MODULARITY FOR THE QUADRATIC THETA FUNCTION

HENRY TWISS

Abstract. The prototypical half-integral weight modular form is the quadratic theta function $\theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}$. While is it often mentioned that the factor of modularity is the theta multiplier $\left(\frac{c}{d}\right)\varepsilon_d^{-1}(cz+d)^{\frac{1}{2}}$, there is not an easily available modern proof. This purpose of this paper is to present a full proof and show that $\theta(z)$ is a weight $\frac{1}{2}$ modular form on $\Gamma_0(4)\backslash\mathbb{H}$.

1. Preliminaries

Let $N \geq 1$ and $k \geq 1$ be odd. A modular form $f : \mathbb{H} \to \mathbb{C}$ of half-integral weight $\frac{k}{2}$ on $\Gamma_0(4N)\backslash\mathbb{H}$ is a function satisfying the following properties:

(i) Holomorphy: f is holomorphic on $\mathbb{H}$,
(ii) Modularity: $f(\gamma z) = \left(\frac{c}{d}\right)\varepsilon_d^{-k}(cz+d)^{\frac{k}{2}} f(z)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$,
(iii) Holomorphy at the cusps: f is holomorphic at the cusps of $\Gamma_0(4N)\backslash\mathbb{H}$ and f is a cuspform if it decays to zero near the cusps.

Condition (ii) is the most important property. The factor of modularity

$$j(\gamma, z) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}(cz+d)^{\frac{1}{2}},$$

is called the theta multiplier. For the theta multiplier, we take the principal branch of the square root (we will always take the principal branch from now on), $\left(\frac{c}{d}\right)$ is the Jacobi symbol (d is necessarily odd since $\det(\gamma) = 1$ and $c \equiv 0 \pmod 4$) with the stipulations $\left(\frac{c}{d}\right) = \left(\frac{-c}{-d}\right)$ (so that we may take $d > 0$) and $\left(\frac{0}{d}\right) = 1$. This last requirement is simply $\left(\frac{0}{1}\right) = 1$ because if $c = 0$ then $\det(\gamma) = 1$ implies $d = \pm 1$. Also, we set $\varepsilon_d = 1, i$ depending on if $d \equiv 1, 3 \pmod 4$.

We define the quadratic theta function $\theta(z)$ by

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}.$$

It obeys a simple relationship to Jacobi's theta function $\vartheta(z) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 z}$:

$$\theta\left(\frac{iz}{2}\right) = \vartheta(z), \tag{1}$$

for $z \in \mathbb{H}$. It is well-known that $\theta(z)$ is holomorphic on $\mathrm{Re}(z) > 0$, and so $\theta(z)$ is holomorphic on $\mathbb{H}$. Moreover, Jacobi's theta function obeys the transformation law

$$\vartheta(z) = \frac{1}{\sqrt{z}}\vartheta\left(\frac{1}{z}\right). \tag{2}$$

2. Modularity of $\theta(z)$

: Poisson summation is the main ingredient needed to show $\theta(z)$ is modular on $\Gamma_0(4)\backslash\mathbb{H}$ of weight $\frac{1}{2}$ with factor of modularity given by the theta multiplier.

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Theorem 2.1. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$. *Then for* $z \in \mathbb{H}$,

$$\theta(\gamma z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz+d}\,\theta(z).$$

Proof. Recall that

$$\Gamma_0(4) = \left\langle \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \quad \text{and} \quad \Gamma_0(2) = \left\langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle. \tag{3}$$

We first argue

$$\Gamma_0(4) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma(2) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

This decomposition follows by Equation (3) and the computations

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

We will now make two reductions to for the modularity condition. For the first reduction, define $\widetilde{\theta}(z) = \theta\left(\frac{z}{2}\right)$. Since $\theta(z)$ is 1-periodic, $\widetilde{\theta}(z)$ is 2-periodic. Moreover, for any $\gamma \in \Gamma_0(4)$, let $\eta = \begin{pmatrix} a & 2b \\ \frac{c}{2} & d \end{pmatrix} \in \Gamma(2)$ so that

$$\gamma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \eta \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\theta(\gamma z) = \widetilde{\theta}\left(2\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \eta \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} z\right) = \widetilde{\theta}(2\eta z). \tag{4}$$

It now suffices to prove a compatible modularity condition for $\widetilde{\theta}(\eta z)$. This is the first reduction. For the second reduction, let $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) z = -\frac{1}{z}$. By Equation (2), we see that

$$\widetilde{\theta}(w) = \vartheta(-iw) = \frac{1}{\sqrt{-iw}}\vartheta\left(-\frac{1}{iw}\right) = \frac{1}{\sqrt{-iw}}\widetilde{\theta}\left(\frac{-1}{w}\right) = \sqrt{\frac{z}{i}}\widetilde{\theta}(z).$$

Since $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^2 = I$, if we let $\eta' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ where η' is defined by

$$\eta' = \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2b & -a \\ d & -\frac{c}{2} \end{pmatrix},$$

then

$$\widetilde{\theta}(\eta z) = \widetilde{\theta}\left(\eta \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}^2 z\right) = \widetilde{\theta}(\eta' w). \tag{5}$$

By Equations (4) and (5) it further suffices to prove a compatible modularity condition for $\widetilde{\theta}(\eta' w)$. This is the second reduction and is what we will prove. To apply Poisson summation we need to sieve out an additive character in $\widetilde{\theta}(\eta' w)$. To acomplish this, observe

$$\eta' w = \frac{a'w + b'}{c'w + d'} = \frac{c'(a'w + b')}{c'(c'w + d')} = \frac{a'c'w + a'd' - 1}{c'(c'w + d')} = \frac{a'(c'w + d') - 1}{c'(c'w + d')} = \frac{a'}{c'} - \frac{1}{c'(c'w + d')}.$$

We can now sieve out the character:

$$\begin{aligned}
\widetilde{\theta}(\eta' w) &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \eta' w} \\
&= \sum_{n\in\mathbb{Z}} e^{\pi i n^2\left(\frac{a'}{c'} - \frac{1}{c'(c'w+d')}\right)} \\
&= \sum_{n\in\mathbb{Z}} e^{\frac{2\pi i n^2 \frac{a'}{2}}{c'}} e^{-\frac{\pi i n^2}{c'(c'w+d')}} \\
&= \sum_{\alpha \pmod{c'}} e^{\frac{2\pi i n^2 \frac{a'}{2}}{c'}} \sum_{m\in\mathbb{Z}} e^{-\frac{\pi i (c'm+\alpha)^2}{c'(c'w+d')}} \\
&= \sum_{\alpha \pmod{c'}} e^{\frac{2\pi i \alpha^2 \frac{a'}{2}}{c'}} \sum_{m\in\mathbb{Z}} e^{-\frac{\pi i c'\left(m+\frac{\alpha}{c'}\right)^2}{c'w+d'}},
\end{aligned} \tag{6}$$

where the second to last equality follows because $e^{\frac{2\pi i n^2 \frac{a'}{2}}{c'}}$ only depends on n modulo c'. Indeed, $\frac{a'}{2} = b$ is an integer and writing $n = mc' + \alpha$ with α taken modulo c' we have $(mc'+\alpha)^2 \equiv \alpha^2 \pmod{c'}$. We will apply Poisson summation to the sum

$$\sum_{m\in\mathbb{Z}} e^{-\frac{\pi i c'\left(m+\frac{\alpha}{c'}\right)^2}{c'w+d'}}.$$

By the identity theorem, it suffices to prove a transformation formula for $\widetilde{\theta}(\eta w)$ on a set containing a limit point. We will prove this on a vertical line in $\mathbb{H}$, so set $w = \frac{iy-d'}{c'}$ for $y > 0$. Now set

$$f(x) = e^{-\frac{\pi i c'\left(x+\frac{\alpha}{c'}\right)^2}{c'w+d'}},$$

and observe $f(x)$ is a Schwarz function. We compute its Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x}\,dx = \int_{-\infty}^{\infty} e^{-\frac{\pi i c'\left(x+\frac{\alpha}{c'}\right)^2}{c'w+d'}} e^{-2\pi i t x}\,dx = \int_{-\infty}^{\infty} e^{-\pi\left(\frac{ic'\left(x+\frac{\alpha}{c'}\right)^2}{c'w+d'} + 2itx\right)}\,dx. \tag{7}$$

Making the change of variables $x \to \sqrt{-\frac{i(c'w+d')}{c'}}x - \frac{\alpha}{c'}$ the last integral in Equation (7) gives

$$\sqrt{-\frac{i(c'w+d')}{c'}} e^{\frac{2\pi i t\alpha}{c'}} \int_{-\infty}^{\infty} e^{-\pi\left(x^2 + 2itx\sqrt{-\frac{i(c'w+d')}{c'}}\right)}\,dx = \sqrt{\frac{c'w+d'}{ic'}} e^{\frac{2\pi i t\alpha}{c'}} \int_{-\infty}^{\infty} e^{-\pi\left(x^2 + 2itx\sqrt{-\frac{i(c'w+d')}{c'}}\right)}\,dx. \tag{8}$$

Complete the square in the exponent of the last integral in Equation (8):

$$-\pi\left(x^2 + 2itx\sqrt{-\frac{i(c'w+d')}{c'}}\right) = -\pi\left(\left(x + it\sqrt{-\frac{i(c'w+d')}{c'}}\right)^2 - \frac{it^2(c'w+d')}{c'}\right),$$

This implies that the last integral in Equation (8) is equal to

$$\sqrt{\frac{c'w+d'}{ic'}} e^{\frac{2\pi i t\alpha}{c'}} e^{\frac{\pi i t^2(c'w+d')}{c'}} \int_{-\infty}^{\infty} e^{-\pi\left(x+it\sqrt{-\frac{i(c'w+d')}{c'}}\right)^2}\,dx.$$

The change of variables $x \to x - it\sqrt{-\frac{i(c'w+d')}{c'}}$ is valid by complexifying the integral, noting that the integrand is entire, and shifting the line of integration. This results in

$$\sqrt{\frac{c'w+d'}{ic'}}e^{\frac{2\pi it\alpha}{c'}}e^{\frac{\pi it^2(c'w+d')}{c'}}\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = \sqrt{\frac{c'w+d'}{ic'}}e^{\frac{2\pi it\alpha}{c'}}e^{\frac{\pi it^2(c'w+d')}{c'}}.$$

where equality follows because the remaining integral is the Gaussian integral. In conclusion,

$$\hat{f}(t) = \sqrt{\frac{c'w+d'}{ic'}}e^{\frac{2\pi it\alpha}{c'}}e^{\frac{\pi it^2(c'w+d')}{c'}}. \tag{9}$$

Poisson summation along with Equations (6) and (9) give the first and second equalities in the following chain:

$$\begin{aligned}
\widetilde{\theta}(\eta'w) &= \sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\alpha^2\frac{a'}{2}}{c'}}\sum_{m\in\mathbb{Z}} f(m)\\
&= \sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\alpha^2\frac{a'}{2}}{c'}}\sum_{t\in\mathbb{Z}} \hat{f}(t)\\
&= \sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\alpha^2\frac{a'}{2}}{c'}}\sum_{t\in\mathbb{Z}} \sqrt{\frac{c'w+d'}{ic'}}e^{\frac{2\pi it\alpha}{c'}}e^{\frac{\pi it^2(c'w+d')}{c'}}\\
&= \sqrt{\frac{c'w+d'}{ic'}}\sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\alpha^2\frac{a'}{2}}{c'}}\sum_{t\in\mathbb{Z}} e^{\frac{2\pi it\alpha}{c'}}e^{\frac{\pi it^2(c'w+d')}{c'}}\\
&= \sqrt{\frac{c'w+d'}{ic'}}\sum_{t\in\mathbb{Z}} e^{\frac{\pi it^2(c'w+d')}{c'}}\sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\alpha^2\frac{a'}{2}}{c'}}e^{\frac{2\pi it\alpha}{c'}}\\
&= \sqrt{\frac{c'w+d'}{ic'}}\sum_{t\in\mathbb{Z}} e^{\pi it^2w}\sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\left(\alpha^2\frac{a'}{2}+t\alpha+t^2\frac{d'}{2}\right)}{c'}}.
\end{aligned} \tag{10}$$

Now $\det(\eta') = 1$ implies $a'd' \equiv 1 \pmod{c'}$. In other words, a' and d' are inverses modulo c'. Then $\frac{a'}{2}(\alpha+td')^2 \equiv \alpha^2\frac{a'}{2}+t\alpha+t^2\frac{d'}{2} \pmod{c'}$, so the last line in Equation (10) is equal to

$$\sqrt{\frac{c'w+d'}{ic'}}\sum_{t\in\mathbb{Z}} e^{\pi it^2w}\sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\frac{a'}{2}(\alpha+td')^2}{c'}}.$$

For fixed t, $\alpha \to \alpha - td'$ is a bijection on $\mathbb{Z}/c'\mathbb{Z}$. Since, $\det(\eta') = 1$ implies $(d', c') = 1$ so that $\alpha \to \alpha d'$ is a bijection on $\mathbb{Z}/c'\mathbb{Z}$ too. Moreover, $a'd' \equiv 1 \pmod{c'}$. These facts justify the first three equalities in the following chain:

$$\sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\frac{a'}{2}(\alpha+td')^2}{c'}} = \sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\frac{a'}{2}\alpha^2}{c'}} = \sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\frac{a'}{2}(\alpha d')^2}{c'}} = \sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\frac{d'}{2}\alpha^2}{c'}} = g\left(\frac{d'}{2}, c'\right). \tag{11}$$

The last equality is the definition of the Gauss sum. The quadratic Gauss sum has explicit evaulation:

$$g\left(\frac{d'}{2}, c'\right) = \left(\frac{\frac{d'}{2}}{c'}\right)\varepsilon_{c'}\sqrt{c'}. \tag{12}$$

Equations (11) and (12) together imply

$$\sqrt{\frac{c'w+d'}{ic'}}\sum_{t\in\mathbb{Z}} e^{\pi it^2w}\sum_{\alpha \pmod{c'}} e^{\frac{2\pi i\frac{a'}{2}(\alpha+td')^2}{c'}} = \left(\frac{\frac{d'}{2}}{c'}\right)\varepsilon_{c'}\sqrt{c'}\sqrt{\frac{c'w+d'}{ic'}}\sum_{t\in\mathbb{Z}} e^{\pi it^2w} = \left(\frac{\frac{d'}{2}}{c'}\right)\varepsilon_{c'}\sqrt{\frac{c'w+d'}{i}}\widetilde{\theta}(w).$$

All together we have shown

$$\widetilde{\theta}(\eta' w) = \left(\frac{\frac{d'}{2}}{c'}\right) \varepsilon_{c'} \sqrt{\frac{c'w + d'}{i}} \widetilde{\theta}(w). \tag{13}$$

We will unpack this modularity condition to one for $\theta(z)$. Rewriting Equation (13) in terms of η, and noting that $\left(\frac{\frac{c}{4}}{d}\right) = \left(\frac{c}{d}\right)$ because $c \equiv 0 \pmod 4$, gives

$$\widetilde{\theta}(\eta z) = \left(\frac{c}{d}\right) \left(\frac{-1}{d}\right) \varepsilon_d \sqrt{\frac{dz - \frac{c}{2}}{i}} \widetilde{\theta}(z). \tag{14}$$

By changing variables $z \to 2z$ and rewriting Equation (14) in terms of $\theta(z)$, we find

$$\theta(\gamma z) = \left(\frac{c}{d}\right) \left(\frac{-1}{d}\right) \varepsilon_d \sqrt{\frac{-\left(\frac{d}{2z} + \frac{c}{2}\right)}{i}} \sqrt{\frac{2z}{i}} \theta(z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz + d}\theta(z),$$

where the last equality follows because $\left(\frac{-1}{d}\right) = 1, -1$ depending on if $d \equiv 1, 3 \pmod 4$ so that $\left(\frac{-1}{d}\right) \varepsilon_d = \varepsilon_d^{-1}$. $\square$

3. The Modular Form $\theta(z)$ & Additional Results

To complete the verification that $\theta(z)$ is a modular form, we need to verify holomorphy at the cusps. There are three cusps of $\Gamma_0(4)$ and they are ∞, 0, and $\frac{1}{2}$ respectively. To see this, we compute the orbits of ∞, 0, and $\frac{1}{2}$:

$$\begin{aligned} \Gamma_0(4)\infty &= \left\{ \frac{a}{c} : (a, c) = 1, c \equiv 0 \pmod 4 \right\} \\ \Gamma_0(4)0 &= \left\{ \frac{b}{d} : (b, d) = 1, d \equiv 1, 3 \pmod 4 \right\} \\ \Gamma_0(4)\frac{1}{2} &= \left\{ \frac{b}{d} : (b, d) = 1, d \equiv 2 \pmod 4 \right\}, \end{aligned}$$

Every element of $\mathbb{Q} \cup \{\infty\}$ belongs to one of these three distinct sets so ∞, 0, and $\frac{1}{2}$ are all the cusps. We only need to verify holomorphy on a set of scaling matrices for the cusps, so let us choose the scaling matrices

$$\sigma_\infty = I, \sigma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_{\frac{1}{2}} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.$$

For the cusp at ∞, Equation (1) implies

$$|\theta(z)| \ll \vartheta(\mathrm{Im}(z)) \ll 1,$$

for large z. For the cusps at 0 and $\frac{1}{2}$, we have analogous estimates:

$$\theta(\sigma_0 z) \ll \sum_{n \in \mathbb{Z}} e^{-\frac{2\pi i n^2 \overline{z}}{|z|^2}} \ll 1 \quad \text{and} \quad \theta(\sigma_{\frac{1}{2}} z) \ll \sum_{n \in \mathbb{Z}} e^{\frac{2\pi i n^2 (2|z| + 3z + 2\overline{z} + 1)}{|2z+3|^2}} \ll 1,$$

This proves $\theta(z)$ is holomorphic at the cusps. Since $\theta(z)$ is holomorphic on $\mathbb{H}$ and at the cusps, Theorem 2.1 implies $\theta(z)$ is a modular form on $\Gamma_0(4)\backslash\mathbb{H}$ of weight $\frac{1}{2}$. We package this into a theorem:

Theorem 3.1. *The quadratic theta function*

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z},$$

is a modular form of of weight $\frac{1}{2}$ on $\Gamma_0(4)\backslash\mathbb{H}$.

If we square the quadratic theta function $\theta^2(z)$ we obtain a modular form of odd weight and nontrivial character:

Corollary 3.1. *$\theta^2(z)$ is a weight 1 modular form on $\Gamma_0(4)\backslash\mathbb{H}$ with character $\left(\frac{-1}{d}\right)$.*

Proof. Squaring the modularity condition in Theorem 2.1 gives

$$\theta^2(\gamma z) = \left(\frac{-1}{d}\right)(cz+d)\theta^2(z).$$

The holomorphy condititions for $\theta^2(z)$ follow from those of $\theta(z)$. $\square$

The most surprising fact about $\theta(z)$ and $\theta^2(z)$ is that their factors of modularity contain (modified) Jacobi symbols. The reason this is surprising is that these symbols have nothing to do with the congruence subgroup $\Gamma_0(4)$, and so the modular forms are "seeing more" than what the congruence subgroup is controlling.

As a last corollary, we have an interesting identity involving Jacobi symbols:

Corollary 3.2. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma_0(4)$. Then*

$$\left(\frac{c'a+d'c}{c'b+d'd}\right)\varepsilon_{d'd}^{-1} = \left(\frac{c'}{d'}\right)\left(\frac{c}{d}\right)\varepsilon_{d'}^{-1}\varepsilon_{d}^{-1}$$

Proof. By Theorem 2.1, the theta multiplier $j(\gamma, z)$ satisfies the coycycle condition. As a consquence, we find

$$\left(\frac{c'a+d'c}{c'b+d'd}\right)\varepsilon_{c'b+d'd}^{-1} = \left(\frac{c'}{d'}\right)\left(\frac{c}{d}\right)\varepsilon_{d'}^{-1}\varepsilon_{d}^{-1}.$$

But $\varepsilon_{c'b+d'd} = \varepsilon_{d'd}$ because $c'b + d'd \equiv d'd \pmod 4$ since $c \equiv 0 \pmod 4$. $\square$