RESEARCH STATEMENT

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Background: I am intersted in studying the simultaneous non-vanshing of products of elliptic L-functions over function fields and twisted by quadratic Dirichlet characters, namely $L(s_1, E_1 \times \chi_d)L(s_2, E_2 \times \chi_d)$, at the central value $s = \frac{1}{2}$. The function field $\mathbb{F}_q(t)$ is the field of rational functions in the variable t with coefficients in the finite field $\mathbb{F}_q$ of q elements. An elliptic curve E over $\mathbb{F}_q(t)$ is the set of solutions (x, y) to the cubic equation

$$y^2 = x^3 + ax + b,$$

where $a, b \in \mathbb{F}_q(t)$. A quadratic Dirichlet character χ_d, with $d \in \mathbb{F}_q[t]$, is a d-periodic multiplicative function whose non-zero values are $\{\pm 1\}$. The elliptic L-function $L(s, E \times \chi_d)$ is a complex function that encodes the arithmetic information about the elliptic curve E and quadratic Dirichlet character χ_d *analytically* into itself. These functions are required to satify Euler products, a functional equation as $s \to 1 - s$ and other regularity properties. Studying the arithmetic information of E is then amenable to understanding the analytic properties of $L(s, E \times \chi_d)$. Similary, there are the L-functions $L(s, E)$ and $L(s, \chi_d)$. This idea of passing from arithmetic to analytic investigations has its roots traced back to Dirichlet and the Riemann zeta function. In the setting of elliptic curves, there is the infamous Birch–Swinnerton-Dyer conjecture which claims that the rank of E is equal to the order of vanishing of $L(s, E)$ at the special value $s = 1$. It is expected that any L-function only vanishes for either trivial or very good reasons at the special values $s = \frac{1}{2}$ and $s = 1$. Simultaneous non-vanshing results are those which state that many L-functions do not vanish at a special value. Complex analytic techniques combined with simultaneous non-vanshing results allow for the analytic non-vanishing information to be translated into arithmetic information leading to often astounding results. For example, non-vanshing at $s = 1$ well-known to be closely connected to primes in arithmetic progressions and prime number theorems (see [1]) while non-vanshing at $s = \frac{1}{2}$ is related to the nonvanishing of theta lifting and the nonexistence of Landau–Siegel zeros (see Todo: [cite]). These applications are responsible for simultaneous non-vanshing drawing much attention in recent years.

Proposal: An idea arose in the 1980's that studying an average over a family of L-functions would provide insight about each individal L-function in the family (see Todo: [cite]). The first instance of this construction was to average the family of quadratic Dirichlet L-functions into a multiple Dirichlet series roughly of shape

$$Z(s, w) = \sum_{d} \frac{L(s, \chi_d)P_d(s)}{|d|^w} = \sum_{m} \frac{L(w, \chi_m)P_m(w)}{|m|^s},$$

where $P_d(s)$ are certain explicit correction polynomials possessing functional equations of shape $s \to 1 - s$. Variations of this series were studied by Todo: [cite]. The multiple Dirichlet series inherits functional equations as $s \to 1 - s$ and $w \to 1 - w$ from those of $L(s, \chi_d)$ and $L(w, \chi_m)$. Composing these functional equations, $Z(s, w)$ satisfies a group of functional equations that is isomorphic to the Weyl group of an A_2 type root system coming from representation theory. This group is isomorphic to the symmetric group S_3 (the symmetries of a triangle). As an almost immediate consquence, $Z(s, w)$ admits analytic continuation to $\mathbb{C}^2$ with a pole at $w = 1$. Taking the residue of $Z\left(\frac{1}{2}, w\right)$ at $w = 1$ and applying a contour integral roughly yields an asymptotic of shape

$$\sum_{|d| \ll X} L\left(\frac{1}{2}, \chi_d\right) P_d\left(\frac{1}{2}\right) = AX \log(X) + BX + o(X),$$

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for some nonzero constants A and B. This implies simultaneous non-vanshing for infinitely many quadratic Dirichlet L-functions $L(s,\chi_d)$ at the central value $s=\frac{1}{2}$. A similar procedure can be done for elliptic L-functions. **I will prove simultaneous non-vanshing for the product of two elliptic L-functions twisted by quadratic Dirichlet characters over function fields at the central value $s=\frac{1}{2}$. Precisely, for any two elliptic curves E_1 and E_2 over $\mathbb{F}_q(t)$ there exists infinitely many $d\in\mathbb{F}_q[t]$ such that**

$$L\left(\frac{1}{2},E_1\times\chi_d\right)L\left(\frac{1}{2},E_2\times\chi_d\right)\neq 0.$$

The underlying idea is to construct a multiple Dirichlet series of shape

$$Z(s_1,s_3,s_5)=\sum_d\frac{L(s_1,E_1\times\chi_d)L(s_3,E_2\times\chi_d)P_d(s_1,s_3;E_1,E_2)}{|d|^{s_5}},$$

where the $P_d(s_1,s_3;E_1,E_2)$ are again explicit correction polynomials, and show that it admits analytic continuation to a region containing the point $\left(\frac{1}{2},\frac{1}{2},1\right)$ with a pole at $s_3=1$. Similar analytic techniques to those for $Z(s,w)$ can then be applied to obtain the simultaneous non-vanishing. Therefore, the key insight is **the analytic continuation of $Z(s_1,s_2,s_3)$ to a suitably large enough domain**. It can be shown that $Z(s_1,s_2,s_3)$ exhibits a group of functional equations isomorphic to the symmetric group S_4. Using these functional equations, $Z(s_1,s_2,s_3)$ exhibits analytic continuation to a tube domain inside $\mathbb{C}^3$ but this domain does not contain the point $\left(\frac{1}{2},\frac{1}{2},1\right)$. To overcome this issue, we must exploit the Euler product expressions for elliptic L-functions to write the four-fold product

$$L(s_1,E_1\times\chi_d)L(s_3,E_2\times\chi_d)=\prod_p\prod_{i=1,3}(1-\alpha_{i,d}(p)p^{-s_i})^{-1}(1-\beta_{i,d}(p)p^{-s_i})^{-1},$$

for some complex numbers $\alpha_{i,d}(p)$ and $\beta_{i,d}(p)$ and over all irreducible moinic polynomials p. We now stratify the four-fold product into four different variables to write

$$Z(s_1,\ldots,s_5)=\sum_d\frac{\prod_p\prod_{1\leq i\leq 4}(1-\gamma_{i,d}(p)p^{-s_i})^{-1}P_d(s_1,\ldots,s_4;E_1,E_2)}{|d|^{s_5}},$$

for some correction polynomials $P_d(s_1,\ldots,s_4;E_1,E_2)$, and where $\gamma_{i,d}(p)\in\{\alpha_{1,d}(p),\beta_{1,d}(p),\alpha_{3,d}(p),\beta_{3,d}(p)\}$. We have the specialization condition

$$Z(s_1,s_1,s_3,s_3,s_5)=Z(s_1,s_3,s_5),\tag{1}$$

and so we must continue $Z(s_1,\ldots,s_5)$ to a region containing $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1\right)$. It is this analytic continuation which can be acomplished. However, the continuation requires an extremely non-trivial argument. The continuation of Z mimics that of the multiple Dirichlet series

$$Z_\chi(s_1,\ldots,s_5)=\sum_d\frac{\prod_{1\leq i\leq 4}L(s_i,\chi_d)P_d(s_1,\ldots,s_4)}{|d|^{s_5}},$$

where $P_d(s_1,\ldots,s_4)$ are certain explicit correction polynomials. The group of functional equations posesed by Z and Z_χ is an infinite Weyl group attached to the affine root system $\widetilde{D_4}$. As a consequence, the polar lines of these multiple Dirichlet series accumulate along the hyperplane $s_1+s_2+s_3+s_4+2s_5=-1$ and create a natural barrier to analytic continuation. The point $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1\right)$ occures before the natural barrier but after the region of continuation obtained by applying the functional equations. That is, it lies in a *dead zone* where continuation should be possible but cannot be obtained by the functional equations directly. In a *tour-de-force* paper, Diaconu-Pasol-Popa (see Todo: [cite]) obtained the analytic continuation of Z_χ to the dead zone by discovering an extra functional equation. This extra functional equation implies that Z_χ factors as

$$Z_\chi(s_1,\ldots,s_5)=F(q^{-s_1-s_2-s_3-s_4-2s_5})Z_W(s_1,\ldots,s_5),$$

where F is a power series and Z_W is a multiple Dirichlet series posessing functional equations isomorphic to Z_χ but is not amenable to the study of L-functions. The series Z_W is known as the Chinta-Gunnells averge (see Todo: [cite]) and the extra functional equation implies that Z_W admits analytic continuation to the natural barrier. From the decomposition, so does Z_χ.

The above idea can be bootstrapped to analytically continue Z which I now outline. In order to ensure $Z(s_1, \ldots, s_5)$ possesses functional equations isomorphic to the Weyl group of $\widetilde{D_4}$ and can be analytically continued to the natural barrier, we must find a formula which relates Z to Z_χ. Such a formula will necessarily depend upon the elliptic curves E_1 and E_2 because Z_χ does not see this data. To find such a formula, we first need to determine the correction polynomials $P_d(s_1, \ldots, s_4; E_1, E_2)$. In order to acomplish this, we adapt the argument in Todo: [cite] to express $Z(s_1, s_3, s_5)$ as:

$$Z(s_1, s_3, s_5) = \sum_{\substack{m_1, m_3 \\ m_1 m_3 = k_0 k_1^2}} \frac{L(s_5, \chi_{m_1 m_3}) P_{n_0 n_1^2}(s_5; E_1, E_2)}{|m_1|^{s_1} |m_3|^{s_3}},$$

where $P_{k_0 k_1^2}(s_5; E_1, E_2)$ is an explicit correction polynomial. Analgously, the same argument may be used to write

$$Z(s_1, s_2, s_3, s_4, s_5) = \sum_{\substack{m_1, m_2, m_3, m_4 \\ m_1 m_2 m_3, m_4 = k_0 k_1^2}} \frac{L(s_5, \chi_{m_1 m_2 m_3 m_4}) P_{n_0 n_1^2}(s_5; E_1, E_2)}{|m_1|^{s_1} |m_2|^{s_2} |m_3|^{s_3} |m_4|^{s_4}},$$

where in this case, $P_{n_0 n_1^2}(s_5; E_1, E_2)$ is some yet to be determined correction polynomial. Miraculously, the specalization condition in Equation (1) implies a recursive relation on the coefficients of these two series which can be written in the form

$$P_{n_0 n_1^2}(s_5; E_1, E_2) = \frac{1}{L(s_5, \chi_{m_1 m_2 m_3 m_4})} \sum_{\substack{m_1 m_2 m_3 m_4 = m_1' m_3' \\ m_1' m_3' = k_0 k_1^2}} L(s_5, \chi_{m_1 m_3}) P_{k_0 k_1^2}(s_5; E_1, E_2).$$

Since the $P_{k_0 k_1^2}(s_5; E_1, E_2)$ are known correction polynomials, this recursive formula allows us to construct the $P_{n_0 n_1^2}(s_5; E_1, E_2)$. The interchange property, namely

$$\sum_{\substack{m_1, m_2, m_3, m_4 \\ m_1 m_2 m_3, m_4 = k_0 k_1^2}} \frac{L(s_5, \chi_{m_1 m_2 m_3 m_4}) P_{n_0 n_1^2}(s_5; E_1, E_2)}{|m_1|^{s_1} |m_2|^{s_2} |m_3|^{s_3} |m_4|^{s_4}} = \sum_d \frac{\prod_p \prod_{1 \le i \le 4} (1 - \gamma_{i,d}(p) p^{-s_i})^{-1} P_d(s_1, \ldots, s_4; E_1, E_2)}{|d|^{s_5}},$$

can then be used to determine the correction polynomials $P_d(s_1, \ldots, s_4; E_1, E_2)$ exactly. This furnishes the analytic continuation of $Z(s_1, \ldots, s_5)$ via its group of functional equations isomorphic to $\widetilde{D_4}$...

Broader Impacts:

References

[1] Murty, M. Ram, and V. Kumar Murty. Non-vanishing of L-functions and applications. Springer Science & Business Media, 2012.