0.1   Todo: [The Kuznetsov Trace Formula]

The Kuznetsov trace formula is an analog of the Petersson trace formula for weight zero Maass forms. From **??**, $\mathcal{L}(N, \chi)$ admits an orthonormal basis of Maass forms for the point spectrum (these forms are generally not Hecke-Maass eigenforms because they need not be Hecke normalized or even cuspidal in the case of the discrete spectrum). However, by **??** and **??** we make take this orthonormal basis to consist of Hecke-Maass eigenforms and the constant function. Denote this basis by $\{u_j\}_{j \geq 0}$ with $u_0(z) = 1$ and let u_j be of type ν_j for $j \geq 1$. In particular, $\{u_j\}_{j \geq 1}$ is an orthonormal basis of Hecke-Maass eigenforms and each such form admits a Fourier series at the $\mathfrak{a}$ cusp given by

$$(u_j | \sigma_{\mathfrak{a}})(z) = \sum_{n \neq 0} a_{j,\mathfrak{a}}(n) \sqrt{y} K_{\nu_j}(2\pi n y) e^{2\pi i n x}.$$

The Kuznetsov trace formula is an equation relating the Fourier coefficients $a_{j,\mathfrak{a}}(n)$ and $a_{j,\mathfrak{b}}(n)$ of the basis $\{u_j\}_{j \geq 1}$ for two cusps $\mathfrak{a}$ and $\mathfrak{b}$ of $\Gamma_0(N) \backslash \mathbb{H}$ to a sum of integral transforms involving test functions and Salié sums. Similar to the Petersson trace formula, we will compute the inner product of two Poincaré series $P_{n,\chi,\mathfrak{a}}(z, \psi)(z)$ and $P_{m,\chi,\mathfrak{b}}(z, \varphi)(z)$ in two different ways. The first will be geometric in nature while the second will be spectral. We first need to compute the Fourier series of such a Poincaré series. Although we will not need it explicitly, we will work over any congruence subgroup:

Proposition 0.1.1. *Let $m \geq 1$, χ be Dirichlet character with conductor dividing the level, $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma \backslash \mathbb{H}$, and $\psi(y)$ be a smooth function such that $\psi(y) \ll_\varepsilon y^{1+\varepsilon}$ as $y \to 0$. The Fourier series of $P_{m,\chi,\mathfrak{a}}(z, \psi)$ on $\Gamma \backslash \mathbb{H}$ at the $\mathfrak{b}$ cusp is given by*

$$(P_{m,\chi,\mathfrak{a}} | \sigma_{\mathfrak{b}})(z, \psi) = \sum_{t \in \mathbb{Z}} \left(\delta_{\mathfrak{a},\mathfrak{b}} \delta_{m,t} \psi(\mathrm{Im}(z)) + \sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \psi(y, m, t, c) S_{\chi,\mathfrak{a},\mathfrak{b}}(m, t, c) \right) e^{2\pi i t z},$$

where $\psi(y, m, t, c)$ is the integral transform given by

$$\psi(y, m, t, c) = \int_{\mathrm{Im}(z) = y} \psi\left(\frac{y}{|cz|^2} \right) e^{-\frac{2\pi i m}{c^2 z} - 2\pi i t z} \, dz.$$

Proof. From the cocycle condition and **??**, we have

$$(P_{m,\chi,\mathfrak{a}} | \sigma_{\mathfrak{b}})(z, \psi) = \delta_{\mathfrak{a},\mathfrak{b}} \psi(\mathrm{Im}(z)) e^{2\pi i m z} + \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}, d \in \mathbb{Z} \\ d \ (\mathrm{mod}\ c) \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(d) \psi\left(\frac{\mathrm{Im}(z)}{|cz + d|^2} \right) e^{2\pi i m \left(\frac{a}{c} - \frac{1}{c^2 z + cd} \right)},$$

where a and b are chosen such that $\det \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \right) = 1$ and we have used the fact that

$$\frac{a}{c} - \frac{1}{c^2 z + cd} = \frac{az + b}{cz + d}.$$

Summing over all pairs (c, d) with $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $d \in \mathbb{Z}$, and $d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is the same as summing over all triples (c, ℓ, r) with $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $\ell \in \mathbb{Z}$, and r taken modulo c with $r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$. Indeed, this is seen by writing $d = c\ell + r$. Moreover, since $ad - bc = 1$ we have $a(c\ell + r) - bc = 1$ which further implies that

$ar \equiv 1 \pmod{c}$. So we may take a to be the inverse for r modulo c. Then

$$\sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}, d \in \mathbb{Z} \\ d \,(\mathrm{mod}\; c) \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(d) \psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + cd}\right)} = \sum_{(c,\ell,r)} \overline{\chi}(c\ell+r) \psi\left(\frac{\mathrm{Im}(z)}{|cz+c\ell+r|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + c^2\ell + cr}\right)}$$

$$= \sum_{(c,\ell,r)} \overline{\chi}(r) \psi\left(\frac{\mathrm{Im}(z)}{|cz+c\ell+r|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + c^2\ell + cr}\right)}$$

$$= \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \sum_{\ell \in \mathbb{Z}} \overline{\chi}(r) \psi\left(\frac{\mathrm{Im}(z)}{|cz+c\ell+r|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + c^2\ell + cr}\right)}$$

$$= \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(r) \sum_{\ell \in \mathbb{Z}} \psi\left(\frac{\mathrm{Im}(z)}{|cz+c\ell+r|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + c^2\ell + cr}\right)},$$

where on the right-hand side it is understood that we are summing over all triples (c,ℓ,r) with the prescribed properties and the second line holds since χ has conductor diving the level and $d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is determined modulo c. Now let

$$I_{c,r}(z,\psi) = \sum_{\ell \in \mathbb{Z}} \psi\left(\frac{\mathrm{Im}(z)}{|cz+c\ell+r|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + c^2\ell + cr}\right)}.$$

We apply the Poisson summation formula to $I_{c,r}(z,\psi)$. This is allowed since the summands are absolutely integrable by **??**, as they exhibit polynomial decay of order $\sigma > 1$ because $\psi(y) \ll_\varepsilon y^{1+\varepsilon}$ as $y \to 0$, and $I_{c,r}(z,\psi)$ is holomorphic because $(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z,\psi)$ is. By the identity theorem it suffices to apply the Poisson summation formula for $z = iy$ with $y > 0$. So let $f(x)$ be given by

$$f(x) = \psi\left(\frac{y}{|cx+r+icy|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 x + cr + ic^2 y}\right)}.$$

As we have just noted, $f(x)$ is absolutely integrable on $\mathbb{R}$. We compute the Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x}\, dx = \int_{-\infty}^{\infty} \psi\left(\frac{y}{|cx+r+icy|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 x + cr + ic^2 y}\right)} e^{-2\pi i t x}\, dx.$$

Complexify the integral to get

$$\int_{\mathrm{Im}(z)=0} \psi\left(\frac{y}{|cz+r+icy|^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{1}{c^2 z + cr + ic^2 y}\right)} e^{-2\pi i t z}\, dz.$$

Now make the change of variables $z \to z - \frac{r}{c} - iy$ to obtain

$$e^{2\pi i m\frac{a}{c} + 2\pi i t\frac{r}{c} - 2\pi t y} \int_{\mathrm{Im}(z)=y} \psi\left(\frac{y}{|cz|^2}\right) e^{-\frac{2\pi i m}{c^2 z} - 2\pi i t z}\, dz.$$

As the remaining integral is $\psi(y,m,t,c)$, it follows that

$$\hat{f}(t) = \psi(y,m,t,c) e^{2\pi i m\frac{a}{c} + 2\pi i t\frac{r}{c} - 2\pi t y}.$$

By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z,\psi) = \sum_{t \in \mathbb{Z}} \left(\psi(y,m,t,c) e^{2\pi i m\frac{a}{c} + 2\pi i t\frac{r}{c}} \right) e^{2\pi i t z},$$

for all $z \in \mathbb{H}$. Substituting this back into the Eisenstein series gives a form of the Fourier series:

$$(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z, \psi) = \delta_{\mathfrak{a},\mathfrak{b}}\psi(\text{Im}(z))e^{2\pi imz} + \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r) \sum_{t \in \mathbb{Z}} \psi(y, m, t, c)e^{2\pi im\frac{a}{c} + 2\pi it\frac{r}{c}}e^{2\pi itz}$$

$$= \sum_{t \in \mathbb{Z}} \left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\text{Im}(z)) + \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r)\psi(y, m, t, c)e^{2\pi im\frac{a}{c} + 2\pi it\frac{r}{c}} \right) e^{2\pi itz}$$

$$= \sum_{t \in \mathbb{Z}} \left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\text{Im}(z)) + \sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \psi(y, m, t, c) \sum_{r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(r)e^{2\pi im\frac{a}{c} + 2\pi it\frac{r}{c}} \right) e^{2\pi itz}.$$

We will simplify the innermost sum. Since a is the inverse for r modulo c, the innermost sum above becomes

$$\sum_{r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(r)e^{2\pi im\frac{a}{c} + 2\pi it\frac{r}{c}} = \sum_{r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(\overline{a})e^{2\pi im\frac{a}{c} + 2\pi it\frac{\overline{a}}{c}} = \sum_{r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \chi(a)e^{\frac{2\pi i(am+\overline{a}t)}{c}} = S_{\chi,\mathfrak{a},\mathfrak{b}}(m, t, c).$$

So at last, we obtain our desired Fourier series:

$$(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t \in \mathbb{Z}} \left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\text{Im}(z)) + \sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \psi(y, m, t, c)S_{\chi,\mathfrak{a},\mathfrak{b}}(m, t, c) \right) e^{2\pi itz}. \qquad \square$$

We can now derive the first half of the Kuznetsov trace formula by computing the inner product between $P_{n,\chi,\mathfrak{a}}(z, \psi)$ and $P_{m,\chi,\mathfrak{b}}(z, \varphi)$:

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), P_{m,\chi,\mathfrak{b}}(\cdot, \varphi)\rangle = \frac{1}{V_{\Gamma_0(N)}} \int_{\mathcal{F}_{\Gamma_0(N)}} P_{n,\chi,\mathfrak{a}}(z, \psi)\overline{P_{m,\chi,\mathfrak{b}}(z, \varphi)}\, d\mu$$

$$= \frac{1}{V_{\Gamma_0(N)}} \int_{\mathcal{F}_{\Gamma_0(N)}} \sum_{\gamma \in \Gamma_{\mathfrak{b}}\backslash\Gamma_0(N)} \chi(\gamma)P_{n,\chi,\mathfrak{a}}(z, \psi)\overline{\varphi(\text{Im}(\sigma_{\mathfrak{b}}^{-1}\gamma z))}e^{-2\pi im\overline{\sigma_{\mathfrak{b}}^{-1}\gamma z}}\, d\mu$$

$$= \frac{1}{V_{\Gamma_0(N)}} \int_{\mathcal{F}_{\Gamma_0(N)}} \sum_{\gamma \in \Gamma_{\mathfrak{b}}\backslash\Gamma_0(N)} P_{n,\chi,\mathfrak{a}}(\gamma z, \psi)\overline{\varphi(\text{Im}(\sigma_{\mathfrak{b}}^{-1}\gamma z))}e^{-2\pi im\overline{\sigma_{\mathfrak{b}}^{-1}\gamma z}}\, d\mu$$

$$= \frac{1}{V_{\Gamma_0(N)}} \int_{\mathcal{F}_{\sigma_{\mathfrak{b}}^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}}} \sum_{\gamma \in \Gamma_{\mathfrak{b}}\backslash\Gamma_0(N)} P_{n,\chi,\mathfrak{a}}(\gamma\sigma_{\mathfrak{b}}z, \psi)\overline{\varphi(\text{Im}(\sigma_{\mathfrak{b}}^{-1}\gamma\sigma_{\mathfrak{b}}z))}e^{-2\pi im\overline{\sigma_{\mathfrak{b}}^{-1}\gamma\sigma_{\mathfrak{b}}z}}\, d\mu$$

$$= \frac{1}{V_{\Gamma_0(N)}} \int_{\mathcal{F}_{\sigma_{\mathfrak{b}}^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}}} \sum_{\gamma \in \Gamma_{\infty}\backslash\sigma_{\mathfrak{b}}^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}^{-1}} P_{n,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}}\gamma z, \psi)\overline{\varphi(\text{Im}(\gamma z))}e^{-2\pi im\overline{\gamma z}}\, d\mu$$

$$= \frac{1}{V_{\Gamma_0(N)}} \int_{\Gamma_{\infty}\backslash\mathbb{H}} (P_{n,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z, \psi)\overline{\varphi(\text{Im}(z))}e^{-2\pi im\overline{z}}\, d\mu,$$

where in the third line we have used the automorphy of $P_{n,\chi,\mathfrak{a}}(z, \psi)$, in the forth and fifth lines we have made the change of variables $z \to \sigma_{\mathfrak{b}}z$ and $\gamma \to \sigma_{\mathfrak{b}}\gamma\sigma_{\mathfrak{b}}^{-1}$ respectively, and in the sixth line we have unfolded. Now substitute in the Fourier series of $P_{n,\chi,\mathfrak{a}}(z, \psi)$ at the $\mathfrak{b}$ cusp to obtain

$$\frac{1}{V_{\Gamma_0(N)}} \int_{\Gamma_{\infty}\backslash\mathbb{H}} \sum_{t \in \mathbb{Z}} \left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,t}\psi(\text{Im}(z)) + \sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \psi(y, n, t, c)S_{\chi,\mathfrak{a},\mathfrak{b}}(n, t, c) \right) \overline{\varphi(\text{Im}(z))}e^{2\pi itz - 2\pi im\overline{z}}\, d\mu,$$

3

which is equivalent to

$$\frac{1}{V_{\Gamma_0(N)}} \int_0^\infty \int_0^1 \sum_{t\geq 1} \left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,t}\psi(y) + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \psi(y,n,t,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(n,t,c) \right) \overline{\varphi(y)}e^{2\pi i(t-m)x}e^{-2\pi(t+m)y}\frac{dx\,dy}{y^2}.$$

By the dominated convergence theorem, we can interchange the sum and the two integrals. Then **??** implies that the inner integral cuts off all of the terms except the diagonal $t = m$. This leaves

$$\frac{1}{V_{\Gamma_0(N)}} \int_0^\infty \left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m}\psi(y) + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \psi(y,n,m,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c) \right) \overline{\varphi(y)}e^{-4\pi my}\frac{dy}{y^2}.$$

Interchanging the integral and the remaining sum by the dominated convergence theorem again, we arrive at

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), P_{m,\chi,\mathfrak{b}}(\cdot,\varphi)\rangle = \delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m}(\psi,\varphi)_{n,m} + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c)V(n,m,c,\psi,\varphi),$$

where we have set

$$(\psi,\varphi)_{n,m} = \frac{1}{V_{\Gamma_0(N)}} \int_0^\infty \psi(y)\overline{\varphi(y)}e^{-2\pi(n+m)y}\frac{dy}{y^2},$$

and

$$V(n,m,c;\psi,\varphi) = \frac{1}{V_{\Gamma_0(N)}} \int_0^\infty \int_{\mathrm{Im}(z)=y} \psi\left(\frac{y}{|cz|^2}\right) \overline{\varphi(y)}e^{-\frac{2\pi im}{c^2z}-2\pi inz-4\pi my}\frac{dz\,dy}{y^2}.$$

This is the first half of the Kuznetsov trace formula. For the second half, **??** gives

$$P_{n,\chi,\mathfrak{a}}(\cdot,\psi) = \sum_{j\geq 0}\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), u_j\rangle u_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^\infty \left\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right) \right\rangle E_{\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)dr,$$

and

$$P_{m,\chi,\mathfrak{a}}(\cdot,\varphi) = \sum_{j\geq 0}\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi), u_j\rangle u_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^\infty \left\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi), E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right) \right\rangle E_{\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)dr.$$

By orthonormality, it follows that

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), P_{m,\chi,\mathfrak{a}}(\cdot,\varphi)\rangle = \sum_j \langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), u_j\rangle \overline{\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi), u_j\rangle}$$

$$+ \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^\infty \left\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right) \right\rangle \overline{\left\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi), E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right) \right\rangle} dr.$$

Now we must simplify the remaining inner products. Let $f \in \mathcal{L}(N,\chi)$ with Fourier series

$$f(z) = a^+(0)y^{\frac{1}{2}+\nu} + a^-(0)y^{\frac{1}{2}-\nu} + \sum_{n\neq 0} a(n)\sqrt{y}K_\nu(2\pi|n|y)e^{2\pi inx}.$$

By unfolding the integral in the Petersson inner product and cutting off everything except the diagonal using **??** exactly as in the case for $\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), P_{m,\chi,\mathfrak{a}}(\cdot,\varphi)\rangle$, we see that

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), f\rangle = \frac{1}{V_\Gamma} \int_0^\infty \overline{a(n)\sqrt{y}K_\nu(2\pi ny)}\psi(y)e^{-4\pi my}\frac{dy}{y^2}.$$

Now set
$$\omega_\nu(n, \psi) = \frac{1}{V_\Gamma} \int_0^\infty \sqrt{y} K_\nu(2\pi|n|y)\overline{\psi(y)}e^{-4\pi my}\frac{dy}{y^2}.$$

Then it follows from the Fourier series of cusp forms and Eisenstein series that
$$\langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), u_j\rangle = \overline{a_j(n)\omega_{\nu_j}(n, \psi)},$$

for $j \geq 1$ and
$$\left\langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), E_\mathfrak{a}\left(\cdot, \frac{1}{2} + ir\right)\right\rangle = \overline{\tau_\mathfrak{a}\left(n, \frac{1}{2} + ir\right)\omega_{ir}(n, \psi)}.$$

In particular, $\langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), u_0\rangle = 0$. So we obtain
$$\langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), P_{m,\chi,\mathfrak{a}}(\cdot, \varphi)\rangle = \sum_{j\geq 1} \overline{a_j(n)}a_j(m)\overline{\omega(n, \psi)}\omega(m, \varphi)$$
$$+ \sum_\mathfrak{a} \frac{1}{4\pi}\int_{-\infty}^\infty \overline{\tau_\mathfrak{a}\left(n, \frac{1}{2} + ir\right)}\tau_\mathfrak{a}\left(m, \frac{1}{2} + ir\right)\overline{\omega(n, \psi)}\omega(m, \varphi)\, dr.$$

This is the second half of the Kuznetsov trace formula. Equating the first and second halves we get the **Kuznetsov trace formula**:
$$\delta_{n,m}(\psi, \varphi) + \sum_{\substack{c\geq 1 \\ c\equiv 0 \ (\mathrm{mod}\ N)}} \frac{1}{c}S_\chi(n, m, c)V(n, m, c, \psi, \varphi) = \sum_{j\geq 1}\overline{a_j(n)}a_j(m)\overline{\omega(n, \psi)}\omega(m, \varphi)$$
$$+ \sum_\mathfrak{a}\frac{1}{4\pi}\int_{-\infty}^\infty \overline{\tau_\mathfrak{a}\left(n, \frac{1}{2} + ir\right)}\tau_\mathfrak{a}\left(m, \frac{1}{2} + ir\right)\overline{\omega(n, \psi)}\omega(m, \varphi)\, dr.$$

The left-hand side is called the **geometric side** and the right-hand side is called the **spectral side**. We collect our work as a theorem:

Theorem 0.1.1 (Kuznetsov trace formula). *Let $\{u_j\}_{j\geq 1}$ be an orthonormal basis of Hecke-Maass eigenforms for $\mathcal{L}(N, \chi)$ of types ν_j with Fourier coefficients $a_j(n)$. Then for any positive integers $n, m \geq 1$, we have*
$$\delta_{n,m}(\psi, \varphi) + \sum_{\substack{c\geq 1 \\ c\equiv 0 \ (\mathrm{mod}\ N)}} \frac{1}{c}S_\chi(n, m, c)V(n, m, c, \psi, \varphi) = \sum_{j\geq 1}\overline{a_j(n)}a_j(m)\overline{\omega(n, \psi)}\omega(m, \varphi)$$
$$+ \sum_\mathfrak{a}\frac{1}{4\pi}\int_{-\infty}^\infty \overline{\tau_\mathfrak{a}\left(n, \frac{1}{2} + ir\right)}\tau_\mathfrak{a}\left(m, \frac{1}{2} + ir\right)\overline{\omega(n, \psi)}\omega(m, \varphi)\, dr.$$

0.2 Misc.

0.3 The Ideal Norm

Let us now prove some properties about the ideal norm. We first show that it respects localization:

Proposition 0.3.1. *Let $\mathcal{O}/\mathfrak{o}$ be a Dedekind extension of separable extension L/K and let $D \subseteq \mathfrak{o} - \{0\}$ be a multiplicative subset. Then for any fractional ideal $\mathfrak{F}$ of $\mathcal{O}$, we have*
$$\mathrm{N}_{\mathcal{O}D^{-1}/\mathfrak{o}D^{-1}}(\mathfrak{F}D^{-1}) = \mathrm{N}_{\mathcal{O}/\mathfrak{o}}(\mathfrak{F})D^{-1}.$$

Proof. Since the ideal norm is multiplicative, it suffices to prove the claim in the case of a prime $\mathfrak{P}$ of $\mathcal{O}$. Then we must show

$$\mathrm{N}_{\mathcal{O}D^{-1}/\mathfrak{o}D^{-1}}(\mathfrak{P}D^{-1}) = \mathrm{N}_{\mathcal{O}/\mathfrak{o}}(\mathfrak{P})D^{-1}.$$

This is immediate from **??** and the definition of the ideal norm. $\qquad\square$

The ideal norm is also compatible with the field trace:

Proposition 0.3.2. *Let $\mathcal{O}/\mathfrak{o}$ be a Dedekind extension of degree n separable extension L/K. Then for any $\lambda \in \mathcal{O}$, we have*

$$\mathrm{N}_{\mathcal{O}/\mathfrak{o}}(\lambda\mathcal{O}) = \mathrm{N}_{L/K}(\lambda)\mathfrak{o}.$$

Proof. In light of Proposition 0.3.1, it suffices to assume $\mathcal{O}/\mathfrak{o}$ is a local Dedekind extension. Therefore $\mathfrak{o}$ is a discrete valuation ring, $\mathcal{O}$ is a principal ideal domain, and $\mathcal{O}/\mathfrak{o}$ admits an integral basis $\alpha_1, \ldots, \alpha_n$ making $\mathcal{O}$ a free $\mathfrak{o}$-module of rank n. Let $\mathfrak{p}$ be the unique prime of $\mathfrak{o}$ and π be a uniformizer so that $\mathfrak{p} = \pi\mathfrak{o}$. Since the ideal norm and the field norm are both multiplicative and $\mathcal{O}$ and $\mathfrak{o}$ are both unique factorization domains, we may assume that λ is prime. Then $\lambda\mathcal{O} = \mathfrak{P}$ for some prime $\mathfrak{P}$ of $\mathcal{O}$. So on the one hand,

$$\mathrm{N}_{\mathcal{O}/\mathfrak{o}}(\lambda\mathcal{O}) = \mathfrak{p}^{f_{\mathfrak{p}}(\mathfrak{P})}.$$

As $\mathfrak{o}$ is a discrete valuation ring, we have the prime factorization $\mathrm{N}_{L/K}(\lambda) = \mu\pi^f$. So on the other hand,

$$\mathrm{N}_{L/K}(\lambda)\mathfrak{o} = \mathfrak{p}^f.$$

It now suffices to show that $f = f_{\mathfrak{p}}(\mathfrak{P})$. Todo: [xxx] $\qquad\square$

The different and discriminant and related to each other via the ideal norm. In particular, the ideal norm of the different is the discriminant:

Proposition 0.3.3. *Let $\mathcal{O}/\mathfrak{o}$ be a Dedekind extension of a degree n separable extension L/K. Then*

$$\mathfrak{d}_{\mathcal{O}/\mathfrak{o}} = \mathrm{N}_{\mathcal{O}/\mathfrak{o}}(\mathfrak{D}_{\mathcal{O}}/\mathfrak{o}).$$

Proof. In view of **??**, we may assume $\mathcal{O}/\mathfrak{o}$ is a local Dedekind extension. Therefore $\mathfrak{o}$ is a discrete valuation ring, $\mathcal{O}$ is a principal ideal domain, and $\mathcal{O}/\mathfrak{o}$ admits an integral basis $\alpha_1, \ldots, \alpha_n$ making $\mathcal{O}$ a free $\mathfrak{o}$-module of rank n. Then $\mathfrak{d}_{\mathcal{O}/\mathfrak{o}}$ is a principal integral ideal where

$$\mathfrak{d}_{\mathcal{O}/\mathfrak{o}} = d_{\mathfrak{o}}(\mathcal{O})\mathfrak{o}.$$

As $\mathcal{O}$ is a principal ideal domain, every fractional ideal is also principal. So on the one hand, $\mathfrak{C}_{\mathcal{O}/\mathfrak{o}} = \lambda\mathcal{O}$ for some nonzero $\lambda \in L$ and $\lambda\alpha_1, \ldots, \lambda\alpha_n$ is a basis of L/K contained in $\mathfrak{C}_{\mathcal{O}/\mathfrak{o}}$. Moreover,

$$d_{L/K}(\lambda\alpha_1, \ldots, \lambda\alpha_n) = \mathrm{N}_{L/K}(\lambda)^2 d_{L/K}(\alpha_1, \ldots, \alpha_n),$$

by **??** and that base change matrix from $\alpha_1, \ldots, \alpha_n$ to $\lambda\alpha_1, \ldots, \lambda\alpha_n$ is the multiplication by λ map. Todo: [xxx] $\qquad\square$