# A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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Abstract. We construct a quadratic double Dirichlet series $Z(s,w)$ built from single variable quadratic Dirichlet $L$-functions $L(s,\chi)$ over $\mathbb{Q}$. We prove that $Z(s,w)$ admits meromorphic continuation to the $(s,w)$-plane and satisfies a group of functional equations.

## 1. Preliminaries

We present an overview of quadratic Dirichlet $L$-functions over $\mathbb{Q}$. We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Now we recall the characters on $\mathbb{Z}$. They are multiplicative functions $\chi : \mathbb{Z} \to \mathbb{C}$. Their image always lands in the roots of unity. The two flavors of characters of interest to use are:

- Dirichlet characters: multiplicative functions $\chi_m : \mathbb{Z} \to \mathbb{C}$ modulo $m \geq 1$ (in that they are $m$-periodic) and such that $\chi_m(n) = 0$ if $(m,n) > 1$.
- Hilbert symbols: Dirichlet characters modulo 1.

If $\chi$ is a character then its conjugate $\overline{\chi}$ is also a character. Moreover, $\overline{\chi}$ is the multiplicative inverse to $\chi$ and the characters modulo $m$ form a group under multiplication. This group is always finite and its order is $\phi(m)$. Characters also satisfy orthogonality properties:

**Theorem 1.1** (Orthogonality relations)**.** *Let $\chi$ and $\psi$ be any two Dirichlets character modulo $m$ and let $a$ and $b$ be any two integers modulo $m$. Then*

*(i)*

$$\frac{1}{\phi(m)} \sum_{a \ (mod\ m)} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

*(ii)*

$$\frac{1}{\phi(m)} \sum_{\chi \ (mod\ m)} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

Todo: [continue here] The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{F}_q[t]$. If $f \in \mathbb{F}_q[t]$ is a monic non-constant irreducible, define the quadratic residue symbol $\chi_f$ by

$$\chi_f(g) = \left(\frac{f}{g}\right) = g^{\frac{|f|-1}{2}} \pmod{f},$$

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for any $g \in \mathbb{F}_q[t]$. Then $\chi_f(g) \in \{\pm 1\}$ provided $f$ and $g$ are relatively prime and $\chi_f(g) = 0$ if $(f, g) > 1$. If $b \in \mathbb{F}^\times$, then we define the quadratic residue symbol $\chi_b$ by

$$\chi_b(g) = \left(\frac{b}{m}\right) = \operatorname{sgn}(b)^{\deg(f)},$$

where $\operatorname{sgn}(b) = \pm 1$ depending on if $b \in (\mathbb{F}^\times)^2$ or not. Moreover, if $d \in \mathbb{F}_q[t]$ then we set $\operatorname{sgn}(d) = \operatorname{sgn}(b_n)$ if $d(t) = b_n t^n + b_{n-1} t^{n+1} + \cdots + b_0$ (with $b_n \neq 0$). Extending $\chi_f$ multiplicativity in $f$, $\chi_f$ is defined for any $f$ not necessarily monic. The quadratic residue symbol also has the following reciprocity property:

**Theorem 1.2** (Quadratic reciprocity)**.** *If $f, g \in \mathbb{F}_q[t]$ are monic, square-free, and relatively prime, then*

$$\left(\frac{f}{g}\right) = (-1)^{\frac{q-1}{2}\deg(f)\deg(g)} \left(\frac{g}{f}\right).$$

Note that if $q \equiv 1 \pmod 4$, the sign in the statement of quadratic reciprocity is always 1 so that the reciprocity is perfect. We now describe the Hilbert symbols on $\mathbb{F}_q[t]$. In fact, there are only two Hilbert symbols, one non-trivial, and one trivial. The non-trivial Hilbert symbol is $\chi_\theta$ where $\theta \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$:

$$\chi_\theta(f) = (-1)^{\deg(f)}.$$

Note that $\overline{\chi_\theta} = \chi_\theta$. The other Hilbert symbol is the trivial character $\chi_\theta^2 = \chi_{\theta\theta} = \chi_1$. In general, we denote a Hilbert symbol by $\chi_a$ where $a \in \{1, \theta\}$.

We can now define the $L$-functions attached to the symbol $\chi_f$ for not necessarily monic $f$. We define the $L$-series $L(s, \chi_f)$ attached to $\chi_f$ by a Dirichlet series or Euler product:

$$L(s, \chi_f) = \sum_{g \text{ monic}} \frac{\chi_f(g)}{|g|^s} = \prod_{P \text{ monic irr}} \left(1 - \frac{\chi_f(P)}{|P|^s}\right)^{-1}.$$

By definition of the quadratic residue symbol, $L(s, \chi_f) \ll \zeta(s)$ for $\operatorname{Re}(s) > 1$ so that $L(s, \chi_f)$ is absolutely uniformly convergent on compacta in this region. $L(s, \chi_f)$ also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ if $f$ is square-free and is analytic otherwise (see [1] for a proof). Moreover, $L(s, \chi_f)$ is a polynomial in $q^{-s}$ of degree at most $\deg(f) - 1$. The completed $L$-function is defined as follows:

$$L^*(s, \chi_f) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_f) & \text{if } \deg(f) \text{ is even,} \\ L(s, \chi_f) & \text{if } \deg(f) \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_f) = \begin{cases} q^{2s-1} |f|^{\frac{1}{2}-s} L^*(1-s, \chi_f) & \text{if } \deg(f) \text{ is even,} \\ q^{2s-1} (q|f|)^{\frac{1}{2}-s} L^*(1-s, \chi_f) & \text{if } \deg(f) \text{ is odd.} \end{cases}$$

Note that in the case $\deg(f)$ is even, the conductor is $|f|$ and in the case $\deg(f)$ is odd, the conductor is $q|f|$. In other words, the gamma factors depend upon the degree of $f$. This will cause a small but important technical issue later when we want to derive functional equations for the quadratic double Dirichlet series.

## The Quadratic Double Dirichlet Series

## The Interchange

## Weighting Terms

## Funcational Equations

## Meromorphic Continuation

## Poles and Residues

## References

[1] Rosen, M. (2002). Number theory in function fields (Vol. 210). Springer Science & Business Media.
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[4] Stanley, R. (2023). Enumerative Combinatorics: Volume 2. Cambridge University Press.