot\left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k\right\rangle, \tag{5}$$

The poles at $w=\frac{1}{2}+it_j$ are computed in the exact same was since $Z(s,w)$ is completely symmetric in s and w.

An Estimate for Short Sums

We will estimate sums of the form

$$T(X,Y)=\sum_{\substack{\frac{X}{2}<m\le X\\ \frac{Y}{2}<h\le Y}} A(m)B(m+h)C(h),$$

and show that they exhibit square root cancellation. First, we require an analytic expression for sums of the form

$$T(X,h)=\sum_{\frac{X}{2}<m\le X} A(m)B(m+h),$$

for fixed $h\ge 1$. We will assume we are working in the region of absolute uniform convergence on compacta of $Z(s,w)$. Let $\psi(t):[0,\infty)\to[0,\infty)$ be a bump function that is identically 1 for $\frac{1}{2}<t\le 1$ and exhibits smooth exponential decay to zero in a unit interval outside of $\left(\frac{1}{2},1\right]$. Denote its Mellin transform by $\Psi(s)$. Then

$$T(X,h)=\sum_{m\ge 1}A(m)B(m+h)\psi\left(\frac{m}{X}\right),$$

and the smoothed version of Perron's formula give

$$T(X,h)=\frac{1}{2\pi i}\int_{(2)}\sum_{m\ge 1}\frac{a(m)b(m+h)}{m^{s+\frac{k-1}{2}}(m+h)^{\frac{k-1}{2}}}\Psi(s)X^s\,ds. \tag{6}$$

Recall the identity,

$$\frac{1}{(1+t)^\beta}=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\beta-u)\Gamma(u)}{\Gamma(\beta)}t^{-u}\,du, \tag{7}$$

for any $0 < c < \beta$. Pulling out a factor of $m^{\frac{k-1}{2}}$ in the denominator of Equation (6) and using Equation (7) with $t = \frac{h}{m}$, $\beta = \frac{k-1}{2}$, and $c = \varepsilon$ gives

$$T(X,h) = \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(2)} \sum_{m \geq 1} \frac{a(m)b(m+h)}{m^{s-u+k-1}h^u} \frac{\Gamma\left(\frac{k-1}{2} - u\right)\Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)} \Psi(s) X^s \, ds \, du. \tag{8}$$

We can use this analytic expression for $T(X,h)$ to obtain one for $T(X,Y)$. First observe that

$$T(X,Y) = \sum_{\frac{Y}{2} < h \leq Y} T(X,h)C(h) = \sum_{h \geq 1} T(X,h)C(h)\psi\left(\frac{h}{Y}\right).$$

Applying the smoothed version of Perron's formula again, and using Equation (8), yields

$$T(X,Y) = \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{(2)} \int_{(2)} \sum_{m \geq 1} \frac{a(m)b(m+h)c(h)}{m^{s-u+k-1}h^{w+u+\frac{k-1}{2}}} \frac{\Gamma\left(\frac{k-1}{2} - u\right)\Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)} \Psi(s)\Psi(w) X^s Y^w \, ds \, dw \, du. \tag{9}$$

In terms of $Z(s,w)$, Equation (9) is expressed as

$$T(X,Y) = \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{(2)} \int_{(2)} Z\left(s-u, w+u-\frac{k-1}{2}\right) \frac{\Gamma\left(\frac{k-1}{2} - u\right)\Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)} \Psi(s)\Psi(w) X^s Y^w \, ds \, dw \, du. \tag{10}$$

We will estimate this triple integral by shifting lines of integration. First, shift the line (2) at s to $\left(\frac{1}{2}\right)$. We pass poles coming from the multiple Dirichlet series when $s = \frac{1}{2} + u + it_j$ so that

$$\begin{aligned} T(X,Y) &= \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{(2)} \int_{\left(\frac{1}{2}\right)} Z\left(s-u, w+u-\frac{k-1}{2}\right) \Psi(s)\Psi(w) X^s Y^w \, ds \, dw \, du \\ &\quad + \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(2)} \sum_{t_j} R_s\left(w+u-\frac{k-1}{2}, u; 0, t_j\right) \Psi\left(\frac{1}{2} + it_j + u\right) \Psi(w) X^{\frac{1}{2}+u+it_j} Y^w \, dw \, du, \end{aligned}$$

where, using Equation (5), we have

$$\begin{aligned} R_s(w,u;\ell,t_j) &= \left[\operatorname*{Res}_{s=\frac{1}{2}-\ell+it_j} Z(s,w)\right] \frac{\Gamma\left(\frac{k-1}{2} - u\right)\Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)} \\ &= \frac{(-1)^\ell}{\ell!} \frac{\Gamma(-\ell+2it_j)\Gamma\left(\frac{1}{2}+\ell-it_j\right)\Gamma\left(\frac{k-1}{2}-u\right)\Gamma(u)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma\left(k-\frac{1}{2}-\ell+it_j\right)\Gamma\left(\frac{k-1}{2}\right)} \frac{L\left(\frac{1}{2}-\ell+it_j+w+\frac{k}{2}-1, u_j \otimes l\right)}{\zeta(1-2\ell+2it_j+2w+k-2)} \\ &\quad \cdot \left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k \right\rangle. \end{aligned}$$

The interchange for $Z(s,w)$ implies that we may swap the roles of s and w in the double Dirichlet series inside of the integrand to obtain

$$\begin{aligned} T(X,Y) &= \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{(2)} \int_{\left(\frac{1}{2}\right)} Z\left(w-u, s+u-\frac{k-1}{2}\right) \Psi(s)\Psi(w) X^s Y^w \, ds \, dw \, du \\ &\quad + \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(2)} \sum_{t_j} R_s\left(w+u-\frac{k-1}{2}, u; 0, t_j\right) \Psi\left(\frac{1}{2} + it_j + u\right) \Psi(w) X^{\frac{1}{2}+u+it_j} Y^w \, dw \, du. \end{aligned}$$

Shifting the lines (2) at w to $\left(\frac{1}{2}+2\varepsilon\right)$ in both integrals, we don't pass by any poles and obtain

$$T(X,Y)=\frac{1}{(2\pi i)^3}\int_{(\varepsilon)}\int_{\left(\frac{1}{2}+2\varepsilon\right)}\int_{\left(\frac{1}{2}\right)} Z\left(w-u,s+u-\frac{k-1}{2}\right)\Psi(s)\Psi(w)X^sY^w\,ds\,dw\,du$$
$$+\frac{1}{(2\pi i)^2}\int_{(\varepsilon)}\int_{\left(\frac{1}{2}+2\varepsilon\right)}\sum_{t_j} R_s\left(w+u-\frac{k-1}{2},u;0,t_j\right)\Psi\left(\frac{1}{2}+it_j+u\right)\Psi(w)X^{\frac{1}{2}+u+it_j}Y^w\,dw\,du.$$

Denote these two integrals by $I_1(X,Y)$ and $I_2(X,Y)$ respectively. We will first estimate

$$I_1(X,Y)=\frac{1}{(2\pi i)^3}\int_{(\varepsilon)}\int_{\left(\frac{1}{2}+2\varepsilon\right)}\int_{\left(\frac{1}{2}\right)} Z\left(w-u,s+u-\frac{k-1}{2}\right)\frac{\Gamma\left(\frac{k-1}{2}-u\right)\Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)}\Psi(s)\Psi(w)X^sY^w\,ds\,dw\,du.$$

To this end, consider the approximation to $Z(s,w)$ given by

$$Z^*(s,w)=Z(s,w)-P(s,w),$$

where

$$P(s,w)=\sum_{0\le\ell\le\frac{k}{2}}\sum_{t_j}\frac{\mathrm{Res}_{s=\frac{1}{2}-\ell+it_j}Z(s,w)}{s-\left(\frac{1}{2}-\ell+it_j\right)}e^{-\left(s-\left(\frac{1}{2}-\ell+it_j\right)\right)^2}+\sum_{0\le\ell\le\frac{k}{2}}\sum_{t_j}\frac{\mathrm{Res}_{w=\frac{1}{2}-\ell+it_j}Z(s,w)}{w-\left(\frac{1}{2}-\ell+it_j\right)}e^{-\left(w-\left(\frac{1}{2}-\ell+it_j\right)\right)^2}.$$

For s and w away from poles of $Z(s,w)$, $P(s,w)$ converges locally absolutely uniformly and is absolutely bounded. Indeed, Sitrling's formula and a standard Lindelöf convexity argument for $L(s,u_j\otimes l)$ together with Equation (5) imply that $\mathrm{Res}_{s=\frac{1}{2}-\ell+it_j}Z(s,w)$ has polynomial growth in t_j and w which is dampened by the exponetial decay from $e^{-\left(s-\left(\frac{1}{2}-\ell+it_j\right)\right)^2}$. The same result holds with the roles of s and w interchanged. Moreover, $P(s,w)$ has the same poles and residues as $Z(s,w)$ so that $Z^*(s,w)$ is holomorphic on $\widehat{\Lambda}$. In the region $(s,w)\in\Lambda_s'-\Lambda_0$, consider

$$Z(s,w)=\sum_{h\ge1}\frac{D(s,h)c(h)}{h^{w+k-1}},$$

where

$$D(s,h)=\sum_{t_j}h^{\frac{1}{2}-s}\overline{\rho_j(-h)}\frac{\Gamma(1-s)\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma(s+k-1)}\left\langle u_j(z),f(z)\overline{g(z)}\mathrm{Im}(z)^k\right\rangle.$$

Provided we are at least ε away from the poles $s=\frac{1}{2}+it_j$, Equation (3) gives

$$Z(s,w)\ll_\varepsilon\sum_{h\ge1}\frac{(1+|s|)^N h^{\frac{1}{2}+\theta-\mathrm{Re}(s)+\varepsilon}}{h^{w+k-1}}\ll_\varepsilon(1+|s|)^N\zeta\left(\frac{k}{2}-\theta\right)\ll(1+|s|)^N,$$

for some $N\ge1$. As $P(s,w)$ is absolutely bounded away from the poles of $Z(s,w)$ and $Z^*(s,w)$ is holomorphic, estimates for $Z(s,w)$ and away from poles will hold for $Z^*(s,w)$ as well. Thus

$$Z^*(s,w)\ll_\varepsilon(1+|s|)^N,$$

in the same region. If $(s,w)\in\Lambda_0$, then $Z^*(s,w)\ll1$. These two estimate together with the Phragmen-Lindelöf convexity principle imply

$$Z^*(s,w)\ll_\varepsilon(1+|s|)^{\frac{2N}{3-k}(1-\mathrm{Re}(s))},$$

for $(s,w)\in\Lambda_s$. Repeating this argument with the roles of s and w interchanged, we have

$$Z^*(s,w)\ll_\varepsilon(1+|w|)^{\frac{2N}{3-k}(1-\mathrm{Re}(w))},$$

for $(s,w) \in \Lambda_w$. The same two estimates hold for $Z(s,w)$ provided we are ε away from the poles. By the Phragmen-Lindelöf convexity principle again, we obtain

$$Z(s,w) \ll_\varepsilon (1+|s|+|w|)^{\frac{2N}{3-k}\min(1-\mathrm{Re}(s),1-\mathrm{Re}(w))}, \tag{11}$$

for $(s,w) \in \widehat{\Lambda}$ and at least ε away from the poles. Using the interchange, the exact same estimate holds for $Z(w,s)$. Indeed,

$$Z(s,w) \ll_\varepsilon (1+|s|+|w|)^{\frac{2N}{3-k}\left(\frac{k-3}{2}-\varepsilon\right)},$$

Applying Equation (11) to $I_1(X,Y)$ yields

$$I_1(X,Y) \ll_\varepsilon \int_{(\varepsilon)}\int_{\left(\frac{1}{2}+2\varepsilon\right)}\int_{\left(\frac{1}{2}\right)} (1+|s|+|w|)^{\frac{2N}{3-k}\left(\frac{k-3}{2}-\varepsilon\right)} \frac{\Gamma\left(\frac{k-1}{2}-u\right)\Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)} \Psi(s)\Psi(w)X^sY^w \, ds\, dw\, du,$$

From Sitrling's formula, the ratio of gamma factors grows like $(1+|u|)^{-\frac{1}{2}}$ in vertical strips and so is polynomially bounded in $|u|$. Since ψ is compactly supported and smooth, $\Psi(s) \ll (1+|s|)^{-N}$ for any $N \geq 1$. So up to a constant, $\Psi(s)$ and $\Psi(w)$ truncate the integrals over s and w to something of bounded support. It follows that the triple integral is uniformly bounded and we have

$$I_1(X,Y) \ll X^{\frac{1}{2}}Y^{\frac{1}{2}+2\varepsilon}. \tag{12}$$

Next we estimate

$$I_2(X,Y) = \frac{1}{(2\pi i)^2}\int_{(\varepsilon)}\int_{\left(\frac{1}{2}+2\varepsilon\right)} \sum_{t_j} R_s\left(w+u-\frac{k-1}{2}, u; 0, t_j\right)\Psi\left(\frac{1}{2}+it_j+u\right)\Psi(w)X^{\frac{1}{2}+u+it_j}Y^w \, dw\, du.$$

Opening up the residue factor gives

$$I_2(X,Y) = \frac{1}{(2\pi i)^2}\int_{(\varepsilon)}\int_{\left(\frac{1}{2}+2\varepsilon\right)} \sum_{t_j} \frac{\Gamma(2it_j)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma\left(\frac{k-1}{2}-u\right)\Gamma(u)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma\left(k-\frac{1}{2}+it_j\right)\Gamma\left(\frac{k-1}{2}\right)} \frac{L(w+u+it_j, u_j\otimes l)}{\zeta(2w+2u+2it_j)}$$
$$\cdot \left\langle u_j(z), f(z)\overline{g(z)}\mathrm{Im}(z)^k\right\rangle \Psi(s)\Psi(w)X^{\frac{1}{2}+u+it_j}Y^w \, dw\, du.$$

We estimate the integrand similar to how we did for $I_1(X,Y)$. Sitrling's formula implies that the ratio of gamma factors grows like $(1+|t_j|)^{\frac{1}{2}-k}(1+|u|)^{-\frac{1}{2}}$ in vertical strips and is polynomially bounded in both $|t_j|$ and $|u|$. Moreover, the zeta function is the denominator is absolutely bounded because $\mathrm{Re}(2w+2u) = 1+6\varepsilon$. As for the Rankin-Selberg convolution, a standard convexity argument and the functional equation together imply $L(w+u+it_j, u_j\otimes l)$ is polynomially bounded in $|w|$, $|u|$, and $|t_j|$ in vertical strips. As before, the factors $\Psi(s)$ and $\Psi(w)$ truncate the integrals over s and w and the sum over t_j to something of bounded support. Thus the double integral and sum are uniformly bounded which gives

$$I_2(X,Y) \ll X^{\frac{1}{2}+\varepsilon}Y^{\frac{1}{2}+2\varepsilon}. \tag{13}$$

Combining Equations (12) and (13), we conclude that

$$T(X,Y) = \sum_{\substack{\frac{X}{2}<m\leq X\\ \frac{Y}{2}<h\leq Y}} A(m)B(m+h)C(h) \ll X^{\frac{1}{2}+\varepsilon}Y^{\frac{1}{2}+\varepsilon}.$$

Of course, a seperate analysis of the continuous spectrum is required for a complete proof. However, this contribution is smaller than that of the discrete spectrum which constitutes the main term.

A Rankin-Selberg Convolution

In this appendix we deduce the fucntional equation and analytic continuation of the Rankin-Selberg convolution

$$L(s,u_j\otimes l)=\zeta(2s)\sum_{h\geq 1}\frac{\overline{\rho_j(-h)}c(h)}{h^{s+\frac{k-1}{2}}},$$

between the Maass cusp form $u_j(z)$ and holomorphic cusp form $l(z)$. Let $\lambda_j=\frac{1}{4}+t_j^2$ be the Laplace eigenvalue of $u_j(z)$. Begin by considering the integral

$$\int_{\Gamma_\infty\backslash\mathbb{H}} u_j(z)\overline{l(z)}\mathrm{Im}(z)^{s+\frac{k}{2}}\,d\mu.$$

Expanding the forms into their Fourier series and interchanging sums we can compute

$$\begin{aligned}\int_{\Gamma_\infty\backslash\mathbb{H}} u_j(z)\overline{l(z)}\mathrm{Im}(z)^{s+\frac{k}{2}}\,d\mu &= \int_0^\infty\int_0^1\sum_{\substack{h\neq 0\\ m\geq 1}} c(m)\overline{\rho_j(-h)}y^{s+\frac{k+1}{2}}K_{it_j}(2\pi|h|y)e^{2\pi ihx}e^{-2\pi im(x-iy)}\,\frac{dx\,dy}{y^2}\\
&=\int_0^\infty\sum_{h\geq 1}c(h)\overline{\rho_j(-h)}y^{s+\frac{k+1}{2}}K_{it_j}(2\pi hy)e^{-2\pi hy}\,\frac{dy}{y^2}\\
&=\sum_{h\geq 1}c(h)\overline{\rho_j(-h)}\int_0^\infty K_{it_j}(2\pi hy)e^{-2\pi hy}y^{s+\frac{k-1}{2}}\,\frac{dy}{y}\\
&=\frac{1}{(2\pi)^{s+\frac{k-1}{2}}}\sum_{h\geq 1}\frac{c(h)\overline{\rho_j(-h)}}{h^{s+\frac{k-1}{2}}}\int_0^\infty K_{it_j}(y)e^{-y}y^{s+\frac{k-1}{2}}\,\frac{dy}{y}\\
&=\frac{1}{(2\pi)^{s+\frac{k-1}{2}}}\frac{L(s,u_j\otimes l)}{\zeta(2s)}\int_0^\infty K_{it_j}(y)e^{-y}y^{s+\frac{k-1}{2}}\,\frac{dy}{y}.\end{aligned}$$

Applying the well-known transform (also stated in [8]),

$$\int_0^\infty K_{it_j}(y)e^{-y}y^{s+\frac{k-1}{2}}\,\frac{dy}{y}=\sqrt{\pi}2^{\frac{1}{2}-s}\frac{\Gamma\left(s+\frac{k-1}{2}+it_j\right)\Gamma\left(s+\frac{k-1}{2}-it_j\right)}{\Gamma\left(s+\frac{k}{2}\right)},$$

we arrive at

$$\int_{\Gamma_\infty\backslash\mathbb{H}} l(z)u_j(z)\mathrm{Im}(z)^{s+\frac{k}{2}}\,d\mu=\frac{\Gamma\left(s+\frac{k-1}{2}+it_j\right)\Gamma\left(s+\frac{k-1}{2}-it_j\right)}{2^s(2\pi)^{s+\frac{k}{2}-1}\Gamma\left(s+\frac{k}{2}\right)\zeta(2s)}L(s,u_j\otimes l).$$

On the other hand, folding the integral yields

$$\begin{aligned}\int_{\Gamma_\infty\backslash\mathbb{H}} l(z)u_j(z)\mathrm{Im}(z)^{s+\frac{k}{2}}\,d\mu &= \int_{\mathcal{F}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} l(\gamma z)u_j(\gamma z)\mathrm{Im}(\gamma z)^{s+\frac{k}{2}}\,d\mu\\
&=\int_{\mathcal{F}} l(z)u_j(z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^k\mathrm{Im}(\gamma z)^{s+\frac{k}{2}}\,d\mu\\
&=\int_{\mathcal{F}} l(z)u_j(z)\mathrm{Im}(z)^{\frac{k}{2}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left(\frac{j(\gamma,z)}{|j(\gamma,z)|}\right)^k\mathrm{Im}(\gamma z)^s\,d\mu\\
&=\int_{\mathcal{F}} l(z)u_j(z)\mathrm{Im}(z)^{\frac{k}{2}}E_\infty(z,s;k)\,d\mu,\end{aligned}$$

where we have introduced the Eisenstein series

$$E_\infty(z,s;k) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left(\frac{j(\gamma,z)}{|j(\gamma,z)|}\right)^k \mathrm{Im}(\gamma z)^s.$$

From [8], the Fourier series can be expressed as

$$E_\infty(z,s;k) = \sum_{n\in\mathbb{Z}} a(n,y,s;k)e^{2\pi i n x},$$

where

$$a(n,y,s) = \begin{cases} y^s\frac{\Gamma\left(s+\frac{k}{2}\right)}{\Gamma(s)} + y^{1-s}\frac{\Gamma\left((1-s)+\frac{k}{2}\right)\Lambda(2s-1,\zeta)}{\Gamma(1-s)\Lambda(2s,\zeta)} & \text{if } n=0,\\ i^k\frac{|n|^{s-1}\sigma_{1-2s}(|n|)}{\Lambda(2s,\zeta)}W_{\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y) & \text{if } n\geq 1,\\ i^k\frac{|n|^{s-1}\sigma_{1-2s}(|n|)\Gamma\left(s+\frac{k}{2}\right)}{\Gamma\left(s-\frac{k}{2}\right)\Lambda(2s,\zeta)}W_{-\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y) & \text{if } n\leq 1,\end{cases}$$

where $W_{\alpha,\nu}(y)$ is the Whittaker function. If we multiply the Eisenstein series by $\Gamma(s)\Gamma(1-s)\Lambda(2s,\zeta)$ and use the fuctional equation for $\Lambda(2s-1,\zeta)$, then the $n=0$ coefficient becomes

$$y^s\Gamma(1-s)\Gamma\left(s+\frac{k}{2}\right)\Lambda(2s,\zeta) + y^{1-s}\Gamma(s)\Gamma\left((1-s)+\frac{k}{2}\right)\Lambda(2(1-s),\zeta),$$

which is invariant under $s\to 1-s$ because the two terms reflect into each other. Therefore the completed Eisenstein series

$$E^*_\infty(z,s;k) = \Gamma(s)\Gamma(1-s)\Lambda(2s,\zeta)E_\infty(z,s;k),$$

is invariant under $s\to 1-s$. Then

$$\Gamma(s)\Gamma(1-s)\Lambda(2s,\zeta)\int_{\mathcal{F}} l(z)u_j(z)\mathrm{Im}(z)^{\frac{k}{2}}E_\infty(z,s;k)\,d\mu = \int_{\mathcal{F}} l(z)u_j(z)\mathrm{Im}(z)^{\frac{k}{2}}E^*_\infty(z,s;k)\,d\mu.$$

Now define the completed Rankin-Selberg convolution

$$L^*(s,u_j\otimes l) = \frac{\Gamma\left(s+\frac{k-1}{2}+it_j\right)\Gamma\left(s+\frac{k-1}{2}-it_j\right)\Gamma(s)\Gamma(1-s)\Lambda(2s,\zeta)}{2^s(2\pi)^{s+\frac{k}{2}-1}\Gamma\left(s+\frac{k}{2}\right)\zeta(2s)}L(s,u_j\otimes l).$$

Then the invariance of $E^*_\infty(z,s;k)$ under $s\to 1-s$ implies

$$L^*(s,u_j\otimes l) = L^*(1-s,u_j\otimes l).$$

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