SUBCONVEXITY FOR GL_2 L-FUNCTIONS COMPUTATIONS

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1. Setup: Sums & Forms

Let $q \geq 1$ and let ψ be a Dirichlet character modulo q. For $m \in \mathbb{Z}$, let

$$c_q(m) = \sum_{a \pmod q} e^{\frac{2\pi i m a}{q}} \quad \text{and} \quad c_\psi(m) = \sum_{a \pmod q} \psi(a) e^{\frac{2\pi i m a}{q}},$$

be the Ramanujan and Gauss sums respectively. For ℓ such that $(\ell, q) = 1$, we have

$$c_\psi(\ell m) = \overline{\psi(\ell)} c_\psi(m),$$

and moreover

$$c_\psi(m) = \overline{\psi(m)} c_\psi(1),$$

provided ψ is primitive. Throughout we will let f and g be two weight k and level 1 holomorphic cusp forms. The admit Fourier series

$$f(z) = \sum_{m \geq 1} a(m) e^{2\pi i m z} = \sum_{m \geq 1} A(m) m^{\frac{k-1}{2}} e^{2\pi i m z} \quad \text{and} \quad g(z) = \sum_{m \geq 1} b(m) e^{2\pi i m z} = \sum_{m \geq 1} B(m) m^{\frac{k-1}{2}} e^{2\pi i m z},$$

and are normalzied so that

$$a(1) = b(1) = 1.$$

In particular, $a(m)$ and $b(m)$ are the m-th Hecke eigenvalues of f and g respectively. We define the L-functions

$$L(s, f \otimes \psi) = \sum_{m \geq 1} \frac{A(m)\psi(m)}{m^s} \quad \text{and} \quad L(s, f \times c_\psi) = \sum_{m \geq 1} \frac{A(m) c_\psi(m)}{m^s}.$$

These two L-functions are most related when ψ is primitive since we have the asymptotic

$$L(s, f \times c_\psi) \sim \sqrt{q} L(s, f \otimes \overline{\psi}).$$

They are least related when $\psi = \psi_{q,0}$ is the trivial character modulo q as

$$L(s, f \times c_\psi) = L^{(q)}(s, f).$$

Morally, one should think of $L(s, f \times c_\psi)$ as an arithmetically smoothed version of $L(s, f \otimes \psi)$. This will allow for some additional saving when studying the second moment of $L(s, f \times c_\psi)$. We will also require Maass cusp forms so let $\{\mu_j\}$ represent an orthonormal basis of Maass cusp forms on $\Gamma_0(\ell_1 \ell_2) \backslash \mathbb{H}$ with spectral parameter t_j for μ_j. They admit Fourier series

$$\mu_j(z) = \sum_{m \neq 0} \rho_j(m) \sqrt{y} K_{it_j}(2\pi |m| y) e^{2\pi i n x},$$

and the Fourier coefficients are normalized so that

$$\rho_j(m) = \rho_j(\mathrm{sgn}(m)) \lambda_j(|m|),$$

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where $\lambda_j(m)$ is the m-th Hecke eigenvalue of μ_j. We will also need the Petersson inner product on $\Gamma_0(\ell_1\ell_2)\backslash\mathbb{H}$, defined by

$$\langle F, G\rangle = \frac{1}{\mathcal{V}}\int_{\Gamma_0(\ell_1\ell_2)\backslash\mathbb{H}} F(z)\overline{G(z)}\,d\mu,$$

where

$$\mathcal{V} = \mathrm{vol}(\Gamma_0(\ell_1\ell_2)\backslash\mathbb{H}) = \frac{\pi}{3}\ell_1\ell_2\prod_{p|\ell_1\ell_2}(1+p^{-1}).$$

Also, define the functions

$$V_{f,g}^{\ell_1,\ell_2} = V_{f,g}^{\ell_1,\ell_2}(z) = \overline{f(\ell_1 z)}g(\ell_2 z)\mathrm{Im}(z)^k \quad \text{and} \quad V_{f,v}^{\ell_1} = V_{f,v}^{\ell_1}(z) = \overline{f(\ell_1 z)}E(z, \text{Todo : [s]}; k)\mathrm{Im}(z)^{\frac{k}{2}}.$$

2. Shifted Dirichlet Series

The Dirichlet Series $D_{f,g}(s;h,\ell_1,\ell_2)$. Let $h \geq 1$. Our first Dirichlet series $D_{f,g}(s;h,\ell_1,\ell_2)$ is given by

$$D_{f,g}(s;h,\ell_1,\ell_2) = \sum_{\ell_1 m = \ell_2 n + h}\frac{a(m)b(n)}{(n\ell_2)^{s+k-1}}.$$

This series is absolutely convergent for $\mathrm{Re}(s) > 1$ and admits meromorphic continuation to $\frac{1-k}{2} - C_1 < \mathrm{Re}(s)$, for any $C_1 > 0$, and in these two regions it satisfies the bounds

$$D_{f,g}(s;h,\ell_1,\ell_2) \ll_{\text{Todo:}[\ell_1,\ell_2]} h^{\frac{k-1}{2}+\varepsilon} \quad \text{and} \quad D_{f,g}(s;h,\ell_1,\ell_2) \ll_{\text{Todo:}[\ell_1,\ell_2]} h^{k+2C_1+\varepsilon},$$

respectively. In the region $\mathrm{Re}(s) < \frac{1-k}{2}$ and ε away from the poles, the meromorphic continuation is given by the absolutely convergent spectral expansion

$$D_{f,g}(s;h,\ell_1,\ell_2) = \sum_{t_j}\overline{\rho_j(-h)\langle V_{f,g}^{\ell_1,\ell_2}, \mu_j\rangle} h^{\frac{1}{2}-s}\frac{\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)\Gamma(1-s)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma(s+k-1)}.$$

In short, $D_{f,g}(s;h,\ell_1,\ell_2)$ has meromorphic continuation to $\mathbb{C}$ but we do not have a repsentation in the strip $\frac{1-k}{2} \leq \mathrm{Re}(s) \leq 1$. The poles occur at $s = \frac{1}{2} - \ell + it_j$ for $\ell \geq 0$ and the residue at this pole is

$$\operatorname*{Res}_{s=\frac{1}{2}-\ell+it_j} D_{f,g}(s;h,\ell_1,\ell_2) = \overline{\rho_j(-h)\langle V_{f,g}^{\ell_1,\ell_2}, \mu_j\rangle} h^{\ell - it_j}\frac{(-1)^\ell}{\ell!}\frac{\Gamma\left(-\ell+2it_j\right)\Gamma\left(\frac{1}{2}+\ell-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma\left(k-\ell-\frac{1}{2}+it_j\right)}$$

$$. .$$

The Dirichlet Series $D_{f,v}(w;n,\ell_1,\ell_2)$. Let $n \geq 1$. Our second Dirichlet series $D_{f,v}(w;n,\ell_1,\ell_2)$ is given by

$$D_{f,v}(w;n,\ell_1,\ell_2) = \sum_{\ell_1 m = \ell_2 n + h}\frac{a(m)\sigma_{1-2v}(h)h^{v-\frac{1}{2}}}{h^{w+\frac{k-1}{2}}}.$$

This series is absolutely convergent for $\mathrm{Re}(w) > \frac{1}{2} + \mathrm{Re}(v)$ and admits meromorphic continuation. To state the meromorphic continuation, let $c > 0$ be such that if v satisfies $\zeta(2v) \neq 0$, then $\mathrm{Re}(v) \geq \frac{1}{2} - \frac{8c}{\log(2+\mathrm{Im}(v))}$. For such a c, we set

$$\delta(s,v,u) = \frac{c}{\log(3+|\mathrm{Im}(s+u)|+|\mathrm{Im}(v)|)} \quad \text{and} \quad \delta_v = \delta(0,0,v).$$

Then we have meromorphic continuation to $\mathrm{Re}(v) \geq \frac{1}{2} - \delta(w,v,u)$ with $\mathrm{Re}(w) > 1 - \frac{k}{2} - \mathrm{Re}(v) - C_2$, for any $C_2 > 0$, and in these two regions satisfies the bounds

Todo : [xxx]

In the region $\mathrm{Re}(w) < \frac{1-k}{2}$ and ε away from the poles, the meromorphic continuation is given by the absolutely convergent spectral expansion

$$D_{f,v}\left(w;n,\ell_1,\ell_2\right) = \sum_{t_j} \overline{\rho_j(-\ell_2 n)\langle V_{f,v}^{\ell_1},\mu_j\rangle}(\ell_2 n)^{\frac{1}{2}-w}\frac{\Gamma\left(w-\frac{1}{2}+it_j\right)\Gamma\left(w-\frac{1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)} \cdot \frac{\Gamma(1-w)\Gamma(w)}{\Gamma\left(w+v+\frac{k}{2}-1\right)\Gamma\left(w-v+\frac{k}{2}\right)},$$

In short, $D_{f,v}\left(w;n,\ell_1,\ell_2\right)$ admits meromorphic continuation to $\mathbb{C}$ but we do not have a representation in the strip $\frac{1-k}{2} \leq \mathrm{Re}(w) \leq \frac{1}{2}+\mathrm{Re}(v)$. The poles occur at $w=\frac{1}{2}-\ell+it_j$ for $\ell \geq 0$ and the residue at this pole is

$$\operatorname*{Res}_{w=\frac{1}{2}-\ell+it_j} D_{f,v}\left(w;n,\ell_1,\ell_2\right) = \overline{\rho_j(-\ell_2 n)\langle V_{f,v}^{\ell_1},\mu_j\rangle}(\ell_2 n)^{\ell-it_j}\frac{(-1)^\ell}{\ell!}\frac{\Gamma\left(\frac{1}{2}+\ell-it_j\right)\Gamma\left(\frac{1}{2}-\ell+it_j\right)}{\Gamma\left(\frac{k-1}{2}-\ell+v+it_j\right)\Gamma\left(\frac{k+1}{2}-\ell-v+it_j\right)} \cdot \frac{\Gamma\left(-\ell+2it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)}.$$

The Multiple Dirichlet Series $Z_{f,g}(s,v,u,\ell_1,\ell_2)$. We now wish to construct a multiple Dirichlet series from $D_{f,g}(s;h,\ell_1,\ell_2)$ and $D_{f,v}(w;n,\ell_1,\ell_2)$. To do this we will suppose

$$\mathrm{Re}(s) > 1, \quad \mathrm{Re}(w) > \frac{1}{2}+\mathrm{Re}(v), \quad \text{and} \quad \mathrm{Re}(v) \geq \frac{1}{2}-\delta(s,v,u).$$

Letting ε be such that $\mathrm{Re}(w) > \frac{1}{2}+\mathrm{Re}(v)+\varepsilon$, both Dirichlet series $D_{f,g}(s;h,\ell_1,\ell_2)$ and $D_{f,v}(w;n,\ell_1,\ell_2)$ converge absolutely and satisfy the estimates

$$D_{f,g}(s;h,\ell_1,\ell_2) \ll_{\ell_1,\ell_2} h^{\frac{k-1}{2}+\varepsilon} \quad \text{and} \quad D_{f,v}(w;n,\ell_1,\ell_2) \ll_{\ell_1,\ell_2} n^{\frac{k-1}{2}+\varepsilon}.$$

Thus for

$$\mathrm{Re}(s) > 1, \quad \mathrm{Re}(u) > \frac{k+1}{2}, \quad \text{and} \quad \mathrm{Re}(v) \geq \frac{1}{2}-\delta(s,v,u),$$

we may define the multiple Dirichlet series $Z_{f,g}(s,v,u;\ell_1,\ell_2)$ by

$$Z_{f,g}(s,v,u;\ell_1,\ell_2) = (\ell_1\ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m=\ell_2 n+h} \frac{a(m)b(n)\sigma_{1-2v}(h)}{(\ell_2 n)^{s+k-1}h^u}.$$

It is absolutely convergent in this region. Moreover, $Z_{f,g}(s,v,u;\ell_1,\ell_2)$ satisfies the interchange

$$Z_{f,g}(s,v,u;\ell_1,\ell_2) = \sum_{h\geq 1}\frac{D_{f,g}(s;h,\ell_1,\ell_2)\sigma_{1-2v}(h)}{h^u} = \sum_{n\geq 1}\frac{D_{f,v}\left(u+v-\frac{k}{2};n,\ell_1,\ell_2\right)b(n)}{(\ell_2 n)^{s+k-1}},$$

where both representations converge absolutely. In the region where $D_{f,g}(s;h,\ell_1,\ell_2)$ admits a spectral expansion, we have an absolutely convergent spectral expansion for $Z_{f,g}(s,v,u;\ell_1,\ell_2)$ given by

$$Z_{f,g}(s,v,u;\ell_1,\ell_2) = \sum_{t_j}\overline{\rho_j(-1)\langle V_{f,g}^{\ell_1,\ell_2},\mu_j\rangle}(\ell_1\ell_2)^{\frac{k-1}{2}}\frac{\Gamma\left(s-\frac{1}{2}+it_j\right)\Gamma\left(s-\frac{1}{2}-it_j\right)\Gamma(1-s)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)\Gamma(s+k-1)} \cdot \frac{L\left(s+u-\frac{1}{2},\mu_j\right)L\left(s+u+2v-\frac{3}{2},\mu_j\right)}{\zeta(2s+2u+2v-2)}.$$

Similarly, in the region where $D_{f,v}\left(u+v-\frac{k}{2};n,\ell_1,\ell_2\right)$ admits a spectral expansion, we have an absolutely convergent spectral expansion for $Z_{f,g}(s,v,u;\ell_1,\ell_2)$ given by

$$Z_{f,g}(s,v,u;\ell_1,\ell_2)=\sum_{t_j}\overline{\rho_j(-1)\langle V_{f,v}^{\ell_1},\mu_j\rangle}\ell_1^{\frac{k-1}{2}}\ell_2^{1-s-v-u}\frac{\Gamma\left(u+v-\frac{k+1}{2}+it_j\right)\Gamma\left(u+v-\frac{k+1}{2}-it_j\right)}{\Gamma\left(\frac{1}{2}+it_j\right)\Gamma\left(\frac{1}{2}-it_j\right)}$$
$$\cdot\frac{\Gamma\left(\frac{k}{2}+1-u-v\right)\Gamma\left(u+v-\frac{k}{2}\right)}{\Gamma(u+2v-1)\Gamma(u)}\frac{L^{(\ell_2)}(s+u+v-1,g\otimes\mu_j)}{\zeta^{(\ell_2)}(2s+2u+2v-2)}\sum_{\alpha\geq 0}\frac{b(\ell_2^{\alpha})\lambda_j(\ell_2^{\alpha+1})}{(\ell_2^{\alpha})^{s+u+v-1+\frac{k-1}{2}}}.$$

The poles of $Z_{f,g}(s,v,u;\ell_1,\ell_2)$ are inherited from the poles of $D_{f,g}(s;h,\ell_1,\ell_2)$ of $D_{f,v}\left(u+v-\frac{k}{2};n,\ell_1,\ell_2\right)$ with corresponding residues.

3. Subconvexity

Setup. Let $G(x)$ be a smooth function with compact support in the interval $[1,2]$ and let $g(s)$ be the Mellin transform. For a Dirichlet character χ modulo Q, we define

$$B_\chi(x)=\sum_{m\geq 1}A(m)\chi(m)G\left(\frac{m}{x}\right)\quad\text{and}\quad B_{c_\chi}(x)=\sum_{m\geq 1}A(m)\overline{c_\chi}(m)G\left(\frac{m}{x}\right).$$

Using a smooth dyadic parition of unity and summation by parts, we we have the bounds

$$L\left(\frac{1}{2},f\otimes\chi\right)\ll Q^{-\frac{1}{2}}\max_{x\ll Q^{1+\varepsilon}}B_\chi(x)\quad\text{and}\quad L\left(\frac{1}{2},f\otimes c_\chi\right)\ll Q^{-\frac{1}{2}}\max_{x\ll Q^{1+\varepsilon}}B_{c_\chi}(x).$$

Since $L\left(s,f\otimes\chi\right)\ll Q^{-\frac{1}{2}}L(s,f\otimes c_\chi)$, we have

$$\left|L\left(\frac{1}{2},f\otimes\chi\right)\right|^2\ll Q^{-2}\max_{x\ll Q^{1+\varepsilon}}|B_{c_\chi}(x)|^2.$$

So to obtain a subconvexity estimate for $L(s,f\otimes\chi)$ at $s=\frac{1}{2}$, it suffices to estimate $B_{c_\chi}(x)$ for $x\ll Q^{1+\varepsilon}$. Now let $q\geq 1$ and ψ be a Dirichlet character modulo q. We define

$$S_\chi(x,q)=\frac{1}{\varphi(q)}\sum_{\psi \pmod q}|B_{c_\psi}(x)|^2\left|\sum_{\ell\sim L}\chi(\ell)\overline{\psi}(\ell)\right|^2,$$

where $\ell\sim L$ means that $\ell\in[L,2L]$ and is prime. As all of the terms in the sum are nonnegative, retaining only the term corresponding to $\psi=\chi$, the prime number theorem gives the lower bound

$$\frac{L^2}{Q\log^2(L)}|B_{c_\chi}(x)|^2\ll S_\chi(x,q).$$

It follows that

$$\frac{L^2Q}{\log^2(L)}\left|L\left(\frac{1}{2},f\otimes\chi\right)\right|^2\ll\frac{L^2}{Q\log^2(L)}\max_{x\ll Q^{1+\varepsilon}}|B_{c_\chi}(x)|^2\ll\max_{x\ll Q^{1+\varepsilon}}\sum_{|q-Q|\ll Q^\varepsilon}S_\chi(x,q),$$

Hence

$$\left|L\left(\frac{1}{2},f\otimes\chi\right)\right|^2\ll\frac{1}{L^{2+\varepsilon}Q}\max_{x\ll Q^{1+\varepsilon}}\sum_{|q-Q|\ll Q^\varepsilon}S_\chi(x,q).$$

Now recall the Mellin inverse

$$\frac{1}{2\pi i}\int_{(2)}\frac{e^{\frac{\pi v^2}{y^2}}Q^{2v}}{y}\,dv=e^{-\frac{y^2\log^2(Q)}{\pi}}\ll\begin{cases}1 & \text{if } |q-Q|\ll\frac{Q^{1+e}}{y},\\ Q^{-A} & \text{if } |q-Q|\gg\frac{Q^{1+\varepsilon}}{y},\end{cases}$$

for any $A \gg 1$. From this integral transform, we conclude that

$$\sum_{|q-Q|\ll Q^{\varepsilon}} S_\chi(x,q) \ll \frac{1}{2\pi i}\int_{(2)} \sum_{q\geq 1} \frac{S_\chi(x,q)}{q^{2v}} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q}\, dv.$$

To estimate the right-hand side, we will rewrite the Dirichlet series over q. To do this, we first expand $S_\chi(x,q)$:

$$\begin{aligned} S_\chi(x,q) &= \frac{1}{\varphi(q)} \sum_{\psi \pmod q} |B_{c_\psi}(x)|^2 \left| \sum_{\ell\sim L} \chi(\ell)\overline{\psi}(\ell)\right|^2 \\ &= \frac{1}{\varphi(q)} \sum_{\ell_1,\ell_2\sim L} \sum_{\psi \pmod q} \sum_{m,n\geq 1} A(m)A(n)G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_\psi(m)c_{\overline{\psi}}(n)\chi(\ell_1)\overline{\psi}(\ell_1)\overline{\chi}(\ell_2)\psi(\ell_2) \\ &= \frac{1}{\varphi(q)} \sum_{\ell_1,\ell_2\sim L} \sum_{\psi \pmod q} \sum_{m,n\geq 1} A(m)A(n)G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_\psi(\ell_1 m)c_{\overline{\psi}}(\ell_2 n)\chi(\ell_1)\overline{\chi}(\ell_2) \\ &= \sum_{\ell_1,\ell_2\sim L} \sum_{m,n\geq 1} A(m)A(n)G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_q(\ell_1 m - \ell_2 n)\chi(\ell_1)\overline{\chi}(\ell_2), \end{aligned}$$

where in the last line we have used the identity

$$\frac{1}{\varphi(q)} \sum_{\psi \pmod q} c_\psi(\ell_1 m)c_{\overline{\psi}}(\ell_2 n) = c_q(\ell_1 m - \ell_2 n).$$

Using the relation

$$\sum_{q\geq 1} \frac{c_q(\ell_1 m - \ell_2 n)}{q^{2v}} = \begin{cases} \frac{\zeta(2v-1)}{\zeta(2v)} & \text{if } \ell_1 m = \ell_2 n, \\ \frac{\sigma_{1-2v}(h)}{\zeta^{2v}} & \text{if } \ell_1 m = \ell_2 n + h, \end{cases}$$

we can express the Dirichlet series over q as a diagonal and off-diagonal term:

$$\begin{aligned} \sum_{q\geq 1} \frac{S_\chi(x,q)}{q^{2v}} &= \sum_{\ell_1,\ell_2\sim L} \sum_{\ell_1 m = \ell_2 n} \frac{\zeta(2v-1)}{\zeta(2v)} A(m)A(n)G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1)\overline{\chi}(\ell_2) \\ &\quad + \sum_{\ell_1,\ell_2\sim L} \sum_{\ell_1 m = \ell_2 n + h} \frac{\sigma_{1-2v}(h)}{\zeta(2v)} A(m)A(n)G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1)\overline{\chi}(\ell_2) \end{aligned}$$

Thus

$$\begin{aligned} &\frac{1}{2\pi i}\int_{(2)} \sum_{q\geq 1} \frac{S_\chi(x,q)}{q^{2v}} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q}\, dv \\ &= \frac{1}{2\pi i}\int_{(2)} \sum_{\ell_1,\ell_2\sim L} \sum_{\ell_1 m = \ell_2 n} \frac{\zeta(2v-1)}{\zeta(2v)} A(m)A(n)G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1)\overline{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q}\, dv \\ &+ \frac{1}{2\pi i}\int_{(2)} \sum_{\ell_1,\ell_2\sim L} \sum_{\ell_1 m = \ell_2 n + h} \frac{\sigma_{1-2v}(h)}{\zeta(2v)} A(m)A(n)G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1)\overline{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q}\, dv. \end{aligned}$$

The Diagonal Contribution. We will estimate

$$\frac{1}{2\pi i}\int_{(2)} \sum_{\ell_1,\ell_2\sim L} \sum_{\ell_1 m = \ell_2 n} \frac{\zeta(2v-1)}{\zeta(2v)} A(m)A(n)G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1)\overline{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q}\, dv.$$

The integral over v is

$$\frac{1}{2\pi i}\int_{(2)}\frac{\zeta(2v-1)}{\zeta(2v)}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv \ll \sum_{|q-Q|\ll Q^{\varepsilon}}\varphi(q)\ll Q^{1+\varepsilon}.$$

Therefore the diagonal contribution is

$$\begin{aligned}
&\ll Q^{1+\varepsilon}\sum_{\ell_1,\ell_2\sim L}\sum_{\ell_1 m=\ell_2 n}A(m)A(n)G\left(\frac{m}{x}\right)G\left(\frac{n}{x}\right)\chi(\ell_1)\overline{\chi}(\ell_2)\\
&\ll Q^{1+\varepsilon}\sum_{\ell_1,\ell_2\sim L}\sum_{\substack{\ell_1 m=\ell_2 n\\ m,n\ll Q^{1+\varepsilon}}}A(m)A(n)\chi(\ell_1)\overline{\chi}(\ell_2)\\
&\ll Q^{1+\varepsilon}\sum_{\ell_1,\ell_2\sim L}\sum_{\substack{d\geq 1\\ d\ll \frac{Q^{1+\varepsilon}}{L}}}A(\ell_1\ell_2 d)A(\ell_1\ell_2 d)\chi(\ell_1)\overline{\chi}(\ell_2)\\
&\ll LQ^{2+\varepsilon}.
\end{aligned}$$

The Off-diagonal Contribution. We will estimate

$$\frac{1}{2\pi i}\int_{(2)}\sum_{\ell_1,\ell_2\sim L}\sum_{\ell_1 m=\ell_2 n+h}\frac{\sigma_{1-2v}(h)}{\zeta(2v)}A(m)A(n)G\left(\frac{m}{x}\right)G\left(\frac{n}{x}\right)\chi(\ell_1)\overline{\chi}(\ell_2)\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv.$$

Applying the Mellin inversion formula to $G\left(\frac{m}{x}\right)$ and $G\left(\frac{n}{x}\right)$, we can express the off-diagonal contribution as

$$\begin{aligned}
\sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)\left(\frac{1}{2\pi i}\right)^3\int_{(2)}\int_{(\sigma_{s_2})}\int_{(\sigma_{s_1})}\frac{1}{\zeta(2v)}\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{m^{s_1+\frac{k-1}{2}}n^{s_2+\frac{k-1}{2}}}\\
\cdot\, g(s_1)g(s_2)x^{s_1+s_2}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,ds_1\,ds_2\,dv,
\end{aligned}$$

with $\sigma_{s_1},\sigma_{s_2}\gg 1$. Now make the following computation:

$$\begin{aligned}
\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{m^{s_1+\frac{k-1}{2}}n^{s_2+\frac{k-1}{2}}} &= \ell_1^{s_1}\ell_2^{s_2}(\ell_1\ell_2)^{\frac{k-1}{2}}\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{(\ell_1 m)^{s_1+\frac{k-1}{2}}(\ell_2 n)^{s_2+\frac{k-1}{2}}}\\
&= \ell_1^{s_1}\ell_2^{s_2}(\ell_1\ell_2)^{\frac{k-1}{2}}\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{(\ell_2 n+h)^{s_1+\frac{k-1}{2}}(\ell_2 n)^{s_2+\frac{k-1}{2}}}\\
&= \ell_1^{s_1}\ell_2^{s_2}(\ell_1\ell_2)^{\frac{k-1}{2}}\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{\left(1+\frac{h}{\ell_2 n}\right)^{s_1+\frac{k-1}{2}}(\ell_2 n)^{s_1+s_2+k-1}}.
\end{aligned}$$

Recall the Mellin inversion formula

$$\frac{1}{(1+t)^{\beta}}=\frac{1}{2\pi i}\int_{(\sigma_u)}\frac{\Gamma(\beta-u)\Gamma(u)}{\Gamma(\beta)}t^{-u}\,du,$$

with $0<\sigma_u<\mathrm{Re}(\beta)$. Applying this formula with $t=\frac{h}{\ell_2 n}$ and $\beta=s_1+\frac{k-1}{2}$, we have

$$\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{m^{s_1+\frac{k-1}{2}}n^{s_2+\frac{k-1}{2}}}=\frac{1}{2\pi i}\int_{(\sigma_u)}\ell_1^{s_1}\ell_2^{s_2}Z_f(s_1+s_2-u,v,u;\ell_1,\ell_2)\frac{\Gamma\left(s_1-u+\frac{k-1}{2}\right)\Gamma(u)}{\Gamma\left(s_1+\frac{k-1}{2}\right)}\,du.$$

Therefore the off-diagonal contribution can be expressed as

$$\sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)\left(\frac{1}{2\pi i}\right)^4\int_{(2)}\int_{(\sigma_u)}\int_{(\sigma_{s_2})}\int_{(\sigma_{s_1})}\frac{1}{\zeta(2v)}\ell_1^{s_1}\ell_2^{s_2}Z_f(s_1+s_2-u,v,u;\ell_1,\ell_2)$$
$$\cdot\frac{\Gamma\left(s_1-u+\frac{k-1}{2}\right)\Gamma(u)}{\Gamma\left(s_1+\frac{k-1}{2}\right)}g(s_1)g(s_2)x^{s_1+s_2}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,ds_1\,ds_2\,du\,dv.$$

Since $\sigma_{s_1},\sigma_{\sigma_2}\gg 1$ and $0<\sigma_u<\sigma_{s_1}+\frac{k-1}{2}$, we may assume

$$\sigma_{s_1}+\sigma_{s_2}-\sigma_u>1,\quad \sigma_u>\frac{k+1}{2},\quad\text{and}\quad 2\geq\frac{1}{2}-\delta(s,v,u).$$

This ensures that $Z_f(s_1+s_2-u,v,u;\ell_1,\ell_2)$ is absolutely convergent. Let us take

$$\sigma_{s_1}=1+\varepsilon_1,\quad \sigma_{s_2}=\frac{k+1}{2}+\varepsilon_2,\quad\text{and}\quad \sigma_u=\frac{k+1}{2}+\varepsilon_3,$$

with $\varepsilon_1+\varepsilon_2-\varepsilon_3>0$. The analytic continuation of $Z_f(s_1+s_2-u,v,u;\ell_1,\ell_2)$ exhibits no poles in the region

$$\sigma_{s_1}+\sigma_{s_2}-\sigma_u>\frac{1}{2}\quad\text{and}\quad\sigma_u>0.$$

Therefore, we may shift the integral in u to $\sigma_u=\varepsilon_4$ without crossing over any poles. Then, we may shift the integrals in s_1 and s_2 to $\sigma_{s_1}+\sigma_{s_2}=\frac{1}{2}+\varepsilon_5$ provided $\varepsilon_5>\varepsilon_4$ without crossing over any poles. Now we shift the integral in u so that

$$\sigma_{s_1}+\sigma_{s_2}-\sigma_u<\frac{1-k}{2}-\varepsilon_6\quad\text{while}\quad\sigma_1-\sigma_u+\frac{k-1}{2}>0.$$

This is possible by moving the integral in u to $\sigma_u=\frac{k}{2}+\varepsilon_5+\varepsilon_6$ and choosing $\sigma_1=\frac{1}{2}+2\varepsilon_5$ and $\sigma_2=-\varepsilon_5$ with $\varepsilon_5>\varepsilon_6$. In doing so, we pass over simple poles of $Z_f(s_1+s_2-u,v,u;\ell_1,\ell_2)$ occuring at $u=s_1+s_2-\frac{1}{2}+\ell-it_j$ for $0\leq\ell\leq\frac{k}{2}$. We do not pass over any simple poles of gamma functions. since $\sigma_u>0$ and $\sigma_1-\sigma_u+\frac{k-1}{2}>0$ throughout. Then the off-diagonal contribution can be expressed as

$$\sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)\left(\frac{1}{2\pi i}\right)^4\int_{(2)}\int_{(\sigma_u)}\int_{(\sigma_{s_2})}\int_{(\sigma_{s_1})}\frac{1}{\zeta(2v)}\ell_1^{s_1}\ell_2^{s_2}(\ell_1\ell_2)^{\frac{k-1}{2}}Z_f(s_1+s_2-u,v,u;\ell_1,\ell_2)$$
$$\cdot\frac{\Gamma\left(s_1-u+\frac{k-1}{2}\right)\Gamma(u)}{\Gamma\left(s_1+\frac{k-1}{2}\right)}g(s_1)g(s_2)x^{s_1+s_2}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,ds_1\,ds_2\,du\,dv$$
$$+\sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)\left(\frac{1}{2\pi i}\right)^3\int_{(2)}\int_{(\sigma_{s_2})}\int_{(\sigma_{s_1})}\sum_{t_j}\frac{1}{\zeta(2v)}\ell_1^{s_1}\ell_2^{s_2}(\ell_1\ell_2)^{\frac{k-1}{2}}\operatorname*{Res}_{u=s_1+s_2-\frac{1}{2}+\ell-it_j}$$
$$\cdot\left[Z_f(s_1+s_2-u,v,u;\ell_1,\ell_2)\frac{\Gamma\left(s_1-u+\frac{k-1}{2}\right)\Gamma(u)}{\Gamma\left(s_1+\frac{k-1}{2}\right)}\right]g(s_1)g(s_2)x^{s_1+s_2}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,ds_1\,ds_2\,dv.$$

Let us concern ourselves with the first term only. Here the first spectral expansion of $Z_f(s_1+s_2-u,v,u;\ell_1,\ell_2)$ is valid. The integral over v is

$$\frac{1}{2\pi i}\int_{(2)}\frac{L\left(s_1+s_2+2v-\frac{3}{2}\right)}{\zeta(2v)\zeta(2s_1+2s_2+2v-2)}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv\ll\sum_{\substack{q=q_1q_2q_3\\|q-Q|\ll Q^{\varepsilon}}}\mu(q_1q_2)q_2^{1-2\varepsilon_5}q_3^{1-s_5}\lambda_j(q_3)\ll Q^{1+\theta+\varepsilon}.$$

Moreover $x\ll Q^{1+\varepsilon}$ so that

$$x^{s_1+s_2}\ll Q^{\frac{1}{2}+\varepsilon}$$

The contribution of L is

$$\ll \sum_{\ell_1,\ell_2\sim L} \chi(\ell_1)\overline{\chi}(\ell_2) \sum_{|t_j|\sim 1} \overline{\rho_j(-1)\langle V_{f,g}^{\ell_1,\ell_2}, \mu_j\rangle \ell_1^{s_1}\ell_2^{s_2}(\ell_1\ell_2)^{\frac{k-1}{2}}} \ll L^{\frac{5}{2}}.$$

In total, the off-diagonal contribution is

$$\ll L^{\frac{5}{2}} Q^{\frac{3}{2}+\theta+\varepsilon}.$$

Balancing. We have the diagonal and off-diagonal estimates

$$\ll LQ^{2+\varepsilon} \quad \text{and} \quad \ll L^{\frac{5}{2}} Q^{\frac{3}{2}+\theta+\varepsilon}.$$

This implies

$$\left|L\left(\frac{1}{2}, f\otimes\chi\right)\right|^2 \ll \frac{1}{L^{2+\varepsilon}Q} \max_{x\ll Q^{1+\varepsilon}} \sum_{|q-Q|\ll Q^{\varepsilon}} S_\chi(x,q) \ll \frac{Q^{1+\varepsilon}}{L^{1+\varepsilon}} + L^{\frac{1}{2}-\varepsilon}Q^{\frac{1}{2}+\theta+\varepsilon}.$$

The terms are balanced when $L = Q^{\frac{1-2\theta}{3}}$. We then have

$$\left|L\left(\frac{1}{2}, f\otimes\chi\right)\right|^2 \ll Q^{\frac{2}{3}+\frac{2\theta}{3}+\varepsilon},$$

and it follows that

$$\left|L\left(\frac{1}{2}, f\otimes\chi\right)\right| \ll Q^{\frac{1}{3}+\frac{\theta}{3}+\varepsilon}.$$

4. Hybrid Subconvexity

Setup. Let $G(x)$ be a smooth function with compact support in the interval $[1,2]$ and let $g(s)$ be the Mellin transform. For a Dirichlet character χ modulo Q and $|t|\le T$, we define

$$B_\chi(x,t) = \sum_{m\ge 1} A(m)\chi(m)m^{-it}G\left(\frac{m}{x}\right) \quad \text{and} \quad B_{c_\chi}(x,t) = \sum_{m\ge 1} A(m)m^{-it}\overline{c_\chi}(m)G\left(\frac{m}{x}\right).$$

Using a smooth dyadic parition of unity and summation by parts, we we have the bounds

$$L\left(\frac{1}{2}+it, f\otimes\chi\right) \ll (QT)^{-\frac{1}{2}} \max_{x\ll (QT)^{1+\varepsilon}} B_\chi(x,t) \quad \text{and} \quad L\left(\frac{1}{2}+it, f\otimes c_\chi\right) \ll (QT)^{-\frac{1}{2}} \max_{x\ll (QT)^{1+\varepsilon}} B_{c_\chi}(x,t).$$

Since $L\left(s, f\otimes\chi\right) \ll Q^{-\frac{1}{2}} L(s, f\otimes c_\chi)$, we have

$$\left|L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \ll Q^{-2}T^{-1} \max_{x\ll (QT)^{1+\varepsilon}} |B_{c_\chi}(x,t)|^2.$$

So to obtain a subconvexity estimate for $L(s, f\otimes\chi)$ at $s=\frac{1}{2}$, it suffices to estimate $B_{c_\chi}(x,t)$ for $x \ll (QT)^{1+\varepsilon}$. Now let $q\ge 1$ and ψ be a Dirichlet character modulo q. We define

$$S_\chi(x,q,t) = \frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} |B_{c_\psi}(x,t)|^2 \left|\sum_{\ell\sim L} \chi(\ell)\overline{\psi}(\ell)\right|^2,$$

where $\ell\sim L$ means that $\ell\in[L,2L]$ and is prime. As all of the terms in the sum are nonnegative, retaining only the term corresponding to $\psi=\chi$, the prime number theorem gives the lower bound

$$\frac{L^2}{Q\log^2(L)} |B_{c_\chi}(x,t)|^2 \ll S_\chi(x,q,t).$$

It follows that

$$\frac{L^2Q}{\log^2(L)}\left|L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \ll \frac{L^2}{QT\log^2(L)} \max_{x\ll (QT)^{1+\varepsilon}} |B_{c_\chi}(x;t)|^2 \ll \frac{1}{T} \max_{x\ll (QT)^{1+\varepsilon}} \sum_{|q-Q|\ll Q^{\varepsilon}} S_\chi(x,q,t).$$

Hence

$$\left| L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \ll \frac{1}{L^{2+\varepsilon}QT} \max_{x\ll (QT)^{1+\varepsilon}} \sum_{|q-Q|\ll Q^{\varepsilon}} S_\chi(x,q,t).$$

Now recall the Mellin inverse

$$\frac{1}{2\pi i}\int_{(2)} \frac{e^{\frac{\pi v^2}{y^2}} Q^{2v}}{y}\, dv = e^{-\frac{y^2\log^2(Q)}{\pi}} \ll \begin{cases} 1 & \text{if } |q-Q| \ll \frac{Q^{1+e}}{y}, \\ Q^{-A} & \text{if } |q-Q| \gg \frac{Q^{1+\varepsilon}}{y}, \end{cases}$$

for any $A \gg 1$. From this integral transform, we conclude that

$$\sum_{|q-Q|\ll Q^{\varepsilon}} S_\chi(x,q,t) \ll \frac{1}{2\pi i}\int_{(2)} \sum_{q\geq 1} \frac{S_\chi(x,q,t)}{q^{2v}} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q}\, dv.$$

To estimate the right-hand side, we will rewrite the Dirichlet series over q. To do this, we first expand $S_\chi(x,q,t)$:

$$\begin{aligned} S_\chi(x,q,t) &= \frac{1}{\varphi(q)} \sum_{\psi \pmod q} |B_{c_\psi}(x,t)|^2 \left| \sum_{\ell\sim L} \chi(\ell)\overline{\psi}(\ell)\right|^2 \\ &= \frac{1}{\varphi(q)} \sum_{\ell_1,\ell_2\sim L} \sum_{\psi \pmod q} \sum_{m,n\geq 1} A(m)A(n)m^{-it}n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_\psi(m) c_{\overline{\psi}}(n)\chi(\ell_1)\overline{\psi}(\ell_1)\overline{\chi}(\ell_2)\psi(\ell_2) \\ &= \frac{1}{\varphi(q)} \sum_{\ell_1,\ell_2\sim L} \sum_{\psi \pmod q} \sum_{m,n\geq 1} A(m)A(n)m^{-it}n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_\psi(\ell_1 m) c_{\overline{\psi}}(\ell_2 n)\chi(\ell_1)\overline{\chi}(\ell_2) \\ &= \sum_{\ell_1,\ell_2\sim L} \sum_{m,n\geq 1} A(m)A(n)m^{-it}n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_q(\ell_1 m - \ell_2 n)\chi(\ell_1)\overline{\chi}(\ell_2), \end{aligned}$$

where in the last line we have used the identity

$$\frac{1}{\varphi(q)} \sum_{\psi \pmod q} c_\psi(\ell_1 m) c_{\overline{\psi}}(\ell_2 n) = c_q(\ell_1 m - \ell_2 n).$$

Using the relation

$$\sum_{q\geq 1} \frac{c_q(\ell_1 m - \ell_2 n)}{q^{2v}} = \begin{cases} \frac{\zeta(2v-1)}{\zeta(2v)} & \text{if } \ell_1 m = \ell_2 n, \\ \frac{\sigma_{1-2v}(h)}{\zeta^{2v}} & \text{if } \ell_1 m = \ell_2 n + h, \end{cases}$$

we can express the Dirichlet series over q as a diagonal and off-diagonal term:

$$\begin{aligned} \sum_{q\geq 1} \frac{S_\chi(x,q,t)}{q^{2v}} &= \sum_{\ell_1,\ell_2\sim L} \sum_{\ell_1 m = \ell_2 n} \frac{\zeta(2v-1)}{\zeta(2v)} A(m)A(n)m^{-it}n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1)\overline{\chi}(\ell_2) \\ &\quad + \sum_{\ell_1,\ell_2\sim L} \sum_{\ell_1 m = \ell_2 n + h} \frac{\sigma_{1-2v}(h)}{\zeta(2v)} A(m)A(n)m^{-it}n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1)\overline{\chi}(\ell_2) \end{aligned}$$

Thus

$$\frac{1}{2\pi i}\int_{(2)}\sum_{q\geq 1}\frac{S_\chi(x,q,t)}{q^{2v}}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv$$
$$=\frac{1}{2\pi i}\int_{(2)}\sum_{\ell_1,\ell_2\sim L}\sum_{\ell_1 m=\ell_2 n}\frac{\zeta(2v-1)}{\zeta(2v)}A(m)A(n)m^{-it}n^{-it}G\left(\frac{m}{x}\right)G\left(\frac{n}{x}\right)\chi(\ell_1)\overline{\chi}(\ell_2)\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv$$
$$+\frac{1}{2\pi i}\int_{(2)}\sum_{\ell_1,\ell_2\sim L}\sum_{\ell_1 m=\ell_2 n+h}\frac{\sigma_{1-2v}(h)}{\zeta(2v)}A(m)A(n)m^{-it}n^{-it}G\left(\frac{m}{x}\right)G\left(\frac{n}{x}\right)\chi(\ell_1)\overline{\chi}(\ell_2)\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv.$$

The Diagonal Contribution. We will estimate

$$\frac{1}{2\pi i}\int_{(2)}\sum_{\ell_1,\ell_2\sim L}\sum_{\ell_1 m=\ell_2 n}\frac{\zeta(2v-1)}{\zeta(2v)}A(m)A(n)m^{-it}n^{-it}G\left(\frac{m}{x}\right)G\left(\frac{n}{x}\right)\chi(\ell_1)\overline{\chi}(\ell_2)\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv.$$

The integral over v is

$$\frac{1}{2\pi i}\int_{(2)}\frac{\zeta(2v-1)}{\zeta(2v)}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv\ll\sum_{|q-Q|\ll Q^\varepsilon}\varphi(q)\ll Q^{1+\varepsilon},$$

Therefore the diagonal contribution is

$$\begin{aligned}
&\ll Q^{1+\varepsilon}\sum_{\ell_1,\ell_2\sim L}\sum_{\ell_1 m=\ell_2 n}A(m)A(n)m^{-it}n^{-it}G\left(\frac{m}{x}\right)G\left(\frac{n}{x}\right)\chi(\ell_1)\overline{\chi}(\ell_2)\\
&\ll Q^{1+\varepsilon}\sum_{\ell_1,\ell_2\sim L}\sum_{\substack{\ell_1 m=\ell_2 n\\ m,n\ll(QT)^{1+\varepsilon}}}A(m)A(n)m^{-it}n^{-it}\chi(\ell_1)\overline{\chi}(\ell_2)\\
&\ll Q^{1+\varepsilon}\sum_{\ell_1,\ell_2\sim L}\sum_{\substack{d\geq 1\\ d\ll\frac{(QT)^{1+\varepsilon}}{L}}}A(\ell_1\ell_2 d)A(\ell_1\ell_2 d)\chi(\ell_1)\overline{\chi}(\ell_2)\\
&\ll LQ^{2+\varepsilon}T^{1+\varepsilon}.
\end{aligned}$$

The Off-diagonal Contribution. We will estimate

$$\frac{1}{2\pi i}\int_{(2)}\sum_{\ell_1,\ell_2\sim L}\sum_{\ell_1 m=\ell_2 n+h}\frac{\sigma_{1-2v}(h)}{\zeta(2v)}A(m)A(n))m^{-it}n^{-it}G\left(\frac{m}{x}\right)G\left(\frac{n}{x}\right)\chi(\ell_1)\overline{\chi}(\ell_2)\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv.$$

Applying the Mellin inversion formula to $G\left(\frac{m}{x}\right)$ and $G\left(\frac{n}{x}\right)$, we can express the off-diagonal contribution as

$$\sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)\left(\frac{1}{2\pi i}\right)^3\int_{(2)}\int_{(\sigma_{s_2})}\int_{(\sigma_{s_1})}\frac{1}{\zeta(2v)}\sum_{\ell_1 m=\ell_2 n+h}\frac{a(m)a(n)\sigma_{1-2v}(h)}{m^{s_1+\frac{k-1}{2}+it}n^{s_2+\frac{k-1}{2}}+it}$$
$$\cdot\, g(s_1)g(s_2)x^{s_1+s_2}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,ds_1\,ds_2\,dv,$$

with $\sigma_{s_1}, \sigma_{s_2} \gg 1$. Now make the following computation:

$$\begin{aligned}\sum_{\ell_1 m=\ell_2 n+h} \frac{a(m)a(n)\sigma_{1-2v}(h)}{m^{s_1+\frac{k-1}{2}+it} n^{s_2+\frac{k-1}{2}+it}} &= \ell_1^{s_1+it}\ell_2^{s_2+it}(\ell_1\ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m=\ell_2 n+h} \frac{a(m)a(n)\sigma_{1-2v}(h)}{(\ell_1 m)^{s_1+\frac{k-1}{2}+it}(\ell_2 n)^{s_2+\frac{k-1}{2}+it}} \\ &= \ell_1^{s_1+it}\ell_2^{s_2+it}(\ell_1\ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m=\ell_2 n+h} \frac{a(m)a(n)\sigma_{1-2v}(h)}{(\ell_2 n+h)^{s_1+\frac{k-1}{2}+it}(\ell_2 n)^{s_2+\frac{k-1}{2}+it}} \\ &= \ell_1^{s_1+it}\ell_2^{s_2+it}(\ell_1\ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m=\ell_2 n+h} \frac{a(m)a(n)\sigma_{1-2v}(h)}{\left(1+\frac{h}{\ell_2 n}\right)^{s_1+\frac{k-1}{2}+it}(\ell_2 n)^{s_1+s_2+k-1+2it}}.\end{aligned}$$

Recall the Mellin inversion formula

$$\frac{1}{(1+t)^\beta} = \frac{1}{2\pi i}\int_{(\sigma_u)} \frac{\Gamma(\beta-u)\Gamma(u)}{\Gamma(\beta)} t^{-u}\, du,$$

with $0<\sigma_u<\mathrm{Re}(\beta)$. Applying this formula with $t=\frac{h}{\ell_2 n}$ and $\beta = s_1+\frac{k-1}{2}+it$, we have

$$\begin{aligned}\sum_{\ell_1 m=\ell_2 n+h} \frac{a(m)a(n)\sigma_{1-2v}(h)}{m^{s_1+\frac{k-1}{2}+it} n^{s_2+\frac{k-1}{2}+it}} = \frac{1}{2\pi i}\int_{(\sigma_u)} &\ell_1^{s_1+it}\ell_2^{s_2+it} Z_f(s_1+s_2-u+2it, v, u; \ell_1, \ell_2) \\ &\cdot \frac{\Gamma\left(s_1-u+\frac{k-1}{2}+2it\right)\Gamma(u)}{\Gamma\left(s_1+\frac{k-1}{2}+2it\right)}\, du.\end{aligned}$$

Therefore the off-diagonal contribution can be expressed as

$$\begin{aligned}\sum_{\ell_1,\ell_2\sim L} \chi(\ell_1)\overline{\chi}(\ell_2)\left(\frac{1}{2\pi i}\right)^4 \int_{(2)}\int_{(\sigma_u)}\int_{(\sigma_{s_2})}\int_{(\sigma_{s_1})} &\frac{1}{\zeta(2v)}\ell_1^{s_1+it}\ell_2^{s_2+it} Z_f(s_1+s_2-u+2it, v, u; \ell_1, \ell_2) \\ &\cdot \frac{\Gamma\left(s_1-u+\frac{k-1}{2}+2it\right)\Gamma(u)}{\Gamma\left(s_1+\frac{k-1}{2}+2it\right)} g(s_1)g(s_2)x^{s_1+s_2}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\, ds_1\, ds_2\, du\, dv.\end{aligned}$$

Since $\sigma_{s_1}, \sigma_{\sigma_2} \gg 1$ and $0<\sigma_u<\sigma_{s_1}+\frac{k-1}{2}$, we may assume

$$\sigma_{s_1}+\sigma_{s_2}-\sigma_u > 1, \quad \sigma_u > \frac{k+1}{2}, \quad \text{and} \quad 2 \geq \frac{1}{2}-\delta(s,v,u).$$

This ensures that $Z_f(s_1+s_2-u+2it, v, u; \ell_1, \ell_2)$ is absolutely convergent. Let us take

$$\sigma_{s_1} = 1+\varepsilon_1, \quad \sigma_{s_2} = \frac{k+1}{2}+\varepsilon_2, \quad \text{and} \quad \sigma_u = \frac{k+1}{2}+\varepsilon_3,$$

with $\varepsilon_1+\varepsilon_2-\varepsilon_3>0$. The analytic continuation of $Z_f(s_1+s_2-u+2it, v, u; \ell_1, \ell_2)$ exhibits no poles in the region

$$\sigma_{s_1}+\sigma_{s_2}-\sigma_u > \frac{1}{2} \quad \text{and} \quad \sigma_u > 0.$$

Therefore, we may shift the integral in u to $\sigma_u = \varepsilon_4$ without crossing over any poles. Then, we may shift the integrals in s_1 and s_2 to $\sigma_{s_1}+\sigma_{s_2} = \frac{1}{2}+\varepsilon_5$ provided $\varepsilon_5>\varepsilon_4$ without crossing over any poles. Now we shift the integral in u so that

$$\sigma_{s_1}+\sigma_{s_2}-\sigma_u < \frac{1-k}{2}-\varepsilon_6 \quad \text{while} \quad \sigma_1-\sigma_u+\frac{k-1}{2} > 0.$$

This is possible by moving the integral in u to $\sigma_u = \frac{k}{2}+\varepsilon_5+\varepsilon_6$ and choosing $\sigma_1 = \frac{1}{2}+2\varepsilon_5$ and $\sigma_2 = -\varepsilon_5$ with $\varepsilon_5 > \varepsilon_6$. In doing so, we pass over simple poles of $Z_f(s_1+s_2-u, v, u; \ell_1, \ell_2)$ occuring at $u =$

$s_1+s_2+2it-\frac{1}{2}+\ell-it_j$ for $0\le \ell\le \frac{k}{2}$. We do not pass over any simple poles of gamma functions. since $\sigma_u>0$ and $\sigma_1-\sigma_u+\frac{k-1}{2}>0$ throughout. Then the off-diagonal contribution can be expressed as

$$\sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)\left(\frac{1}{2\pi i}\right)^4\int_{(2)}\int_{(\sigma_u)}\int_{(\sigma_{s_2})}\int_{(\sigma_{s_1})}\frac{1}{\zeta(2v)}\ell_1^{s_1+it}\ell_2^{s_2+it}(\ell_1\ell_2)^{\frac{k-1}{2}}Z_f(s_1+s_2-u+2it,v,u;\ell_1,\ell_2)$$
$$\cdot\frac{\Gamma\left(s_1-u+\frac{k-1}{2}+2it\right)\Gamma(u)}{\Gamma\left(s_1+\frac{k-1}{2}+2it\right)}g(s_1)g(s_2)x^{s_1+s_2}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,ds_1\,ds_2\,du\,dv$$
$$+\sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)\left(\frac{1}{2\pi i}\right)^3\int_{(2)}\int_{(\sigma_{s_2})}\int_{(\sigma_{s_1})}\sum_{t_j}\frac{1}{\zeta(2v)}\ell_1^{s_1+it}\ell_2^{s_2+it}(\ell_1\ell_2)^{\frac{k-1}{2}}\operatorname*{Res}_{u=s_1+s_2+2it-\frac{1}{2}+\ell-it_j}$$
$$\cdot\left[Z_f(s_1+s_2-u+2it,v,u;\ell_1,\ell_2)\frac{\Gamma\left(s_1-u+\frac{k-1}{2}+2it\right)\Gamma(u)}{\Gamma\left(s_1+\frac{k-1}{2}+2it\right)}\right]g(s_1)g(s_2)x^{s_1+s_2}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,ds_1\,ds_2\,dv.$$

Let us concern ourselves with the first term only. Here the first spectral expansion of $Z_f(s_1+s_2-u+2it,v,u;\ell_1,\ell_2)$ is valid. The integral over v is

$$\frac{1}{2\pi i}\int_{(2)}\frac{L\left(s_1+s_2+2v-\frac{3}{2}+2it\right)}{\zeta(2v)\zeta(2s_1+2s_2+2v-2+4it)}\frac{e^{\frac{\pi v^2}{Q^2}}Q^{2v}}{Q}\,dv\ll\sum_{\substack{q=q_1q_2q_3\\|q-Q|\ll Q^\varepsilon}}\mu(q_1q_2)q_2^{1-2\varepsilon_5}q_3^{1-s_5}\lambda_j(q_3)\ll Q^{1+\theta+\varepsilon}.$$

Moreover $x\ll(QT)^{1+\varepsilon}$ so that

$$x^{s_1+s_2}\ll(QT)^{\frac{1}{2}+\varepsilon}$$

The contribution of L is

$$\ll\sum_{\ell_1,\ell_2\sim L}\chi(\ell_1)\overline{\chi}(\ell_2)\sum_{|t_j|\sim 1}\overline{\rho_j(-1)\langle V_{f,g}^{\ell_1,\ell_2},\mu_j\rangle}\ell_1^{s_1+it}\ell_2^{s_2+it}(\ell_1\ell_2)^{\frac{k-1}{2}}\ll L^{\frac{5}{2}}.$$

In total, the off-diagonal contribution is

$$\ll L^{\frac{5}{2}}Q^{\frac{3}{2}+\theta+\varepsilon}T^{\frac{1}{2}+\varepsilon}.$$

Balancing. We have the diagonal and off-diagonal estimates

$$\ll LQ^{2+\varepsilon}T^{1+\varepsilon}\quad\text{and}\quad\ll L^{\frac{5}{2}}Q^{\frac{3}{2}+\theta+\varepsilon}T^{\frac{1}{2}+\varepsilon}.$$

This implies

$$\left|L\left(\frac{1}{2},f\otimes\chi\right)\right|^2\ll\frac{1}{L^{2+\varepsilon}QT}\max_{x\ll(QT)^{1+\varepsilon}}\sum_{|q-Q|\ll Q^\varepsilon}S_\chi(x,q)\ll\frac{Q^{1+\varepsilon}T^\varepsilon}{L^{1+\varepsilon}}+\frac{L^{\frac{1}{2}-\varepsilon}Q^{\frac{1}{2}+\theta+\varepsilon}}{T^{\frac{1}{2}-\varepsilon}}.$$

The terms are balanced when $L=Q^{\frac{1-2\theta}{3}}$. We then have

$$\left|L\left(\frac{1}{2},f\otimes\chi\right)\right|^2\ll Q^{\frac{2}{3}+\frac{2\theta}{3}+\varepsilon},$$

and it follows that

$$\left|L\left(\frac{1}{2},f\otimes\chi\right)\right|\ll Q^{\frac{1}{3}+\frac{\theta}{3}+\varepsilon}.$$