# A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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Abstract. We construct a quadratic double Dirichlet series $Z(s,w)$ built from single variable quadratic Dirichlet $L$-functions $L(s,\chi)$ over $\mathbb{Q}$. We prove that $Z(s,w)$ admits meromorphic continuation to the $(s,w)$-plane and satisfies a group of functional equations.

## 1. Preliminaries

We present an overview of quadratic Dirichlet $L$-functions over $\mathbb{Q}$. We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m\geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Now we recall characters on $\mathbb{Z}$. They are multiplicative functions $\chi : \mathbb{Z} \to \mathbb{C}$. They form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions $\chi_d : \mathbb{Z} \to \mathbb{C}$ modulo $d \geq 1$ (in that they are $d$-periodic) and such that $\chi_d(m) = 0$ if $(m,d) > 1$.
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. If $\chi$ is a Dirichlet character then its conjugate $\overline{\chi}$ is also a Dirichlet character. Moreover, $\overline{\chi}$ is the multiplicative inverse to $\chi$ and the Dirichlet characters modulo $m$ form a group under multiplication. This group is always finite and its order is $\phi(d) = |(\mathbb{Z}/d\mathbb{Z})^*|$. Dirichlet characters also satisfy orthogonality relations:

**Theorem 1.1** (Orthogonality relations)**.**

*(i) For any two Dirichlet characters $\chi$ and $\psi$ modulo $d$,*

$$\frac{1}{\phi(d)} \sideset{}{'}\sum_{a \pmod{d}} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

*(ii) For any $a, b \in (\mathbb{Z}/d\mathbb{Z})^*$,*

$$\frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

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The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{Z}$. First let us recall this symbol. For any odd prime $p$ and any $m \geq 1$, we define the quadratic residue symbol $\left(\frac{m}{p}\right)$ by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod p = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod p \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod p \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod p. \end{cases}$$

This symbol only depends upon $m$ modulo $p$ and is multiplicative in $m$. We can extend the quadratic residue symbol multiplicatively in the denomator. If $d = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of $d$, then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \leq i \leq k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any odd $d \geq 1$. We can extend this symbol further and allow $d \geq 1$ to be even. To this end, we define

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod 8, \\ -1 & \text{if } m \equiv 3, 5 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

and extend $\left(\frac{m}{d}\right)$ multiplatively in $d$ when $d$ is even. Now the quadratic residue symbol makes sense for any $m, d \geq 1$. Moreover, it is multiplicative in both $m$ and $d$ but no longer depends upon only $m$ modulo $d$ (it also depends upon $m$ modulo 8). In particular,

$$\left(\frac{-1}{d}\right) = \begin{cases} 1 & d \equiv 1 \pmod 4, \\ -1 & d \equiv 3 \pmod 4, \\ 0 & d \equiv 0 \pmod 2, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = \begin{cases} 1 & d \equiv 1, 7 \pmod 8, \\ -1 & d \equiv 3, 5 \pmod 8, \\ 0 & d \equiv 0 \pmod 2, \end{cases}$$

and if $d \not\equiv 0 \pmod 2$, we can compactly write

$$\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}} = \begin{cases} 1 & d \equiv 1 \pmod 4, \\ -1 & d \equiv 3 \pmod 4, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = (-1)^{\frac{d^2-1}{8}} = \begin{cases} 1 & d \equiv 1, 7 \pmod 8, \\ -1 & d \equiv 3, 5 \pmod 8. \end{cases}$$

The quadratic residue symbol also admits the following reciprocity law:

**Theorem 1.2** (Quadratic reciprocity)**.** *If $d, m \geq 1$ are relatively prime, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2} \frac{m^{(2)}-1}{2}} \left(\frac{m}{d}\right),$$

*where $d^{(2)}$ and $m^{(2)}$ are the parts of $d$ and $m$ relatively prime to 2 respectively.*

We can now define the quadratic Dirichlet characters. For any odd square-free $d \geq 1$, define the quadratic Dirichlet character $\chi_d$ by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod 4, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension $\mathbb{Q}(\sqrt{d})$. We extend $\chi_d$ multiplicatively in the denominator so that $\chi_d$ makes sense for any odd $d \geq 1$. In particular, $\chi_d(m) = \pm 1$ provided $d$ and $m$ are relatively prime and $\chi_d(m) = 0$ if $(m, d) > 1$. Quadratic reciprocity implies that $\chi_d$ is a Dirichlet character modulo $d$ if $d \equiv 1 \pmod 4$ and is a Dirichlet character modulo $4d$ if $d \equiv 2, 3 \pmod 4$. Indeed, if $d \equiv 1 \pmod 4$ then $d^{(2)} = d$ and the sign is always 1. If $d \equiv 3 \pmod 4$, then $d^{(2)} = d$ and the sign is $\left(\frac{-1}{m}\right)$ which is a character modulo 4. If $d \equiv 2 \pmod 4$, then $d^{(2)} \equiv 1, 3 \pmod 4$ and we are reduced to one of the previous two cases.

We now discuss the Hilbert characters. We will only need four of them: the quadratic Dirichlet characters modulo 8. We define them as follows:

$$\psi_1(m) = \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod 2, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases} \qquad \psi_{-1}(m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4, \\ -1 & \text{if } m \equiv 3 \pmod 4, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

$$\psi_2(m) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod 8, \\ -1 & \text{if } m \equiv 3, 5 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases} \qquad \psi_{-2}(m) = \begin{cases} 1 & \text{if } m \equiv 1, 3 \pmod 8, \\ -1 & \text{if } m \equiv 5, 7 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

We can write $\psi_{-1}$ and $\psi_2$ in terms of Legendre symbols:

$$\psi_{-1}(m) = \left(\frac{-1}{m}\right) \quad \text{and} \quad \psi_2(m) = \left(\frac{m}{2}\right).$$

Moreover, these characters satisfy the relations

$$\psi_{-2}(m) = \psi_{-1}(m)\psi_2(m), \quad \psi_1(m) = \psi_{-1}(m)\psi_{-1}(m), \quad \text{and} \quad \psi_{-1}(m) = \psi_2(m)\psi_{-2}(m).$$

These facts together with quadratic reciprocity imply

$$\chi_d(m) = \begin{cases} \chi_m(d) & \text{if } d \equiv 1 \pmod 4 \text{ or } m \equiv 1 \pmod 4, \\ \psi_{-1}(m)\chi_m(d) & \text{if } d \equiv 3 \pmod 4, \\ \psi_{-1}(d)\chi_m(d) & \text{if } m \equiv 3 \pmod 4, \\ \psi_2\left(\frac{d}{2}\right)\chi_m(d) & \text{if } d \equiv 2 \pmod 4 \text{ and } m \equiv 3 \pmod 4, \\ \psi_2\left(\frac{m}{2}\right)\chi_m(d) & \text{if } m \equiv 2 \pmod 4 \text{ and } d \equiv 3 \pmod 4, \\ 0 & \text{if } (m, d) > 1. \end{cases}$$

With the Dirichlet characters and Hilbert characters introduced, we are ready to discuss the $L$-functions associated to quadratic Dirichlet characters. We define the $L$-function $L(s, \chi_d)$ attached to $\chi_d$ by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \text{ monic}} \frac{\chi_d(m)}{|m|^s} = \prod_{P \text{ monic irr}} \left(1 - \frac{\chi_d(P)}{|P|^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character, $L(s, \chi_d) \ll \zeta(s)$ for $\mathrm{Re}(s) > 1$ so that $L(s, \chi_d)$ is locally absolutely uniformly convergent in this region. $L(s, \chi_d)$ also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ if $d$ is a perfect square and is analytic otherwise (see [**?**] for a proof). Moreover, $L(s, \chi_d)$ is a polynomial in $q^{-s}$ of degree at most $\deg(d) - 1$. The completed $L$-function is defined as follows:

$$L^*(s, \chi_d) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_d) & \text{if } \deg(d) \text{ is even}, \\ L(s, \chi_d) & \text{if } \deg(d) \text{ is odd}, \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_d) = \begin{cases} q^{2s-1}|d|^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \deg(d) \text{ is even}, \\ q^{2s-1}(q|d|)^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \deg(d) \text{ is odd}. \end{cases}$$

Note that in the case $\deg(d)$ is even, the conductor is $|d|$ and in the case $\deg(d)$ is odd, the conductor is $q|d|$. In other words, the gamma factors depend upon the degree of $d$. This will cause a small but important technical issue later when we want to derive functional equations for the quadratic double Dirichlet series.

## The Quadratic Double Dirichlet Series

## The Interchange

## Weighting Terms

## Functional Equations

## Meromorphic Continuation

## Poles and Residues

## References

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