A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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Abstract. We construct a quadratic double Dirichlet series $Z(s,w)$ built from single variable quadratic Dirichlet L-functions $L(s,\chi)$ over $\mathbb{Q}$. We prove that $Z(s,w)$ admits meromorphic continuation to the (s,w)-plane and satisfies a group of functional equations.

1. Preliminaries

We present an overview of quadratic Dirichlet L-functions over $\mathbb{Q}$. We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m\geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Now we recall characters on $\mathbb{Z}$. They are multiplicative functions $\chi : \mathbb{Z} \to \mathbb{C}$. The two flavors of characters of interest to use are:

- Dirichlet characters: multiplicative functions $\chi_d : \mathbb{Z} \to \mathbb{C}$ modulo $d \geq 1$ (in that they are d-periodic) and such that $\chi_d(m) = 0$ if $(m,d) > 1$.
- Hilbert symbols: Dirichlet characters modulo 1.

In either case, the image always lands in the roots of unity. If χ is a Dirichlet character then its conjugate $\overline{\chi}$ is also a Dirichlet character. Moreover, $\overline{\chi}$ is the multiplicative inverse to χ and the Dirichlet characters modulo m form a group under multiplication. This group is always finite and its order is $\phi(d) = |(\mathbb{Z}/d\mathbb{Z})^*|$. Dirichlet characters also satisfy orthogonality relations:

Theorem 1.1 (Orthogonality relations)**.**

(i) For any two Dirichlet characters χ and ψ modulo d,

$$\frac{1}{\phi(d)} \sideset{}{'}\sum_{a \pmod{d}} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

(ii) For any $a, b \in (\mathbb{Z}/d\mathbb{Z})^*$,

$$\frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{Z}$. First let us recall this symbol. For any odd prime p and any $m \geq 1$, we define the quadratic residue

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symbol $\left(\frac{m}{p}\right)$ by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod{p} = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod{p} \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod{p} \text{ is not solvable, .} \\ 0 & \text{if } m \equiv 0 \pmod{p}. \end{cases}$$

This symbol is multiplicative in m. Moreover,

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & p \equiv 1 \pmod{4}, \\ -1 & p \equiv 3 \pmod{4}, \end{cases} \quad \text{and} \quad \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & p \equiv 1,7 \pmod{8}, \\ -1 & p \equiv 3,5 \pmod{8}. \end{cases}$$

We can extend the quadratic residue symbol multiplicatively in the denomator. If $d = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of d, then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \le i \le k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any odd $d \geq 1$. The quadratic residue symbol also has the following reciprocity property:

Theorem 1.2 (Quadratic reciprocity)**.** *If $d, m \geq 1$ are odd and relatively prime, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d-1}{2}\frac{m-1}{2}} \left(\frac{m}{d}\right).$$

We will require a version of quadratic reciprocity when $d, m \geq 1$ are relatively prime but not necessarily odd. To this end, we define the symbol $\left(\frac{m}{2}\right)$ by

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1,7 \pmod{8}, \\ -1 & \text{if } m \equiv 3,5 \pmod{8}, , \\ 0 & \text{if } m \equiv 0 \pmod{2}. \end{cases}$$

This definition is chosen in accordance with $\left(\frac{2}{m}\right)$. Indeed, we have the reciprocity law

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2}\frac{m^{(2)}-1}{2}} \left(\frac{m}{d}\right),$$

Todo: [xxx]

We can now define the quadratic Dirichlet characters. For any non-zero $d \in \mathbb{F}_q[t]$, define the quadratic Dirichlet character χ_d by the following quadratic residue symbol:

$$\chi_d(m) = \left(\frac{d}{m}\right) = m^{\frac{|d|-1}{2}} \pmod{d},$$

for any non-zero monic $m \in \mathbb{F}_q[t]$. Then $\chi_d(m) \in \{\pm 1\}$ provided d and m are relatively prime and $\chi_d(m) = 0$ if $(m, d) > 1$. Note that by quadratic reciprocity, χ_d is a Dirichlet character modulo d if d is monic. Since the quadratic residue symbols are multiplicative, χ_d is multiplicative in d. Therefore, factoring out a constant if necessary, we may always force d to be monic. Moreover, for $b \in \mathbb{F}_q^\times$, we see that χ_b is a Hilbert symbol:

$$\chi_b(m) = \left(\frac{b}{m}\right) = \operatorname{sgn}(b)^{\deg(m)},$$

where $m \in \mathbb{F}_q[t]$ is a non-zero monic. The only Hilbet symbols we will need are those given by the quadratic reisdue symbol. There are only two of them: one nontrivial and one trivial. The nontrivial Hilbert symbol is χ_θ where $\theta \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$:

$$\chi_\theta(m) = (-1)^{\deg(m)},$$

where $m \in \mathbb{F}_q[t]$ is a non-zero monic. Note that $\overline{\chi_\theta} = \chi_\theta$. The other Hilbert symbol is the trivial character $\chi_\theta^2 = \chi_{\theta\theta} = \chi_1$. In general, we denote a Hilbert symbol by χ_a where $a \in \{1, \theta\}$.

With the Dirichlet characters and Hilbert symbols introduced, we are ready to discuss the L-functions associated to quadratic Dirichlet characters. We define the L-function $L(s, \chi_d)$ attached to χ_d by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \text{ monic}} \frac{\chi_d(m)}{|m|^s} = \prod_{P \text{ monic irr}} \left(1 - \frac{\chi_d(P)}{|P|^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character, $L(s, \chi_d) \ll \zeta(s)$ for $\mathrm{Re}(s) > 1$ so that $L(s, \chi_d)$ is locally absolutely uniformly convergent in this region. $L(s, \chi_d)$ also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ if d is a perfect square and is analytic otherwise (see [**?**] for a proof). Moreover, $L(s, \chi_d)$ is a polynomial in q^{-s} of degree at most $\deg(d) - 1$. The completed L-function is defined as follows:

$$L^*(s, \chi_d) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_d) & \text{if } \deg(d) \text{ is even,} \\ L(s, \chi_d) & \text{if } \deg(d) \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_d) = \begin{cases} q^{2s-1} |d|^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \deg(d) \text{ is even,} \\ q^{2s-1} (q|d|)^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \deg(d) \text{ is odd.} \end{cases}$$

Note that in the case $\deg(d)$ is even, the conductor is $|d|$ and in the case $\deg(d)$ is odd, the conductor is $q|d|$. In other words, the gamma factors depend upon the degree of d. This will cause a small but important technical issue later when we want to derive functional equations for the quadratic double Dirichlet series.

The Quadratic Double Dirichlet Series

The Interchange

Weighting Terms

Funcational Equations

Meromorphic Continuation

Poles and Residues

References

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