

A MULTIPLE DIRICHLET SERIES APPROACH TO SHIFTED CONVOLUTION SUMS

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ABSTRACT. A generalization of the classical additive divisor problem is that of shifted convolution sums for holomorphic cusp forms: find asymptotics for sums of the form

$$S(X, h) = \sum_{1 < m \leq X} A(m) \overline{B(m+h)},$$

where $A(m)$ and $B(m)$ are the Hecke normalized Fourier coefficients of weight k holomorphic cusp forms f and g on say $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$. One method of obtaining good estimates for $S(X, h)$ when $h \geq 1$ involves a detailed study of the spectral expansion of the shifted Dirichlet series

$$D(s, h) = \sum_{m \geq 1} \frac{a(m)b(m+h)}{m^{s+k-1}},$$

where $a(m)$ and $b(m)$ are the unnormalized Fourier coefficients. In the following, we discuss the analytic properties of the multiple Dirichlet series

$$Z(s, w) = \sum_{m, h \geq 1} \frac{a(m)b(m+h)c(h)}{m^{s+k-1}h^{w+k-1}},$$

where $c(h)$ are the Fourier coefficients of another weight k holomorphic cusp form l . We exploit the analytic properties of this multiple Dirichlet series to sketch the proof of estimates for the following triple shifted convolution sums:

$$T(X, Y) = \sum_{\substack{\frac{X}{2} < m \leq X \\ \frac{Y}{2} < h \leq Y}} A(m)B(m+h)C(h),$$

where $C(h)$ are the Hecke normalized Fourier coefficients of l .

1. MOTIVATING SHIFTED CONVOLUTION SUMS

The prototypical example of shifted convolution sums is when $a(m) = b(m) = \tau_2(n)$ is the usual divisor function. Obtaining estimates for the sum

$$D_2(X, h) = \sum_{m \leq X} \tau_2(m)\tau_2(m+h),$$

is the well-known **binary additive divisor problem**. More generally, sums of the form

$$D_{k,\ell}(X, h) = \sum_{m \leq X} \tau_k(m)\tau_\ell(m+h),$$

where τ_k is the k -th divisor function, are called **additive divisor sums**. These objects are of interest because $D_k(X, h) = D_{k,k}(X, h)$ is attached to the $2k$ -th moment of the Riemann zeta function, defined by

$$I_k(T) = \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^{2k} dt.$$

The first appearance of this phenomena occurred in 1926 when Ingham (see [3]) showed that $D_2(X, h)$ was related to the 4-th moment by using estimate for $D_2(X, h)$ to establish the asymptotic

$$I_2(T) \sim \frac{T}{2\pi^2} (\log T)^4.$$

The essential ingredient in Ingham's proof was an approximation function equation for $|\zeta(\frac{1}{2} + it)|^4$ involving the sums $D_2(X, h)$. Since Ingham's proof, many others have used results about additive divisor sums to establish asymptotics for moments (for example, see [5] and [6]). Unfortunately, not much is known about $D_k(X, h)$ when $k > 2$.

On the other hand, when $k = 2$, additive divisor sums are also attached to the spectral theory of automorphic forms. This is because $\tau_2(m)$ appears as the m -th Fourier coefficient of the Eisenstein series $E(z, s)$ on $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$ when $s = \frac{1}{2}$. When the $a(m)$ and $b(m)$ are Fourier coefficients of automorphic forms, the associated sum

$$S(X, h) = \sum_{m \leq X} A(m)B(m+h),$$

with Hecke normalized Fourier coefficients, is called a **shifted convolution sum**. If the automorphic forms are non-cuspidal and the Fourier coefficients are Hecke normalized, we have

$$S(X, h) = XP(\log(X)) + O_{h,\varepsilon}(X^{\frac{2}{3}+\varepsilon}),$$

where P is some polynomial. Note that the error term here has cube root cancellation. Conjecturally, we should have square root cancellation in the error term (with an additional h^ε factor). When the automorphic forms are cuspidal, there is no main term, but the error term is of the same size as in the non-cuspidal case. Again, the correct order of magnitude should have square root cancellation. The specific triple shifted convolution sums we will discuss below are when $a(m)$, $b(m)$, and $c(m)$, are Fourier coefficients of weight k holomorphic cuspforms

$$f(z) = \sum_{m \geq 1} a(m)e^{2\pi imz}, \quad g(z) = \sum_{m \geq 1} b(m)e^{2\pi imz}, \quad \text{and} \quad l(z) = \sum_{m \geq 1} c(m)e^{2\pi imz},$$

on $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$. We will denote the Hecke normalized coefficients by $A(m)$, $B(m)$, and $C(m)$, respectively.

2. SPECTRAL EXPANSIONS FOR SHIFTED DIRICHLET SERIES

Let $h \geq 1$. We will estimate our shifted convolution sums using the shifted Dirichlet series

$$D(s, h) = \sum_{m \geq 1} \frac{a(m)b(m+h)}{m^{s+k-1}}.$$

The Ramanujan conjecture implies $a(m) \ll m^{\frac{k-1}{2}}$ and $b(m+h) \ll (m+h)^{\frac{k-1}{2}} \ll_h m^{\frac{k-1}{2}}$ and so $D(s, h)$ converges locally absolutely uniformly for $\mathrm{Re}(s) > 1$. It will be necessary to analytically continue $D(s, h)$ to \mathbb{C} and this is provided by its spectral expansion. To obtain this spectral expansion, we consider a Poincaré series $P_{h,Y}(z, s; \delta)$ which can be thought of as a δ -deformed version of the more classical Poincaré $P_h(z, s)$ (studied by Selberg) and truncated outside of $Y^{-1} \leq \mathrm{Im}(\gamma z) \leq Y$. We compute $I_{Y,\delta}(s, h)$ given by

$$I_{Y,\delta}(s, h) = \left\langle P_{h,Y}(z, s; \delta), f(z)\overline{g(z)}\mathrm{Im}(z)^k \right\rangle,$$

in two ways. After unfolding and computing the spectral expansion, one analytically continues $I_{Y,\delta}(s, h)$. Upon taking the limit $Y \rightarrow \infty$, we arrive at

$$I_\delta(s, h) = \lim_{Y \rightarrow \infty} I_{Y,\delta}(s, h) = \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} D(s, h; \delta),$$

where $D(s, h; \delta)$ is a δ -deformed version of $D(s, h)$ and $I_\delta(s, h)$ admits a spectral expansion. The analytic continuation of $I_\delta(s, h)$ gives the analytic continuation of $D(s, h; \delta)$ to all of \mathbb{C} . Taking the limit as $\delta \rightarrow 0$

results in the analytic continuation of $D(s, h)$ to the region $\operatorname{Re}(s) < \frac{1-k}{2}$. A detailed derivation of the spectral expansion can be found in [2]. Explicitly, the spectral expansion modulo constants and the continuous spectrum is

$$D(s, h) = \sum_{t_j} h^{\frac{1}{2}-s} \overline{\rho_j(-h)} \frac{\Gamma(1-s) \Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(s+k-1)} \left\langle u_j(z), f(z) \overline{g(z)} \operatorname{Im}(z)^k \right\rangle, \quad (1)$$

which is valid when $\operatorname{Re}(s) < \frac{1-k}{2}$. Since $D(s, h) \ll 1$ for $\operatorname{Re}(s) > 1$, we see that $D(s, h)$ admits meromorphic continuation to \mathbb{C} but we do not have an explicit expression for $D(s, h)$ in the strip $\frac{1-k}{2} \leq \operatorname{Re}(s) \leq 1$. As for residues, the residue of $D(s, h)$ at $s = \frac{1}{2} - \ell + it_j$ for $0 \leq \ell \leq \frac{k}{2}$ is

$$\operatorname{Res}_{s=\frac{1}{2}-\ell+it_j} D(s, h) = \frac{(-1)^\ell}{\ell!} \frac{\Gamma(-\ell + 2it_j) \Gamma(\frac{1}{2} + \ell - it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(k - \frac{1}{2} - \ell + it_j)} \frac{\overline{\rho_j(-h)}}{h^{it_j-\ell}} \cdot \left\langle u_j(z), f(z) \overline{g(z)} \operatorname{Im}(z)^k \right\rangle. \quad (2)$$

We also have an estimate in vertical strips. For $\frac{1-k}{2} \leq \operatorname{Re}(s) \leq 1$, there exists $N \geq 1$ such that

$$D(s, h) \ll_\varepsilon (1 + |s|)^N h^{\frac{1}{2} + \theta - \operatorname{Re}(s) + \varepsilon}, \quad (3)$$

where θ is the best bound toward the Ramanujan-Petersson conjecture, provided s is at least distance ε away from the closest pole of $D(s, h)$.

3. A DOUBLE DIRICHLET SERIES FOR SHIFTED CONVOLUTION SUMS

Let

$$f(z) = \sum_{m \geq 1} a(m) e^{2\pi i m z} \quad \text{and} \quad g(z) = \sum_{m \geq 1} b(m) e^{2\pi i m z} \quad \text{and} \quad l(z) = \sum_{m \geq 1} c(m) e^{2\pi i m z},$$

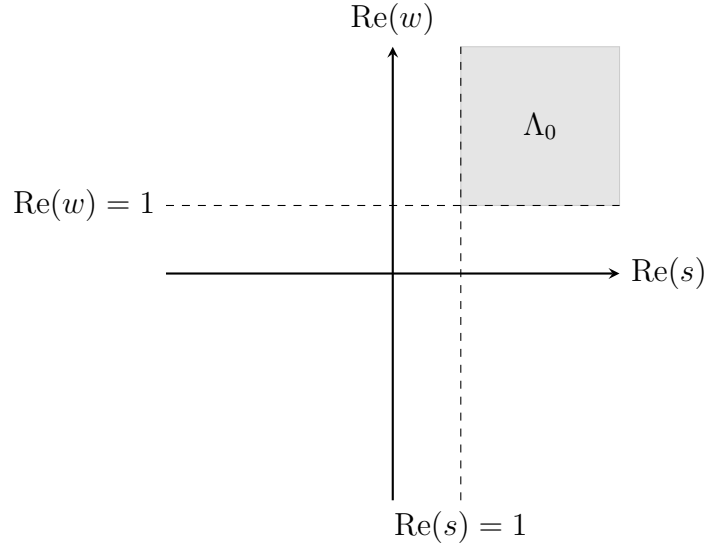
be weight k holomorphic cusp forms on $\Gamma = \operatorname{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$. Define the double Dirichlet series

$$Z(s, w) = \sum_{m, h \geq 1} \frac{a(m) b(m+h) c(h)}{m^{s+k-1} h^{w+k-1}} = \sum_{h \geq 1} \frac{D(s, h) c(h)}{h^{w+k-1}} = \sum_{m \geq 1} \frac{a(m) D(w, m)}{m^{s+k-1}}.$$

As either of the latter two expressions converge absolutely uniformly on compacta provided $\operatorname{Re}(s) > 1$ and $\operatorname{Re}(w) > 1$, we see that $Z(s, w)$ is converges locally absolutely uniformly convergence in the region $\Lambda_0 = \{(s, w) \in \mathbb{C}^2 : \operatorname{Re}(s) > 1, \operatorname{Re}(w) > 1\}$ and all of the equalities are justified. The last equality deserves as special name and is known as **the interchange** for $Z(s, w)$:

$$\sum_{h \geq 1} \frac{D(s, h) c(h)}{h^{w+k-1}} = \sum_{m \geq 1} \frac{a(m) D(w, m)}{m^{s+k-1}}.$$

Representing the point (s, w) by $(\operatorname{Re}(s), \operatorname{Re}(w))$, graphically we have continuation to the region



To obtain meromorphic continuation to a larger region in s , suppose $\text{Re}(s) < \frac{1-k}{2}$ and replace $D(s, h)$ with its spectral expansion (modulo constants and the continuous spectrum) given in Equation (1) to obtain

$$Z(s, w) = \sum_{h \geq 1} \sum_{t_j} h^{\frac{1}{2}-s} \overline{\rho_j(-h)} \frac{\Gamma(1-s) \Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(s+k-1)} \left\langle u_j(z), f(z) \overline{g(z)} \text{Im}(z)^k \right\rangle \frac{c(h)}{h^{w+k-1}},$$

Interchanging sums and collecting terms yields

$$Z(s, w) = \sum_{t_j} \frac{\Gamma(1-s) \Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(s+k-1)} \left\langle u_j(z), f(z) \overline{g(z)} \text{Im}(z)^k \right\rangle \sum_{h \geq 1} \frac{\overline{\rho_j(-h)} c(h)}{h^{s+w+k-\frac{3}{2}}}.$$

Now the sum over h is the Rankin-Selberg convolution of l and u_j up to a zeta factor. Indeed,

$$L(s, u_j \otimes l) = \zeta(2s) \sum_{h \geq 1} \frac{\overline{\rho_j(-h)} C(h)}{h^s} = \zeta(2s) \sum_{h \geq 1} \frac{\overline{\rho_j(-h)} c(h)}{h^{s+\frac{k-1}{2}}},$$

so that

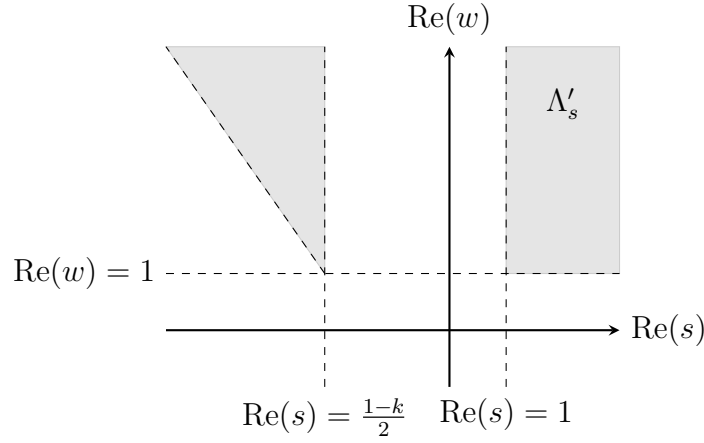
$$L\left(s + w + \frac{k}{2} - 1, u_j \otimes l\right) = \zeta(2s + 2w + k - 2) \sum_{h \geq 1} \frac{\overline{\rho_j(-h)} c(h)}{h^{s+w+k-\frac{3}{2}}}.$$

This L -function admits analytic continuation to \mathbb{C} . Moreover, as the zeta factor has no zeros provided $\text{Re}(2s + 2w + k - 2) > 1$, it follows that the sum over h is holomorphic for $\text{Re}(s + w + \frac{k}{2} - 1) > \frac{1}{2}$ or equivalently $\text{Re}(s + w) > \frac{3-k}{2}$. Therefore $Z(s, w)$ admits meromorphic continuation to the region

$$\Lambda'_s = \Lambda_0 \cup \left\{ (s, w) \in \mathbb{C}^2 : \text{Re}(s + w) > \frac{3-k}{2}, \text{Re}(s) < \frac{1-k}{2} \right\},$$

where in the region $\Lambda'_s - \Lambda_0$, $Z(s, w)$ can be expressed as

$$Z(s, w) = \sum_{t_j} \frac{\Gamma(1-s) \Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(s+k-1)} \left\langle u_j(z), f(z) \overline{g(z)} \text{Im}(z)^k \right\rangle \frac{L(s + w + \frac{k}{2} - 1, u_j \otimes l)}{\zeta(2s + 2w + k - 2)}.$$

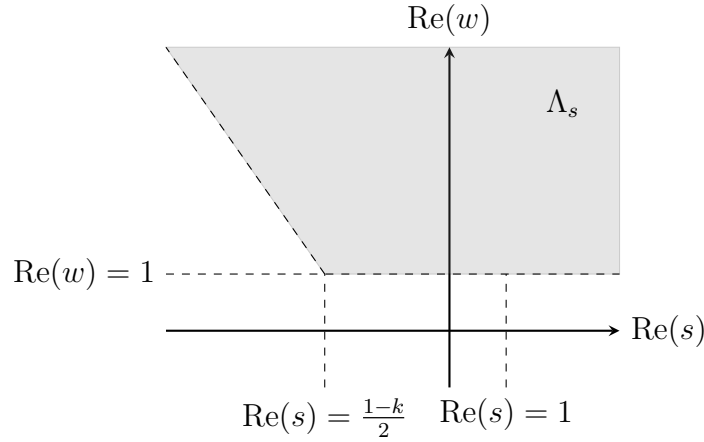


We can actually obtain continuation to a slightly large region at the expense of an explicit formula. To see this, using Equation (3) we have

$$Z(s, w) \ll_{\varepsilon} \sum_{h \geq 1} \frac{(1 + |s|)^N c(h)}{h^{\operatorname{Re}(s+w) + k - \theta - \varepsilon - \frac{3}{2}}},$$

for $\frac{1-k}{2} < \operatorname{Re}(s) < 1$ and $\operatorname{Re}(w) > 1$ provided s is at least distance ε away from the closest pole. The latter expression converges in this region if $\operatorname{Re}(s + w) > \frac{5}{2} - k + \theta + \varepsilon$, or equivalently, $k > 2 + \theta + \varepsilon$ which holds because $k \geq 4$ and $\theta < 1$. Since $D(s, h)$ admits meromorphic continuation to \mathbb{C} , the above bound implies that we obtain continuation to the region

$$\Lambda_s = \Lambda_0 \cup \left\{ (s, w) \in \mathbb{C}^2 : \operatorname{Re}(s + w) > \frac{3}{2}, \operatorname{Re}(w) > 1 \right\}.$$



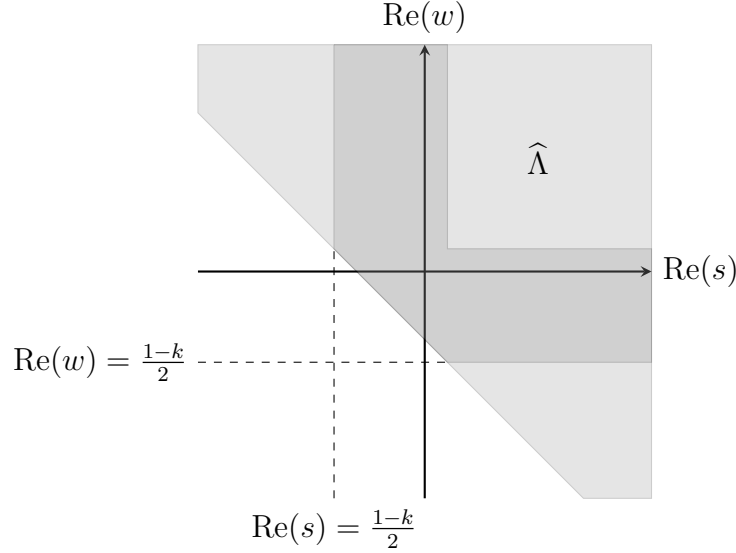
Using the interchange, we can perform the same procedure with the roles of s and w flipped to obtain meromorphic continuation to the region

$$\Lambda'_w = \Lambda_0 \cup \left\{ (s, w) \in \mathbb{C}^2 : \operatorname{Re}(s + w) > \frac{3-k}{2}, \operatorname{Re}(w) < \frac{1-k}{2} \right\},$$

and hence to the larger region

$$\Lambda_w = \Lambda_0 \cup \left\{ (s, w) \in \mathbb{C}^2 : \operatorname{Re}(s + w) > \frac{3-k}{2}, \operatorname{Re}(s) > 1 \right\}.$$

Since the region $\Lambda = \Lambda_s \cup \Lambda_w$ is a connected tube domain, we can appeal to Bochner's continuation theorem to at last obtain meromorphic continuation to the convex hull $\widehat{\Lambda}$.



The primary benifit of meromorphic continuation to $\hat{\Lambda}$ is that we obtain continuation in a region containing the poles at $s = \frac{1}{2} + it_j$ (or $w = \frac{1}{2} + it_j$) provided w (or s) has large enough real part. In particular, the representations

$$Z(s, w) = \sum_{m, h \geq 1} \frac{a(m)b(m+h)c(h)}{m^{s+k-1}h^{s+k-1}},$$

$$Z(s, w) = \sum_{t_j} \frac{\Gamma(1-s) \Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(s+k-1)} \left\langle u_j(z), f(z) \overline{g(z)} \text{Im}(z)^k \right\rangle \frac{L(s+w+\frac{k}{2}-1, u_j \otimes l)}{\zeta(2s+2w+k-2)},$$

$$Z(s, w) = \sum_{t_j} \frac{\Gamma(1-w) \Gamma(w - \frac{1}{2} + it_j) \Gamma(w - \frac{1}{2} - it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(w+k-1)} \left\langle u_j(z), f(z) \overline{g(z)} \text{Im}(z)^k \right\rangle \frac{L(s+w+\frac{k}{2}-1, u_j \otimes l)}{\zeta(2s+2w+k-2)},$$

are valid on the corresponding regions

$$\Lambda_0 = \{(s, w) \in \mathbb{C}^2 : \text{Re}(s) > 1, \text{Re}(w) > 1\},$$

$$\Lambda'_s - \Lambda_0 = \cup \left\{ (s, w) \in \mathbb{C}^2 : \text{Re}(s+w) > \frac{3-k}{2}, \text{Re}(s) < \frac{1-k}{2} \right\},$$

$$\Lambda'_w - \Lambda_0 = \cup \left\{ (s, w) \in \mathbb{C}^2 : \text{Re}(s+w) > \frac{3-k}{2}, \text{Re}(s) < \frac{1-k}{2} \right\},$$

and in general, $Z(s, w)$ admits meromorphic continuation to the region

$$\hat{\Lambda} = \left\{ (s, w) \in \mathbb{C}^2 : \text{Re}(s+w) > \frac{3-k}{2} \right\}.$$

Accordingly, the darker shaded region in the figure above is exactly where we have meromorphic continuation of $Z(s, w)$ but do not have an expression for $Z(s, w)$. This region is

$$\hat{\Lambda} - \Lambda'_s - \Lambda'_w,$$

and it includes some of the poles of $Z(s, w)$. In general, the poles of the shifted Dirichlet series $D(s, h)$ and $D(w, m)$ induce poles of $Z(s, w)$. No additional poles arise from the Rankin-Selberg convolutions appearing in the meromorphic continuation. Indeed, these convolutions are of a Maass cusp form and a modular form which are orthogonal with respect to the Petersson inner product. Let us compute the

residue of $Z(s, w)$ at $s = \frac{1}{2} - \ell + it_j$ where $0 \leq \ell \leq \frac{k}{2}$ provided $\operatorname{Re}(w) > 1$. We use the representation

$$Z(s, w) = \sum_{h \geq 1} \frac{D(s, h)c(h)}{h^{w+k-1}}.$$

As each term in the numerator has a pole at $s = \frac{1}{2} - \ell + it_j$ via the spectral expansion, using Equation (2) we obtain

$$\begin{aligned} \operatorname{Res}_{s=\frac{1}{2}-\ell+it_j} Z(s, w) &= \sum_{h \geq 1} \frac{(-1)^\ell}{\ell!} \frac{\Gamma(-\ell + 2it_j)\Gamma(\frac{1}{2} + \ell - it_j)}{\Gamma(\frac{1}{2} + it_j)\Gamma(\frac{1}{2} - it_j)\Gamma(k - \frac{1}{2} - \ell + it_j)} \frac{\overline{\rho_j(-h)}c(h)}{h^{w+k-1-\ell+it_j}} \\ &\quad \cdot \left\langle u_j(z), f(z)\overline{g(z)}\operatorname{Im}(z)^k \right\rangle, \end{aligned} \quad (4)$$

Collecting the sum over h gives the simplified expression

$$\begin{aligned} \operatorname{Res}_{s=\frac{1}{2}-\ell+it_j} Z(s, w) &= \frac{(-1)^\ell}{\ell!} \frac{\Gamma(-\ell + 2it_j)\Gamma(\frac{1}{2} + \ell - it_j)}{\Gamma(\frac{1}{2} + it_j)\Gamma(\frac{1}{2} - it_j)\Gamma(k - \frac{1}{2} - \ell + it_j)} \frac{L(\frac{1}{2} - \ell + it_j + w + \frac{k}{2} - 1, u_j \otimes l)}{\zeta(1 - 2\ell + 2it_j + 2w + k - 2)} \\ &\quad \cdot \left\langle u_j(z), f(z)\overline{g(z)}\operatorname{Im}(z)^k \right\rangle, \end{aligned} \quad (5)$$

The poles at $w = \frac{1}{2} + it_j$ are computed in the exact same way since $Z(s, w)$ is completely symmetric in s and w .

4. AN ESTIMATE FOR SHORT SUMS

We will estimate sums of the form

$$T(X, Y) = \sum_{\substack{\frac{X}{2} < m \leq X \\ \frac{Y}{2} < h \leq Y}} A(m)B(m+h)C(h),$$

and show that they exhibit square root cancellation. First, we require an analytic expression for sums of the form

$$T(X, h) = \sum_{\frac{X}{2} < m \leq X} A(m)B(m+h),$$

for fixed $h \geq 1$. We will assume we are working in the region of absolute uniform convergence on compacta of $Z(s, w)$. Let $\psi(t) : [0, \infty) \rightarrow [0, \infty)$ be a bump function that is identically 1 for $\frac{1}{2} < t \leq 1$ and exhibits smooth exponential decay to zero in a unit interval outside of $(\frac{1}{2}, 1]$. Denote its Mellin transform by $\Psi(s)$. Then

$$T(X, h) = \sum_{m \geq 1} A(m)B(m+h)\psi\left(\frac{m}{X}\right),$$

and the smoothed version of Perron's formula give

$$T(X, h) = \frac{1}{2\pi i} \int_{(2)} \sum_{m \geq 1} \frac{a(m)b(m+h)}{m^{s+\frac{k-1}{2}}(m+h)^{\frac{k-1}{2}}} \Psi(s) X^s ds. \quad (6)$$

Recall the identity,

$$\frac{1}{(1+t)^\beta} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(\beta-u)\Gamma(u)}{\Gamma(\beta)} t^{-u} du, \quad (7)$$

for any $0 < c < \beta$. Pulling out a factor of $m^{\frac{k-1}{2}}$ in the denominator of Equation (6) and using Equation (7) with $t = \frac{h}{m}$, $\beta = \frac{k-1}{2}$, and $c = \varepsilon$ gives

$$T(X, h) = \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(2)} \sum_{m \geq 1} \frac{a(m)b(m+h)}{m^{s-u+k-1}h^u} \frac{\Gamma\left(\frac{k-1}{2} - u\right) \Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)} \Psi(s) X^s ds du. \quad (8)$$

We can use this analytic expression for $T(X, h)$ to obtain one for $T(X, Y)$. First observe that

$$T(X, Y) = \sum_{\frac{Y}{2} < h \leq Y} T(X, h) C(h) = \sum_{h \geq 1} T(X, h) C(h) \psi\left(\frac{h}{Y}\right).$$

Applying the smoothed version of Perron's formula again, and using Equation (8), yields

$$T(X, Y) = \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{(2)} \int_{(2)} \sum_{m \geq 1} \frac{a(m)b(m+h)c(h)}{m^{s-u+k-1}h^{w+u+\frac{k-1}{2}}} \frac{\Gamma\left(\frac{k-1}{2} - u\right) \Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)} \Psi(s) \Psi(w) X^s Y^w ds dw du. \quad (9)$$

In terms of $Z(s, w)$, Equation (9) is expressed as

$$T(X, Y) = \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{(2)} \int_{(2)} Z\left(s - u, w + u - \frac{k-1}{2}\right) \frac{\Gamma\left(\frac{k-1}{2} - u\right) \Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)} \Psi(s) \Psi(w) X^s Y^w ds dw du. \quad (10)$$

We will estimate this triple integral by shifting lines of integration. First, shift the line (2) at s to $\left(\frac{1}{2}\right)$. We pass poles coming from the multiple Dirichlet series when $s = \frac{1}{2} + u + it_j$ so that

$$\begin{aligned} T(X, Y) &= \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{(2)} \int_{\left(\frac{1}{2}\right)} Z\left(s - u, w + u - \frac{k-1}{2}\right) \Psi(s) \Psi(w) X^s Y^w ds dw du \\ &\quad + \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(2)} \sum_{t_j} R_s\left(w + u - \frac{k-1}{2}, u; 0, t_j\right) \Psi\left(\frac{1}{2} + it_j + u\right) \Psi(w) X^{\frac{1}{2}+u+it_j} Y^w dw du, \end{aligned}$$

where, using Equation (5), we have

$$\begin{aligned} R_s(w, u; \ell, t_j) &= \left[\operatorname{Res}_{s=\frac{1}{2}-\ell+it_j} Z(s, w) \right] \frac{\Gamma\left(\frac{k-1}{2} - u\right) \Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)} \\ &= \frac{(-1)^\ell}{\ell!} \frac{\Gamma(-\ell + 2it_j) \Gamma\left(\frac{1}{2} + \ell - it_j\right) \Gamma\left(\frac{k-1}{2} - u\right) \Gamma(u)}{\Gamma\left(\frac{1}{2} + it_j\right) \Gamma\left(\frac{1}{2} - it_j\right) \Gamma\left(k - \frac{1}{2} - \ell + it_j\right) \Gamma\left(\frac{k-1}{2}\right)} \frac{L\left(\frac{1}{2} - \ell + it_j + w + \frac{k}{2} - 1, u_j \otimes l\right)}{\zeta(1 - 2\ell + 2it_j + 2w + k - 2)} \\ &\quad \cdot \left\langle u_j(z), f(z) \overline{g(z)} \operatorname{Im}(z)^k \right\rangle. \end{aligned}$$

The interchange for $Z(s, w)$ implies that we may swap the roles of s and w in the double Dirichlet series inside of the integrand to obtain

$$\begin{aligned} T(X, Y) &= \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{(2)} \int_{\left(\frac{1}{2}\right)} Z\left(w - u, s + u - \frac{k-1}{2}\right) \Psi(s) \Psi(w) X^s Y^w ds dw du \\ &\quad + \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(2)} \sum_{t_j} R_s\left(w + u - \frac{k-1}{2}, u; 0, t_j\right) \Psi\left(\frac{1}{2} + it_j + u\right) \Psi(w) X^{\frac{1}{2}+u+it_j} Y^w dw du. \end{aligned}$$

Shifting the lines (2) at w to $(\frac{1}{2} + 2\varepsilon)$ in both integrals, we don't pass by any poles and obtain

$$\begin{aligned} T(X, Y) &= \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{(\frac{1}{2}+2\varepsilon)} \int_{(\frac{1}{2})} Z\left(w-u, s+u-\frac{k-1}{2}\right) \Psi(s) \Psi(w) X^s Y^w ds dw du \\ &\quad + \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\frac{1}{2}+2\varepsilon)} \sum_{t_j} R_s\left(w+u-\frac{k-1}{2}, u; 0, t_j\right) \Psi\left(\frac{1}{2}+it_j+u\right) \Psi(w) X^{\frac{1}{2}+u+it_j} Y^w dw du. \end{aligned}$$

Denote these two integrals by $I_1(X, Y)$ and $I_2(X, Y)$ respectively. We will first estimate

$$I_1(X, Y) = \frac{1}{(2\pi i)^3} \int_{(\varepsilon)} \int_{(\frac{1}{2}+2\varepsilon)} \int_{(\frac{1}{2})} Z\left(w-u, s+u-\frac{k-1}{2}\right) \frac{\Gamma\left(\frac{k-1}{2}-u\right) \Gamma(u)}{\Gamma\left(\frac{k-1}{2}\right)} \Psi(s) \Psi(w) X^s Y^w ds dw du.$$

To this end, consider the approximation to $Z(s, w)$ given by

$$Z^*(s, w) = Z(s, w) - P(s, w),$$

where

$$P(s, w) = \sum_{0 \leq \ell \leq \frac{k}{2}} \sum_{t_j} \frac{\text{Res}_{s=\frac{1}{2}-\ell+it_j} Z(s, w)}{s - (\frac{1}{2} - \ell + it_j)} e^{-(s - (\frac{1}{2}-\ell+it_j))^2} + \sum_{0 \leq \ell \leq \frac{k}{2}} \sum_{t_j} \frac{\text{Res}_{w=\frac{1}{2}-\ell+it_j} Z(s, w)}{w - (\frac{1}{2} - \ell + it_j)} e^{-(w - (\frac{1}{2}-\ell+it_j))^2}.$$

For s and w away from poles of $Z(s, w)$, $P(s, w)$ converges locally absolutely uniformly and is absolutely bounded. Indeed, Sitriling's formula and a standard Lindelöf convexity argument for $L(s, u_j \otimes l)$ together with Equation (5) imply that $\text{Res}_{s=\frac{1}{2}-\ell+it_j} Z(s, w)$ has polynomial growth in t_j and w which is dampened by the exponential decay from $e^{-(s - (\frac{1}{2}-\ell+it_j))^2}$. The same result holds with the roles of s and w interchanged. Moreover, $P(s, w)$ has the same poles and residues as $Z(s, w)$ so that $Z^*(s, w)$ is holomorphic on $\hat{\Lambda}$. In the region $(s, w) \in \Lambda'_s - \Lambda_0$, consider

$$Z(s, w) = \sum_{h \geq 1} \frac{D(s, h) c(h)}{h^{w+k-1}},$$

where

$$D(s, h) = \sum_{t_j} h^{\frac{1}{2}-s} \overline{\rho_j(-h)} \frac{\Gamma(1-s) \Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(s+k-1)} \left\langle u_j(z), f(z) \overline{g(z)} \text{Im}(z)^k \right\rangle.$$

Provided we are at least ε away from the poles $s = \frac{1}{2} + it_j$, Equation (3) gives

$$Z(s, w) \ll_{\varepsilon} \sum_{h \geq 1} \frac{(1+|s|)^N h^{\frac{1}{2}+\theta-\text{Re}(s)+\varepsilon}}{h^{w+k-1}} \ll_{\varepsilon} (1+|s|)^N \zeta\left(\frac{k}{2} - \theta\right) \ll (1+|s|)^N,$$

for some $N \geq 1$. As $P(s, w)$ is absolutely bounded away from the poles of $Z(s, w)$ and $Z^*(s, w)$ is holomorphic, estimates for $Z(s, w)$ and away from poles will hold for $Z^*(s, w)$ as well. Thus

$$Z^*(s, w) \ll_{\varepsilon} (1+|s|)^N,$$

in the same region. If $(s, w) \in \Lambda_0$, then $Z^*(s, w) \ll 1$. These two estimate together with the Phragmen-Lindelöf convexity principle imply

$$Z^*(s, w) \ll_{\varepsilon} (1+|s|)^{\frac{2N}{3-k}(1-\text{Re}(s))},$$

for $(s, w) \in \Lambda_s$. Repeating this argument with the roles of s and w interchanged, we have

$$Z^*(s, w) \ll_{\varepsilon} (1+|w|)^{\frac{2N}{3-k}(1-\text{Re}(w))},$$

for $(s, w) \in \Lambda_w$. The same two estimates hold for $Z(s, w)$ provided we are ε away from the poles. By the Phragmen-Lindelöf convexity principle again, we obtain

$$Z(s, w) \ll_{\varepsilon} (1 + |s| + |w|)^{\frac{2N}{3-k} \min(1-\operatorname{Re}(s), 1-\operatorname{Re}(w))}, \quad (11)$$

for $(s, w) \in \widehat{\Lambda}$ and at least ε away from the poles. Using the interchange, the exact same estimate holds for $Z(w, s)$. Indeed,

$$Z(s, w) \ll_{\varepsilon} (1 + |s| + |w|)^{\frac{2N}{3-k} (\frac{k-3}{2} - \varepsilon)},$$

Applying Equation (11) to $I_1(X, Y)$ yields

$$I_1(X, Y) \ll_{\varepsilon} \int_{(\varepsilon)} \int_{(\frac{1}{2}+2\varepsilon)} \int_{(\frac{1}{2})} (1 + |s| + |w|)^{\frac{2N}{3-k} (\frac{k-3}{2} - \varepsilon)} \frac{\Gamma(\frac{k-1}{2} - u) \Gamma(u)}{\Gamma(\frac{k-1}{2})} \Psi(s) \Psi(w) X^s Y^w ds dw du,$$

From Sitrling's formula, the ratio of gamma factors grows like $(1 + |u|)^{-\frac{1}{2}}$ in vertical strips and so is polynomially bounded in $|u|$. Since ψ is compactly supported and smooth, $\Psi(s) \ll (1 + |s|)^{-N}$ for any $N \geq 1$. So up to a constant, $\Psi(s)$ and $\Psi(w)$ truncate the integrals over s and w to something of bounded support. It follows that the triple integral is uniformly bounded and we have

$$I_1(X, Y) \ll X^{\frac{1}{2}} Y^{\frac{1}{2}+2\varepsilon}. \quad (12)$$

Next we estimate

$$I_2(X, Y) = \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\frac{1}{2}+2\varepsilon)} \sum_{t_j} R_s \left(w + u - \frac{k-1}{2}, u; 0, t_j \right) \Psi \left(\frac{1}{2} + it_j + u \right) \Psi(w) X^{\frac{1}{2}+u+it_j} Y^w dw du.$$

Opening up the residue factor gives

$$I_2(X, Y) = \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\frac{1}{2}+2\varepsilon)} \sum_{t_j} \frac{\Gamma(2it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(\frac{k-1}{2} - u) \Gamma(u)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(k - \frac{1}{2} + it_j) \Gamma(\frac{k-1}{2})} \frac{L(w + u + it_j, u_j \otimes l)}{\zeta(2w + 2u + 2it_j)} \\ \cdot \left\langle u_j(z), f(z) \overline{g(z)} \operatorname{Im}(z)^k \right\rangle \Psi(s) \Psi(w) X^{\frac{1}{2}+u+it_j} Y^w dw du.$$

We estimate the integrand similar to how we did for $I_1(X, Y)$. Sitrling's formula implies that the ratio of gamma factors grows like $(1 + |t_j|)^{\frac{1}{2}-k} (1 + |u|)^{-\frac{1}{2}}$ in vertical strips and is polynomially bounded in both $|t_j|$ and $|u|$. Moreover, the zeta function in the denominator is absolutely bounded because $\operatorname{Re}(2w + 2u) = 1 + 6\varepsilon$. As for the Rankin-Selberg convolution, a standard convexity argument and the functional equation together imply $L(w + u + it_j, u_j \otimes l)$ is polynomially bounded in $|w|$, $|u|$, and $|t_j|$ in vertical strips. As before, the factors $\Psi(s)$ and $\Psi(w)$ truncate the integrals over s and w and the sum over t_j to something of bounded support. Thus the double integral and sum are uniformly bounded which gives

$$I_2(X, Y) \ll X^{\frac{1}{2}+\varepsilon} Y^{\frac{1}{2}+2\varepsilon}. \quad (13)$$

Combining Equations (12) and (13), we conclude that

$$T(X, Y) = \sum_{\substack{\frac{X}{2} < m \leq X \\ \frac{Y}{2} < h \leq Y}} A(m) B(m+h) C(h) \ll X^{\frac{1}{2}+\varepsilon} Y^{\frac{1}{2}+\varepsilon}.$$

Of course, a separate analysis of the continuous spectrum is required for a complete proof. However, this contribution is smaller than that of the discrete spectrum which constitutes the main term.

APPENDIX A. A RANKIN-SELBERG CONVOLUTION

In this appendix we deduce the functional equation and analytic continuation of the Rankin-Selberg convolution

$$L(s, u_j \otimes l) = \zeta(2s) \sum_{h \geq 1} \frac{\overline{\rho_j(-h)} c(h)}{h^{s + \frac{k-1}{2}}},$$

between the Maass cusp form $u_j(z)$ and holomorphic cusp form $l(z)$. Let $\lambda_j = \frac{1}{4} + t_j^2$ be the Laplace eigenvalue of $u_j(z)$. Begin by considering the integral

$$\int_{\Gamma_\infty \backslash \mathbb{H}} u_j(z) \overline{l(z)} \operatorname{Im}(z)^{s + \frac{k}{2}} d\mu.$$

Expanding the forms into their Fourier series and interchanging sums we can compute

$$\begin{aligned} \int_{\Gamma_\infty \backslash \mathbb{H}} u_j(z) \overline{l(z)} \operatorname{Im}(z)^{s + \frac{k}{2}} d\mu &= \int_0^\infty \int_0^1 \sum_{\substack{h \neq 0 \\ m \geq 1}} c(m) \overline{\rho_j(-h)} y^{s + \frac{k+1}{2}} K_{it_j}(2\pi|h|y) e^{2\pi i h x} e^{-2\pi i m(x-iy)} \frac{dx dy}{y^2} \\ &= \int_0^\infty \sum_{h \geq 1} c(h) \overline{\rho_j(-h)} y^{s + \frac{k+1}{2}} K_{it_j}(2\pi h y) e^{-2\pi h y} \frac{dy}{y^2} \\ &= \sum_{h \geq 1} c(h) \overline{\rho_j(-h)} \int_0^\infty K_{it_j}(2\pi h y) e^{-2\pi h y} y^{s + \frac{k-1}{2}} \frac{dy}{y} \\ &= \frac{1}{(2\pi)^{s + \frac{k-1}{2}}} \sum_{h \geq 1} \frac{c(h) \overline{\rho_j(-h)}}{h^{s + \frac{k-1}{2}}} \int_0^\infty K_{it_j}(y) e^{-y} y^{s + \frac{k-1}{2}} \frac{dy}{y} \\ &= \frac{1}{(2\pi)^{s + \frac{k-1}{2}}} \frac{L(s, u_j \otimes l)}{\zeta(2s)} \int_0^\infty K_{it_j}(y) e^{-y} y^{s + \frac{k-1}{2}} \frac{dy}{y}. \end{aligned}$$

Applying the well-known transform (also stated in [8]),

$$\int_0^\infty K_{it_j}(y) e^{-y} y^{s + \frac{k-1}{2}} \frac{dy}{y} = \sqrt{\pi} 2^{\frac{1}{2}-s} \frac{\Gamma(s + \frac{k-1}{2} + it_j) \Gamma(s + \frac{k-1}{2} - it_j)}{\Gamma(s + \frac{k}{2})},$$

we arrive at

$$\int_{\Gamma_\infty \backslash \mathbb{H}} l(z) u_j(z) \operatorname{Im}(z)^{s + \frac{k}{2}} d\mu = \frac{\Gamma(s + \frac{k-1}{2} + it_j) \Gamma(s + \frac{k-1}{2} - it_j)}{2^s (2\pi)^{s + \frac{k}{2} - 1} \Gamma(s + \frac{k}{2}) \zeta(2s)} L(s, u_j \otimes l).$$

On the other hand, folding the integral yields

$$\begin{aligned} \int_{\Gamma_\infty \backslash \mathbb{H}} l(z) u_j(z) \operatorname{Im}(z)^{s + \frac{k}{2}} d\mu &= \int_{\mathcal{F}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} l(\gamma z) u_j(\gamma z) \operatorname{Im}(\gamma z)^{s + \frac{k}{2}} d\mu \\ &= \int_{\mathcal{F}} l(z) u_j(z) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, z)^k \operatorname{Im}(\gamma z)^{s + \frac{k}{2}} d\mu \\ &= \int_{\mathcal{F}} l(z) u_j(z) \operatorname{Im}(z)^{\frac{k}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left(\frac{j(\gamma, z)}{|j(\gamma, z)|} \right)^k \operatorname{Im}(\gamma z)^s d\mu \\ &= \int_{\mathcal{F}} l(z) u_j(z) \operatorname{Im}(z)^{\frac{k}{2}} E_\infty(z, s; k) d\mu, \end{aligned}$$

where we have introduced the Eisenstein series

$$E_\infty(z, s; k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left(\frac{j(\gamma, z)}{|j(\gamma, z)|} \right)^k \text{Im}(\gamma z)^s.$$

From [8], the Fourier series can be expressed as

$$E_\infty(z, s; k) = \sum_{n \in \mathbb{Z}} a(n, y, s; k) e^{2\pi i n x},$$

where

$$a(n, y, s) = \begin{cases} y^s \frac{\Gamma(s + \frac{k}{2})}{\Gamma(s)} + y^{1-s} \frac{\Gamma((1-s) + \frac{k}{2}) \Lambda(2s-1, \zeta)}{\Gamma(1-s) \Lambda(2s, \zeta)} & \text{if } n = 0, \\ i^k \frac{|n|^{s-1} \sigma_{1-2s}(|n|)}{\Lambda(2s, \zeta)} W_{\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y) & \text{if } n \geq 1, \\ i^k \frac{|n|^{s-1} \sigma_{1-2s}(|n|) \Gamma(s + \frac{k}{2})}{\Gamma(s - \frac{k}{2}) \Lambda(2s, \zeta)} W_{-\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y) & \text{if } n \leq -1, \end{cases}$$

where $W_{\alpha, \nu}(y)$ is the Whittaker function. If we multiply the Eisenstein series by $\Gamma(s)\Gamma(1-s)\Lambda(2s, \zeta)$ and use the functional equation for $\Lambda(2s-1, \zeta)$, then the $n = 0$ coefficient becomes

$$y^s \Gamma(1-s) \Gamma\left(s + \frac{k}{2}\right) \Lambda(2s, \zeta) + y^{1-s} \Gamma(s) \Gamma\left((1-s) + \frac{k}{2}\right) \Lambda(2(1-s), \zeta),$$

which is invariant under $s \rightarrow 1-s$ because the two terms reflect into each other. Therefore the completed Eisenstein series

$$E_\infty^*(z, s; k) = \Gamma(s)\Gamma(1-s)\Lambda(2s, \zeta)E_\infty(z, s; k),$$

is invariant under $s \rightarrow 1-s$. Then

$$\Gamma(s)\Gamma(1-s)\Lambda(2s, \zeta) \int_{\mathcal{F}} l(z) u_j(z) \text{Im}(z)^{\frac{k}{2}} E_\infty(z, s; k) d\mu = \int_{\mathcal{F}} l(z) u_j(z) \text{Im}(z)^{\frac{k}{2}} E_\infty^*(z, s; k) d\mu.$$

Now define the completed Rankin-Selberg convolution

$$L^*(s, u_j \otimes l) = \frac{\Gamma\left(s + \frac{k-1}{2} + it_j\right) \Gamma\left(s + \frac{k-1}{2} - it_j\right) \Gamma(s)\Gamma(1-s)\Lambda(2s, \zeta)}{2^s (2\pi)^{s+\frac{k}{2}-1} \Gamma\left(s + \frac{k}{2}\right) \zeta(2s)} L(s, u_j \otimes l).$$

Then the invariance of $E_\infty^*(z, s; k)$ under $s \rightarrow 1-s$ implies

$$L^*(s, u_j \otimes l) = L^*(1-s, u_j \otimes l).$$

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