Analytic Number Theory

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Contents

Part I

An Introduction to Modular Forms & L-functions

Chapter 1

Preliminaries

There is quite a bit of knowledge that most authors assume one is fluent in when writing any text on analytic number theory that is not necessarily standard material every reader knowns. A good selection is the following:

- Asymptotic Notation,
- Dirichlet Characters,
- Special Sums,
- Decay & Integral Transforms,
- The Gamma Function.

This is not an extensive list (depending on whom is writing), but it is a decent one for sure. In the interest of keeping this text mostly self-contained, this chapter is dedicated to the basics of these topics as they are the gadgets that will take center stage in our later investigations. A well-versed reader is encouraged to skim these sections for completeness. On the other hand, readers who are not completely comfortable these topics are encouraged to read this chapter in full and accept the material mentioned without proof as black box. The mathematics presented in this chapter belongs to an analytic number theorist's tool box rather than being pure analytic number theory. In order to improve the readability of the remainder of the text we will use the results presented here without reference unless it is a matter of clarity. As for standard knowledge, we assume familiarity with basic number theory, complex analysis, real analysis, functional analysis, topology, and algebra. We have also outsourced specific subtopics to the appendix and we will reference them when necessary.

1.1 Asymptotic Notation

Much of the language of analytic number theory is given in asymptotic notation as it allows us to discuss approximate growth and dispense with superfluous constants. For this reason, asymptotic notation is the first material that we will present. The asymptotic notations that we will cover are listed in the following table:

Estimate	Notation
Big O	$f(z) = O(g(z))$
Vinogradov's symbol	$f(z) \ll g(z)$
Order of magnitude symbol	$f(z) \asymp g(z)$
Little o	$f(z) = o(g(z))$
Asymptotic equivalence	$f(z) \sim g(z)$
Omega symbol	$f(z) = \Omega(g(z))$

Implicit in all of these estimates is some limiting process $z \to z_0$ where z_0 is some complex number or ∞ (and $\pm\infty$ for the real case respectively). If z_0 is finite, then it is understood that the estimate is assumed to hold for all z such that $|z - z_0| < \delta$ for some real $\delta > 0$. If z_0 is infinite, then the estimate is assumed to hold for all sufficiently large values of z. That is, $|z| > z_0$ for some z_0 (and $z > z_0$ or $z < z_0$ in the real case for $\pm\infty$ respectively). If the limiting process is not explicitly mentioned, it is assumed to be as $z \to \infty$ (or as $+\infty$ or $-\infty$ for the real case depending upon the context and even then we may exclude the sign for brevity).

Remark 1.1.1. *Suppose $f : \mathbb{N} \to \mathbb{C}$ is a function. Extending f by making it linear between $f(n)$ and $f(n+1)$ so that $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ is piecewise continuous, we can consider estimates with n in place of z. All of the following theory still holds.*

O-estimates & Symbols

We say $f(z)$ **is of order** $g(z)$ or $f(z)$ is $O(g(z))$ as $z \to z_0$ and write $f(z) = O(g(z))$ if there is some positive constant c such that

$$|f(z)| \leq c|g(z)|,$$

holds as $z \to z_0$. We call this a O**-estimate**.

Remark 1.1.2. *Many authors assume that $g(z)$ is a nonnegative function so that the absolute values on $g(z)$ can be dropped. As we require other forms of asymptotic notation that will be used more generally, we do not make this assumption since one could very well replace $O(g(z))$ with $O(|g(z)|)$. In practice this deviation causes no issue.*

As a symbol, let $O(g(z))$ stand for a function $f(z)$ that is $O(g(z))$. Then we may use the O-estimates in algebraic equations. Note that this extends the definition of the symbol because $f(z) = O(g(z))$ means $f(z)$ is $O(g(z))$. Colloquially, the O-estimate says that for z close to z_0 the size of $f(z)$ grows like $g(z)$. It doesn't say much about the size of $g(z)$ compared to $f(z)$. The constant c is not unique as any $c' > c$ also works. Any such constant is called the **implicit constant** of the O-estimate. The implicit constant may depend on one or more parameters, ε, σ, etc. If so, we use subscripts O_ε, O_σ, $O_{\varepsilon,\sigma}$, etc. to indicate the dependence of the implicit constant on these parameters. If it is possible to choose the implicit constant independent of a certain parameter then we say that the estimate is **uniform** with respect to that parameter. Moreover, we say that an implicit constant is **effective** if the constant is numerically computable and **ineffective** otherwise. Many important problems in number theory are about computing effective constants with high degrees of precision or finding proofs where the implicit constants are effective.
The symbol $\ll$ is known as **Vinogradov's symbol** is an alternative way to express O-estimates. We write $f(z) \ll g(z)$ as $z \to z_0$ if $f(z) = O(g(z))$ as $z \to z_0$. We also write $f(z) \gg g(z)$ as $z \to z_0$ to mean

$g(z) \ll f(z)$ as $z \to z_0$. If there is a dependence of the implicit constant on parameters, we use subscripts to denote dependence on these parameters. If both $f(z) \ll g(z)$ and $g(z) \ll f(z)$ as $z \to z_0$, then we say $f(z)$ and $g(z)$ have the **same order of magnitude** and write $f(z) \asymp g(z)$ as $z \to z_0$. This is different from the O-estimate in the respect that for z close to z_0, $f(z)$ grows like $g(z)$ and conversely $g(z)$ grows like $f(z)$. In other words, $f(z)$ and $g(z)$ grow the same. If there is a dependence of the implicit constant on parameters, we use subscripts to denote dependence on these parameters. From the definition of the O-estimate, this is equivalent to the existence of positive constants c_1 and c_2 such that

$$c_1|g(z)| \leq |f(z)| \leq c_2|g(z)|.$$

Equivalently, we can interchange $f(z)$ and $g(z)$ in the above equation.

o-estimates & Symbols

We say $f(z)$ **is of smaller order than** $g(z)$ or $f(z)$ is $o(g(z))$ as $z \to z_0$ and write $f(z) = o(g(z))$ if

$$\lim_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| = 0,$$

provided $g(z) \neq 0$ for all z sufficiently close to z_0. We call this a ***o*-estimate**. The o-estimate is saying that for z close to z_0 the growth of $g(z)$ dominates the growth of $f(z)$. If $f(z) = o(g(z))$ as $z \to z_0$, then $f(z) = O(g(z))$ as $z \to z_0$ where the implicit constant can be taken arbitrarily small by definition of the o-estimate. Therefore, o-estimates are stronger than O-estimates. As a symbol, let $o(g(z))$ stand for a function $f(z)$ that is $o(g(z))$. Then we may use the o-estimates in algebraic equations. Note that this extends the definition of the symbol because $f(z) = o(g(z))$ means $f(z)$ is $o(g(z))$.
We say $f(z)$ **is asymptotic to** $g(z)$ or $f(z)$ and $g(z)$ are **asymptotically equivalent** as $z \to z_0$ and write $f(z) \sim g(z)$ if

$$\lim_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| = 1,$$

provided $g(z) \neq 0$ for all z sufficiently close to z_0. It is useful to think of asymptotic equivalence as $f(z)$ and $g(z)$ being the same size in the limit as $z \to z_0$. Immediately from the definition, we see that this is an equivalence relation on functions. In particular, if $f(z) \sim g(z)$ and $g(z) \sim h(z)$ then $f(z) \sim h(z)$. Also, if $f(z) \sim g(z)$ as $z \to z_0$, then $f(z) \asymp g(z)$ as $z \to z_0$ with $c_1 \leq 1 \leq c_2$. So asymptotic equivalence is stronger than being of the same order of magnitude. Also note that $f(x) \sim g(x)$ is equivalent to $f(x) = g(x)(1 + o(1))$ and hence implies $f(x) = g(x)(1 + O(1))$. We write $f(z) = \Omega(g(z))$ as $z \to z_0$ if

$$\limsup_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| > 0.$$

This is precisely the negation of $f(z) = o(g(z))$, so that $f(z) = \Omega(g(z))$ means $f(z) = o(g(z))$ is false. In other words, for z near z_0 the growth of $g(z)$ does not dominate the growth of $f(z)$. This is weaker than $f(z) \gg g(z)$ because $f(z) = \Omega(g(z))$ means $|f(z)| \geq c|g(z)|$ for values of z arbitrarily close to z_0 whereas $f(z) \gg g(z)$ means $|f(z)| \geq c|g(z)|$ for all values of z sufficiently close to z_0.

Algebraic Manipulation for O-estimates and o-estimates

Asymptotic estimates become increasingly more useful when we can use them in equations to represent approximations. We catalogue some of the most useful algebraic manipulations for O-estimates and

o-estimates. Most importantly, if an algebraic equation involves a O-estimate or o-estimate then it is understood that the equation is not symmetric and is interpreted to be read from left to right. That is, any function of the form satisfying the estimate on the left-hand side also satisfies the estimate on the right-hand side too. We begind with O-estimates. The trivial algebraic manipulations are collected in the proposition below:

Proposition 1.1.1. *The following O-estimates hold as $z \to z_0$:*

(i) *If $f(z) = O(g(z))$ and $g(z) = O(h(z))$, then $f(z) = O(h(z))$. Equivalently, $O(O(h(z))) = O(h(z))$.*

(ii) *If $f_i(z) = O(g_i(z))$ for $i = 1, 2$, then $f_1(z)f_2(z) = O(g_1(z)g_2(z))$.*

(iii) *If $f(z) = O(g(z)h(z))$, then $f(z) = g(z)O(h(z))$.*

(iv) *If $f_i(z) = O(g_i(z))$ for $i = 1, 2, \ldots, n$, then $\sum_{1 \le i \le n} f_i(z) = O\left(\sum_{1 \le i \le n} |g_i(z)|\right)$.*

(v) *If $f_n(z) = O(g_n(z))$ for $n \ge 1$, then $\sum_{n \ge 1} f_n(z) = O\left(\sum_{n \ge 1} |g_n(z)|\right)$ provided both $\sum_{n \ge 1} f_n(z)$ and $\sum_{n \ge 1} |g_n(z)|$ converge.*

(vi) *If $f(z) = O(g(z))$ as $z \to z_0$ and $h(z)$ is such that $h(z) \to z_0$ as $z \to z_0$, then $(f \circ h)(z) = O((g \circ h)(z))$.*

(vii) *If $f(z) = O(g(z))$, then $\mathrm{Re}(f(z)) = O(g(z))$ and $\mathrm{Im}(f(z)) = O(g(z))$.*

Proof. Statements (i)-(iii) and (vi) follow immediately from the definition of the O-estimate. Statement (iv) follows from the definition and the triangle inequality. Statement (v) follows in the same way as (iv) given that both sums converge. Statement (vii) follows from the definition and the bounds $|\mathrm{Re}(z)| \le |z|$ and $|\mathrm{Im}(z)| \le |z|$. $\square$

O-estimates also behave well with respect to integrals provided the functions involved are of a real variable:

Proposition 1.1.2. *Suppose $f(z)$ and $g(z)$ are functions of a real variable, $f(z) = O(g(z))$ as $z \to \infty$, $f(z)$ and $g(z)$ are integrable on the region where this estimate holds, and let $[z_1, z_2]$ belong to this region. Then*

$$\int_{z_1}^{z_2} f(z)\,dz = O\left(\int_{z_1}^{z_2} |g(z)|\,dz\right).$$

Proof. This follows immediately from the definition of the O-estimate. $\square$

The next proposition is a collection of some useful expressions for simplifying equations involving O-estimates:

Proposition 1.1.3. *Let $f(z)$ be a function such that $f(z) \to 0$ as $z \to 0$. The following O-estimates hold as $z \to 0$:*

(i) $\frac{1}{1+O(f(z))} = 1 + O(f(z))$.

(ii) $(1 + O(f(z)))^p = 1 + O(f(z))$ *for any complex number p.*

(iii) $\log(1 + O(f(z))) = O(f(z))$.

(iv) $e^{1+O(f(z))} = 1 + O(f(z))$.

Proof. Taking the first-order Taylor series and applying Taylor's theorem, we have the O-estimates

(i) $\frac{1}{1+z} = 1 + O(z)$.

(ii) $(1+z)^p = 1 + O(z)$.

(iii) $\log(1+z) = O(z)$.

(iv) $e^z = 1 + O(z)$.

Now apply Proposition 1.1.1 (v) with $h(z) = O(f(z))$ to each of these estimates, and use Proposition 1.1.1 (i). $\square$

For o-estimates, the following properties are useful:

Proposition 1.1.4. *The following o-estimates hold as $z \to z_0$:*

(i) If $f(z) = o(g(z))$ and $g(z) = o(h(z))$, then $f(z) = o(h(z))$. Equivalently, $o(o(h(z))) = o(h(z))$.

(ii) If $f_i(z) = o(g_i(z))$ for $i = 1, 2$, then $f_1(z)f_2(z) = o(g_1(z)g_2(z))$.

(iii) If $f(z) = o(g(z)h(z))$, then $f(z) = g(z)o(h(z))$.

(iv) If $f_i(z) = o(g_i(z))$ for $i = 1, 2, \ldots, n$, then $\sum_{1\le i\le n} f_i(z) = o\left(\sum_{1\le i\le n} |g_i(z)|\right)$.

(v) If $f(z) = o(g(z))$ as $z \to z_0$ and $h(z)$ is such that $h(z) \to z_0$ as $z \to z_0$, then $(f \circ h)(z) = o((g \circ h)(z))$.

Proof. Statements (i)-(iii) and (v) follow immediately from the definition of the o-estimate and basic properties of limits. Statement (iv) follows from the definition and that $\sum_{1\le i\le n} |g_i(z)| \ge |g_i(z)|$ for any $1 \le i \le n$. $\square$

1.2 Dirichlet Characters

The most important multiplicative periodic functions for an analytic number theorist are the Dirichlet characters. These have a rich theory of their own which we will discuss, and naturally appear in many other places. It will be convienent to setup some notation for the remainder of the text. If a is a residue class that is invertible, we let $\overline{a}$ denote the inverse class. For example, if a is taken modulo m and $(a, m) = 1$ then $\overline{a}$ is the residue class modulo m such that $a\overline{a} \equiv 1 \pmod{m}$. Moreover, a $'$ on a sum over a set of integers modulo m will indicate that the sum is over all $a \pmod{m}$ such that $(a, m) = 1$.
A **Dirichlet character** χ modulo $m \ge 1$ (or of modulus $m \ge 1$) is an m-periodic completely multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ such that $\chi(a) = 0$ if and only if $(a, m) > 1$. Sometimes we will also write χ_m to denote a Dirichlet character modulo m if we need to express the dependence upon the modulus. For any m, there is always the **principal Dirichlet character** modulo m which we denote by $\chi_{m,0}$ (sometimes also seen as $\chi_{0,m}$ or the ever more confusing χ_0) and is defined by

$$\chi_{m,0}(a) = \begin{cases} 1 & (a, m) = 1, \\ 0 & (a, m) > 1. \end{cases}$$

When $m = 1$, the principal Dirichlet character is identically 1 and we call this the **trivial Dirichlet character**. This is also the only Dirichlet character modulo 1, so $\chi_1 = \chi_{1,0}$. In general, we say a Dirichlet character χ is **principal** if it only takes values 0 or 1.
We now discuss some basic facts of Dirichlet characters. Since $a^{\phi(m)} \equiv 1 \pmod{m}$ by Euler's little theorem, where ϕ is Euler's totient function, the multiplicativity of χ implies $\chi(a)^{\phi(m)} = 1$. Therefore

the nonzero values of χ_m are $\phi(m)$-th roots of unity. In particular, there are only finitely many Dirichlet characters of any fixed modulus m. Given two Dirichlet character χ and ψ modulo m, we define $\chi\psi$ by $\chi\psi(a) = \chi(a)\psi(a)$. This is also a Dirichlet character modulo m, so the Dirichlet characters modulo m form an abelian group denoted by X_m. If we have a Dirichlet character χ modulo m, then $\overline{\chi}$ defined by $\overline{\chi}(a) = \overline{\chi(a)}$ is also a Dirichlet character modulo m and is called the **conjugate Dirichlet character** of χ. Since the nonzero values of χ are roots of unity, if $(a, m) = 1$ then $\overline{\chi}(a) = \chi(a)^{-1}$. So $\overline{\chi}$ is the inverse of χ.
This is all strikingly similar to characters on $(\mathbb{Z}/m\mathbb{Z})^*$ (see Appendix C.1), and there is a connection. To see it, by the periodicity of χ, it's nonzero values are uniquely determined by $(\mathbb{Z}/m\mathbb{Z})^*$. Then since χ is multiplicative, it descends to a character χ of $(\mathbb{Z}/m\mathbb{Z})^*$ (we abuse notation here). Conversely, if we are given a character χ of $(\mathbb{Z}/m\mathbb{Z})^*$ we can extend it to a Dirichlet character by defining it to be m-periodic and declaring $\chi(a) = 0$ if $(a, m) > 1$. We call this extension the **zero extension**. So in other words, Dirichlet characters modulo m are the zero extensions of group characters on $(\mathbb{Z}/m\mathbb{Z})^*$. Clearly zero-extension respects multiplication of characters. As groups are isomorphic to their character groups (see Appendix C.1), we deduce that the group of Dirichlet characers modulo m is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^*$. That is, $X_m \cong \widehat{(\mathbb{Z}/m\mathbb{Z})^*} \cong (\mathbb{Z}/m\mathbb{Z})^*$. In particular, there are $\phi(m)$ Dirichlet characters modulo m. From now on we identify Dirichlet characters modulo m with their corresponding group characters of $(\mathbb{Z}/m\mathbb{Z})^*$. We now state two very useful relations called **orthogonality relations** for Dirichlet characters (this follows from the more general orthogonality relations in Appendix C.1 but we wish to give a direct proof):

Proposition 1.2.1.

(i) For any two Dirichlet characters χ and ψ modulo m,

$$\frac{1}{\phi(m)} \sideset{}{'}\sum_{a \pmod m} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

(ii) For any $a, b \in (\mathbb{Z}/m\mathbb{Z})^*$,

$$\frac{1}{\phi(m)} \sum_{\chi \pmod m} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

Proof. We will prove the statements separately.

(i) Denote the left-hand side by S and let b be such that $(b, m) = 1$. Then $a \to ab^{-1}$ is a bijection on $(\mathbb{Z}/m\mathbb{Z})^*$ so that

$$\frac{\chi(b)\overline{\psi}(b)}{\phi(m)} \sideset{}{'}\sum_{a \pmod m} \chi(a)\overline{\psi}(a) = \frac{1}{\phi(m)} \sideset{}{'}\sum_{a \pmod m} \chi(ab)\overline{\psi}(ab) = \frac{1}{\phi(m)} \sideset{}{'}\sum_{a \pmod m} \chi(a)\overline{\psi}(a).$$

Consquently $\chi(b)\overline{\psi}(b)S = S$ so that $S = 0$ unless $\chi(b)\overline{\psi}(b) = 1$ for all b such that $(b, m) = 1$. This happens if and only if $\psi = \chi$ in which case $S = 1$. This proves (i).

(ii) Denote the left-hand side by S. Let ψ be any Dirichlet character modulo m. As $\chi \to \chi\overline{\psi}$ is a bijection on X_m, we have

$$\frac{\psi(a)\overline{\psi}(b)}{\phi(m)} \sum_{\chi \pmod m} \chi(a)\overline{\chi}(b) = \frac{1}{\phi(m)} \sum_{\chi \pmod m} \psi\chi(a)\overline{\psi\chi}(b) = \frac{1}{\phi(m)} \sum_{\chi \pmod m} \chi(a)\overline{\chi}(b).$$

Therefore $\psi(a)\overline{\psi}(b)S = S$ so that $S = 0$ unless $\psi(a)\overline{\psi}(b) = \psi(a\overline{b}) = 1$ for all Dirichlet characters ψ modulo m. If this happens, then $a\overline{b} = 1 \pmod{m}$, or equivalently, $a \equiv b \pmod{m}$. Indeed, let $m = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ be the prime factorization of m. By the classification theorem for finite abelian groups,

$$(\mathbb{Z}/m\mathbb{Z})^* \cong (\mathbb{Z}/p_1^{r_1}\mathbb{Z})^* \times (\mathbb{Z}/p_2^{r_2}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_k^{r_k}\mathbb{Z})^*.$$

Now let n_i be a generator for the cyclic group $(\mathbb{Z}/p_i^{r_i}\mathbb{Z})^*$ and let ω_i be a primitive $p_i^{k_i}$-th root of unity for $1 \le i \le k$. Writing $a\overline{b} = n_1^{f_1}n_2^{f_2}\cdots n_k^{f_k}$, consider the Dirichlet character ψ modulo m defined by

$$\psi(n_1^{e_1}n_2^{e_2}\cdots n_k^{e_k}) = \omega_1^{e_1f_1}\omega_2^{e_2f_2}\cdots\omega_r^{e_rf_r}.$$

We have

$$\psi(1) = \omega_1^{f_1}\omega_2^{f_2}\cdots\omega_r^{f_r}.$$

As w_i has order $p_i^{k_i}$ and $0 \le f_i < p_i^{k_i} - 1$ for all i, the only way $\psi(1) = 1$ is if $f_i = 0$ for all i. Therefore $a\overline{b} = 1 \pmod{m}$. In this case $S = 1$. This proves (ii). □

In many practical settings, the orthogonality relations are often used in the following form:

Corollary 1.2.1.

(i) For any Dirichlet character χ modulo m,

$$\frac{1}{\phi(m)}\sideset{}{'}\sum_{a \pmod{m}} \chi(a) = \delta_{\chi,\chi_{m,0}}.$$

(ii) For any $a \in (\mathbb{Z}/m\mathbb{Z})^*$,

$$\frac{1}{\phi(m)}\sum_{\chi \pmod{m}} \chi(a) = \delta_{a,1}.$$

Proof. For (i), take $\psi = \chi_{m,0}$ in Proposition 1.2.1 (i). For (ii), take $b \equiv 1 \pmod{m}$ in Proposition 1.2.1 (ii). □

Now that we understand the basics of Dirichlet characters, we might be interested in computing them. This is not hard to do by hand for small m. For example, the table below gives the Dirichlet characters modulo 5 where i in $\chi_{5,i}$ is an indexing variable:

	0	1	2	3	4
$\chi_{5,0}$	0	1	1	1	1
$\chi_{5,1}$	0	1	i	$-i$	-1
$\chi_{5,2}$	0	1	$-i$	i	-1
$\chi_{5,3}$	0	1	-1	-1	1

If the modulus is large this is of course more difficult. However, there is a way to build Dirichlet characters of modulus m_2 from those of modulus m_1. Let χ_{m_1} be a Dirichlet character modulo m_1. If $m_1 \mid m_2$ then $(a, m_2) = 1$ impies $(a, m_1) = 1$. Therefore we can define a Dirichlet character χ_{m_2} by

$$\chi_{m_2}(a) = \begin{cases} \chi_{m_1}(a) & \text{if } (a, m_2) = 1, \\ 0 & \text{if } (a, m_2) > 1. \end{cases}$$

In this case, we say χ_{m_2} is **induced** from χ_{m_1} or that χ_{m_1} **lifts** to χ_{m_2}. All that is happening is χ_{m_2} is a Dirichlet character modulo m_2 whose values are given by those that χ_{m_1} takes. Clearly every Dirichlet character is induced from itself. On the other hand, provided there is a prime p dividing m_2 and not m_1 (so m_2 is a larger modulus), χ_{m_2} will be different from χ_{m_1}. For instance, $\chi_{m_2}(p) = 0$ but $\chi_{m_1}(p) \neq 0$. In general, we say a Dirichlet character is **primitive** if it is not induced by any character other than itself. Notice that the principal Dirichlet characters are precisely those Dirichlet characters induced from the trivial Dirichlet character, and the only primitive one is the trivial Dirichlet character. In any case, we can determine when Dirichlet characters are induced:

Proposition 1.2.2. *A Dirichlet character χ_{m_2} is induced from a Dirichlet character χ_{m_1} if and only if χ_{m_2} is constant on the residue classes in $(\mathbb{Z}/m_2\mathbb{Z})^*$ that are congruent modulo m_1. When this happens, χ_{m_1} is uniquely determined.*

Proof. For the foward implication, if χ_{m_2} is induced from χ_{m_1}, then χ_{m_2} is constant on the residue classes in $(\mathbb{Z}/m_2\mathbb{Z})^*$ that are congruent modulo m_1 because χ_{m_1} is. For the reverse implication, we first show that the reduction modulo m_1 map $\mathbb{Z}/m_2\mathbb{Z} \to \mathbb{Z}/m_1\mathbb{Z}$ induces a surjective homomorphism $\varphi : (\mathbb{Z}/m_2\mathbb{Z})^* \to (\mathbb{Z}/m_1\mathbb{Z})^*$. For $u_1 \in \mathbb{Z}/m_1\mathbb{Z}$ a unit, let a be the product of all primes dividing $\frac{m_2}{m_1}$ but not u_1. Then $u_2 = u_1 + m_1 a$ is not divisible by any prime p dividing m_1 or $\frac{m_2}{m_1}$. Hence $(u_2, m_2) = 1$ so u_2 is a unit. Note that a is uniquely determined by u_1 so that u_2 is uniquely determined and hence φ is unique. It's also a homomorphism because reduction modulo m_1 is. Now suppose χ_{m_2} is constant on the residue classes in $(\mathbb{Z}/m_2\mathbb{Z})^*$ that are congruent modulo m_1. Surjectivity of φ implies χ_{m_2} induces a unique group character on $(\mathbb{Z}/m_1\mathbb{Z})^*$ and hence a unique Dirichlet character modulo m_1. By construction χ_{m_2} is induced from χ_{m_1}. □

Why might we be interested in primitive Dirichlet characters? The reason is that the primitive Dirichlet character are the building blocks for all Dirichlet characters:

Theorem 1.2.1. *Every Dirichlet character χ is induced from a primitive Dirichlet character $\widetilde{\chi}$ that is uniquely determined by χ.*

Proof. Let the modulus of χ be m. Define a partial ordering on the set of Dirichlet characters where $\psi \leq \chi$ if χ is induced from ψ. This ordering is clearly reflexive, and it is transitive by Proposition 1.2.2. Set $X = \{\psi : \psi \leq \chi\}$. This set is nonempty, and is finite by Proposition 1.2.2. Now suppose $\chi_{m_1}, \chi_{m_2} \in X$. Setting $m_3 = (m_1, m_2)$, and we have a commuting square

$$\begin{array}{ccc} (\mathbb{Z}/m\mathbb{Z})^* & \xrightarrow{\varphi} & (\mathbb{Z}/m_1\mathbb{Z})^* \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ (\mathbb{Z}/m_2\mathbb{Z})^* & \longrightarrow & (\mathbb{Z}/m_3\mathbb{Z})^* \end{array}$$

where φ is as in Proposition 1.2.2. Also from Proposition 1.2.2, χ is constant on the residue classes of $(\mathbb{Z}/m\mathbb{Z})^*$ that are congruent modulo m_1 or m_2 and hence also m_3. Therefore Proposition 1.2.2 implies there is a unique Dirichlet character χ_{m_3} of modulus m_3 that lifts to χ_{m_1} and χ_{m_2}. We have now shown that every pair $\chi_{m_1}, \chi_{m_2} \in X$ has a lower bound χ_{m_3}. Hence X contains a primitive Dirichlet character $\widetilde{\chi}$ that is minimal with respect to this partial ordering. There is only one such element. Indeed, since $m_3 \leq m_1, m_2$ the partial ordering is compatible with the total ordering by period. Thus $\widetilde{\chi}$ is unique. □

In light of Theorem 1.2.1, we define **conductor** q of a Dirichlet character χ modulo m to be the period of the unique primitive character $\widetilde{\chi}$ that induces χ. This is the most important data of a Dirichlet character

since it tells us how χ is built. Note that χ is primitive if and only if its conductor and modulus are equal. Also observe that if χ has conductor q, then χ is q-periodic (necessarily $q \mid m$), and the nonzero values of χ are all q-th roots of unity because those are the nonzero values of $\widetilde{\chi}$. Moreover, $\chi = \widetilde{\chi}\chi_{\frac{m}{q},0}$ by the definition of induced Dirichlet characters.
We would also like to distinguish Dirichlet characters whose nonzero values are real or imaginary. We say χ is **real** if it is real-valued. Hence the nonzero values of χ are 1 or -1 since they must be roots of unity. We say χ is an **complex** if it is not real. More commonly, we distinguish Dirichlet characters of modulus m by their order as an element of $(\mathbb{Z}/m\mathbb{Z})^*$. If χ is of order 2, 3, etc in $(\mathbb{Z}/m\mathbb{Z})^*$ then we say it is **quadratic**, **cubic**, etc. In particular, a Dirichlet character is quadratic if and only if it is real. For example, if m is odd then the Jacobi symbol $\left(\frac{\cdot}{m}\right)$ is a quadratic Dirichlet character modulo m. When it is clear that the character is quadratic, we let χ_m denote the quadratic Dirichlet character modulo m defined by the Jacobi symbol. For any Dirichlet character χ, $\chi(-1) = \pm 1$ because $\chi(-1)^2 = 1$. We would like to distinguish this parity. Accordingly, we say χ is **even** if $\chi(-1) = 1$ and **odd** if $\chi(-1) = -1$. Clearly even Dirichlet characters are even functions and odd Dirichlet characters are odd functions. Moreover, χ and $\overline{\chi}$ have the same parity and any lift of χ has the same parity as χ.

1.3 Special Sums

Analytic number theory does not come without its class of special sums that appear natrually. They play the role of discrete countparts to continuous objects (there is a rich underpinning here). Without a sufficient understanding of these sums, they would cause a discrete obstruction to an analytic problem that we wish to solve.

Ramanujan Sums

Let's begin with the Ramanujan sum. For a positive integer m and any integer b, the **Ramanujan sum** $r(b; m)$ is defined by

$$r(b; m) = \sideset{}{'}\sum_{a \pmod{m}} e^{\frac{2\pi i a b}{m}}.$$

Note that the Ramanujan sum is a finite sum of m-th roots of unity on the unit circle. Clearly $r(0; m) = \phi(m)$ where ϕ is Euler's totient function. Ramanujan sums can be computed explicitly by means of the Möbius function (see Appendix A.1):

Proposition 1.3.1. *For any positive integer m and any nonzero $b \in \mathbb{Z}$,*

$$r(b; m) = \sum_{\ell \mid (b,m)} \ell \mu\left(\frac{m}{\ell}\right).$$

Proof. This is a computation:

$$\begin{aligned}
r(b;m) &= \sideset{}{'}\sum_{a \pmod m} e^{\frac{2\pi i ab}{m}} \\
&= \sum_{a \pmod m} e^{\frac{2\pi i ab}{m}} \sum_{d|(a,m)} \mu(d) && \text{Proposition A.2.1} \\
&= \sum_{d|m} \mu(d) \sum_{\substack{a \pmod m \\ d|a}} e^{\frac{2\pi i ab}{m}} \\
&= \sum_{d|m} \mu(d) \sum_{kd \pmod m} e^{\frac{2\pi i kdb}{m}} && a \to kd. \\
&= \sum_{d|m} \mu(d) \sum_{k \left(\mathrm{mod}\ \frac{m}{d}\right)} e^{\frac{2\pi i kb}{\frac{m}{d}}}.
\end{aligned}$$

Now if $\frac{m}{d} \mid b$ the inner sum is $\frac{m}{d}$, and otherwise it is zero because $k \to k\overline{b}$ is a bijection on $\mathbb{Z}/\frac{m}{d}\mathbb{Z}$ and thus we are summing over all $\left(\frac{m}{d}\right)$-th roots of unity. So the double sum above reduces to

$$\sum_{\substack{\frac{m}{d}|b \\ d|m}} \frac{m}{d}\mu(d) = \sum_{\ell|(b,m)} \ell\mu\left(\frac{m}{\ell}\right),$$

upon performing the change of variables $\frac{m}{d} \to \ell$. $\square$

Gauss Sums

When we take a Ramanujan sum and introduce a Dirichlet character we get a Gauss sums. Let χ be a Dirichlet character modulo m. For any $b \in \mathbb{Z}$, the **Gauss sum** $\tau(b,\chi)$ attached to χ is given by

$$\tau(b,\chi) = \sum_{a \pmod m} \chi(a) e^{\frac{2\pi i ab}{m}} = \sideset{}{'}\sum_{a \pmod m} \chi(a) e^{\frac{2\pi i ab}{m}},$$

where the last equality follows because $\chi(a) = 0$ unless $(a,m) = 1$. If $b = 1$ we will write $\tau(\chi)$ instead. That is, $\tau(\chi) = \tau(1,\chi)$. Gauss sums are interesting because the Dirichlet character is a multiplicative character while the exponenetial is an additive one. So the Gauss sums is a convolution between a multiplicative and additive character. This is the fundamental reason that makes them difficult to study as one needs to seperate the additive and multiplicative structures. Observe that if $m = 1$ then χ is the trivial character and $\tau(b,\chi) = 1$. So the interesting cases are when $m \geq 2$. There are some basic properties of Gauss sums that are very useful:

Proposition 1.3.2. *Let χ and ψ be Dirichlet characters modulo $m, n \geq 2$ respectively and let $b \in \mathbb{Z}$. Then the following hold:*

(i) $\overline{\tau(b,\overline{\chi})} = \chi(-1)\tau(b,\chi)$.

(ii) If $(b,m) = 1$, then $\tau(b,\chi) = \overline{\chi}(b)\tau(\chi)$.

(iii) If $(b,m) > 1$ and χ is primitive, then $\tau(b,\chi) = 0$.

(iv) If $(m,n) = 1$, then $\tau(b,\chi\psi) = \chi(n)\psi(m)\tau(b,\chi)\tau(b,\psi)$.

(v) *Let q be the conductor of χ and let $\widetilde{\chi}$ be the primitive Dirichlet character that lifts to χ. Then*

$$\tau(\chi) = \mu\left(\frac{m}{q}\right)\widetilde{\chi}\left(\frac{m}{q}\right)\tau(\widetilde{\chi}).$$

Proof. We will prove the statements separately.

(i) Observe that $a \to -a$ is an isomorphism of $\mathbb{Z}/m\mathbb{Z}$ since $(-1, m) = 1$. Thus

$$\begin{aligned}
\overline{\tau(b, \overline{\chi})} &= \overline{\sum_{a \pmod m} \overline{\chi}(a) e^{\frac{2\pi iab}{m}}} \\
&= \sum_{a \pmod m} \chi(a) e^{-\frac{2\pi iab}{m}} \\
&= \sum_{a \pmod m} \chi(-a) e^{\frac{2\pi iab}{m}} \\
&= \chi(-1) \sum_{a \pmod m} \chi(a) e^{\frac{2\pi iab}{m}} \\
&= \chi(-1)\tau(b, \chi),
\end{aligned}$$

and (i) follows.

(ii) The map $a \to a\overline{b}$ is an isomorphism of $\mathbb{Z}/m\mathbb{Z}$ since $(b, m) = 1$. Therefore

$$\tau(b, \chi) = \sum_{a \pmod m} \chi(a) e^{\frac{2\pi iab}{m}} = \sum_{a \pmod m} \chi(a\overline{b}) e^{\frac{2\pi ia}{m}} = \overline{\chi}(b) \sum_{a \pmod m} \chi(a) e^{\frac{2\pi ia}{m}} = \overline{\chi}(b)\tau(\chi),$$

and (ii) is proven.

(iii) Now fix a divisor $d < m$ of m and choose an integer c such that $c \equiv 1 \pmod m$. Then necessarily $(c, m) = 1$. As $d \mid m$, $c \equiv 1 \pmod d$ and $(c, d) = 1$. Moreover, there is such a c with the additional property that $\chi(c) \neq 1$. For if not, χ is induced from $\chi_{d,0}$ which contradicts χ being primitive. Now set $d = \frac{m}{(b,m)} < m$ and choose c as above. Since $(c, m) = 1$, $a \to a\overline{c}$ is a bijection on $\mathbb{Z}/m\mathbb{Z}$, so that

$$\chi(c)\tau(b, \chi) = \sum_{a \pmod m} \chi(ac) e^{\frac{2\pi iab}{m}} = \sum_{a \pmod m} \chi(a) e^{\frac{2\pi iab\overline{c}}{m}}.$$

As $e^{\frac{2\pi ib}{m}}$ is a d-th root of unity, and $\overline{c} \equiv 1 \pmod d$ (because c is and $d \mid m$) we have $e^{\frac{2\pi iab\overline{c}}{m}} = e^{\frac{2\pi iab}{m}}$. Thus the last sum above is $\tau(b, \chi)$. So altogether $\chi(c)\tau(b, \chi) = \tau(b, \chi)$. Since $\chi(c) \neq 1$, we conclude $\tau(b, \chi) = 0$ proving (iii).

There exists some integer c with $(c, m) = 1$ such that $\chi(c) \neq 1$, for if not χ would be principal of modulus $m \geq 2$ and therefore not primitive.

(iv) Since $(m, n) = 1$, the Chinese remainder theorem implies that $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/mn\mathbb{Z})$ via the isomorphim $(a,b) \to an + a'm$ with a taken modulo m and a' taken modulo n. Therefore

$$\begin{aligned}
\tau(b,\chi\psi) &= \sum_{an+a'm \pmod{mn}} \chi\psi(an+a'm)e^{\frac{2\pi i(an+a'm)b}{mn}} \\
&= \sum_{a \pmod m} \sum_{a' \pmod n} \chi\psi(an+a'm)e^{\frac{2\pi i(an+a'm)b}{mn}} \\
&= \sum_{a \pmod m} \sum_{a' \pmod n} \chi(an+a'm)\psi(an+a'm)e^{\frac{2\pi i(an+a'm)b}{mn}} \\
&= \sum_{a \pmod m} \sum_{a' \pmod n} \chi(an)\psi(a'm)e^{\frac{2\pi i(an+a'm)b}{mn}} \\
&= \chi(n)\psi(m) \sum_{a \pmod m} \sum_{a' \pmod n} \chi(a)\psi(a')e^{\frac{2\pi i ab}{m}}e^{\frac{2\pi i a'b}{n}} \\
&= \chi(n)\psi(m) \sum_{a \pmod m} \chi(a)e^{\frac{2\pi i ab}{m}} \sum_{a' \pmod n} \psi(a')e^{\frac{2\pi i a'b}{n}} \\
&= \chi(n)\psi(m)\tau(b,\chi)\tau(b,\psi).
\end{aligned}$$

This proves (iv).

(v) If $\left(\frac{m}{q},q\right) > 1$, then $\widetilde{\chi}\left(\frac{m}{q}\right) = 0$ so we need to show $\tau(\chi) = 0$. As $\left(\frac{m}{q},q\right) > 1$, there exists a prime p such that $p \mid \frac{m}{q}$ and $p \mid q$. By Euclidean division we may write any a modulo m in the form $a = a'\frac{m}{p} + a''$ with a' taken modulo p and a'' taken modulo $\frac{m}{p}$. Then

$$\tau(\chi) = \sum_{a \pmod m} \chi(a)e^{\frac{2\pi i a}{m}} = \sum_{\substack{a' \pmod p \\ a'' \left(\bmod \frac{m}{p}\right)}} \chi\left(a'\frac{m}{p} + a''\right)e^{\frac{2\pi i\left(a'\frac{m}{p}+a''\right)}{m}}. \tag{1.1}$$

Since $p \mid \left(\frac{m}{q},q\right)$, we have $p^2 \mid m$. Therefore $\left(a'\frac{m}{p} + a'', m\right) = 1$ if and only if $\left(a'\frac{m}{p} + a'', \frac{m}{p}\right) = 1$ and this latter condition is equivalent to $\left(a'', \frac{m}{p}\right) = 1$. Thus the last sum in Equation (1.1) is

$$\sum_{\substack{a' \pmod p \\ a'' \left(\bmod \frac{m}{p}\right) \\ \left(a'',\frac{m}{p}\right)=1}} \chi\left(a'\frac{m}{p} + a''\right)e^{\frac{2\pi i\left(a'\frac{m}{p}+a''\right)}{m}}.$$

As $p \mid \frac{m}{q}$, we know $q \mid \frac{m}{p}$ so that $a'\frac{m}{p} + a'' \equiv a'' \pmod q$. Then Proposition 1.2.2 implies $\chi\left(a'\frac{m}{p} + a''\right) = \widetilde{\chi}(a'')$ and this sum is further reduced to

$$\sideset{}{'}\sum_{a'' \left(\bmod \frac{m}{p}\right)} \widetilde{\chi}(a'')e^{\frac{2\pi i a''}{m}} \sum_{a' \pmod p} e^{\frac{2\pi i a'}{p}}. \tag{1.2}$$

The inner sum in Equation (1.2) vanishes since it is the sum over all p-th roots of unity and thus $\tau(\chi) = 0$. Now suppose $\left(\frac{m}{q},q\right) = 1$. Then (iv) implies

$$\tau(\chi) = \tau(\widetilde{\chi}\chi_{\frac{m}{q},0}) = \widetilde{\chi}\left(\frac{m}{q}\right)\chi_{\frac{m}{q},0}(q)\tau(\widetilde{\chi})\tau(\chi_{\frac{m}{q},0}) = \tau(\chi_{\frac{m}{q},0})\widetilde{\chi}\left(\frac{m}{q}\right)\tau(\widetilde{\chi}).$$

Now observe that $\tau(\chi_{\frac{m}{q},0}) = r\left(1;\frac{m}{q}\right)$. By Proposition 1.3.1 we see that $r\left(1;\frac{m}{q}\right) = \mu\left(\frac{m}{q}\right)$ and (v) follows. $\square$

Notice that Proposition 1.3.2 reduces the evaluation of the Gauss sum $\tau(b,\chi)$ to that of $\tau(\chi)$ at least when χ is primitive. When χ is not primitive and $(b,m)>1$ we need to appeal to evaulating $\tau(b,\chi)$ by more direct means. Evaluating $\tau(\chi)$ for general characters χ turns out to be a very difficult problem and is still open. However, it is not difficult to determine the modulus of $\tau(\chi)$ when χ is primitive:

Theorem 1.3.1. *Let χ be a primitive Dirichlet character modulo $m \geq 2$. Then*

$$|\tau(\chi)| = \sqrt{m}.$$

Proof. This is just a computation:

$$\begin{aligned}
|\tau(\chi)|^2 &= \tau(\chi)\overline{\tau(\chi)} \\
&= \sum_{a \pmod m} \tau(\chi)\overline{\chi}(a)e^{-\frac{2\pi i a}{m}} \\
&= \sum_{a \pmod m} \tau(a,\chi)e^{-\frac{2\pi i a}{m}} && \text{Proposition 1.3.2 (i) and (ii)} \\
&= \sum_{a \pmod m}\left(\sum_{a' \pmod m} \chi(a')e^{\frac{2\pi i a a'}{m}}\right)e^{-\frac{2\pi i a}{m}} \\
&= \sum_{a,a' \pmod m} \chi(a')e^{\frac{2\pi i a(a'-1)}{m}} \\
&= \sum_{a' \pmod m} \chi(a')\left(\sum_{a \pmod m} e^{\frac{2\pi i a(a'-1)}{m}}\right).
\end{aligned}$$

Let $S(a')$ denote the inner sum. For the a' such that $a'-1 \equiv 0 \pmod m$, $S(a') = m$. Otherwise $a \to a(a'-1)$ is a bijection on $\mathbb{Z}/m\mathbb{Z}$ (we assumed $m \neq 1$) so that $S(a') = 0$ because it is the sum of all m-th roots of unity. It follows that the double sum is $\chi(1)m = m$. So altogether $|\tau(\chi)|^2 = m$ and hence $|\tau(\chi)| = \sqrt{m}$. $\square$

As an almost immediate corollary to Theorem 1.3.1, we deduce a useful expression for primitive Dirichlet characters modulo m in terms of additive characters on $(\mathbb{Z}/m\mathbb{Z})$:

Corollary 1.3.1. *Let χ be a primitive Dirichlet character modulo m Then*

$$\tau(n,\chi) = \overline{\chi}(n)\tau(\chi),$$

for all $n \in \mathbb{Z}$. In particular,

$$\chi(n) = \frac{1}{\tau(\overline{\chi})}\sum_{a \pmod m} \overline{\chi}(a)e^{\frac{2\pi i a n}{m}},$$

for all $n \in \mathbb{Z}$.

Proof. If $(n,m)=1$, then the first identity is Proposition 1.3.2 (ii). If $(n,m)>1$, then the first identity follows from Proposition 1.3.2 (iii) and that $\overline{\chi}(n)=0$. This proves the first identity in full. For the second identity, first note that $\tau(\chi)\neq 0$ by Theorem 1.3.1. Replacing χ with $\overline{\chi}$, dividing the first identity by $\tau(\chi)$, and expanding the Gauss sum, gives the second identity. $\square$

In light of Theorem 1.3.1 we define the **epsilon factor** ε_χ for a Dirichlet character χ of modulus m by

$$\varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{m}}.$$

Theorem 1.3.1 says that this value lies on the unit circle when χ is primitive and not the trivial character. In any case, the question of the evaluation of Gauss sums further boils down to determining what value the epsilon factor is. This is the real difficultly as the epsilon factor is very hard to calculate and its value is not known for general Dirichlet characters. When χ is primitive, there is a simple relationship between ε_χ and $\varepsilon_{\overline{\chi}}$:

Proposition 1.3.3. *Let χ be a primitive Dirichlet character modulo $m \geq 2$. Then*

$$\varepsilon_\chi \varepsilon_{\overline{\chi}} = \chi(-1).$$

Proof. By Proposition 1.3.2 (iii) and that ε_χ lies on the unit circle,

$$\varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{m}} = \chi(-1)\overline{\frac{\tau(\chi)}{\sqrt{m}}} = \chi(-1)\varepsilon_{\overline{\chi}}^{-1},$$

from whence the statement follows. $\square$

Quadratic Gauss Sums

Another class of sums is a generalization of the Gauss sum in the case where the Dirichlet character is quadratic and given by the Jacobi symbol. For a positive integer m and any $b \in \mathbb{Z}$, the **quadratic Gauss sum** $g(b, m)$ is defined by

$$g(b,m) = \sum_{a \pmod{m}} e^{\frac{2\pi i a^2 b}{m}}.$$

If $b = 1$ we write $g(m)$ instead. That is, $g(m) = g(m)$. Our language is somehwat abusive since if χ is the quadratic Dirichlet character modulo m (necessarily odd) given by the Jacobi symbol, then we have the two quadratic Gauss sums $g(b, m)$ and $\tau(b, \chi)$. It turns out that the sum $g(b, m) = \tau(b, \chi)$ when m is odd and χ is the quadratic Dirichlet character modulo m given by the Jacobi symbol. This will take a little work to prove. We first reduce to the case when $(b, m) = 1$:

Proposition 1.3.4. *Let m be a positive odd integer and let $b \in \mathbb{Z}$. Then*

$$g(b,m) = (b,m)g\left(\frac{b}{(b,m)}, \frac{m}{(b,m)}\right).$$

Proof. By Euclidean divison write any a modulo m in the form $a = a'\frac{m}{(b,m)} + a''$ with a' take modulo (b, m)

and a'' take modulo $\frac{m}{(b,m)}$. Then

$$\begin{aligned}
g(b,m) &= \sum_{a \pmod m} e^{\frac{2\pi i a^2 b}{m}} \\
&= \sum_{\substack{a' \pmod{(b,m)} \\ a'' \left(\bmod \frac{m}{(b,m)}\right)}} e^{\frac{2\pi i \left(a' \frac{m}{(b,m)} + a''\right)^2 b}{m}} \\
&= \sum_{a'' \left(\bmod \frac{m}{(b,m)}\right)} e^{\frac{2\pi i (a'')^2 b}{m}} \sum_{a' \pmod{(b,m)}} e^{\frac{2\pi i \left(2a''a' \frac{m}{(b,m)} + \left(a' \frac{m}{(b,m)}\right)^2\right) b}{m}} \\
&= \sum_{a'' \left(\bmod \frac{m}{(b,m)}\right)} e^{\frac{2\pi i (a'')^2 \frac{b}{(b,m)}}{\frac{m}{(b,m)}}} \sum_{a' \pmod{(b,m)}} e^{\frac{2\pi i \left(2a''a' \frac{m}{(b,m)} + \left(a' \frac{m}{(b,m)}\right)^2\right) \frac{b}{(b,m)}}{\frac{m}{(b,m)}}} \\
&= (b,m) \sum_{a'' \left(\bmod \frac{m}{(b,m)}\right)} e^{\frac{2\pi i (a'')^2 \frac{b}{(b,m)}}{\frac{m}{(b,m)}}},
\end{aligned}$$

where the last line follows because $\left(2a''a' \frac{m}{(b,m)} + \left(a' \frac{m}{(b,m)}\right)^2\right) \equiv 0 \left(\bmod \frac{m}{(b,m)}\right)$ and thus the second sum is (b,m). The remaining sum is $g\left(\frac{b}{(b,m)}, \frac{m}{(b,m)}\right)$ which finishes the proof. $\square$

As a consequence of Proposition 1.3.4, we may always assume $(b,m) = 1$. Now we give an equivalent formulation of the Gauss sum attached to quadratic characters given by Jacobi symbols and show that in the case $m = p$ an odd prime, our two notions of quadratic Gauss sums agree:

Proposition 1.3.5. *Let m be a positive odd integer and let $b \in \mathbb{Z}$ such that $(b,m) = 1$. Also let χ be the quadratic Dirichlet character modulo m given by the Jacobi symbol. Then*

$$\tau(b,\chi) = \sum_{a \pmod m} \left(1 + \left(\frac{a}{m}\right)\right) e^{\frac{2\pi i a b}{m}}.$$

Moreover, when $m = p$ is prime,

$$\tau(b,\chi) = g(b,p).$$

Proof. To prove the first statement, observe

$$\sum_{a \pmod m} \left(1 + \left(\frac{a}{m}\right)\right) e^{\frac{2\pi i a b}{m}} = \sum_{a \pmod m} e^{\frac{2\pi i a b}{m}} + \sum_{a \pmod m} \left(\frac{a}{m}\right) e^{\frac{2\pi i a b}{m}}.$$

The first sum on the right-hand side is zero as it is the sum over all m-th roots of unity since $(b,m) = 1$. This proves the first claim. Now let $m = p$ be an odd prime. From the definition of the Legendre symbol we see that $1 + \left(\frac{a}{p}\right) = 2, 0$ depending on if a is a quadratic residue modulo p or not provided $a \not\equiv 0 \pmod p$. If $a \equiv 0 \pmod p$, then $1 + \left(\frac{a}{p}\right) = 1$. Moreover, if a is a quadratic residue modulo p, then $a \equiv (a')^2 \pmod p$ for some a'. So one the one hand,

$$\tau(b,\chi) = \sum_{a \pmod p} \left(1 + \left(\frac{a}{p}\right)\right) e^{\frac{2\pi i a b}{p}} = 1 + 2 \sum_{\substack{a \pmod p \\ a \equiv (a')^2 \pmod p \\ a \not\equiv 0 \pmod p}} e^{\frac{2\pi i (a')^2 b}{p}}.$$

On the other hand,

$$g(b,p) = 1 + \sum_{\substack{a \pmod p \\ a \not\equiv 0 \pmod p}} e^{\frac{2\pi i a^2 b}{p}},$$

but this last sum counts every quadratic residue twice because $(-a)^2 = a^2$. Hence the two previous sums are equal. □

Proposition 1.3.5 gives an equivalence between our two notions of quadratic Gauss sums, but we would like a similar result when m is not prime. In this direction, a series of reduction properties will be helpful:

Proposition 1.3.6. *Let m and n be positive integers, p be an odd prime, and let $b \in \mathbb{Z}$. Then the following hold:*

(i) If $(b,p) = 1$, then $g(b,p^r) = pg(b,p^{r-2})$ for all $r \in \mathbb{Z}$ with $r \geq 2$.

(ii) If $(m,n) = 1$ and $(b,mn) = 1$, then $g(b,mn) = g(bn,m)g(bm,n)$.

(iii) If m is odd and $(b,m) = 1$, then $g(b,m) = \left(\frac{b}{m}\right)g(m)$ where $\left(\frac{b}{m}\right)$ is the Jacobi symbol.

Proof. We will prove the statements separately.

(i) First notice that

$$g(b,p^r) = \sum_{a \pmod{p^r}} e^{\frac{2\pi i a^2 b}{p^r}} = \sum_{a \pmod{p^r}}{}' e^{\frac{2\pi i a^2 b}{p^r}} + \sum_{a \pmod{p^{r-1}}} e^{\frac{2\pi i a^2 b}{p^{r-2}}},$$

since every a modulo p satisfies $(a,p) = 1$ or not. By Euclidean division every element a modulo p^{r-1} is of the form $a = a'p^{r-2} + a''$ with a' taken modulo p and a'' taken modulo p^{r-2}. Since $(a'p^{r-2} + a'') \equiv a'' \pmod{p^{r-2}}$, every a'' is counted p times modulo p^{r-2}. Along with the fact that $(a'p^{r-2} + a'')^2 \equiv (a'')^2 \pmod{p^{r-2}}$, these facts give the middle equality in the following chain:

$$\sum_{a \pmod{p^{r-1}}} e^{\frac{2\pi i a^2 b}{p^{r-2}}} = \sum_{\substack{a' \pmod p \\ a'' \left(\bmod p^{r-2}\right)}} e^{\frac{2\pi i \left(a'p^{r-2}+a''\right)^2 b}{p^{r-2}}} = p \sum_{a'' \pmod p} e^{\frac{2\pi i (a'')^2 b}{p^{r-2}}} = pg(b,p^{r-2}).$$

It remains to show

$$\sum_{a \pmod{p^r}}{}' e^{\frac{2\pi i a^2 b}{p^r}},$$

is zero. This sum is exactly $r(b;p^r)$ so by Proposition 1.3.1, and that $(b,p) = 1$, we conclude

$$\sum_{a \pmod{p^r}}{}' e^{\frac{2\pi i a^2 b}{p^r}} = \mu(p^r) = 0,$$

because $r \geq 2$. This proves (i).

(ii) Observe

$$g(bn,m)g(bm,n) = \left(\sum_{a \pmod m} e^{\frac{2\pi i a^2 bn}{m}}\right)\left(\sum_{a' \pmod n} e^{\frac{2\pi i (a')^2 bm}{n}}\right) = \sum_{\substack{a \pmod m \\ a' \pmod n}} e^{\frac{2\pi i \left((an)^2 + (a'm)^2\right)b}{mn}}.$$

Note that $e^{\frac{2\pi i\left((an)^2+(a'm)^2\right)b}{mn}}$ only depends upon $(an)^2+(a'm)^2$ modulo mn. Clearly $(an+a'm)^2 \equiv (an)^2+(a'm)^2 \pmod{mn}$, so set $a''=an+a'm$ taken modulo mn. Since $(m,n)=1$, the Chinese remainder theorem implies that $(\mathbb{Z}/m\mathbb{Z})\times(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/mn\mathbb{Z})$ via the isomorphim $(a,a') \to an+a'm$. Thus the last sum above is equal to

$$\sum_{a'' \pmod{mn}} e^{\frac{2\pi i (a'')^2 b}{mn}},$$

which is precisely $g(b,mn)$. So (ii) is proven.

(iii) If $m=p$, then Proposition 1.3.5, Proposition 1.3.2 (ii), and that quadratic characters are their own conjugate altogether imply the claim. Now let $r \geq 1$ and assume by strong induction that the claim holds when $m=p^{r'}$ for all positive integers r' such that $r'<r$. Then by (i), we have

$$g(b,p^r) = pg(b,p^{r-2}) = \left(\frac{b}{p^{r-2}}\right) pg(p^{r-2}) = \left(\frac{b}{p^{r-2}}\right) g(p^r) = \left(\frac{b}{p^r}\right) g(p^r). \tag{1.3}$$

It now suffices to prove the claim when $m=p^rq^s$ where q is another odd prime and $s \geq 1$. Then by (ii) and Equation (1.3), we compute

$$\begin{aligned} g(b,p^rq^s) &= g(bq^s,p^r)g(bp^r,q^s) \\ &= \left(\frac{bq^s}{p^r}\right)\left(\frac{bp^r}{q^s}\right) g(p^r)g(q^s) \\ &= \left(\frac{b}{p^r}\right)\left(\frac{q^s}{p^r}\right)\left(\frac{b}{q^s}\right)\left(\frac{p^r}{q^s}\right) g(p^r)g(q^s) \\ &= \left(\frac{b}{p^rq^s}\right)\left(\frac{q^s}{p^r}\right)\left(\frac{p^r}{q^s}\right) g(p^r)g(q^s) \\ &= \left(\frac{b}{p^rq^s}\right) g(q^s,p^r)g(p^r,q^s) \\ &= \left(\frac{b}{p^rq^s}\right) g(p^rq^s). \end{aligned}$$

This proves (iii). ∎

At last we can prove when our two notions of quadratic Gauss sums agree:

Theorem 1.3.2. *Let χ be the quadratic Dirichlet character modulo m given by the Jacobi symbol with conductor q and let $b \in \mathbb{Z}$ such that $(b,m)=1$. Then*

$$\tau(b,\chi) = g(b,q).$$

Proof. Since χ is quadratic, it suffices to prove the claim when $b=1$ by Proposition 1.3.2 (ii) and Proposition 1.3.6 (iii). Now let $m=p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ be the prime decomposition of m. Upon observing $\left(\frac{\cdot}{p}\right)^2 = \chi_{p,0}$, we see that $\chi = \widetilde{\chi}\chi_{p_1p_2\cdots p_k,0}$ where $\widetilde{\chi}$ is the primitive and quadratic Dirichlet character given by the Jacobi symbol with modulus q being the product of those primes p_i with e_i odd. Then by Proposition 1.3.2 (v), we have

$$\tau(\chi) = \mu\left(\frac{m}{q}\right)\widetilde{\chi}\left(\frac{m}{q}\right)\tau(\widetilde{\chi}) = \tau(\widetilde{\chi}),$$

where the latter equality follows because the defining property of q implies $\frac{m}{q}$ is a product of even powers of primes and $\widetilde{\chi}$ is quadratic. Letting χ_p denote the quadratic Dirichlet character modulo p given by the Legendre symbol, multiplicativity of the Jacobi symbol and repeated application of Proposition 1.3.2 (iv) gives the first equality in the chain

$$\tau(\widetilde{\chi}) = \prod_{\substack{1\le i,j\le k \\ r_i,r_j\equiv 1 \pmod 2}} \chi_{p_i}(p_j) \prod_{\substack{1\le i\le k \\ r_i\equiv 1 \pmod 2}} \tau(\chi_{p_i}) = \prod_{\substack{1\le i,j\le k \\ r_i,r_j\equiv 1 \pmod 2}} \left(\frac{p_j}{p_i}\right) \prod_{\substack{1\le i\le k \\ r_i\equiv 1 \pmod 2}} g(p_i) = g(q).$$

The middle equality comes from Proposition 1.3.5 and the last equality from repeated application of Proposition 1.3.6 (ii). So altogether we conclude $\tau(\chi) = g(q)$. □

Now let's turn to Proposition 1.3.6 and the evaluation of the quadratic Gauss sum. Proposition 1.3.6 (ii) and (iii) reduce the evaluation of $g(b, m)$ for odd m and $(b, m) = 1$ to computing $g(p)$ for p an odd prime. As with the Gauss sum, it is not difficult to compute the modulus of the quadratic Gauss sum:

Theorem 1.3.3. *Let m be a positive odd integer. Then*

$$|g(m)| = \sqrt{m}.$$

Proof. By Proposition 1.3.6 (ii), it suffices to prove this when $m = p^r$ is a power of an odd prime. By Euclidean division write $r = 2n + r'$ for some positive integer n and with $r' = 0, 1$ depending on if r is even or odd respectively. Then Proposition 1.3.6 (i) implies

$$|g(p^r)|^2 = p^{2n}|g(p^{r'})|^2.$$

If $r' = 0$, then $2n = r$ so that $p^{2n} = p^r$. Thus $|g(p^r)| = \sqrt{p^r}$. If $r' = 1$, then Theorem 1.3.1 and Proposition 1.3.5 together imply $|g(p^{r'})|^2 = p$ so that the right-hand side above is $p^{2n+1} = p^r$ and again we have $|g(p^r)| = \sqrt{p^r}$. □

Accordingly, we define the **epsilon factor** ε_m for a positive integer m by

$$\varepsilon_m = \frac{g(m)}{\sqrt{m}}.$$

Theorem 1.3.3 says that this value lies on the unit circle when m is odd. Thus the question of the evaluation of quadratic Gauss sums reduces to determining what the epsilon factor is. This was completely resolved and the original proof is due to Gauss in 1808 (see [Gau08]). He actually treated the case m is even as well. We have avoided discussing this because we will not need it in the following and many of the previous proofs need to be augmented when m is even (see [Lan94] for a treatment of the even case). As for the evaluation, one of the cleanest proofs uses analytic techniques (see [Lan94]) and the precise statement is the following:

Theorem 1.3.4. *Let m be a positive integer. Then*

$$\varepsilon_m = \begin{cases} (1+i) & \text{if } m \equiv 0 \pmod 4, \\ 1 & \text{if } m \equiv 1 \pmod 4, \\ 0 & \text{if } m \equiv 2 \pmod 4, \\ i & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

As an immediate corollary, this implies the evaluation of the epsilon factor ε_χ where χ is the quadratic Dirichlet character modulo p given by the Legendre symbol for an odd prime p:

Corollary 1.3.2. *Let p be an odd prime and χ be the quadratic Dirichlet character modulo p given by the Legendre symbol. Then*

$$\varepsilon_\chi = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ i & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. The statement follows immediately from Theorem 1.3.4 and Proposition 1.3.5. □

Kloosterman & Salié Sums

Our last class of sums generalize Ramanujan and Gauss sums. For a positive integer c and any integers n and m, the **Kloosterman sum** $K(n,m;c)$ is defined by

$$K(n,m;c) = \sum_{\substack{a \pmod c \\ (a,c)=1}} e^{\frac{2\pi i(an+\overline{a}m)}{c}} = \sideset{}{'}\sum_{a \pmod c} e^{\frac{2\pi i(an+\overline{a}m)}{c}}.$$

Notice that if either $n=0$ or $m=0$ then the Kloosterman sum reduces to a Ramanujan sum. Kloosterman sums have similar properties to those of Ramanujan and Gauss sums, but we will not need them. The only result we will need is a famous bound, often called the **Weil bound** for Kloosterman sums, proved by Weil (see [Wei48] for a proof):

Theorem 1.3.5 (Weil bound). *Let c be a positive integer and n and m be integers. Then*

$$|K(n,m;c)| \le \sigma_0(c)\sqrt{(n,m,c)}\sqrt{c},$$

where $\sigma_0(c)$ is the divisor function.

Lastly, Salié sums are Kloosterman sums with Dirichlet characters. To be precise, for a positive integer c, any integers n and m, and a Dirichlet character χ with conductor $q \mid c$, the **Salié sum** $S_\chi(n,m;c)$ is defined by

$$S_\chi(n,m;c) = \sum_{\substack{a \pmod c \\ (a,c)=1}} \chi(a)e^{\frac{2\pi i(an+\overline{a}m)}{c}} = \sideset{}{'}\sum_{a \pmod c} \chi(a)e^{\frac{2\pi i(an+\overline{a}m)}{c}}.$$

If either $n=0$ or $m=0$ then the Salié sum reduces to a Gauss sum.

1.4 Decay & Integral Transforms

Exponential Decay

Integrals are a core backbone of analytic number theory and all too often the domain we are integrating over is unbounded. Accordingly, the functions we would like to work with should be integrable over these domains. One way to "have our cake and eat it too" would be to require that our functions decay very rapidly near the unbounded regions. More precisely, we consider functions f such that $f(x)-\alpha = o(x^c)$ as $x \to x_0$ for all $c \in \mathbb{R}$. In this case we say f has **exponential decay** to α as $x \to x_0$. So save the constant term, f decays faster than any polynomial in x. Often we work with function f where f has exponential decay to zero, or we will work with functions $f(z,x)$ where for fixed z, f has exponential decay to zero in x. These functions guarantee that many integrals we take will be locally absolutely uniformly bounded under very mild conditions:

Method 1.4.1. Suppose we are given an integral

$$\int_D f(z,x)\,dx,$$

where D is an unbounded domain and such that the following hold:

(i) $f(z,x)$ is analytic on $\Omega \times D$ for some reigion Ω.

(ii) For any fixed z, we can decompose $D = D' \cup (D - D')$ into where D' is compact, f is integrable on D', and f vanishes at infinity with exponential decay on $(D - D')$.

Then the integral is holomorpic in z. By (i) and Theorem B.1.2 it suffices to show that the integral is locally absolutely uniformly bounded. We argue by splitting the region of integration into $D = D' \cup (D - D')$ where D' is compact. Choosing some compact subset K of Ω, the exponential decay of f implies that there is some c such that $f(z,x) = o(x^c)$. Say, take c to be the smallest c over all $z \in K$ with $c < 1$. Then

$$\int_D |f(z,x)|\,d\mu = \int_{D'} |f(z,x)|\,dx + \int_{D-D'} |f(z,x)|\,dx \ll MV + \int_{D-D'} |x^c|\,dx,$$

where $V = \mathrm{Vol}(D')$ is finite because D' is compact, and M is the supremum of $|f(z,x)|$ on the compact set $K \times D'$ which exists since $f(z,x)$ is analytic. As, $c < 1$ the last intergal is bounded by the integral test.

Notice that if $f(z,s) = f(z)$ the same argument shows that the integral is absolutely bounded. This method we will use repeatedly throughout the remainder of the text, and one should become familar with it.

The Fourier Transform

The first type of integral transform we will need is the Fourier transform. Suppose $f : \mathbb{R} \to \mathbb{C}$ is absolutely integrable. The **Fourier transform** $\hat{f} : \mathbb{R} \to \mathbb{C}$ of f is defined by

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i t x}\,dx.$$

This integral is absolutely convergent precisely because f is absolutely integrable. As a first application of Method 1.4.1, if f is analytic and has exponential decay to zero then its Fourier transform exists. In practical settings however, we usually restrict f to be a **Schwarz function** which means that f is smooth and $f^{(n)}(x) = o(x^c)$ for all $c \in \mathbb{R}$. In other words, f is a smooth function such that it and all of its derivatives have exponential decay to zero. For example, e^{-x^2} is a Schwarz function. Schwarz functions are very important because they are functions for which the **Poisson summation formula** applies:

Theorem 1.4.1 (Poisson summation formula). *Suppose f is a Schwarz function. Then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{t \in \mathbb{Z}} \hat{f}(t).$$

Proof. Set
$$F(x) = \sum_{n \in \mathbb{Z}} f(x+n).$$
Since f has exponential decay to zero, F is locally absolutely uniformly convergent by the Weirstrass M-test. Actually since f is Schwarz, F and all of its derivatives are locally absolutely uniformly convergent. By the uniform limit theorem F is smooth. Since F is also 1-periodic it admits a Fourier series (see Appendix B.4). The t-th Fourier coefficient of F is
$$\begin{aligned}
\hat{F}(t) &= \int_0^1 F(x)e^{-2\pi i t x}\,dx \\
&= \int_0^1 \sum_{n \in \mathbb{Z}} f(x+n)e^{-2\pi i t x}\,dx \\
&= \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n)e^{-2\pi i t x}\,dx && \text{DCT} \\
&= \sum_{n \in \mathbb{Z}} \int_n^{n+1} f(x)e^{-2\pi i t x}\,dx \\
&= \int_{-\infty}^{\infty} f(x)e^{-2\pi i t x}\,dx && \text{DCT} \\
&= \hat{f}(t).
\end{aligned}$$
Therefore the Fourier series of F is
$$F(x) = \sum_{t \in \mathbb{Z}} \hat{f}(t)e^{2\pi i t x}.$$
Setting $x = 0$ gives the desired identity. □

The Poisson summation formula is immensely useful (this cannot be overstated). It's sort of like a train in that it takes one where one wants to got to and back when one is done. Our primary application is that the Poisson summation formula will allow us to derive very striking transformation laws which are the key ingredient for analytically continuing many important objects.

The Mellin Transform

Like the Fourier transform, the Mellin transform is another type of integral transform. If f is some continuous function on $\mathbb{R}_{>0}$, then the **Mellin transform** $\{\mathcal{M}f\}(s)$ of $f(x)$ is given by
$$\{\mathcal{M}f\}(s) = \int_0^\infty f(x)x^s\,\frac{dx}{x}.$$
If $f(x)$ is a sufficiently nice function then the integral will be bounded in some half-plane in s. For example, this happens if $f(x)$ has exponential decay to zero as $x \to \infty$ and remains bounded as $x \to 0$. In this case, the integral is absolutely uniformly bounded for $\mathrm{Re}(s) > 0$. The classical example is when $f(x) = e^{-x}$ so that $\{\mathcal{M}e^{-x}\}(s) = \Gamma(s)$. Taking the Mellin transform of generalizations of e^x, namely theta functions (to follow), give L-functions. If the Mellin transform $g(s)$ is sufficiently nice, then the initital function can be recovered via means of the **inverse Mellin transform** $\{\mathcal{M}^{-1}g\}(x)$:
$$\{\mathcal{M}^{-1}g\}(x) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} g(s)x^{-s}\,ds.$$

It is not immediately clear that this integral converges or is independent of c. The following theorem makes precise what properties $g(s)$ needs to satisfty and for which c the inverse Mellin transform recovers the original function $f(x)$ (see [DB15] for a proof):

Theorem 1.4.2 (The Mellin inversion formula). *Let a and b be distinct real numbers. Suppose $g(s)$ is analytic in the strip $a < \mathrm{Re}(s) < b$, tends to zero uniformly as $\mathrm{Im}(s) \to \infty$ along any line $\mathrm{Re}(s) = c$ for $a < c < b$, and that the integral of $g(s)$ along this line is locally absolutely uniformly bounded. Then if*

$$f(x) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c} g(s) x^{-s}\, ds,$$

this integral is independent of c and moreover $g(s) = \{\mathcal{M}f\}(s)$. Conversely, suppose $f(x)$ is piecewise continuous on $\mathbb{R}_{>0}$ such that its value is halfway between the limit values at any jump discontinuity and

$$g(s) = \int_0^\infty f(x) x^s \, \frac{dx}{x},$$

is absolutely bounded for $a < \mathrm{Re}(s) < b$. Then $f(x) = \{\mathcal{M}^{-1}g\}(x)$.

1.5 The Gamma Function

The gamma function is ubiquitous in analytic number theory and the better one understands it the better one will be at seeing the forst for the trees in any problem involving analytic number theory. The **gamma function** $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\, dx,$$

for $\mathrm{Re}(z) > 0$. The integral is locally absolutely uniformly bounded in this region. Indeed, if K is a compact subset in the region $\mathrm{Re}(z) > 0$, then upon splitting the integral we have

$$\Gamma(z) = \int_0^1 e^{-x} x^{z-1}\, dx + \int_1^\infty e^{-x} x^{z-1}\, dx. \tag{1.4}$$

By Method 1.4.1, the second integral in Equation (1.4) is locally absolutely uniformly bounded in this region. As for the first integral, let K be a compact set in the region $\mathrm{Re}(z) > 0$. Set $\beta = \inf_{z\in K}\{\mathrm{Re}(z)\}$. Then

$$\int_0^1 |e^{-x} x^{z-1}|\, dx \le \int_0^1 x^{\mathrm{Re}(z)-1}\, dx = \left.\frac{x^{\mathrm{Re}(z)}}{\mathrm{Re}(z)}\right|_0^1 \le \frac{1}{\beta}. \tag{1.5}$$

Equation (1.5) implies that the first integral in Equation (1.4) is locally absolutely uniformly bounded too. Altogether, this means $\Gamma(z)$ is as well. Also note that for real $z > 0$, $\Gamma(z)$ is real. The most basic properties of $\Gamma(z)$ are the following:

Proposition 1.5.1. $\Gamma(z)$ *satisfies the following properties:*

(i) $\Gamma(1) = 1$.

(ii) $\Gamma(z + 1) = z\Gamma(z)$.

Proof. We obtain (i) by direct computation:

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = (-e^{-x})\Big|_0^\infty = 1.$$

An application of integration by parts gives (ii):

$$\Gamma(z+1) = \int_0^\infty e^{-x}x^z\,dx = (-e^{-x}x^z)\Big|_0^\infty + z\int_0^\infty e^{-x}x^{z-1}\,dx = z\int_0^\infty e^{-x}x^{z-1}\,dx = z\Gamma(z).$$

□

From Proposition 1.5.1 we see that for $z = n$ a poitive integer, $\Gamma(n) = (n-1)!$. So $\Gamma(z)$ can be thought of as a holomorphic extension of the factorial function. We can use property (ii) of Proposition 1.5.1 to extended $\Gamma(z)$ to a meromorphic function on all of $\mathbb{C}$:

Theorem 1.5.1. *$\Gamma(z)$ admits analytic continuation to a meromorphic function on $\mathbb{C}$ with poles at $z = -n$ for $n \geq 0$. All of these poles are simple and with residue $\frac{(-1)^n}{n!}$ at $z = -n$.*

Proof. Using Proposition 1.5.1, (ii) repeatedly, for any integer $n \geq 0$ we have

$$\Gamma(z) = \frac{\Gamma(z+1+n)}{z(z+1)\cdots(z+n)}.$$

The right-hand side defines an meromorphic function in the region $\mathrm{Re}(z) > -n$ and away from the points $0, -1, \ldots, -n$. Letting n be arbitrary, we see that $\Gamma(z)$ has meromorphic continuation to $\mathbb{C}$ with poles at $0, -1, -2, \ldots$. We now compute the residue at $z = -n$. Around this point $\Gamma(z)$ admits analytic continuation with representation

$$\frac{\Gamma(z+1+n)}{z(z+1)\cdots(z+n)},$$

where all of the factors except for $z+n$ are holomorphic at $z = -n$. Thus the pole is simple, and

$$\operatorname*{Res}_{z=-n}\Gamma(z) = \lim_{z\to -n}\frac{\Gamma(z+1+n)(z+n)}{z(z+1)\cdots(z+n)} = \frac{\Gamma(1)}{(-n)(1-n)\cdots(-1)} = \frac{(-1)^n}{n!}.$$

□

In particular, Theorem 1.5.1 implies $\mathrm{Res}_{z=0}\,\Gamma(z) = 1$ and $\mathrm{Res}_{z=1}\,\Gamma(z) = -1$. Also, since $\Gamma(z)$ is real for real $z > 0$ we have $\overline{\Gamma(z)} = \Gamma(\overline{z})$ and then the identity theorem implies that this holds everywhere $\Gamma(z)$ is analytic. There are a few other properties of the gamma function that are famous and which we will use frequently. The first of which is the **Legendre duplication formula** (see [Rem98] for a proof):

Theorem 1.5.2 (Legendre duplication formula). *For any $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$,*

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

As a first application, we can use this formula to compute $\Gamma\left(\frac{1}{2}\right)$. Letting $z = \frac{1}{2}$ in the Legendre duplication formula and recalling $\Gamma(1) = 1$, we see that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. There is also the important Hadamard factorization of the reciprocial of $\Gamma(z)$ (see [SSS03] for a proof):

Proposition 1.5.2. *For all $z \in \mathbb{C}$,*

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n\geq 1}\left(1 + \frac{z}{n}\right)e^{-\frac{z}{n}},$$

where γ is the Euler-Mascheroni constant.

In particular, $\frac{1}{\Gamma(z)}$ is entire so that $\Gamma(z)$ is nowhere vanishing on $\mathbb{C}$. Also, $\frac{1}{\Gamma(z)}$ is of order 1 (see Appendix B.5). In particular, this means that $\Gamma(z)$ is also order 1 for $\mathrm{Re}(z) > 0$. We call $\frac{\Gamma'}{\Gamma}(z)$ the **digamma function**. Equivalently, the digamma function is the logarithmic derivative of the gamma function. If we take the logarithmic derivative of the Hadamard factorization for $\frac{1}{z\Gamma(z)}$, we obtain a useful expression for the digamma function:

Corollary 1.5.1. *For all $z \in \mathbb{C} - \mathbb{Z}_{\leq -1}$,*

$$\frac{\Gamma'}{\Gamma}(z+1) = -\gamma + \sum_{n \geq 1} \left(\frac{1}{n} - \frac{1}{z+n} \right),$$

where γ is the Euler-Mascheroni constant.

Proof. By Proposition 1.5.1 (ii), $\frac{1}{\Gamma(z+1)} = \frac{1}{z\Gamma(z)}$. Taking the logarithmic derivative using Proposition 1.5.2 we obtain

$$-\frac{\Gamma'}{\Gamma}(z+1) = \gamma + \sum_{n \geq 1} \left(\frac{1}{z+n} - \frac{1}{n} \right),$$

provided z is away from the poles of $\Gamma(z)$. This finishes the proof. □

We will also require a well-known approximation for the gamma function known as **Stirling's formula** (see [Rem98] for a proof):

Theorem 1.5.3 (Stirling's formula).

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z},$$

provided $|\arg(z)| < \pi - \varepsilon$ and $|z| > \delta$ for some $\varepsilon, \delta > 0$.

If the real part of z lies inside some comapct set then Stirling's formula gives a useful estimate showing that $\Gamma(z)$ exhibits exponenetial decay as $z \to \infty$ (see [Rem98] for a proof):

Corollary 1.5.2. *Let $|\arg(z)| < \pi - \varepsilon$ and $|z| > \delta$ for some $\varepsilon, \delta > 0$. Then if $\mathrm{Re}(z)$ is restricted to some compact set with $z = x + iy$, we have*

$$\Gamma(z) \sim \sqrt{2\pi} |y|^{x-\frac{1}{2}} e^{-\frac{\pi}{2}|y|}.$$

From our earlier discussion on asymptotics, directly weaker than the asymptotic equivalence in Stirling's formula is the estimate

$$\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}(1 + O(1)). \tag{1.6}$$

Taking the logarithm (since $|\arg(z)| < \pi - \varepsilon$ the logarithm is defined) of this estimate gives

$$\log \Gamma(z) = \frac{1}{2} \log(2\pi) + \left(z - \frac{1}{2} \right) \log(z) - z + O(1), \tag{1.7}$$

which will be useful. Using Equation (1.7) we can obtain a useful asymptotic formula for the digamma function:

Proposition 1.5.3.

$$\frac{\Gamma'}{\Gamma}(z) = \log(z) + O(1),$$

provided $|\arg(z)| < \pi - \varepsilon$ and $|z| > \delta$ for some $\varepsilon, \delta > 0$.

Proof. Since $z - \frac{1}{2} \sim z$, Equation (1.7) give the simplifed asymptotic

$$\log \Gamma(z) = \frac{1}{2}\log(2\pi) + z\log(z) - z + O(1).$$

Set $g(z) = \frac{1}{2}\log(2\pi) + z\log(z) - z$ so that $\log\Gamma(z) = g(z) + O(1)$. Then $\log\Gamma(z) - g(z) = O(1)$, and by Cauchy's integral formula, we have

$$\begin{aligned}
\frac{\Gamma'}{\Gamma}(z) &= \frac{d}{dz}\left(g(z) + O(1)\right) \\
&= g'(z) + \frac{d}{dz}(\log\Gamma(z) - g(z)) \\
&= \log(z) + \frac{1}{2\pi i}\oint_{|w-z|=r} \frac{\log\Gamma(w) - g(w)}{(w-z)^2}\, dw,
\end{aligned}$$

for some sufficiently small radius $r > 0$. Therefore

$$\left|\frac{\Gamma'}{\Gamma}(z) - \log(z)\right| \leq \frac{1}{2\pi}\int_{|w-z|=r} \frac{|\log\Gamma(w) - g(w)|}{r^2}\, |dw| \ll 1,$$

where the last equality follows because $\log\Gamma(z) - g(z) = O(1)$. □

The last result we will need is an explicit representation for $\log\Gamma(z)$ known as **Binet's log gamma formula**:

Proposition 1.5.4 (Binet's log gamma formula).

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\Gamma(z) - z + \frac{1}{2}\log(2\pi) + 2\int_0^\infty \frac{\arctan\left(\frac{x}{z}\right)}{e^{2\pi x} - 1}\, dx.$$

Chapter 2

Holomorphic & Maass Forms

Holomorphic and Maass forms are special classes of functions on the upper half-space $\mathbb{H}$ of the complex plane. The former are holomorphic, have a transformation law with respect to a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, and satisfy a growth condition. The latter are real-analytic, eigenfunctions with respect to the Laplace operator, invariant with respect to a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, and satisfty a growth condition. We will introduce both of these forms in a general context as well as the more general automorphic forms. Before we introduce the forms themselves however, it is useful to discuss some of the theory about congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ and modular curves. This small discussion will make the setup of holomorphic, Maass, and automorphic forms more natural.

2.1 Congruence Subgroups & Modular Curves

Congruence Subgroups

The **modular group** is $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. That is, the modular group is the set of matrices with integer entries and of determinant 1 determined up to sign. The reason we are only interested in these matrices up to sign is because the modular group has a natural action on the upper half-space $\mathbb{H}$ and this action will be invariant under a change in sign. The first result usually proved about the modular group is that it is generated by two matrices:

Proposition 2.1.1.

$$\mathrm{PSL}_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Proof. Set S and T to be the first and second generators respectively. Clearly they belong to $\mathrm{PSL}_2(\mathbb{Z})$. Also, S and T^n for $n \in \mathbb{Z}$ acts on $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$ by

$$S\gamma = S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \quad \text{and} \quad T^n\gamma = T^n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}.$$

In particular, S interchanges the upper left and lower left entries of γ up to sign and T^n adds an n multiple of the lower left entry to the upper left entry. We have to show $\gamma \in \langle S, T\rangle$ and we will acomplish this by showing that the inverse is in $\langle S, T\rangle$. If $|c| = 0$ then γ is the identity since $\det(\gamma) = 1$ so suppose $|c| \neq 0$. By Euclidean division we can write $a = qc + r$ for some $q \in \mathbb{Z}$ and $|r| < |c|$. Then

$$T^{-q}\gamma = \begin{pmatrix} a-qc & b-qd \\ c & d \end{pmatrix} = \begin{pmatrix} r & b-qd \\ c & d \end{pmatrix}.$$

Multiplying by S yields

$$ST^{-q}\gamma = S\begin{pmatrix} r & b-qd \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ r & b-qd \end{pmatrix},$$

and this matrix has the upper left entry at least as large as the lower left entry in norm. Actually the upper left entry is strictly larger since $|c| > |r|$ by Euclidean divison. Therefore if we repeatedly apply this procedure, it must terminate with the lower left entry vanishing. But then we have reached the identity matrix. Therefore we have show γ has an inverse in $\langle S, T\rangle$. $\square$

We will also be intersted in special subgroups of the modular group defined by congruence conditions on their entries. For $N \geq 1$, set

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

Then $\Gamma(N)$ is the kernel of the natural homomorphism $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$ so it is a normal subgroup with finite index. We call $\Gamma(N)$ the **principal congruence subgroup** of level N. For $\Gamma \leq \mathrm{PSL}_2(\mathbb{Z})$, we say Γ is a **congruence subgroup** if $\Gamma(N) \leq \Gamma$ for some N and the minimal such N is called the **level** of Γ. Note that if $M \mid N$, then $\Gamma(N) \leq \Gamma(M)$. Thus if Γ is a congruence subgroup of level N, then $\Gamma(kN) \leq \Gamma$ for all $k \geq 1$. This implies that congruence subgroups are closed under intersection. Also, it turns out that the natural homomorphism $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective:

Proposition 2.1.2. *The natural homomorphism* $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$ *is surjective.*

Proof. Suppose $\left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Then $\overline{a}\overline{d} - \overline{b}\overline{c} \equiv 1 \pmod N$ so by Bézout's identity (generalized to three integers) $(\overline{c}, \overline{d}, N) = 1$. We claim that there exists s and t such that $c = \overline{c} + sN$, $d = \overline{d} + tN$ with $(c, d) = 1$. Set $g = (\overline{c}, \overline{d})$. Then $(g, N) = 1$ because $(\overline{c}, \overline{d}, N) = 1$. If $\overline{c} = 0$ then set $s = 0$ so $c = 0$ and choose t such that $t \equiv 1 \pmod p$ for any prime $p \mid g$ and $t \equiv 0 \pmod p$ for any prime $p \nmid g$ and $p \mid \overline{d}$. Such a t exists by the Chinese remainder theorem. Now if $p \mid (c, d)$, then either $p \mid g$ or $p \nmid g$. If $p \mid g$, then $p \mid d - \overline{d} = tN$ which is absurd since $t \equiv 1 \pmod p$ and $(t, N) = 1$. If $p \nmid g$, then $p \nmid d - \overline{d} = tN$ but this is also absurd since $t \equiv 0 \pmod p$. Therefore $(c, d) = 1$ as claimed. If $\overline{c} = 0$ then $\overline{d} \neq 0$, and we can proceed similarly. Since $(c, d) = 1$ there exists a and b such that $ad - bc = 1$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and maps onto $\left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right)$. This proves surjectivity. $\square$

By Proposition 2.1.2, $[\mathrm{PSL}_2(Z) : \Gamma(N)] = |\mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})|$. Since $\Gamma(N) \leq \Gamma$ and $\Gamma(N)$ has finite index in $\mathrm{PSL}_2(\mathbb{Z})$ so does Γ. The subgroups

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\},$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\},$$

are particularly important and are congruence subgroups of level N. The latter subgroup is called the **Hecke congruence subgroup** of level N. Note that $\Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N)$. If Γ is a general congruence subgroup, it is useful to find a generating set for Γ in order to reduce results about Γ to that of the generators. This is usually achieved by performing some sort of Euclidean division argument on the enrties of a matrix $\gamma \in \Gamma$ using the supposed generating set to construct the inverse for γ. For example, this was the proof stragety employed in Proposition 2.1.1.

Modular Curves

Recall that $\mathrm{GL}_2^+(\mathbb{Q})$ naturally acts on the Riemann sphere $\hat{\mathbb{C}}$ by Möbius transformations. Explicitly, any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ acts on $z \in \hat{\mathbb{C}}$ by

$$\gamma z = \frac{az+b}{cz+d},$$

where $\gamma\infty = \frac{a}{c}$ and $\gamma\left(-\frac{d}{c}\right) = \infty$. Moreover, recall that this action is a group action, is invariant under scalar multiplication, and acts as automorphisms of $\hat{\mathbb{C}}$. Now observe

$$\mathrm{Im}(\gamma z) = \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{az+b}{cz+d}\frac{c\overline{z}+d}{c\overline{z}+d}\right) = \mathrm{Im}\left(\frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2}\right) = \det(\gamma)\frac{\mathrm{Im}(z)}{|cz+d|^2},$$

where the last equality follows because $\mathrm{Im}(\overline{z}) = -\mathrm{Im}(z)$ and $\det(\gamma) = ad - bc$. Since $\deg(\gamma) > 0$ and $|cz+d|^2 > 0$, γ preserves the sign of the imaginary part of z. So γ preserves the upper half-space $\mathbb{H}$, the lower half-space $\overline{\mathbb{H}}$, and the extended real line $\hat{\mathbb{R}}$ respectively. Moreover, γ restricts to an automorphism on these subspaces since Möbius transformations are automorphisms. In particular, $\mathrm{PSL}_2(\mathbb{Z})$ naturally acts on $\hat{\mathbb{C}}$ by Möbius transformations and preserves the upper half-space. Certain actions of subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ also play important roles. A **Fushian group** is any subgroup of $\mathrm{PSL}_2(\mathbb{R})$ that acts properly discontinuously on $\mathbb{H}$. It turns out that the modular group is a Fushian group (see [DS05] for a proof):

Proposition 2.1.3. *The modular group is a Fushian group.*

Note that Proposition 2.1.3 immediately implies that any subgroup of the modular group is also Fushian. In particular, all congruence subgroups are Fushian. This lets us say a little more about the action of a congruence subgroup Γ on $\mathbb{H}$. Indeed, points in $\mathbb{H}$ are closed since $\mathbb{H}$ is Hausdorff. Now Γ is a Fushian group so it acts properly discontinuously on $\mathbb{H}$. These two facts together imply that the Γ-orbit of any point in $\mathbb{H}$ is a discrete set. We will make use of this property later on, but for now we are ready to introduce modular curves.

A **modular curve** is a quotient $\Gamma\backslash\mathbb{H}$ of the upper half-space $\mathbb{H}$ by a congruence subgroup Γ. We give $\Gamma\backslash\mathbb{H}$ the quotient topology induced from $\mathbb{H}$ as a subset of the Riemann sphere $\hat{\mathbb{C}}$. This gives $\Gamma\backslash\mathbb{H}$ some nice topological properties (see [DS05] for a proof):

Proposition 2.1.4. *For any congruence subgroup Γ, the modular curve $\Gamma\backslash\mathbb{H}$ is connected and Hausdorff.*

A **fundamental domain** for $\Gamma\backslash\mathbb{H}$ is a closed set $\mathcal{F}_\Gamma \subseteq \mathbb{H}$ satisfing the following conditions:

(i) Any point in $\mathbb{H}$ is Γ-equivalent to a point in $\mathcal{F}_\Gamma$.

(ii) If two points in $\mathcal{F}_\Gamma$ are Γ-equivalent via a non-identity element, then they lie on the boundary of $\mathcal{F}_\Gamma$.

(iii) The interior of $\mathcal{F}_\Gamma$ is a domain.

In other words, $\mathcal{F}_\Gamma$ is a complete set of representatives (possibly with overlap on the boundary) for $\Gamma\backslash\mathbb{H}$ that has a nice topological structure with respect to $\mathbb{H}$. Note that if $\mathcal{F}_\Gamma$ is a fundamental domain then so is $\gamma\mathcal{F}_\Gamma$ for any $\gamma \in \Gamma$ and moreover $\mathbb{H} = \bigcup_{\gamma\in\Gamma} \gamma\mathcal{F}_\Gamma$. So the choice of $\mathcal{F}_\Gamma$ is not unique. Intuitively, a fundamental domain is a geometric realization of $\Gamma\backslash\mathbb{H}$ which is often more fruitful than thinking of $\Gamma\backslash\mathbb{H}$ as an abstract set of equivalence classes. Moreover, it's suggestive that we can give $\Gamma\backslash\mathbb{H}$ a geometric structure and indeed we can (see [DS05] for more). Property (iii) is usually not included in the definition of a fundamental domain for many authors. The reason that we impose this additional property is because we will integrate over $\mathcal{F}_\Gamma$ and so we want $\mathcal{F}_\Gamma$ to genuinely represent a domain as a subset of $\mathbb{H}$.

Proposition 2.1.5.

$$\mathcal{F} = \left\{ z \in \mathbb{H} : |\mathrm{Re}(z)| \le \frac{1}{2}, |z| \ge 1 \right\},$$

is a fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$*.*

Proof. Set $\mathrm{PSL}_2(\mathbb{Z}) = \langle S, T \rangle$ where S and T are as in Proposition 2.1.1. We first show any point in $\mathbb{H}$ is $\mathrm{PSL}_2(\mathbb{Z})$-equivalent to a point in $\mathcal{F}$. Let $z = x + iy \in \mathbb{H}$. Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$, we have

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2} = \frac{y}{(cx+d)^2 + (cy)^2}.$$

Since $\det(\gamma) = 1$ we cannot have $c = d = 0$. Then as $y \ne 0$, $|cz+d|^2$ is bounded away from zero and moreover there are finitely many pairs (c, d) such that $|cz+d|^2$ is less than any given upper bound. Therefore there exists $\gamma_0 \in \mathrm{PSL}_2(\mathbb{Z})$ that minimizes $|cz+d|^2$ and hence maximizes $\mathrm{Im}(\gamma_0 z)$. In particular,

$$\mathrm{Im}(S\gamma_0 z) = \frac{\mathrm{Im}(\gamma_0 z)}{|\gamma_0 z|^2} \le \mathrm{Im}(\gamma_0 z).$$

The inequality above implies $|\gamma_0 z| \ge 1$. Since $\mathrm{Im}(T^n \gamma_0 z) = \mathrm{Im}(\gamma_0 z)$ for all $n \in \mathbb{Z}$, repeating the argument above with $T^n\gamma_0$ in place of γ_0, we see that $|T^n \gamma_0 z| \ge 1$. But T shifts the real part by 1 so we can choose n such that $|\mathrm{Re}(T^n\gamma_0 z)| \le \frac{1}{2}$. Therefore $T^n\gamma_0 \in \mathrm{PSL}_2(\mathbb{Z})$ sends z into $\mathcal{F}$ as desired. We will now show that if two points in $\mathcal{F}$ are $\mathrm{PSL}_2(\mathbb{Z})$-equivalent via a non-identity element, then they lie on the boundary of $\mathcal{F}$. Since $\mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by automorphisms, by Proposition 2.1.1 it suffices to show that S and T map $\mathcal{F}$ outside of $\mathcal{F}$ except for possibly the boundary. This is clear for T since it maps the left boundary line $\{z : \mathrm{Re}(z) = -\frac{1}{2}, |z| \ge 1\}$ to the right boundary line $\{z : \mathrm{Re}(z) = \frac{1}{2}, |z| \ge 1\}$ and every other point of $\mathcal{F}$ is mapped to the right of this line. For S, note that it maps the semicircle $\{z : |z| = 1\}$ to itself (although not identically) and maps ∞ to zero. Since Möbius transformations send circles to circles and lines to lines it follows that every other point of $\mathcal{F}$ is taken to a point enclosed by the semicircle $\{z : |z| = 1\}$. Lastly, the interior of $\mathcal{F}$ is a domain since it is open and path-connected. This finishes the proof. □

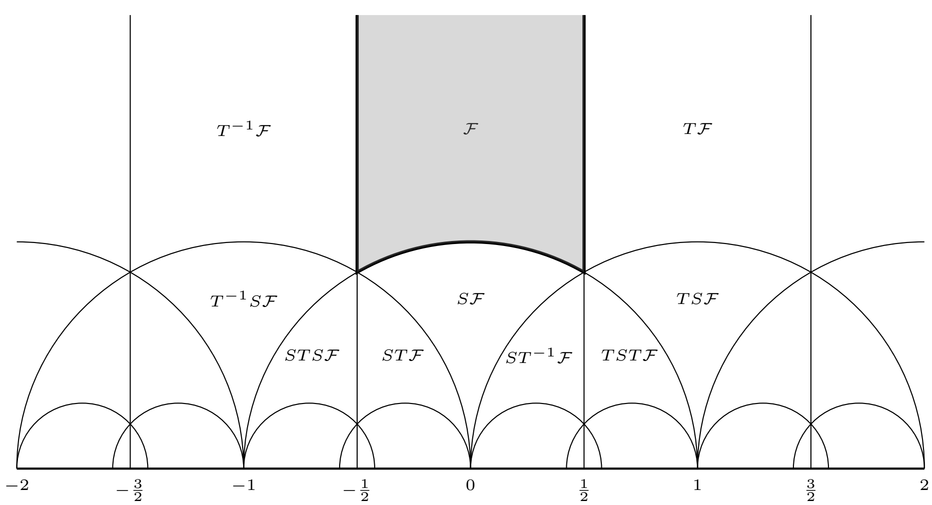

Figure 2.1: The standard fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$.

The shaded region in Figure 2.1 is the fundamental domain in Proposition 2.1.5 and we call it the **standard fundamental domain**. Figure 2.1 also displays how this fundamental domain changes under the actions of the generators of $\mathrm{PSL}_2(\mathbb{Z})$ as in Proposition 2.1.1. A fundamental domain for any other modular curve can be built from the standard fundamental domain as the following proposition shows (see [Kil15] for a proof):

Proposition 2.1.6. *Let Γ be any congruence subgroup. Then*

$$\mathcal{F}_\Gamma = \bigcup_{\gamma \in \Gamma\backslash \mathrm{PSL}_2(\mathbb{Z})} \gamma\mathcal{F},$$

is a fundamental domain for $\Gamma\backslash\mathbb{H}$.

We might notice that $\mathcal{F}$ in Figure 2.1 is unbounded as it doesnt contain the point ∞. However, if we consider $\mathcal{F} \cup \{\infty\}$ then it would appear that this space is compact. The point ∞ is an example of a cusp and we now make this idea precise. Since any $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ preserves $\hat{\mathbb{R}}$ and γ has integer entries, γ also preserves $\mathbb{Q} \cup \{\infty\}$. A **cusp** of $\Gamma\backslash\mathbb{H}$ is an element of of $\Gamma\backslash(\mathbb{Q} \cup \{\infty\})$. As Γ has finite index in the modular group, there can only be finitely many cusps and the number of cusps is at most the index of Γ. In particular, the Γ-orbit of ∞ is a cusp of $\Gamma\backslash\mathbb{H}$. We denote cusps by gothic characters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ or by representatives of their equivalence classes. For example, we let ∞ denote the cusp $\Gamma\infty$.

Remark 2.1.1. *It turns out that the cusps can be represented as the points needed to make a fundmantal domain $\mathcal{F}_\Gamma$ compact as a subset of $\hat{\mathbb{C}}$. To see this, suppose $\mathfrak{a}$ is a limit point of $\mathcal{F}_\Gamma$ that does not belong to $\mathcal{F}_\Gamma$. Then $\mathfrak{a} \in \hat{\mathbb{R}}$. In the case of the standard fundmantal domain $\mathcal{F}$, $\mathfrak{a} = \infty$ which is a cusp. Otherwise, $\mathcal{F}_\Gamma$ is a union of images of $\mathcal{F}$ by Proposition 2.1.6 and since $\mathrm{PSL}_2(\mathbb{Z})\infty = \mathbb{Q} \cup \{\infty\}$, we find that $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$.*

Let $\Gamma_\mathfrak{a} \leq \Gamma$ denote the stabilizer subgroup of the cusp $\mathfrak{a}$. For the ∞ cusp, we can describe Γ_∞ explicitly. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ stabilizes ∞, then necessarily $c = 0$ and since $\det(\gamma) = 1$ we must have $a = d = 1$. Therefore $\gamma = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some $b \in \mathbb{Z}$ and γ acts on $\mathbb{H}$ by translation by b. Of course, not every translation is guaranteed to belong to Γ. Letting t be the smallest positive integer such that $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, we have $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \rangle$. In particular, Γ_∞ is an infinite cyclic group. We say that Γ is **reduced at infinity** if $t = 1$ so that $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$. In particular, $\Gamma_1(N)$ and $\Gamma_0(N)$ are reduced at infinity.

Remark 2.1.2. *If Γ is of level N, then N is the smallest positive integer such that $\Gamma(N) \leq \Gamma$ so that $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$ is the minimal translation guaranteed to belong to Γ. However, there may be smaller translations so in general $t \leq N$.*

Actually, for any cusp $\mathfrak{a}$, $\Gamma_\mathfrak{a}$ is also an infinite cyclic group. To see this, if $\mathfrak{a} = \frac{a}{c}$ with $(a, c) = 1$ is a cusp of $\Gamma\backslash\mathbb{H}$ not equivalent to ∞, then there exists an $\sigma_\mathfrak{a} \in \mathrm{PSL}_2(\mathbb{Z})$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$. Indeed, there exists integers d and b such that $ad - bc = 1$ by Bézout's identity and then $\sigma_\mathfrak{a} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is such a matrix. It follows that $\Gamma_\mathfrak{a} = \sigma_\mathfrak{a}\Gamma_\infty\sigma_\mathfrak{a}^{-1}$ and since Γ_∞ is infinite cyclic so is $\Gamma_\mathfrak{a}$. We call any matrix $\sigma_\mathfrak{a} \in \mathrm{PSL}_2(\mathbb{Z})$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$ a **scaling matrix** for the cusp $\mathfrak{a}$. Note that $\sigma_\mathfrak{a}$ is determined up to composition on the right by an element of Γ_∞ or composition on the left by an element of $\Gamma_\mathfrak{a}$. Scaling matrices are useful because they allow us to transfer information at the cusp $\mathfrak{a}$ to the cusp at ∞.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$ with scaling matrices $\sigma_\mathfrak{a}$ and $\sigma_\mathfrak{b}$ respectively. When investigating holomorphic forms, it will be useful to have a double coset decomposition for sets of the form $\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{b}$. This is referred to as the **Bruhat decomposition** for Γ:

Theorem 2.1.1 (Bruhat decomposition). *Let Γ be any congruence subgroup and let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$ with scaling matrices $\sigma_\mathfrak{a}$ and $\sigma_\mathfrak{b}$ respectively. Then we have the disjoint decomposition*

$$\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{b} = \delta_{\mathfrak{a},\mathfrak{b}}\Omega_\infty \cup \bigcup_{\substack{c \geq 1 \\ d \ (mod\ c)}} \Omega_{d/c},$$

where

$$\Omega_\infty = \left\{ \begin{pmatrix} * & b \\ 0 & * \end{pmatrix} : \begin{pmatrix} * & b \\ 0 & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\},$$

and

$$\Omega_{d/c} = \Gamma_\infty \omega_{d/c} \Gamma_\infty,$$

for some $\omega_{d/c} = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ *with* $c \geq 1$.

Proof. We first show that Ω_∞ is nonempty if and only if $\mathfrak{a} = \mathfrak{b}$. Indeed, if $\omega_\infty \in \Omega_\infty$ then $\omega_\infty = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}$ for some $\gamma \in \Gamma$. Then

$$\gamma\mathfrak{b} = \sigma_{\mathfrak{a}}\omega_\infty\sigma_{\mathfrak{b}}^{-1}\mathfrak{b} = \sigma_{\mathfrak{a}}\omega_\infty\infty = \sigma_{\mathfrak{a}}\infty = \mathfrak{a}.$$

This shows that $\mathfrak{a} = \mathfrak{b}$. Conversely, suppose $\mathfrak{a} = \mathfrak{b}$. Then $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ contains $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$ so that Ω_∞ is nonempty. So Ω_∞ is nonempty if and only if $\mathfrak{a} = \mathfrak{b}$. In this case, for any two elements $\omega_\infty = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}$ and $\omega'_\infty = \sigma_{\mathfrak{a}}^{-1}\gamma'\sigma_{\mathfrak{a}}$ of Ω_∞, we have

$$\gamma'\gamma^{-1}\mathfrak{a} = \sigma_{\mathfrak{a}}\omega'_\infty w_\infty^{-1}\sigma_{\mathfrak{a}}^{-1}\mathfrak{a} = \sigma_{\mathfrak{a}}\omega'_\infty w_\infty^{-1}\infty = \sigma_{\mathfrak{a}}\mathfrak{a}.$$

Hence $\gamma'\gamma^{-1} \in \Gamma_{\mathfrak{a}}$ which implies $\omega'_\infty w_\infty^{-1} = \sigma_{\mathfrak{a}}^{-1}\gamma'\gamma^{-1}\sigma_{\mathfrak{a}} \in \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$. Therefore

$$\Omega_\infty = \Gamma_\infty w_\infty = w_\infty\Gamma_\infty = \Gamma_\infty w_\infty \Gamma_\infty,$$

where the latter two equalities hold because w_∞ is a translation and translations commute. Every other element of $\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}$ belongs to one of the double cosets $\Omega_{d/c}$ with $c \geq 1$ (since we are working in $\mathrm{PSL}_2(\mathbb{Z})$). The relation

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} * & * \\ c & d \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c & d + cm \end{pmatrix},$$

shows that $\Omega_{d/c}$ is determined uniquely by c and $d \pmod{c}$. This completes the proof of the theorem. $\square$

As a first application, take $\mathfrak{a} = \mathfrak{b} = \infty$ and $\sigma_{\mathfrak{a}} = \sigma_{\mathfrak{b}} = I$. Then the Bruhat decomposition implies

$$\Gamma_\infty\backslash\Gamma \cong I \cup \bigcup_{\substack{c \geq 1 \\ d \pmod{c}}} \omega_{d/c}\Gamma_\infty,$$

since $\Gamma_\infty\backslash\Omega_\infty = \Gamma_\infty\backslash\Gamma_\infty \cong I$. This shows that every element of $\Gamma_\infty\backslash\Gamma$ corresponds to a unique $(c, d) \in \mathbb{Z}^2 - \{\mathbf{0}\}$ with $c \geq 0$ and d determined only modulo c except when $c = 0$ in which case $d = 1$ (the pair $(1, 0)$ corresponds to I). Of course, this correspondence might not be surjective since the double coset $\Omega_{d/c}$ may be empty if there is no $\omega_{d/c} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$. For example, if Γ is of level N then every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ satisfies $c \equiv 0 \pmod{N}$ so that

$$\Gamma_\infty\backslash\Gamma = I \cup \bigcup_{\substack{c \geq 1 \\ c \equiv 0 \pmod{N} \\ d \pmod{c}}} \omega_{d/c}\Gamma_\infty.$$

However, we still may have other cosets which are empty since Γ being of level N only guarantees that $c \equiv 0 \pmod{N}$. To figure out for what subset of pairs (c, d) the correspondence is a bijection it suffices to determine the admissible d not just the admissible $d \pmod{c}$.

Remark 2.1.3. *An exceptionally important case is for the Hecke congruence subgroups* $\Gamma_0(N)$ *with* $\mathfrak{a} = \mathfrak{b} = \infty$. *Suppose* $c \geq 1$ *with* $c \equiv 0 \pmod{N}$ *and* $d \pmod{c}$. *By Bezout's identity there exists integers* a *and* b *with* $ad - bc = 1$ *so that* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ *if and only if* $(c, d) = 1$. *So we must have* $d \equiv 1 \pmod{c}$. *Then the following correspondence implied by the Bruhat decomposition is a bijection:*

$$\Gamma_\infty\backslash\Gamma = I \cup \bigcup_{\substack{c \geq 1, d \in \mathbb{Z} \\ c \equiv 0 \pmod{N} \\ (c,d) = 1}} \omega_{d/c}\Gamma_\infty.$$

2.2 The Theory of Holomorphic Forms

Holomorphic Forms

Let Γ be a congruence subgroup of level N and let χ be a Dirichlet character of conductor $q \mid N$. Define $\chi(\gamma) = \chi(d)$ if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is **holomorphic form** (or **modular form**) of weight $k \geq 1$, level N, and **character** (or **nebentypus** or **twist**) χ, on $\Gamma \backslash \mathbb{H}$ if the following properties are satisfied:

(i) f is holomorphic on $\mathbb{H}$.

(ii) $f(\gamma z) = \chi(\gamma)(cz+d)^k f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

(iii) $f(\alpha z)$ remains bounded as $\mathrm{Im}(z) \to \infty$ for all $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$.

Property (ii) is called the **modularity condition** and we say f is **modular**. We set $j(\gamma, z) = (cz+d)$ and call $j(\gamma, z)$ the **factor of modularity** of f. Property (iii) is called the **growth condition** and we say f is **holomorphic at the cusps**. Also, we say f is a **(holomorphic) cusp form** if in addition,

(iv) $f(\alpha z) \to 0$ as $\mathrm{Im}(z) \to \infty$ for all $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$.

Holomorphic forms also admit Fourier series. Let $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ be the smallest translation belonging to Γ. By modularity,

$$f(z+t) = f\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} z\right) = f(z),$$

so that f is t-periodic.

Remark 2.2.1. *If f is t-periodic, then we may think of f as a function on $\Gamma_\infty \backslash \mathbb{H}$. Taking representatives, $\Gamma_\infty \backslash \mathbb{H}$ is a strip in $\mathbb{H}$ with bounded real part so that f is a function on a domain with boudned real part.*

As f is t-periodic it admits a Fourier series

$$f(z) = \sum_{n \in \mathbb{Z}} a(n,y) e^{\frac{2\pi i n x}{t}}.$$

Actually, the Fourier coefficients $a(n,y)$ do not depend on y. To see this, since f is holomorphic we know

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

Substiuting in the Fourier series and equating coefficients we obtain the ODE

$$2\pi n a(n,y) + a'(n,y) = 0,$$

where the $'$ indicates differentiation with respect to y. Solving this ODE (by separation of variables) we see that there exists an $a(n)$ such that

$$a(n,y) = a(n) e^{-2\pi n y}.$$

Using these constant coefficients instead, we say f has a **Fourier series at the ∞ cusp**:

$$f(z) = \sum_{n \in \mathbb{Z}} a(n) e^{\frac{2\pi i n z}{t}}.$$

Accordingly, we say f is **holomorphic at infinity** if

$$\lim_{z\to\infty}\sum_{n\in\mathbb{Z}} a(n)e^{\frac{2\pi inz}{t}},$$

exists and is finite. This is equivalent to $f(z)$ remaining bounded as $\mathrm{Im}(z)\to\infty$ and therefore the Fourier series is actually over all $n\geq 0$ (the negative terms do not exhibit decay as $\mathrm{Im}(z)\to\infty$ unless their Fourier coefficients vanish). Moreover, the limit above is $a(0)$ and f will have exponential decay to $a(0)$ as $\mathrm{Im}(z)\to\infty$. Writing $z=x+iy$ and fixing any $y>0$, f restricts to a t-periodic smooth function on $\mathbb{R}$ that is integrable on $[0,t]$, as it is holomorphic, so we have

$$a(n)=\int_0^t f_\infty(x+iy)e^{-\frac{2\pi inx}{t}}\,dx.$$

For a general cusp $\mathfrak{a}$ of $\Gamma\backslash\mathbb{H}$, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for $\mathfrak{a}$. That is, $\sigma_{\mathfrak{a}}\infty=\mathfrak{a}$. Then $f(\sigma_{\mathfrak{a}}z)$ is holomorphic at infinity precisely when $f(\sigma_{\mathfrak{a}}z)$ remains bounded as $\mathrm{Im}(z)\to\infty$ which is to say that $f(z)$ remains bounded as $z\to\mathfrak{a}$. This motivates the saying that f is holomorphic at the cusps. More generally, f has a **Fourier series at the $\mathfrak{a}$ cusp**:

$$f(\sigma_{\mathfrak{a}}z)=\sum_{n\geq 0} a_{\mathfrak{a}}(n)e^{\frac{2\pi inz}{t}}.$$

Also, the Fourier series above is independent of the scaling matrix because f is Γ_∞-invariant and the set of scaling matrices is stable under multiplication from $\Gamma_{\mathfrak{a}}$ on the right. For ease of communication, if it is clear what cusp we are working at we will mention the Fourier series and its coefficients without regard to the cusp. The Fourier coefficients $a_{\mathfrak{a}}(n)$ are then given by

$$a_{\mathfrak{a}}(n)=\int_0^t f(\sigma_{\mathfrak{a}}(x+iy))e^{-\frac{2\pi inx}{t}}\,dx,$$

for any fixed $y>0$. Notice that f is a cusp form precisely if $a_{\mathfrak{a}}(0)=0$ for every Fourier series of f at every cusp $\mathfrak{a}$. In particular, f exhibits exponential decay to zero near the cusps. Since the Fourier series is independent of the scaling matrix used, we only need to check condition (iii) for a set of scaling matrices for the cusps since every element of $\mathrm{PSL}_2(\mathbb{Z})$ is either a scaling matrix for a cusp or belongs to Γ_∞ depending on if it fixes ∞ or not. From now on, we will always assume that our congruence subgroups are reduced at infinity so that $t=1$.

Sometimes we will be interested in holomorphic forms with nontrivial characters χ on $\Gamma\backslash\mathbb{H}$. To ensure that our holomorphic forms are not identically zero, the weight k depends on χ. Indeed, if f is a weight k holomorphic form on $\Gamma\backslash\mathbb{H}$, modularity implies

$$f(z)=f\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix}z\right)=\chi(-1)(-1)^kf(z),$$

So f is identically zero unless $\chi(-1)=(-1)^k$. In other words, if χ is odd then k must be odd and if χ is even then k must be even.

We will now discuss a very useful property that the factor of modularity satisfies. Let $\gamma,\gamma'\in\Gamma$ and suppose $j(\gamma,z)$ is the factor of modularity of a holomorphic form f on $\Gamma\backslash\mathbb{H}$. Set $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ and $\gamma'=\left(\begin{smallmatrix}a'&b'\\c'&d'\end{smallmatrix}\right)$ so that

$\gamma'\gamma = \left(\begin{smallmatrix} a'a+b'c & a'b+b'b \\ c'a+d'c & c'b+d'd \end{smallmatrix}\right)$. Then

$$\begin{aligned} j(\gamma', \gamma z)j(\gamma, z) &= \left(c'\frac{az+b}{cz+d} + d'\right)(cz+d) \\ &= \left(c'\frac{az+b}{cz+d} + d'\right)(cz+d) \\ &= (c'(az+b) + d'(cz+d)) \\ &= (c'a + d'c)z + c'b + d'd) \\ &= j(\gamma'\gamma, z). \end{aligned}$$

In short,

$$j(\gamma'\gamma, z) = j(\gamma', \gamma z)j(\gamma, z),$$

and this is called the **cocycle condition** for $j(\gamma, z)$. The cocycle condition immediately implies that the factor of modularity is determined by any set of generators for Γ. This is a very useful fact to remember because it is often easier to verify modularity on a generating set of a congruence subgroup.

Eisenstein & Poincaré Series

Let Γ be a congruence subgroup of level N. We will introduce two important classes of holomorphic forms on $\Gamma\backslash\mathbb{H}$ namely the Eisenstein series and the Poincaré series. For any weight $k \geq 4$ and Dirichlet character χ with conductor $q \mid N$, we define the **(holomorphic) Eisenstein series** $E_{k,\chi}$ of weight k twised by χ on $\Gamma\backslash\mathbb{H}$ by

$$E_{k,\chi}(z) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma)j(\gamma, z)^{-k},$$

Remark 2.2.2. *The reason why we restrict to $k \geq 4$ is because for $k = 0, 2$ the Eisenstein series do not converge (see Proposition B.8.1).*

The Bruhat decomposition applied to $\Gamma_\infty\backslash\Gamma$ implies

$$E_{k,\chi}(z) \ll \sum_{(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \frac{1}{|cz+d|^k},$$

and as $k \geq 4$, this latter series is locally absolutely uniformly convergent for $z \in \mathbb{H}$ by Proposition B.8.1. Hence $E_{k,\chi}(z)$ does too and so it is holomorphic on $\mathbb{H}$.

Remark 2.2.3. *In the case of the modular group, the Bruhat decomposition implies that a set of representatives for the quotient $\Gamma_\infty\backslash\Gamma$ is*

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} * & * \\ c & d \end{pmatrix} : c \geq 1, d \in \mathbb{Z}, (c, d) = 1\right\}.$$

Then $E_{k,\chi}$ is given by

$$E_{k,\chi}(z) = 1 + \sum_{\substack{c \geq 1, d \in \mathbb{Z} \\ (c,d)=1}} \frac{\overline{\chi}(d)}{(cz+d)^k}.$$

We now show that $E_{k,\chi}$ is modular. This is just a computation:

$$\begin{aligned}
E_{k,\chi}(\gamma z) &= \sum_{\gamma' \in \Gamma_\infty \backslash \Gamma} \overline{\chi}(\gamma') j(\gamma', \gamma z)^{-k} \\
&= \sum_{\gamma' \in \Gamma_\infty \backslash \Gamma} \overline{\chi}(\gamma') \left(\frac{j(\gamma'\gamma, z)}{j(\gamma, z)} \right)^{-k} && \text{cocycle condition} \\
&= j(\gamma, z) \sum_{\gamma' \in \Gamma_\infty \backslash \Gamma} \overline{\chi}(\gamma') j(\gamma'\gamma, z)^{-k} \\
&= \chi(\gamma) j(\gamma, z) \sum_{\gamma' \in \Gamma_\infty \backslash \Gamma} \overline{\chi}(\gamma') \overline{\chi}(\gamma) j(\gamma'\gamma, z)^{-k} \\
&= \chi(\gamma) j(\gamma, z) \sum_{\gamma' \in \Gamma_\infty \backslash \Gamma} \overline{\chi}(\gamma'\gamma) j(\gamma'\gamma, z)^{-k} \\
&= \chi(\gamma) j(\gamma, z) \sum_{\gamma' \in \Gamma_\infty \backslash \Gamma} \overline{\chi}(\gamma') j(\gamma', z)^{-k} && \gamma' \to \gamma'\gamma^{-1} \text{ bijection on } \Gamma \\
&= \chi(\gamma) j(\gamma, z) E_{k,\chi}(z).
\end{aligned}$$

To verify holomorphy at the cusps, we will need a technical lemma:

Lemma 2.2.1. *Let $a, b > 0$ be real numbers and consider the half-strip*

$$S_{a,b} = \{z \in \mathbb{H} : |\mathrm{Re}(z)| \leq a, \mathrm{Im}(z) \geq b\}.$$

Then there is a $\delta \in (0, 1)$ such that

$$|nz + m| \geq \delta |ni + m|,$$

for all $n, m \in \mathbb{Z}$ and all $z \in S_{a,b}$.

Proof. If $n = 0$ then any δ is sufficient and this δ is independent of z. If $n \neq 0$, then the desired inequality is equivalent to

$$\left| \frac{z + \frac{m}{n}}{i + \frac{n}{m}} \right| \geq \delta.$$

So consider the function

$$f(z, x) = \left| \frac{z + x}{i + x} \right|,$$

for $z \in S_{a,b}$ and real x. It suffices to show $f(z, x) \geq \delta$. As $z \in \mathbb{H}$, $z - x \neq 0$ so that $f(z, x)$ is continuous and $f(z, x) > 0$ for all z and x. Now let $Y > b$ and consider the region

$$S_{a,b}^Y = \{z \in \mathbb{H} : |\mathrm{Re}(z)| \leq a, b \leq \mathrm{Im}(z) \leq Y\}.$$

We claim that there exists a Y such that if $\mathrm{Im}(z) > Y$ and $|x| > Y$ then $f(z, x)^2 > \frac{1}{4}$. Indeed, we compute

$$f(z, x)^2 = \frac{(z + x)(\overline{z} + x)}{(i + x)(-i + x)} = \frac{|z|^2 + 2\mathrm{Re}(z)x + x^2}{1 + x^2} \geq \frac{\mathrm{Im}(z) + x^2}{1 + x^2},$$

where in the inequality we have used that $|z|^2 \geq \mathrm{Im}(z)$ and $\mathrm{Re}(z)$ is bounded. Now $\frac{x^2}{1+x^2} \to 1$ as $x \to \pm\infty$ so choosing Y such that $|x| > Y$ implies $\frac{x^2}{1+x^2} \geq \frac{1}{4}$ yields

$$\frac{\mathrm{Im}(z) + x^2}{1 + x^2} \geq \frac{\mathrm{Im}(z)}{1 + x^2} + \frac{x^2}{1 + x^2} \geq \frac{\mathrm{Im}(z)}{1 + x^2} + \frac{1}{4} > \frac{1}{4}.$$

It follows that $f(z,x) > \frac{1}{2}$ outside of $S_{a,b}^{Y} \times [-Y,Y]$. But this latter region is compact and so $f(z,x)$ obtains a minimum δ' on it. Then set $\delta = \min\{\frac{1}{2}, \delta'\}$ and we are done. $\square$

We can now show that $E_{k,\chi}$ is holomorphic at the cusps. Let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$. Then

$$E_{k,\chi}(\sigma_{\mathfrak{a}} z) \ll \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{|c\sigma_{\mathfrak{a}} z + d|^k}.$$

Now decompose this last sum as

$$\sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{|c\sigma_{\mathfrak{a}} z + d|^k} = \sum_{d\neq 0} \frac{1}{d^k} + \sum_{c\neq 0}\sum_{d\in\mathbb{Z}} \frac{1}{|c\sigma_{\mathfrak{a}} z + d|^k} = 2\sum_{d\geq 1} \frac{1}{d^k} + 2\sum_{c\geq 1}\sum_{d\in\mathbb{Z}} \frac{1}{|c\sigma_{\mathfrak{a}} z + d|^k}.$$

Since the first sum is bounded, it suffices to show that the double sum is bounded as $\mathrm{Im}(z) \to \infty$. To see this, let $\mathrm{Im}(z) \geq 1$ and δ be as in Lemma 2.2.1. Then for any integer $N \geq 1$ we can write

$$\begin{aligned} \sum_{c\geq 1}\sum_{d\in\mathbb{Z}} \frac{1}{|c\sigma_{\mathfrak{a}} z + d|^k} &= \sum_{c+|d|\leq N} \frac{1}{|c\sigma_{\mathfrak{a}} z + d|^k} + \sum_{c+|d|>N} \frac{1}{|c\sigma_{\mathfrak{a}} z + d|^k} \\ &\leq \sum_{c+|d|\leq N} \frac{1}{|c\sigma_{\mathfrak{a}} z + d|^k} + \sum_{c+|d|>N} \frac{1}{(\delta|ci + d|)^k} \\ &\leq \sum_{c+|d|\leq N} \frac{1}{|c\sigma_{\mathfrak{a}} z + d|^k} + \frac{1}{\delta^k}\sum_{c+|d|>N} \frac{1}{|ci + d|^k}. \end{aligned}$$

Since $\sum_{c\geq 1}\sum_{d\in\mathbb{Z}} \frac{1}{|ci+d|^k}$ converges by Proposition B.8.1, the second sum tends to zero as $N \to \infty$. As for the first sum, it is finite and each term tends to a finite value as $\mathrm{Im}(z) \to \infty$. Holomorphy at $\mathfrak{a}$ follows. We collect all of this work as a theorem:

Theorem 2.2.1. *Let $k \geq 4$ and χ be Dirichlet character with conductor $q \mid N$. The Eisenstein series*

$$E_{k,\chi}(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma) j(\gamma,z)^{-k},$$

is a weight k holomorphic form with character χ on $\Gamma\backslash\mathbb{H}$.

Remark 2.2.4. *The intuition behind the construction of the Eisenstein series $E_{k,\chi}$ is that $E_{k,\chi}$ is built by averaging the character and factor of modularity over Γ. Actually, this average happens over $\Gamma_\infty\backslash\Gamma$ because Γ_∞ acts trivially since $j(\gamma,z) = 1$ if $\gamma \in \Gamma_\infty$ as $\gamma = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for some $n \in \mathbb{Z}$.*

Now we will discuss our second class of holomorphic forms. For every $m \geq 1$, weight $k \geq 4$ and Dirichlet character χ with conductor $q \mid N$, we define the m-th **(holomorphic) Poincaré series** $P_{m,k,\chi}$ of weight k with character χ on $\Gamma\backslash\mathbb{H}$ by

$$P_{m,k,\chi}(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma) j(\gamma,z)^{-k} e^{2\pi i m\gamma z}.$$

We first need to verify that $P_{m,k,\chi}$ is actually well-defined. This amounts to checking $e^{2\pi i m\gamma z}$ is independent of the representative γ. Indeed, if γ' represents the same element as γ in $\Gamma_\infty\backslash\Gamma$, then they differ on the left by an element of $\eta_\infty \in \Gamma_\infty$. So suppose $\gamma' = \eta_\infty\gamma$ with $\eta_\infty = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_\infty$. Then

$$e^{2\pi i m\gamma' z} = e^{2\pi i m\eta_\infty\gamma z} = e^{2\pi i m(\gamma z + n)} = e^{2\pi i m\gamma z} e^{2\pi i m n z} = e^{2\pi i m\gamma z},$$

and hence $P_{m,k,\chi}$ is well-defined. To see that $P_{m,k,\chi}$ is holomorphic on $\mathbb{H}$, first note that $|e^{2\pi i m\gamma z}| = e^{-2\pi m \mathrm{Im}(\gamma z)} < 1$ for all $\gamma \in \Gamma$. Applying the Bruhat decomposition to $\Gamma_\infty\backslash\Gamma$, we see that

$$P_{m,k,\chi}(z) \ll \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{|cz+d|^k}.$$

As $k \geq 4$, this latter series is locally absolutely uniformly convergent for $z \in \mathbb{H}$ by Proposition B.8.1. Hence $P_{m,k,\chi}(z)$ does too and so it is holomorphic on $\mathbb{H}$.

Remark 2.2.5. *In the case of the modular group, the Bruhat decomposition implies that a set of representatives for the quotient* $\Gamma_\infty\backslash\Gamma$ *is*

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} * & * \\ c & d \end{pmatrix} : c \geq 1, d \in \mathbb{Z}, (c,d) = 1\right\}.$$

as in Remark 2.2.3. Then $P_{m,k,\chi}$ *is given by*

$$P_{m,k,\chi}(z) = e^{2\pi i m z} + \sum_{\substack{c\geq 1, d\in\mathbb{Z} \\ (c,d)=1}} \overline{\chi}(d)\frac{e^{2\pi i m\left(\frac{az+b}{cz+d}\right)}}{(cz+d)^k},$$

where a *and* b *are chosen such that* $\det\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = 1$.

We now show that $P_{m,k,\chi}$ is modular. This is just a computation:

$$\begin{aligned}
P_{m,k,\chi}(\gamma z) &= \sum_{\gamma'\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma')j(\gamma',\gamma z)^{-k}e^{2\pi i m\gamma'\gamma z} \\
&= \sum_{\gamma'\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma')\left(\frac{j(\gamma'\gamma,z)}{j(\gamma,z)}\right)^{-k} e^{2\pi i m\gamma'\gamma z} && \text{cocycle condition} \\
&= j(\gamma,z)\sum_{\gamma'\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma')j(\gamma'\gamma,z)^{-k}e^{2\pi i m\gamma'\gamma z} \\
&= \chi(\gamma)j(\gamma,z)\sum_{\gamma'\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma')\overline{\chi}(\gamma)j(\gamma'\gamma,z)^{-k}e^{2\pi i m\gamma'\gamma z} \\
&= \chi(\gamma)j(\gamma,z)\sum_{\gamma'\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma'\gamma)j(\gamma'\gamma,z)^{-k}e^{2\pi i m\gamma'\gamma z} \\
&= \chi(\gamma)j(\gamma,z)\sum_{\gamma'\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma')j(\gamma',z)^{-k}e^{2\pi i m\gamma' z} && \gamma' \to \gamma'\gamma^{-1} \text{ bijection on } \Gamma \\
&= \chi(\gamma)j(\gamma,z)P_{m,k,\chi}(z),
\end{aligned}$$

To verify holomorphy at the cusps, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$. First note that $|e^{2\pi i m\gamma\sigma_{\mathfrak{a}} z}| = e^{-2\pi m\mathrm{Im}(\gamma\sigma_{\mathfrak{a}} z)} < 1$. Then

$$P_{m,k,\chi}(\sigma_{\mathfrak{a}} z) \ll \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{|c\sigma_{\mathfrak{a}} z+d|^k},$$

and now we can proceed in exactly the same way used to show that $E_{k,\chi}$ was holomorphic at the cusps. We collect all of this work as a theorem:

Theorem 2.2.2. *Let* $k \geq 4$ *and* χ *be a Dirichlet character with conductor* $q \mid N$. *The Poincaré series*

$$P_{m,k,\chi}(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^{-1}e^{2\pi i m\gamma z},$$

is a weight k *holomorphic form with character* χ *on* $\Gamma\backslash\mathbb{H}$.

Spaces of Holomorphic Forms

Let Γ be a congruence subgroup of level N. Let $\mathcal{M}_k(\Gamma,\chi)$ denote the space of all weight k holomorphic forms on $\Gamma\backslash\mathbb{H}$ with character χ. Let $\mathcal{S}_k(\Gamma,\chi)$ denote the associated subspace of cusp forms. Moreover, if the character χ is the trivial character, we will surpress the dependence upon χ. If Γ_1 and Γ_2 are two congruence subgroups such that $\Gamma_1 \le \Gamma_2$, then we have the inclusion

$$\mathcal{M}_k(\Gamma_2) \subseteq \mathcal{M}_k(\Gamma_1).$$

So in general, the smaller the congruence subgroup the more holomorphic forms there are. We will need a dimensionality result regarding the space of holomorphic forms of a fixed weight. However, it will suffice to only require the result for untwised forms. The result is that $\mathcal{M}_k(\Gamma)$ is never too large (see [DS05] for a proof):

Theorem 2.2.3. *Then $\mathcal{M}_k(\Gamma)$ is finite dimensional.*

Since $\mathcal{S}_k(\Gamma)$ is a subspace of $\mathcal{M}_k(\Gamma)$, Theorem 2.2.3 implies that $\mathcal{S}_k(\Gamma)$ is also finite dimensional. It turns out that $\mathcal{S}_k(\Gamma)$ is naturally an inner product space. To see this, we define an operator on $\mathcal{S}_k(\Gamma)\times\mathcal{M}_k(\Gamma)$. First we require a measure. Our choice of measure will be the **hyperbolic measure** $d\mu(z) = \frac{dx\,dy}{y^2}$. The most important property that we will use frequently is that this measure is $\mathrm{GL}_2^+(\mathbb{R})$-invariant (see [DS05] for a proof):

Proposition 2.2.1. *The hyperbolic measure $d\mu$ is $\mathrm{GL}_2^+(\mathbb{R})$-invariant.*

As this fact will be used so frequently any time we integrate, we will not mention it explicitely. Indeed, the particular use is that if Γ is a congruence subgroup, $d\mu$ is Γ-invariant. Now let $\mathcal{F}_\Gamma$ be a fundamental domain for $\Gamma\backslash\mathbb{H}$ and set

$$V_\Gamma = \int_{\mathcal{F}_\Gamma} d\mu.$$

We call V_Γ the **volume** of $\Gamma\backslash\mathbb{H}$. Also, if $\mathcal{F}_\Gamma = \mathcal{F}$ we write $V_\Gamma = V$. In other words, V is the volume of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. Since the integrand is Γ-invariant, V_Γ is independent of the choice of fundamental domain. There is also a simple relation between V_Γ and the index of Γ in $\mathrm{PSL}_2(\mathbb{Z})$:

$$V_\Gamma = [\mathrm{PSL}_2(\mathbb{Z}):\Gamma]V, \tag{2.1}$$

which follows immeditely from Proposition 2.1.6. Using Proposition 2.1.5,

$$V = \int_{\mathcal{F}} d\mu = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\,dy\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x)\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\pi}{3}.$$

Therefore V is finite. Furthermore so is V_Γ for every congruence subgroup Γ by Equation (2.1) and that congruence subgroups have finite index in the modular group. Now for $f \in \mathcal{S}_k(\Gamma)$ and $g \in \mathcal{M}_k(\Gamma)$, define their **Petersson inner product** by

$$\langle f, g\rangle_\Gamma = \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu.$$

If the congruence subgroup is clear from context we will supress the dependence upon Γ. Since f is a cusp form, the integral is absolutely bounded by Method 1.4.1. The integrand is also Γ-invariant so that the integral independent of the choice of fundamental domain. These two facts together imply that the Petersson inner product is well-defined. Actually, we have the following proposition:

Proposition 2.2.2. *$\mathcal{S}_k(\Gamma)$ is a Hilbert space with respect to Petersson inner product.*

Proof. Let $f, g \in \mathcal{S}_k(\Gamma)$. Linearity of the integral immediately implies that the Petersson inner product is linear on $\mathcal{S}_k(\Gamma)$. It is also positive definite since

$$\langle f, f\rangle = \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{f(z)}\mathrm{Im}(z)^k\,d\mu = \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |f(z)|^2\mathrm{Im}(z)^k\,d\mu \geq 0,$$

with equality if and only if f is identically zero because $|f(z)|^2\mathrm{Im}(z)^k \geq 0$. To see that it is conjugate symetric, observe

$$\begin{aligned}
\overline{\langle g, f\rangle} &= \overline{\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} g(z)\overline{f(z)}\mathrm{Im}(z)^k\,d\mu} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \overline{g(z)\overline{f(z)}\mathrm{Im}(z)^k\,d\mu} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \overline{g(z)}f(z)\mathrm{Im}(z)^k\,\overline{d\mu} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \overline{g(z)}f(z)\mathrm{Im}(z)^k\,d\mu \qquad\qquad d\mu = \frac{dx\,dy}{y^2} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu \\
&= \langle f, g\rangle.
\end{aligned}$$

So the Petersson inner product is a Hermitian inner product on $\mathcal{S}_k(\Gamma)$. Since $\mathcal{S}_k(\Gamma)$ is finite dimensional by Theorem 2.2.3, it follows immediately that $\mathcal{S}_k(\Gamma)$ is a Hilbert space. □

Remark 2.2.6. *As a consequence of Proposition 2.2.2, the Petersson inner product is also non-degenerate on $\mathcal{S}_k(\Gamma)$. Actually, for the exact same reasoning this holds on $\mathcal{S}_k(\Gamma) \times \mathcal{M}_k(\Gamma)$ wherever the Petersson inner product is defined.*

Now suppose $f \in \mathcal{S}_k(\Gamma)$ with Fouier coefficients $a_n(f)$. Define linear functionals $\phi_{m,k} : \mathcal{S}_k(\Gamma) \to \mathbb{C}$, for every $m \geq 1$, by

$$\phi_{m,k}(f) = a_m(f).$$

Since $\mathcal{S}_k(\Gamma)$ is a finite dimensional Hilbert space, the Riesz representation theorem implies that there exists a unique $v_{m,k} \in \mathcal{S}_k(\Gamma)$ such that

$$\langle f, v_{m,k}\rangle = \phi_{m,k}(f) = a_m(f).$$

We would like to know what these cusp forms are. It turns out that $v_{m,k}$ will be the Poincaré series $P_{m,k,\chi}$

up to a normalization factor. To see this, we compute the inner product:

$$
\begin{aligned}
\langle f, P_{m,k,\chi}\rangle &= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{P_{m,k,\chi}(z)}\mathrm{Im}(z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\chi(\gamma)\overline{j(\gamma,z)^{-k}}e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\frac{\chi(\gamma)j(\gamma,z)^k}{|j(\gamma,z)|^{2k}}e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(\gamma z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\frac{1}{|j(\gamma,z)^k|^{2k}}e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(z)^k\,d\mu && \text{modularity} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(\gamma z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(\gamma z)^k\,d\mu && \mathrm{Im}(\gamma z)^k = \frac{\mathrm{Im}(z)^k}{|j(\gamma,z)|^{2k}}.
\end{aligned}
$$

We put this last integral into a different form. The idea is to rewrite the integral over the fundamental domain $\mathcal{F}_\Gamma$ as an integral over the strip $\Gamma_\infty\backslash\mathbb{H}$. First, observe

$$
\begin{aligned}
\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(\gamma z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(\gamma z)^k\,d\mu &= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma z)e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(\gamma z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\mathcal{F}_\Gamma} f(\gamma z)e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(\gamma z)^k\,d\mu && \text{DCT} \\
&= \frac{1}{V_\Gamma}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma\mathcal{F}_\Gamma} f(z)e^{-2\pi i m\overline{z}}\mathrm{Im}(z)^k\,d\mu && z\to\gamma^{-1}z.
\end{aligned}
$$

Now $\mathbb{H} = \bigcup_{\gamma\in\Gamma}\gamma\mathcal{F}_\Gamma$ so that $\Gamma_\infty\backslash\mathbb{H} = \bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma\mathcal{F}_\Gamma$. Therefore we can write

$$
\frac{1}{V_\Gamma}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma\mathcal{F}_\Gamma} f(z)e^{-2\pi i m\overline{z}}\mathrm{Im}(z)^k\,d\mu = \frac{1}{V_\Gamma}\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)e^{-2\pi i m\overline{z}}\mathrm{Im}(z)^k\,d\mu.
$$

The rest is a computation:

$$
\begin{aligned}
\frac{1}{V_\Gamma}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma\mathcal{F}_\Gamma} f(z)e^{-2\pi i m\overline{z}}\mathrm{Im}(z)^k\,d\mu &= \frac{1}{V_\Gamma}\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)e^{-2\pi i m\overline{z}}\mathrm{Im}(z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_0^\infty\int_0^1 f(x+iy)e^{-2\pi i m\overline{(x+iy)}}y^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_0^\infty\int_0^1\sum_{n\geq 1}a_n(f)e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^k\,\frac{dx\,dy}{y^2} \\
&= \frac{1}{V_\Gamma}\int_0^\infty\sum_{n\geq 1}\int_0^1 a_n(f)e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^k\,\frac{dx\,dy}{y^2} && \text{DCT} \\
&= \frac{1}{V_\Gamma}\int_0^\infty a_m(f)e^{-4\pi m y}y^k\,\frac{dy}{y^2},
\end{aligned}
$$

where the last line follows because

$$
\int_0^1 e^{2\pi i(n-m)x}\,dx = \delta_{n-m,0}, \tag{2.2}
$$

so that the inner integral cuts off all the terms except for the $n = m$ term. Then

$$\begin{aligned}
\frac{1}{V_\Gamma}\int_0^\infty a_m(f)e^{-4\pi my}y^k\,\frac{dy}{y^2} &= \frac{a_m(f)}{V_\Gamma}\int_0^\infty e^{-4\pi my}y^{k-1}\,\frac{dy}{y} \\
&= \frac{a_m(f)}{V_\Gamma}\int_0^\infty e^{-4\pi my}y^{k-1}\,\frac{dy}{y} && y \to \frac{y}{4\pi m} \\
&= \frac{a_m(f)}{V_\Gamma(4\pi m)^{k-1}}\int_0^\infty e^{-y}y^{k-1}\,\frac{dy}{y} \\
&= \frac{\Gamma(k-1)}{V_\Gamma(4\pi m)^{k-1}}a_m(f) && \text{definition of } \Gamma(k-1).
\end{aligned}$$

In conclusion,

$$\langle f, P_{m,k,\chi}\rangle = \frac{\Gamma(k-1)}{V_\Gamma(4\pi m)^{k-1}}a_m(f). \tag{2.3}$$

Now set $\widetilde{P_{m,k,\chi}} = \frac{V_\Gamma(4\pi m)^{k-1}}{\Gamma(k-1)}P_{m,k,\chi}$. For all cusp forms f (actually any holomorphic form where the Petersson inner product is defined),

$$\langle f, \widetilde{P_{m,k,\chi}} - v_{m,k}\rangle = 0.$$

By Remark 2.2.6 the Petersson inner product is non-degenerate so we conclude $v_{m,k} = \widetilde{P_{m,k,\chi}}$. In particular, this shows that the Poincaré series $P_{m,k,\chi}$ are cusp forms. We usually work with the Poincaré series $P_{m,k,\chi}$ instead of their normalized counterparts $\widetilde{P_{m,k,\chi}}$. In any case, with Equation (2.3) in hand we can prove the following result:

Theorem 2.2.4. *The Poincaré series span $\mathcal{S}_k(\Gamma)$.*

Proof. Let $f \in \mathcal{S}_k(\Gamma)$ with Fourier coefficients $a_n(f)$. Since $\Gamma(k-1) \neq 0$, Equation (2.3) implies $\langle f, P_{m,k,\chi}\rangle = 0$ if and only if $a_m(f) = 0$. So f is orthogonal to all the Poincaré series if and only if every Fourier coefficient $a_m(f)$ is zero. But this happens if and only if f is identically zero. $\square$

Let's return to the previous computation for a breif moment. The procedure of changing an integral over a fundamental domain $\mathcal{F}_\Gamma$ into an integral over the strip $\Gamma_\infty\backslash\mathbb{H}$, or vice versa, works in a more general setting and is called **unfolding the integral** or **folding the integral** respectively:

Method 2.2.1 (Unfolding/folding the integral). Suppose Γ is any congruence subgroup with fundamental domain $\mathcal{F}_\Gamma$, and we are given an absolutely convergent integral of the form

$$\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma'\backslash\Gamma} f(\gamma z)\,d\mu,$$

where $\Gamma' \leq \Gamma$ is a subgroup (f is not necessarily a holomorphic form). Then the above integral can be written in the form

$$\int_{\Gamma'\backslash\mathbb{H}} f(z)\,d\mu.$$

To do this use DCT to interchange the sum and integral and then make the change of variales $z \to \gamma^{-1}z$ to put the integral in the form

$$\sum_{\gamma\in\Gamma'\backslash\Gamma}\int_{\gamma\mathbb{F}_\Gamma} f(z)\,d\mu.$$

Then observe $\mathbb{H} = \bigcup_{\gamma \in \Gamma} \gamma \mathcal{F}_\Gamma$ so that $\Gamma' \backslash \mathbb{H} = \bigcup_{\gamma \in \Gamma' \backslash \Gamma} \gamma \mathcal{F}_\Gamma$. The desired result follows. Of course, there is equality everywhere so we can also run the procedure in reverse.

Double Coset Operators

We are ready to introduce a class of general operators, depending upon double cosets, on a congruence subgroup Γ of level N. We will use these operators to define the diamond and Hecke operators. Recall that $j(\gamma, z) = (cz + d)$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, is the factor of modularity. Actually, $j(\gamma, z)$ makes sense for any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ and we extend it accordingly. Let Γ_1 and Γ_2 be two congruence subgroups (not necessarily of the same level). For $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ consider the double coset

$$\Gamma_1 \alpha \Gamma_2 = \{\gamma_1 \alpha \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}.$$

Then Γ_1 acts on the set $\Gamma_1 \alpha \Gamma_2$ by left multiplication so that it decomposes into a disjoin union of orbit spaces. Thus

$$\Gamma_1 \alpha \Gamma_2 = \bigcup_{\beta \in \Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2} \Gamma_1 \beta,$$

where the sum is over the orbit representatives β. However, in order for these operators to be well-defined it is necessary that the orbit decomposition above is a finite union. This is indeed the case and we will require two lemmas. The first is that congruence subgroups are preserved under conjugation by elements of $\mathrm{GL}_2^+(\mathbb{Q})$ provided we restrict to those elements in $\mathrm{PSL}_2(\mathbb{Z})$:

Lemma 2.2.2. *Let Γ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then $\alpha^{-1}\Gamma\alpha \cap \mathrm{PSL}_2(\mathbb{Z})$ is a congruence subgroup.*

Proof. Recall that if Γ is of level M, then $\Gamma(kM) \le \Gamma$ for every $k \ge 1$. Thus there is an integer $\widetilde{N}$ such that $\Gamma(\widetilde{N}) \le \Gamma$, $\widetilde{N}\alpha \in \mathrm{GL}_2^+(\mathbb{Z})$, and $\widetilde{N}\alpha \in \mathrm{GL}_2^+(\mathbb{Z})$. Now let $N = \widetilde{N}^3$ and notice that any $\gamma \in \Gamma(N)$ is of the form

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix},$$

for $k_1, \dots, k_4 \in \mathbb{Z}$. Therefore $\Gamma(N) \subseteq I + N\mathrm{Mat}_2(\mathbb{Z})$. Thus

$$\alpha\Gamma(N)\alpha^{-1} \le \alpha(I + N\mathrm{Mat}_2(\mathbb{Z}))\alpha^{-1} = I + \widetilde{N}\mathrm{Mat}_2(\mathbb{Z}).$$

As every matrix in $\alpha\Gamma(N)\alpha^{-1}$ has determinant 1 and $\Gamma(\widetilde{N}) \subseteq I + \widetilde{N}\mathrm{Mat}_2(\mathbb{Z})$, it follows that $\alpha\Gamma(N)\alpha^{-1} \le \Gamma(\widetilde{N})$. As $\Gamma(\widetilde{N}) \le \Gamma$, we conclude

$$\Gamma(N) \le \alpha^{-1}\Gamma\alpha,$$

and intersecting with $\mathrm{PSL}_2(\mathbb{Z})$ completes the proof. □

Note that by Lemma 2.2.2, if $\alpha^{-1}\Gamma\alpha \subset \mathrm{SL}_2(\mathbb{Z})$ then $\alpha^{-1}\Gamma\alpha$ is a congruence subgroup of Γ is. Moreover, since congruence subgroups are closed under intersection, Lemma 2.2.2 further implies that $\alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$ is a congruence subgroup for and two congruence subgroups Γ_1 and Γ_2 any any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Our second lemma gives a way to describe the orbit representatives for $\Gamma_1\alpha\Gamma_2$ in terms of coset representatives:

Lemma 2.2.3. *Let Γ_1 and Γ_2 be congruence subgroups and let let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Set $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$. Then left multiplication map*

$$\Gamma_2 \to \Gamma_1\alpha\Gamma_2 \qquad \gamma_2 \mapsto \alpha\gamma_2,$$

induces a bijection from the coset space $\Gamma_3\backslash\Gamma_2$ to the orbit space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$.

Proof. We will show that the induced map is both surjective and injective. For surjectivity, the orbit representatives β of $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ are of the form $\beta = \gamma_1\alpha\gamma_2$ for some $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. Since Γ_1 is acting on $\Gamma_1\alpha\Gamma_2$ by left multiplication, β can be written as $\beta = \alpha\gamma_2'$ for some $\gamma_2' \in \Gamma_2$. This shows that the induced map is a surjection. To prove injectivity, let $\gamma_2, \gamma_2' \in \Gamma_2$ be such that the orbit space representatives $\alpha\gamma_2$ and $\alpha\gamma_2'$ are equivalent. That is,

$$\Gamma_1\alpha\gamma_2 = \Gamma_1\alpha\gamma_2'.$$

This implies $\alpha\gamma_2(\gamma_2')^{-1} \in \Gamma_1\alpha$ and so $\gamma_2(\gamma_2')^{-1} \in \alpha^{-1}\Gamma_1\alpha$. But we also have $\gamma_2(\gamma_2')^{-1} \in \Gamma_2$ and these two facts together imply $\gamma_2(\gamma_2')^{-1} \in \Gamma_3$. Hence

$$\Gamma_3\gamma_2 = \Gamma_3\gamma_2',$$

which shows that the induced map is also an injection. □

With these lemmas in hand, we can prove that the orbit decomposition of $\Gamma_1\alpha\Gamma_2$ is finite:

Proposition 2.2.3. *Let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then the orbit decomposition*

$$\Gamma_1\alpha\Gamma_2 = \bigcup_j \Gamma_1\beta,$$

with respect to the action of Γ_1 by left multiplication, is a finite union.

Proof. Let $\Gamma_3 = \alpha\Gamma_1\alpha^{-1} \cap \Gamma_2$. Then Γ_3 acts on Γ_2 by left multiplication. By Lemma 2.2.3, the number of orbits of $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ is the same as the number of cosets of $\Gamma_3\backslash\Gamma_2$ which is $[\Gamma_2 : \Gamma_3]$. By Lemma 2.2.2, $\alpha^{-1}\Gamma_1\alpha \cap \mathrm{PSL}_2(\mathbb{Z})$ is a congruence subgroup and hence $[\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma_1\alpha \cap \mathrm{PSL}_2(\mathbb{Z})]$ is finite. As $\Gamma_2 = \mathrm{PSL}_2(\mathbb{Z}) \cap \Gamma_2$ and $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \mathrm{PSL}_2(\mathbb{Z}) \cap \Gamma_2$, it follows that $[\Gamma_2 : \Gamma_3] \le [\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma_1\alpha \cap \mathrm{PSL}_2(\mathbb{Z})]$ completing the proof. □

In light of Proposition 2.2.3, we will denote the orbit representatives by β_j to make it clear that there are finitely many. We can now introduce our operators. Fix some congruence subgroup Γ and consider $\mathcal{M}_k(\Gamma)$. Then for $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we define the operator $[\alpha]_k$ on $\mathcal{M}_k(\Gamma)$ to be the linear operator given by

$$(f[\alpha]_k)(z) = \det(\alpha)^{k-1} j(\alpha, z)^{-k} f(\alpha z),$$

for any $f \in \mathcal{M}_k(\Gamma)$. Moreover, $[\alpha]_k$ is multiplicative. Indeed, if $\alpha, \alpha' \in \mathrm{GL}_2^+(\mathbb{Q})$, then

$$\begin{aligned}((f[\alpha']_k)[\alpha]_k)(z) &= \det(\alpha)^{k-1} j(\alpha, z)^{-k} (f[\alpha']_k)(\alpha z) \\ &= \det(\alpha')^{k-1} \det(\alpha)^{k-1} j(\alpha', \alpha z)^{-k} j(\alpha, z)^{-k} f(\alpha'\alpha z) \\ &= \det(\alpha'\alpha)^{k-1} j(\alpha'\alpha, z)^{-k} f(\alpha'\alpha z) && \text{cocycle condition} \\ &= (f[\alpha'\alpha]_k)(z).\end{aligned}$$

Also, if $\gamma \in \Gamma$ and we choose the representative with $\det(\gamma) = 1$, then the chain of equalities

$$(f[\gamma]_k)(z) = j(\gamma, z)^{-k} f(\gamma z) = f(z),$$

is equivalent to the modularity of f on $\Gamma\backslash\mathbb{H}$ with trivial character. Thus f is holomorphic form on $\Gamma\backslash\mathbb{H}$ with trivial character if and only if $f[\gamma]_k = f$ for all $\gamma \in \Gamma$ where γ is chosen to be the representative with positive determinant. Now let Γ_1 and Γ_2 be two congruence subgroups and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. We define the **double coset operator** $[\Gamma_1\alpha\Gamma_2]_k$ on $\mathcal{M}_k(\Gamma_1)$ to be the linear operator given by

$$(f[\Gamma_1\alpha\Gamma_2]_k)(z) = \sum_j (f[\beta_j]_k)(z) = \sum_j \det(\beta_j)^{k-1} j(\beta_j, z)^{-k} f(\beta_j z),$$

for any $f \in \mathcal{M}_k(\Gamma_1)$. By Proposition 2.2.3 this sum is finite. It remains to check that $f[\Gamma_1\alpha\Gamma_2]_k$ is well-defined. Indeed, if β_j and β_j' belong to the same orbit, then $\beta_j'\beta_j^{-1} \in \Gamma_1$. But then as $f \in \mathcal{M}_k(\Gamma_1)$, is it invariant under the $[\beta_j'\beta_j^{-1}]_k$ operator so that

$$(f[\beta_j]_k)(z) = ((f[\beta_j'\beta_j^{-1}]_k)[\beta_j]_k)(z) = (f[\beta_j']_k)(z),$$

and therefore the $[\Gamma_1\alpha\Gamma_2]_k$ operator is well-defined. Actually, the map $[\Gamma_1\alpha\Gamma_2]_k$ preserves holomorphic forms:

Proposition 2.2.4. *For any two congruence subgroups Γ_1 and Γ_2, $[\Gamma_1\alpha\Gamma_2]_k$ maps $\mathcal{M}_k(\Gamma_1)$ into $\mathcal{M}_k(\Gamma_2)$. Moreover, $[\Gamma_1\alpha\Gamma_2]_k$ preserves the subspace of cusp forms.*

Proof. Suppose $f \in \mathcal{M}_k(\Gamma_1)$ and $\gamma \in \Gamma_2$. Then

$$\begin{aligned}
(f[\Gamma_1\alpha\Gamma_2]_k)(\gamma z) &= \sum_j \det(\beta_j)^{k-1} j(\beta_j, \gamma z)^{-k} f(\beta_j\gamma z) \\
&= \sum_j \det(\beta_j\gamma)^{k-1} j(\beta_j, \gamma z)^{-k} f(\beta_j\gamma z) && \det(\gamma) = 1 \\
&= \sum_j \det(\beta_j\gamma)^{k-1} \left(\frac{j(\gamma, z)}{j(\beta_j\gamma, z)}\right)^k f(\beta_j\gamma z) && \text{cocycle condition} \\
&= j(\gamma, z)^k \sum_j \det(\beta_j\gamma)^{k-1} j(\beta_j\gamma, z)^{-k} f(\beta_j\gamma z) \\
&= j(\gamma, z)^k \sum_j \det(\beta_j)^{k-1} j(\beta_j, z)^{-k} f(\beta_j z) && \beta_j \to \beta_j\gamma^{-1} \text{ bijection on } \Gamma_1\alpha\Gamma_2 \\
&= j(\gamma, z)^k \sum_j (f[\beta_j]_k)(z) \\
&= j(\gamma, z)^k (f[\Gamma_1\alpha\Gamma_2]_k)(z),
\end{aligned}$$

So $f[\Gamma_1\alpha\Gamma_2]_k$ is modular. Holomorphy and the growth condition are immediate since the sum is finite by Proposition 2.2.3. Therefore $f[\Gamma_1\alpha\Gamma_2]_k \in \mathcal{M}_k(\Gamma_2)$. To see that $[\Gamma_1\alpha\Gamma_2]_k$ preserves the subspace of cusp forms, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$ of $\Gamma_2\backslash\mathbb{H}$ and let $f \in \mathcal{S}_k(\Gamma_1)$. For any orbit representative β_j, $\beta_j\sigma_{\mathfrak{a}}$ takes ∞ to an element of $\mathbb{Q} \cup \{\infty\}$ since $\beta_j \in \mathrm{GL}_2^+(\mathbb{Q})$. In other words, $\beta_j\sigma_{\mathfrak{a}}\infty = \mathfrak{b}$ for some cusp $\mathfrak{b}$ of $\Gamma_1\backslash\mathbb{H}$. Now choose an integer $r \geq 1$ such that $r\alpha \in \mathrm{GL}_2^+(\mathbb{Z})$. As a consequence, there exist integers $n, m \geq 1$ such that $rj(\beta_j, \sigma_{\mathfrak{a}}z) = |n\sigma_{\mathfrak{a}}z + m|$. Then letting $\mathrm{Im}(z) > 1$ and δ be as in Lemma 2.2.1, we have

$$j(\beta_j, \sigma_{\mathfrak{a}}z) = \frac{|n\sigma_{\mathfrak{a}}z + m|}{r} \geq \frac{\delta|ni + m|}{r}.$$

From this bound, we see that

$$(f[\Gamma_1\alpha\Gamma_2]_k)(\sigma_{\mathfrak{a}}z) \ll \left(\frac{\delta|ni + m|}{r}\right)^{-k} \sum_j \det(\beta_j)^{k-1} f(\beta_j\sigma_{\mathfrak{a}}z).$$

As f is a cusp form, it has exponential decay to zero near the cusps so that the sum on the right-hand side tends to zero as $\mathrm{Im}(z) \to \infty$. Then $f[\Gamma_1\alpha\Gamma_2]_k$ is a cusp form too. $\square$

The double coset operators are the most basic types of operators on holomorphic forms. They are the building blocks needed to define the more important diamond and Hecke operators.

Diamond & Hecke Operators

The diamond and Hecke operators are special linear operators that are used to construct a linear theory of holomorphic forms. They will also help us understand the Fourier coefficients. Throughout this discussion, we will obtain corresponding results for holomorphic forms with nontrivial characters. We will discuss the diamond operator first. To define them, we need to consider both the congruence subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$. Recall that $\Gamma_1(N) \le \Gamma_0(N)$ and consider the map

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^* \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to d \pmod{N},$$

(d is invertible modulo N since $c \equiv 0 \pmod{N}$ and $ad-bc = 1$). This is a surjective homomorphism and its kernel is exaclty $\Gamma_1(N)$ so that $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ and $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$. Letting $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $f \in \mathcal{M}_k(\Gamma_1)$, consider $(f\,[\Gamma_1(N)\alpha\Gamma_1(N)]_k)\,(z)$. This is only dependent upon the lower-right entry d of α taken modulo N. To see this, since $\Gamma_1(N)$ is normal in $\Gamma_0(N)$, $\Gamma_1(N)\alpha = \alpha\Gamma_1(N)$ so that $\Gamma_1(N)\alpha\Gamma_1(N) = \alpha\Gamma_1(N)$ and hence there is only one representative for the orbit decomposition. Therefore

$$(f\,[\Gamma_1(N)\alpha\Gamma_1(N)]_k)\,(z) = \sum_j (f[\beta]_k)(z) = (f[\alpha]_k)(z).$$

This induces an action of $\Gamma_0(N)$ on $\mathcal{M}_k(\Gamma_1)$ and since $\Gamma_1(N)$ acts trivially, this is really an action of $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$. In other words, we have an induced action that depends only upon the lower-right entry d of α taken modulo N. So for any d modulo N, we define the **diamond operator** $\langle d\rangle : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$ to be the linear operator given by

$$(\langle d\rangle f)(z) = (f[\alpha]_k)(z),$$

for any $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Our discussion above has already shown that the diamond operators $\langle d\rangle$ are well-defined. Moreover, the diamond operators are also invertible with $\langle \overline{d}\rangle$ serving as an inverse and α^{-1} as a representative for the definition. Also, since the operator $[\alpha]_k$ is multiplicative and

$$\begin{pmatrix} * & * \\ 0 & d \end{pmatrix}\begin{pmatrix} * & * \\ 0 & e \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & de \end{pmatrix} \pmod{N},$$

the diamond operators are multiplicative. One reason the diamond operators are useful is that they decompose $\mathcal{M}_k(\Gamma_1(N))$ into eigenspaces. For any Dirichlet character χ modulo N, we let

$$\mathcal{M}_k(N,\chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) : \langle d\rangle f = \chi(d)f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^*\},$$

be the χ-eigenspace. Also let $\mathcal{S}_k(N,\chi)$ be the corresponding subspace of cusp forms. Then $\mathcal{M}_k(\Gamma_1(N))$ admits a decomposition into these eigenspaces:

Proposition 2.2.5. *We have the direct sum decomposition*

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi \pmod{N}} \mathcal{M}_k(N,\chi).$$

Moreover, this direct sum decomposition respects the subspace of cusp forms.

Proof. We have a representation of $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$ over $\mathcal{M}_k(\Gamma_1(N))$ given by the diamond operators. Explicitely,

$$\Phi : (\mathbb{Z}/N\mathbb{Z})^* \times \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N)) \qquad (d, f) \to \langle d \rangle f.$$

But any representation of a finite abelian group over $\mathbb{C}$ is completely reducible with repect to the characters of the group and every irreducible subrepresentation is 1-dimensional (see Appendix C.2). Since the characters of $(\mathbb{Z}/N\mathbb{Z})^*$ are given by Dirichlet characters, the decomposition follows. The decomposition respects the subspace of cusp forms because the double coset operators do. $\square$

If $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and we choose the representative with positive determinant, then $\chi(\gamma) = \chi(d)$ and the chain of equalities

$$(\langle d \rangle f)(z) = (f[\gamma]_k)(z) = j(\gamma, z)^{-k} f(\gamma z) = \chi(d) f(z),$$

is equivalent to the modularity of f with character χ on $\Gamma_0(N)\backslash\mathbb{H}$. Thus f is a holomorphic form with character χ on $\Gamma_0(N)\backslash\mathbb{H}$ if and only if $f[\gamma]_k = \chi(\gamma) f$ for all $\gamma \in \Gamma_0(N)$ where γ is chosen to be the representative with positive determinant. It follows that the diamond operators sieve holomorphic forms on $\Gamma_1(N)\backslash\mathbb{H}$ with trivial character in terms of holomorphic forms on $\Gamma_0(N)\backslash\mathbb{H}$ with nontrivial characters. In particular, $\mathcal{M}_k(N, \chi) = \mathcal{M}_k(\Gamma_0(N), \chi)$ and $\mathcal{S}_k(N, \chi) = \mathcal{S}_k(\Gamma_0(N), \chi)$. So by Proposition 2.2.5, we have

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi \pmod{N}} \mathcal{M}_k(\Gamma_0(N), \chi),$$

and this decomposition respects the subspace of cusp forms. This fact clarifies why it is necessary to consider holomorphic forms with nontrivial characters.
Now it is time to define the Hecke operators. For a prime p, we define the p-th **Hecke operator** $T_p : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$ to be the linear operator given by

$$(T_p f)(z) = \left(\left[\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) \right]_k f \right)(z).$$

By Proposition 2.2.4, T_p preserves the subspace of cusp forms. We will start discussing properties of the diamond and Hecke operators, but we first state an important lemma that will be used throughout (see [DS05] for a proof):

Lemma 2.2.4. *As sets,*

$$\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) = \left\{ \gamma \in \mathrm{Mat}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \ (mod\ N), \det(\gamma) = p \right\}.$$

With Lemma 2.2.4, it is not too hard to see that the diamond and Hecke operators commute:

Proposition 2.2.6. *For every $d \in (\mathbb{Z}/N\mathbb{Z})^*$ and prime p, the diamond operator $\langle d \rangle$ and the Hecke operator T_p on $\mathcal{M}_k(\Gamma_1(N))$ commute:*

$$\langle d \rangle T_p = T_p \langle d \rangle$$

Proof. Let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we have

$$\gamma\alpha\gamma^{-1} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & p \end{pmatrix} \pmod{N},$$

because $c \equiv 0 \pmod N$, $ad - bc = 1$, and $ad \equiv 1 \pmod N$. By Lemma 2.2.4, $\gamma\alpha\gamma^{-1} \in \Gamma_1(N)\alpha\Gamma_1(N)$ and so we can use this representative in place of α. On the one hand,

$$\Gamma_1(N)\alpha\Gamma_1(N) = \bigcup_j \Gamma_1(N)\beta_j.$$

On the other hand, using $\gamma\alpha\gamma^{-1}$ in place of α and the normality of $\Gamma_1(N)$ in $\Gamma_0(N)$, we have

$$\Gamma_1(N)\alpha\Gamma_1(N) = \Gamma_1(N)\gamma\alpha\gamma^{-1}\Gamma_1(N) = \gamma\Gamma_1(N)\alpha\Gamma_1(N)\gamma^{-1} = \gamma\bigcup_j \Gamma_1(N)\beta_j\gamma^{-1} = \bigcup_j \Gamma_1(N)\gamma\beta_j\gamma^{-1}.$$

Upon comparing these two decompositions of $\Gamma_1(N)\alpha\Gamma_1(N)$ gives

$$\bigcup_j \Gamma_1(N)\beta_j = \bigcup_j \Gamma_1(N)\gamma\beta_j\gamma^{-1}.$$

Now let $f \in \mathcal{M}_k(\Gamma_1(N))$. Then this equivalence of unions implies

$$\langle d\rangle T_p f = \sum_j f[\beta_j\gamma]_k = \sum_j f[\gamma\beta_j]_k = T_p\langle d\rangle f.$$

□

Using Lemma 2.2.4 we can obtain an explicit description of the Hecke operator T_p:

Proposition 2.2.7. *Let $f \in \mathcal{M}_k(\Gamma_1(N))$. Then the Hecke operator T_p acts on f as follows:*

$$(T_p f)(z) = \begin{cases} \displaystyle\sum_{j \pmod p} \left(f\left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right]_k\right)(z) + \left(f\left[\begin{pmatrix} m & n \\ N & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k\right)(z) & \text{if } p \nmid N, \\ \displaystyle\sum_{j \pmod p} \left(f\left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right]_k\right)(z) & \text{if } p \mid N, \end{cases}$$

where m and n are chosen such that $\det\left(\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)\right) = 1$.

Proof. Set $\Gamma_3 = \alpha^1\Gamma_1(N)\alpha \cap \Gamma_1(N)$ where $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. Define

$$\beta_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \quad \text{and} \quad \beta_\infty = \begin{pmatrix} m & n \\ N & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} pm & n \\ pN & p \end{pmatrix},$$

for j taken modulo p and where where m and n are chosen such that $\det\left(\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)\right) = 1$. It suffices to show $\{\beta_1, \ldots, \beta_{p-1}\}$ and $\{\beta_1, \ldots, \beta_{p-1}, \beta_\infty\}$ are complete sets of orbit representatives for $\Gamma_1(N)\backslash\Gamma_1(N)\alpha\Gamma_1(N)$ depending on if $p \nmid N$ or not. To acomplish this, we will find a complete set of coset representatives for $\Gamma_3\backslash\Gamma_1(N)$ and then use Lemma 2.2.3. First we require an explicit description of Γ_3. Let

$$\Gamma^0(p) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod p\right\},$$

and define

$$\Gamma_1^0(N, p) = \Gamma_1(N) \cap \Gamma^0(p).$$

We claim $\Gamma_3 = \Gamma_1^0(N, p)$. For the forward inclusion, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ and observe that

$$\alpha^{-1}\gamma\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} a & pd \\ p^{-1}c & d \end{pmatrix}.$$

If $\alpha^{-1}\gamma\alpha \in \Gamma_3$, then $\alpha^{-1}\gamma\alpha \in \Gamma_1(N)$ and thus $p \mid c$ so that $\alpha^{-1}\gamma\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Moreover, the previous computation implies

$$\alpha^{-1}\gamma\alpha = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod{p}.$$

This shows $\Gamma_3 \subseteq \Gamma_1^0(N,p)$. For the reverse inclusion, suppose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^1(N,p)$. Then $b = pk$ for some $k \in \mathbb{Z}$. Now observe

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\begin{pmatrix} a & k \\ pc & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \alpha^{-1}\gamma\alpha,$$

where $\gamma = \left(\begin{smallmatrix} a & k \\ pc & d \end{smallmatrix}\right)$. As $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ we conclude $\gamma \in \Gamma_1(N)$ as well. Now let

$$\gamma_j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_\infty = \begin{pmatrix} pm & n \\ N & 1 \end{pmatrix},$$

for j taken modulo p and where m and n are as before. Clearly $\gamma_j \in \Gamma_1(N)$ for all j. As $pm - Nn = 1$, we have $pm \equiv 1 \pmod{N}$ so that $\gamma_\infty \in \Gamma_1(N)$ as well. We claim that $\{\gamma_1, \ldots, \gamma_{p-1}\}$ and $\{\gamma_1, \ldots, \beta_{p-1}, \gamma_\infty\}$ are sets of coset representatives for $\Gamma_3\backslash\Gamma_1(N)$ depending on if $p \nmid N$ or not. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ and consider

$$\gamma\gamma_j^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b-aj \\ c & d-cj \end{pmatrix}.$$

As $\gamma\gamma_j^{-1} \in \Gamma_1(N)$ because both γ and γ_i are, $\gamma\gamma_j^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$ for some i if and only if

$$b \equiv aj \pmod{p}.$$

First suppose $p \nmid a$. Then a is invertible modulo p so we may take $j = \overline{a}b \pmod{p}$. Now suppose $p \mid a$. If there is some i satisfying $b \equiv ai \pmod{p}$, then we also have $p \mid b$. But as $ad - bc = 1$, this is impossible and so no such i exists. As $a \equiv 1 \pmod{N}$, $p \mid a$ if and only if $p \nmid N$. In this case consider instead

$$\gamma\gamma_\infty^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -n \\ -N & pm \end{pmatrix} = \begin{pmatrix} a-Nb & pmb-na \\ c-Nd & pmd-nc \end{pmatrix}.$$

Since $p \mid a$, we have $pmb - na \equiv 0 \mod p$ so that $\gamma\gamma_\infty^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$. Altogether, we have shown that $\{\gamma_1, \ldots, \gamma_{p-1}\}$ and $\{\gamma_1, \ldots, \beta_{p-1}, \gamma_\infty\}$ are sets of coset representatives for $\Gamma_3\backslash\Gamma_1(N)$ depending on if $p \nmid N$ or not. To show they are complete sets, we need to show that no two representatives belong to the same coset. To this end, suppose j and j' are distinct, taken modulo p, and consider

$$\gamma_j\gamma_{j'}^{-1} = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & j' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & j-j' \\ 0 & 1 \end{pmatrix}.$$

Then $\gamma_j\gamma_{j'}^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$ if and only if $j - j' \equiv 0 \pmod{p}$. This is impossible since j and j' are distinct. Hence γ_j and $\gamma_{j'}$ represent distinct cosets. Now consider

$$\gamma_j\gamma_\infty^{-1} = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -n \\ -N & pm \end{pmatrix} = \begin{pmatrix} 1-Nj & pmj-n \\ -N & pm \end{pmatrix}.$$

Now $\gamma_j\gamma_\infty^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$ if and only if $pmj - n \equiv 0 \pmod{p}$. This is impossible sice $pm - Nn = 1$ implies $p \nmid n$. Therefore γ_j and γ_∞ represent distinct cosets. It follows that $\{\gamma_1, \ldots, \gamma_{p-1}\}$ and $\{\gamma_1, \ldots, \beta_{p-1}, \gamma_\infty\}$ are complete sets of coset representatives completing the proof. As

$$\alpha\gamma_j = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} = \beta_j \quad \text{and} \quad \alpha\gamma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} pm & n \\ N & 1 \end{pmatrix} = \begin{pmatrix} pm & n \\ pN & p \end{pmatrix} = \beta_\infty,$$

Lemma 2.2.3 finishes the proof. $\square$

This explicit definition of T_p can be used to compute how the Hecke operators act on the Fourier coefficients of a holomorphic form:

Proposition 2.2.8. *Let $f \in \mathcal{M}_k(\Gamma_1(N))$ with Fourier coefficients $a_n(f)$. Then for primes p with $(p, N) = 1$,*

$$(T_p f)(z) = \sum_{n \geq 0} \left(a_{np}(f) + \chi_{N,0}(p) p^{k-1} a_{\frac{n}{p}}(\langle p \rangle f) \right) e^{2\pi i n z},$$

is the Fourier series of $T_p f$ where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid n$. Moreover, if $f \in \mathcal{M}_k(N, \chi)$ then $T_p f \in \mathcal{M}_k(N, \chi)$ and

$$(T_p f)(z) = \sum_{n \geq 0} \left(a_{np}(f) + \chi(p) p^{k-1} a_{\frac{n}{p}}(f) \right) e^{2\pi i n z},$$

is the Fourier series of $T_p f$ where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid n$.

Proof. In view of Proposition 2.2.5 and the linearity of the Hecke operators, it suffices to assume $f \in \mathcal{M}_k(N, \chi)$ and thus only the second formula needs to be verified. Observe that

$$\left(f \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \right)(z) = \frac{1}{p} f\left(\frac{z+j}{p} \right) = \frac{1}{p} \sum_{n \geq 0} a_n(f) e^{\frac{2\pi i n(z+j)}{p}}.$$

Summing over all j modulo p gives

$$\sum_{j \pmod p} \left(f \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \right)(z) = \sum_{j \pmod p} \frac{1}{p} \sum_{n \geq 0} a_n(f) e^{\frac{2\pi i n(z+j)}{p}} = \sum_{n \geq 0} a_n(f) e^{\frac{2\pi i n z}{p}} \frac{1}{p} \sum_{j \pmod p} e^{\frac{2\pi n i j}{p}}.$$

If $p \nmid N$ then the inner sum vanishes because it is the sum over all p-th roots of unity. If $p \mid N$ then the sum is p. Therefore

$$\sum_{j \pmod p} \left(f \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \right)(z) = \sum_{n \geq 0} a_{np}(f) e^{2\pi i n z}.$$

If $p \mid N$, then Proposition 2.2.7 implies

$$(T_p f)(z) = \sum_{n \geq 0} a_{np}(f) e^{2\pi i n z},$$

which is the claimed Fourier series of $T_p f$. $p \nmid N$, then we have the additional term

$$\begin{aligned}
\left(f \left[\begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right]_k \right)(z) &= \left(\langle p \rangle f \left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right]_k \right)(z) \\
&= p^{k-1} (\langle p \rangle f)(pz) \\
&= \sum_{n \geq 0} p^{k-1} a_n(\langle p \rangle f) e^{2\pi i p n z} \\
&= \sum_{n \geq 0} \chi(p) p^{k-1} a_n(f) e^{2\pi i p n z},
\end{aligned}$$

where the first equality holds because $\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right) \in \Gamma_0(N)$ and the last equality holds because $\langle p \rangle f = \chi(p) f$. In this case, Proposition 2.2.7 gives

$$(T_p f)(z) = \sum_{n \geq 0} a_{np}(f) + \chi(p) p^{k-1} a_{\frac{n}{p}}(f) e^{2\pi i n z}.$$

Since $\chi(p) = 0$ if $p \mid N$, these two cases can be expressed together as

$$(T_p f)(z) = \sum_{n \geq 0} \left(a_{np}(f) + \chi(p)p^{k-1} a_{\frac{n}{p}}(f) \right) e^{2\pi i n z}.$$

□

We now mention the crucial result about Hecke operators which is that they form a simultaneously commuting family with the diamond operators:

Proposition 2.2.9. *Let p and q be primes and $d, e \in (\mathbb{Z}/N\mathbb{Z})^*$. The Hecke operators T_p and T_q and diamond operators $\langle d \rangle$ and $\langle e \rangle$ on $\mathcal{M}_k(\Gamma_1(N))$ form a simultaneously commuting family:*

$$T_p T_q = T_q T_p, \quad \langle d \rangle T_p = T_p \langle d \rangle, \quad \text{and} \quad \langle d \rangle \langle e \rangle = \langle e \rangle \langle d \rangle.$$

Proof. Showing the diamond and Hecke operators commute was Proposition 2.2.6. To show commutativity of the diamond operators, let $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\eta = \left(\begin{smallmatrix} * & * \\ * & e \end{smallmatrix}\right) \in \Gamma_0(N)$. Then

$$\gamma\eta \equiv \begin{pmatrix} * & * \\ * & de \end{pmatrix} \equiv \begin{pmatrix} * & * \\ * & ed \end{pmatrix} \equiv \eta\gamma \pmod{N}.$$

Therefore $[\gamma\eta]_k = [\eta\gamma]_k$ and so for any $f \in \mathcal{M}_k(\Gamma_1(N))$, we have

$$\langle d \rangle \langle e \rangle f = f[\gamma\eta]_k = f[\eta\gamma]_k = \langle e \rangle \langle d \rangle f.$$

We now show that the Hecke operators commute. In view of Proposition 2.2.5 and linearity of the Hecke operators, it suffices to prove this for $f \in \mathcal{M}_k(N, \chi)$. Applying Proposition 2.2.8 twice, for any $n \geq 1$ we compute

$$\begin{aligned} a_n(T_p T_q f) &= a_{np}(T_q f) + \chi(p)p^{k-1} a_{\frac{n}{p}}(T_q f) \\ &= a_{npq}(f) + \chi(q)q^{k-1} a_{\frac{np}{q}}(f) + \chi(p)p^{k-1}(a_{\frac{nq}{p}}(f) + \chi(q)q^{k-1} a_{\frac{n}{pq}}(f)) \\ &= a_{npq}(f) + \chi(q)q^{k-1} a_{\frac{np}{q}}(f) + \chi(p)p^{k-1} a_{\frac{nq}{p}}(f) + \chi(pq)(pq)^{k-1} a_{\frac{n}{pq}}(f). \end{aligned}$$

The last expression is symmetric in p and q so that $a_n(T_p T_q f) = a_n(T_q T_p f)$ for all $n \geq 1$. Since all of the Fourier coefficeints are equal, we get

$$T_p T_q f = T_q T_p f.$$

□

We can use Proposition 2.2.9 to construct diamond operators $\langle m \rangle$ and Hecke operators T_m for all $m \geq 1$. The **diamond operator** $\langle m \rangle : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$ is defined to be the linear operator given by

$$\langle m \rangle = \begin{cases} \langle m \rangle \text{ with } m \text{ taken modulo } N & \text{if } (m, N) = 1, \\ 0 & \text{if } (m, N) > 1. \end{cases}$$

Now for the Hecke operators. If $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the prime decomposition of m, then we define the m-th **Hecke operator** $T_m : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$ to be the linear operator given by

$$T_m = \prod_{1 \leq i \leq k} T_{p_i^{r_i}},$$

where T_{p^r} is defined inductively by

$$T_{p^r} = \begin{cases} T_p T_{p^{r-1}} - p^{k-1} \langle p \rangle T_{p^{r-2}} & \text{if } p \nmid N, \\ T_p^r & \text{if } p \mid N, \end{cases}$$

for all $r \geq 2$. Then by Proposition 2.2.9, the Hecke operators T_m are multiplicative but not completely multiplicative in m. Moreover, they commute with the diamond operators $\langle m \rangle$. Using these definitions, Propositions 2.2.8 and 2.2.9, a more general formula for how the Hecke operators T_m act on Fourier coefficients can be derived:

Proposition 2.2.10. *Let $f \in \mathcal{M}_k(\Gamma_1(N))$ with Fourier coefficients $a_n(f)$. Then for $m \geq 1$ with $(m, N) = 1$,*

$$(T_m f)(z) = \sum_{n \geq 0} \left(\sum_{d \mid (n,m)} d^{k-1} a_{\frac{nm}{d^2}}(\langle d \rangle f) \right) e^{2\pi i n z},$$

is the Fourier series of $T_m f$. Moreover, if $f \in \mathcal{M}_k(N, \chi)$, then

$$(T_m f)(z) = \sum_{n \geq 0} \left(\sum_{d \mid (n,m)} \chi(d) d^{k-1} a_{\frac{nm}{d^2}}(f) \right) e^{2\pi i n z}.$$

Proof. In view of Proposition 2.2.5 and linearity of the Hecke operators, we may assume $f \in \mathcal{M}_k(N, \chi)$. Therefore we only need to verify the second formula. When $m = 1$ the result is obvious and when $m = p$, we have

$$\sum_{d \mid (n,p)} \chi(d) d^{k-1} a_{\frac{np}{d^2}}(f) = a_{np}(f) + \chi(p) p^{k-1} a_{\frac{n}{p}}(f),$$

which is the result obtained from Proposition 2.2.8. By induction assume that the desired formula holds for $m = 1, p, \ldots, p^{r-1}$. Using the definition of T_{p^r} and Proposition 2.2.8, for any $n \geq 1$ we compute

$$\begin{aligned} a_n(T_{p^r} f) &= a_n(T_p T_{p^{r-1}} f) - \chi(p) p^{k-1} a_n(T_{p^{r-2}} f) \\ &= a_{np}(T_{p^{r-1}} f) + \chi(p) p^{k-1} a_{\frac{n}{p}}(T_{p^{r-1}} f) - \chi(p) p^{k-1} a_n(T_{p^{r-2}} f). \end{aligned}$$

By our induction hypothesis, this last expression is

$$\sum_{d \mid (np, p^{r-1})} \chi(d) d^{k-1} a_{\frac{np^r}{d^2}}(f) + \chi(p) p^{k-1} \sum_{d \mid \left(\frac{n}{p}, p^{r-1}\right)} \chi(d) d^{k-1} a_{\frac{np^{r-2}}{d^2}}(f) - \chi(p) p^{k-1} \sum_{d \mid (n, p^{r-2})} \chi(d) d^{k-1} a_{\frac{np^{r-2}}{d^2}}(f).$$

Write the first sum as

$$\sum_{d \mid (np, p^{r-1})} \chi(d) d^{k-1} a_{\frac{np^r}{d^2}}(f) = a_{np^r}(f) + \sum_{d \mid (n, p^{r-2})} \chi(d) d^{k-1} a_{\frac{np^{r-2}}{d^2}}(f),$$

and observe that the sum on the right-hand side cancels the entire third term above. Therefore our expression reduces to

$$\begin{aligned} a_{np^r}(f) + \chi(p) p^{k-1} \sum_{d \mid \left(\frac{n}{p}, p^{r-1}\right)} \chi(d) d^{k-1} a_{\frac{np^{r-2}}{d^2}}(f) &= a_{np^r}(f) + \sum_{d \mid \left(\frac{n}{p}, p^{r-1}\right)} \chi(dp)(dp)^{k-1} a_{\frac{np^{r-2}}{d^2}}(f) \\ &= a_{np^r}(f) + \sum_{\substack{d \mid (n, p^r) \\ d \neq 1}} \chi(d) d^{k-1} a_{\frac{np^r}{d^2}}(f) \\ &= \sum_{d \mid (n, p^r)} \chi(d) d^{k-1} a_{\frac{np^r}{d^2}}(f), \end{aligned}$$

where in the second line we have performed the change of variables $dp \to d$ in the sum. This proves the claim when $m = p^r$ for all $r \geq 0$. By multiplicativity of the Hecke operators, it suffices to prove the claim when $m = p^r q^s$ for another prime q and some $s \geq 0$. We compute

$$\begin{aligned}
a_n(T_{p^r q^s} f) &= a_n(T_{p^r} T_{q^s} f) \\
&= \sum_{d_1 \mid (n, p^r)} \chi(d_1) d_1^{k-1} a_{\frac{np^r}{d_1^2}}(T_{q^s} f) \\
&= \sum_{d_1 \mid (n, p^r)} \chi(d_1) d_1^{k-1} \sum_{d_2 \mid \left(\frac{np^r}{d_1^2}, q^s\right)} \chi(d_2) d_2^{k-1} a_{\frac{np^r q^s}{(d_1 d_2)^2}}(f) \\
&= \sum_{d_1 \mid (n, p^r)} \sum_{d_2 \mid \left(\frac{np^r}{d_1^2}, q^s\right)} \chi(d_1 d_2)(d_1 d_2)^{k-1} a_{\frac{np^r q^s}{(d_1 d_2)^2}}(f).
\end{aligned}$$

Summing over pairs (d_1, d_2) of divisors of (n, p^r) and $\left(\frac{np^r}{d^2}, q^s\right)$ respectively is the same as summing over divisors d of $(n, p^r q^s)$. Indeed, because p and q are relative prime, any such d is of the form $d = d_1 d_2$ where $d_1 \mid (n, p^r)$ and $d_2 \mid \left(\frac{np^r}{d_2^2}, q^s\right)$. Therefore the double sum becomes

$$\sum_{d \mid (n, p^r q^s)} \chi(d) d^{k-1} a_{\frac{np^r q^s}{d^2}}(f).$$

This completes the proof. $\square$

The diamond and Hecke operators turn out to be normal with respect to the Petersson inner product on the subspace of cusp forms. To prove this fact, we will require a lemma:

Lemma 2.2.5. *Let Γ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then the following are true:*

(i) If $\alpha^{-1}\Gamma\alpha \subseteq \mathrm{SL}_2(\mathbb{Z})$, then $V_{\alpha^{-1}\Gamma\alpha} = V_\Gamma$ and $[\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma\alpha] = [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma]$.

(ii) There exist $\beta_1, \ldots, \beta_n \in \mathrm{GL}_2^+(\mathbb{Q})$, where $n = [\Gamma : \alpha^{-1}\Gamma\alpha \cap \Gamma] = [\Gamma : \alpha\Gamma\alpha^{-1} \cap \Gamma]$, and such that

$$\Gamma\alpha\Gamma = \bigcup_j \Gamma\beta_j = \bigcup_j \beta_j\Gamma.$$

Proof. For (i), we claim

$$V_{\alpha^{-1}\Gamma\alpha} = \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} d\mu = \int_{\mathcal{F}_\Gamma} d\mu = V_\Gamma,$$

where the second equality is justified by first making the change of variables $z \to \alpha z$ and then noting that α acts as an automorphism on $\mathbb{H}$ so that $\mathcal{F}_{\Gamma\alpha} = \mathcal{F}_\Gamma$. The second statement now follows from Equation (2.1). For (ii), apply (i) with the congruence subgroup $\alpha\Gamma\alpha^{-1} \cap \Gamma$ in place of Γ to get

$$[\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma\alpha \cap \Gamma] = [\mathrm{PSL}_2(\mathbb{Z}) : \alpha\Gamma\alpha^{-1} \cap \Gamma].$$

Dividing both sides by $[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma]$ gives

$$[\Gamma : \alpha^{-1}\Gamma\alpha \cap \Gamma] = [\Gamma : \alpha\Gamma\alpha^{-1} \cap \Gamma].$$

Therefore we can find coset representatives $\gamma_1, \dots, \gamma_n \in \Gamma$ and $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n \in \Gamma$ such that

$$\Gamma = \bigcup_j (\alpha^{-1}\Gamma\alpha \cap \Gamma)\gamma_j = \bigcup_j (\alpha^{-1}\Gamma\alpha \cap \Gamma)\widetilde{\gamma}_j^{-1}.$$

Invoking Lemma 2.2.3 twice, we can express each of these coset decompositiions as an orbit decomposition:

$$\bigcup_j (\alpha^{-1}\Gamma\alpha \cap \Gamma)\gamma_j = \bigcup_j \Gamma\alpha\gamma_j \quad \text{and} \quad \bigcup_j (\alpha^{-1}\Gamma\alpha \cap \Gamma)\widetilde{\gamma}_j^{-1} = \bigcup_j \Gamma\alpha^{-1}\widetilde{\gamma}_j^{-1}.$$

It follows that

$$\Gamma = \bigcup_j \Gamma\alpha\gamma_j = \bigcup_j \widetilde{\gamma}_j\alpha\Gamma.$$

For each j the orbit spaces $\Gamma\alpha\gamma_j$ and $\widetilde{\gamma}_j\alpha\Gamma$ have nonempty intersection. For if they did we would have $\Gamma\alpha\gamma_j \subseteq \bigcup_{i\neq j} \widetilde{\gamma}_i\alpha\Gamma$ and thus $\Gamma\alpha\Gamma \subseteq \bigcup_{i\neq j} \widetilde{\gamma}_i\alpha\Gamma$. This contradicts the previous decomposition of Γ. Therefore we can find representatives $\beta_j \in \Gamma\alpha\gamma_j \cap \widetilde{\gamma}_j\alpha\Gamma$ for every j. Then β_j

$$\Gamma = \bigcup_j \Gamma\beta_j = \bigcup_j \beta_j\Gamma.$$

□

We can use Lemma 2.2.5 to compute adjoints:

Proposition 2.2.11. *Let Γ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Set $\alpha' = \det(\alpha)\alpha^{-1}$. Then the following are true:*

(i) If $\alpha^{-1}\Gamma\alpha \subseteq \mathrm{SL}_2(\mathbb{Z})$, then for all $f \in \mathcal{S}_k(\Gamma)$ and $g \in \mathcal{S}_k(\alpha^{-1}\Gamma\alpha)$, we have

$$\langle f[\alpha]_k, g\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g[\alpha']_k\rangle_\Gamma.$$

(ii) For all $f, g \in \mathcal{S}_k(\Gamma)$, we have

$$\langle f[\Gamma\alpha\Gamma]_k, g\rangle = \langle f, g[\Gamma\alpha'\Gamma]_k\rangle.$$

In particular, if $\alpha^{-1}\Gamma\alpha = \Gamma$ then $[\alpha]_k^* = [\alpha']_k$ and $[\Gamma\alpha\Gamma]_k^* = [\Gamma\alpha'\Gamma]_k$.

Proof. To prove (i) we first compute

$$\begin{aligned}
\langle f[\alpha]_k, g\rangle_{\alpha^{-1}\Gamma\alpha} &= \frac{1}{V_{\alpha^{-1}\Gamma\alpha}} \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} (f[\alpha]_k)(z)\overline{g(z)}\mathrm{Im}(z)^k \, d\mu \\
&= \frac{1}{V_{\alpha^{-1}\Gamma\alpha}} \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} \det(\alpha)^{k-1} j(\alpha, z)^{-k} f(\alpha z)\overline{g(z)}\mathrm{Im}(z)^k \, d\mu \\
&= \frac{1}{V_\Gamma} \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} \det(\alpha)^{k-1} j(\alpha, z)^{-k} f(\alpha z)\overline{g(z)}\mathrm{Im}(z)^k \, d\mu \qquad \text{Lemma 2.2.5 (i).}
\end{aligned}$$

Making the change of variables $z \to \alpha' z$, using the fact $\mathcal{F}_{\alpha'\alpha^{-1}\Gamma\alpha} = \mathcal{F}_{\Gamma\alpha'} = \mathcal{F}_\Gamma$ (because $\alpha'\alpha^{-1}$ lies in the center of $\mathrm{PSL}_2(\mathbb{Z})$ and α' acts as an automorphism on $\mathbb{H}$), and that α' acts as α^{-1} on $\mathbb{H}$ (they differ by a scalar) together, gives

$$\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} \det(\alpha)^{k-1} j(\alpha, \alpha' z)^{-k} f(z)\overline{g(\alpha' z)}\mathrm{Im}(\alpha' z)^k \, d\mu.$$

Using the cocycle relation, and the identities $\mathrm{Im}(\alpha' z) = \det(\alpha')\frac{\mathrm{Im}(z)}{|j(\alpha',z)|^2}$, $j(\alpha\alpha', z) = \det(\alpha)$, and $\det(\alpha') = \det(a')$ together, we can further rewrite the integral as

$$\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \det(\alpha')^{k-1}\overline{j(\alpha',z)^{-k}}f(z)\overline{g(\alpha' z)}\mathrm{Im}(z)^k\,d\mu.$$

Reversing the computation in the start of the proof shows that that

$$\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \det(\alpha')^{k-1}\overline{j(\alpha',z)^{-k}}f(z)\overline{g(\alpha' z)}\mathrm{Im}(z)^k\,d\mu = \langle f, g[\alpha']_k\rangle_\Gamma,$$

completing the proof of (i). To prove (ii), the coset decomposition $\Gamma\alpha\Gamma = \bigcup_j \Gamma\beta_j$ from Lemma 2.2.5 (ii) implies we can use the β_j as representatives in the definition of the $[\Gamma\alpha\Gamma]_k$ operator. As the β_j also satisfy $\Gamma\alpha\Gamma = \bigcup_j \beta_j\Gamma$, upon inverting β_j and noting that $\beta_j \in \Gamma\alpha$, we obtain $\Gamma\alpha^{-1}\Gamma = \bigcup_j \Gamma\beta_j^{-1}$. Since scalar multiplication commutes with matrices and the matrices in Γ have determinant 1, we conclude that $\Gamma\alpha'\Gamma = \bigcup_j \Gamma\beta_j'$ where $\beta_j' = \det(\beta_j)\beta_j^{-1}$ (also $\det(\beta_j) = \det(\alpha)$). So we can use the β_j' as representatives in the definition of the $[\Gamma\alpha'\Gamma]_k$ operator. The statement now follows from (i). The last statement is now obvious. ☐

We can now prove that the diamond and Hecke operators are normal:

Proposition 2.2.12. *On $\mathcal{S}_k(\Gamma_1(N))$, the diamond operators $\langle m\rangle$ and Hecke operators T_m are normal for all $m \ge 1$ with $(m, N) = 1$ with respect to the Petersson inner product. Moreover, their adjoints are given by*

$$\langle m\rangle^* = \langle \overline{m}\rangle \quad \text{and} \quad T_p^* = \langle \overline{p}\rangle T_p.$$

Proof. As taking the adjoint is a linear operator, the definition of the diamond and Hecke operators and Proposition 2.2.9 imply that it suffices to prove the two adjoint formulas for the when $m = p$ is prime. We will first prove the adjoint formula for $\langle p\rangle$. Let $\alpha = \left(\begin{smallmatrix} * & * \\ * & p\end{smallmatrix}\right) \in \Gamma_0(N)$. As $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ and $\det(\alpha) = 1$, Proposition 2.2.11 gives

$$\langle p\rangle^* = [\alpha]_k^* = [\alpha']_k = [\alpha^{-1}]_k = \langle \overline{p}\rangle.$$

This proves the adjoint formula for the diamond operators and normality follows from multiplicativity. For the Hecke operator T_p, let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p\end{smallmatrix}\right)$ and note that $\alpha' = \left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)$. Now Proposition 2.2.11 implies

$$T_p^* = [\Gamma_1(N)\alpha\Gamma_1(N)]_k^* = [\Gamma_1(N)\alpha'\Gamma_1(N)]_k.$$

Let m and n be such that $pm - Nn = 1$. Then

$$\begin{pmatrix} 1 & n \\ N & pm\end{pmatrix}\alpha' = \begin{pmatrix} 1 & n \\ N & pm\end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} p & n \\ pN & pm\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}\begin{pmatrix} p & n \\ N & m\end{pmatrix} = \alpha\begin{pmatrix} p & n \\ N & m\end{pmatrix},$$

which implies that $\alpha' = \left(\begin{smallmatrix} 1 & n \\ N & pm\end{smallmatrix}\right)^{-1}\alpha\left(\begin{smallmatrix} p & n \\ N & m\end{smallmatrix}\right)$. As $\left(\begin{smallmatrix} 1 & n \\ N & pm\end{smallmatrix}\right) \in \Gamma_1(N)$ (note that $pm \equiv 1 \pmod N$ since $pm - Nn = 1$) and $\left(\begin{smallmatrix} p & n \\ N & m\end{smallmatrix}\right) \in \Gamma_0(N)$, substiuting the triple product expression for α' into $[\Gamma_1(N)\alpha'\Gamma_1(N)]_k$ and noting that $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ yields

$$\Gamma_1(N)\alpha'\Gamma_1(N) = \Gamma_1(N)\alpha\Gamma_1(N)\begin{pmatrix} p & n \\ N & m\end{pmatrix}.$$

Now if the β_j are representatives for $[\Gamma_1\alpha\Gamma_1(N)]_k$, then $\Gamma_1\alpha\Gamma_1(N) = \bigcup_j \Gamma_1(N)\beta_j$. Therefore the formula above implies $\Gamma_1\alpha'\Gamma_1(N) = \bigcup_j \Gamma_1(N)\beta_j\left(\begin{smallmatrix} p & n \\ N & m\end{smallmatrix}\right)$ and so the $\beta_j\left(\begin{smallmatrix} p & n \\ N & m\end{smallmatrix}\right)$ can be used as representatives for $[\Gamma_1\alpha'\Gamma_1(N)]_k$. As $pm - Nn = 1$, $m \equiv \overline{p} \pmod N$ and so Proposition 2.2.9 implies that $T_p^* = \langle\overline{p}\rangle T_p$. This proves the adjoint formula for the Hecke operators and normality follows from multiplicativity. ☐

In the case of the modular group, Proposition 2.2.12 says that all of the diamond and Hecke operators are normal. Now suppose f is a non-constant holomorphic form with Fourier coefficients $a_n(f)$. If f is a simultaneous eigenfunction for all diamond operators $\langle m\rangle$ and Hecke operators T_m with $(m,N)=1$, we call f an **eigenform**. If the condition $(m,N)=1$ can be dropped, so that f is a simultaneous eigenfunction for all diamond and Hecke operators, then we say f is a **Hecke eigenform**. In particular, on $\mathrm{PSL}_2(\mathbb{Z})$ all eigenforms are Hecke eigenforms. Let the eigenvalue of T_m for f be $\lambda_f(m)$. Then Proposition 2.2.10 immediately implies that the first Fourier coefficient of T_mf is $a_m(f)$ and so $a_m(f)=\lambda_f(m)a_1(f)$ for all $m\geq 1$ with $(m,N)=1$. Therefore we cannot have $a_1(f)=0$ for this would mean f is constant. We can normalize f by dividing by $a_1(f)$ so that the Fourier series has constant term 1. It follows that the m-th Fourier coefficient of f, when $(m,N)=1$, is precisely the eigenvalue $\lambda_f(m)$. This normalization is called the **Hecke normalization** of f. The **Petersson normalization** of f is where we normalize so that $\langle f,f\rangle=1$. From the spectral theorem we derive an important corollary:

Theorem 2.2.5. *$\mathcal{S}_k(\Gamma_1(N))$ admits an orthonormal basis of eigenforms.*

Proof. This follows from the spectral theorem along with Propositions 2.2.9 and 2.2.12. □

As a near immediate consequence of Theorem 2.2.5, we can show that the Fourier coefficients of eigenforms satisfy certain relations known as the **Hecke relations**:

Proposition 2.2.13 (Hecke relations, holomorphic version). *Let $f\in\mathcal{S}_k(N,\chi)$ be a Hecke eigenform with Fourier coefficients $\lambda_f(n)$. Then the Fourier coefficients are multiplicative and satisfy*

$$\lambda_f(n)\lambda_f(m)=\sum_{d\mid(n,m)}\chi(d)d^{k-1}\lambda_f\left(\frac{nm}{d^2}\right)\quad\text{and}\quad\lambda_f(nm)=\sum_{d\mid(n,m)}\mu(d)\chi(d)d^{k-1}\lambda_f\left(\frac{n}{d}\right)\lambda_f\left(\frac{m}{d}\right),$$

for all $n,m\geq 1$ with $(nm,N)=1$.

Proof. By Theorem 2.2.5 it suffices to verify this when f is a Hecke eigenform. We may also assume f is Hecke normalized without any harm. The multiplicativity of the Fourier coefficients now follows from the multiplicity of the Hecke operators. Moreover, the first identity follows immeditely from computing the n-th Fourier coefficeint of T_mf in two different ways. On the one hand, use that f is a Hecke eigenform. On the other hand, use Proposition 2.2.10. For the second identity, we have

$$\chi(p)p^{k-1}=\lambda_f(p)^2-\lambda_f(p^2),$$

provided $(p,N)=1$, by computing the p-th Fourier coefficeint of T_pf in two different ways as we just did above. The second identity now follows from the first because $\lambda_f(n)$ is a specially multiplicative function in n (see Appendix A.2). □

As an immedite consequence of the Hecke relations, the Hecke operators satisfy analogous relations:

Corollary 2.2.1. *For all $n,m\geq 1$ with $(nm,N)=1$, we have*

$$T_nT_m=\sum_{d\mid(n,m)}\chi(d)d^{k-1}T_{\frac{nm}{d^2}}\quad\text{and}\quad T_{nm}=\sum_{d\mid(n,m)}\mu(d)\chi(d)d^{k-1}T_{\frac{n}{d}}T_{\frac{m}{d}}.$$

Proof. This is immedite from Theorem 2.2.5 and the Hecke relations. □

The identities in Corollary 2.2.1 can also be established directly. Moreover, the first identity is symmetric in n and m so it can be used to show that the Hecke operators commute.

Atkin–Lehner Theory

So far, our entire theory of holomorphic forms has started with a fixed congruence subgroup of some level. Atkin–Lehner theory, or the theory of oldforms & newforms, allows us to discuss holomorphic forms in the context of moving between levels. In this setting, we will only deal with congruence subgroups of the form $\Gamma_1(N)$ and cusp forms on $\Gamma_1(N)\backslash\mathbb{H}$. The easiest way lift a holomorphic form from a smaller level to a larger level is to observe that if $M \mid N$, then $\Gamma_1(N) \le \Gamma_1(M)$ so there is a natural inclusion $\mathcal{S}_k(\Gamma_1(M)) \subseteq \mathcal{S}_k(\Gamma_1(N))$. There is a less trivial way of lifting from $\mathcal{S}_k(\Gamma_1(M))$ to $\mathcal{S}_k(\Gamma_1(N))$. For any $d \mid \frac{N}{M}$, let $\alpha_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $f \in \mathcal{S}_k(\Gamma_1(M))$, then

$$(f[\alpha_d]_k)(z) = \det(\alpha_d)^{k-1} j(\alpha_d, z)^{-k} f(\alpha_d z) = d^{k-1} f(dz).$$

In fact, $[\alpha_d]_k$ maps $\mathcal{S}_k(\Gamma_1(M))$ into $\mathcal{S}_k(\Gamma_1(N))$ and preserves the subspace of cusp forms:

Proposition 2.2.14. *Let M and N be positive integers such that $M \mid N$. For any $d \mid \frac{N}{M}$, $[\alpha_d]_k$ maps $\mathcal{S}_k(\Gamma_1(M))$ into $\mathcal{S}_k(\Gamma_1(N))$.*

Proof. It is clear that holomorphy and the growth condition are satisfied for $f[\alpha_d]_k$. It is also clear that $f[\alpha_d]_k$ is a cusp form if f is. So all that is left to verify is the modularity condition. To see this, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d' \end{smallmatrix}\right) \in \Gamma_1(N)$, we have

$$\alpha_d \gamma \alpha_d^{-1} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d' \end{pmatrix}\begin{pmatrix} d^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & bd \\ d^{-1}c & d' \end{pmatrix} = \gamma', \tag{2.4}$$

where $\gamma' = \left(\begin{smallmatrix} a & bd \\ d^{-1}c & d' \end{smallmatrix}\right)$. Since $c \equiv 0 \pmod N$ and $d \mid \frac{N}{M}$, we deduce that $d^{-1}c \equiv 0 \pmod M$. So $\gamma' \in \Gamma_1(M)$ and therefore $\alpha_d \Gamma_1(N) \alpha_d^{-1} \subseteq \Gamma_1(M)$, or equivalently, $\Gamma_1(N) \subseteq \alpha_d^{-1}\Gamma_1(M)\alpha_d$. Writing $\gamma = \alpha_d^{-1}\gamma'\alpha_d$, Equation (2.4) implies that $j(\gamma', \alpha_d z) = j(\gamma, z)$. Then

$$\begin{aligned}
(f[\alpha_d]_k)(\gamma z) &= d^{k-1} f(d\gamma z) \\
&= d^{k-1} f(d\alpha_d^{-1}\gamma'\alpha_d z) \\
&= d^{k-1} f(\gamma'\alpha_d z) && d\alpha_d^{-1} z = z \\
&= j(\gamma', \alpha_d z) d^{k-1} f(\alpha_d z) \\
&= j(\gamma, z) d^{k-1} f(dz) && j(\gamma', \alpha_d z) = j(\gamma, z) \text{ and } \alpha_d z = dz \\
&= j(\gamma, z)(f[\alpha_d]_k)(z).
\end{aligned}$$

This verifies $f[\alpha_d]_k$ is modular and so $f[\alpha_d]_k \in \mathcal{S}_k(\Gamma_1(N))$. □

We can now define oldforms and newforms. For each divisor d of N, set

$$i_d : \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{d}\right)\right) \times \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{d}\right)\right) \to \mathcal{S}_k(\Gamma_1(N)) \qquad (f, g) \mapsto f + g[\alpha_d]_k.$$

This map is well-defined by Proposition 2.2.14. The subspace of **oldforms of level** N is

$$\mathcal{S}_k(\Gamma_1(N))^{\text{old}} = \sum_{p \mid N} \operatorname{Im}(i_p),$$

and the subspace of **newforms of level** N is

$$\mathcal{S}_k(\Gamma_1(N))^{\text{new}} = \left(\mathcal{S}_k(\Gamma_1(N))^{\text{old}}\right)^{\perp},$$

where the orthogonal complement is taken with respect to the Petersson inner product. The elements of such subspaces are called **oldforms** and **newforms** respectively. Both subspaces are invairant under the diamond and Hecke operators (see [DS05] for a proof):

Proposition 2.2.15. *The spaces $\mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ and $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ are invariant under the diamond operators $\langle m \rangle$ and Hecke operators T_m for all $m \geq 1$.*

As a corollary, we deduce that these subspaces admit orthogonal bases of eigenforms:

Corollary 2.2.2. *$\mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ and $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ admit orthonormal bases of eigenforms.*

Proof. This follows immediately from Theorem 2.2.5 and Proposition 2.2.15 $\square$

Something quite amazing happens for the subspace in $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$; the condition $(m, N) = 1$ for eigenforms in a base can be removed. Therefore the eigenforms are actually eigenfunctions for all of the diamond and Hecke operators. We require a preliminary result whose proof is quite invovled but it is not beyond the scope of this text (see [DS05] for a proof):

Lemma 2.2.6. *If $f \in \mathcal{S}_k(\Gamma_1(N))$ with Fourier coefficients $a_n(f)$ and such that $a_n(f) = 0$ whenever $(n, N) = 1$, then*

$$f = \sum_{p|N} p^{k-1} f_p[\alpha_p],$$

for some $f_p \in \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{p}\right)\right)$.

The important observation to make about Lemma 2.2.6 is that if $f \in \mathcal{S}_k(\Gamma_1(N))$ is such that its n-th Fourier coefficients vanish when n is relatively prime to the level, then f must be an oldform. With this lemma we can prove the main theorem about $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$. The introduction of some language will be useful for the statement and its proof. We say that f is a **primitive Hecke eigenform** if it is a Hecke normalized Hecke eigenform in $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$. We can now prove the main result about newforms:

Theorem 2.2.6. *Let $f \in \mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ be an eigenform. Then the following hold:*

(i) f is a Hecke eigenform (the condition $(m, N) = 1$ can be dropped).

(ii) If $\widetilde{f}$ satisfies the same conditions as f and has the same eigenvalues for the Hecke operators, then $\widetilde{f} = cf$ for some nonzero constant c.

Moreover, the primitive Hecke eigenforms in $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ form an orthogonal basis. Each such primitive Hecke eigenform f lies in an eigenspace $\mathcal{S}_k(N, \chi)$ for a Dirichlet character χ modulo N and $T_m f = \lambda_f(m) f$ for all $m \geq 1$.

Proof. Let f have Fourier coefficients $a_n(f)$. Suppose $f \in \mathcal{S}_k(\Gamma_1(N))$ is an eigenform. For such m there exists $\lambda_f(m), \mu_f(m) \in \mathbb{C}$ such that $T_m f = \lambda_f(m)$ and $\langle m \rangle f = \mu_f(m)$. If we set $\chi(n) = \mu_f(m)$, then χ is a Dirichlet character modulo N. This follows because multiplicativity of $\langle m \rangle$ implies the same for χ and χ is N-periodic since $\langle m \rangle$ is N-periodic ($\langle m \rangle$ is defined by m taken modulo N if $(m, N) = 1$ and $\langle m \rangle$ is the zero operator if $(m, N) > 1$ so that $\mu_f(m) = 0$). But then $\langle m \rangle f = \chi(m) f$ so that $f \in \mathcal{S}_k(N, \chi)$. As f is an eigenform, $a_m(f) = \lambda_f(m) a_1(f)$ when $(m, N) = 1$. So if $a_1(f) = 0$, Lemma 2.2.6 implies $f \in \mathcal{S}_k(\Gamma_1(N))^{\text{old}}$. We can now prove the statements.

(i) The claim is trivial if f is zero, so assume otherwise. Suppose $f \in \mathcal{S}_k(\Gamma_1(N))^{\text{new}}$. Then $f \notin \mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ and so by what we have shown $a_1(f) \neq 0$. Therefore we may algebraically normalize f so that $a_1(f) = 1$ and $a_m(f) = \lambda_f(m)$. Now set $g_m = T_m f - \lambda_f(m) f$ for any $m \geq 1$. By Proposition 2.2.15, $g_m \in \mathcal{S}_k(\Gamma_1(N))^{\text{new}}$. Moreover, g_m is an eigenform with respect to the same operators as f and, in particular, its first Fourier coefficient is zero. But then $g_m \in \mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ too and so $g_m = 0$ because $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ and $\mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ are orthogonal subspaces. This means $T_m f = \lambda_f(m) f$ for any $m \geq 1$.

(ii) Suppose $\widetilde{f}$ satisfies the same conditions as f with the same eigenvalues for the Hecke operators. By (i) we may assume f and $\widetilde{f}$ are Hecke eigenforms. After Hecke normalization, f and $\widetilde{f}$ have the same Fourier coefficients and so are identical. Therefore before Hecke normalization $f = c\widetilde{f}$ for some nonzero constant c.

Note that our intital remarks together with (i) show that each primitive Hecke eigenform f belongs to some eigenspace $\mathcal{S}_k(N, \chi)$ and $T_m f = \lambda_f(m) f$ for all $m \geq 1$. By Proposition 2.2.5, the primitive Hecke eigenforms are mutually orthogonal. So all we need to show is that they are linearly independently. So suppose, to the contrary, that

$$\sum_{1 \leq i \leq r} c_i f_i = 0, \tag{2.5}$$

for some primitive Hecke eigenforms f_i, constants c_i not all zero, and with r minimal. Note that $r \geq 2$ for else we do not have a nontrivial linear relation. Letting p be prime, applying the operator $T_p - \lambda_{f_1}(p)$ to Equation (2.5) gives

$$\sum_{2 \leq i \leq r} c_i(\lambda_{f_i}(p) - \lambda_{f_1}(p)) f_i = 0.$$

Since r was chosen to be minimal, we must have $\lambda_{f_i}(p) - \lambda_{f_1}(p)$ for all i. But p was arbitrary, so $f_i = f_1$ for all i by since eigenforms are determined by their Fourier coefficients at primes (see the definition of T_n and Proposition 2.2.9). Hence $r = 1$ which is a contradiction. $\square$

Statement (i) in Theorem 2.2.6 says that newforms are Hecke eigenforms. This shows that newforms are the generalization of eigenforms for the congruence subgroups $\Gamma_1(N)$. In particular, the Hecke relations for primitive Hecke eigenforms hold for all $n, m \geq 1$. Statement (ii) in Theorem 2.2.6 is referred to as the **multiplicity one theorem** for holomorphic forms. It can be interpreted as saying that a basis of newforms for $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ contains one element per "eigenvalue" where we mean a set of eigenvalues one for each Hecke operator T_m. With a stronger version of the multiplicity one theorem, often called the **strong multiplicity one theorem** for holomorphic forms, one can prove that the Fourier coefficeints of primitive Hecke eigenforms are real. This result takes a lot of work to show (see [DS05] for a note and appropriate references) and while we will not need it in the following, we state it for convenience:

Theorem 2.2.7. *The Fourier coefficeints of primitive Hecke eigenforms are real.*

We now require one last linear operator. Let

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

and note that $\det(W_N) = N$. We define the **Atkin–Lehner involution** $\omega_N : \mathcal{S}_k(\Gamma_1(N)) \to \mathcal{S}_k(\Gamma_1(N))$ to be the linear operator given by

$$(\omega_N f)(z) = (\sqrt{N}z)^{-k} f\left(W_N z\right) = (\sqrt{N}z)^{-k} f\left(-\frac{1}{Nz}\right).$$

To see that ω_N is well-defined, first note that holomorphy and the growth condition are obvious. For modularity, note that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$, we have

$$W_N\gamma = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ Na & Nb \end{pmatrix} = \begin{pmatrix} d & -N^{-1}c \\ -Nb & a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \gamma' W_N,$$

where $\gamma' = \left(\begin{smallmatrix} d & -N^{-1}c \\ -Nb & a \end{smallmatrix}\right) \in \Gamma_1(N)$. It follows that

$$\begin{aligned}
(\omega_N f)(\gamma z) &= (\sqrt{N}\gamma z)^{-k} f(W_N \gamma z) \\
&= (\sqrt{N}\gamma z)^{-k} f(\gamma' W_N z) \\
&= \left(\sqrt{N}\frac{az+b}{cz+d}\right)^{-k} \left(\frac{b}{z}+a\right)^{k} f\left(-\frac{1}{Nz}\right) \\
&= \left(\sqrt{N}\frac{az+b}{cz+d}\right)^{-k} \left(\frac{z}{az+b}\right)^{-k} f\left(-\frac{1}{Nz}\right) \\
&= (cz+d)^k(\sqrt{N}z)^{-k} f\left(-\frac{1}{Nz}\right) \\
&= j(\gamma,z)^k(\omega_N f)(z).
\end{aligned}$$

This verifies modularity for $\omega_N f$. Lastly, it is clear that $\omega_N f$ is a cusp form because f is. Altogether, this shows that $\omega_N f$ is well-defined. It is also an involution because $\omega_N(\omega_N f) = f$. The important fact we need is that the Hecke operators commute with the Atkin–Lehner involution and the Atkin–Lehner involution is self-adjoint with respect to the Petersson inner product (see [Miy89] or [DS05] for a proof):

Proposition 2.2.16. *For every $m \geq 1$, the Hecke operator T_m commutes with the Atkin-Lehner involution ω_N on $\mathcal{S}_k(\Gamma_1(N))$:*

$$T_m\omega_N = \omega_N T_m.$$

Moreover, the Atkin-Lehner involution is normal with respect to the Petersson inner product:

$$\omega_N^* = \omega_N.$$

By the spectral theorem, we can refine the eigenbasis of $\mathcal{S}_k(\Gamma_1(N))$:

Theorem 2.2.8. *$\mathcal{S}_k(\Gamma_1(N))$ admits an orthonormal basis of eigenforms that are also eigenfunctions for the Atkin-Lehner involution.*

Proof. This follows from the spectral theorem along with Theorem 2.2.5 and Proposition 2.2.16. □

Note that as ω_N is an involution, its only possible eigenvalues are ± 1. Therefore by Theorem 2.2.8 we may assume that any eigenform $f \in \mathcal{S}_k(\Gamma_1(N))$ in a basis satisfies

$$\omega_N f = \omega_N(f) f,$$

where $\omega_N(f) = \pm 1$.

The Ramanujan Conjecture

We will now discuss a famous conjecture about the size of the Fourier coefficients of primitive Hecke eigenforms. Historically the conjecture was born from conjectures made about the **modular discriminant** Δ given by

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2),$$

which is a weight 12 primitive Hecke eigenform on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ (see [DS05]). Therefore it is natural to begin our discussion here. It can be shown that the Fourier series of the modular discriminant is

$$\Delta(z) = \sum_{n \geq 1} \tau(n) e^{2\pi i n z},$$

(see [Apo76] for a proof) where the $\tau(n)$ are integers with $\tau(1) = 1$ and $\tau(2) = -24$. The function $\tau : \mathbb{N} \to \mathbb{Z}$ is called **Ramanujan's τ function**. Ramanujan himself studied this function in his 1916 paper (see [Ram16]), and computed $\tau(n)$ for $1 \le n \le 30$. From these computations he conjectured the following three properties τ should satisfy:

(i) If $(n,m) = 1$, then $\tau(nm) = \tau(n)\tau(m)$.

(ii) $\tau(p^n) = \tau(p^{n-1})\tau(p) - p^{11}\tau(p^{n-2})$ for all prime p.

(iii) $|\tau(p)| \le 2p^{\frac{11}{2}}$ for all prime p.

Note that (i) and (ii) are strikingly similar to the properties satisfied by the Hecke operators. In fact, (i) and (ii) are special cases of the properties of Hecke operators since Δ is a primitive Hecke eigenform on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ (note that $\Gamma_1(1) = \mathrm{PSL}_2(\mathbb{Z})$) with eigenvalue $\tau(n)$ for the n-th Hecke operator. This ends our commentary on properties (i) and (ii). Property (iii) turned out to be drastically more difficult to prove and is known as the **classical Ramanujan conjecture**. To state the Ramanujan conjecture, suppose $f \in \mathcal{S}_k(N,\chi)$ is a primitive Hecke eigenform with Fourier coefficients $\lambda_f(n)$. For each prime p, consider the polynomial

$$1 - \lambda_f(p)p^{-\frac{k-1}{2}}p^{-s} + \chi(p)p^{-2s},$$

and let $\alpha_1(p)$ and $\alpha_2(p)$ denote the roots. From this quadratic, we have

$$\alpha_1(p) + \alpha_2(p) = \lambda_f(p)p^{-\frac{k-1}{2}} \quad \text{and} \quad \alpha_1(p)\alpha_2(p) = \chi(p).$$

The more general **Ramanujan conjecture** is following statement:

Theorem 2.2.9 (Ramanujan conjecture). *Let $f \in \mathcal{S}_k(N,\chi)$ be a primitive Hecke eigenform. Denote the Fourier coefficients by $\lambda_f(n)$ and let $\alpha_1(p)$ and $\alpha_2(p)$ be the roots of $1 - \lambda_f(p)p^{-\frac{k-1}{2}}p^{-s} + \chi(p)p^{-2s}$. Then for all primes p,*

$$|\lambda_f(p)| \le 2p^{\frac{k-1}{2}}.$$

Moreover, if $p \nmid N$, then

$$|\alpha_1(p)| = |\alpha_2(p)| = 1.$$

In the 1970's Deligne proved the Ramanujan conjecture (see [Del71] and [Del74] for the full proof). The argument is significantly beyond the scope of this text, and in actuality follows from Deligne's work on the Weil conjectures (except in the case $k = 1$ which requires a modified argument). This requires understanding classical algebraic topology and ℓ-acid cohomology in addition to the basic analytic number theory. As such, the proof of the Ramanujan conjecture has been one of the biggest advances in analytic number theory in recent decades. Nevertheless, it is not difficult to get close to the Ramanujan conjeture without much effort. Indeed, suppose $f \in \mathcal{S}_k(N,\chi)$ be a primitive Hecke eigenform. Then $\left|f(z)\mathrm{Im}(z)^{\frac{k}{2}}\right|$ is $\Gamma_0(N)$-invariant because $\mathrm{Im}(\gamma z)^{\frac{k}{2}} = \frac{\mathrm{Im}(z)^{\frac{k}{2}}}{|j(\gamma,z)|^k}$ and $|\chi(\gamma)| = 1$. Moreover, it is bounded on $\mathcal{F}_{\Gamma_0(N)}$ because f is a cusp form. Then $\Gamma_0(N)$-invariance implies $\left|f(z)\mathrm{Im}(z)^{\frac{k}{2}}\right|$ is bounded on $\mathbb{H}$. From the definition of the Fourier coefficients, it follows that

$$\lambda_f(n)y^{\frac{k}{2}} = \int_0^1 f(x+iy)y^{\frac{k}{2}}e^{-2\pi in(x+iy)}\,dx \le e^{2\pi ny}\int_0^1 \left|f(x+iy)y^{\frac{k}{2}}\right|\,dx \ll e^{2\pi ny}.$$

Setting $y = \frac{1}{n}$ implies

$$\lambda_f(n) \ll n^{\frac{k}{2}}.$$

This bound is known as the **Hecke bound** for holomorphic forms.

2.3 The Theory of Maass Forms

Automorphic Forms

Let Γ be a congruence subgroup of level N and let χ be a Dirichlet character of conductor $q \mid N$. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is **automorphic** on $\Gamma\backslash\mathbb{H}$, with character χ, if $f(\gamma z) = \chi(\gamma)f(z)$ for all $\gamma \in \Gamma$. We call this condition the **automorphy condition**.

Remark 2.3.1. *Holomorphic forms are obtained by relaxing the automorphy condition (and imposing some additional mild conditions).*

The **Laplace operator** Δ on $\mathbb{H}$ is the linear operator on C^2-functions on $\mathbb{H}$ given by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

This operator behaves well with respect to the $\mathrm{PSL}_2(\mathbb{Z})$ action on $\mathbb{H}$ as the following proposition shows (see [Mot97] for a proof):

Proposition 2.3.1. *The Laplace operator Δ is $\mathrm{PSL}_2(\mathbb{Z})$-invariant. That is, if f is a C^2-function on $\mathbb{H}$ and $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, then*

$$\Delta(f(\gamma z)) = (\Delta f)(\gamma z).$$

Notice that Proposition 2.3.1 implies the Laplace operator Δ is Γ-invariant. Recall that $f : \mathbb{H} \to \mathbb{C}$ is automorphic on $\Gamma\backslash\mathbb{H}$ if it is Γ-invariant. If $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ is the minimal translation belonging to Γ, then any automorphic function is t-periodic. So if $\mathfrak{a}$ is a cusp of $\Gamma\backslash\mathbb{H}$, f has a **Fourier series at the $\mathfrak{a}$ cusp**:

$$f(\sigma_{\mathfrak{a}} z) = \sum_{n\in\mathbb{Z}} a_{\mathfrak{a}}(n, y)e^{\frac{2\pi inx}{t}}.$$

Note that the Fourier series above is independent of the scaling matrix because f is Γ_∞-invariant and the set of scaling matrices is stable under multiplication from $\Gamma_{\mathfrak{a}}$ on the right. Moreover, the sum is over all $n \in \mathbb{Z}$ since we do not require holomorphy at the cusps. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is an **automorphic form** on $\Gamma\backslash\mathbb{H}$, with character χ, if f is automorphic on $\Gamma\backslash\mathbb{H}$, with character χ, and is an eigenfunction for the Laplace operator Δ. If in addition $a_{\mathfrak{a}}(n, y) = 0$ for all cusps $\mathfrak{a}$, then we say that f is an **(automorphic) cusp form**. It also turns out that automorphic forms are smooth. This is because Δ is an elliptic operator and any eigenfunction of an elliptic operator is automatically real-analytic and hence smooth (see [Eva22] for a proof).

We will now compute the Fourier series of automorphic forms. If f is an automorphic form on $\Gamma\backslash\mathbb{H}$ with eigenvalue $\lambda = s(1-s)$, then the Fourier coefficients are mostly determined by Δ. To see this first note that the Fourier series at the $\mathfrak{a}$ cusp cusp converges absolutely, since f is smooth (see Appendix B.4), so we may differentiate termwise. Then the fact that f is an eigenfunction for Δ with eigenvalue λ gives the ODE

$$-y^2 a_{\mathfrak{a}}''(n, y) + 4\pi^2 n^2 y^2 a_{\mathfrak{a}}(n, y) = \lambda a_{\mathfrak{a}}(n, y),$$

upon differentiating termwise. In the ODE the $'$ indicates differentiation with respect to y. If $n \neq 0$, this is a modified Bessel equation. To see this, first we put the ODE in homogeneous form

$$y^2 a_{\mathfrak{a}}''(n, y) - (4\pi^2 n^2 y^2 - \lambda)a_{\mathfrak{a}}(n, y) = 0.$$

Make the change of variables $y \to \frac{y}{2\pi n}$ with $a_{\mathfrak{a}}(n,y) \to a_{\mathfrak{a}}(n, 2\pi n y)$ to get

$$y^2 a_{\mathfrak{a}}''(n,y) - (y^2 - \lambda)a_{\mathfrak{a}}(n,y) = 0.$$

Again, change variables $a_{\mathfrak{a}}(n,y) \to \sqrt{y}a_{\mathfrak{a}}(n,y)$ and divide by $\sqrt{y}$ to obtain

$$y^2 a_{\mathfrak{a}}''(n,y) + y a_{\mathfrak{a}}'(n,y) - \left(y^2 - \left(\lambda - \frac{1}{4}\right)\right) a_{\mathfrak{a}}(n,y) = 0.$$

Upon setting $\nu = \sqrt{\lambda - \frac{1}{4}}$, the above equation becomes

$$y^2 a_{\mathfrak{a}}''(n,y) + y a_{\mathfrak{a}}'(n,y) - (y^2 + (i\nu)^2) a_{\mathfrak{a}}(n,y) = 0,$$

which is a modified Bessel equation. Lastly, since

$$i\nu = \sqrt{\frac{1}{4} - \lambda} = \sqrt{\frac{1}{4} - s(1-s)} = \sqrt{s^2 - s + \frac{1}{4}} = \sqrt{\left(s - \frac{1}{2}\right)^2} = s - \frac{1}{2},$$

the general solution takes the form

$$a_{\mathfrak{a}}(n,y) = a_{\mathfrak{a}}(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y) + b_{\mathfrak{a}}(n)I_{s-\frac{1}{2}}(2\pi|n|y),$$

for some coefficients $a_{\mathfrak{a}}(n)$ and $b_{\mathfrak{a}}(n)$ possibly depending upon s (see Appendix B.7). But $I_{s-\frac{1}{2}}(2\pi|n|y)$ grows exponentially in y (see Appendix B.7) and since f has moderate growth at the cusps we must have $b_{\mathfrak{a}}(n) = 0$ for all $n \neq 0$. If $n = 0$, then the differential equation is a second order linear ODE which in homogeneous form is

$$y^2 a_{\mathfrak{a}}''(0,y) + \lambda a_{\mathfrak{a}}(0,y) = 0.$$

This is a Cauchy-Euler equation, and since s and $1-s$ are the two roots of $z^2 - z + \lambda$, the general solution is

$$a_{\mathfrak{a}}(0,y) = a_{\mathfrak{a}}(0)y^s + b_{\mathfrak{a}}(0)y^{1-s},$$

for some coefficients $a_{\mathfrak{a}}(0)$ and $b_{\mathfrak{a}}(0)$ possibly depending upon s. Altogether, the Fourier series at $\mathfrak{a}$ is of the form

$$f(\sigma_{\mathfrak{a}}z) = a_{\mathfrak{a}}(0)y^s + b_{\mathfrak{a}}(0)y^{1-s} + \sum_{n\neq 0} a_{\mathfrak{a}}(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}.$$

The coefficients $a_{\mathfrak{a}}(n)$, $a_{\mathfrak{a}}(0)$, and $b_{\mathfrak{a}}(0)$ are the only part of the Fourier series that actually depend on the implicit congruence subgroup Γ. We collect our result as a theorem:

Theorem 2.3.1. *Let $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ be the minimal translation belonging to Γ, and let f be an automorphic form on $\Gamma\backslash\mathbb{H}$ with eigenvalue $\lambda = s(1-s)$. Then the Fourier series of f at the $\mathfrak{a}$ cusp is of the form*

$$f(\sigma_{\mathfrak{a}}z) = a_{\mathfrak{a}}(0)y^s + b_{\mathfrak{a}}(0)y^{1-s} + \sum_{n\neq 0} a_{\mathfrak{a}}(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{\frac{2\pi i n x}{t}}.$$

We can simplify this Fourier series in some cases. We say f is **even** if $f(-\overline{z}) = f(z)$ and is **odd** if $f(-\overline{z}) = -f(z)$. Since $-\overline{z} = -x + iy$, this means that f is even or odd with respect to $\mathrm{Re}(z) = x$. Note that if f is odd it must be a cusp form. As $e^{2\pi i n x} = \cos(nx) + i\sin(nx)$, if f has eigenvalue $\lambda = s(1-s)$ and $\sigma_{\mathfrak{a}}$ is a scaling matrix for the $\mathfrak{a}$ cusp, then the Fourier series of f at the $\mathfrak{a}$ cusp is

$$f(\sigma_{\mathfrak{a}}z) = a_{\mathfrak{a}}(0)y^s + b_{\mathfrak{a}}(0)y^{1-s} + \sum_{n\neq 0} a_{\mathfrak{a}}(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)\cos\left(\frac{2\pi n x}{t}\right),$$

if f is even, and

$$f(z) = \sum_{n \neq 0} a_{\mathfrak{a}}(n) i \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y) \sin\left(\frac{2\pi nx}{t}\right),$$

if f is odd. Therefore, in even case $a_{\mathfrak{a}}(-n) = a_{\mathfrak{a}}(n)$ for all $n \geq 1$ and in the odd case $-a_{\mathfrak{a}}(-n) = a_{\mathfrak{a}}(n)$ for all $n \geq 1$. In either case, f admits a Fourier series of the form

$$f(\sigma_{\mathfrak{a}} z) = a_{\mathfrak{a}}(0)y^s + b_{\mathfrak{a}}(0)y^{1-s} + \sum_{n \geq 1} a_{\mathfrak{a}}(n)\sqrt{y} K_{s-\frac{1}{2}}(2\pi ny) \,\mathrm{SC}\left(\frac{2\pi nx}{t}\right),$$

for some different Fourier coefficeints $a_{\mathfrak{a}}(n)$, $a_{\mathfrak{a}}(0)$, and $b_{\mathfrak{a}}(0)$ and where $\mathrm{SC}(x) = \cos(x)$ if f is even and $\mathrm{SC}(x) = \sin(x)$ if f is odd. If f is odd, or more generally a cusp form, then necessarily $a_{\mathfrak{a}}(0) = b_{\mathfrak{a}}(0) = 0$. From now on, if we are discussing an even or odd automorphic form, we will always mean this Fourier series. For any cusp form, we can derive an important growth condition. Indeed, Theorem 2.3.1 implies that the Foruier series of f at the $\mathfrak{a}$ cusp is given by

$$f(\sigma_{\mathfrak{a}} z) = \sum_{n \neq 0} a_{\mathfrak{a}}(n)\sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y) e^{\frac{2\pi inx}{t}}.$$

Now $K_{s-\frac{1}{2}}(2\pi|n|y) = o_s(e^{-2\pi|n|y})$ (see Appendix B.7), so that

$$f(\sigma_{\mathfrak{a}} z) = o_s\left(\sqrt{y}\sum_{n \neq 0} e^{-2\pi|n|y}\right) = o_s\left(\sqrt{y}\sum_{n \in \mathbb{Z}} e^{-2\pi|n|y}\right) = o_s\left(\frac{2\sqrt{y}}{1-e^{-2\pi y}}\right) = o_s(e^{2\pi y}). \tag{2.6}$$

Equation (2.6) implies that cusp forms are bounded on $\mathbb{H}$. We will make more use of this estimate soon.

Maass Forms

Maass forms are essentially Γ-invariant analogues to holomorphic forms. They are also eigenfunctions for the Laplace operator. Let Γ be a congruence subgroup of level N and let χ be a Dirichlet character of conductor $q \mid N$. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is a **Maass form** on $\Gamma\backslash\mathbb{H}$ with eigenvalue λ and **character** (or **nebentypus** or **twist**) χ if the following properties are satisfied:

(i) f is smooth on $\mathbb{H}$.

(ii) f is automorphic on $\Gamma\backslash\mathbb{H}$ with character χ.

(iii) f is an eigenfunction for Δ with eigenvalue λ.

(iv) $f(\alpha z) = o(e^{2\pi \mathrm{Im}(z)})$ as $\mathrm{Im}(z) \to \infty$ for all $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$.

As we have already mentioned in the case for automorphic forms, property (i) is implied by property (iii). Property (iv) is called the **growth condition** for f and we say f is of **moderate growth at the cusps**. In particular, f is bounded on $\mathbb{H}$. Now let $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ be the smallest translation belonging to Γ so that $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \rangle$. By automorphy, we have

$$f(z+t) = f\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} z\right) = f(z),$$

so that f is t-periodic.

Remark 2.3.2. *If f is t-periodic, then we may think of f as a function on $\Gamma_\infty\backslash\mathbb{H}$. Taking representatives, $\Gamma_\infty\backslash\mathbb{H}$ is a strip in $\mathbb{H}$ with bounded real part so that f is a function on a domain with boudned real part.*

Then exactly as for holomorphic forms, if $\mathfrak{a}$ is a cusp of $\Gamma\backslash\mathbb{H}$ and $\sigma_\mathfrak{a}$ is a scaling matrix for $\mathfrak{a}$, f has a **Fourier series at the $\mathfrak{a}$ cusp**:

$$f(\sigma_\mathfrak{a} z) = \sum_{n\in\mathbb{Z}} a_\mathfrak{a}(n,y)e^{\frac{2\pi inx}{t}}.$$

As usual, the Fourier series is independent of the scaling matrix because f is Γ_∞-invariant and the set of scaling matrices is stable under multiplication from $\Gamma_\mathfrak{a}$ on the right. The sum is still over all $n \in \mathbb{Z}$, as for automorphic forms, since the growth condition is weaker than holomorphy at the cusps. For ease of communication, if it is clear what cusp we are working at we will mention the Fourier series and its coefficients without regard to the cusp. As the Fourier series is independent of the scaling matrix, we only need to check the growth condition on a set of scaling matrices for the cusps. Moreover, the Fourier coefficients are given by

$$a_\mathfrak{a}(n,y) = \int_0^t f(\sigma_\mathfrak{a}(x+iy))e^{-\frac{2\pi inx}{t}}\,dx.$$

Also, we say that f is **even** if $f(-\overline{z}) = f(z)$ and is **odd** if $f(-\overline{z}) = -f(z)$. We say f is a **(Maass) cusp form** if in addition,

(v) For all cusps $\mathfrak{a}$,

$$\int_0^t f(\sigma_\mathfrak{a}(x+iy))\,dx = 0.$$

Note that the integral in condition (v) is exactly the constant coefficient of the Fourier series of f at the $\mathfrak{a}$ cusp. So f is a cusp form precisely if $a_\mathfrak{a}(0,y) = 0$ for every Fourier series of f at every cusp $\mathfrak{a}$. As we have already mentioned, automorphic forms are smooth since Δ is an elliptic operator. This fact with Equation (2.6) implies that Maass forms, and automorphic forms, and their corresponding cusp forms, are equivalent. Therefore the Fourier series of Maass forms are exactly the same as the Fourier series of automorphic forms. Lastly, from now on we will always assume that our congruence subgroups are reduced at infinity so that $t = 1$.

Eisenstein Series & Poincaré Series with Test Functions

We will be interested in studying two classes of automorphic forms. The first is a class of Maass forms naturally defined on $\Gamma\backslash\mathbb{H}$. These Maass forms are actually functions defined on a large space $\mathbb{H}\times\Omega$ for some domain Ω and hence are functions of two variables say z and s. These functions are Maass forms for every fixed $s \in \Omega$. For every cusp $\mathfrak{a}$ of $\Gamma\backslash\mathbb{H}$, there is an associated Maass form. So let $\sigma_\mathfrak{a}$ be a scaling matrix for the cusp $\mathfrak{a}$. The Maass form $E_\mathfrak{a}(z,s)$ we are interested in is called the **(real-analytic) Eisenstein series** on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp and is defined as

$$E_\mathfrak{a}(z,s) = \sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma} \mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)^s.$$

To see that $E_\mathfrak{a}(z,s)$ is independent of the scaling matrix $\sigma_\mathfrak{a}$, suppose $\sigma'_\mathfrak{a}$ is another scaling matrix for the $\mathfrak{a}$ cusp. Then $\sigma'_\mathfrak{a} = \eta_\mathfrak{a}\sigma_\mathfrak{a}\eta_\infty$ for some $\eta_\mathfrak{a} \in \Gamma_\mathfrak{a}$ and $\eta_\infty \in \Gamma_\infty$. But then $(\sigma'_\mathfrak{a})^{-1} = \eta_\infty^{-1}\sigma_\mathfrak{a}^{-1}\eta_\mathfrak{a}^{-1}$ and as $\eta_\mathfrak{a}^{-1}\gamma$ is in the same equivalence class as γ and the action of η_∞^{-1} does not affect the imaginary part (as it acts by translation), we conclude $\mathrm{Im}((\sigma'_\mathfrak{a})^{-1}\gamma z) = \mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)$. Hence $E_\mathfrak{a}(z,s)$ is independent of the scaling matrix used. We now show that this series is locally absolutely uniformly convergent for $z \in \mathbb{H}$ and $\mathrm{Re}(s) > 1$. To see this, applying the Bruhat decomposition to $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\backslash\Gamma = \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma$ yields

$$E_{\mathfrak{a}}(z,s) \ll \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}}.$$

The latter series is locally absolutely uniformly convergent for $\mathrm{Re}(s) > 1$ and $z \in \mathbb{H}$ by Proposition B.8.1. Therefore $E_{\mathfrak{a}}(z,s)$ does too. To see that it is automorphic on $\Gamma\backslash\mathbb{H}$, let $\gamma \in \Gamma$. Then

$$E_{\mathfrak{a}}(\gamma z,s) = \sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s = \sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma' z)^s = E_{\mathfrak{a}}(z,s),$$

where the second equality follows because $\gamma' \to \gamma'\gamma^{-1}$ is a bijection on Γ. Now we show that it is an eigenfunction for Δ. Observe

$$\Delta(y^s) = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(y^s) = s(1-s)y^s.$$

Therefore $\mathrm{Im}(z)^s = y^s$ is an eigenfunction for Δ with eigenvalue $s(1-s)$. Since Δ is Γ-invariant,

$$\Delta(\mathrm{Im}(\gamma z)^s) = (\Delta\mathrm{Im}(\cdot)^s)(\gamma z) = s(1-s)\mathrm{Im}(\gamma z)^s,$$

so that $\mathrm{Im}(\gamma z)^s$ is also an eigenfunction for Δ with eigenvalue $s(1-s)$. It follows immediately that

$$\Delta(E_{\mathfrak{a}}(z,s)) = s(1-s)E_{\mathfrak{a}}(z,s),$$

which shows $E_{\mathfrak{a}}(z,s)$ is also an eigenfunction for Δ with eigenvalue $s(1-s)$. We now verify the growth condition for $E_{\mathfrak{a}}(z,s)$. Let $\sigma_{\mathfrak{b}}$ be a scaling matrix for the cusp $\mathfrak{b}$ and write $\sigma_{\mathfrak{b}} = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. Then

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \ll \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{\mathrm{Im}(\sigma_{\mathfrak{b}}z)^s}{|c\sigma_{\mathfrak{b}}z+d|^{2s}} = \frac{\mathrm{Im}(z)^s}{|c'z+d'|^{2s}} \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{|c\sigma_{\mathfrak{b}}z+d|^{2s}}.$$

Now decompose this last sum as

$$\sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{|c\sigma_{\mathfrak{b}}z+d|^{2s}} = \sum_{d\neq 0}\frac{1}{d^{2s}} + \sum_{c\neq 0}\sum_{d\in\mathbb{Z}} \frac{1}{|c\sigma_{\mathfrak{b}}z+d|^{2s}} = 2\sum_{d\geq 1}\frac{1}{d^{2s}} + 2\sum_{c\geq 1}\sum_{d\in\mathbb{Z}} \frac{1}{|c\sigma_{\mathfrak{b}}z+d|^{2s}}.$$

Notice that $\sum_{c\geq 1}\sum_{d\in\mathbb{Z}}\frac{1}{d^{2s}}$ converges provided $\mathrm{Re}(s) > 1$. Moreover, the exact same argument as for holomorphic form Eisenstein series shows that $\sum_{c\geq 1}\sum_{d\in\mathbb{Z}}\frac{1}{|c\sigma_{\mathfrak{b}}z+d|^{2s}}$ converges in this region as well. Now for all $\mathrm{Im}(z) \geq 1$ and $\mathrm{Re}(s) > 1$,

$$\frac{\mathrm{Im}(z)^s}{|c'z+d'|^{2s}} \ll \frac{\mathrm{Im}(z)^s}{|c'\mathrm{Im}(z)|^{2s}} \ll \mathrm{Im}(z)^s.$$

So altogether,

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \ll \mathrm{Im}(z)^s = o(e^{2\pi\mathrm{Im}(z)}),$$

provided $\mathrm{Im}(z) \geq 1$ and $\mathrm{Re}(s) > 1$. This verifies $E_{\mathfrak{a}}(z,s)$ is of moderate growth at the cusps. We collect all of this work as a theorem:

Theorem 2.3.2. *For* $\mathrm{Re}(s) > 1$*, the Eisenstein series*

$$E_{\mathfrak{a}}(z,s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \mathrm{Im}(\gamma z)^s,$$

is a Maass form on $\Gamma\backslash\mathbb{H}$ *with eigenvalue* $\lambda = s(1-s)$*.*

The second class of automorphic forms on $\Gamma\backslash\mathbb{H}$ are the Poincaré series with test functions. To build these series we first need the notion of a test function. If $\psi : \mathbb{R}^+ \to \mathbb{C}$ is a smooth function such that

$$\psi(y) \ll y^{1+\varepsilon}$$

for some $\varepsilon > 0$ as $y \to 0$. We call ψ a **test function**. We will consider a Poincaré series modified by ψ. Let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the $\mathfrak{a}$ cusp. Then the **(automorphic) Poincaré series** $P_{\mathfrak{a},m}(z,\psi)$ at the $\mathfrak{a}$ cusp with test function ψ is defined by

$$P_{\mathfrak{a},m}(z,\psi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)) e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z}.$$

We first show $P_{\mathfrak{a},m}(z,\psi)$ is well-defined. To do this we need to check that $e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z}$ is independent of the representative γ used. If γ' represents the same element as γ in $\Gamma_{\mathfrak{a}}\backslash\Gamma$, then they differ on the left by an element of $\eta_{\mathfrak{a}} \in \Gamma_{\mathfrak{a}}$. So suppose $\gamma' = \eta_{\mathfrak{a}}\gamma$. As $\Gamma_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}$, $\eta_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\eta_\infty\sigma_{\mathfrak{a}}^{-1}$ for some $\eta_\infty \in \Gamma_\infty$ say with $\eta_\infty = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_\infty$. Then

$$e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma' z} = e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\eta_{\mathfrak{a}}\gamma z} = e^{2\pi i m \eta_\infty \sigma_{\mathfrak{a}}^{-1}\gamma z} = e^{2\pi i m (\sigma_{\mathfrak{a}}^{-1}\gamma z + n)} = e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z} e^{2\pi i m n z} = e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z},$$

and hence $P_{\mathfrak{a},m}(z,\psi)$ is well-defined. We now show that $P_{\mathfrak{a},m}(z,\psi)$ is also independent of the scaling matrix. To see this, first note that $\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)$ is independent of the scaling matrix in exactly the same way as we showed for the Eisenstein series. It now suffices to show that $e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z}$ is independent of the scaling matrix too. So suppose $\sigma_{\mathfrak{a}}'$ is another scaling matrix for the $\mathfrak{a}$ cusp. Then $\sigma_{\mathfrak{a}}' = \eta_{\mathfrak{a}}\sigma_{\mathfrak{a}}\eta_\infty$ for some $\eta_{\mathfrak{a}} \in \Gamma_{\mathfrak{a}}$ and $\eta_\infty \in \Gamma_\infty$ say with $\eta_\infty = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_\infty$. But then $(\sigma_{\mathfrak{a}}')^{-1} = \eta_\infty^{-1}\sigma_{\mathfrak{a}}^{-1}\eta_{\mathfrak{a}}^{-1}$ and as $\eta_\infty^{-1} = \left(\begin{smallmatrix} 1 & -n \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$e^{2\pi i m (\sigma_{\mathfrak{a}}')^{-1}\gamma z} = e^{2\pi i m \eta_\infty^{-1}\sigma_{\mathfrak{a}}^{-1}\eta_{\mathfrak{a}}^{-1}\gamma z} = e^{2\pi i m (\sigma_{\mathfrak{a}}^{-1}\eta_{\mathfrak{a}}^{-1}\gamma z - n)} = e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\eta_{\mathfrak{a}}^{-1}\gamma z} e^{-2\pi i m n} = e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\eta_{\mathfrak{a}}^{-1}\gamma z} = e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z},$$

where the last equality follows since $\eta_{\mathfrak{a}}^{-1}\gamma$ is in the same equivalence class as γ and that $e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z}$ is independent of the representative γ used. It follows that $P_{\mathfrak{a},m}(z,\psi)$ is independent of the scaling matrix. Moreover, $P_{\mathfrak{a},m}(z,\psi)$ is locally absolutely uniformly convergent for $z \in \mathbb{H}$, but to see this we first require a technical lemma:

Lemma 2.3.1. *For any compact subset* K *of* $\mathbb{H}$*, there are finitely many pairs* $(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}$*, with* $c \neq 0$*, for which*

$$\frac{\mathrm{Im}(z)}{|cz+d|^2} > 1,$$

for all $z \in K$*.*

Proof. Let $\beta = \sup_{z\in K} |z|$. As $|cz+d| \geq |cz| > 0$ and $\mathrm{Im}(z) < |z|$, we have

$$\frac{\mathrm{Im}(z)}{|cz+d|^2} \leq \frac{1}{|c|^2|z|} \leq \frac{1}{|c|^2\beta}.$$

So if $\frac{\mathrm{Im}(z)}{|cz+d|^2} > 1$, then $\frac{1}{|c|^2\beta} > 1$ which is to say $|c| < \frac{1}{\sqrt{\beta}}$ and therefore $|c|$ is bounded. On the other hand, $|cz+d| \geq |d| \geq 0$. Excluding the finitely many terms $(c,0)$, we may assume $|d| > 0$. In this case, similarly

$$\frac{\mathrm{Im}(z)}{|cz+d|^2} \leq \frac{|z|}{|d|^2} \leq \frac{\beta}{|d|^2}.$$

So if $\frac{\mathrm{Im}(z)}{|cz+d|^2} > 1$, then $\frac{\beta}{|d|^2} > 1$ which is to say $|d| < \sqrt{\beta}$. So $|d|$ is also bounded. Since both $|c|$ and $|d|$ are bounded, the claim follows. $\square$

Now we are ready to show that $P_{\mathfrak{a},m}(z,\psi)$ is locally absolutely uniformly convergent. Let K be a compact subset of $\mathbb{H}$. Then it suffices to show $P_{\mathfrak{a},m}(z,\psi)$ is uniformly convergent on K. Applying the Bruhat decomposition to $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\backslash\Gamma = \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma$ gives

$$P_{\mathfrak{a},m}(z,\psi) \ll \psi(\mathrm{Im}(z)) + \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right).$$

It now further suffices to show that the series above is uniformly convergent on K. By Lemma 2.3.1 there are all but finitely many terms in the sum with $\psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right) \ll \left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)^{1+\varepsilon}$. But the finitely many other terms are all uniformly bounded on K because ψ is continuous (as it is smooth). Therefore

$$\sum_{\substack{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}\\ d \pmod{c}}} \psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right) \ll \sum_{\substack{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}\\ d \pmod{c}}} \left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)^{1+\varepsilon} \ll \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)^{1+\varepsilon},$$

and this last series is locally absolutely uniformly convergent for $z \in \mathbb{H}$ by Proposition B.8.1. It follows that $P_{\mathfrak{a},m}(z,\psi)$ does too. We now show that $P_{\mathfrak{a},m}(z,\psi)$ is automorphic on $\Gamma\backslash\mathbb{H}$. So letting $\gamma \in \Gamma$, we have

$$P_{\mathfrak{a},m}(\gamma z,\psi) = \sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z))e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} = \sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma' z))e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma' z} = P_{\mathfrak{a},m}(z,\psi),$$

where the second equality follows because $\gamma' \to \gamma'\gamma^{-1}$ is a bijection on Γ. Therefore $P_{\mathfrak{a},m}(z,\psi)$ is automorphic on $\Gamma\backslash\mathbb{H}$. We collect this work as a theorem:

Theorem 2.3.3. *Let ψ be a test function. The Poincaré series*

$$P_{\mathfrak{a},m}(z,\psi) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z},$$

is an automorphic form on $\Gamma\backslash\mathbb{H}$.

L^2-integrable Automorphic Functions

We will describe the space of automorphic functions and appropriate subspaces of interest. Let $\mathcal{A}(\Gamma)$ denote the space of all automorphic functions on $\Gamma\backslash\mathbb{H}$. We will usually restrict our interest to the subspace

$$\mathcal{L}(\Gamma) = \{f \in \mathcal{A}(\Gamma) : ||f|| < \infty\},$$

of L^2-integrable automorphic functions over $\mathcal{F}_\Gamma$, where the norm is given by

$$||f|| = \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |f(z)|^2\,d\mu\right)^{\frac{1}{2}}.$$

Since f is automorphic, the integral is independent of the choice of fundamental domain. Since this is an L^2-space, $\mathcal{L}(\Gamma)$ is an induced inner product space (because the parallelogram law is satisfied). In particular, for any $f, g \in \mathcal{L}(\Gamma)$ we define their **Petersson inner product** to be

$$\langle f, g \rangle_\Gamma = \frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}\, d\mu.$$

If the congruence subgroup is clear from context we will supress the dependence upon Γ. The integral is absolutely bounded by the Cauchy–Schwarz inequality and that $f, g \in \mathcal{L}(\Gamma)$. As f and g are automorphic, the integral is independent of the choice of fundamental domain. In fact, we can do better:

Theorem 2.3.4. *$\mathcal{L}(\Gamma)$ is a Hilbert space with respect to the Petersson inner product.*

Proof. To show that the Petersson inner product is a Hermitian inner product on $\mathcal{L}(\Gamma)$, just mimic the corresponding part of proof of Proposition 2.2.2 with $k = 0$. We now show that $\mathcal{L}(\Gamma)$ is complete. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{L}(\Gamma)$. Then $||f_n - f_m|| \to 0$ as $n, m \to \infty$. But

$$||f_n - f_m|| = \left(\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} |f_n(z) - f_m(z)|^2\, d\mu \right)^{\frac{1}{2}},$$

and this integral tends to zero if and only if $|f_n(z) - f_m(z)| \to 0$ as $n, m \to \infty$. Therefore $\lim_{n\to\infty} f_n(z)$ exists and we define the limiting function f by $f(z) = \lim_{n\to\infty} f_n(z)$. We claim that f is automorphic. Indeed, as the f_n are automorphic, we have

$$f(\gamma z) = \lim_{n\to\infty} f_n(\gamma z) = \lim_{n\to\infty} f_n(z) = f(z),$$

for any $\gamma \in \Gamma$. Also, $||f|| < \infty$. To see this, since $(f_n)_{n\geq 1}$ is Cauchy we know $(||f_n||)_{n\geq 1}$ converges. In particular, $\lim_{n\to\infty} ||f_n|| < \infty$. But

$$\lim_{n\to\infty} ||f_n|| = \lim_{n\to\infty} \left(\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} |f_n(z)|^2\, d\mu \right)^{\frac{1}{2}} = \left(\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} \left| \lim_{n\to\infty} f_n(z) \right|^2 d\mu \right)^{\frac{1}{2}} = \left(\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} |f(z)|^2\, d\mu \right)^{\frac{1}{2}} = ||f||,$$

where the second equality holds by the dominated convergence theorem. Hence $||f|| < \infty$ as desired and so $f \in \mathcal{L}(\Gamma)$. We now show that $f_n \to f$ in the L^2-norm. Indeed,

$$||f(z) - f_n(z)|| = \left(\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} |f(z) - f_n(z)|^2\, d\mu \right)^{\frac{1}{2}},$$

and it follows that $||f(z) - f_n(z)|| \to 0$ as $n \to \infty$ so that the Cauchy sequence $(f_n)_{n\geq 1}$ converges. $\square$

Now consider the subspaces

$$\mathcal{D}(\Gamma) = \{f \in \mathcal{A}(\Gamma) : f \text{ and } \Delta f \text{ are smooth and bounded}\},$$
$$\mathcal{B}(\Gamma) = \{f \in \mathcal{A}(\Gamma) : f \text{ is smooth and bounded}\}.$$

Since boundedness on $\mathbb{H}$ implies square-integrability over $\mathcal{F}_\Gamma$, we have the following chain of inclusions:

$$\mathcal{D}(\Gamma) \subseteq \mathcal{B}(\Gamma) \subseteq \mathcal{L}(\Gamma) \subseteq \mathcal{A}(\Gamma).$$

Moreover, $\mathcal{D}(\Gamma)$ is almost all of $\mathcal{L}(\Gamma)$ as the following proposition shows:

Proposition 2.3.2. *$\mathcal{D}(\Gamma)$ is dense in $\mathcal{L}(\Gamma)$.*

Proof. Note that $\mathcal{D}(\Gamma)$ is an algebra of functions that vanish at infinity. We will show that $\mathcal{D}(\Gamma)$ is nowhere vanishing, seperates points, and is self-adjoint. For nowhere vanishing, observe that for each $z \in \mathbb{H}$, $\mathcal{D}(\Gamma)$ contains a smooth bump function φ_z defined on some neighborhood U_z of z. We now show $\mathcal{D}(\Gamma)$ also seperates points. To see this consider two distinct points $z, w \in \mathbb{H}$. Let $U_{z,w}$ be a small neighborhood of z not containing w. Then $\varphi_z\mid_{U_{z,w}}$ belongs to $\mathcal{D}(\Gamma)$ with $\varphi_z\mid_{U_{z,w}}(z) = 1$ and $\varphi_z\mid_{U_{z,w}}(w) = 0$. To see why $\mathcal{D}(\Gamma)$ is self-adjoint, recall that complex conjugation is smooth and commutes with partial derivatives so that if f belongs to $\mathcal{D}(\Gamma)$ then so does $\overline{f}$. Therefore the Stone–Weierstrass theorem for complex functions defined on locally compact Hausdorff spaces (as $\mathbb{H}$ is a locally compact Hausdorff space) implies that $\mathcal{D}(\Gamma)$ is dense in $C_0(\mathbb{H})$ with the supremum norm. Note that $\mathcal{L}(\Gamma) \subseteq C_0(\mathbb{H})$ on the level of sets. Now we show $\mathcal{D}(\Gamma)$ is dense in $\mathcal{L}(\Gamma)$. Let $f \in \mathcal{L}(\Gamma)$. By what we have just show, there exists a sequence $(f_n)_{n\geq 1}$ in $\mathcal{D}(\Gamma)$ converging to f in the supremum norm. But

$$||f - f_n|| = \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |f(z) - f_n(z)|^2\, d\mu\right)^{\frac{1}{2}} \leq \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \sup_{z\in\mathcal{F}_\Gamma} |f(z) - f_n(z)|^2\, d\mu\right)^{\frac{1}{2}},$$

and the last expression tends to zero as $n \to \infty$ because $f_n \to f$ in the supremum norm. □

As $\mathcal{D}(\Gamma) \subseteq \mathcal{B}(\Gamma)$, Proposition 2.3.2 implies that $\mathcal{B}(\Gamma)$ is dense in $\mathcal{L}(\Gamma)$ too. It can be shown that the Laplace operator Δ is positive and symmetric on $\mathcal{D}(\Gamma)$ and hence admits a self-adjoint extension to $\mathcal{L}(\Gamma)$ (see [Iwa02] for a proof):

Theorem 2.3.5. *On $\mathcal{L}(\Gamma)$, the Laplace operator Δ is positive semi-definite and self-adjoint.*

If we suppose $f \in \mathcal{L}(\Gamma)$ is an eigenfunction for Δ with eigenvalue λ, then Theorem 2.3.5 implies λ is real and positive. Writing $\lambda = s(1-s)$, we see that s and $1-s$ are the roots of the polynomial $z^2 - z + \lambda$. As λ is real, these roots are either conjugate symmetric or real. In the former case, $s = 1 - \overline{s}$ so that if $s = \sigma + it$, we find

$$\sigma = 1 - \sigma \quad \text{and} \quad t = t.$$

Therefore $s = \frac{1}{2} + it$. In the later case, s is real and since λ is positive we must have $s \in (0,1)$. It follows that in either case, we may write $\lambda = \frac{1}{4} + t^2$ with either $t \in \mathbb{R}$ or $it \in [0, \frac{1}{2})$. We refer to t as the **spectral parameter** of f. Note that we can then write

$$\lambda = s(1-s) = \frac{1}{4} + t^2.$$

Unfortunately the Eisenstein series $E_\mathfrak{a}(z,s)$ are not L^2-integrable over $\mathcal{F}_\Gamma$ and so do not belong to $\mathcal{L}(\Gamma)$. To obtain integrable automorphic functions we will agument these Eisenstein series so that they have compact support. Let $\psi : \mathbb{R}^+ \to \mathbb{C}$ be a smooth compactly supported function and $\sigma_\mathfrak{a}$ be a scaling matrix for the $\mathfrak{a}$ cusp. We define the **(incomplete) Eisenstein series** $E_\mathfrak{a}(z,\psi)$ at the $\mathfrak{a}$ cusp with respect to ψ by

$$E_\mathfrak{a}(z,\psi) = \sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma} \psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)).$$

This series is locally absolutely uniformly convergent since only finitely many of the terms are nonzero. Indeed, $\sigma_\mathfrak{a}^{-1}\Gamma$ is a Fushian group because it is a subset of the modular group. So using $\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\backslash\Gamma = \Gamma_\infty\backslash\sigma_\mathfrak{a}^{-1}\Gamma$ we see that $\{\sigma_\mathfrak{a}^{-1}\gamma z : \gamma \in \Gamma_\mathfrak{a}\backslash\Gamma\}$ is discrete. Since $\mathrm{Im}(z)$ is an open map it takes discrete sets to discrete sets so that $\{\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z) : \gamma \in \Gamma_\mathfrak{a}\backslash\Gamma\}$ is also discerte. Now $\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))$ is nonzero if and only if $\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z) \in$

$\mathrm{Supp}(\psi)$ and $\{\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z) : \gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma\} \cap \mathrm{Supp}(\psi)$ is finite as it is a discrete subset of a compact set (since ψ has compact support). Hence finitely many of the terms are nonzero. Now $E_{\mathfrak{a}}(z,\psi)$ is also automorphic on $\Gamma\backslash\mathbb{H}$:

$$E_{\mathfrak{a}}(\gamma z,\psi) = \sum_{\gamma' \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)) = \sum_{\gamma' \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma' z)) = E_{\mathfrak{a}}(z,\psi),$$

where the second equality follows because $\gamma' \to \gamma'\gamma^{-1}$ is a bijection on Γ. Now the compact support of ψ implies that $E_{\mathfrak{a}}(z,\psi)$ is also compactly supported (since the function $\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))$ is continuous and $\mathbb{C}$ is Hausdorff) and hence bounded on $\mathbb{H}$. As a consequence, $E_{\mathfrak{a}}(z,\psi)$ is L^2-integrable and therefore belongs to $\mathcal{L}(\Gamma)$. This is the advantage of these Eisenstein series. We collect this work as a theorem:

Theorem 2.3.6. *For any smooth compactly supported function $\psi : \mathbb{R}^+ \to \mathbb{C}$, the Eisenstein series*

$$E_{\mathfrak{a}}(z,\psi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)),$$

is automorphic on $\Gamma\backslash\mathbb{H}$ and belongs to $\mathcal{L}(\Gamma)$.

Unfortunately, the Eisenstein series $E_{\mathfrak{a}}(z,\psi)$ fail to be Maass forms because they are not eigenfunctions for the Laplace operator. This is because compactly supported functions cannot be real-analytic (which as we have already mentioned is implied for any eigenfunction of the Laplace operator). However, there is an important interaction between incomplete Eisenstein series and Maass cusp forms. Let $\mathcal{E}(\Gamma)$ and $\mathcal{C}(\Gamma)$ denote the spaces generated by such forms respectively. Moreover, let $\mathcal{A}(\Gamma)$ be the space of all Maass forms. Note that $\mathcal{E}(\Gamma)$ and $\mathcal{C}(\Gamma)$ are are subspaces of $\mathcal{B}(\Gamma)$ but $\mathcal{A}(\Gamma)$ need not be. Let $\mathcal{E}_t(\Gamma)$, $\mathcal{C}_t(\Gamma)$, $\mathcal{A}_t(\Gamma)$, and $\mathcal{B}_t(\Gamma)$ denote the corresponding subspaces of such forms whose eigenvalue is $\lambda = \frac{1}{4} + t^2$. For completeness, let $\mathcal{C}_t(\Gamma,\chi)$, $\mathcal{A}_t(\Gamma,\chi)$ denote the subspaces of such forms with character χ. Moreover, if the character χ is the trivial character, we will surpress the dependence upon χ. As we will now show, $\mathcal{E}(\Gamma)$ and $\mathcal{C}(\Gamma)$ constitute all of $\mathcal{B}(\Gamma)$. Indeed, let $f \in \mathcal{B}(\Gamma)$ with $E_{\mathfrak{a}}(\cdot,\psi) \in \mathcal{E}(\Gamma)$. We compute their inner product:

$$\begin{aligned}
\langle f, E_{\mathfrak{a}}(\cdot,\psi)\rangle &= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{E_{\mathfrak{a}}(z,\psi)}\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))}\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \overline{\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))}\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(\gamma^{-1}\sigma_{\mathfrak{a}} z)\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \overline{\psi(\mathrm{Im}(z))}\,d\mu && z \to \gamma^{-1}\sigma_{\mathfrak{a}} z \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(\sigma_{\mathfrak{a}} z)\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \overline{\psi(\mathrm{Im}(z))}\,d\mu && \text{automorphy} \\
&= \int_{\Gamma_\infty\backslash\mathbb{H}} f(\sigma_{\mathfrak{a}} z)\overline{\psi(\mathrm{Im}(z))}\,d\mu && \text{unfolding} \\
&= \int_{\Gamma_\infty\backslash\mathbb{H}} f(\sigma_{\mathfrak{a}} z)\overline{\psi(\mathrm{Im}(z))}\,d\mu \\
&= \int_0^\infty \int_0^1 f(\sigma_{\mathfrak{a}}(x+iy))\overline{\psi(y)}\,\frac{dx\,dy}{y^2} \\
&= \int_0^\infty \left(\int_0^1 f(\sigma_{\mathfrak{a}}(x+iy))\,dx\right)\overline{\psi(y)}\,\frac{dy}{y^2}.
\end{aligned}$$

The inner integral is precisely the constant term in the Fourier series of f at the $\mathfrak{a}$ cusp. It follows that f is orthogonal to $\mathcal{E}(\Gamma)$ if and only if f is a cusp form. By what we have just shown,

$$\mathcal{B}(\Gamma) = \mathcal{E}(\Gamma) \oplus \mathcal{C}(\Gamma) \quad \text{and} \quad \mathcal{B}_t(\Gamma) = \mathcal{E}_t(\Gamma) \oplus \mathcal{C}_t(\Gamma),$$

and since $\mathcal{B}(\Gamma)$ is dense in $\mathcal{L}(\Gamma)$ we have

$$\mathcal{L}(\Gamma) = \overline{\mathcal{E}(\Gamma)} \oplus \overline{\mathcal{C}(\Gamma)},$$

where the closure is with respect to the topology induced by the L^2-norm.

Spectral Theory of the Laplace Operator

We are now ready to discuss the spectral theory of the Laplace operator Δ. What we want to do is to decompose $\mathcal{L}(\Gamma)$ into subspaces invariant under Δ such that on each subspace Δ has either pure point spectrum, absolutely continuous spectrum, or residual spectrum. Although the proof is beyond the scope of this text, the spectral resolution of the Laplace operator on $\mathcal{C}(\Gamma)$ is as follows (see [Iwa02] for a proof):

Theorem 2.3.7. *The Laplace operator Δ has pure point spectrum on $\mathcal{C}(\Gamma)$. The corresponding subspaces $\mathcal{C}_t(\Gamma)$ have finite dimension and are mutually orthogonal. Letting $\{u_j\}$ be any orthonormal basis of cusp forms for $\mathcal{C}(\Gamma)$, every $f \in \mathcal{C}(\Gamma)$ admits a series of the form*

$$f(z) = \sum_j \langle f, u_j \rangle u_j(z),$$

which is locally absolutely uniformly convergent if $f \in \mathcal{D}(\Gamma)$ and converges in the L^2-norm otherwise.

We will now discuss the spectrum of the Laplace operator on $\mathcal{E}(\Gamma)$. Essential is the meromorphic continuation of the Eisenstein series $E_{\mathfrak{a}}(z,s)$ (see [Iwa02] for a complete argument):

Theorem 2.3.8. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$. The Eisenstein series $E_{\mathfrak{a}}(z,s)$ admits meromorphic continuation to $\mathbb{C}$ in the s-plane, via a Fourier series at the $\mathfrak{b}$ cusp given by*

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) = \delta_{\mathfrak{a},\mathfrak{b}} y^s + \tau_{\mathfrak{a},\mathfrak{b}}(s) y^{1-s} + \sum_{n \neq 0} \tau_{\mathfrak{a},\mathfrak{b}}(n,s) \sqrt{y} K_{it}(2\pi |n| y) e^{2\pi i n x},$$

where $\tau_{\mathfrak{a},\mathfrak{b}}(s)$ and $\tau_{\mathfrak{a},\mathfrak{b}}(n,s)$ are meromorphic functions. The poles of $\tau_{\mathfrak{a},\mathfrak{b}}(s)$ are simple and belong to the segment $(\frac{1}{2}, 1]$. A pole of $\tau_{\mathfrak{a},\mathfrak{b}}(s)$ is also a pole of $\tau_{\mathfrak{a},\mathfrak{a}}(s)$, and the poles of $E_{\mathfrak{a}}(z,s)$ are among the poles of $\tau_{\mathfrak{a},\mathfrak{a}}(s)$. Also, the residues of the Eisenstein series at these poles are Maass forms in $\mathcal{E}(\Gamma)$.

To begin decomposing the space $\mathcal{E}(\Gamma)$, consider the subspace $C_0^\infty(\mathbb{R}^+)$ of $\mathcal{L}^2(\mathbb{R}^+)$ with the normalized standard inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^\infty f(r) \overline{g(r)} \, dr,$$

for any $f, g \in C_0^\infty(\mathbb{R}^+)$. For each cups $\mathfrak{a}$ of $\Gamma\backslash\mathbb{H}$ we associate the **Eisenstein transform** $E_{\mathfrak{a}} : C_0^\infty(\mathbb{R}^+) \to \mathcal{A}(\Gamma)$ defined by

$$(E_{\mathfrak{a}} f)(z) = \frac{1}{4\pi} \int_0^\infty f(r) E_{\mathfrak{a}}\left(z, \frac{1}{2} + ir\right) dr.$$

Clearly $E_{\mathfrak{a}} f$ is automorphic because $E_{\mathfrak{a}}(z,s)$ is. It is not too hard to show the following (see [Iwa02] for a proof):

Proposition 2.3.3. *If $f \in C_0^\infty(\mathbb{R}^+)$, then $E_\mathfrak{a} f$ is L^2-integrable over $\mathcal{F}_\Gamma$. That is, $E_\mathfrak{a} : C_0^\infty(\mathbb{R}^+) \to \mathcal{L}(\Gamma)$. Moreover,*

$$\langle E_\mathfrak{a} f, E_\mathfrak{b} g\rangle = \delta_{\mathfrak{a},\mathfrak{b}}\langle f, g\rangle,$$

for any $f, g \in C_0^\infty(\mathbb{R}^+)$ and any two cusps $\mathfrak{a}$ and $\mathfrak{b}$.

We let $\mathcal{E}_\mathfrak{a}(\Gamma)$ denote the image of the Eisenstein transform $E_\mathfrak{a}$ and call $\mathcal{E}_\mathfrak{a}(\Gamma)$ the **Eisenstein space** of $E_\mathfrak{a}(z,s)$. An immediate consequence of Proposition 2.3.3 is that the Eisenstein spaces for distinct cusps are orthogonal. Moreover, since $E_\mathfrak{a}\left(z, \frac{1}{2}+ir\right)$ is an eigenfunction for the Laplace operator with eigenvalue $\lambda = \frac{1}{4} + r^2$ and f and $E_\mathfrak{a}\left(z, \frac{1}{2}+ir\right)$ are smooth, the Leibniz integral rule implies

$$\Delta E_\mathfrak{a} = E_\mathfrak{a} M,$$

where $M : C_0^\infty(\mathbb{R}^+) \to C_0^\infty(\mathbb{R}^+)$ is the multiplication operator given by

$$(Mf)(r) = \left(\frac{1}{4} + r^2\right) f(r),$$

for all $f \in C_0^\infty(\mathbb{R}^+)$. Therefore if $E_\mathfrak{a} f$ belongs to $\mathcal{E}_\mathfrak{a}(\Gamma)$ then so does $E_\mathfrak{a}(Mf)$. But as $f, Mf \in C_0^\infty(\mathbb{R}^+)$, this means $\mathcal{E}_\mathfrak{a}(\Gamma)$ is invariant under the Laplace operator. While the Eisenstein spaces are invairant, they do not make up all of $\mathcal{E}(\Gamma)$. By Theorem 2.3.8, the residues of the Eisenstein series belong to $\mathcal{E}(\Gamma)$. Let $\mathcal{R}(\Gamma)$ denote the subspace spanned by the residues of all the Eisenstein series. We call any element of $\mathcal{R}(\Gamma)$ a **(residual) Maass form** (by Theorem 2.3.8 they are Maass forms). Also let $\mathcal{R}_{s_j}(\Gamma)$ denote the subspace spanned by those residues taken at $s = s_j$. Since there are finitely many cusps of $\Gamma\backslash\mathbb{H}$, each $\mathcal{R}_{s_j}(\Gamma)$ is finite dimensional. As the number of residues in $(\frac{1}{2}, 1]$ is finite by Theorem 2.3.8, it follows that $\mathcal{R}(\Gamma)$ is finite dimensional too. So $\mathcal{R}(\Gamma)$ decomposes as

$$\mathcal{R}(\Gamma) = \bigoplus_{\frac{1}{2} < s_j \le 1} \mathcal{R}_{s_j}(\Gamma).$$

This decomposition is orthogonal because the Maass forms belonging to distinct subspaces $\mathcal{R}_{s_j}(\Gamma)$ have distinct eigenvalues and eigenfunctions of self-adjoint operators are orthogonal (recall that Δ is self-adjoint by Theorem 2.3.5). Also, each subspace $\mathcal{R}_{s_j}(\Gamma)$ is clearly invariant under the Laplace operator because its elements are Maass forms. We are now ready for the spectral resolution. Although the proof is beyond the scope of this text, the spectral resolution of the Laplace operator on $\mathcal{E}(\Gamma)$ is as follows (see [Iwa02] for a proof):

Theorem 2.3.9. *$\mathcal{E}(\Gamma)$ admits the orthogonal decomposition*

$$\mathcal{E}(\Gamma) = \mathcal{R}(\Gamma) \bigoplus_{\mathfrak{a}} \mathcal{E}(\Gamma),$$

where the direct sum is over the cusps of $\Gamma\backslash\mathbb{H}$. The Laplace operator Δ has discrete spectrum on $\mathcal{R}(\Gamma)$ in the segment $[0, \frac{1}{4})$ and has pure continuous spectrum on each Eisenstein space $\mathcal{E}_\mathfrak{a}(\Gamma)$ covering the segment $\left[\frac{1}{4}, \infty\right)$ uniformly with multiplicity 1. Letting $\{u_j\}$ be any orthonormal basis residual Maass forms for $\mathcal{R}(\Gamma)$, every $f \in \mathcal{E}_\mathfrak{a}(\Gamma)$ admits a decomposition of the form

$$f(z) = \sum_j \langle f, u_j\rangle u_j(z) + \sum_\mathfrak{a} \frac{1}{4\pi}\int_{-\infty}^{\infty} \left\langle f, E_\mathfrak{a}\left(\cdot, \frac{1}{2}+ir\right)\right\rangle E_\mathfrak{a}\left(z, \frac{1}{2}+ir\right)\, dr,$$

where the sum is over the cusps of $\Gamma\backslash\mathbb{H}$. The series is locally absolutely uniformly convergent and the integrals are locally absolutely uniformly bounded if $f \in \mathcal{D}(\Gamma)$ and converges in the L^2-norm otherwise.

Combining Theorems 2.3.7 and 2.3.9 gives the full spectral resolution of $\mathcal{L}(\Gamma)$.

Theorem 2.3.10. $\mathcal{B}(\Gamma)$ *admits the orthogonal decomposition*

$$\mathcal{B}(\Gamma) = \mathcal{C}(\Gamma) \oplus \mathcal{R}(\Gamma) \bigoplus_{\mathfrak{a}} \mathcal{E}(\Gamma),$$

where the sum is over all cusps of $\Gamma\backslash\mathbb{H}$*. The Laplace operator has pure point spectrum on* $\mathcal{C}(\Gamma)$*, discrete spectrum on* $\mathcal{R}(\Gamma)$*, and absolutely continuous spectrum on* $\mathcal{E}(\Gamma)$*. Letting* $\{u_j\}$ *be any orthonormal basis of Maass forms for* $\mathcal{C}(\Gamma) \oplus \mathcal{R}(\Gamma)$*, any* $f \in \mathcal{L}(\Gamma)$ *has a series of the form*

$$f(z) = \sum_j \langle f, u_j \rangle u_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle f, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + ir\right)\right\rangle E_{\mathfrak{a}}\left(z, \frac{1}{2} + ir\right) dr,$$

which is locally absolutely uniformly convergent if $f \in \mathcal{D}(\Gamma)$ *and in the* L^2*-norm otherwise. Moreover,*

$$\mathcal{L}(\Gamma) = \overline{\mathcal{C}(\Gamma)} \oplus \overline{\mathcal{R}(\Gamma)} \bigoplus_{\mathfrak{a}} \overline{\mathcal{E}(\Gamma)},$$

where the closure is with respect to the topology induced by the L^2*-norm.*

Proof. Combine Theorems 2.3.7 and 2.3.9 and use the fact that $\mathcal{B}(\Gamma) = \mathcal{E}(\Gamma) \oplus \mathcal{C}(\Gamma)$ for the first statement. The last statement holds because $\mathcal{B}(\Gamma)$ is dense in $\mathcal{L}(\Gamma)$. □

Double Coset Operators

We can extend the theory of double coset operators to Maass forms just as we did for holomorphic forms. Indeed, for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we define the operator $[\alpha]$ on $\mathcal{A}_t(\Gamma)$ to be the linear operator given by

$$(f[\alpha])(z) = \det(\alpha)^{-\frac{1}{2}} f(\alpha z),$$

Clearly $[\alpha]$ is multiplicative. Moreover, if $\gamma \in \Gamma$ and we choose the representative with positive determinant, then the chain of equalities

$$(f[\gamma])(z) = f(\gamma z) = f(z),$$

is equivalent to the automorphy of f on $\Gamma\backslash\mathbb{H}$. For any two congruence subgroups Γ_1 and Γ_2 (not necessarily of the same level) and any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we define the **double coset operator** $[\Gamma_1\alpha\Gamma_2]$ to be the linear operator on $\mathcal{A}_t(\Gamma_1)$ given by

$$(f[\Gamma_1\alpha\Gamma_2])(z) = \sum_j (f[\beta_j])(z) = \sum_j \det(\beta_j)^{-\frac{1}{2}} f(\beta_j z).$$

for any $f \in \mathcal{A}_t(\Gamma_1)$. As was the case for holomorphic forms, Proposition 2.2.3 implies that this sum is finite. Mimicing the same argument for $[\Gamma_1\alpha\Gamma_2]_k$, we see that $[\Gamma_1\alpha\Gamma_2]$ is also well-defined. There is also an analogous statement about the double coset operators for Maass forms:

Proposition 2.3.4. *For any two congruence subgroups* Γ_1 *and* Γ_2*,* $[\Gamma_1\alpha\Gamma_2]$ *maps* $\mathcal{A}_t(\Gamma_1)$ *into* $\mathcal{A}_t(\Gamma_2)$*. Moreover,* $[\Gamma_1\alpha\Gamma_2]$ *preserves the subspace of cusp forms.*

Proof. Mimicing the proof of Proposition 2.2.4, smoothness replacing holomorphy, and the analogous growth condition for Maass forms, the only piece left to verify is that $f[\Gamma_1\alpha\Gamma_2]$ is an eigenfunction for Δ with eigenvalue λ if f is. This is easy, since

$$\Delta(f[\Gamma_1\alpha\Gamma_2])(z) = \sum_j \Delta((f[\beta_j])(z)) = \lambda \sum_j \det(\beta_j)^{-\frac{1}{2}} f(\beta_j z) = \lambda(f[\Gamma_1\alpha\Gamma_2])(z).$$

Thus $f[\Gamma_1\alpha\Gamma_2]$ is an eigenfunction for Δ with eigenvalue λ. This completes the proof. □

Diamond & Hecke Operators

Extending the theory of diamond operators and Hecke operators to Maass forms is fairly straightforward. To see this, we have already shown that $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ so

$$(f\left[\Gamma_1(N)\alpha\Gamma_1(N)\right])(z) = \sum_j (f[\beta_j])(z) = (f[\alpha])(z),$$

for any $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Therefore, for any d taken modulo N, we define the **diamond operator** $\langle d \rangle : \mathcal{A}_t(\Gamma_1(N)) \to \mathcal{A}_t(\Gamma_1(N))$ to be the linear operative given by

$$(\langle d \rangle f)(z) = (f[\alpha])(z),$$

for any $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$. As for holomorphic forms, the diamond operators are multiplicative and invertible. They also decompose $\mathcal{A}_t(\Gamma_1(N))$ into eigenspaces. For any Dirichlet character modulo N, let

$$\mathcal{A}_t(N,\chi) = \{f \in \mathcal{A}_t(\Gamma_1(N)) : \langle d \rangle f = \chi(d) f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^*\},$$

be the χ-eigenspace. Also let $\mathcal{C}_t(N,\chi)$ be the corresponding subspace of cusp forms. Then $\mathcal{A}_t(\Gamma_1(N))$ admits a decomposition into these eigenspaces:

Proposition 2.3.5. *We have the direct sum decomposition*

$$\mathcal{A}_t(\Gamma_1(N)) = \bigoplus_{\chi \ (mod\ N)} \mathcal{A}_t(N,\chi).$$

Moreover, this direct sum decomposition respects the subspace of cusp forms.

Proof. Mimic the proof of Proposition 2.2.5. □

If $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and we choose the representative with positive determinant, then $\chi(\gamma) = \chi(d)$ and the chain of equalities

$$(\langle d \rangle f)(z) = (f[\gamma])(z) = f(\gamma z) = \chi(d) f(z),$$

is equivalent to the automorphy of f with character χ on $\Gamma_0(N)\backslash\mathbb{H}$. So f is a Maass form with character χ on $\Gamma_0(N)\backslash\mathbb{H}$ if and only if $f[\gamma] = \chi(\gamma) f$ for all $\gamma \in \Gamma_0(N)$ where γ is chosen to be the representative with positive determinant. This means that if $\mathcal{A}_t(\Gamma,\chi)$ denotes the space of Maass forms on $\Gamma\backslash\mathbb{H}$ with spectral parameter t and character χ, and $\mathcal{C}_t(\Gamma,\chi)$ denotes the corresponding subspace of cusp forms, then $\mathcal{A}_t(N,\chi) = \mathcal{A}_t(\Gamma_0(N),\chi)$ and $\mathcal{C}_t(N,\chi) = \mathcal{C}_t(\Gamma_0(N),\chi)$. So by Proposition 2.3.5, we have

$$\mathcal{A}_t(\Gamma_1(N)) = \bigoplus_{\chi \ (\mathrm{mod}\ N)} \mathcal{A}_t(\Gamma_0(N),\chi),$$

and this decomposition respects the subspace of cusp forms. As for holomorphic forms, this decomposition helps clarify why we consider Maass forms with nontrivial characters.

We define the Hecke operators in the same way as for holomorphic forms. For a prime p, we define the p-th **Hecke operator** $T_p : \mathcal{A}_t(\Gamma_1(N)) \to \mathcal{A}_t(\Gamma_1(N))$ to be the linear operator given by

$$(T_p f)(z) = \left(f\left[\Gamma_1(N)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\Gamma_1(N)\right]\right)(z).$$

By Proposition 2.3.4, T_p preserves the subspaces of Maass forms and cusp forms. The diamond and Hecke operators commute:

Proposition 2.3.6. *For every $d \in (\mathbb{Z}/N\mathbb{Z})^*$ and prime p, the diamond operator $\langle d \rangle$ and the Hecke operator T_p on $\mathcal{A}_t(\Gamma_1(N))$ commute:*

$$\langle d \rangle T_p = T_p \langle d \rangle$$

Proof. Mimic the proof of Proposition 2.3.6. □

Exaclty as for holomorphic forms, Lemma 2.2.4 will give an explicit description of the Hecke operator T_p:

Proposition 2.3.7. *Let $f \in \mathcal{A}_t(\Gamma_1(N))$. Then the Hecke operator T_p acts on f as follows:*

$$(T_p f)(z) = \begin{cases} \displaystyle\sum_{j \pmod p} \left(f \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right] \right)(z) + \left(f \left[\begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right] \right)(z) & \text{if } p \nmid N, \\ \displaystyle\sum_{j \pmod p} \left(f \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right] \right)(z) & \text{if } p \mid N, \end{cases}$$

where m and n are chosen such that $\det\left(\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right) \right) = 1$.

Proof. Mimic the proof of Proposition 2.2.7. □

We use Proposition 2.3.7 to understand how the Hecke operators act on the Fourier coefficeints of Maass forms:

Proposition 2.3.8. *Let $f \in \mathcal{A}_t(\Gamma_1(N))$ be even or odd with Fourier coefficients $a_n(f)$, $a_0(f)$, and $b_0(f)$. Then for primes p with $(p, N) = 1$, $T_p f$ is even or odd respectively and*

$$(T_p f)(z) = a_0(T_p f) y^s + b_0(T_p f) y^{1-s} + \sum_{n \geq 1} \left(a_{np}(f) + \chi_{N,0}(p) a_{\frac{n}{p}}(\langle p \rangle f) \right) \sqrt{y} K_{it}(2\pi |n| y) \operatorname{SC}(2\pi n x),$$

is the Fourier series of $T_p f$ where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid n$. Moreover, if $f \in \mathcal{A}_t(N, \chi)$ then $T_p f \in \mathcal{A}_t(N, \chi)$ and

$$(T_p f)(z) = a_0(T_p f) y^s + b_0(T_p f) y^{1-s} + \sum_{n \geq 1} \left(a_{np}(f) + \chi(p) a_{\frac{n}{p}}(f) \right) \sqrt{y} K_{it}(2\pi |n| y) \operatorname{SC}(2\pi n x),$$

is the Fourier series of $T_p f$ where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid n$.

Proof. Mimic the proof of Proposition 2.2.8. □

As for holomorphic forms, the Hecke operators form a simultaneously commuting family with the diamond operators:

Proposition 2.3.9. *Let p and q be primes and $d, e \in (\mathbb{Z}/N\mathbb{Z})^*$. Then the Hecke operators T_p and T_q and diamond operators $\langle d \rangle$ and $\langle e \rangle$ on $\mathcal{A}_t(\Gamma_1(N))$ form a simultaneously commuting family:*

$$T_p T_q = T_q T_p, \quad \langle d \rangle T_p = T_p \langle d \rangle, \quad \text{and} \quad \langle d \rangle \langle e \rangle = \langle e \rangle \langle d \rangle.$$

Proof. Mimic the proof of Proposition 2.2.9. □

We use Proposition 2.3.9 to construct diamond operators $\langle m \rangle$ and Hecke operators T_m for all $m \geq 1$ exactly as for holomorphic forms. Explicitely, the **diamond operator** $\langle m \rangle : \mathcal{A}_t(\Gamma_1(N)) \to \mathcal{A}_t(\Gamma_1(N))$ is defined to be the linear operator given by

$$\langle m \rangle = \begin{cases} \langle m \rangle \text{ with } m \text{ taken modulo } N & \text{if } (m, N) = 1, \\ 0 & \text{if } (m, N) > 1. \end{cases}$$

For the Hecke operators, if $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the prime decomposition of m, then the m-th **Hecke operator** $T_m : \mathcal{A}_t(\Gamma) \to \mathcal{A}_t(\Gamma)$ is the linear operator given by

$$T_m = \prod_{1 \leq i \leq k} T_{p_i^{r_i}},$$

where T_{p^r} is defined inductively by

$$T_{p^r} = \begin{cases} T_p T_{p^{r-1}} - \langle p \rangle T_{p^{r-2}} & \text{if } p \nmid N, \\ T_p^r & \text{if } p \mid N, \end{cases}$$

for all $r \geq 2$. By Proposition 2.3.9, the Hecke operators T_m are multiplicative but not completely multiplicative in m and they commute with the diamond operators $\langle m \rangle$. Moreover, a more general formula for how the Hecke operators T_m act on the Fourier coefficients can be derived:

Proposition 2.3.10. *Let $f \in \mathcal{A}_t(\Gamma_1(N))$ be even or odd with Fourier coefficients $a_n(f)$, $a_0(f)$, and $b_0(f)$. Then for $m \geq 1$ with $(m, N) = 1$, $T_m f$ is even or odd respectively and*

$$(T_m f)(z) = a_0(T_m f) y^s + b_0(T_m f) y^{1-s} + \sum_{n \geq 1} \left(\sum_{d \mid (n,m)} a_{\frac{nm}{d^2}}(\langle d \rangle f) \right) \sqrt{y} K_{it}(2\pi |n| y) \,\mathrm{SC}(2\pi i n x),$$

is the Fourier series of $T_m f$. Moreover, if $f \in \mathcal{A}_t(N, \chi)$, then

$$(T_m f)(z) = a_0(T_m f) y^s + b_0(T_m f) y^{1-s} + \sum_{n \geq 1} \left(\sum_{d \mid (n,m)} \chi(d) a_{\frac{nm}{d^2}}(f) \right) \sqrt{y} K_{it}(2\pi |n| y) \,\mathrm{SC}(2\pi i n x).$$

Proof. Mimic the proof of Proposition 2.2.10. ☐

The diamond and Hecke operators turn out to be normal with respect to the Petersson inner product on the subspace of cusp forms. We can use Lemma 2.2.5 to compute adjoints:

Proposition 2.3.11. *Let Γ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then the following are true:*

(i) If $\alpha^{-1} \Gamma \alpha \subseteq \mathrm{SL}_2(\mathbb{Z})$, then for all $f \in \mathcal{C}_t(\Gamma)$ and $g \in \mathcal{C}_t(\alpha^{-1} \Gamma \alpha)$, we have

$$\langle f[\alpha], g \rangle_{\alpha^{-1} \Gamma \alpha} = \langle f, g[\alpha^{-1}] \rangle_{\Gamma}.$$

(ii) For all $f, g \in \mathcal{C}_t(\Gamma)$, we have

$$\langle f[\Gamma \alpha \Gamma], g \rangle = \langle f, g[\Gamma \alpha^{-1} \Gamma] \rangle.$$

In particular, if $\alpha^{-1} \Gamma \alpha = \Gamma$ then $[\alpha]^* = [\alpha^{-1}]$ and $[\Gamma \alpha \Gamma]^* = [\Gamma \alpha^{-1} \Gamma]$.

Proof. Mimic the proof of Proposition 2.2.11. $\square$

We can now prove that the diamond and Hecke operators are normal:

Proposition 2.3.12. *On $\mathcal{C}_t(\Gamma_1(N))$, the diamond operators $\langle m\rangle$ and Hecke operators T_m are normal for all $m \geq 1$ with $(m, N) = 1$ with respect to the Petersson inner product. Moreover, their adjoints are given by*

$$\langle m\rangle^* = \langle \overline{m}\rangle \quad \textit{and} \quad T_p^* = \langle \overline{p}\rangle T_p.$$

Proof. Mimic the proof of Proposition 2.2.12. $\square$

When we are considering Maass forms on the full modular group, Proposition 2.3.12 says that all of the diamond and Hecke operators are normal. Before we discuss the spectral theory, we need one last operator. Define $T_{-1} : \mathcal{A}_t(\Gamma_1(N)) \to \mathcal{A}_t(\Gamma_1(N))$ to be the linear operator given by

$$(T_{-1}f)(z) = f(-\overline{z}),$$

for any $f \in \mathcal{A}_t(\Gamma_1(N))$. Clearly T_{-1} preserves the subspace of cusp forms. We also have the following proposition:

Proposition 2.3.13.

1. *On $\mathcal{A}_t(\Gamma_1(N))$, the operator T_{-1} commutes with the diamond operators $\langle m\rangle$ and Hecke operators T_n for all $m, n \geq 1$.*

2. *On $\mathcal{C}_t(\Gamma_1(N))$ the operator T_{-1} is normal with respect to the Petersson inner product and the adjoint is given by*
$$T_{-1}^* = -T_{-1}.$$

Proof. To prove (i), for any matrix $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ observe that $-\overline{\alpha z} = \alpha(-\overline{z})$. Hence T_{-1} commutes with the $[\alpha]$ operator. Since T_{-1} is linear, it follows that it commutes with the double coset operators and hence the diamond operators and Hecke operators as well. For (ii), $z \to -\overline{z}$ is an automorphism of $\mathbb{H}$ sending $d\mu \to -d\mu$. So for any $f, g \in \mathcal{C}_t(\Gamma_1(N))$, we have

$$\begin{aligned}
\langle T_{-1}f, g\rangle &= \frac{1}{V_{\Gamma_1(N)}} \int_{\mathcal{F}_{\Gamma_1(N)}} (T_{-1}f)(z)\overline{g(z)}\, d\mu \\
&= \frac{1}{V_{\Gamma_1(N)}} \int_{\mathcal{F}_{\Gamma_1(N)}} f(-\overline{z})\overline{g(z)}\, d\mu \\
&= -\frac{1}{V_{\Gamma_1(N)}} \int_{\mathcal{F}_{\Gamma_1(N)}} f(z)\overline{g(-\overline{z})}\, d\mu \\
&= \langle f, -T_{-1}g\rangle.
\end{aligned}$$

This proves the adjoint formula $T_{-1}^* = -T_{-1}$ and normality is now clear. $\square$

Similar to the case for holomorphic forms, if f is a Maass form on $\Gamma_1(N)\backslash\mathbb{H}$ that is a simultaneous eigenfunction for the operator T_{-1} and all diamond operators $\langle m\rangle$ and Hecke operators T_m with $(m, N) = 1$, we call f an **eigenform**. If the condition $(m, N) = 1$ can be dropped, so that f is a simultaneous eigenfunction for all diamond and Hecke operators, then we say f is a **Hecke-Maass eigenform**. In particular, on $\mathrm{PSL}_2(\mathbb{Z})$ all eigenforms are Hecke-Maass eigenforms. Denote the eigenvalue of T_m for f by

$\lambda_f(m)$. As for holomorphic forms, Proposition 2.3.10 implies that the first Fourier coefficient of $T_m f$ is $a_m(f)$ and so $a_m(f) = \lambda_f(m)a_1(f)$ for all $m \geq 1$ with $(m, N) = 1$. Therefore we cannot have $a_1(f) = 0$ and so we can normalize f by dividing by $a_1(f)$ which guarantees that the Fourier series has constant term 1. Then the m-th Fourier coefficient of f, when $(m, N) = 1$, is precisely the eigenvalue $\lambda_f(m)$. This normalization is called the **Hecke normalization** of f. The **Petersson normalization** of f is where we normalize so that $\langle f, f\rangle = 1$. From the spectral theorem we have an analogous corollary as for holomorphic forms:

Theorem 2.3.11. *$\mathcal{C}_t(\Gamma_1(N))$ admits an orthonormal basis of eigenforms.*

Proof. This follows from the spectral theorem along with Propositions 2.3.9, 2.3.12 and 2.3.13. $\square$

In particular, Theorem 2.3.11 implies that the orthonormal basis in Theorem 2.3.7 can be taken to be an orthonormal basis of eigenforms since it is clear that the T_{-1} operator, diamond operators $\langle m\rangle$, and Hecke operators T_m all commute with Δ. Also, just as in the holomorphic setting, we have **Hecke relations**:

Proposition 2.3.14 (Hecke relations, Maass version). *Let $f \in \mathcal{C}_t(N, \chi)$ be a Hecke-Maass eigenform with Fourier coefficients $\lambda_f(n)$. Then the Fourier coefficients are multiplicative and satisfy*

$$\lambda_f(n)\lambda_f(m) = \sum_{d\mid(n,m)} \chi(d)\lambda_f\left(\frac{nm}{d^2}\right) \quad \text{and} \quad \lambda_f(nm) = \sum_{d\mid(n,m)} \mu(d)\chi(d)\lambda_f\left(\frac{n}{d}\right)\lambda_f\left(\frac{m}{d}\right),$$

for all $n, m \geq 1$ with $(nm, N) = 1$.

Proof. Mimic the proof of the Hecke relations for holomorphic forms. $\square$

As an immedite consequence of the Hecke relations, the Hecke operators satisfy analogous relations:

Corollary 2.3.1. *For all $n, m \geq 1$ with $(nm, N) = 1$, we have*

$$T_nT_m = \sum_{d\mid(n,m)} \chi(d)T_{\frac{nm}{d^2}} \quad \text{and} \quad T_{nm} = \sum_{d\mid(n,m)} \mu(d)\chi(d)T_{\frac{n}{d}}T_{\frac{m}{d}}.$$

Proof. Mimic the proof of Corollary 2.2.1. $\square$

Just as for holomorphic forms, the identities in Corollary 2.3.1 can also be established directly and the first identity can be used to show that the Hecke operators commute.

Atkin–Lehner Theory

There is also an Atkin–Lehner theoy for Maass forms. As with holomorphic forms, we will only deal with congruence subgroups of the form $\Gamma_1(N)$ and cusp forms on $\Gamma_1(N)\backslash\mathbb{H}$. The trivial way to lift Maass forms from a smaller level to a larger level is via the natural inclusion $\mathcal{C}_t(\Gamma_1(M)) \subseteq \mathcal{C}_t(\Gamma_1(N))$ provided $M \mid N$ which follows from $\Gamma_1(N) \leq \Gamma_1(M)$. Alternatively, for any $d \mid \frac{N}{M}$, let $\alpha_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $f \in \mathcal{C}_t(\Gamma_1(M))$, then

$$(f[\alpha_d])(z) = f(\alpha_d z) = f(dz).$$

Similar to holomorphic forms, $[\alpha_d]$ maps $\mathcal{C}_t(\Gamma_1(M))$ into $\mathcal{C}_t(\Gamma_1(N))$:

Proposition 2.3.15. *Let M and N be positive integers such that $M \mid N$. For any $d \mid \frac{N}{M}$, $[\alpha_d]$ maps $\mathcal{C}_t(\Gamma_1(M))$ into $\mathcal{C}_t(\Gamma_1(N))$.*

Proof. Mimicing the proof of Proposition 2.2.14 with $k = 0$, smoothness replacing holomorphy, and the analogous growth condition for Maass forms, the only piece left to verify is that $f[\alpha_d]$ is an eigenfunction for Δ with eigenvalue λ if f is. This is easy, for

$$\Delta(f[\alpha_d])(z) = (\Delta f)(dz) = \lambda f(dz) = \lambda(f[\alpha_d])(z).$$

Therefore $f[\alpha_d]$ is an eigenfunction for Δ with eigenvalue λ. This completes the proof. $\square$

We can now define oldforms and newforms. For each divisor d of N, set

$$i_d : \mathcal{C}_t\left(\Gamma_1\left(\frac{N}{d}\right)\right) \times \mathcal{C}_t\left(\Gamma_1\left(\frac{N}{d}\right)\right) \to \mathcal{C}_t(\Gamma_1(N)) \qquad (f, g) \mapsto f + g[\alpha_d].$$

This map is well-defined by Proposition 2.3.15. The subspace of **oldforms of level** N is

$$\mathcal{C}_t(\Gamma_1(N))^{\text{old}} = \sum_{p|N} \operatorname{Im}(i_p),$$

and the subspace of **newforms of level** N is

$$\mathcal{C}_t(\Gamma_1(N))^{\text{new}} = \left(\mathcal{C}_t(\Gamma_1(N))^{\text{old}}\right)^{\perp},$$

where the orthogonal complement is taken with respect to the Petersson inner product. The elements of such subspaces are called **oldforms** and **newforms** respectively. Both subspaces are invairant under the diamond and Hecke operators (mimic the proof of Proposition 2.2.15 given in [DS05]):

Proposition 2.3.16. *The spaces $\mathcal{C}_t(\Gamma_1(N))^{\text{old}}$ and $\mathcal{C}_t(\Gamma_1(N))^{\text{new}}$ are invariant under the diamond operators $\langle m \rangle$ and Hecke operators T_m for all $m \geq 1$.*

As a corollary, these subspaces admit orthogonal bases of eigenforms:

Corollary 2.3.2. *$\mathcal{C}_t(\Gamma_1(N))^{\text{old}}$ and $\mathcal{C}_t(\Gamma_1(N))^{\text{new}}$ admit orthonormal bases of eigenforms.*

Proof. This follows immediately from Theorem 2.3.11 and Proposition 2.3.16 $\square$

We can remove the condition $(m, N) = 1$ for eigenforms in a basis of $\mathcal{C}_t(\Gamma_1(N))^{\text{new}}$ so that the eigenforms are eigenfunctions for all of the diamond and Hecke operators. As for holomorphic forms, we need a preliminary result (mimic the proof of Lemma 2.2.6 as given in [DS05]):

Lemma 2.3.2. *If $f \in \mathcal{C}_t(\Gamma_1(N))$ with Fourier coefficients $a_n(f)$ and such that $a_n(f) = 0$ whenever $(n, N) = 1$, then*

$$f = \sum_{p|N} f_p[\alpha_p],$$

for some $f_p \in \mathcal{C}_t\left(\Gamma_1\left(\frac{N}{p}\right)\right)$.

As was the case for holomorphic forms, we observe from Lemma 2.3.2 that if $f \in \mathcal{C}_t(\Gamma_1(N))$ is such that its n-th Fourier coefficients vanish when n is relatively prime to the level, then f must be an oldform. The main theorem about $\mathcal{C}_t(\Gamma_1(N))^{\text{new}}$ can now be proved. We say that f is a **primitive Hecke-Maass eigenform** if it is a Hecke normalized Hecke-Maass eigenform in $\mathcal{C}_t(\Gamma_1(N))^{\text{new}}$. We can now prove the main result about newforms:

Theorem 2.3.12. *Let $f \in \mathcal{C}_t(\Gamma_1(N))^{\text{new}}$ be an eigenform. Then the following hold:*

(i) f is a Hecke-Maass eigenform (the condition $(m, N) = 1$ can be dropped).

(ii) If $\widetilde{f}$ satisfies the same conditions as f and has the same eigenvalues for the Hecke operators, then $\widetilde{f} = cf$ for some nonzero constant c.

Moreover, the primitive Hecke-Maass eigenforms in $\mathcal{C}_t(\Gamma_1(N))^{\text{new}}$ form an orthogonal basis. Each such primitive Hecke-Maass eigenform f lies in an eigenspace $\mathcal{C}_t(N, \chi)$ for a Dirichlet character χ modulo N and $T_m f = \lambda_f(m) f$ for all $m \geq 1$.

Proof. Mimic the proof of Theorem 2.2.6. □

Statement (i) in Theorem 2.3.12 says that newforms are Hecke-Maass eigenforms. So again, newforms are the generalization of eigenforms for the congruence subgroups $\Gamma_1(N)$. In particular, primitive Maass-Hecke eigenforms satisfy the Hecke relations for all $n, m \geq 1$. Statement (ii) in Theorem 2.3.12 is referred to as the **multiplicity one theorem** for Maass forms. So as is the case for holomorphic forms, $\mathcal{C}_t(\Gamma_1(N))^{\text{new}}$ contains one element per "eigenvalue" where we mean a set of eigenvalues one for each Hecke operator T_m. Interestingly, unlike holomorphic forms it is unknown if the Fourier coefficients of Maass forms are real or even algebraic in general.
We require one last piece of machinery. As for holomorphic forms, we have an involution on the space $\mathcal{C}_t(N, \chi)$. Recall the matrix

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

with $\det(W_N) = N$. We define the **Atkin–Lehner involution** $\omega_N : \mathcal{C}_t(\Gamma_1(N)) \to \mathcal{C}_t(\Gamma_1(N))$ to be the linear operator given by

$$(\omega_N f)(z) = f\left(W_N z\right) = f\left(-\frac{1}{Nz}\right).$$

To see that ω_N is well-defined, first note that smoothness and the growth condition are obvious. For automorphy, recall that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$, we have

$$W_N \gamma = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ Na & Nb \end{pmatrix} = \begin{pmatrix} d & -N^{-1}c \\ -Nb & a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \gamma' W_N,$$

with $\gamma' = \left(\begin{smallmatrix} d & -N^{-1}c \\ -Nb & a \end{smallmatrix}\right) \in \Gamma_1(N)$. Then

$$(\omega_N f)(\gamma z) = f(W_N \gamma z) = f(\gamma' W_N z) = f\left(-\frac{1}{Nz}\right) = (\omega_N f)(z).$$

This verifies automorphy for $\omega_N f$. Also, it is clear that $\omega_N f$ is a cusp form because f is. Hence $\omega_N f$ is well-defined. Moreover, it is an involution because $\omega_N(\omega_N f) = f$. The crucial fact we need is that the Hecke operators commute with the Atkin–Lehner involution and the Atkin–Lehner involution is self-adjoint with respect to the Petersson inner product (mimic the proof for holomorphic forms given in [Miy89] or [DS05]):

Proposition 2.3.17. *For every $m \geq 1$, the Hecke operator T_m commutes with the Atkin-Lehner involution ω_N on $\mathcal{C}_t(\Gamma_1(N))$:*

$$T_m \omega_N = \omega_N T_m.$$

Moreover, the Atkin-Lehner involution is normal with respect to the Petersson inner product:

$$\omega_N^* = \omega_N.$$

By the spectral theorem, we can improve the eigenbasis of $\mathcal{C}_t(\Gamma_1(N))$:

Theorem 2.3.13. *$\mathcal{C}_t(\Gamma_1(N))$ admits an orthonormal basis of eigenforms that are also eigenfunctions for the Atkin-Lehner involution.*

Proof. This follows from the spectral theorem along with Theorem 2.3.11 and Proposition 2.3.17. □

As ω_N is an involution, its only possible eigenvalues are ± 1. So by Theorem 2.3.13 we may assume that any eigenform $f \in \mathcal{C}_t(\Gamma_1(N))$ in a basis satisfies

$$\omega_N f = \omega_N(f) f,$$

where $\omega_N(f) = \pm 1$.

The Ramanujan-Petersson Conjecture

As for the size of the Fourier coefficients of Maass forms, much is currently unknown. But there is an analogous conjecture, known as the Ramanujan-Petersson conjecture, with some partial progress. To state it, suppose $f \in \mathcal{C}_t(N, \chi)$ is a primitive Hecke-Maass eigenform with Fourier coefficients $\lambda_f(n)$. For each prime p, consider the polynomial

$$1 - \lambda_f(p)p^{-s} + \chi(p)p^{-2s},$$

and let $\alpha_1(p)$ and $\alpha_2(p)$ denote the roots. Then

$$\alpha_1(p) + \alpha_2(p) = \lambda_f(p) \quad \text{and} \quad \alpha_1(p)\alpha_2(p) = \chi(p).$$

The more general **Ramanujan-Petersson conjecture** is following statement:

Theorem 2.3.14 (Ramanujan-Petersson conjecture). *Let $f \in \mathcal{C}_t(N, \chi)$ be a primitive Hecke-Maass eigenform. Denote the Fourier coefficients by $\lambda_f(n)$ and let $\alpha_1(p)$ and $\alpha_2(p)$ be the roots of $1 - \lambda_f(p)p^{-s} + \chi(p)p^{-2s}$. Then for all primes p,*

$$|\lambda_f(p)| \le 2.$$

Moreover, if $p \nmid N$, then

$$|\alpha_1(p)| = |\alpha_2(p)| = 1.$$

The Ramanujan-Petersson conjecture has not been proven, but there has been partial progress toward the conjecture. The currest best bound is $|\lambda_f(p)| \le 2p^{\frac{7}{64}}$ and is due to Kim and Sarnak (see [KRS03] for the proof). Nevertheless, we can get close to the Ramanujan-Petersson conjecture without much work. Let $f \in \mathcal{C}_t(N, \chi)$ be a primitive Hecke-Maass eigenform. Fix some $Y > 0$ and consider

$$\int_Y^\infty \int_0^1 |f(x+iy)|^2 \, \frac{dx\,dy}{y^2}.$$

Since f is a cusp form, this integral is absolutely bounded by Method 1.4.1. Moreover, this integral can be expressed as

$$\int_Y^\infty \int_0^1 \sum_{n,m \neq 0} \lambda_f(n)\overline{\lambda_f(m)} K_{it}(2\pi|n|y)\overline{K_{it}(2\pi|m|y)} e^{2\pi i(n-m)x} e^{-2\pi(n+m)y} \, \frac{dx\,dy}{y}.$$

Appealing to the dominated convergence theorem, we can interchange the sum and the inner integral. Then Equation (2.2) implies that the inner integral cuts off all of the terms except the diagional resulting in

$$\int_Y^\infty \sum_{n\neq 0} |\lambda_f(n)|^2 |K_{it}(2\pi|n|y)|^2 e^{-4\pi ny} \frac{dx\,dy}{y}.$$

As this expression is finite, we have

$$|\lambda_f(n)|^2 \int_Y^\infty K_{it}(2\pi|n|y)|^2 e^{-4\pi ny} \frac{dx\,dy}{y} \ll \int_Y^\infty \int_0^1 |f(x+iy)|^2 \frac{dx\,dy}{y^2},$$

where we note that the integral on the left-hand side is finite. Now f is bounded on $\mathbb{H}$ so that

$$\int_Y^\infty \int_0^1 |f(x+iy)|^2 \frac{dx\,dy}{y^2} \ll \int_Y^\infty \int_0^1 \frac{dx\,dy}{y^2} \ll \frac{1}{Y}.$$

Putting these two estimates together gives

$$|\lambda_f(n)|^2 \int_Y^\infty K_{it}(2\pi|n|y)|^2 e^{-4\pi ny} \frac{dx\,dy}{y} \ll \frac{1}{Y}.$$

Taking $Y = \frac{1}{n}$ results in

$$\lambda_f(n) \ll n^{\frac{1}{2}}.$$

This bound is known as the **Hecke bound** for Maass forms. It turns out that the Ramanujan-Petersson conjecture is tightly connected to another conjecture of Selberg about the smallest possible eigenvalue of Maass forms on $\Gamma\backslash\mathbb{H}$. Note that the possible eigenvalues are discrete by Theorem 2.3.10 and so there exists a smallest eigenvalue. To state it, recall that if f is a Maass form with eigenvalue λ on $\Gamma\backslash\mathbb{H}$, then $\lambda = \frac{1}{4}+t^2$ with either $t \in \mathbb{R}$ or $it \in [0, \frac{1}{2})$. **Selberg's conjecture** claims that the second case never occurs:

Conjecture 2.3.1 (Selberg's conjecture). *If λ is the smallest eigenvalue for Maass forms on $\Gamma\backslash\mathbb{H}$, then*

$$\lambda \geq \frac{1}{4}.$$

Selberg was able to achieve a remarkable lower bound using the analytic continuation of a certain Dirichlet series and the Weil bound for Kloosterman sums (see [Iwa02] for a proof):

Theorem 2.3.15. *If λ is the smallest eigenvalue for Maass forms on $\Gamma\backslash\mathbb{H}$, then*

$$\lambda \geq \frac{3}{16}.$$

In the language of automorphic representations, these two conejctures are a consequence of a much larger conjecture (see [BB13] for details).

Chapter 3

L-functions

We start our discussion of L-functions with Dirichlet series. Dirichlet series are essential tools in analytic number theory because they are a way of analytically encoding arithmetic information. If the Dirichlet series possesses sufficiently large analytic continuation we call it an L-function and from the analytic properties of L-functions we can extract number theoritic results. After discussing Dirichlet series we will define a specific class of L-functions: Selberg class L-functions. Next we show that several natural Dirichlet series are actually Selberg class L-functions. Specifically, we discuss the Riemann zeta function, L-functions attached to Dirichlet characters, and L-functions formed from holomorphic forms. In the case of holomorphic forms we also describe a method of Rankin and Selberg for constructing new L-functions from old ones.

3.1 The General Setup for L-functions

Dirichlet Series

A **Dirichlet series** $D(s)$ is a sum of the form

$$D(s) = \sum_{n \geq 1} \frac{a(n)}{n^s},$$

with $a(n) \in \mathbb{C}$. We exclude the case $a(n) = 0$ for all $n \geq 1$ so that $D(s)$ is not identically zero. We would first like to understand where this series converges. It does not take much for $D(s)$ to converge uniformly in a sector:

Theorem 3.1.1. *Suppose $D(s)$ is a Dirichlet series with coefficient $a(n)$ and that $D(s)$ converges at $s_0 = \sigma_0 + it_0$. Then for any $H > 0$, $D(s)$ converges uniformly in the sector*

$$\{s = \sigma + it \in \mathbb{C} : \sigma \geq \sigma_0, |t - t_0| \leq H(\sigma - \sigma_0)\}.$$

Proof. Set $R(u) = \sum_{n \geq u} \frac{a(n)}{n^{-s_0}}$ so that $a(n) = (R(n) - R(n+1))n^{s_0}$. Then for any two positive integers N and M with $1 \leq M < N$, partial summation (see Appendix B.3) implies

$$\sum_{M \leq n \leq N} \frac{a(n)}{n^s} = R(M)M^{s_0 - s} - R(N)N^{s_0 - s} - \sum_{M+1 \leq n \leq N} R(n)((n-1)^{s_0 - s} - n^{s_0 - s}). \tag{3.1}$$

We will now express the sum on the right-hand side as an integral. To do this, observe that

$$(n-1)^{s_0 - s} - n^{s_0 - s} = -(s_0 - s) \int_{n-1}^{n} u^{s_0 - s - 1} \, du.$$

Therefore

$$\begin{aligned}\sum_{M+1\le n\le N} R(n)((n-1)^{s_0-s} - n^{s_0-s}) &= -(s_0-s)\sum_{M+1\le n\le N} R(n)\int_{n-1}^{n} u^{s_0-s-1}\,du \\ &= -(s_0-s)\sum_{M+1\le n\le N}\int_{n-1}^{n} R(u)u^{s_0-s-1}\,du \\ &= -(s_0-s)\int_{M}^{N} R(u)u^{s_0-s-1}\,du,\end{aligned} \tag{3.2}$$

where the second to last line follows because $R(u)$ is constant on the interval $[u, u+1)$. Combining Equations (3.1) and (3.2) gives

$$\sum_{M\le n\le N}\frac{a(n)}{n^s} = R(M)M^{s_0-s} - R(N)N^{s_0-s} + (s_0-s)\int_{M}^{N} R(u)u^{s_0-s-1}\,du. \tag{3.3}$$

Now for any $\varepsilon > 0$ there exists an M such that $|R(u)| < \varepsilon$ for all $u \ge M$ because $D(s)$ is convergent at s_0. In particular, $|R(u)u^{s_0-s}| < \varepsilon$ for all $u \ge M$ becuase $\sigma \ge \sigma_0$. Moreover for s in the prescribed sector,

$$|s-s_0| \le (\sigma-\sigma_0) + |t-t_0| \le (H+1)(\sigma-\sigma_0).$$

These estimates and Equation (3.3) together imply

$$\left|\sum_{M\le n\le N}\frac{a(n)}{n^s}\right| = 2\varepsilon + \varepsilon|s-s_0|\int_{M}^{N} u^{\sigma_0-\sigma-1}\,du \le 2\varepsilon + \varepsilon(H+1)(\sigma-\sigma_0)\int_{M}^{N} u^{\sigma_0-\sigma-1}\,du.$$

Since the integral is finite, $\sum_{M\le n\le N}\frac{a(n)}{n^s}$ can be made arbitrarily small uniformly for s in the desired sector. The claim now follows by the uniform version of Cauchy's criterion. $\square$

By taking $H \to \infty$ in Theorem 3.1.1 we see that $D(s)$ converges in the half-plane $\mathrm{Re}(s) > \sigma_0$. Let σ_c be the infimum of all $\mathrm{Re}(s)$ for which $D(s)$ converges. We call σ_c the **abscissa of convergence** of $D(s)$. Similarly, let σ_a be the infimum of all $\mathrm{Re}(s)$ for which $D(s)$ converges absolutely. Since the terms of $D(s)$ are holomorphic, the convergence is locally absolutely uniform (actually uniform in sectors) for $\mathrm{Re}(s) > \sigma_a$. It follows that $D(s)$ is holomorphic in the region $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. We call σ_a the **abscissa of absolute convergence** of $D(s)$. One should think of σ_c and σ_a as the boundaries of convergence and absolute convergence respectively. Of course, anything can happen at $\mathrm{Re}(s) = \sigma_c$ and $\mathrm{Re}(s) = \sigma_a$, but to the right of these lines we have convergence and absolute convergence of $D(s)$ respectively. It turns out that σ_a is never far from σ_c provided σ_c is finite:

Theorem 3.1.2. *If $D(s)$ is a Dirichlet series with finite abscissa of convergence σ_c, then*

$$\sigma_c \le \sigma_a \le \sigma_c + 1.$$

Proof. The first inequality is trivial since absolute convergence implies convergence. For the second inequality, let $\varepsilon > 0$. Since $D(s)$ converges at $\sigma_c+\varepsilon$, the terms $a(n)n^{-(\sigma_c+\varepsilon)}$ tend to zero as $n \to \infty$. Therefore $a(n) \ll_\varepsilon n^{\sigma_c+\varepsilon}$ where the implicit constant is independent of n. But then $a(n)n^{-(\sigma_c+\varepsilon)} \ll_\varepsilon 1$ which implies $\sum_{n\ge1} a(n)n^{-(\sigma_c+1+2\varepsilon)}$ is absolutely convergent by the comparison test with respect to $\sum_{n\ge1} n^{-(1+\varepsilon)}$. In terms of $D(s)$, this means $\sigma_a \le \sigma_c + 1 + 2\varepsilon$ and taking $\varepsilon \to 0$ gives the second inequality. $\square$

We will now introduce several convergence theorems for Dirichlet series. It will be useful to setup some notation first. If $D(s)$ is a Dirichlet series with coefficients $a(n)$, then for any real X, we set

$$A(X) = \sum_{n \le X} a(n).$$

This is the partial sum of the coefficients $a(n)$ up to X. Our first convergence theorem relates boundeness of $A(X)$ to the value of σ_c:

Proposition 3.1.1. *Suppose $D(s)$ is a Dirichlet series and that $A(X) \ll 1$. Then $\sigma_c \le 0$.*

Proof. Fix s be such that $\mathrm{Re}(s) > 0$. Since $A(X) \ll 1$, $A(X)X^s \to 0$ as $X \to \infty$. Abel's summation formula (see Appendix B.3) then implies

$$D(s) = s\int_1^\infty A(u)u^{-(s+1)}\,du.$$

But because $A(u) \ll 1$, we have

$$s\int_1^\infty A(u)u^{-(s+1)}\,du \ll s\int_1^\infty u^{-(s+1)}\,du = -u^{-s}\Big|_1^\infty = 1.$$

Therefore the integral converges for $\mathrm{Re}(s) > 0$ and hence $D(s)$ does too. It follows that $\sigma_c \le 0$. $\square$

Our next theorem states that if the coefficients of $D(s)$ are polynomially bounded, we can obtain an upper bound for the abscissa of absolute convergence:

Proposition 3.1.2. *Suppose $D(s)$ is a Dirichlet series whose coefficients $a(n)$ satisfy $|a(n)| \ll n^\alpha$ for some real α. Then the abscissa of absolute convergence satisfies $\sigma_a \le 1 + \alpha$.*

Proof. It suffices to show that $D(s)$ is absolutely convergent in the region $\mathrm{Re}(s) > 1 + \alpha$. For s is in this region, the polynomial bound gives

$$|D(s)| \le \sum_{n \ge 1}\left|\frac{a(n)}{n^s}\right| \ll \sum_{n \ge 1}\frac{1}{n^{s-\alpha}}.$$

The latter series converges by the integral test because $\mathrm{Re}(s) - \alpha > 1$. Therefore $D(s)$ is absolutely convergent. $\square$

Obtaining polynomial bounds on coefficients of Dirichlet series are, in most cases, not hard to establish. So the assumption in Proposition 3.1.2 is mild. Actually, there is a partial converse to Proposition 3.1.2 which gives an approximate size to $A(X)$:

Proposition 3.1.3. *Suppose $D(s)$ is a Dirichlet series with coefficients $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for any $\varepsilon > 0$,*

$$A(X) \ll_\varepsilon X^{\sigma_a + \varepsilon}.$$

Proof. By Abel's summation formula (see Appendix B.3),

$$\sum_{n\leq X}\frac{a(n)}{n^{\sigma_a+\varepsilon}}=A(X)X^{-(\sigma_a+\varepsilon)}+(\sigma_a+\varepsilon)\int_0^X A(u)u^{-(\sigma_a+\varepsilon+1)}\,du. \tag{3.4}$$

If we set $R(u)=\sum_{n\geq u}\frac{a(n)}{n^{\sigma_a+\varepsilon}}$, then $a(n)=(R(n)-R(n+1))n^{\sigma_a+\varepsilon}$ and it follows that

$$A(u)=\sum_{n\leq u}(R(n)-R(n+1))n^{\sigma_a+\varepsilon}.$$

Substituting this into Equation (3.4), we obtain

$$\int_0^X\sum_{n\leq u}(R(n)-R(n+1))n^{\sigma_a+\varepsilon}u^{-(\sigma_a+\varepsilon+1)}\,du.$$

As $R(n)$ is constant on the interval $[n,n+1)$, linearity of the integral implies

$$\int_0^X\sum_{n\leq u}(R(n)-R(n+1))n^{\sigma_a+\varepsilon}u^{-(\sigma_a+\varepsilon+1)}\,du=\sum_{0\leq n\leq X}(R(n)-R(n+1))n^{\sigma_a+\varepsilon}\int_n^{n+1}u^{-(\sigma_a+\varepsilon+1)}+O_\varepsilon(1),$$

where the O-estimate is present since X may not be an integer. Now $R(n)\ll_\varepsilon 1$ since it is the tail of $D(\sigma_a+\varepsilon)$ and moreover,

$$\int_n^{n+1}u^{-(\sigma_a+\varepsilon+1)}=-\frac{u^{-(\sigma_a+\varepsilon)}}{\sigma_a+\varepsilon}\bigg|_n^{n+1}=\frac{n^{-(\sigma_a+\varepsilon)}}{\sigma_a+\varepsilon}-\frac{(n+1)^{-(\sigma_a+\varepsilon)}}{\sigma_a+\varepsilon}\ll_\varepsilon 1,$$

because $\sigma_a+\varepsilon>0$. So

$$\int_0^X A(u)u^{-(\sigma_a+\varepsilon+1)}\,du=\int_0^X\sum_{n\leq u}(R(n)-R(n+1))n^{\sigma_a+\varepsilon}u^{-(\sigma_a+\varepsilon+1)}\,du\ll_\varepsilon 1.$$

Also, $\sum_{n\leq X}\frac{a(n)}{n^{\sigma_a+\varepsilon}}\ll_\varepsilon 1$ because $D(\sigma_a+\varepsilon)$ converges. We conclude

$$A(X)X^{-(\sigma_a+\varepsilon)}=\sum_{n\leq X}\frac{a(n)}{n^{\sigma_a+\varepsilon}}-(\sigma_a+\varepsilon)\int_0^X A(u)u^{-(\sigma_a+\varepsilon+1)}\,du.\ll_\varepsilon 1,$$

which is equivalent to the desired estimate. □

A way to think about Proposition 3.1.3 is that if the abscissa of absolute convergence is $\sigma_a\geq 0$ then the size of the coefficients $a(n)$ is at most $n^{\sigma_a+\varepsilon}$ on average for any $\varepsilon>0$. Of corse, if $a(n)\ll n^\alpha$ then Proposition 3.1.2 implies that $\sigma_a\leq 1+\alpha$ and so Proposition 3.1.3 gives the significantly weaker estimate $A(X)\ll_\varepsilon X^{1+\alpha+\varepsilon}$. However, if we only have a bound of the form $A(X)\ll X^\alpha$ we can still obtain an upper estimate for the abscissa of absolute convergence:

Proposition 3.1.4. *Suppose $D(s)$ is a Dirichlet series with coefficients $a(n)$ such that $A(X)\ll X^\alpha$ for some real $\alpha\geq 0$. Then the abscissa of absolute convergence satisfies $\sigma_a\leq\alpha$.*

Proof. It suffices to show that $D(s)$ is absolutely convergent in the region $\mathrm{Re}(s) > \alpha$. Let s be in this region and set $\mathrm{Re}(s) = \beta$. Note that $\beta > \alpha$. Then

$$|D(s)| \le \sum_{n\ge1}\left|\frac{a(n)}{n^s}\right| = \sum_{n\ge1}\frac{|a(n)|}{n^\beta}.$$

By Abel's summation formula (see Appendix B.3),

$$\sum_{n\le N}|a(n)|n^{-\beta} = |a(N)|N^{-\beta} - |a(1)| + \beta\int_1^N A^*(u)u^{-(\beta+1)}\,du,$$

where $A^*(u) = \sum_{n\le u}|a(n)|$. Now $A(u) \ll u^\alpha$, which is to say that $A^*(u) \ll u^\alpha$. Therefore there is some N' such that $A^*(N) \ll N^\alpha$ for all $N \ge N'$. In partcular $a(N) \ll N^\alpha$. So for such N, we estimate as follows:

$$\begin{aligned}\sum_{n\le N}|a(n)|n^{-\beta} &= |a(N)|N^{-\beta} - |a(1)| + \beta\int_1^N A^*(u)u^{-(\beta+1)}\,du \\ &= |a(N)|N^{-\beta} - |a(1)| + \beta\int_1^{N'} A^*(u)u^{-(\beta+1)}\,du + \beta\int_{N'}^N A^*(u)u^{-(\beta+1)}\,du \\ &\ll |a(N)|N^{-\beta} - |a(1)| + \beta\int_1^{N'} A^*(u)u^{-(\beta+1)}\,du + \beta\int_{N'}^N u^{\alpha-(\beta+1)}\,du.\end{aligned}$$

As $N \to \infty$, the left-hand side tends towards $\sum_{n\ge1}\frac{|a(n)|}{n^\beta}$. As for the right-hand side, the first term tends to zero since $\beta > \alpha$. The second and third terms remain bounded as they are independent of N. For the last term, we compute

$$\int_{N'}^N u^{\alpha-(\beta+1)}\,du = \left.\frac{u^{\alpha-\beta}}{\alpha-\beta}\right|_{N'}^N = \frac{N^{\alpha-\beta}}{\alpha-\beta} - \frac{(N')^{\alpha-\beta}}{\alpha-\beta}.$$

But $\beta > \alpha$ so this term is also bounded as $N \to \infty$. This finishes the proof. □

Do not be fooled; Proposition 3.1.4 is in general weaker than Proposition 3.1.2. For example, from our comments following Proposition 3.1.3, if $D(s)$ is a Dirichlet series with coefficients $a(n)$ and we have the estimate $A(X) \ll_\varepsilon X^\beta$ for some real β then Proposition 3.1.4 only says that $\sigma_a \le \beta$. If α is very small compared to β, this is a significantly worse upper bound for the abscissa of absolute convergence than what Proposition 3.1.2 would imply if $a(n) \ll n^\alpha$. Actually, the question of sharp polynomial bounds for Dirichlet coefficients can be very deep. However, if the coefficients $a(n)$ are always nonnegative, then **Landau's theorem** provides a way of obtaining a lower bound for polynomially growth as well as describing a singularity of $D(s)$:

Theorem 3.1.3 (Landau's theorem). *Suppose $D(s)$ is a Dirichlet series with nonnegative coefficients $a(n)$ and finite abscissa of absolute convergence σ_a. Then σ_a is a singularity of $D(s)$.*

Proof. If we replace $a(n)$ by $a(n)n^{-\sigma_a}$ then we may assume $\sigma_a = 0$. Now suppose to the contrary that $D(s)$ was holomorphic at $s = 0$. Therefore for some $\delta > 0$, $D(s)$ is holomorphic in the domain

$$\mathcal{D} = \{s : \sigma_a > 0\} \cup \{|s| < \delta\}.$$

Write $D(s)$ as a power series at $s = 1$:

$$P(s) = \sum_{k\ge0} c_k(s-1)^k,$$

where

$$c_k = \frac{D^{(k)}(1)}{k!} = \frac{1}{k!}\sum_{n\geq 1}\frac{a(n)(-\log(n))^k}{n},$$

because $D(s)$ is holomorphic and so we can differentiate termwise. The radius of convergence of $P(s)$ is the distance from $s = 1$ to the nearest singularity of $P(s)$. Since $P(s)$ is holomorphic on $\mathcal{D}$, the closest points are $\pm i\delta$. Therefore, the radius of convergence is at least $|1 \pm \delta| = \sqrt{1+\delta^2}$. We can write $\sqrt{1+\delta^2} = 1 + \delta'$ for some $\delta' > 0$. Then for $|s-1| < 1+\delta'$, write $P(s)$ as

$$P(s) = \sum_{k\geq 0}\frac{(s-1)^k}{k!}\sum_{n\geq 1}\frac{a(n)(-\log(n))^k}{n} = \sum_{k\geq 0}\frac{(1-s)^k}{k!}\sum_{n\geq 1}\frac{a(n)(\log(n))^k}{n}.$$

If $s < 1$ then this last double sum is a sum of positive terms because $a(n) \geq 0$. Moreover, since $D(s)$ is convergent here the two sums can be interchanged by the dominated convergence theorem. Intechanging sums we see that

$$P(s) = \sum_{n\geq 1}\frac{a(n)}{n}\sum_{k\geq 0}\frac{(1-s)^k(\log(n))^k}{k!} = \sum_{n\geq 1}\frac{a(n)}{n}e^{(1-s)\log(n)} = \sum_{n\geq 1}\frac{a(n)}{n^s} = D(s),$$

for $-\delta' < s < 1$. As $\delta' > 0$, this implies that $D(s)$ converges absolutely (since $a(n) \geq 0$) for some s with $\mathrm{Re}(s) < 0$ (say $s = -\frac{\delta'}{2}$) which contradicts $\sigma_a = 0$. □

What Landau's theorem implies is that if $D(s)$ is a Dirichlet series with nonnegative coefficients then $a(n) \not\ll n^{\sigma_a-(1+\varepsilon)}$ for any $\varepsilon > 0$ because otherwise Proposition 3.1.2 implies $\sigma_a \leq \sigma_a - \varepsilon$. So it gives a lower bound for polynomial growth. Actually, Landau's theorem also gives the lower bound $A(X) \not\ll X^{\sigma_a-\varepsilon}$ for otherwise Proposition 3.1.4 would similarly imply $\sigma_a \leq \sigma_\alpha - \varepsilon$. When we come across Dirichlet series whose coefficients are polynomially bounded or polynomially bounded on average, we will invoke these results without mention, except for Landau's theorem, as this is also common practice in the literature.
If the coefficients $a(n)$ are chosen at random, $D(s)$ will not usually posses any good properties outside of convergence in some region (it might not even posess that). However, the Dirichlet series we will encounter, and a fair few in the wild, will have multiplicitive coefficients. In this case, the Dirichlet series admits an infinite product expression:

Proposition 3.1.5. *Suppose the coefficients $a(n)$ of a Dirichlet series $D(s)$ are multiplicative and satisfty $|a(n)| \ll n^\alpha$ for some real $\alpha \geq 0$. Then*

$$D(s) = \prod_p\left(\sum_{k\geq 0}\frac{a(p^k)}{p^{ks}}\right),$$

for $\mathrm{Re}(s) > 1+\alpha$. *Conversely, suppose that there are coefficients $a(n)$ such that*

$$\prod_p\left(\sum_{k\geq 0}\left|\frac{a(p^k)}{p^{ks}}\right|\right),$$

converges for $\mathrm{Re}(s) > 1+\alpha$. *Then the equality above defines a Dirichlet series $D(s)$ that converges absolutely in this region too. Moreover, if the coefficients $a(n)$ are completely multiplicative, then*

$$D(s) = \prod_p(1-a(p)p^{-s})^{-1},$$

for $\mathrm{Re}(s) > 1+\alpha$.

Proof. Since $|a(n)| \ll n^{\alpha}$, Proposition 3.1.2 implies that $D(s)$ converges absolutely for $\mathrm{Re}(s) > 1+\alpha$. Let s be such that $\mathrm{Re}(s) > 1+\alpha$. Since

$$\sum_{k\geq 0}\left|\frac{a(p^k)}{p^{ks}}\right| < \sum_{n\geq 1}\left|\frac{a(n)}{n^s}\right|,$$

the infinite series on the left converges because the right does by the absolute convergence of $D(s)$. Now let $N > 0$ be an integer. Then by the fundamental theorem of arithmetic

$$\prod_{p<N}\left(\sum_{k\geq 0}\frac{a(p^k)}{p^{ks}}\right) = \sum_{n<N}\frac{a(n)}{n^s} + \sum_{n\geq N}{}^{*}\,\frac{a(n)}{n^s}, \tag{3.5}$$

where the $*$ denotes that we are summing over only those additional terms $\frac{a(n)}{n^s}$ that appear in the expanded product on the left-hand side with $n \geq N$. As $N \to \infty$, the first sum on the right-hand side tends to $D(s)$ and the second sum tends to zero because it is part of the tail of $D(s)$ (which tends to zero by convergence). This proves that the product converges, and is equal to $D(s)$. Equation (3.5) also holds absolutely in the sense that

$$\prod_{p<N}\left(\sum_{k\geq 0}\left|\frac{a(p^k)}{p^{ks}}\right|\right) = \sum_{n<N}\left|\frac{a(n)}{n^s}\right| + \sum_{n\geq N}{}^{*}\left|\frac{a(n)}{n^s}\right|, \tag{3.6}$$

since $D(s)$ converges absolutely. For the converse statement, since the product

$$\prod_{p}\left(\sum_{k\geq 0}\left|\frac{a(p^k)}{p^{ks}}\right|\right),$$

converges for $\mathrm{Re}(s) > 1+\alpha$ each factor is necessarily finite. That is, for each prime p the series $\sum_{k\geq 0}\frac{a(p^k)}{p^{ks}}$ converges absolutely in this region. Now fix an integer $N > 0$. Then Equation (3.6) holds. Taking $N \to \infty$ in Equation (3.6), the left-hand side converges by assumption. Therefore the right-hand sides does too. But the first sum on the right-hand side tends to

$$\sum_{n\geq 1}\left|\frac{a(n)}{n^s}\right|,$$

and the second sum is part of its tail. So the first sum must convergence hence defining an absolutely convergent Dirichlet series in $\mathrm{Re}(s) > 1+\alpha$, and the second sum must tend to zero. Lastly, if the $a(n)$ are completely multiplicative, then since $\mathrm{Re}(s) > 1+\alpha$, we have

$$\left|\frac{a(p)}{p^s}\right| \ll \left|\frac{1}{p^{s-\alpha}}\right| < \left|\frac{1}{p}\right| < 1.$$

So the formula for a geometric series gives

$$\prod_{p}\left(\sum_{k\geq 0}\frac{a(p^k)}{p^{ks}}\right) = \prod_{p}\left(\sum_{k\geq 0}\left(\frac{a(p)}{p^s}\right)^k\right) = \prod_{p}(1-a(p)p^{-s})^{-1}.$$

$\square$

Note that in Proposition 3.1.5, the requirement for a product to define an absolutely convergent Dirichlet series is that the series defining the factors in the product must be absolutely convergent. Thankfully, this is always the case for geometric series. Now suppose $D(s)$ is a Dirichlet series that has the product expression

$$D(s) = \prod_p (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1} \cdots (1 - \alpha_d(p)p^{-s})^{-1}.$$

We call this product the **Euler product** of $D(s)$, and it is said to be of **degree** d. In Proposition 3.1.5, complete multiplicativity of the coefficients is enough to guarantee that $D(s)$ has an Euler product of degree 1, but in general $D(s)$ will admit an Euler product of degree $d > 1$ if the coefficients are only multiplicative but satisfy additional properties like a recurrence relation. When we come across Dirichlet series whose coefficients are multiplicative or we are given an Euler product we will use Proposition 3.1.5 without mention as this is common practice in the literature. Lastly, if $D(s)$ has an Euler product then for any $N \geq 1$ we let $D^{(N)}(s)$ denote the Dirichlet series with the factors $p \mid N$ in the Euler product removed. That is,

$$D^{(N)}s = D(s) \prod_{p \mid N} (1 - \alpha_1(p)p^{-s})(1 - \alpha_2(p)p^{-s}) \cdots (1 - \alpha_d(p)p^{-s}).$$

L-functions

We are now ready to discuss Selberg class L-functions. In the following, we will denote an L-functions by $L(s, f)$, for the moment, f will carry no formal meaning and is only used to suggest that the L-function is attached to some interesting arithmetic object f. When we discuss specific L-functions, f will carry a formal meaning. In contexts where the arithmetic data f is unimportant, we drop the f and write $L(s)$ instead. An L-**series** $L(s, f)$ is a Dirichlet series

$$L(s, f) = \sum_{n \geq 1} \frac{a_f(n)}{n^s},$$

where the $a_f(n) \in \mathbb{C}$ are coefficeints usually attached to some arithmetic object f. If $L(s, f)$ admits meromorphic continuation, we call the continuation, also denoted $L(s, f)$, an L-**function**. We say that an L-function $L(s, f)$ belongs to the **Selberg class** if the following are satisfied:

(i) For $\mathrm{Re}(s) > 1$, $L(s, f)$ has the degree d Euler product

$$L(s, f) = \prod_p (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1} \cdots (1 - \alpha_d(p)p^{-s})^{-1},$$

with $a_f(1) = 1$, $a_f(n), \alpha_i(p) \in \mathbb{C}$, and $a_f(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$. We call

$$L_p(s, f) = (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1} \cdots (1 - \alpha_d(p)p^{-s})^{-1},$$

the **local factor** at p, and the $\alpha_i(p)$ are called the **local roots** or **local parameters** of $L(s, f)$ at p.

(ii) There exists a factor

$$\gamma(s, f) = \pi^{-\frac{ds}{2}} \prod_{i=1}^{d} \Gamma\left(\frac{s + \kappa_i}{2}\right),$$

with $\kappa_i \in \mathbb{C}$ that are either real or appear in conjugate pairs. We also require $\mathrm{Re}(\kappa_i) \geq -1$. The κ_i are called the **local parameters at infinity** of $L(s, f)$.

(iii) An integer $q(f) \geq 1$ called the **conductor** such that $\alpha_i(p) \neq 0$ for all prime p such that $p \nmid q(f)$. If $p \mid q(f)$, then we say p is **ramified** for $L(s, f)$ and **unramified** otherwise.

(iv) We call the function

$$\Lambda(s,f) = q(f)^{\frac{s}{2}}\gamma(s,f)L(s,f),$$

the **completed** L-function of $L(s,f)$. It must satisfy the functional equation

$$\Lambda(s,f) = \varepsilon(f)\Lambda(1-s,\overline{f}),$$

where $\varepsilon(f)$ is a complex number with $|\varepsilon(f)| = 1$ called the **root number** of $L(s,f)$, and $\overline{f}$ is an object associated to f called the **dual** of f such that $L(s,\overline{f})$ satisfies $a_{\overline{f}}(n) = \overline{a_f(n)}$, $\gamma(s,\overline{f}) = \gamma(s,f)$, and $q(\overline{f}) = q(f)$ for $L(s,\overline{f})$. We call $L(s,\overline{f})$ the **dual** of $L(s,f)$. If $\overline{f} = f$ in the functional equation we say $L(s,f)$ is **self-dual**.

(v) $L(s,f)$ admits meromorphic continuation to $\mathbb{C}$ with at most pole at $s=1$, and must be of order 1 (see Appendix B.5) after clearing the polar divisors.

Suppose we are given two L-functions $L(s,f)$ and $L(s,g)$. We define the L-series $L(s,f\times g)$ by

$$L(s,f\times g) = \sum_{n\geq 1}\frac{a_f(n)a_g(n)}{n^s}.$$

We say $L(s,f\otimes g)$ is the **Rankin-Selberg convolution** of $L(s,f)$ and $L(s,g)$ if there exists an L-series $L(s,f\otimes g)$ that factors as a product of L-series one of which is $L(s,f\times g)$. In the case $g=f$, we call $L(s,f\otimes f)$ the **Rankin-Selberg square** of $L(s,f)$. Now suppose $L(s,f)$ and $L(s,g)$ are Selberg class L-functions with degree d and e Euler products, local parameters κ_i and ν_i, and local roots $\alpha_i(p)$ and $\beta_j(p)$ respectively. We say that the Rankin-Selberg convolution $L(s,f\otimes g)$ belongs to the **Selberg class** if it satisfies properties (i)-(v) of Selberg class L-functions with degree de Euler product and with the following adjustments:

(i) It's Euler product takes the form

$$L(s,f\otimes g) = \prod_{p\nmid q(f)q(g)} L_p(s,f\otimes g)\prod_{p\mid q(f)q(g)} H_p(p^{-s}),$$

where

$$L_p(s,f\otimes g) = \prod_{\substack{1\leq i\leq d\\ 1\leq j\leq e}}\left(1-\alpha_i(p)\overline{\beta_j(p)}\right)^{-1} \quad \text{and} \quad H_p(p^{-s}) = \prod_{1\leq i\leq de}(1-\gamma_i(p)p^{-s}),$$

for some $\gamma_i(p)\in\mathbb{C}$ with $|\gamma_i(p)| < p$.

(ii) The gamma factor takes the form

$$\gamma(s,f\otimes g) = (\pi)^{-\frac{des}{2}}\prod_{i,j}\Gamma\left(\frac{s+\mu_{i,j}}{2}\right),$$

where the local parameter at infinity $\mu_{i,j}$ corresponding to (κ_i,ν_j) satisfies the addition bounds $\mathrm{Re}(\mu_{i,j}) \leq \mathrm{Re}(\kappa_i)+\mathrm{Re}(\nu_j)$ and $|\mu_{i,j}| \leq |\kappa_i| + |\nu_j|$.

(iii) The conductor $q(f\otimes g)$ divides $q(f)^d q(g)^e$.

(iv) $L(s,f\otimes g)$ has a pole at $s=1$ if $g=f$.

An additional comment is in order. If an L-function has a functional equation of shape $s \to 1-s$, then the **critical strip** is the strip
$$\left\{ s \in \mathbb{C} : \left| \mathrm{Re}(s) - \frac{1}{2} \right| \leq \frac{1}{2} \right\}.$$
This is precisely the region where we cannot determine the value of the L-function from its representation as a Dirichlet series using the functional equation. It turns out that much of the important information about $L(s,f)$ is contained inside of the critical strip. The **critical line** is the vertical line that bisects the critical strip. So the critical line is $\mathrm{Re}(s) = \frac{1}{2}$ as displayed in Figure 3.1.

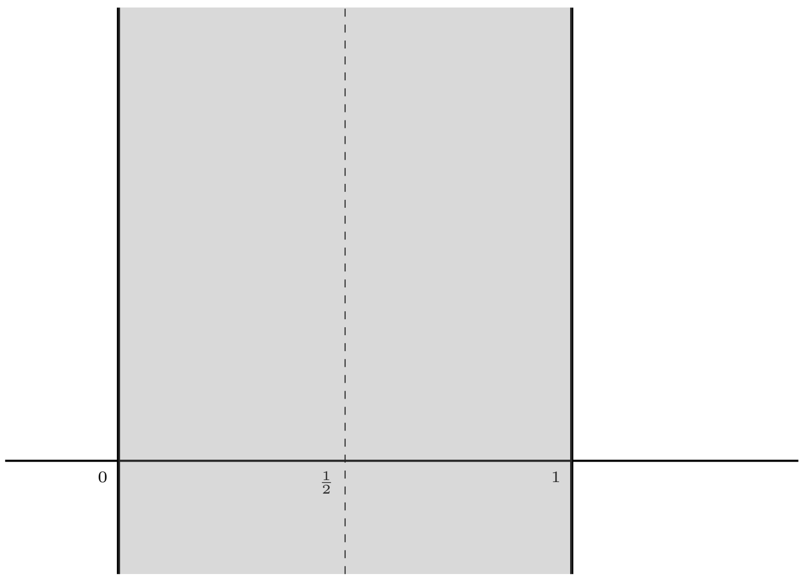

Figure 3.1: The critical strip and critical line.

3.2 The Riemann Zeta Function

The Definition & Euler Product of ζ(s)

The **Riemann zeta function** or simply the **zeta function** $\zeta(s)$ is defined as an L-series:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

This is the prototypical example of a Dirichlet series as all the coefficients are 1. Our main goal is to show that $\zeta(s)$ belongs to the Selberg class. As the coefficients are trivially polynomially bounded, $\zeta(s)$ is locally absolutely uniformly convergent for $\mathrm{Re}(s) > 1$. Also note that $\zeta(s)$ is necessarily nonzero in this region. Determining the Euler product is also an easy matter. As the coefficients are obviously completely multiplicative, we have the degree 1 Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

in this region as well. The local factor at p is $(1-p^{-s})^{-1}$ with local root 1. In particular, we have shown that the zeta function satisfies property (i) of the Selberg class and we package this into a theorem:

Theorem 3.2.1. *For* $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

is locally absolutely uniformly convergent with degree 1 *Euler product.*

The Integral Representation of $\zeta(s)$: Part I

Riemann's ingenious insight was to analytically continue $\zeta(s)$. By this, he sought to find a representation of $\zeta(s)$ defined on a larger region than $\mathrm{Re}(s) > 1$. This is the approach we will take, and the argument follows the same line of reasoning as that of Riemann. We consider the gamma function $\Gamma\left(\frac{s}{2}\right)$:

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-x} x^{\frac{s}{2}} \,\frac{dx}{x}.$$

Remark 3.2.1. *We have chosen to express the gamma function in terms of the measure* $\frac{dx}{x}$ *instead of* dx*. This is a tactical change for two reasons. The first is that* $\frac{dx}{x}$ *is invariant under the change of variables* $x \to Cx$ *for any constant* C*. The second is that under the change of variables* $x \to \frac{1}{x}$ *we have* $\frac{dx}{x} \to -\frac{dx}{x}$ *but the bounds of integration are also flipped. So we may leave the measure invariant provided we dont flip the bounds of integration. These types of change of variables are essential in the study of L-functions which motivates the use of this measure.*

Performing the change of variables $x \to \pi n^2 x$ for fixed $n \geq 1$ yields

$$\Gamma\left(\frac{s}{2}\right) = \pi^{\frac{s}{2}} n^s \int_0^\infty e^{-\pi n^2 x} x^{\frac{s}{2}} \,\frac{dx}{x}. \tag{3.7}$$

Dividing by $\pi^{\frac{s}{2}} n^s$ and summing over $n \geq 1$, we see that for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \sum_{n \geq 1} \int_0^\infty e^{-\pi n^2 x} x^{\frac{s}{2}} \,\frac{dx}{x} \\
&= \int_0^\infty \sum_{n \geq 1} e^{-\pi n^2 x} x^{\frac{s}{2}} \,\frac{dx}{x} && \text{DCT} \\
&= \int_0^\infty \omega(x) x^{\frac{s}{2}} \,\frac{dx}{x},
\end{aligned}$$

where we set

$$\omega(x) = \sum_{n \geq 1} e^{-\pi n^2 x}.$$

Therefore we have an integral representation

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty \omega(x) x^{\frac{s}{2}} \,\frac{dx}{x}. \tag{3.8}$$

This was essentially Riemann's insight: rewrite the zeta function in terms of a Mellin transform. Unfortunately, we cannot proceed until we understand $\omega(x)$. So we will make a slight detour and come back to the integral representation after.

Jacobi's Theta Function $\vartheta(s)$

Jacobi's theta function $\vartheta(s)$ is defined for $\mathrm{Re}(s) > 0$ by

$$\vartheta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s} = 1 + 2\sum_{n\geq 1} e^{-\pi n^2 s}.$$

It is locally absolutely uniformly convergent in this region by the ratio test. It's relation to $\omega(s)$ is the identity

$$\omega(s) = \frac{\vartheta(s)-1}{2}. \tag{3.9}$$

The essential fact about Jacobi's theta function we will need is the **transformation law for Jacobi's theta funtion** that was known to Riemann:

Theorem 3.2.2 (Transformation law for Jacobi's theta function). *For* $\mathrm{Re}(s) > 0$*,*

$$\vartheta(s) = \frac{1}{\sqrt{s}}\vartheta\left(\frac{1}{s}\right).$$

Proof. By the identity theorem it suffices to prove this on a set containg a limit point. We will prove this on the right-half of the real line, so take s real with $s > 0$. Set $f(x) = e^{-\pi x^2 s}$. Then $f(x)$ is a Schwarz function. We compute its Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx = \int_{-\infty}^{\infty} e^{-\pi x^2 s}e^{-2\pi itx}\,dx = \int_{-\infty}^{\infty} e^{-\pi(x^2 s+2itx)}\,dx.$$

Making the change of variables $x \to \frac{x}{\sqrt{s}}$, the last integral above becomes

$$\frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} e^{-\pi\left(x^2+\frac{2itx}{\sqrt{s}}\right)}.$$

Complete the square in the exponent by noticing

$$-\pi\left(x^2+\frac{2itx}{\sqrt{s}}\right) = -\pi\left(\left(x+\frac{it}{\sqrt{s}}\right)^2+\frac{t^2}{s^2}\right).$$

Taking exponentials, this implies that the previous integral is equal to

$$\frac{e^{-\frac{\pi t^2}{s}}}{\sqrt{s}}\int_{-\infty}^{\infty} e^{-\pi\left(x+\frac{it}{\sqrt{s}}\right)^2}\,dx.$$

We now treat this last integral as a complex integral. That is,

$$\int_{-\infty}^{\infty} e^{-\pi\left(x+\frac{it}{\sqrt{s}}\right)^2}\,dx = \int_{\mathrm{Im}(z)=0} e^{-\pi\left(z+\frac{it}{\sqrt{s}}\right)^2}\,dz = \int_{\mathrm{Im}(z)=\frac{t}{\sqrt{s}}} e^{-\pi z^2}\,dz, \tag{3.10}$$

where in the last equality we have made the change of variables $z \to z - \frac{it}{\sqrt{s}}$. Now fix $T > 0$ and let R_T be the positively oriented rectangle bounded by the lines $\mathrm{Im}(z) = 0$, $\mathrm{Im}(z) = \frac{t}{\sqrt{s}}$, $\mathrm{Re}(z) = -T$ and $\mathrm{Re}(z) = T$. Consider

$$\lim_{T\to\infty}\int_{R_T} e^{-\pi z^2}\,dz.$$

On the one hand, the residue theorem implies that the integral is the sum of a $2\pi i$ multiple of the residues in the rectangle R_T and so the limit is the sum of a $2\pi i$ multiple of the residues in the strip bounded by $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = \frac{t}{\sqrt{s}}$. Since $e^{-\pi\left(z+\frac{it}{\sqrt{s}}\right)^2}$ is entire, this is zero. On the other hand, we can decompose the integral as

$$\int_{\mathrm{Im}(z)=0} e^{-\pi z^2}\,dz + \lim_{T\to\infty}\int_0^{\frac{t}{\sqrt{s}}} e^{-\pi(iz+T)^2}\,dz - \int_{\mathrm{Im}(z)=\frac{t}{\sqrt{s}}} e^{-\pi z^2}\,dz - \lim_{T\to\infty}\int_0^{\frac{t}{\sqrt{s}}} e^{-\pi(iz-T)^2}\,dz.$$

Since $(iz\pm T)^2 \ll_z T^2$, the integrands of the second and fourth terms decay to zero as $T\to\infty$. Hence the corresponding limits of integrals is zero. Putting these two remarks together shows

$$\int_{\mathrm{Im}(z)=\frac{t}{\sqrt{s}}} e^{-\pi z^2}\,dz = \int_{\mathrm{Im}(z)=0} e^{-\pi z^2}\,dz = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx. \tag{3.11}$$

So back to the integral at hand, Equations (3.10) and (3.11) together imply

$$\frac{e^{-\frac{\pi t^2}{s}}}{\sqrt{s}}\int_{-\infty}^{\infty} e^{-\pi\left(x+\frac{it}{\sqrt{s}}\right)^2}\,dx = \frac{e^{-\frac{\pi t^2}{s}}}{\sqrt{s}}\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = \frac{e^{-\frac{\pi t^2}{s}}}{\sqrt{s}},$$

where the last equality follows because the last integral above is 1 since it is the Gaussian integral (see Appendix D.1). The Poisson summation formula and the identity theorem together finish the proof. □

Take note that the key ingredient in the proof was the Poisson summation formula. This is typical of a larger relaity when one needs to prove transformation laws. Also, the method of changing the line of integration from $\mathrm{Im}(z) = \frac{t}{\sqrt{s}}$ to $\mathrm{Im}(z) = 0$ works in a much more general setting and is a very useful analytic technique called **shifting the line of integration**:

Method 3.2.1 (Shifting the line of integration). Suppose we are given an integral

$$\int_{\mathrm{Re}(z)=a} f(z)\,dz \quad \text{or} \quad \int_{\mathrm{Im}(z)=a} f(z)\,dz,$$

and some real b with $b < a$ in the first case and $b > a$ in the second case. Also suppose the following conditions hold corresponding to each case:

(i) f is meromorphic on a strip containing $\mathrm{Re}(z) = a, b$ or $\mathrm{Im}(z) = a, b$ respectively.

(ii) f is holomorphic about $\mathrm{Re}(z) = a, b$ or $\mathrm{Im}(z) = a, b$ respectively.

(iii) $f(z) \to 0$ as $\mathrm{Im}(z) \to \infty$ or $f(z) \to 0$ as $\mathrm{Re}(z) \to \infty$ respectively.

To collect these cases, let (a) stand for the line $\mathrm{Re}(z) = a$ or $\mathrm{Im}(z) = a$ respectively with positive orientation. Then the line of integration (a) can be shifted to the line of integration (b) with the possible addition of residues. Take a rectangle R_T given positive orientation and with its edges on (a) and (b) respectively and consider

$$\lim_{T\to\infty}\int_{R_T} f(z)\,dz.$$

On the one hand, the residue theorem implies the integral is a sum of a $2\pi i$ multiple of the residues r_i in the rectangle R_T and hence the limit is a sum of a $2\pi i$ multiple of the residues in the strip bounded by (a) and (b). Denote the corresponding set of poles inside this strip by P. On the other hand, the integral can be decomposed into a sum of four integrals along the edges of R_T and by taking the limit the edges other than (a) and (b) will tend to zero because of the assumptions on f. The remaining two pieces is the difference between the integral along (a) and (b). So in total,

$$\int_{(a)} f(z)\,dz = \int_{(b)} f(z)\,dz + 2\pi i \sum_{\rho\in P} \operatorname*{Res}_{z=\rho} f(z).$$

A particular application of interest is when the integral in question is real and over the entire real line, the integrand is entire as a complex function, and one is trying to shift the line of integration of the complexified integral to $\mathrm{Im}(z) = a$. In this case, shifting the line of integration amounts to making the change of variables $x \to x - ia$ without affecting the initial line of integration. For example, in the proof of the transformation law for Jacobi's theta function we can make the change of variables $x \to \frac{x}{\sqrt{s}} - \frac{it}{\sqrt{s}}$ without affecting the line of integration. We will appeal to this application often when proving transformation laws and one should become familiar with it. This completes our interest in Jacobi's theta function.

The Integral Representation of $\zeta(s)$: Part II

Returning to the zeta function, we split the integral in Equation (3.8) into two pieces

$$\int_0^\infty \omega(x)x^{\frac{s}{2}}\,\frac{dx}{x} = \int_0^1 \omega(x)x^{\frac{s}{2}}\,\frac{dx}{x} + \int_1^\infty \omega(x)x^{\frac{s}{2}}\,\frac{dx}{x}. \tag{3.12}$$

Since $\omega(x)$ has exponential decay to zero as $x \to \infty$, the second piece is locally absolutely uniformly bounded for $\mathrm{Re}(s) > 1$ by Method 1.4.1. Hence it defines an analytic function there. The idea now is to rewrite the first piece in the same form and symmetrize the result as much as possible. We being by performing a change of variables $x \to \frac{1}{x}$ to the first piece to obtain

$$\int_1^\infty \omega\left(\frac{1}{x}\right)x^{-\frac{s}{2}}\,\frac{dx}{x}$$

Now the transformation law for $\vartheta(x)$ and Equation (3.9) together imply

$$\omega\left(\frac{1}{x}\right) = \frac{\vartheta\left(\frac{1}{x}\right) - 1}{2} = \frac{\sqrt{x}\vartheta(x) - 1}{2} = \frac{\sqrt{x}(2\omega(x)+1) - 1}{2} = \sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}. \tag{3.13}$$

Equation (3.13) gives the first equality in the following chain:

$$\begin{aligned}
\int_1^\infty \omega\left(\frac{1}{x}\right)x^{-\frac{s}{2}}\,\frac{dx}{x} &= \int_1^\infty \left(\sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}\right)x^{-\frac{s}{2}}\,\frac{dx}{x} \\
&= \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\,\frac{dx}{x} + \int_1^\infty \frac{x^{\frac{1-s}{2}}}{2}\,\frac{dx}{x} - \int_1^\infty \frac{x^{-\frac{s}{2}}}{2}\,\frac{dx}{x} \\
&= \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\,\frac{dx}{x} + \frac{1}{1-s} - \frac{1}{s} \\
&= \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\,\frac{dx}{x} - \frac{1}{s(1-s)}.
\end{aligned}$$

Substituting this result back into Equation (3.12) with Equation (3.8) yields the following result:

Theorem 3.2.3. *For* $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left[-\frac{1}{s(1-s)} + \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\frac{dx}{x} + \int_1^\infty \omega(x)x^{\frac{s}{2}}\frac{dx}{x}\right].$$

This integral representation will give analytic continuation. To see this, first observe that we know everything outside the backets is entire. Everything inside the brackets except for the first integral is analytic for $\mathrm{Re}(s) > 1$. Thus the first integral must be analytic in this region too. Since the two integrals are interchanged as $s \to 1-s$ and the rational term $-\frac{1}{s(1-s)}$ is invariant, the right-hand side is analytic for $\mathrm{Re}(s) < 0$. This gives the analytic continuation of $\zeta(s)$ to the region

$$\left\{s \in \mathbb{C} : \left|\mathrm{Re}(s) - \frac{1}{2}\right| > \frac{1}{2}\right\}.$$

The Functional Equation, Critical Strip & Residue of $\zeta(s)$

An immediate consequence of the symmetry of the integral representation is the functional equation:

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}}\zeta(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\pi^{\frac{1-s}{2}}}\zeta(1-s).$$

We identify the gamma factor as

$$\gamma(s,\zeta) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right),$$

with $\kappa = \frac{s}{2}$ the only local parameter at infinity. Clearly it satisfies the required bounds. The conductor is $q(\zeta) = 1$ so no primes ramify. The completed zeta function is

$$\Lambda(s,\zeta) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

with functional equation

$$\Lambda(s,\zeta) = \Lambda(1-s,\zeta).$$

This is the functional equation of $\zeta(s)$ and in this case is just a reformulation of the previous functional equation. From it we find that the root number is $\varepsilon(\zeta) = 1$ and that ζ is self-dual. Altogether, we have shown that ζ satisfies properties (ii)-(iv) of the Selberg class.
Having obtained the funtional equation, we now use the integral representation to obtain meromorphic continuation of $\zeta(s)$ inside the critical strip. From the integral representation, we have

$$\zeta(s) = \frac{1}{\gamma(s,\zeta)}\left[-\frac{1}{s(1-s)} + \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\frac{dx}{x} + \int_1^\infty \omega(x)x^{\frac{s}{2}}\frac{dx}{x}\right]. \tag{3.14}$$

To get continuation inside of the critical strip, we argue that the two integrals are locally absolutely uniformly bounded for $|\mathrm{Re}(s)-\frac{1}{2}| \leq \frac{1}{2}$. This follows by Method 1.4.1 (we could have gotten this continuation earlier but we didn't need it until now). If we assume $s \neq 0, 1$, then the analytic continuation to the inside of the critical strip follows since the fractional term $\frac{1}{s(1-s)}$ is holomorphic there. The cases $s = 0, 1$ require separate inspection. When $s = 0$, $\gamma(s,\zeta)$ has a simple pole coming from the gamma factor, and therefore its reciprocal has a simple zero. This cancels the corresponding simple pole of $\frac{1}{s(1-s)}$ so that $\zeta(s)$ has a

removable singularity and thus is holomorphic at $s = 0$. At $s = 1$, $\gamma(s, \zeta)$ is nonzero, and so $\zeta(s)$ has a simple pole. Therefore $\zeta(s)$ has meromorphic continuation to all of $\mathbb{C}$ with a simple pole at $s = 1$.
We can now show that the order of $\zeta(s)$ is 1 and conclude that it satisfies property (v) of the Selberg class, and is therefore an L-function belonging to the Selberg class. As there is only a simple pole at $s = 1$, multiply by $(s - 1)$ to clear the polar divisor. Now the integral in the integral representation is absolutely bounded, so computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1 and $\mathrm{Re}(s)$ is bounded, we have

$$\frac{1}{\gamma(s,\zeta)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}, \tag{3.15}$$

for any $\varepsilon > 0$. So the reciprocal of the gamma factor is of the same order. Then Equations (3.14) and (3.15) together imply

$$(s-1)\zeta(s) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So $(s - 1)\zeta(s)$ is of order 1, and thus $\zeta(s)$ is as well after removing the polar factor. Having shown the analytic continuation of $\zeta(s)$, and verified that it belongs to the Selberg class, there is only one thing left to do. This is to compute the residue of $\zeta(s)$ at $s = 1$:

Proposition 3.2.1.

$$\operatorname*{Res}_{s=1} \zeta(s) = 1.$$

Proof. The only term in the integral representation of $\zeta(s)$ contributing to the pole is $-\frac{1}{\gamma(s,\zeta)}\frac{1}{s(1-s)}$. Observe

$$\lim_{s\to 1} \frac{1}{\gamma(s,\zeta)} = \lim_{s\to 1} \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} = 1,$$

because $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Therefore

$$\operatorname*{Res}_{s=1} \zeta(s) = \operatorname*{Res}_{s=1} -\frac{1}{\gamma(s,\zeta)}\frac{1}{s(1-s)} = \operatorname*{Res}_{s=1} -\frac{1}{s(1-s)} = \lim_{s\to 1} -\frac{(s-1)}{s(1-s)} = 1.$$

□

We summarize all of our work into the following theorem:

Theorem 3.2.4. *$\zeta(s)$ is a Selberg class L-function. It admits meromorphic continuation to $\mathbb{C}$ via the integral representation*

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left[-\frac{1}{s(1-s)} + \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\frac{dx}{x} + \int_1^\infty \omega(x)x^{\frac{s}{2}}\frac{dx}{x}\right],$$

with functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Lambda(s,\zeta) = \Lambda(1-s,\zeta),$$

and there is a simple pole at $s = 1$ of residue 1.

Lastly, we note that by virtue of the functional equation we can also compute $\zeta(0)$ as well. Indeed, since $\mathrm{Res}_{s=1}\,\zeta(s) = 1$, we have

$$\lim_{s\to 1}(s-1)\Lambda(s,\zeta) = \operatorname*{Res}_{s=1}\zeta(s)\lim_{s\to 1}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) = 1.$$

In other words, $\Lambda(s, \zeta)$ has a simple pole at $s = 1$ with residue 1 too. Since the completed zeta function is completely symmetric as $s \to 1 - s$, it has a simple pole at $s = 0$ with residue 1. Hence

$$1 = \lim_{s\to 1}(s-1)\Lambda(1-s,\zeta) = \operatorname*{Res}_{s=1}\Gamma\left(\frac{1-s}{2}\right)\lim_{s\to 1}\pi^{-\frac{1-s}{2}}\zeta(1-s) = -2\zeta(0),$$

because $\mathrm{Res}_{s=0}\,\Gamma(s) = 1$. Therefore $\zeta(0) = -\frac{1}{2}$.

3.3 Dirichlet L-functions

The Definition & Euler Product of $L(s,\chi)$

To every Dirichlet character χ there is an associated L-function. Throughout we will let m denote the modulus and q the conductor of χ respectively. The **Dirichlet L-function** $L(s,\chi)$ attached to the Dirichlet character χ is defined as an L-series:

$$L(s,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n^s}.$$

Since $\chi(n)=0$ if $(n,m)>1$, the above sum can be restricted to all integers relatively prime to m. We first obtain convergence in a half-plane. As $|\chi(n)|\ll 1$, $L(s,\chi)$ is locally absolutely uniformly convergent for $\mathrm{Re}(s)>1$ just as is the case for $\zeta(s)$. Because χ is completely multiplicative we also have the degree 1 Euler product,

$$L(s,\chi)=\prod_p(1-\chi(p)p^{-s})^{-1}=\prod_{p\nmid m}(1-\chi(p)p^{-s})^{-1},$$

in this region as well. The last equality holds because if $p\mid m$ we have $\chi(p)=0$. So for $p\mid m$, the local factor at p is 1 with local root 0. For $p\nmid m$ the local factor at p is $(1-\chi(p)p^{-s})^{-1}$ with local root $\chi(p)$. Now if χ is induced by $\widetilde{\chi}$, then $\chi(p)=\widetilde{\chi}(p)$ if $p\nmid q$ and $\chi(p)=0$ if $p\mid m$ so that

$$L(s,\chi)=\prod_{p\nmid m}(1-\widetilde{\chi}(p)p^{-s})^{-1}=\prod_p(1-\widetilde{\chi}(p)p^{-s})^{-1}\prod_{p\mid m}(1-\widetilde{\chi}(p)p^{-s})=L(s,\widetilde{\chi})\prod_{p\mid m}(1-\widetilde{\chi}(p)p^{-s}). \tag{3.16}$$

Therefore $L(s,\chi)$ belongs to the Selberg class if and only if $L(s,\widetilde{\chi})$ does. So we may assume χ is primitive. We may assume further that $q>1$ because if not χ is principal which means $\widetilde{\chi}$ is trivial so that $L(s,\widetilde{\chi})=\zeta(s)$, and this L-function already belongs to the Selberg class. Our main goal is now to show that $L(s,\chi)$ is a Selberg class L-function when χ is primitive and $q>1$. We have already shown that $L(s,\chi)$ satisfies property (i) of the Selberg class and we package this into a theorem:

Theorem 3.3.1. *Let χ be a primitive Dirichlet character with conductor $q>1$. For $\mathrm{Re}(s)>1$,*

$$L(s,\chi)=\prod_p(1-\chi(p)p^{-s})^{-1}=\prod_{p\nmid q}(1-\chi(p)p^{-s})^{-1},$$

is locally absolutely uniformly convergent with degree 1 Euler product.

The Integral Representation of $L(s,\chi)$: Part I

To find the integral representation for $L(s,\chi)$, we argue in a manner similar to $\zeta(s)$. However, the following will depend if χ is even or odd, so to handle both cases simultaneously let $\mathfrak{a}=0,1$ according to whether χ is even or odd. In particular, $\chi(-1)=(-1)^{\mathfrak{a}}$. Note that $\mathfrak{a}$ takes the same value for both χ and $\overline{\chi}$. Making the substitution $s\to s+\mathfrak{a}$ in Equation (3.7) and multiplying by $\chi(n)$ yields

$$\chi(n)\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)=\pi^{\frac{s+\mathfrak{a}}{2}}n^s\int_0^\infty\chi(n)n^{\mathfrak{a}}e^{-\pi n^2x}x^{\frac{s+\mathfrak{a}}{2}}\frac{dx}{x},$$

after moving the $n^{\mathfrak{a}}$ on the inside of the integral. Dividing by $\pi^{\frac{s+\mathfrak{a}}{2}}n^s$ and summing over $n \geq 1$, we see that for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
\pi^{-\frac{s+\mathfrak{a}}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi) &= \sum_{n\geq 1}\int_0^\infty \chi(n)n^{\mathfrak{a}}e^{-\pi n^2 x}x^{\frac{s+\mathfrak{a}}{2}}\,\frac{dx}{x} \\
&= \int_0^\infty \sum_{n\geq 1}\chi(n)n^{\mathfrak{a}}e^{-\pi n^2 x}x^{\frac{s+\mathfrak{a}}{2}}\,\frac{dx}{x} \qquad \text{DCT} \\
&= \int_0^\infty \omega_\chi(x)x^{\frac{s+\mathfrak{a}}{2}}\,\frac{dx}{x},
\end{aligned}$$

where we set

$$\omega_\chi(x) = \sum_{n\geq 1}\chi(n)n^{\mathfrak{a}}e^{-\pi n^2 x}.$$

Therefore we have an integral representation

$$L(s,\chi) = \frac{\pi^{\frac{s+\mathfrak{a}}{2}}}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}\int_0^\infty \omega_\chi(x)x^{\frac{s+\mathfrak{a}}{2}}\,\frac{dx}{x}, \tag{3.17}$$

and just like $\zeta(s)$ we need to find a transformation law for $\omega_\chi(x)$ before we can proceed.

The Dirichlet Theta Function $\vartheta_\chi(s)$

The **Dirichlet theta function** $\vartheta_\chi(s)$, attached to the character χ, is defined for $\mathrm{Re}(s) > 0$ by

$$\vartheta_\chi(s) = \sum_{n\in\mathbb{Z}}\chi(n)n^{\mathfrak{a}}e^{-\pi n^2 s} = 2\sum_{n\geq 1}\chi(n)n^{\mathfrak{a}}e^{-\pi n^2 s}.$$

It is locally absolutely uniformly convergent in this region by the ratio test. Notice that the term corresponding to $n = 0$ vanishes because $\chi(0) = 0$, and $\chi(n)n^{\mathfrak{a}} = \chi(-n)(-n)^{\mathfrak{a}}$ so that the n-th and $(-n)$-th terms agree. Therefore the relationship between the twisted theta function and $\omega_\chi(s)$ is

$$\omega_\chi(s) = \frac{\vartheta_\chi(s)}{2}. \tag{3.18}$$

Remark 3.3.1. *Equation* (3.18) *is a slightly less complex relationship that Equation* (3.9). *This is because assuming $q > 1$ means $\chi(0) = 0$.*

The essential fact about the twisted theta function we will need is the following transformation law similar to the transformation law for Jacobi's theta function:

Theorem 3.3.2. *Let χ be a primitive Dirichlet character with conductor $q > 1$. For* $\mathrm{Re}(s) > 0$,

$$\vartheta_\chi(s) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}(qs)^{\frac{1}{2}+\mathfrak{a}}}\vartheta_{\overline{\chi}}\left(\frac{1}{q^2 s}\right).$$

Proof. By the identity theorem it suffices to prove this on a set containg a limit point. We will prove this on the right-half of the real line, so take s real with $s > 0$. Since χ is q-periodic, we can write

$$\vartheta_\chi(s) = \sum_{a \pmod q}\chi(a)\sum_{m\in\mathbb{Z}}(mq+a)^{\mathfrak{a}}e^{-\pi(mq+a)^2 s}.$$

Set $f(x) = (xq + a)^{\mathfrak{a}} e^{-\pi(xq+a)^2 s}$. Then $f(x)$ is a Schwarz function. We compute its Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x}\, dx = \int_{-\infty}^{\infty} (xq + a)^{\mathfrak{a}} e^{-\pi(xq+a)^2 s} e^{-2\pi i t x}\, dx = \int_{-\infty}^{\infty} (xq + a)^{\mathfrak{a}} e^{-\pi((xq+a)^2 s + 2itx)}\, dx.$$

By performing the change of variables $x \to \frac{x}{q\sqrt{s}} - \frac{a}{q}$, the last integral above becomes

$$\frac{e^{\frac{2\pi i a t}{q}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \int_{-\infty}^{\infty} x^{\mathfrak{a}} e^{-\pi\left(x^2 + \frac{2itx}{q\sqrt{s}}\right)}\, dx.$$

Complete the square in the exponent by observing

$$-\pi\left(x^2 + \frac{2itx}{q\sqrt{s}}\right) = -\pi\left(\left(x + \frac{it}{q\sqrt{s}}\right)^2 + \frac{t^2}{q^2 s}\right).$$

Taking exponentials, this implies that the previous integral is equal to

$$\frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \int_{-\infty}^{\infty} x^{\mathfrak{a}} e^{-\pi\left(x + \frac{it}{q\sqrt{s}}\right)^2}\, dx.$$

The change of variables $x \to x - \frac{it}{q\sqrt{s}}$ is permitted without affecting the line of integration by viewing the integral as a complex integral, noting that the integrand is entire as a complex function, and shifting the line of integration. This gives

$$\frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \int_{-\infty}^{\infty} \left(x - \frac{it}{q\sqrt{s}}\right)^{\mathfrak{a}} e^{-\pi x^2}\, dx = \frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \int_{-\infty}^{\infty} \left(x + \frac{t}{iq\sqrt{s}}\right)^{\mathfrak{a}} e^{-\pi x^2}\, dx.$$

If $\mathfrak{a} = 0$, we obtain

$$\frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = \frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}}, \tag{3.19}$$

where the equality holds because the integral is 1 since it is the Gaussian integral (see Appendix D.1). If $\mathfrak{a} = 1$, then by direct computation

$$\int_{-\infty}^{\infty} x e^{-\pi x^2}\, dx = -\frac{1}{2\pi} e^{-\pi x^2}\Big|_{-\infty}^{\infty} = 0,$$

and thus

$$\frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \int_{-\infty}^{\infty} \left(\frac{t}{iq\sqrt{s}}\right) e^{-\pi x^2}\, dx = \frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \left(\frac{t}{iq\sqrt{s}}\right) \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = \frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \left(\frac{t}{iq\sqrt{s}}\right), \tag{3.20}$$

where the last equality follows because the last integral is the Gaussian integral again. Since $\left(\frac{t}{iq\sqrt{s}}\right)^{\mathfrak{a}} = 1$ if $\mathfrak{a} = 0$, we can compactly express the right-hand sides of Equations (3.19) and (3.20) in the form

$$\frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \left(\frac{t}{iq\sqrt{s}}\right)^{\mathfrak{a}}.$$

The Poisson summation formula then implies

$$\begin{aligned}
\vartheta_\chi(s) &= \sum_{a \pmod q} \chi(a) \sum_{t \in \mathbb{Z}} \frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \left(\frac{t}{iq\sqrt{s}} \right)^{\mathfrak{a}} \\
&= \frac{1}{i^{\mathfrak{a}} q^{1+\mathfrak{a}} s^{\frac{1}{2}+\mathfrak{a}}} \sum_{a \pmod q} \chi(a) \sum_{t \in \mathbb{Z}} t^{\mathfrak{a}} e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}} \\
&= \frac{1}{i^{\mathfrak{a}} q^{1+\mathfrak{a}} s^{\frac{1}{2}+\mathfrak{a}}} \sum_{t \in \mathbb{Z}} t^{\mathfrak{a}} e^{-\frac{\pi t^2}{q^2 s}} \sum_{a \pmod q} \chi(a) e^{\frac{2\pi i a t}{q}} \\
&= \frac{1}{i^{\mathfrak{a}} q^{1+\mathfrak{a}} s^{\frac{1}{2}+\mathfrak{a}}} \sum_{t \in \mathbb{Z}} t^{\mathfrak{a}} e^{-\frac{\pi t^2}{q^2 s}} \tau(t, \chi) && \text{definition of } \tau(t,\chi),
\end{aligned}$$

and this holds for all $\mathrm{Re}(s) > 0$ by the identity theorem. Since χ is primitive, $\tau(t, \chi) = \overline{\chi}(t)\tau(\chi)$ for all $t \in \mathbb{Z}$ by Corollary 1.3.1. Therefore we have

$$\frac{1}{i^{\mathfrak{a}} q^{1+\mathfrak{a}} s^{\frac{1}{2}+\mathfrak{a}}} \sum_{t \in \mathbb{Z}} t^{\mathfrak{a}} e^{-\frac{\pi t^2}{q^2 s}} \tau(t, \chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}} q^{1+\mathfrak{a}} s^{\frac{1}{2}+\mathfrak{a}}} \sum_{t \in \mathbb{Z}} \overline{\chi}(t) t^{\mathfrak{a}} e^{-\frac{\pi t^2}{q^2 s}} = \frac{\tau(\chi)}{i^{\mathfrak{a}} q^{1+\mathfrak{a}} s^{\frac{1}{2}+\mathfrak{a}}} \vartheta_{\overline{\chi}} \left(\frac{1}{q^2 s} \right).$$

Recalling that $\varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{q}}$, we conclude

$$\vartheta_\chi(s) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}} (qs)^{\frac{1}{2}+\mathfrak{a}}} \vartheta_{\overline{\chi}} \left(\frac{1}{q^2 s} \right).$$

□

The most striking property about this transformation law is that it relates $\vartheta_\chi(s)$ and $\vartheta_{\overline{\chi}}(s)$. Regardless, we can now exploit it to analytically continue $L(s, \chi)$.

The Integral Representation of $L(s, \chi)$: Part II

Returning to $L(s, \chi)$, split the integral in Equation (3.17) into two pieces

$$\int_0^\infty \omega_\chi(x) x^{\frac{s+\mathfrak{a}}{2}} \frac{dx}{x} = \int_0^{\frac{1}{q}} \omega_\chi(x) x^{\frac{s+\mathfrak{a}}{2}} \frac{dx}{x} + \int_{\frac{1}{q}}^\infty \omega_\chi(x) x^{\frac{s+\mathfrak{a}}{2}} \frac{dx}{x}. \tag{3.21}$$

Since $\omega_\chi(x)$ has exponential decay to zero as $x \to \infty$, the second piece is locally absolutely uniformly bounded for $\mathrm{Re}(s) > 1$ by Method 1.4.1. Hence it defines an analytic function there. We now rewrite the first piece in the same form and symmetrize the result as much as possible. Start by performing a change of variables $x \to \frac{1}{q^2 x}$ to the first piece to obtain

$$q^{-(s+\mathfrak{a})} \int_{\frac{1}{q}}^\infty \omega_\chi \left(\frac{1}{q^2 x} \right) x^{-\frac{s+\mathfrak{a}}{2}} \frac{dx}{x}.$$

Now the transformation law for $\vartheta_\chi(x)$ and Equation (3.18) together imply

$$\begin{aligned}
\omega_\chi\left(\frac{1}{q^2x}\right) &= \frac{\vartheta_\chi\left(\frac{1}{q^2x}\right)}{2} \\
&= \frac{i^{\mathfrak{a}}(qx)^{\frac{1}{2}+\mathfrak{a}}}{\varepsilon_{\overline{\chi}}}\frac{\vartheta_{\overline{\chi}}(x)}{2} \\
&= \varepsilon_\chi(-i)^{\mathfrak{a}}(qx)^{\frac{1}{2}+\mathfrak{a}}\frac{\vartheta_{\overline{\chi}}(x)}{2} \quad \text{Proposition 1.3.3 and } \chi(-1)=(-1)^{\mathfrak{a}} \\
&= \frac{\varepsilon_\chi(qx)^{\frac{1}{2}+\mathfrak{a}}}{i^{\mathfrak{a}}}\frac{\vartheta_{\overline{\chi}}(x)}{2} \\
&= \frac{\varepsilon_\chi(qx)^{\frac{1}{2}+\mathfrak{a}}}{i^{\mathfrak{a}}}\omega_{\overline{\chi}}(x).
\end{aligned} \tag{3.22}$$

Equation (3.22) gives the first equality in the following chain:

$$\begin{aligned}
q^{-(s+\mathfrak{a})}\int_{\frac{1}{q}}^{\infty}\omega_\chi\left(\frac{1}{q^2x}\right)x^{-\frac{s+\mathfrak{a}}{2}}\frac{dx}{x} &= q^{-(s+\mathfrak{a})}\int_{\frac{1}{q}}^{\infty}\left(\frac{\varepsilon_\chi(qx)^{\frac{1}{2}+\mathfrak{a}}}{i^{\mathfrak{a}}}\omega_{\overline{\chi}}(x)\right)x^{-\frac{s+\mathfrak{a}}{2}}\frac{dx}{x} \\
&= \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1}{2}-s}\int_{\frac{1}{q}}^{\infty}\omega_{\overline{\chi}}(x)x^{\frac{(1-s)+\mathfrak{a}}{2}}\frac{dx}{x}.
\end{aligned}$$

Substituting this last expression back into Equation (3.21) with Equation (3.17) gives the following result:

Theorem 3.3.3. *Let χ be a primitive Dirichlet character with conductor $q > 1$. For* $\mathrm{Re}(s) > 1$,

$$L(s,\chi) = \frac{\pi^{\frac{s+\mathfrak{a}}{2}}}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}\left[\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1}{2}-s}\int_{\frac{1}{q}}^{\infty}\omega_{\overline{\chi}}(x)x^{\frac{(1-s)+\mathfrak{a}}{2}}\frac{dx}{x} + \int_{\frac{1}{q}}^{\infty}\omega_\chi(x)x^{\frac{s+\mathfrak{a}}{2}}\frac{dx}{x}\right].$$

This integral representation will give analytic continuation. Indeed, we know everything outside the brackets is entire and the latter of the two integrals inside the brackets is analytic for $\mathrm{Re}(s) > 1$. Thus the first integral inside the brackets must be analytic in this region too. Since the two integrals are interchanged as $s \to 1-s$, save for the term $\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1}{2}-s}$, the right-hand side is analytic for $\mathrm{Re}(s) < 0$. This gives the analytic continuation of $L(s,\chi)$ to the region

$$\left\{s \in \mathbb{C} : \left|\mathrm{Re}(s) - \frac{1}{2}\right| > \frac{1}{2}\right\}.$$

The Functional Equation & Critical Strip of $L(s,\chi)$

An immediate consequence of the symmetry of the integral representation is the functional equation:

$$q^{\frac{s}{2}}\frac{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}{\pi^{\frac{s+\mathfrak{a}}{2}}}L(s,\chi) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1-s}{2}}\frac{\Gamma\left(\frac{(1-s)+\mathfrak{a}}{2}\right)}{\pi^{\frac{(1-s)+\mathfrak{a}}{2}}}L(1-s,\overline{\chi}).$$

We identify the gamma factor as

$$\gamma(s,\chi) = \pi^{-\frac{s+\mathfrak{a}}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right),$$

with $\kappa = \frac{s+\mathfrak{a}}{2}$ the only local parameter at infinity. Clearly it satisfies the required bounds. The conductor is $q(\chi) = q$ and if p is an unramified prime then the local root is $\chi(p) \neq 0$. The completed L-function is

$$\Lambda(s,\chi) = q^{\frac{s}{2}}\pi^{-\frac{s+\mathfrak{a}}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi),$$

with functional equation

$$\Lambda(s,\chi) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}\Lambda(1-s,\overline{\chi}).$$

From it we see that the root number is $\varepsilon(\chi) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}$ and that $L(s,\chi)$ has dual $L(s,\overline{\chi})$. In total, $L(s,\chi)$ satisfies properties (ii)-(iv) of the Selberg class.
We now analytically continue $L(s,\chi)$ inside the critical strip and therefore to all of $\mathbb{C}$. From the integral representation, we have

$$L(s,\chi) = \frac{1}{\gamma(s,\chi)}\left[\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1}{2}-s}\int_{\frac{1}{q}}^{\infty}\omega_{\overline{\chi}}(x)x^{\frac{(1-s)+\mathfrak{a}}{2}}\frac{dx}{x} + \int_{\frac{1}{q}}^{\infty}\omega_\chi(x)x^{\frac{s+\mathfrak{a}}{2}}\frac{dx}{x}\right]. \tag{3.23}$$

To get continuation inside the critical strip it suffices that the two integrals are locally absolutely uniformly bounded for $|\mathrm{Re}(s)-\frac{1}{2}| \leq \frac{1}{2}$. This follows by Method 1.4.1 (we could have deduced this continuation earlier but we didn't need it until now) and thus gives analytic continuation to the critical strip and hence to all of $\mathbb{C}$. In particular, we have shown that $L(s,\chi)$ has no poles.
All we are left to show is that $L(s,\chi)$ is of order 1 to conclude that it satisfies property (v) of the Selberg class and therefore is an L-function belonging to the Selberg class. Since $L(s,\chi)$ has no poles, we do not need to clear any polar divisors. As the integral in the representation is absolutely bounded, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1 and $\mathrm{Re}(s)$ is bounded, we have

$$\frac{1}{\gamma(s,\chi)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}, \tag{3.24}$$

for any $\varepsilon > 0$. So the reciprocal of the gamma factor is of the same order. Then Equations (3.23) and (3.24) together imply

$$L(s,\chi) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So $L(s,\chi)$ is of order 1. We summarize all of our work into the following theorem:

Theorem 3.3.4. *For any primitive Dirichlet character χ with conductor $q > 1$, $L(s,\chi)$ is a Selberg class L-function. It admits analytic continuation to $\mathbb{C}$ via the integral representation*

$$L(s,\chi) = \frac{\pi^{\frac{s+\mathfrak{a}}{2}}}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}\left[\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1}{2}-s}\int_{\frac{1}{q}}^{\infty}\omega_{\overline{\chi}}(x)x^{\frac{(1-s)+\mathfrak{a}}{2}}\frac{dx}{x} + \int_{\frac{1}{q}}^{\infty}\omega_\chi(x)x^{\frac{s+\mathfrak{a}}{2}}\frac{dx}{x}\right],$$

and with functional equation

$$q^{\frac{s}{2}}\pi^{-\frac{s+\mathfrak{a}}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi) = \Lambda(s,\chi) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}\Lambda(1-s,\overline{\chi}).$$

3.4 Hecke L-functions

It is time to start our investigation into the L-functions of Hecke eigenforms. Actually, we will only be interested in the L-functions of holomorphic primitive Hecke eigenform on $\Gamma_1(N)\backslash\mathbb{H}$ as these belong to the Selberg class. The theory can be extended to more general modular and Maass forms but some of the properies of Selberg class L-functions are lost or currently unknown.

The Definition & Euler Product of $L(s,f)$

Let $f \in \mathcal{S}_k(N,\chi)$ be a primitive Hecke eigenform and denote its Fourier series by

$$f(z) = \sum_{n\geq 1} \lambda_f(n) e^{2\pi i n z}.$$

The **Hecke L-function** $L(s,f)$ of f is defined as an L-series:

$$L(s,f) = \sum_{n\geq 1} \frac{a_f(n)}{n^s} = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^{s+\frac{k-1}{2}}},$$

where $a_f(n) = \lambda_f(n) n^{-\frac{k-1}{2}}$ (here we are abusing notation for the Fourier coefficeints of holomorphic forms). Our goal now is to show that this L-function belongs to the Selberg class. First, we need to ensure that the L-function is locally absolutely uniformly convergent. By the Ramanujan conjecture, we have $\lambda_f(n) \ll n^{\frac{k-1}{2}+\varepsilon}$ for any $\varepsilon > 0$ because the Fourier coefficients are multiplicative and $\sigma_0(n) \ll n^\varepsilon$ (see [MV06] for a proof). In particular, $a_f(n) \ll n^\varepsilon$. So $L(s,f)$ is locally absolutely uniformly convergent for $\mathrm{Re}(s) > 1+\varepsilon$ and hence locally absolutely uniformly convergent for $\mathrm{Re}(s) > 1$. The L-function will also have an Euler product since f is a primitive Hecke eigenform. In this case, the Hecke relations imply that the coefficients $a_f(n)$ are multiplicative and satisfy

$$a_f(p^n) = \begin{cases} a_f(p^{n-1})a_f(p) - \chi(p)a_f(p^{n-2}) & \text{if } p \nmid N, \\ (a_f(p))^n & \text{if } p \mid N, \end{cases} \tag{3.25}$$

for all primes p and $n \geq 2$. Because $L(s,f)$ converges absolutely in the region $\mathrm{Re}(s) > 1$, multiplicativity of the Fourier coefficients implies

$$L(s,f) = \prod_p \left(\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} \right),$$

in this region. We now simplify the factor inside the product using this Equation (3.25). On the one hand, if $p \nmid N$:

$$\begin{aligned} \sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} &= 1 + \frac{a_f(p)}{p^s} + \sum_{n\geq 2} \frac{a_f(p^n)}{p^{ns}} \\ &= 1 + \frac{a_f(p)}{p^s} + \sum_{n\geq 2} \frac{a_f(p^{n-1})a_f(p) - \chi(p)a_f(p^{n-2})}{p^{ns}} \\ &= 1 + \frac{a_f(p)}{p^s} + \frac{a_f(p)}{p^s} \sum_{n\geq 1} \frac{a_f(p^n)}{p^{ns}} - \frac{\chi(p)}{p^{2s}} \sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} \\ &= 1 + \left(\frac{a_f(p)}{p^s} - \frac{\chi(p)}{p^{2s}} \right) \sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}}. \end{aligned}$$

By isolating the sum we find

$$\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} = \left(1 - \frac{a_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}} \right)^{-1}. \tag{3.26}$$

On the other hand, if $p \mid N$:

$$\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} = \sum_{n\geq 0} \frac{(a_f(p))^n}{p^{ns}} = \left(1 - a_f(p)p^{-s}\right)^{-1}. \tag{3.27}$$

Therefore Equations (3.26) and (3.27) together give

$$L(s,f) = \prod_{p\nmid N}(1 - a_f(p)p^{-s} + \chi(p)p^{-2s})^{-1} \prod_{p\mid N}(1 - a_f(p)p^{-s})^{-1}.$$

If $p \nmid N$, let $\alpha_1(p)$ and $\alpha_2(p)$ be the roots of $1 - a_f(p)p^{-s} + \chi(p)p^{-2s}$. That is,

$$(1 - \alpha_1(p)p^{-s})(1 - \alpha_2(p)p^{-s}) = (1 - a_f(p)p^{-s} + \chi(p)p^{-2s}). \tag{3.28}$$

If $p \mid N$, let $\alpha_1(p) = a_f(p)$ and $\alpha_2(p) = 0$. We can then express $L(s,f)$ as a degree 2 Euler product:

$$L(s,f) = \prod_p (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1}.$$

The local factor at p is $(1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1}$ with local roots $\alpha_1(p)$ and $\alpha_2(p)$. Upon applying partial fraction decomposition to the local factor, we find

$$\frac{1}{1-\alpha_1(p)p^{-s}}\frac{1}{1-\alpha_2(p)p^{-s}} = \frac{\frac{\alpha_1(p)}{\alpha_1(p)-\alpha_2(p)}}{1-\alpha_1(p)p^{-s}} + \frac{\frac{-\alpha_2(p)}{\alpha_1(p)-\alpha_2(p)}}{1-\alpha_2(p)p^{-s}}.$$

Expanding both sides as series in p^{-s}, and comparing coefficients gives

$$a_f(p^n) = \frac{\alpha_1(p)^{n+1} - \alpha_2(p)^{n+1}}{\alpha_1(p) - \alpha_2(p)}. \tag{3.29}$$

Altogether, $L(s,f)$ satisfies property (i) for belonging to the Selberg class. We package this into a theorem:

Theorem 3.4.1. *Let $f \in \mathcal{S}_k(N,\chi)$ be a primitive Hecke eigenform. For* $\mathrm{Re}(s) > 1$,

$$L(s,f) = \prod_{p\nmid N}(1 - a_f(p)p^{-s} + \chi(p)p^{-2s})^{-1} \prod_{p\mid N}(1 - a_f(p)p^{-s})^{-1} = \prod_p (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1},$$

is locally absolutely uniformly convergent with degree 2 Euler product.

The Integral Representation of $L(s,f)$

We now look to represent $L(s,f)$ as a symmetric integral under $s \to 1-s$. The integral we want to consider is a Mellin tranform:

$$\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\frac{dy}{y}.$$

This time, we don't know *a priori* that this integral defines an analytic function for $\mathrm{Re}(s) > 1$. In any case, we compute

$$\begin{aligned}
\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y} &= \int_0^\infty \sum_{n\geq 1}\lambda_f(n)e^{-2\pi ny}y^{s+\frac{k-1}{2}}\,\frac{dy}{y} \\
&= \sum_{n\geq 1}\lambda_f(n)\int_0^\infty e^{-2\pi ny}y^{s+\frac{k-1}{2}}\,\frac{dy}{y} && \text{DCT} \\
&= \sum_{n\geq 1}\frac{\lambda_f(n)}{(2\pi)^{s+\frac{k-1}{2}}n^{s+\frac{k-1}{2}}}\int_0^\infty e^{-y}y^{s+\frac{k-1}{2}}\,\frac{dy}{y} && y\to\frac{y}{2\pi n} \\
&= \frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}}\sum_{n\geq 1}\frac{\lambda_f(n)}{n^{s+\frac{k-1}{2}}} \\
&= \frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}}L(s,f).
\end{aligned}$$

As this last expression defines an analytic function for $\mathrm{Re}(s) > 1$, the integral does too. Rewriting, we have an integral representation

$$L(s,f) = \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma\left(s+\frac{k-1}{2}\right)}\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}. \tag{3.30}$$

Now split the integral on the right-hand side into two pieces

$$\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y} = \int_0^{\frac{1}{\sqrt{N}}} f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y} + \int_{\frac{1}{\sqrt{N}}}^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}. \tag{3.31}$$

Since $f(iy)$ has exponential decay to zero as $y\to\infty$, the second piece is locally absolutely uniformly bounded for $\mathrm{Re}(s) > 1$ by Method 1.4.1. Hence it defines an analytic function there. Now we will rewrite the first piece in the same form and symmetrize the result as much as possible. Begin by performing the change of variables $y \to \frac{1}{Ny}$ to the first piece to obtain

$$\int_{\frac{1}{\sqrt{N}}}^\infty f\left(\frac{i}{Ny}\right)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y}.$$

Rewriting in terms of the Atkin–Lehner involution $\omega_N f$ and recalling that $\omega_N f = \omega_N(f)f$ by Theorem 2.2.8, we have

$$\begin{aligned}
\int_{\frac{1}{\sqrt{N}}}^\infty f\left(\frac{i}{Ny}\right)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y} &= \int_{\frac{1}{\sqrt{N}}}^\infty f\left(-\frac{1}{iNy}\right)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y} \\
&= \int_{\frac{1}{\sqrt{N}}}^\infty \left(\sqrt{N}iy\right)^k(\omega_N f)(iy)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y} \\
&= \int_{\frac{1}{\sqrt{N}}}^\infty \left(\sqrt{N}iy\right)^k\omega_N(f)f(iy)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y} \\
&= \omega_N(f)i^kN^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^\infty f(iy)y^{(1-s)-\frac{k-1}{2}}\,\frac{dy}{y}.
\end{aligned}$$

Substituting this result back into Equation (3.31) with Equation (3.30) yields the following result:

Theorem 3.4.2. *Let $f \in \mathcal{S}_k(N,\chi)$ be a primitive Hecke eigenform. For* $\mathrm{Re}(s) > 1$,

$$L(s,f) = \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma\left(s+\frac{k-1}{2}\right)}\left[\omega_N(f)i^kN^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^{\infty} f(iy)y^{(1-s)+\frac{k-1}{2}}\,\frac{dy}{y} + \int_{\frac{1}{\sqrt{N}}}^{\infty} f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}\right].$$

This integral will give analytic continuation. To see this, we know everything outside the brackets is entire and the second of the two integrals inside the brackets is analytic for $\mathrm{Re}(s) > 1$. Thus the first integral inside the brackets must be analytic in this region since $L(s,f)$ is too. Now the two integrals, save for the term $\omega_N(f)i^kN^{\frac{1}{2}-s}$, are interchanged as $s \to 1-s$. Hence the right-hand side is analytic for $\mathrm{Re}(s) < 0$ as well. Thus we have analytic continuation to the region

$$\left\{s \in \mathbb{C} : \left|\mathrm{Re}(s) - \frac{1}{2}\right| > \frac{1}{2}\right\}.$$

The Functional Equation & Critical Strip of $L(s,f)$

An immediate consequence of the symmetry of the integral representation is the functional equation:

$$\frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}}L(s,f) = \omega_N(f)i^kN^{-\frac{s}{2}}\frac{\Gamma\left((1-s)+\frac{k-1}{2}\right)}{(2\pi)^{(1-s)+\frac{k-1}{2}}}L(1-s,f).$$

Using the Legendre duplication formula for the gamma function we find that

$$\begin{aligned}\frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}} &= \frac{1}{(2\pi)^{s+\frac{k-1}{2}}2^{1-\left(s+\frac{k-1}{2}\right)}\sqrt{\pi}}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)\\ &= \frac{1}{2\pi^{s+\frac{1}{2}}}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)\\ &= \frac{1}{\sqrt{4\pi}}\pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right).\end{aligned}$$

The constant factor in front is independent of s and so can be canceled in the functional equation. Therefore we identify the gamma factor as

$$\gamma(s,f) = \pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right),$$

with $\kappa_1 = \frac{k-1}{2}$ and $\kappa_2 = \frac{k+1}{2}$ the local parameters at infinity. Clearly they satisfy the required bounds. The conductor is $q(f) = N$, so the primes dividing the level ramify, and by the Ramanujan conjecture $\alpha_i(p) \neq 0$ for $i = 1, 2$ and all primes $p \nmid N$. The completed L-function is

$$\Lambda(s,f) = N^{-\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)L(s,f),$$

with functional equation

$$\Lambda(s,f) = \omega_N(f)i^k\Lambda(1-s,f).$$

This is the functional equation of $L(s,f)$. From it, the root number is $\varepsilon(f) = \omega_N(f)i^k$ and we see that $L(s,f)$ is self-dual. Altogether, this shows that $L(s,f)$ satisfies properties (ii)-(iv) of the Selberg class.

We now analytically continue $L(s,f)$ inside the critical strip and hence to all of $\mathbb{C}$. From the integral representation, we know

$$L(s,f)=\sqrt{4\pi}\frac{1}{\gamma(s,f)}\left[\omega_N(f)i^kN^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^{\infty}f(iy)y^{(1-s)+\frac{k-1}{2}}\frac{dy}{y}+\int_{\frac{1}{\sqrt{N}}}^{\infty}f(iy)y^{s+\frac{k-1}{2}}\frac{dy}{y}\right]. \tag{3.32}$$

To get continuation inside the critical strip it suffices that the two integrals are locally absolutely uniformly bounded for $|\mathrm{Re}(s)-\frac{1}{2}|\le\frac{1}{2}$. This follows by Method 1.4.1 as we already know. This gives analytic continuation to the critical strip and hence to all of $\mathbb{C}$. In particular, we have shown that $L(s,f)$ has no poles.

At last, all we need to show is that $L(s,f)$ is of order 1 to conclude that it satisfies property (v) of the Selberg class and therefore is an L-function belonging to the Selberg class. Since $L(s,f)$ has no poles, we do not need to clear any polar divisors. As the integral in the representation is absolutely bounded, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1 and $\mathrm{Re}(s)$ is bounded, we have

$$\frac{1}{\gamma(s,f)}\ll_\varepsilon e^{|s|^{1+\varepsilon}}, \tag{3.33}$$

for any $\varepsilon>0$. So the reciprocal of the gamma factor is of the same order. Then Equations (3.32) and (3.33) together imply

$$L(s,f)\ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So $L(s,f)$ is of order 1. We summarize all of our work into the following theorem:

Theorem 3.4.3. *For any primitive Hecke eigenform $f\in\mathcal{S}_k(N,\chi)$, $L(s,f)$ is a Selberg class L-function. It admits analytic continuation to $\mathbb{C}$ via the integral representation*

$$L(s,f)=\frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma\left(s+\frac{k-1}{2}\right)}\left[\omega_N(f)i^kN^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^{\infty}f(iy)y^{(1-s)+\frac{k-1}{2}}\frac{dy}{y}+\int_{\frac{1}{\sqrt{N}}}^{\infty}f(iy)y^{s+\frac{k-1}{2}}\frac{dy}{y}\right],$$

and with functional equation

$$N^{-\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)L(s,f)=\Lambda(s,f)=\omega_N(f)i^k\Lambda(1-s,f).$$

3.5 The Rankin-Selberg Method

We are ready to describe the Rankin-Selberg method. This is a process by which we can construct new L-functions from old ones. We discuss the method only in the simplest case for two reasons. The first is that many technical difficulties arise in the fully general setting. The second is that in the simplest case the Rankin-Selberg convolution involves a Maass form we have studied and this establishes a direct connection between holomorphic forms and Maass forms. To combat not working in full generality, we make remarks where the theory needs to be adjusted in more general settings.

The Defintion & Euler Product of $L(s, f \otimes g)$

We will start by discussing two different but related L-functions. Let $f, g \in \mathcal{S}_k(\mathrm{PSL}_2(\mathbb{Z}))$ be primitive Hecke eigenforms. Let the Fourier series for f and g be given by

$$f(z) = \sum_{n \geq 1} \lambda_f(n) e^{2\pi i n z} \quad \text{and} \quad g(z) = \sum_{n \geq 1} \lambda_g(n) e^{2\pi i n z}.$$

The L-function $L(s, f \times g)$ of f and g is defined as an L-series:

$$L(s, f \times g) = \sum_{n \geq 1} \frac{a_{f \times g}(n)}{n^s} = \sum_{n \geq 1} \frac{a_f(n)\overline{b_g(n)}}{n^s} = \sum_{n \geq 1} \frac{\lambda_f(n)\overline{\lambda_g(n)}}{n^{s+k-1}},$$

where $a_{f \times g}(n) = a_f(n)\overline{b_g(n)}$. The **Rankin-Selberg convolution** $L(s, f \otimes g)$ of f and g is defined as an L-series:

$$L(s, f \otimes g) = \sum_{n \geq 1} \frac{a_{f \otimes g}(n)}{n^s} = \zeta(2s) L(s, f \times g),$$

where $a_{f \otimes g}(n) = \sum_{m\ell^2 = n} a_f(n)\overline{b_g(n)}$.

Remark 3.5.1. *If $f \in \mathcal{S}_k(N, \chi)$ and $g \in \mathcal{S}_\ell(N, \psi)$, one needs to use the Dirichlet L-function $L(2s, \chi\psi)$ instead of $\zeta(2s)$.*

Our main goal is to show that $L(s, f \otimes g)$ is actually the Rankin-Selberg convolution of $L(s, f)$ and $L(s, g)$. The first step is to prove local absolute uniform convergence for $\mathrm{Re}(s) > 1$. To see this, notice that $\zeta(2s)$ does so it suffices to show $L(s, f \times g)$ does too. In exactly the same way as we remarked for Hecke L-functions, $a_{f \times g}(n) \ll n^\varepsilon$. Hence $L(s, f \times g)$ is locally absolutely uniformly convergent for $\mathrm{Re}(s) > 1 + \varepsilon$ and hence locally absolutely uniformly convergent for $\mathrm{Re}(s) > 1$.
The L-function will have an Euler product since both f and g are primitive Hecke eigenforms. In this case, let $\alpha_i(p)$ and $\beta_j(p)$ for $1 \leq i, j \leq 2$ be the local roots of $L(s, f)$ and $L(s, g)$ respectively. Since $L(s, f \otimes g)$ converges absolutely in the region $\mathrm{Re}(s) > 1$, for s in this region, multiplicativity of the Fourier coefficients implies

$$L(s, f \otimes g) = \zeta(2s) L(s, f \times g) = \prod_{p \nmid NM} (1 - p^{-2s})^{-1} \prod_p \left(\sum_{n \geq 0} \frac{a_f(p^n)\overline{b_g(p^n)}}{p^{ns}} \right).$$

We now simplify the factor inside the latter product using Equation (3.29):

$$\begin{aligned}
\sum_{n \geq 0} \frac{a_f(p^n)\overline{b_g(p^n)}}{p^{ns}} &= \sum_{n \geq 0} \left(\frac{\alpha_1(p)^{n+1} - \alpha_2(p)^{n+1}}{\alpha_1(p) - \alpha_2(p)} \right) \left(\frac{(\overline{\beta_1(p)})^{n+1} - (\overline{\beta_2(p)})^{n+1}}{\overline{\beta_1(p)} - \overline{\beta_2(p)}} \right) p^{-ns} \\
&= (\alpha_1(p) - \alpha_2(p))^{-1} \left(\overline{\beta_1(p)} - \overline{\beta_2(p)} \right)^{-1} \\
&\cdot \left[\sum_{n \geq 1} \frac{\alpha_1(p)^n (\overline{\beta_1(p)})^n}{p^{(n-1)s}} + \frac{\alpha_2(p)^n (\overline{\beta_2(p)})^n}{p^{(n-1)s}} - \frac{\alpha_1(p)^n (\overline{\beta_2(p)})^n}{p^{(n-1)s}} - \frac{\alpha_2(p)^n (\overline{\beta_1(p)})^n}{p^{(n-1)s}} \right] \\
&= (\alpha_1(p) - \alpha_2(p))^{-1} \left(\overline{\beta_1(p)} - \overline{\beta_2(p)} \right)^{-1} \left[\alpha_1(p)\overline{\beta_1(p)} \left(1 - \alpha_1(p)\overline{\beta_1(p)} p^{-s} \right)^{-1} \right. \\
&+ \alpha_2(p)\overline{\beta_2(p)} \left(1 - \alpha_2(p)\overline{\beta_2(p)} p^{-s} \right)^{-1} - \alpha_1(p)\overline{\beta_2(p)} \left(1 - \alpha_1(p)\overline{\beta_2(p)} p^{-s} \right)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&- \alpha_2(p)\overline{\beta_1(p)}\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)^{-1}\Bigg] \\
&= (\alpha_1(p)-\alpha_2(p))^{-1}\left(\overline{\beta_1(p)}-\overline{\beta_2(p)}\right)^{-1}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)^{-1} \\
&\cdot\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)^{-1}\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)^{-1}\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)^{-1} \\
&\cdot\Bigg[\alpha_1(p)\overline{\beta_1(p)}\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right) \\
&+ \alpha_2(p)\overline{\beta_2(p)}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right) \\
&- \alpha_1(p)\overline{\beta_2(p)}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right) \\
&- \alpha_2(p)\overline{\beta_1(p)}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)\Bigg].
\end{aligned}
$$

The term in the brackets simplifies to

$$
\left(1-\alpha_1(p)\alpha_2(p)\overline{\beta_1(p)\beta_2(p)}p^{-2s}\right)(\alpha_1(p)-\alpha_2(p))\left(\overline{\beta_1(p)}-\overline{\beta_2(p)}\right),
$$

because all of the other terms are killed by symmetry in $\alpha_1(p)$, $\alpha_2(p)$, $\overline{\beta_1(p)}$, and $\overline{\beta_2(p)}$. The Ramanujan conjecture implies $\alpha_1(p)\alpha_2(p)\overline{\beta_1(p)\beta_2(p)} = 1$. Therefore the corresponding factor above is $(1-p^{-2s})$. This factor cancels the local factor at p in the Euler product of $\zeta(2s)$, so that

$$
\sum_{n\ge 0}\frac{a_f(p^n)\overline{b_g(p^n)}}{p^{ns}} = \prod_{1\le i,j\le 2}\left(1-\alpha_i(p)\overline{\beta_j(p)}p^{-s}\right)^{-1}.
$$

So in total we have a degree 4 Euler product,

$$
L(s, f\otimes g) = \prod_{1\le i,j\le 2}\left(1-\alpha_i(p)\overline{\beta_j(p)}p^{-s}\right)^{-1}.
$$

Clearly $a_{f\otimes g}(1) = 1$ because this is the constant term in the Euler product. In particular, we have verified property (i) and adjustment (i) for Rankin-Selberg convolutions and we package this into a theorem:

Theorem 3.5.1. *Let $f, g \in \mathcal{S}_k(\mathrm{PSL}_2(\mathbb{Z}))$ be primitive Hecke eigenforms. For $\mathrm{Re}(s) > 1$,*

$$
L(s, f\otimes g) = \prod_{1\le i,j\le 2}\left(1-\alpha_i(p)\overline{\beta_j(p)}p^{-s}\right)^{-1},
$$

is locally absolutely uniformly convergent with degree 4 Euler product.

Remark 3.5.2. *If $f \in \mathcal{S}_k(N,\chi)$ and $g \in \mathcal{S}_\ell(N,\psi)$, the Euler product becomes more difficult to compute since the local p factors for $p \mid NM$ change as either $\alpha_2(p) = 0$ or $\beta_2(p) = 0$. Moreover, the situation is increasing complicated if $(N, M) > 1$. In fact, if $(N, M) > 1$ then the conductor of $L(s, f\otimes g)$ can be smaller than NM which is known as* ***conductor dropping*** *and can cause serious obstructions to the theory.*

The Integral Representation of $L(s, f \otimes g)$: Part I

We now look for a way to express $L(s, f \otimes g)$ as an integral symmetric as $s \to 1-s$. The integral we want to consider is

$$\int_{\Gamma_\infty \backslash \mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\, d\mu,$$

where Γ_∞ is the associated subgroup corresponding to $\mathrm{PSL}_2(\mathbb{Z})$. This will turn out to be a Mellin transform as we will soon see.

Remark 3.5.3. *If $f \in \mathcal{S}_k(N, \chi)$ and $g \in \mathcal{S}_\ell(N, \psi)$, then Γ_∞ is the associated subgroup corresponding to $\Gamma_0(NM)$. Also, for $k \neq \ell$ the power of $\mathrm{Im}(z)$ in the integrand needs to be adjusted.*

The region of convergence of this integral is not immediately clear because we cannot appeal to Method 1.4.1 directly. Indeed,

$$\Gamma_\infty \backslash \mathbb{H} = \{z \in \mathbb{H} : 0 \leq \mathrm{Re}(z) \leq 1\},$$

intersects infinitely many fundamental domains for Γ. In any case, we have

$$\begin{aligned}
\int_{\Gamma_\infty \backslash \mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\, d\mu &= \int_0^\infty \int_0^1 f(x+iy)\overline{g(x+iy)}y^{s+k}\,\frac{dx\,dy}{y^2} \\
&= \int_0^\infty \int_0^1 \sum_{n,m\geq 1} a(n)\overline{b(m)}e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^{s+k}\,\frac{dx\,dy}{y^2} \\
&= \int_0^\infty \sum_{n,m\geq 1} \int_0^1 a(n)\overline{b(m)}e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^{s+k}\,\frac{dx\,dy}{y^2} \qquad \text{DCT} \\
&= \int_0^\infty \sum_{n\geq 1} a(n)\overline{b(n)}e^{-4\pi ny}y^{s+k}\,\frac{dy}{y^2},
\end{aligned}$$

where the last line follows by Equation (2.2). Notice that the last integral is a Mellin transform. The rest is a computation:

$$\begin{aligned}
\int_0^\infty \sum_{n\geq 1} a(n)\overline{b(n)}e^{-4\pi ny}y^{s+k}\,\frac{dy}{y^2} &= \sum_{n\geq 1} a(n)\overline{b(n)} \int_0^\infty e^{-4\pi ny}y^{s+k}\,\frac{dy}{y^2} \qquad \text{DCT} \\
&= \sum_{n\geq 1} \frac{a(n)\overline{b(n)}}{(4\pi n)^{s+k-1}} \int_0^\infty e^{-y}y^{s+k-1}\,\frac{dy}{y} \qquad y \to \frac{y}{4\pi n} \\
&= \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{n\geq 1} \frac{a(n)\overline{b(n)}}{n^{s+k-1}} \\
&= \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} L(s, f\times g).
\end{aligned}$$

This last expression is locally absolutely uniformly convergent for $\mathrm{Re}(s) > 1$ because the L-function is and the gamma factor is holomorphic in this region. Therefore the original integral is locally absolutely uniformly bounded. At this point we have an integral representation

$$L(s, f\times g) = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)} \int_{\Gamma_\infty \backslash \mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\, d\mu.$$

We rewrite the integral as follows:

$$\begin{aligned}\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\,d\mu &= \int_{\mathcal{F}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma z)\overline{g(\gamma z)}\mathrm{Im}(\gamma z)^{s+k}\,d\mu && \text{folding}\\ &= \int_{\mathcal{F}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^k\overline{j(\gamma,z)^k}f(z)\overline{g(z)}\mathrm{Im}(\gamma z)^{s+k}\,d\mu && \text{modularity}\\ &= \int_{\mathcal{F}} f(z)\overline{g(z)}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}|j(\gamma,z)|^{2k}\mathrm{Im}(\gamma z)^{s+k}\,d\mu\\ &= \int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\gamma z)^{s}\,d\mu\\ &= \int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k E_\infty(z,s)\,d\mu.\end{aligned}$$

Note that $E_\infty(z,s)$ is the Eisenstein series on $\Gamma\backslash\mathbb{H}$ at the ∞ cusp. Altogether this gives the integral representation

$$L(s,f\times g) = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)}\int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k E_\infty(z,s)\,d\mu. \tag{3.34}$$

which is valid for $\mathrm{Re}(s)>1$. We cannot investigate the integral any further until we understand the Fourier coefficients of $E_\infty(z,s)$. Therefore we will take a necessary detour and return to the integral after.

Remark 3.5.4. *For $f\in\mathcal{S}_k(N,\chi)$ and $g\in\mathcal{S}_\ell(N,\psi)$, the integral representation involves a different Eisenstein series other than $E_\infty(z,s)$ because we can not easily combine the modularity factors $\chi(\gamma)j(\gamma,z)^k$ and $\overline{\psi(\gamma)j(\gamma,z)^\ell}$ that appear after folding.*

The Fourier Series of $E_\infty(z,s)$

Let

$$E_\infty(z,s) = \sum_{n\in\mathbb{Z}} a(n,y,s)e^{2\pi i n x}.$$

be the Fourier series of $E_\infty(z,s)$. Our task now is to compute these coefficients. To do this we will need the following technical lemma:

Lemma 3.5.1. *Let $M\geq 1$ be square-free. Then for $\mathrm{Re}(s)>1$ and $b\in\mathbb{Z}$,*

$$\sum_{m\geq 1}\frac{r(b;m)}{m^{2s}} = \begin{cases}\frac{\zeta(2s-1)}{\zeta(2s)} & \text{if } b=0,\\ \frac{\sigma_{1-2s}(|b|)}{\zeta(2s)} & \text{if } b\neq 0,\end{cases}$$

where $\sigma_s(b)$ is the generalized sum of divisors function.

Proof. If $\mathrm{Re}(s)>1$ then the desired evaulation of the sum is locally absolutely uniformly convergent because the zeta function is in that region. Hence the sum will be too provided we prove the identity. Suppose $b=0$. Then $r(0;m)=\phi(m)$. Since $\phi(m)$ is multiplicative we have

$$\sum_{m\geq 1}\frac{\phi(m)}{m^{2s}} = \prod_p\left(\sum_{k\geq 0}\frac{\phi(p^k)}{p^{k(2s)}}\right). \tag{3.35}$$

Recalling that $\phi(p^k) = p^k - p^{k-1}$ for $k \geq 1$, make the following computation:

$$\begin{aligned}
\sum_{k\geq 0} \frac{\phi(p^k)}{p^{k(2s)}} &= 1 + \sum_{k\geq 1} \frac{p^k - p^{k-1}}{p^{k(2s)}} \\
&= \sum_{k\geq 0} \frac{1}{p^{k(2s-1)}} - \frac{1}{p} \sum_{k\geq 1} \frac{1}{p^{k(2s-1)}} \\
&= \sum_{k\geq 0} \frac{1}{p^{k(2s-1)}} - p^{-2s} \sum_{k\geq 0} \frac{1}{p^{k(2s-1)}} \\
&= (1 - p^{-2s}) \sum_{k\geq 0} \frac{1}{p^{k(2s-1)}} \\
&= \frac{1 - p^{-2s}}{1 - p^{-(2s-1)}}.
\end{aligned} \tag{3.36}$$

Combining Equations (3.35) and (3.36) gives

$$\sum_{m\geq 1} \frac{\phi(m)}{m^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)}.$$

Now suppose $b \neq 0$, Proposition 1.3.1 gives the first equality in the following chain:

$$\begin{aligned}
\sum_{m\geq 1} \frac{r(b;m)}{m^{2s}} &= \sum_{m\geq 1} m^{-2s} \sum_{\ell|(b,m)} \ell\mu\left(\frac{m}{\ell}\right) \\
&= \sum_{\ell|b} \ell \sum_{m\geq 1} \frac{\mu(m)}{(m\ell)^{2s}} \\
&= \left(\sum_{\ell|b} \ell^{1-2s}\right) \left(\sum_{m\geq 1} \frac{\mu(m)}{m^{2s}}\right) \\
&= \sigma_{1-2s}(b) \sum_{m\geq 1} \frac{\mu(m)}{m^{2s}} \\
&= \sigma_{1-2s}(|b|) \sum_{m\geq 1} \frac{\mu(m)}{m^{2s}} \\
&= \frac{\sigma_{1-2s}(|b|)}{\zeta(2s)} \qquad \text{Proposition A.2.2.}
\end{aligned}$$

□

We can now compute the Fourier coefficients $a(n, y, s)$:

Proposition 3.5.1. *The Fourier coefficients $a(n, y, s)$ are given by*

$$a(n, y, s) = \begin{cases} y^s + y^{1-s} \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} & \text{if } n = 0, \\ 2\pi^s \frac{|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)}{\Gamma(s)\zeta(2s)} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y) & \text{if } n \neq 0, \end{cases}$$

where $K_s(y)$ is the K-Bessel function.

Proof. By Remark 2.1.3, we have

$$E_\infty(z,s) = y^s + \sum_{\substack{c\geq 1, d\in\mathbb{Z}\\(c,d)=1}} \frac{y^s}{|cx+icy+d|^{2s}}.$$

So the definition of the Fourier coefficients gives

$$a(n,y,s) = \int_0^1 E(x+iy,s)e^{-2\pi inx}\,dx = y^s\delta_{n,0} + \int_0^1 \sum_{\substack{c\geq 1, d\in\mathbb{Z}\\(c,d)=1}} \frac{y^s}{|cx+icy+d|^{2s}}e^{-2\pi inx}\,dx, \tag{3.37}$$

where we have used Equation (2.2). We are now reduced to computing

$$I(z,s) = \int_0^1 \sum_{\substack{c\geq 1, d\in\mathbb{Z}\\(c,d)=1}} \frac{y^s}{|cx+icy+d|^{2s}}e^{-2\pi inx}\,dx.$$

Make the following observation: summing over all pairs (c,d) with $c\geq 1$, $d\in\mathbb{Z}$, $c\equiv 0 \pmod{NM}$, and $(c,d)=1$ is the same as summing over all triples (c,ℓ,r) with $c\geq 1$, $\ell\in\mathbb{Z}$, $r \pmod{c}$, with $c\equiv 0 \pmod{NM}$ and $(r,c)=1$. This is seen by writing $d = c\ell + r$. Therefore

$$\begin{aligned}
I(z,s) &= \int_0^1 \sum_{\substack{c\geq 1, d\in\mathbb{Z}\\(c,d)=1}} \frac{y^s}{|cx+icy+d|^{2s}}e^{-2\pi inx}\,dx \\
&= \int_0^1 \sum_{(c,\ell,r)} \frac{y^s}{|c(x+\ell)+r+icy|^{2s}}e^{-2\pi inx}\,dx \\
&= \sum_{(c,\ell,r)} \int_0^1 \frac{y^s}{|c(x+\ell)+r+icy|^{2s}}e^{-2\pi inx}\,dx && \text{DCT} \\
&= \sum_{(c,\ell,r)} \frac{1}{c^{2s}} \int_0^1 \frac{y^s}{\left|x+\ell+\frac{r}{c}+iy\right|^{2s}}e^{-2\pi inx}\,dx \\
&= \sum_{(c,\ell,r)} \frac{1}{c^{2s}} \int_\ell^{\ell+1} \frac{y^s}{\left|x+\frac{r}{c}+iy\right|^{2s}}e^{-2\pi inx}\,dx && x\to x-\ell \\
&= \sideset{}{'}\sum_{\substack{c\geq 1\\ r \pmod{c}}} \frac{1}{c^{2s}} \int_{-\infty}^{\infty} \frac{y^s}{\left|x+\frac{r}{c}+iy\right|^{2s}}e^{-2\pi inx}\,dx && \text{DCT} \\
&= \sum_{c\geq 1} \frac{1}{c^{2s}} \sideset{}{'}\sum_{r \pmod{c}} e^{\frac{2\pi inr}{c}} \int_{-\infty}^{\infty} \frac{y^s}{(x^2+y^2)^s}e^{-2\pi inx}\,dx && x\to x-\frac{r}{c} \\
&= \sum_{c\geq 1} \frac{r(n;c)}{c^{2s}} \int_{-\infty}^{\infty} \frac{y^s}{(x^2+y^2)^s}e^{-2\pi inx}\,dx \\
&= \sum_{c\geq 1} \frac{r(n;c)}{c^{2s}} y^{1-s}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^s}e^{-2\pi inxy}\,dx && x\to xy,
\end{aligned}$$

where on the right-hand side it is understood we are summing over all triples (c,ℓ,r) with the prescribed properties. Using Lemma 3.5.1 yields

$$I(z,s) = \begin{cases} \frac{\zeta(2s-1)}{\zeta(2s)} y^{1-s}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^s}\,dx & \text{if } n=0, \\ \frac{\sigma_{1-2s}(|n|)}{\zeta(2s)} y^{1-s}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^s}e^{-2\pi inxy}\,dx & \text{if } n\neq 0. \end{cases}$$

Appealing to Appendix D.1 for this last integral and substiuting the result back into Equation (3.37) finishes the proof. $\square$

The Completed Real-analytic Eisenstein Series $E^*(z,s)$

We would like to analytically continue $E_\infty(z,s)$ in s past the region $\mathrm{Re}(s)>1$ (this follows from Theorem 2.3.8 but we will demonstrate a full proof for the single Eisenstein series on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$). From the computation of the Fourier coefficients we will have possible poles coming from the denominator of the Fourier coefficients. To remove this difficultly, we will multiply by a factor to clear the poles. In turn, this will give us a functional equation as $s\to 1-s$. The factor will be the completed zeta function $\Lambda(2s,\zeta)=\pi^{-s}\Gamma(s)\zeta(2s)$ (scaled by 2). We define the the **completed real-analytic Eisenstein series** $E^*(z,s)$ by

$$E^*(z,s)=\Lambda(2s,\zeta)E_\infty(z,s)=\pi^{-s}\Gamma(s)\zeta(2s)E_\infty(z,s).$$

From Proposition 3.5.1, the Fourier coefficients $a^*(n,y,s)$ of $E^*(z,s)$ are given by

$$a^*(n,y,s)=\begin{cases} y^s\pi^{-s}\Gamma(s)\zeta(2s)+y^{1-s}\pi^{-(s-\frac{1}{2})}\Gamma(s-\frac{1}{2})\zeta(2s-1) & \text{if } n=0,\\ 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y) & \text{if } n\neq 0.\end{cases}$$

Our goal is to now derive a functional equation for $E^*(z,s)$. Using the definition and functional equation for $\Lambda(2s-1,\zeta)$, we can rewrite the second term in the $n=0$ coefficient to get

$$a^*(n,y,s)=\begin{cases} y^s\Lambda(2s,\zeta)+y^{1-s}\Lambda(2(1-s),\zeta) & \text{if } n=0,\\ 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y) & \text{if } n\neq 0.\end{cases} \tag{3.38}$$

Now observe that the $n=0$ coefficient is invariant under $s\to 1-s$. Each $n\neq 0$ coefficeint is also invariant under $s\to 1-s$. To see this we will use two facts. First, from Appendix B.7, $K_s(y)$ is invariant under $s\to -s$ and $s-\frac{1}{2}\mapsto \frac{1}{2}-s$ under $s\to 1-s$. Therefore $K_{s-\frac{1}{2}}(2\pi|n|y)$ is invariant as $s\to 1-s$. Second, for $n\geq 1$ we have

$$n^{s-\frac{1}{2}}\sigma_{1-2s}(n)=n^{\frac{1}{2}-s}n^{2s-1}\sigma_{1-2s}(n)=n^{\frac{1}{2}-s}n^{2s-1}\sum_{d\mid n}d^{1-2s}=n^{\frac{1}{2}-s}\sum_{d\mid n}\left(\frac{n}{d}\right)^{2s-1}=n^{\frac{1}{2}-s}\sigma_{2s-1}(n),$$

where the second to last equality follows by writing $n^{2s-1}=\left(\frac{n}{d}\right)^{2s-1}d^{2s-1}$ for each $d\mid n$. These two facts together give the invariance of the $n\neq 0$ coefficients under $s\to 1-s$. Altogether, we have shown the following functional equation for $E^*(z,s)$:

$$E^*(z,s)=E^*(z,1-s).$$

It follows that for fixed $z\in\mathbb{H}$, $E^*(z,s)$ is holomorphic on the region

$$\left\{s\in\mathbb{C}:\left|\mathrm{Re}(s)-\frac{1}{2}\right|>\frac{1}{2}\right\}.$$

Remark 3.5.5. *For $f\in\mathcal{S}_k(N,\chi)$ and $g\in\mathcal{S}_\ell(N,\psi)$, we use a modified version of the Eisenstein series $E_\infty(z,s)$ on $\Gamma_0(NM)\backslash\mathbb{H}$. But on this congruence subgroup there is more than the cusp at ∞. In particular, there is also a cusp at 0. The functional equation of the modified $E_\infty(z,s)$ becomes much more complicated as it reflects into a modified version of $E_0(z,s)$ as $s\to 1-s$ so that we need to understand the Fourier coefficients of both of these Eisenstein series.*

We now obtain meromorphic continuation of $E^*(z,s)$ inside the critical strip and therefore meromorphic continuation in s to all of $\mathbb{C}$. We first write $E^*(z,s)$ as a Fourier series using Equation (3.38):

$$E^*(z,s) = y^s\Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta) + \sum_{n\neq 0} 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}.$$

This is valid for any $y > 0$ and $|\mathrm{Re}(s) - \frac{1}{2}| > \frac{1}{2}$. Since $\Lambda(2s,\zeta)$ is meromorphic in the critical strip $|\mathrm{Re}(s) - \frac{1}{2}| \leq \frac{1}{2}$, the constant term of $E^*(z,s)$ is meromorphic in this region. To prove the meromorphic continuation of $E^*(z,s)$ it now suffices to show

$$\sum_{n\neq 0} 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx},$$

is meromorphic as well. We will actually prove it is locally absolutely uniformly convergent. To achieve this we need two bounds, one for $\sigma_{1-2s}(|n|)$ and one for $K_{s-\frac{1}{2}}(2\pi|n|y)$. For the first bound, we use the estimate $\sigma_0(n) \ll n^{\varepsilon}$ (see [MV06] for a proof). Therefore we have the crude bound

$$\sigma_{1-2s}(|n|) = \sum_{d|n} d^{1-2s} < \sigma_0(|n|)|n|^{1-2s} \ll_{\varepsilon} |n|^{1-2s+\varepsilon}. \tag{3.39}$$

For the second estimate, Lemma B.7.2 and that $\mathrm{Re}(s)$ is bounded give

$$K_{s-\frac{1}{2}}(2\pi|n|y) \ll e^{-2\pi|n|y}. \tag{3.40}$$

Then Equations (3.39) and (3.40) together imply

$$\sum_{n\neq 0} 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx} \ll_{\varepsilon} \sum_{n\geq 1} 4n^{\frac{1}{2}-s+\varepsilon}\sqrt{y}e^{-2\pi ny}, \tag{3.41}$$

and this latter series is locally absolutely uniformly convergent by the ratio test. The meromorphic continuation of $E^*(z,s)$ to $\mathbb{C}$ in s follows. It remains to investigate the poles and residues which we now do. We will acomplish this from direct inspection of the Fourier coefficients:

Proposition 3.5.2. *$E^*(z,s)$ has simple poles at $s = 0$ and $s = 1$, and*

$$\operatorname*{Res}_{s=0} E^*(z,s) = -\frac{1}{2} \quad \text{and} \quad \operatorname*{Res}_{s=1} E^*(z,s) = \frac{1}{2}.$$

Proof. Since the constant term in the Fourier series of $E^*(z,s)$ is the only non-holomorphic term, poles of $E^*(z,s)$ can only come from that term. So we are reduced to understanding the poles of

$$y^s\Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta). \tag{3.42}$$

Notice $\Lambda(2s,\zeta)$ has simple poles at $s = 0$, $s = \frac{1}{2}$ (one from the zeta function and one from the gamma factor) and no others. It follows that $E^*(z,s)$ has a simple pole at $s = 0$ coming from the y^s term in Equation (3.42), and by the functional equation there is also a pole at $s = 1$ coming from the y^{1-s} term. At $s = \frac{1}{2}$, both terms in Equation (3.42) have simple poles and we will show that the singularity there is removable. Recall $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Also, by Proposition 3.2.1, $\mathrm{Res}_{s=\frac{1}{2}}\,\zeta(2s) = \frac{1}{2}$ and $\mathrm{Res}_{s=\frac{1}{2}}\,\zeta(2(1-s)) = -\frac{1}{2}$. So altogether

$$\operatorname*{Res}_{s=\frac{1}{2}} E^*(z,s) = \operatorname*{Res}_{s=\frac{1}{2}}[y^s\Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta)] = \frac{1}{2}y^{\frac{1}{2}} - \frac{1}{2}y^{\frac{1}{2}} = 0.$$

Hence the singularity at $s = \frac{1}{2}$ is removable. As for the residues at $s = 0$ and $s = 1$, the functional equation implies that they are negatives of each other. So it suffices to compute the residue at $s = 0$. Recall $\zeta(0) = -\frac{1}{2}$ and $\operatorname{Res}_{s=0} \Gamma(s) = 1$. Then together we find

$$\operatorname*{Res}_{s=0} E^*(z,s) = \operatorname*{Res}_{s=0} y^s \Lambda(2s,\zeta) = -\frac{1}{2}. \qquad \square$$

This completes our study of the real-analytic Eisenstein series $E_\infty(z,s)$.

The Integral Representation of $L(s, f \otimes g)$: Part II

We now have enough information to further understand the Rankin-Selberg convolution $L(s, f\otimes g)$. Writing Equation (3.34) in terms of $E^*(z,s)$ and $L(s, f \otimes g)$ gives the following result:

Theorem 3.5.2. *Let $f, g \in \mathcal{S}_k(\mathrm{PSL}_2(\mathbb{Z}))$ be primitive Hecke eigenforms. For $\mathrm{Re}(s) > 1$,*

$$L(s, f \otimes g) = \frac{(4\pi)^{s+k-1}\pi^s}{\Gamma(s+k-1)\Gamma(s)} \int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k E^*(z,s)\, d\mu.$$

This integral will give analytic continuation. To see this, note that the gamme factors are analytic for $\mathrm{Re}(s) < 0$. By the functional equation for $E^*(z,s)$, the integral is invariant as $s \to 1-s$. These two facts together give analytic continuation to the region

$$\left\{ s \in \mathbb{C} : \left| \mathrm{Re}(s) - \frac{1}{2} \right| > \frac{1}{2} \right\}.$$

The Functional Equation, Critical Strip & Residues of $L(s, f \otimes g)$

An immediate consequence of the symmetry of integral representation is the functional equation:

$$\frac{\Gamma(s+k-1)\Gamma(s)}{(4\pi)^{s+k-1}\pi^s} L(s, f\otimes g) = \frac{\Gamma((1-s)+k-1)\Gamma(1-s)}{(4\pi)^{(1-s)+k-1}\pi^{1-s}} L(1-s, f \otimes g).$$

Applying the Legendre duplication formula for the gamma function twice we see that

$$\begin{aligned} \frac{\Gamma(s+k-1)\Gamma(s)}{(4\pi)^{s+k-1}\pi^s} &= \frac{2^{2s+k-3}}{(4\pi)^{s+k-1}\pi^{s+1}} \Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) \\ &= \frac{1}{2^{k+1}\pi^k}\pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right). \end{aligned} \tag{3.43}$$

The factor in front is independent of s and can therefore be canceled in the functional equation. We identify the gamma factor as:

$$\gamma(s, f\otimes g) = \pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right),$$

with $\mu_{1,1} = k-1$, $\mu_{2,2} = k$, $\mu_{1,2} = 0$, and $\mu_{2,1} = 1$ the local parameters at infinity. Clearly they satisfy the required bounds. The completed L-function is

$$\Lambda(s, f\otimes g) = \pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) L(s, f\otimes g),$$

so the conductor is $q(f\otimes g)=1$ and no primes ramify. Clearly, $q(f\otimes g)\mid q(f)^2q(g)^2$. Then

$$\Lambda(s,f\otimes g)=\Lambda(1-s,f\otimes g),$$

is the functional equation of $L(s,f\otimes g)$. In particular, the root number $\varepsilon(f\otimes g)=1$, and $L(s,f\otimes g)$ is self-dual. In total, we have verified properties (ii)-(iv) and adjustments (ii) and (iii) for Rankin-Selberg convolutions.

We now want to get continuation inside of the critical strip. However, we will only be able to obtain meromorphic continuation inside the critical strip because of the presence of poles for $E^*(z,s)$. From the integral representation, we know

$$L(s,f\otimes g)=2^{k+1}\pi^k\frac{1}{\gamma(s,f\otimes g)}\int_{\mathcal{F}}f(z)\overline{g(z)}\mathrm{Im}(z)^kE^*(z,s)\,d\mu.$$

Substituting the Fourier series for $E^*(z,s)$ gives

$$\begin{aligned}L(s,f\otimes g)=2^{k+1}\pi^k\frac{1}{\gamma(s,f\otimes g)}\Bigg[&\int_{\mathcal{F}}f(x+iy)\overline{g(x+iy)}y^k(y^s\Lambda(2s,\zeta)+y^{1-s}\Lambda(2(1-s),\zeta))\,\frac{dx\,dy}{y^2}\\&+\int_{\mathcal{F}}f(x+iy)\overline{g(x+iy)}y^k\sum_{n\neq0}2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}\,\frac{dx\,dy}{y^2}\Bigg],\end{aligned}\tag{3.44}$$

and we are reduced to showing that both integrals are locally absolutely uniformly bounded in this region away from poles. To this end, let s be such that $|\mathrm{Re}(s)-\frac{1}{2}|\leq\frac{1}{2}$ and is away the points 0 and 1 so that the completed zeta functions are holomorphic. Then the first integral is locally absolutely uniformly bounded in this region by Method 1.4.1. As for the second integral, Equation (3.41) implies that it is

$$O_\varepsilon\left(\int_{\mathcal{F}}f(x+iy)\overline{g(x+iy)}y^k\sum_{n\geq1}4n^{\frac{1}{2}-s+\varepsilon}\sqrt{y}e^{-2\pi ny}\,\frac{dx\,dy}{y^2}\right).$$

As the sum in the integrand is holomorphic, we can now appeal to Method 1.4.1. The meromorphic continuation to the critical strip and hence to all of $\mathbb{C}$ follows.

Since the poles of $E^*(z,s)$ are simple poles at $s=0$ and $s=1$, $L(s,f\otimes g)$ has at most simple poles here too. Actually, $\gamma(s,f\otimes g)$ has a simple pole at $s=0$ coming from the gamma factors and therefore its reciprocal has a simple zero. This cancels the possible simple pole at $s=0$ coming from $E^*(z,s)$ and therefore $L(s,f\otimes g)$ is actually holomorphic at $s=0$. So there is at most a simple pole at $s=1$.

We can now show that $L(s,f\otimes g)$ is of order 1 and conclude that it satisfies property (v) of Rankin-Selberg convolutions. Since the pole at $s=1$ is at worst simple, multiplying by $(s-1)$ clears the polar divisor. As the integral in the integral representation is absolutely bounded, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1 and $\mathrm{Re}(s)$ is bounded, we have

$$\frac{1}{\gamma(s,f\otimes g)}\ll_\varepsilon e^{|s|^{1+\varepsilon}},\tag{3.45}$$

for any $\varepsilon>0$. So the reciprocal of the gamma factor is of the same order. Then Equations (3.44) and (3.45) together imply

$$(s-1)L(s,f\otimes g)\ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

Thus $(s-1)L(s,f\otimes g)$ is of order 1, and so $L(s,f\otimes g)$ is as well after removing the polar factor. At last, we compute the residue of $L(s,f\otimes g)$ at $s=1$:

Proposition 3.5.3. *Let $f,g \in \mathcal{S}_k(\mathrm{PSL}_2(\mathbb{Z}))$ be primitive Hecke eigenforms. Then*

$$\operatorname*{Res}_{s=1} L(s, f \otimes g) = \frac{4^k \pi^{k+1}}{2\Gamma(k)} \langle f, g \rangle,$$

where $\langle f, g \rangle$ is the Petersson inner product.

Proof. From Equation (3.43), we see that

$$\lim_{s \to 1} 2^{k+1}\pi^k \frac{1}{\gamma(s, f \otimes g)} = \frac{4^k \pi^{k+1}}{\Gamma(k)}.$$

Then Proposition 3.5.2 implies

$$\operatorname*{Res}_{s=1} L(s, f \otimes g) = \frac{4^k \pi^{k+1}}{\Gamma(k)} \operatorname*{Res}_{s=1} \int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k E^*(z,s)\, d\mu = \frac{4^k \pi^{k+1}}{2\Gamma(k)} \langle f, g \rangle.$$ □

Notice that if $g = f$, then $\langle f, f \rangle \neq 0$ and therefore the residue at $s = 1$ is not zero and hence there is a genuine pole. Actually, by Theorem 2.2.6 the primitive Hecke eigenforms are orthogonal so that $\langle f, g \rangle = 0$ unless $f = g$. This is the only instance in which there is a pole. This verifies adjustment (iv) for Rankin-Selberg convolutions, and therefore we have shown altogether that $L(s, f \otimes g)$ is the Rankin-Selberg convolution of $L(s, f)$ and $L(s, g)$. We summarize all of our work into the following theorem:

Theorem 3.5.3. *For any two primitive Hecke eigenforms $f, g \in \mathcal{S}_k(\mathrm{PSL}_2(\mathbb{Z}))$, $L(s, f \otimes g)$ is a Selberg class L-function. It admits meromorphic continuation to $\mathbb{C}$ via the integral representation*

$$L(s, f \otimes g) = \frac{(4\pi)^{s+k-1}\pi^s}{\Gamma(s+k-1)\Gamma(s)} \int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k E^*(z,s)\, d\mu,$$

with functional equation

$$\pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) L(s, f \otimes g) = \Lambda(s, f \otimes g) = \Lambda(1-s, f \otimes g),$$

and if $f = g$ there is simple pole at $s = 1$ of residue $\frac{4^k \pi^{k+1}}{2\Gamma(k)} \langle f, g \rangle$.

3.6 Theta Functions

At this point we have shown that the zeta function, L-functions attached to primitive Dirichlet characters (of modulus $q > 1$), and L-functions attached to primitive Hecke eigenforms all admit meromorphic continuation to $\mathbb{C}$ and satisfy a functional equation of shape $s \to 1 - s$. In all of these cases, the idea was to find an integral representation that is meromorphic on $\mathbb{C}$ and symmetric under $s \to 1 - s$. There is a unifying idea which encompasses all of these cases and more. That idea is lifting a transformation law of a theta function by taking its Mellin transform. To connect the Mellin transform to L-functions, we require theta functions. For our purposes, a **theta function** is an infinite series indexed over a lattice whose terms are exponentials. We also require the theta function to be holomorphic on $\mathbb{C}$ and admit exponential decay to zero near ∞. Each of the L-functions we have studied, excluding the Rankin-Selberg convolution, is associated to a theta function:

$$\zeta(s) \longleftrightarrow \vartheta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s},$$
$$L(s,\chi) \longleftrightarrow \vartheta_\chi(s) = \sum_{n\in\mathbb{Z}} \chi(n) n^{\mathfrak{a}} e^{-\pi n^2 s},$$
$$L(s,f) \longleftrightarrow f(iy) = \sum_{n\in\mathbb{Z}} a(n) e^{-\pi n y},$$

where in the last case we note that $a(n) = 0$ for $n < 0$ because f is holomorphic and $a(0) = 0$ because f is cuspidal. Now on the one hand, all of these theta functions can be written as sums over $n \geq 1$: the first two cases by symmetry of the n and $-n$ terms and the last case by the above comment. Isolating the subsum over $n \geq 1$ and specializing at a nonnegative real variable we get $\omega(x)$, $\omega_\chi(x)$, and $f(iy)$ respectively. Taking the Mellin transform of these latter functions reproduced the associated L-functions up to gamma factors. We then decomposed the Mellin transform into two pieces and symmetrized the result by using transformation laws for the theta functions. The most difficult part of all of these arguments was cooking up the theta function that corresponds to the L-function solely from its representation as a Dirichlet series in the region of absolute convergence. For the Riemann zeta function this was essentially "Riemann's insight": start with the Gamma function, apply a change of variables, and sum over all $n \geq 1$ to obtain the Mellin transform of the corresponding theta function. For $L(s,\chi)$, the argument is adapted from that of Riemann although it is more complicated since the associated theta function depends on if the character χ is even or odd (odd being the more difficult case). On the other hand, $L(s,f)$ has the advantage that the theta function is easy to guess outright. It's the Fourier series of f along the upper-half imaginary axis. Moreover, it comes equip with the necessary transformation law via the modularity of f. In more general settings, we need to know something algebraic or geometric about a given theta function in order to deduce a transformation law that can be used to prove meromorphic continuation and a functional equation for its associated L-function. In general the method of meromorphic continuation is as follows:

Method 3.6.1. Suppose we are given an L-series

$$L(s,f) = \sum_{n\geq 1} \frac{a(n)}{n^s},$$

that is locally absolutely uniformly convergent for $\mathrm{Re}(s) > 1$, and there is a theta function $\omega_f(s)$ such that $L(s,f)$ is approximately the Mellin transform of $\omega_f(s)$. That is,

$$L(s,f) \approx \int_0^\infty \omega_f(s) x^s \, \frac{dx}{x}.$$

Also suppose $\omega_f(s)$ satisfies a tranformation law approximately of the form

$$\omega_f(x) \approx \omega_f\left(\frac{1}{q(f)^2 x}\right),$$

for some constant $q(f)$ (that will be the conductor of the L-function). Then $L(s,f)$ admits meromorphic continuation to $\mathbb{C}$. To acomplish this, first decompose the Mellin transform into two pieces:

$$\int_0^\infty \omega_f(x) x^s \, \frac{dx}{x} = \int_0^{\frac{1}{q(f)}} \omega_f(x) x^s \, \frac{dx}{x} + \int_{\frac{1}{q(f)}}^\infty \omega_f(x) x^s \, \frac{dx}{x}.$$

Then apply the transformation law for $\omega_f(s)$ to the first piece and symmetrize the result to obtain an integral representation of the following form:

$$L(s,f) \approx \text{polar factor} + \int_{\frac{1}{q(f)}}^{\infty} \omega_f(x)x^{1-s}\,\frac{dx}{x} + \int_{\frac{1}{q(f)}}^{\infty} \omega_f(x)x^{s}\,\frac{dx}{x},$$

where the polar factor will appear if there is a constant term in $\omega_f(x)$. Both of the integrals will be locally absolutely uniformly bounded by the exponential decay of $\omega_f(x)$. Since the integral representation is symmetric under $s \to 1-s$, this gives the meromorphic continuation to $\mathbb{C}$.

Chapter 4

Additional Results

4.1 Perron Formulas

With the Mellin inversion formula, it is not hard to prove a very useful integral expression for the sum of coefficients of a Dirichlet series. First, we setup some general notation. If $D(s)$ is a Dirichlet series with coefficients $a(n)$, then for any real X, we set

$$A'(X) = \sum_{n \le X}{}' a(n),$$

where the ' indicates that the last term is multiplied by $\frac{1}{2}$ if X is an integer. We would like to relate $A'(X)$ to an integral involving the entire Dirichlet series $D(s)$. In particular, this integral is a type of inverse Mellin transform. Any formula that relates a finite sum of coefficeints of a Dirichlet series to an integral involving the entire Dirichlet series is called a **Perron type formula**. We will see several of them, the first being **Perron's formula** which is a consequence of Abel's summation formula and the Mellin inversion formula applied to Dirichlet series:

Theorem 4.1.1 (Perron's formula). *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for any $c > \sigma_a$,*

$$A'(X) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c} D(s) X^s \, \frac{ds}{s}.$$

Proof. Let s be such that $\mathrm{Re}(s) > \sigma_a$. By Abel's summation formula (see Appendix B.3),

$$\sum_{n \ge 1} \frac{a(n)}{n^s} = \lim_{Y \to \infty} A'(Y) Y^{-s} + s \int_1^\infty A'(u) u^{-(s+1)} \, du.$$

Now $A'(Y) \le A(Y)$ and $A(Y) \ll_\varepsilon Y^{\sigma_a + \varepsilon}$ for any $\varepsilon > 0$ by Proposition 3.1.3. So that $A'(Y)Y^{-s} \ll Y^{\sigma_a + \varepsilon - \mathrm{Re}(s)}$. Choosing $\varepsilon < \mathrm{Re}(s) - \sigma_a$, this latter term tends to zero as $Y \to \infty$, which implies that $A'(Y)Y^{-s}$ also tends to zero as $Y \to \infty$. Therefore we can write the equation above as

$$D(s) s^{-1} = \int_1^\infty A'(u) u^{-(s+1)} \, du = \int_0^\infty A'(u) u^{-(s+1)} \, du,$$

where the second equality follows because $A(u) = 0$ in the interval $[0, 1)$. The Mellin inversion formula immediately gives the result. $\square$

We would like to relate this sum to an integral involving the entire Dirichlet series $D(s)$. In particular, this integral is a type of inverse Mellin transform. Any formula that resembles
Perron's formula is particularly useful because it allows one examine a sum of Dirichlet coefficients, a discrete object, by means of a complex integral where analytic techniques are at our disposal. There is also a truncated version of Perron's formula which is often more useful for estimates rather than abstract results. To state it, we need to setup some notation and will require a lemma. For any $c > 0$, consider the discontinuous integral (see [Dav80])

$$\delta(y) = \frac{1}{2\pi i}\int_{(c)} y^s \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$$

Also, for any $T > 0$, let

$$I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} y^s \frac{ds}{s},$$

be $\delta(y)$ truncated outside of height T. The lemma we require gives an approximation for how close $I(y,T)$ is to $\delta(y)$ (see [Dav80] for a proof):

Lemma 4.1.1. *For any $c > 0$, $y > 0$, and $T > 0$, we have*

$$I(y,T) - \delta(y) = \begin{cases} O\left(y^c \min\left(1, \frac{1}{T|\log(y)|}\right)\right) & \text{if } y \neq 1, \\ O\left(\frac{c}{T}\right) & \text{if } y = 1. \end{cases}$$

We can now state and prove the truncated version of Perron's formula:

Theorem 4.1.2 (Perron's formula, truncated version). *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for any $c > \sigma_a$ and $T > 0$,*

$$A'(X) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} D(s)X^s \frac{ds}{s} + O\left(X^c \sum_{\substack{n\geq 1\\ n\neq X}} \frac{a(n)}{n^c}\min\left(1, \frac{1}{T\left|\log\left(\frac{X}{n}\right)\right|}\right) + \delta_X \frac{c}{T}\right),$$

where $\delta_X = 1, 0$ according to if X is an integer or not.

Proof. By Appendix D.1, we have

$$A'(X) = \sum_{n\geq 1} a(n)\delta\left(\frac{X}{n}\right).$$

Now using Lemma 4.1.1, we may replace $\delta\left(\frac{X}{n}\right)$ to obtain

$$A'(X) = \sum_{n\geq 1} a(n)\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{X^s}{n^s}\frac{ds}{s} + \sum_{\substack{n\geq 1\\ n\neq X}} a(n)\left[O\left(\frac{X^c}{n^c}\min\left(1, \frac{1}{T\left|\log\left(\frac{X}{n}\right)\right|}\right) + \delta_X \frac{c}{T}\right)\right].$$

Since $D(s)$ converges absolutely we may move the sum inside of the big O-estimate, and moreover, the dominated convergence theorem implies we may interchange the sum and the integral. The statement of the lemma follows. $\square$

There is a slighly weaker variant of the truncated version of Perron's formula that follows as a corollary:

Corollary 4.1.1. *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for any $c > \sigma_a$ and $T > 0$,*

$$A'(X) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} D(s)X^s \frac{ds}{s} + O_c\left(\frac{X^c}{T}\right),$$

Proof. For sufficiently large X, we have

$$\min\left(1, \frac{1}{T\left|\log\left(\frac{X}{n}\right)\right|}\right) <_c \frac{X^c}{T}.$$

The statement now follows from the truncated version of Perron's formula. □

There is also a version of Perron's formula where we add a smoothing function. For any real X, we set

$$A_\psi(X) = \sum_{n\geq 1} a(n)\psi\left(\frac{n}{X}\right),$$

where $\psi : [0,\infty) \to [0,\infty)$ is a compactly supported smooth function. This is most useful in two cases. The first is when we choose $\psi(x)$ to be a smooth bump function. In this setting, the bump function has applies weight 1 or 0 to the Dirichlet coefficeints $a(n)$ and we can estimate sums like

$$\sum_{\frac{X}{2}\leq n<X} a(n) \quad \text{or} \quad \sum_{X\leq n<X+Y} a(n),$$

for some X and Y with $Y < X$. Sums of this type are called **unweighted**. As an example of an unweighted sum, let $\psi : [0,\infty) \to [0,\infty)$ be a smooth bump function that is identically 1 on $[0,1]$ and decays to zero in the interval $\left[1, \frac{X+1}{X}\right]$. Explicitely,

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ e^{-\frac{1-t}{\frac{X+1}{X}-t}} & \text{if } 1 < t \leq \frac{X+1}{X}, \\ 0 & \text{if } t \geq \frac{X+1}{X}. \end{cases}$$

Then

$$A_\psi(X) = \sum_{n\geq 1} a(n)\psi\left(\frac{n}{X}\right) = \sum_{n\leq X} a(n).$$

In the second case, we want ψ to dampen the $a(n)$ with a weight other than 1 or 0. Sums of this type are called **weighted**. In any case, suppose the support of $\psi(x)$ is contained in $[0, C]$. These conditions will force the Mellin transform $\Psi(s)$ of $\psi(x)$ to exist and have nice properties. To see that $\Psi(s)$ exists, let K be a compact set in the region $\mathrm{Re}(s) > 0$ and let $\beta = \inf_{s\in K}\{\mathrm{Re}(s)\}$. Note that $\psi(x)$ is bounded because it is compactly supported. Then for $s \in K$,

$$\Psi(s) = \int_0^\infty \psi(x)x^s \frac{dx}{x} \ll \int_0^C x^{\mathrm{Re}(s)-1}\,dx = \left.\frac{x^{\mathrm{Re}(s)}}{\mathrm{Re}(s)}\right|_0^C \leq \frac{C^{\mathrm{Re}(s)}}{\beta}.$$

Therefore $\Psi(s)$ is locally absolutely bounded for $\mathrm{Re}(s) > 0$. In particular, the Mellin inversion formula implies that $\psi(x)$ is the Mellin inverse of $\Psi(s)$. As for nice properties, $\Psi(s)$ does not grow too fast in vertical strips:

Proposition 4.1.1. *Let $\psi : [0, \infty) \to [0, \infty)$ to be a compactly supported smooth function and let $\Psi(s)$ denote its Mellin transform. Then for any $N \geq 1$,*

$$\Psi(s) \ll s^{-N},$$

provided s is contained in the vertical strip $a < \mathrm{Re}(s) < b$, for any a and b with $0 < a < b$.

Proof. Fix a and b with $a < b$. Also, let the support of $\psi(x)$ is contained in $[0, C]$. Now consider

$$\Psi(s) = \int_0^\infty \psi(x) x^s \,\frac{dx}{x}.$$

Since $\psi(x)$ is compactly supported, integrating by parts yields

$$\Psi(s) = \frac{1}{s} \int_0^\infty \psi'(x) x^{s+1} \,\frac{dx}{x}.$$

Repeatedly integrating by parts $N \geq 1$ times, we arrive at

$$\Psi(s) = \frac{1}{s(s+1)\cdots(s+N-1)} \int_0^\infty \psi^{(N)}(x) x^{s+N} \,\frac{dx}{x}.$$

As $s + k - 1 \sim s$ for $1 \leq k \leq N$, we have

$$\Psi(s) \ll s^{-N} \int_0^\infty \psi^{(N)}(x) x^{s+N} \,\frac{dx}{x}.$$

The claim will follow if we can show that the integral is bounded by a constant. Since $\psi(x)$ is compactly supported so is $\psi^{(N)}(x)$. In particular, this implies $\psi^{(N)}(x)$ is bounded. Therefore

$$\int_0^\infty \psi^{(N)}(x) x^{s+N} \,\frac{dx}{x} \ll \int_0^C x^{s+N} \,\frac{dx}{x} = \left.\frac{x^{s+N}}{s+N}\right|_0^C = \frac{C^{s+N}}{s+N} \ll \frac{C^{b+N}}{N} \ll 1,$$

where the second to last estimate follows because $a < \mathrm{Re}(s) < b$ with $0 < a < b$. So the integral is bounded by a constant and the claim follows. □

The following theorem is the smoothed version of Perron's formula:

Theorem 4.1.3 (Perron's formula, smoothed version). *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Let $\psi : [0, \infty) \to [0, \infty)$ be a compactly supported smooth function. Then for any $c > \sigma_a$,*

$$A_\psi(X) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c} D(s)\Psi(s) X^s \,ds,$$

where Ψ is the Mellin transform of ψ.

Proof. This is just a computation:

$$\begin{aligned}
A_\psi(X) &= \sum_{n \geq 1} a(n) \psi\left(\frac{n}{X}\right) \\
&= \sum_{n \geq 1} \frac{a(n)}{2\pi i} \int_{\mathrm{Re}(s)=c} \Psi(s) \left(\frac{n}{X}\right)^{-s} ds && \text{inverse mellin transform} \\
&= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\sigma} \sum_{n \geq 1} a(n) \Psi(s) \left(\frac{n}{X}\right)^{-s} ds && \text{DCT} \\
&= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c} D(s)\Psi(s) X^s \,ds.
\end{aligned}$$

□

The smoothed version of Perron's formula is useful because it is often more versatile than working with Perron's formula directly. Indeed, from Proposition 4.1.1 the convergence of the integral in the smoothed version improves and we can use analytic techniques to estimate this integral.

4.2 The Petersson Trace Formula

From Theorem 2.2.6, $\mathcal{S}_k(N,\chi)$ admits an orthonormal basis of primitive Hecke eigenforms. Call this basis $\{u_j\}_{1\le j\le r}$ where r is the dimension of $\mathcal{S}_k(N,\chi)$. Each of these cusp forms admits a Fourier series

$$u_j(z)=\sum_{n\ge 1}\lambda_j(n)e^{2\pi inz}.$$

The Petersson trace formula is an equation relating the Fourier coefficients $\lambda_j(n)$ of the basis $\{u_j\}_{1\le j\le r}$ to a sum of J-Bessel functions and Salié sums. To prove the Petersson trace formula we compute the inner product of two Poincaré series $P_{n,k,\chi}$ and $P_{m,k,\chi}$ in two different ways. One way is geometric in nature using the unfolding method and the other uses the spectral theory of $\mathcal{S}_k(N,\chi)$. Since Equation (2.3) says that $\langle P_{n,k,\chi},P_{m,k,\chi}\rangle$ essentially extacts the m-th Fourier coefficeint of $P_{n,k,\chi}$, the Petersson trace formula amounts to computing the m-th Fourier coefficient of $P_{n,k,\chi}$ in two different ways.
We will begin with the geometric method first which utilizes Equation (2.3). In order to make use of this equation we need to compute the Forier series of $P_{n,k,\chi}$ and this is achieved by the Poisson summation formula. From Remark 2.1.3, we have

$$P_{n,k,\chi}(z)=e^{2\pi inz}+\sum_{\substack{c\ge 1,d\in\mathbb{Z}\\ c\equiv 0 \pmod N\\ (c,d)=1}}\overline{\chi}(d)\frac{e^{2\pi in\left(\frac{a}{c}-\frac{1}{c^2z+cd}\right)}}{(cz+d)^k},$$

where a and b are chosen such that $\det\left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\right)=1$ and we have used the fact that

$$\frac{a}{c}-\frac{1}{c^2z+cd}=\frac{az+b}{cz+d}.$$

Now summing over all pairs (c,d) with $c\ge 1$, $d\in\mathbb{Z}$, $c\equiv 0 \pmod N$, and $(c,d)=1$ is the same as summing over all triples (c,ℓ,r) with $c\ge 1$, $\ell\in\mathbb{Z}$, $r \pmod c$, with $c\equiv 0 \pmod N$ and $(r,c)=1$. Indeed, this is seen by writing $d=c\ell+r$. Moreover, since $ad-bc=1$ we have $a(c\ell+r)-bc=1$ which further implies that $ar \equiv 1 \pmod{c}$. So we may take a to be the inverse for r modulo c. Then

$$\sum_{\substack{c\geq 1, d\in\mathbb{Z}\\(c,d)=1\\c\equiv 0 \pmod{N}}} \overline{\chi}(d)\frac{e^{2\pi in\left(\frac{a}{c}-\frac{1}{c^2z+cd}\right)}}{(cz+d)^k} = \sum_{(c,\ell,r)} \overline{\chi}(c\ell+r)\frac{e^{2\pi in\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k}$$

$$= \sum_{(c,\ell,r)} \overline{\chi}(r)\frac{e^{2\pi in\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k}$$

$$= \sideset{}{'}\sum_{\substack{c\geq 1\\c\equiv 0 \pmod{N}\\r \pmod{c}}} \sum_{\ell\in\mathbb{Z}} \overline{\chi}(r)\frac{e^{2\pi in\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k}$$

$$= \sideset{}{'}\sum_{\substack{c\geq 1\\c\equiv 0 \pmod{N}\\r \pmod{c}}} \overline{\chi}(r)\sum_{\ell\in\mathbb{Z}} \frac{e^{2\pi in\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k},$$

where the second line holds since χ has conductor $q \mid N$ and $c \equiv 0 \pmod{N}$. We will now apply the Poisson summation formula to the innermost sum. Set

$$I_{c,r}(z) = \sum_{\ell\in\mathbb{Z}} \frac{e^{2\pi in\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k}.$$

We will prove a transformation law for $I_{c,r}(z)$. Note that this function is holomorphic on $\mathbb{H}$ because $P_{n,k,\chi}(z)$ is holomorphic on $\mathbb{H}$. So by the identity theorem we may verify a transformation law on a set containing a limit point. Accordingly, set $z = iy$ for $y > 1$ and define

$$f(x) = \frac{e^{2\pi in\left(\frac{a}{c}-\frac{1}{c^2x+cr+ic^2y}\right)}}{(cx+r+icy)^k}.$$

To see that $f(x)$ is Schwarzm first observe

$$\mathrm{Im}\left(\frac{a}{c}-\frac{1}{c^2x+cr+ic^2y}\right) = \mathrm{Im}\left(-\frac{1}{c^2x+cr+ic^2y}\right) = \mathrm{Im}\left(\frac{c^2x+cr-ic^2y}{|c^2x+cr+ic^2y|}\right) = \frac{c^2y}{|c^2x+cr+ic^2y|}.$$

It follows that $\mathrm{Im}\left(\frac{a}{c}-\frac{1}{c^2x+cr+ic^2y}\right)$ tends to zero as $x \to \pm\infty$. Moreover, $|cx+r+icy| \geq |icy| \geq c$ so that

$$f(x) \ll \frac{e^{-2\pi n\mathrm{Im}\left(\frac{a}{c}-\frac{1}{c^2x+cr+ic^2y}\right)}}{c},$$

as $x \to \pm\infty$. Since the right-hand side of this estimate has exponential decay to zero, $f(x)$ is Schwarz. We now compute the Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx = \int_{-\infty}^{\infty} \frac{e^{2\pi in\left(\frac{a}{c}-\frac{1}{c^2x+cr+ic^2y}\right)}}{(cx+r+icy)^k}e^{-2\pi itx}\,dx.$$

Complexify the integral to get

$$\int_{\mathrm{Im}(z)=0} \frac{e^{2\pi in\left(\frac{a}{c}-\frac{1}{c^2z+cr+ic^2y}\right)}}{(cz+r+icy)^k} e^{-2\pi itz}\,dz.$$

Now make the change of variables $z \to z - \frac{r}{c} - iy$ to obtain

$$e^{2\pi in\frac{a}{c}} e^{2\pi it\left(iy+\frac{r}{c}\right)} \int_{\mathrm{Im}(z)=y} \frac{e^{-\frac{2\pi in}{c^2z}}}{(cz)^k} e^{-2\pi itz}\,dz.$$

The integrand is meromorphic with a pole only at $z = 0$. Therefore by shifting the line of integration we may take the limit as $\mathrm{Im}(z) \to \infty$ without picking up additional residues. However

$$\left| e^{-\frac{2\pi in}{c^2z}} \right| = e^{2\pi n\left(\frac{\mathrm{Im}(z)}{|cz|^2}\right)} \quad \text{and} \quad \left| e^{-2\pi itz} \right| = e^{2\pi t\mathrm{Im}(z)}.$$

So as $\mathrm{Im}(z) \to \infty$, the first expression has exponential decay to zero and the second expression does to if and only if $t < 0$. Moreover, when $t = 0$ the second expression is bounded. Altogether this means that the integral vanishes if $t \leq 0$. It remains to compute the integral for $t > 0$. To do this, make the change of variables $z \to -\frac{z}{2\pi it}$ to the to obtain

$$\begin{aligned}
-\frac{1}{2\pi it}\int_{\mathrm{Re}(x)=2\pi ty} \frac{e^{-\frac{4\pi^2nt}{c^2z}}}{\left(-\frac{cz}{2\pi it}\right)^k} e^z\,dz &= -\frac{1}{2\pi it}\int_{\mathrm{Re}(z)=2\pi ty} \left(-\frac{2\pi it}{cz}\right)^k e^{z-\frac{4\pi^2nt}{c^2z}}\,dz \\
&= \frac{(-2\pi it)^{k-1}}{c^k}\int_{\mathrm{Re}(z)=2\pi ty} z^{-k} e^{z-\frac{4\pi^2nt}{c^2z}}\,dz \\
&= \frac{(-2\pi it)^{k-1}}{c^k}\int_{-\infty}^{(0^+)} z^{-k} e^{z-\frac{4\pi^2nt}{c^2z}}\,dz \\
&= \frac{2\pi i^{-k}}{c}\left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right),
\end{aligned}$$

where in the second to last line we have homotoped the line of integration to a Hankle contour about the negative real axis and in the last line we have used the Schläflin integral representation for the J-Bessel function (see Appendix B.7). So in total we obtain

$$\left(\frac{2\pi i^{-k}}{c}\left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right) e^{2\pi in\frac{a}{c}+2\pi it\frac{r}{c}}\right) e^{2\pi t(iy)},$$

when $t > 0$. By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z) = \sum_{t>0}\left(\frac{2\pi i^{-k}}{c}\left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right) e^{2\pi in\frac{a}{c}+2\pi it\frac{r}{c}}\right) e^{2\pi itz},$$

for all $z \in \mathbb{H}$. Plugging this back into the Poincaré series gives a form of the Fourier series

$$P_{n,k,\chi}(z) = e^{2\pi inz} + \sideset{}{'}\sum_{\substack{c\geq 1 \\ c\equiv 0 \pmod{N} \\ r \pmod{c}}} \overline{\chi}(r) \sum_{t>0} \left(\frac{2\pi i^{-k}}{c} \left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right) e^{2\pi in\frac{a}{c}+2\pi it\frac{r}{c}} \right) e^{2\pi inz}$$

$$= \sum_{t>0} \left(\delta_{n,t} + \left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{k-1} \sideset{}{'}\sum_{\substack{c\geq 1 \\ c\equiv 0 \pmod{N} \\ r \pmod{c}}} \overline{\chi}(r) \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right) e^{2\pi in\frac{a}{c}+2\pi it\frac{r}{c}} \right) e^{2\pi itz}$$

$$= \sum_{t>0} \left(\delta_{n,t} + \left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{k-1} \sum_{\substack{c\geq 1 \\ c\equiv 0 \pmod{N}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right) \sideset{}{'}\sum_{r \pmod{c}} \overline{\chi}(r) e^{2\pi in\frac{a}{c}+2\pi it\frac{r}{c}} \right) e^{2\pi itz}.$$

We will simplify the innermostsum. Since a is the inverse for r modulo c, the innermost sum above becomes

$$\sideset{}{'}\sum_{r \pmod{c}} \overline{\chi}(r) e^{2\pi in\frac{a}{c}+2\pi it\frac{r}{c}} = \sideset{}{'}\sum_{r \pmod{c}} \overline{\chi}(\overline{a}) e^{2\pi in\frac{a}{c}+2\pi it\frac{\overline{a}}{c}} = \sideset{}{'}\sum_{a \pmod{c}} \chi(a) e^{\frac{2\pi i(an+\overline{a}t)}{c}} = S_\chi(n,t;c).$$

So at last, we obtain our desired Fourier series

$$P_{n,k,\chi}(z) = \sum_{t>0} \left(\delta_{n,t} + \left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{k-1} \sum_{\substack{c\geq 1 \\ c\equiv 0 \pmod{N}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nt}}{c}\right) S_\chi(n,t;c) \right) e^{2\pi itz}.$$

We can now derive the first half of the Petersson trace formula. Using Equation (2.3) we obtain

$$\langle P_{n,k,\chi}, P_{m,k,\chi}\rangle = \frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}} \left(\delta_{n,m} + \left(\frac{\sqrt{m}}{\sqrt{n}}\right)^{k-1} \sum_{\substack{c\geq 1 \\ c\equiv 0 \pmod{N}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_\chi(n,m;c) \right).$$

To obtain the second half of the Petersson trace formula, we use the fact that $\{u_j\}_{1\leq j\leq r}$ is an orthonormal basis and Equation (2.3) to write

$$\begin{aligned}
P_{n,k,\chi}(z) &= \sum_{1\leq j\leq r} \langle P_{n,k,\chi}, u_j\rangle u_j(z) \\
&= \sum_{1\leq j\leq r} \overline{\langle P_{n,k,\chi}, u_j\rangle} u_j(z) \\
&= \frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}} \sum_{1\leq j\leq r} \lambda_j(n) u_j(z) \\
&= \frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}} \sum_{1\leq j\leq r} \lambda_j(n) \sum_{t\geq 1} \lambda_j(t) e^{2\pi itz} \\
&= \sum_{t\geq 1} \left(\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}} \sum_{1\leq j\leq r} \overline{\lambda_j(n)} \lambda_j(t) \right) e^{2\pi itz}.
\end{aligned}$$

The last expression is an alternative representation of the Fourier series of $P_{n,k,\chi}$. Using Equation (2.3) again but aplied to this reprensentation, we obtain the second half of the Petersson trace formula

$$\langle P_{n,k,\chi}, P_{m,k,\chi}\rangle = \left(\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}}\right)^2 \sum_{1\le j\le r} \overline{\lambda_j(n)}\lambda_j(m).$$

Equating the first and second half and canceling the common $\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}}$ factor gives

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}} \sum_{1\le j\le r} \overline{\lambda_j(n)}\lambda_j(m) = \delta_{n,m} + \left(\frac{\sqrt{m}}{\sqrt{n}}\right)^{k-1} \sum_{\substack{c\ge 1\\ c\equiv 0 \pmod N}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_\chi(n,m;c).$$

Now when $n = m$, $\left(\frac{\sqrt{m}}{\sqrt{n}}\right)^{k-1} = 1$ so we can factor this term out of the entire right-hand side and cancel it resulting in the **Petersson trace formula**:

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi \sqrt{nm})^{k-1}} \sum_{1\le j\le r} \overline{\lambda_j(n)}\lambda_j(m) = \delta_{n,m} + \sum_{\substack{c\ge 1\\ c\equiv 0 \pmod N}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_\chi(n,m;c).$$

We refer to the left-hand side side as the **spectral side** and the right-hand side as the **geometric side**. We collect our work as a theorem:

Theorem 4.2.1 (Petersson trace formula). *Let $\{u_j\}_{1\le j\le r}$ be an orthonormal basis of primitive Hecke eigenforms for $\mathcal{S}_k(\Gamma_0(N),\chi)$ with Fourier coefficeints $\lambda_j(n)$. Then for any positive integers $n, m \ge 1$, we have*

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi \sqrt{nm})^{k-1}} \sum_{1\le j\le r} \overline{\lambda_j(n)}\lambda_j(m) = \delta_{n,m} + \sum_{\substack{c\ge 1\\ c\equiv 0 \pmod N}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_\chi(n,m;c).$$

4.3 The Ramanujan Conjecture on Average

Let f be a weight k primitive Hecke eigenform with Fourier coefficients $\lambda_f(n)$. As an application of the Rankin-Selberg method, it is possible to show the slightly weaker result that $\lambda_f(n) \ll n^{\frac{k-1}{2}+\varepsilon}$ holds on average, for any $\varepsilon > 0$, without assuming the Ramanujan conjecture. To see this, for any real X, we have

$$\left(\sum_{n\le X} \lambda_f(n)\right)^2 \le X \sum_{n\le X} |\lambda_f(n)|^2, \tag{4.1}$$

by Cauchy-Schwarz. Now, without assuming the Ramanujan conjecture, the Rankin-Selberg square $L(s, f\otimes f)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{3}{2}$. So while the critical strip is wider, it still admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$. By Landau's theorem, the abscissa of absolute convergence of $L(s, f\otimes f)$, and hence $L(s, f\times f)$ too, is 1 so that $\sum_{n\le X} |\lambda_f(n)|^2 \ll_\varepsilon X^{k+\varepsilon}$. Substiuting this bound into Equation (4.1), we obtain

$$\left(\sum_{n\le X} \lambda_f(n)\right)^2 \ll_\varepsilon X^{k-1+\varepsilon}.$$

Upon taking the square root,

$$\sum_{n \leq X} \lambda_f(n) \ll_\varepsilon X^{\frac{k-1}{2}+\varepsilon},$$

for some $\varepsilon > 0$. This bound should be compared with the implication $\lambda_f(n) \ll n^{\frac{k-1}{2}+\varepsilon}$ that follows from the Ramanujan conjecture.

Part II

Analytic Methods and L-functions

Chapter 5

Non-vanishing Results for L-functions

This chapter is special in that it is the first time we see how L-functions can be used to prove interesting arithmetic results. Namely, we prove two crowing germs of analytic number theory: Dirichlet's theorem on primes in arithmetic progressions and the prime number theorem. The first of these two theorems, is a consequence of a non-vanishing results for Dirichlet L-functions at $s = 1$. This should give sufficient motivation for the study of non-vanishing results so after proving Dirichlet's result, we prove two stronger non-vanishing theorems: one for the Riemann zeta function and the other for Dirichlet L-functions. We then use the non-vanishing result for the Riemann zeta function to prove the other crowning gem of analytic number theory: the prime number theorem. These results are important not only because they are famous and were sought after, but because many other analytic techniques we will devlope will be used to prove generalizations and sharper versions of both Dirichlet theorem on primes in arithmetic progressions and the prime number theorem. For example, after devloping some additional machinery in the later chapters we will give another proof of the prime number theorem that is more easily generalizable to other settings. As for what a non-vanishing result is, given an L-function $L(s)$ admitting meromorphic continuation to $\mathbb{C}$ and a functional equation of shape $s \to 1 - s$, it is an interesting question to ask for which s, $L(s) \neq 0$. A result that says for $s = s_0$, $L(s_0) \neq 0$ is known as a **non-vanishing result**. Non-vanishing results tend to have very important consequences about the arithmetic of $\mathbb{Z}$ since L-functions are encoding arithmetic information. Some particularly important cases are when $s = \frac{1}{2}$, $s = 1$, or more generally when s is either on the critical line or the right boundary of the critical strip (the left boundary is immediate from the functional equation).

5.1 Dirichlet's Theorem on Primes in Arithmetic Progressions

One of the more well-known arithmetic results proved using L-functions is **Dirichlet's theorem on primes in arithmetic progressions**:

Theorem 5.1.1 (Dirichlet's theorem on primes in arithmetic progressions). *Let a and m be positive integers such that $(a, m) = 1$. Then the arithmetic progression $\{a + km \mid k \in \mathbb{N}\}$ contain infinitely many primes.*

We will delay the proof for the moment, for it is well-worth understanding the some of the motivation behind why this theorem is interesting and how exactly Dirichlet used the analytic techniques of L-functions to attack this purely arithmetic statement. We being by recalling Euclid's famous theorem on the infitude of the primes. Euclid's proof is completely elementary and arithmetic in nature. He argues that if there were finitely many primes $p_1, p_2, \ldots, p_k$ then a short consideration of $(p_1 p_2 \cdots p_k) + 1$ shows that this number must either be divisible by a prime not in our list or must be prime itself. As primes are the multiplicative building blocks of arithmetic, Euclid assures us that we have an ample amount of prime clay to work with. Now there is a slightly stronger result due to Euler (see [Eul44]) requiring analytic techniques (this result was introduced in Chapter 3):

Theorem 5.1.2. *The series*

$$\sum_p \frac{1}{p},$$

diverges.

Proof. For $\mathrm{Re}(s) > 1$, $\zeta(s)$ is holomorphic and admits the Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

Taking the logarithm, we get

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s}).$$

The Taylor series of the logarithm gives

$$\log(1 - p^{-s}) = \sum_{k \geq 1} (-1)^{k-1} \frac{(-p^{-s})^k}{k} = \sum_{k \geq 1} (-1)^{2k-1} \frac{1}{kp^{ks}},$$

so that

$$\log \zeta(s) = \sum_p \sum_{k \geq 1} \frac{1}{kp^{ks}}. \tag{5.1}$$

The double sum restricted to $k \geq 2$ is uniformly bounded for $\mathrm{Re}(s) > 1$. To see this, first observe

$$\left| \sum_{k \geq 2} \frac{1}{kp^{ks}} \right| \leq \sum_{k \geq 2} \left| \frac{1}{kp^{ks}} \right| \leq \sum_{k \geq 2} \left| \frac{1}{p^{ks}} \right| \leq \sum_{k \geq 2} \frac{1}{p^k} = \frac{1}{p^2} \sum_{k \geq 0} \frac{1}{p^k} = \frac{1}{p^2}(1 - p^{-1})^{-1} \leq \frac{2}{p^2},$$

where the last inequality follows because $p \geq 2$. Then

$$\left| \sum_p \sum_{k \geq 2} \frac{1}{kp^{ks}} \right| \leq 2 \sum_p \frac{1}{p^2} < 2 \sum_{n \geq 1} \frac{1}{n^2} = 2\zeta(2). \tag{5.2}$$

So by equation Equations (5.1) and (5.2),

$$\left| \log \zeta(s) - \sum_p \frac{1}{p^s} \right| = \left| \sum_p \sum_{k \geq 2} \frac{1}{kp^{ks}} \right|,$$

remains bounded as $s \to 1$. The claim now follows since $\zeta(s)$ has a simple pole at $s = 1$. $\square$

Theorem 5.1.2 tells us that there are infinitely many primes, but also that the primes are not too "sparce" in the integers for otherwise the series would converge. The idea Dirichlet used to prove his result on primes in arithmetic progressions was in a very similar spirit. He sought out to prove the divergence of the series

$$\sum_{p \equiv a \pmod{m}} \frac{1}{p},$$

for positive integers a and m with $(a, m) = 1$ as the divergence immediately implies there are infinitely many primes p of the form $p \equiv a \pmod{m}$. In the case $a = 1$ and $m = 2$ we recover Theorem 5.1.2 exactly since every prime is odd.

Dirichlet's proof proceeds in a similar way to that of Theorem 5.1.2 and this is where Dirichlet used what are now known as Dirichlet characters and Dirichlet L-functions. The proof can be broken into three steps. The first is to proceed as Euler did, but with the Dirichlet L-function $L(s, \chi)$ where χ has modulus m. That is, write $L(s, \chi)$ as a sum over primes and a bounded term as $s \to 1$. The next step is to use the orthogonality relations of the characters to sieve out the correct sum. The last step is to show the non-vanishing result $L(1, \chi) \neq 0$ for all non-principal characters χ. This is the essential part of the proof as it is what assures us that the sum diverges. We will prove this non-vanishing result first and then prove Dirichlet's theorem on primes in arithmetic progressions.

Theorem 5.1.3. *For any non-principal Dirichlet character χ, $L(1, \chi) \neq 0$.*

Proof. Choose a positive integer $m > 1$. It will be enough to prove this for all Dirichlet characters χ modulo m. We establish a preliminary result first. We claim there exist positive integers f_p and g_p with $f_p g_p = \phi(m)$ such that

$$\prod_{\chi} L(s, \chi) = \prod_{p \nmid m} (1 - p^{-f_p s})^{-g_p}.$$

To see this, the map $\chi \to \chi(p)$ is a homomorphism from the group of Dirichlet characters modulo m into μ_m the group of m-th roots of unity. Since μ_m is cyclic, the image of this map is a cyclic group of order say f_p. In other words, the image is exactly μ_{f_p}. Letting g_p be the order of the kernel, $f_p g_p = \phi(m)$ because $X_m \cong (\mathbb{Z}/m\mathbb{Z})^*$. In other words, for every $\omega \in \mu_{f_p}$ there are g_p characters χ such that $\chi(p) = \omega$. So for fixed $p \nmid m$, we compute

$$\prod_{\chi} (1 - \chi(p) p^{-s}) = \prod_{\omega \in \mu_m} (1 - \omega p^{-s})^{g_p} = (1 - p^{-f_p s})^{g_p}, \tag{5.3}$$

where the last equality follows since the product is over all f_p-th roots of unity. Then Equation (5.3) implies

$$\prod_{\chi} L(s, \chi) = \prod_{\chi} \prod_{p \nmid m} (1 - \chi(p) p^{-s})^{-1} = \prod_{p \nmid m} \prod_{\chi} (1 - \chi(p) p^{-s})^{-1} = \prod_{p \nmid m} (1 - p^{-f_p s})^{-g_p}. \tag{5.4}$$

This establishes the preliminary result. Upon expanding the last product in Equation (5.4), we see that $\prod_\chi L(s, \chi)$ defines a Dirichlet series with positive coefficients and constant term 1. Therefore it takes positive values larger than 1 along the part of the real line in the region $\mathrm{Re}(s) > 1$. We will now show $L(1, \chi) \neq 0$ for non-principal χ, and we will seperate the cases χ is real or complex. Suppose χ is complex. If $L(1, \chi) = 0$ then the functional equation implies $L(1, \overline{\chi}) = 0$. Now on the other hand, $L(s, \chi_{m,0})$ has a simple pole at $s = 1$ (coming from the $\zeta(s)$ factor) so altogether $\prod_\chi L(s, \chi)$ is zero at $s = 1$. This contradicts that it takes positive values larger than 1 along the part of the real line in the region $\mathrm{Re}(s) > 1$ and so $L(1, \chi) \neq 0$. Now suppose χ is real and consider

$$\frac{L(s, \chi_{m,0}) L(s, \chi)}{L(2s, \chi_{m,0})} = \prod_{p \nmid m} \frac{(1 - p^{-s})^{-1} (1 - \chi(p) p^{-s})^{-1}}{(1 - p^{-2s})^{-1}}.$$

If $\chi(p) = -1$ then the corresponding factor on the right-hand side is 1. If $\chi(p) = 1$, then

$$\frac{(1 - p^{-s})^{-1} (1 - \chi(p) p^{-s})^{-1}}{(1 - p^{-2s})^{-1}} = \frac{(1 - p^{-s})^{-2}}{(1 - p^{-2s})^{-1}} = \frac{(1 + p^{-s})}{(1 - p^{-s})} = 1 + 2 \sum_{k \geq 1} \frac{1}{p^{ks}}.$$

These facts together imply that $\frac{L(s,\chi_{m,0})L(s,\chi)}{L(2s,\chi_{m,0})}$ defines a Dirichlet series with positive coefficients and constant term 1. Therefore it takes positive values larger than 1 along the part of the real line in the region $\mathrm{Re}(s) > 1$. If $L(1, \chi) = 0$ then $\frac{L(s,\chi_{m,0})L(s,\chi)}{L(2s,\chi_{m,0})}$ is zero at $s = 1$ because the zero of $L(s, \chi)$ cancels the simple pole of $L(s, \chi_{m,0})$ at $s = 1$ and $L(2, \chi_{m,0}) \neq 0$ because $L(s, \chi_{m,0})$ is defined by a Dirichlet series with positive coefficients in the region $\mathrm{Re}(s) > 1$. As in the complex case, this gives a contradiction. So we have shown $L(1, \chi) \neq 0$ for all non-principal χ which completes the proof. $\square$

We now have enough machinery to prove Dirichlet's theorem on primes in arithmetic progressions:

Proof of Dirichlet's theorem on primes in arithmetic progressions. Let χ be a Dirichlet character modulo m. Then for $\mathrm{Re}(s) > 1$, $L(s, \chi)$ is holomorphic and admits the Euler product

$$L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}.$$

Taking the logarithm gives

$$\log L(s, \chi) = -\sum_p \log(1 - \chi(p)p^{-s}).$$

The Taylor series of the logarithm implies

$$\log(1 - \chi(p)p^{-s}) = \sum_{k\geq 1} (-1)^{k-1} \frac{(-\chi(p)p^{-s})^k}{k} = \sum_{k\geq 1} (-1)^{2k-1} \frac{\chi(p^k)}{kp^{ks}},$$

so that

$$\log L(s, \chi) = \sum_p \sum_{k\geq 1} \frac{\chi(p^k)}{kp^{ks}}. \tag{5.5}$$

The double sum restricted to $k \geq 2$ is uniformly bounded for $\mathrm{Re}(s) > 1$. Indeed, first observe

$$\left| \sum_{k\geq 2} \frac{\chi(p^k)}{kp^{ks}} \right| \leq \sum_{k\geq 2} \left| \frac{\chi(p^k)}{kp^{ks}} \right| \leq \sum_{k\geq 2} \left| \frac{1}{p^{ks}} \right| \leq \sum_{k\geq 2} \frac{1}{p^k} = \frac{1}{p^2} \sum_{k\geq 0} \frac{1}{p^k} = \frac{1}{p^2}(1 - p^{-1})^{-1} \leq \frac{2}{p^2},$$

where the last inequality follows because $p > 2$. Then

$$\left| \sum_p \sum_{k\geq 2} \frac{\chi(p^k)}{kp^{ks}} \right| \leq 2\sum_p \frac{1}{p^2} < 2\sum_{n\geq 1} \frac{1}{n^2} = 2\zeta(2),$$

as desired. Now using Equation (5.5), we have

$$\sum_{\chi \pmod m} \overline{\chi(a)} \log L(s, \chi) = \sum_{\chi \pmod m} \sum_p \frac{\overline{\chi(a)}\chi(p)}{p^s} + \sum_{\chi \pmod m} \overline{\chi(a)} \sum_p \sum_{k\geq 2} \frac{\chi(p^k)}{kp^{ks}}. \tag{5.6}$$

By the orthogonality relations (Proposition 1.2.1 (ii)), we find that

$$\sum_{\chi \pmod m} \sum_p \frac{\overline{\chi(a)}\chi(p)}{p^s} = \sum_p \frac{1}{p^s} \sum_{\chi \pmod m} \overline{\chi(a)}\chi(p) = \phi(m) \sum_{p\equiv a \pmod m} \frac{1}{p^s}, \tag{5.7}$$

and so combining Equations (5.6) and (5.7) gives

$$\sum_{\chi \pmod m} \overline{\chi(a)} \log L(s,\chi) - \sum_{\chi \pmod m} \overline{\chi(a)} \sum_p \sum_{k \geq 2} \frac{\chi(p^k)}{kp^{ks}} = \phi(m) \sum_{p \equiv a \pmod m} \frac{1}{p^s}.$$

The latter sum on the left-hand side is uniformly bounded for $\mathrm{Re}(s) > 1$ because the inner double sum is and there are finitely many Dirichlet characters modulo m. Therefore it suffices to show that the first sum on the left-hand side diverges as $s \to 1$. For $\chi = \chi_{m,0}$,

$$L(s,\chi_{m,0}) = \zeta(s) \prod_{p|m} (1-p^{-s}).$$

So the corresponding term in the sum is

$$\overline{\chi_{m,0}}(a) \log L(s,\chi_{m,0}) = \log L(s,\chi_{m,0}) = \log \left(\zeta(s) \prod_{p|m} (1-p^{-s}) \right) = \log \zeta(s) + \sum_{p|m} \log(1-p^{-s}),$$

which diverges as $s \to 1$ because $\zeta(s)$ has a simple pole at $s = 1$. We will be done if $\log L(s,\chi)$ remains bounded as $s \to 1$ for all $\chi \neq \chi_{m,0}$. So assume χ is not principal. Then if $\widetilde{\chi}$ is the character inducing χ, we have

$$L(s,\chi) = L(s,\widetilde{\chi}) \prod_{p|m} (1-\widetilde{\chi}(p)p^{-s}),$$

where $L(s,\widetilde{\chi})$ is holomorphic. Therefore $L(s,\chi)$ is holomorphic too so it further suffices to show $L(1,\chi) \neq 0$. This follows from Theorem 5.1.3 and thus the proof is complete. □

5.2 Non-vanishing on $\mathrm{Re}(s) = 1$

Here we provide proofs that the Riemann zeta function and Dirichlet L-functions do not vanish on the line $\mathrm{Re}(s) = 1$. The second of these two results can be regarded as a stronger version of Theorem 5.1.3. While both will play a role in understanding the zeros of these L-functions, the non-vanishing result for the Riemann zeta function is the key ingredient in the proof of the prime number theorem. We will proof the non-vanishing result for $\zeta(s)$ first, but we need a lemma that will be immensely useful in other investigations:

Lemma 5.2.1. *For any nonzero real t, set*

$$\eta(s) = \zeta(s)^3 \zeta(s+it)^4 \zeta(s+2it).$$

Then for $s = \sigma > 1$,

$$\mathrm{Re}\left(\frac{\eta'}{\eta}(\sigma)\right) = \mathrm{Re}\left(3\frac{\zeta'}{\zeta}(\sigma) + 4\frac{\zeta'}{\zeta}(\sigma+it) + \frac{\zeta'}{\zeta}(\sigma+2it)\right) \leq 0.$$

Proof. In the region $\mathrm{Re}(s) > 1$, $\zeta(s)$ is holomorphic and admits the Euler product

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}.$$

Taking the logarithmic derivative of $\zeta(s)$ gives

$$\frac{\zeta'}{\zeta}(s) = -\sum_{p} \frac{\log(p)p^{-s}}{1-p^{-s}} = -\sum_{p}\sum_{k\geq 1} \frac{\log(p)}{p^{ks}} = -\sum_{n\geq 1} \frac{\Lambda(n)}{n^s}, \tag{5.8}$$

where $\Lambda(n)$ is the von Mangoldt function (see Appendix A.1). Now fix $s = \sigma + it$ with $\sigma > 1$ and observe

$$\begin{aligned}\sum_{n\geq 1} \frac{\Lambda(n)}{n^s} &= \sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma n^{it}} \\ &= \sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma} e^{-it\log(n)} \\ &= \sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma}\Big(\cos(t\log(n)) - i\sin(t\log(n))\Big).\end{aligned}$$

We conclude

$$\operatorname{Re}\left(\frac{\zeta'}{\zeta}(s)\right) = -\sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma}\cos(t\log(n)). \tag{5.9}$$

Since $\cos(2\theta) = 2\cos(\theta) - 1$ for any θ, we have

$$3 + 4\cos(\theta) + \cos(2\theta) = 2(1+\cos(\theta))^2 \geq 0, \tag{5.10}$$

provided θ is real. As $\frac{\Lambda(n)}{n^\sigma} \geq 0$ for all $n \geq 1$, so Equation (5.10) implies

$$\sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma}\Big(3 + 4\cos(t\log(n)) + \cos(2t\log(n))\Big) \geq 0, \tag{5.11}$$

for any real t. $\square$

The non-vanishing result follows very easily from Lemma 5.2.1:

Theorem 5.2.1. *$\zeta(s) \neq 0$ on the line* $\operatorname{Re}(s) = 1$.

Proof. We will now show $\zeta(s) \neq 0$ on the line $\operatorname{Re}(s) = 1$. We may assume $s \neq 1$ because we know $\zeta(s)$ has a simple pole there. So fix a real nonzero t and consider the function

$$\eta(s) = \zeta(s)^3\zeta(s+it)^4\zeta(s+2it),$$

Suppose $\zeta(1+it) = 0$. Then at $s = 1$, $\eta(s)$ would have a zero, either simple or of order 2, depending on if $\zeta(s+2it)$ was zero or not since $\zeta(s)^3(s+it)^4$ has a simple zero at $s = 1$. Therefore it suffices to show $\eta(s)$ is nonzero at $s = 1$. Let d be the order of the zero of $\eta(s)$ at $s = 1$. Note that $d \geq 1$. Then $\eta(s) = (s-1)^d\eta_1(s)$ with $\eta_1(s)$ holomorphic at $s = 1$ and such that $\eta_1(1) \neq 0$. Upon taking the logarithmic derivative, we obtain

$$\frac{\eta'}{\eta}(s) = \frac{d}{s-1} + \frac{\eta_1'}{\eta_1}(s).$$

Then for $\sigma > 1$, it follows that

$$\lim_{\sigma\to 1}(\sigma - 1)\frac{\eta'}{\eta}(\sigma) = d.$$

But from Lemma 5.2.1, $(\sigma-1)\frac{\eta'}{\eta}(\sigma)$ has nonpositive real part and so the limit cannot be the positive integer d. This is a contradiction. Therefore $\eta(s)$ is nonzero at $s = 1$ and thus $\zeta(1+it) \neq 0$. $\square$

There is a completely analogous lemma and argument for Dirichlet L-functions. First the lemma:

Lemma 5.2.2. *For any non-principal Dirichlet character χ of conductor $q > 1$, and nonzero real t, set*

$$\eta(s, \chi) = L(s, \chi_{q,0})^3 L(s + it, \chi)^4 L(s + 2it, \chi^2).$$

Then for $s = \sigma > 1$,

$$\operatorname{Re}\left(\frac{\eta'}{\eta}(\sigma, \chi)\right) = \operatorname{Re}\left(3\frac{L'}{L}(\sigma, \chi_{q,0}) + 4\frac{L'}{L}(\sigma + it, \chi) + \frac{L'}{L}(\sigma + 2it, \chi^2)\right) \leq 0.$$

Proof. In the region $\operatorname{Re}(s) > 1$, $L(s, \chi)$ is holomorphic and admits the Euler product

$$L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}.$$

Taking the logarithmic derivative of $L(s, \chi)$ yields

$$\frac{L'}{L}(s, \chi) = -\sum_p \frac{\chi(p)\log(p)p^{-s}}{1 - \chi(p)p^{-s}} = -\sum_p \sum_{k\geq 1} \frac{\chi(p^k)\log(p)}{p^{ks}} = -\sum_{n\geq 1} \frac{\chi(n)\Lambda(n)}{n^s}, \tag{5.12}$$

where $\Lambda(n)$ is the von Mangoldt function. Setting $s = \sigma + it$ with $\sigma > 1$ and writing $\chi(n) = e^{i\varphi}$, we have

$$\begin{aligned}
\sum_{n\geq 1} \frac{\chi(n)\Lambda(n)}{n^s} &= \sum_{n\geq 1} \frac{\chi(n)\Lambda(n)}{n^\sigma n^{it}} \\
&= \sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma} e^{i(\varphi - t\log(n))} \\
&= \sum_{n\geq 1} \frac{\chi(n)\Lambda(n)}{n^\sigma}\Big(\cos(\varphi - t\log(n)) + i\sin(\varphi - t\log(n))\Big).
\end{aligned}$$

It follows that

$$\operatorname{Re}\left(\frac{L'}{L}(s, \chi)\right) = -\sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma}\cos(\varphi - t\log(n)). \tag{5.13}$$

Because $\chi^2(n) = e^{2i\varphi}$ and $\chi_{q,0}(n) = 1$, we obtain analogous equations to Equation (5.13):

$$\operatorname{Re}\left(\frac{L'}{L}(s, \chi^2)\right) = -\sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma}\cos(2\varphi - t\log(n)), \tag{5.14}$$

and

$$\operatorname{Re}\left(\frac{L'}{L}(s, \chi_{q,0})\right) = -\sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma}\cos(t\log(n)). \tag{5.15}$$

Since $\frac{\Lambda(n)}{n^\sigma} \geq 0$ for all $n \geq 1$, Equation (5.10) implies

$$\sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma}\Big(3 + 4\cos(\varphi - t\log(n)) + 4\cos(2\varphi - 2t\log(n))\Big), \tag{5.16}$$

for any real t. Now for $s = \sigma$ with $\sigma > 1$, upon taking the logarithmic derivative of $\eta(\sigma, \chi)$, Equations (5.13) to (5.16) together imply

$$\operatorname{Re}\left(\frac{\eta'}{\eta}(\sigma, \chi)\right) = \operatorname{Re}\left(3\frac{L'}{L}(\sigma, \chi_{q,0}) + 4\frac{L'}{L}(\sigma + it, \chi) + \frac{L'}{L}(\sigma + 2it, \chi^2)\right) \leq 0.$$

□

The non-vanishing result for $L(s,\chi)$ is proven in a manner similar to $\zeta(s)$ using Lemma 5.2.2:

Theorem 5.2.2. *For any non-principal Dirichlet charachter χ, $L(s,\chi) \neq 0$ on the line* $\mathrm{Re}(s) = 1$.

Proof. Let t be real and nonzero and consider

$$\eta(s,\chi) = L(s,\chi_{q,0})^3 L(s+it,\chi)^4 L(s+2it,\chi^2),$$

Recall that $L(s,\chi_{q,0})$ has a simple pole at $s=1$. Now suppose $L(1+it,\chi)=0$. Then at $s=1$, $\eta(s,\chi)$ would have a zero, either simple or of order 2, depending on if $L(s+2it,\chi)$ was zero or not since $L(s,\chi_{q,0})^3 L(s+it,\chi)^4$ has a simple zero at $s=1$. Therefore we need to show $\eta(s,\chi)$ is nonzero at $s=1$. Let d be the order of the zero of $\eta(s,\chi)$ at $s=1$. Then $d \geq 1$ and we can write $\eta(s,\chi) = (s-1)^d \eta_1(s,\chi)$ with $\eta_1(s,\chi)$ holomorphic at $s=1$ and such that $\eta_1(1,\chi) \neq 0$. Taking the logarithmic derivative, we find

$$\frac{\eta'}{\eta}(s,\chi) = \frac{d}{s-1} + \frac{\eta_1'}{\eta_1}(s,\chi).$$

Then for $\sigma > 1$, we have

$$\lim_{\sigma\to 1}(\sigma-1)\frac{\eta'}{\eta}(\sigma,\chi) = d.$$

But from Lemma 5.2.2, $(\sigma-1)\frac{\eta'}{\eta}(\sigma,\chi)$ has nonpositive real part and so the limit cannot be d. This gives a contradiction. Hence $\eta(s,\chi)$ is nonzero at $s=1$ and so $L(1+it,\chi) \neq 0$. $\square$

5.3 The Prime Number Theorem

The **prime counting function** $\pi(x)$ is defined by

$$\pi(x) = \sum_{p \leq x} 1,$$

for a real x. So $\pi(x)$ counts the number of primes that no larger than x. Euclid's infitude of the primes is equivalent to $\pi(x) \to \infty$ as $x \to \infty$. A more interesting question is to ask how the primes are distributed among the integers. The **prime number theorem** answers this question and the precise statement is the following:

Theorem 5.3.1 (Prime number theorem).

$$\pi(x) \sim \frac{x}{\log(x)}.$$

As with Dirichlet's theorem on primes in arithmetic progressions, we will delay the proof for the moment and give some intuition and historical context to the result. Intuitively, the prime number theorem is a result about how dense the primes are in the integers. To see this, notice that the result is equivalent to the asymptotic

$$\frac{\pi(x)}{x} \sim \frac{1}{\log(x)}.$$

Letting $x \geq 1$, the left-hand side is the probability that a radomly chosen positive integer no larger than x is prime. Thus the asymptotic result says that for large enough x, the probability that a randomly chosen integer no larger than x is prime is approximately $\frac{1}{\log(x)}$. We can also interpret this as saying that the average gap between primes no larger than x is approximately $\frac{1}{\log(x)}$. As a consequence, a positive integer with at most $2n$ digits is about half as likely to be prime than a positive integer with at most n digits. Indeed, there are $10^n - 1$ numbers with at most n digits, $10^{2n} - 1$ with at most $2n$ digits, and $\log(10^{2n} - 1)$ is approximately $2\log(10^n)$. Note that the prime number theorem says nothing about the exact error $\pi(x) - \frac{x}{\log(x)}$ as $x \to \infty$. The theorem only says that the relative error tends to zero:

$$\lim_{x\to\infty} \frac{\pi(x) - \frac{x}{\log(x)}}{\frac{x}{\log(x)}} = 0.$$

Now for some hisotrical context. While Gauss was not the first to put forth a conjectural form of the prime number theorem, he was known for compiling extensive tables of primes and he suspected that the density of the primes up to x was roughly $\frac{1}{\log(x)}$. How might one suspect this is the correct density? Well, let $d\delta_p$ be the weighted point measure that assigns $\frac{1}{p}$ at the prime p and zero everywhere else. Then

$$\sum_{p\leq x} \frac{1}{p} = \int_1^x d\delta_p(u).$$

We can interpret the integral as integrating the density $d\delta_p$ over the volume $[1, x]$. Let's try and find a more explicit expression for the density $d\delta_p$. Now Euler (see [Eul44]), argued that

$$\sum_{p\leq x} \frac{1}{p} \sim \log\log(x).$$

But notice that

$$\log\log(x) = \int_1^{\log(x)} \frac{du}{u} = \int_e^x \frac{1}{u}\frac{du}{\log u},$$

where in the second equality we have made the change of variables $u \to \log(u)$. So altogether,

$$\sum_{p\leq x} \frac{1}{p} \sim \int_e^x \frac{1}{u}\frac{du}{\log u}.$$

This is an asymptotic formula that gives a more explicit representation of the density $d\delta_p$. Notice that both sides of this asymptotic are weighted the same, the left-hand side by $\frac{1}{p}$, and the right-hand side by $\frac{1}{u}$. If we remove these weight (this is not strictly allowed), then we might hope

$$\pi(x) = \sum_{p\leq x} 1 \sim \int_e^x \frac{1}{\log(u)} du.$$

Interpreting the integral as an integral of density over volume, then for large x the density of primes up to x is approximately $\frac{1}{\log(x)}$ which is what the prime number theorem claims. Legendre was the first to put forth a conjectural form of the prime number theorem. In 1798 (see [Leg98]), he claimed that $\pi(x)$ was of the form

$$\frac{x}{A\log(x) + B},$$

for some constants A and B. In 1808 (see [Leg08]) he refined his conjecture by claiming

$$\frac{x}{\log(x) + A(x)},$$

where $\lim_{x\to\infty} A(x) \approx 1.08366$. Also in 1808 (see [Leg08]), Legendre conjectured what is now known as Dirichlet's theorem on primes in arithmetic progressions. As we have seen, Dirichlet's idea used complex analytic methods to resolve an arithmetic question. A similar type of idea is essential in proving the prime number theorem, and so Dirichlet's ideas are certainly due credit. It was not until 1896 that the prime number theorem was proved independently by Hadamard and de la Vallée Poussin (see [Had96, Pou97]). Their proofs, as well as every proof thereon out until 1949, used complex analytic methods in an essential way (there are now elementary proofs due to Erdös and Selberg). The proof we present uses Theorem 5.2.1 and requires a few different preliminary results. Many of these results are somewhat disconnected, so we will prove them seperately and then prove the prime number theorem. However, we will outline the overall idea. Start with the **Tchebychef functions**:

$$\theta(x) = \sum_{p\leq x} \log(p) \quad \text{and} \quad \psi(x) = \sum_{n\leq x} \Lambda(n),$$

defined for a real x, where $m \geq 1$ is an integer, and where $\Lambda(n)$ is the von Mangoldt function. Since $\frac{\log(p^m)}{\log(p)} = m$ and $\frac{\log(x)}{\log(p)}$ is continuous, for $x > 0$ we may write

$$\psi(x) = \sum_{n\leq x} \Lambda(n) = \sum_{p^m\leq x} \log(p) = \sum_{p\leq x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p). \tag{5.17}$$

This is often a more useful representation. We will first reduce the asymptotis of $\pi(x)$ to that of the Tchebychef functions, in particular, $\psi(x)$. We will then show $\psi(x) = O(x)$ which is a weaker statement than the prime number theorem. After, we introduce a technical result that will be needed in the proof of the prime number theorem. Once all of this is done we will be ready to prove the theorem itself. This will be acomplished by relating $\zeta(s)$ to $\psi(x)$ and using the technical theorem to deduce asymptotics for $\psi(x)$ which will complete the proof. Our first result, as we have mentioned, relates the asymptotics of $\pi(x)$, $\theta(x)$, and $\psi(x)$. Actually, it is an equivalence:

Lemma 5.3.1. *The following are equivalent:*

(i) $\pi(x) \sim \frac{x}{\log(x)}$.

(ii) $\theta(x) \sim x$.

(iii) $\psi(x) \sim x$.

Proof. Let $x > 0$. Then

$$\theta(x) = \sum_{p\leq x} \log(p) \leq \sum_{p\leq x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p) \leq \sum_{p\leq x} \frac{\log(x)}{\log(p)} \log(p) \leq \sum_{p\leq x} \log(x) = \pi(x)\log(x).$$

This chain of inequalities and Equation (5.17) together imply

$$\frac{\theta(x)}{x} \leq \frac{\psi(x)}{x} \leq \frac{x\log(x)}{x}.$$

Therefore we have

$$\lim_{x\to\infty} \frac{\theta(x)}{x} \leq \lim_{x\to\infty} \frac{\psi(x)}{x} \leq \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x}. \tag{5.18}$$

Now fix an α with $0 < \alpha < 1$ and let $x > 1$. Then

$$\theta(x) = \sum_{p \leq x} \log(p) \geq \sum_{x^\alpha < p \leq x} \log(p) \geq \sum_{x^\alpha < p \leq x} \alpha \log(x) = \alpha \log(x)(\pi(x) - \pi(x^\alpha)) > \alpha \log(x)(\pi(x) - x^\alpha),$$

where the last inequality follows because $\pi(x) < x$ provided $x > 0$. This chain of inequalities implies

$$\frac{\theta(x)}{x} \geq \alpha \frac{\pi(x)\log(x)}{x} - \alpha x^{\alpha-1}\log(x).$$

Note that $x^{\alpha-1}\log(x) \to 0$ as $x \to \infty$ because $0 < \alpha < 1$. Then

$$\lim_{x\to\infty} \frac{\theta(x)}{x} \geq \alpha \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x},$$

and letting $\alpha \to 1$ we conclude

$$\lim_{x\to\infty} \frac{\theta(x)}{x} \geq \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x}. \tag{5.19}$$

So Equations (5.18) and (5.19) together give

$$\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} \leq \lim_{x\to\infty} \frac{\theta(x)}{x} \leq \lim_{x\to\infty} \frac{\psi(x)}{x} \leq \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x}.$$

This completes the proof. □

We now prove the weaker asymptotic $\psi(x) = O(x)$:

Proposition 5.3.1.

$$\psi(x) = O(x).$$

Proof. Let $m > 1$ be an integer and fix an $x > 0$ such that $2^m < x \leq 2^{m+1}$. By Equation (5.17)

$$\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p).$$

Then by our choice of m,

$$\begin{aligned} \psi(x) &= \psi(x) + \psi(2^m) - \psi(2^m) \\ &\leq \psi(2^m) + \psi(2^{m+1}) - \psi(2^m) \\ &= \sum_{p \leq 2^m} \left\lfloor \frac{\log(2^m)}{\log(p)} \right\rfloor \log(p) + \sum_{2^m < p \leq 2^{m+1}} \left\lfloor \frac{\log(2^{m+1})}{\log(p)} \right\rfloor \log(p). \end{aligned} \tag{5.20}$$

We will now discuss two general estimates and then return to the two sums in Equation (5.20). For the first estimate, if $n \geq 1$ is an integer and p is a prime such that $n < p \leq 2n$, then p divides $\frac{(2n)!}{n!} = n!\binom{2n}{n}$. Since p does not divide $n!$ it must divide $\binom{2n}{n}$ so that

$$\prod_{n < p \leq 2n} p \leq \binom{2n}{n} < (1+1)^{2n} = 2^{2n},$$

where the last inequality follows by the binomial theorem. In particular,

$$\sum_{n<p\leq 2n} \log(p) = \log\left(\prod_{n<p\leq 2n} p\right) < \log(2^{2n}) = 2n\log(2).$$

Therefore

$$\sum_{p\leq 2^m} \log(p) = \sum_{1\leq k\leq m}\left(\sum_{2^{k-1}<p\leq 2^k} \log(p)\right) < \sum_{1\leq k\leq m} 2^k\log(2) < 2^{m+1}\log(2). \tag{5.21}$$

For our second estimate, if $p \leq x$ is a prime such that $\left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor > 1$ then $\left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \geq 2$ so that $x \geq p^2$ and hence $\sqrt{x} \geq p$. So

$$\sum_{p\leq\sqrt{x}} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p) \leq \sum_{p\leq\sqrt{x}} \frac{\log(x)}{\log(p)}\log(p) = \log(x)\sum_{p\leq\sqrt{x}} 1 = \pi(\sqrt{x})\log(x). \tag{5.22}$$

Returning to the first of our two sums in Equation (5.20) and recalling that $2^m < x \leq 2^{m+1}$, Equations (5.21) and (5.22) imply

$$\begin{aligned}\sum_{p\leq 2^m} \left\lfloor \frac{\log(2^m)}{\log(p)} \right\rfloor \log(p) &= \sum_{p\leq\sqrt{2^m}} \left\lfloor \frac{\log(2^m)}{\log(p)} \right\rfloor \log(p) + \sum_{\sqrt{2^m}<p\leq 2^m} \log(p) \\ &\leq \sum_{p\leq\sqrt{2^m}} \left\lfloor \frac{\log(2^m)}{\log(p)} \right\rfloor \log(p) + \sum_{p\leq 2^m} \log(p) \\ &< \pi(\sqrt{2^m})\log(2^m) + 2^{m+1}\log(2) \\ &= \pi(\sqrt{x})\log(x) + 2^{m+1}\log(2).\end{aligned} \tag{5.23}$$

As for the second sum in Equation (5.20), $p > 2^m$ implies $p > \sqrt{2^{m+1}}$ because $m > 1$. Therefore $\left\lfloor \frac{\log(2^{m+1})}{\log(p)} \right\rfloor = 1$ so from Equation (5.21)

$$\sum_{2^m<p\leq 2^{m+1}} \left\lfloor \frac{\log(2^{m+1})}{\log(p)} \right\rfloor \log(p) = \sum_{2^m<p\leq 2^{m+1}} \log(p) < 2^{m+1}\log(2). \tag{5.24}$$

Altogether, Equations (5.20), (5.23) and (5.24) give the first inequality in the following chain:

$$\begin{aligned}\psi(x) &< \pi(\sqrt{x})\log(x) + 2^{m+1}\log(2) + 2^{m+1}\log(2) \\ &= \pi(\sqrt{x})\log(x) + 4(2^m)\log(2) \\ &< \pi(\sqrt{x})\log(x) + 4x\log(2) \\ &< \sqrt{x}\log(x) + 4x\log(2) \\ &= \left(\frac{1}{\sqrt{x}}\log(x) + 4\log(2)\right)x.\end{aligned}$$

Since $\frac{1}{\sqrt{x}}\log(x) \to 0$ as $x \to \infty$, there is a positive M such that $\left|\frac{1}{\sqrt{x}}\log(x)\right| < M$ for all $x \geq 0$. Hence

$$\psi(x) < (M + 4\log(2))x,$$

for all $x \geq 0$. But this is to say that $\psi(x) = O(x)$. $\square$

We now discuss our technical result. A **Tauberian theorem** is a theorem which gives conditions for when a series or integral converges at some part of the boundary of its domain of definition. Our technical theorem is of this kind and is due to Newman (see [MSV24] for a proof):

Theorem 5.3.2. *Let $f(x)$ be bounded and locally integrable function on $[1,\infty)$. Moreover, suppose*

$$g(s) = \int_1^\infty f(x)x^{-(s+1)}\,dx,$$

defines an analytic function for $\mathrm{Re}(s) > 0$ *and admits analytic continuation to a neighborhood of* $\mathrm{Re}(s) = 0$. *Then $\int_1^\infty \frac{f(x)}{x}\,dx$ exists and*

$$\int_1^\infty \frac{f(x)}{x}\,dx = g(0).$$

Theorem 5.3.2 is interesting because the analytic continuation of $g(s)$ is not necessarily given by its defining integral, but this theorem guarantees that it is at $s = 0$. We are now ready to prove the prime number theorem:

Proof of the prime number theorem. By Lemma 5.3.1 it suffices to show $\psi(x) \sim x$. This is what we will prove. Consider the integral

$$s\int_1^\infty \psi(x)x^{-(s+1)}\,dx,$$

as a function of a complex variable s. So by Proposition 5.3.1, we have the estimate

$$s\int_1^\infty \psi(x)x^{-(s+1)}\,dx = O\left(\int_1^\infty x^{-s}\,dx\right).$$

For $\mathrm{Re}(s) > 1$, the integral in the right-hand side of the O-estimate above is locally absolutely uniformly bounded so that the left-hand side is too. Thus the left-hand side defines a holomorphic function for $\mathrm{Re}(s) > 1$. We now derive an alternaive description for this function. Recalling Equation (5.8) and that $\psi(x) = \sum_{n\le x}\Lambda(n)$, we can apply the Mellin inversion formula to Perron's formula (or follow the proof of Perron's formula) to obtain

$$-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \psi(x)x^{-(s+1)}\,dx,$$

for $\mathrm{Re}(s) > 1$. Now consider

$$-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1} = s\int_1^\infty \psi(x)x^{-(s+1)}\,dx - \frac{1}{s-1}.$$

We claim that $-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$ has analytic contunuation to a neighborhood of $\mathrm{Re}(s) = 1$. By the meromorphic continuation of $\zeta(s)$, $-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$ is holomorphic everywhere except possibly at the points where $\zeta(s)$ has zeros or at the pole $s = 1$. By Theorem 5.2.1, $\frac{1}{\zeta(s)}$ is defined on $\mathrm{Re}(s) = 1$ and hence is holomorphic in a neighborhood of $\mathrm{Re}(s) = 1$ since $\zeta(s)$ is. Therefore $-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$ is holomorphic except possibly at $s = 1$. In this case, the pole is simple so $\zeta(s) = (s-1)\zeta_1(s)$ where $\zeta_1(s)$ is holomorphic and such that $\zeta_1(1) \neq 0$ and hence is nonzero is a neighborhood of 1. Upon taking the logarithmic derivative of $\zeta(s)$ we see that

$$-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1} = \frac{\zeta_1'}{\zeta_1}(s).$$

It follows that $-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$ is holomorphic at $s = 1$ too. Thus $-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$ is holomorphic on the line $\mathrm{Re}(s) = 1$ and therefore admits analytic continuation to a neighborhood of $\mathrm{Re}(s) = 1$. In particular,

$$s \int_1^\infty \psi(x) x^{-(s+1)}\, dx - \frac{1}{s-1}.$$

is analytic in a neighborhood of $\mathrm{Re}(s) = 1$. We will now use the Tauberian theorem. Observe that in a neighborhood of $\mathrm{Re}(s) = 1$ with $\mathrm{Re}(s) > 0$, we have

$$\begin{aligned}
\int_1^\infty \left(\frac{\psi(x) - x}{x}\right) x^{-(s+1)}\, dx &= \int_1^\infty \left(\frac{\psi(x)}{x} - 1\right) x^{-(s+1)}\, dx \\
&= \int_1^\infty \psi(x) x^{-(s+2)}\, dx - \int_1^\infty x^{-(s+1)}\, dx \\
&= \int_1^\infty \psi(x) x^{-(s+2)}\, dx - \frac{1}{s} \\
&= \frac{1}{s+1}(s+1) \int_1^\infty \psi(x) x^{-(s+2)}\, dx - \frac{1}{s} \\
&= \frac{1}{s+1}\left((s+1) \int_1^\infty \psi(x) x^{-(s+2)}\, dx - \frac{1}{s} - 1\right),
\end{aligned} \tag{5.25}$$

where the second line follows because $\mathrm{Re}(s) > 0$ and the last line follows because $\frac{1}{s+1}\left(\frac{1}{s} - 1\right) = \frac{1}{s}$. But

$$(s+1) \int_1^\infty \psi(x) x^{-(s+2)}\, dx - \frac{1}{s},$$

admits anaytic continuation to a neighborhood of $\mathrm{Re}(s) = 0$ by what we have already shown and so the last expression in Equation (5.25) is also analytic in a neighborhood of $\mathrm{Re}(s) = 0$. Hence,

$$\int_1^\infty \left(\frac{\psi(x) - x}{x}\right) x^{-(s+1)}\, dx,$$

admits analytic continuation to a neighborhood of $\mathrm{Re}(s) = 0$. Since $\psi(x) = O(x)$, $\frac{\psi(x)-x}{x}$ is bounded on $[1, \infty)$. Also, $\psi(x)$ has finitely many jump discontinuities so that it is locally integrable on $[1, \infty)$ and therefore $\frac{\psi(x)-x}{x}$ is too. So all of the assumptions of Theorem 5.3.2 are satisfied and we conclude that

$$\int_1^\infty \frac{\psi(x) - x}{x^2}\, dx,$$

exists. The existance of this integral will imply $\psi(x) \sim x$ which finishes the proof. Indeed, if this asymptotic does not hold then $\lim_{x\to\infty} \left|\frac{\psi(x)}{x}\right| \neq 1$ so that either $\left|\frac{\psi(x)}{x}\right| > 1$ or $\left|\frac{\psi(x)}{x}\right| < 1$ for arbitrarily large values of x. As x is positive and $\psi(x)$ is a positive function, either $\frac{\psi(x)}{x} > 1$ or $\frac{\psi(x)}{x} < 1$ for arbitrarily large values of x. In the first case, $\frac{\psi(x)}{x} > 1$ is equivalent to the existance of a positive $\lambda > 1$ such that $\psi(x) \geq \lambda x$. This inequality together with the fact that $\psi(x)$ is monotonic increasing together imply the inequality in the following chain:

$$\int_x^{\lambda x} \frac{\psi(t) - t}{t^2}\, dt = \int_x^{\lambda x} \frac{\psi(t) - t}{t} \frac{dt}{t} \geq \int_x^{\lambda x} \frac{\lambda x - t}{t} \frac{dt}{t} = \int_1^\lambda \frac{\lambda x - tx}{tx} \frac{dt}{t} = \int_1^\lambda \frac{\lambda - t}{t^2}\, dt > 0,$$

where in the second equality we have used the change of variables $t \to xt$. Since this lower bound is independent of x, $\int_x^{\lambda x} \frac{\psi(t)-t}{t^2}\, dt$ is bounded away from zero by a constant for arbitrarily large values of x.

This gives a contradiction since the existance of $\int_1^\infty \frac{\psi(x)-x}{x^2}\,dx$ implies that $\int_x^{\lambda x} \frac{\psi(t)-t}{t^2}\,dt \to 0$ as $x \to \infty$. In the second case, $\frac{\psi(x)}{x} < 1$ is equivalent to the existance of a positive $\lambda < 1$ such that $\psi(x) \le \lambda x$. Analogous to the first case,

$$\int_{\lambda x}^{x} \frac{\psi(t)-t}{t^2}\,dt = \int_{\lambda x}^{x} \frac{\psi(t)-t}{t}\frac{dt}{t} \le \int_{\lambda x}^{x} \frac{\lambda x - t}{t}\frac{dt}{t} = \int_{\lambda}^{1} \frac{\lambda x - tx}{tx}\frac{dt}{t} = \int_{\lambda}^{1} \frac{\lambda - t}{t^2}\,dt < 0.$$

Since this upper bound is independent of x, $\int_{\lambda x}^{x} \frac{\psi(t)-t}{t^2}\,dt$ is bounded away from zero by a constant for arbitrarily large values of x. Again, this gives a contradiction since the existance of $\int_1^\infty \frac{\psi(x)-x}{x^2}\,dx$ implies that $\int_{\lambda x}^{x} \frac{\psi(t)-t}{t^2}\,dt \to 0$ as $x \to \infty$. So finally $\psi(x) \sim x$ and the theorem is proved. $\square$

A more classical proof of the prime number theorem is observing, as we did, that $-\frac{\zeta'}{\zeta}(s) = \sum_{n\ge1} \frac{\Lambda(n)}{n^s}$ and $\psi(x) = \sum_{n\le x} \Lambda(x)$ and then appealing to Perron's formula to write

$$\psi(x) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} -\frac{\zeta'}{\zeta}(s)x^s\frac{ds}{s}.$$

One then performs a very careful analysis of the latter integral to conclude $\psi(x) \sim x$. This involves making several estimates about the growth rate of $\frac{\zeta'}{\zeta}(s)$ as well as other convergence arguments. However, not much is needed beyond a good understanding of complex analysis. We have deviated from this style of proof for technical simplicity even though the underlying idea is, perhapse, more straightforward.

Remark 5.3.1. *It's interesting to note that our use of non-vanishing results in the proofs of the prime number theorem and Dirichlet's theorem on primes in arithmetic progressions served different purposes. In the proof of the prime number theorem the non-vanishing result was used to establish analytic continuation of $\frac{\zeta'}{\zeta}(s)$. On the other hand, in the proof of Dirichlet's theorem on primes in arithmetic progressions the non-vanishing result was used to conclude that $\log L(s,\chi)$ was bounded provided the Dirichlet character χ was non-principal.*

We will introduce one last function before we are done discussing the prime number theorem. That function is the **logarithmic integral** $\mathrm{Li}(x)$ defined by

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)},$$

for $x \ge 2$. Notice that $\mathrm{Li}(x) \sim \frac{x}{\log x}$ because

$$\lim_{x\to\infty}\left|\frac{\mathrm{Li}(x)}{\frac{x}{\log x}}\right| = \lim_{x\to\infty}\left|\frac{\int_2^x \frac{dt}{\log(t)}}{\frac{x}{\log x}}\right| = \lim_{x\to\infty}\left|\frac{\frac{1}{\log(x)}}{\frac{\log(x)-1}{\log^2(x)}}\right| = \lim_{x\to\infty}\left|\frac{\log(x)}{\log(x)-1}\right| = 1.$$

where in the second equality we have used l'Hôpital's rule. So an equivalent version of the prime number theorem is the following:

Theorem 5.3.3 (Prime number theorem).

$$\pi(x) \sim \mathrm{Li}(x).$$

The advantage of using $\mathrm{Li}(x)$ is that it is a better numerical approximation to $\pi(x)$ than $\frac{x}{\log(x)}$. To make this statment precise, first observe that we have yet another equivalent version of the prime number theorem:

Theorem 5.3.4 (Prime number theorem).

$$\pi(x) = \frac{x}{\log(x)} + o\left(\frac{x}{\log(x)}\right).$$

So, in particular, the prime number theorem implies the weaker asymptotic

$$\pi(x) = \frac{x}{\log(x)} + O\left(\frac{x}{\log(x)}\right),$$

which says that the extact error between $\pi(x)$ and $\frac{x}{\log(x)}$ grows no faster than $\frac{x}{\log(x)}$. However, there is the following result of de la Vallée Poussin (see [Pou99]):

Proposition 5.3.2. *For some $a > 0$,*

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\log(x)}}\right),$$

Since $xe^{-a\sqrt{\log(x)}} \to 0$ as $x \to \infty$ and $\frac{x}{\log(x)} \to \infty$ as $x \to \infty$, $xe^{-a\sqrt{\log(x)}} < \frac{x}{\log(x)}$ for sufficiently large x. Therefore the extact error $\pi(x) - \mathrm{Li}(x)$ grows slower than $\pi(x) - \frac{x}{\log x}$ for sufficiently large x. This is what we mean when we say $\mathrm{Li}(x)$ is a better numerical approximation to $\pi(x)$ than $\frac{x}{\log(x)}$. There is also the following result due to Hardy and Littlewood (see [HL16]):

Proposition 5.3.3. $\pi(x) - \mathrm{Li}(x)$ *changes sign infinitely often as $x \to \infty$.*

So in addition, Proposition 5.3.3 implies that $\mathrm{Li}(x)$ never underestimates or overestimates $\pi(x)$ continuously. On the other hand, the exact error $\pi(x) - \frac{x}{\log(x)}$ is positive provided $x \geq 17$ (see [RS62]).

Chapter 6

Hypotheses of L-functions

In this chapter we discuss two hypotheses of L-functions. The first is the more important open question in number theory and perhapse all of mathematics: the Riemann hypothesis. It discusses the distribution of the zeros of the Riemann zeta function. After surveying the Riemann hypothesis we introduce its weaker cousin the Lindelöf hypothesis which is about the growth rate of the Riemann zeta function along the crtitical line. In our survey of the Lindelöf hypothesis we also discuss the classical convexity argument. Analogs of these hypotheses to other L-functions are also mentioned.

6.1 The Riemann Hypothesis & Nontrivial Zeros

Along with non-vanishing results, it turns out that the zeros of L-functions are also extremely important but this requires some level of discussion to be convincing. Let's start with our prototypical L-function the Riemann zeta function. We would like to understand its zeros. Reall that for $\mathrm{Re}(s) > 1$, $\zeta(s)$ admits an Euler product:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

This product vanishes if and only if one of its factors are zero. But in the region $\mathrm{Re}(s) > 1$, $(1-p^{-s})^{-1} \neq 0$ so $\zeta(s)$ has no zeros in this region. The functional equation will allow us to understand the zeros in the region $\mathrm{Re}(s) < 0$. Indeed, we can rewrite the functional equation for $\zeta(s)$ as

$$\zeta(1-s) = \pi^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \zeta(s). \tag{6.1}$$

We want to understand when the right-hand side of Equation (6.1) vanishes in the region $\mathrm{Re}(s) \geq 1$. We just showed that $\zeta(s)$ has no zeros in the region $\mathrm{Re}(s) > 1$ and by Theorem 5.2.1 this extends to $\mathrm{Re}(s) \geq 1$. So if there is a zero it comes from either of the gamma factors. This happens exactly at poles of $\Gamma\left(\frac{1-s}{2}\right)$ for $\mathrm{Re}(s) \geq 1$ which by Theorem 1.5.1 are all simple and occur at $s = 1 + 2n$ for any integer $n \geq 0$. But at $s = 0$, $\Gamma\left(\frac{s}{2}\right)$ has a simple pole and this cancels the simple zero of the other gamma factor. In terms of the region $\mathrm{Re}(s) \leq 0$, $\zeta(s)$ has simple zeros at $s = -2n$ for $n \geq 1$. So we have shown that $\zeta(s)$ has simple zeros at negative even integers.

Let χ be a primitive Dirichlet character of modulus $q > 1$. We can repeat the same procedure for $L(s, \chi)$. Indeed, for $\mathrm{Re}(s) > 1$, $L(s, \chi)$ has the Euler product

$$L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1},$$

and this product vanishes if and only if one of the factors are zero. But for $\mathrm{Re}(s) > 1$, $(1-\chi(p)p^{-s})^{-1} \neq 0$ so $L(s,\chi)$ has no zeros in this region. We can rewrite the functional equation for $L(s,\chi)$ as

$$L(1-s,\chi) = \frac{i^{\mathfrak{a}}}{\varepsilon_\chi} q^{s-\frac{1}{2}} \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{(1-s)+\mathfrak{a}}{2}\right)} L(s,\overline{\chi}). \tag{6.2}$$

Just as for $\zeta(s)$, we want to understand when the right-hand side of Equation (6.2) vanishes for $\mathrm{Re}(s) \geq 1$. Now $L(s,\chi)$ and $L(s,\overline{\chi})$ don't have zeros in the region $\mathrm{Re}(s) > 1$ and by Theorem 5.2.2 this extends to $\mathrm{Re}(s) \geq 1$. Hence the right-hand side is zero if and only if one of the gamma factors vanish. The zeros come from the poles of $\Gamma\left(\frac{(1-s)+\mathfrak{a}}{2}\right)$ for $\mathrm{Re}(s) \geq 1$. By Theorem 1.5.1 these are all simple and occur at $s = 1+\mathfrak{a}+2n$ for $n \geq 0$. In terms of the region $\mathrm{Re}(s) \leq 0$, $L(s,\chi)$ has simple zeros at $s = -2n$ for $n \geq 0$ if χ is even and at $s = -(2n+1)$ for $n \geq 0$ if χ is odd. We can state our results compactly as follows: $L(s,\chi)$ has simple zeros at nonpositive even integers if χ is even and at negative odd integers if χ is odd. In particular, if χ is even there is a zero at $s = 0$ (a boundary point of the critical strip).
For a general Selberg class L-function $L(s)$, the situation is analogous. From the Euler product, $L(s)$ will have no zeros in the region $\mathrm{Re}(s) > 1$. The functional equation then shows that the zeros $L(s)$ in the region $\mathrm{Re}(s) < 0$ come from the the poles of the gamma functions. Some additional care is needed at the boundary lines $\mathrm{Re}(s) = 0, 1$. The zeros of $L(s)$ outside or at the boundary of the critial strip are called **trivial zeros**. The zeros of $L(s)$ inside the critical strip are called **nontrivial zeros**.
Returning to the Riemann zeta function, let ρ be a nontrivial zero. Inside the critial strip the gamma factors in the functional equation are holomorphic and non-vanishing. So the functional equation implies $1-\rho$ is also a nontrivial zero. Actually, we can do slighly better. Since $\zeta(s)$ takes real values for real $s > 1$ ($\zeta(s)$ is defined by a Dirichlet series there), the Schwarz reflection principle implies $\zeta(\overline{s}) = \overline{\zeta(s)}$ and that $\zeta(s)$ takes real values on the entire real axis (save for the pole). So $\overline{\rho}$ and $1-\overline{\rho}$ are nontrivial zeros too and therefore the nontrivial zeros of $z(s)$ come in sets of four:

$$\rho, \quad \overline{\rho}, \quad 1-\rho, \quad \text{and} \quad 1-\overline{\rho}.$$

Almost the same type of symmetry holds for $L(s,\chi)$. Let ρ be a nontrivial zero of $L(s,\chi)$. Note that $\overline{L(s,\chi)} = L(\overline{s},\overline{\chi})$ by the identity theorem and that this equality holds for $\mathrm{Re}(s) > 1$ (where $L(s,\chi)$ is defined by a Dirichlet series). This implies $\overline{\rho}$ is a nontrivial zero of $L(s,\overline{\chi})$. Inside the critical strip, the gamma factors in the functional equation are nonzero and holomorphic so we conclude that $1-\overline{\rho}$ is also a nontrivial zero of $L(s,\chi)$. Unfortunately, if χ is complex $L(s,\chi)$ does not necessarily take real values for real $s > 1$ and so we cannot conclude that $\overline{\rho}$ is a nontrivial zero. However, if χ is quadratic then it only takes real values and so $\overline{\rho}$ will also be a nontrivial zero. In the case of $\zeta(s)$, the Riemann hypothesis says that this symmetry of zeros is as simple as it could possibly be:

Theorem 6.1.1 (Riemann hypothesis). *All of the nontrivial zeros of $\zeta(s)$ lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

This is one of, if not the most, famous and important open problems in mathematics. It has resisted all attempts of a proof by every great mathematician over the last century and a half. The Clay Mathematics Institute has also named it one of the millennium pize problems which means that anyone who can give a proof (or disproof) will recieve a \$1 million cash prize. Riemann's original motivation for the conjecture came from looking for an explicit formula for $\pi(x)$ which is the purpose of his 1859 manuscript (see [Rie59]). This explicit formula for $\pi(x)$ involves the nontrivial zeros of Riemann zeta function. Riemann computed a few of them, found that they were on the critical line, and conjectured that it is very likely that all of them lie on the critical line.

The Riemann hypothesis is important because, if true, it tells us a lot of information about how the primes are distributed among the positive integers. In particular, Koch in 1901 showed that the Riemann hypothesis implies an asymptotic estimate for the exact error between $\pi(x)$ and $\mathrm{Li}(x)$ (see [Koc01]):

Proposition 6.1.1. *Under the assumption of the Riemann hypothesis,*

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log(x)).$$

The Riemann hypothesis also implies the Lindelöf hypothesis although it is not an immediate consequence. In 1919 Backlund showed the following equivalence for the Lindelöf hypothesis (see [Bac18]):

Proposition 6.1.2. *The Lindelöf hypothesis is equivalent to the following statement: Fix an $\varepsilon > 0$, a real T, and set*

$$Z_{\varepsilon,T} = \left\{ s \in \mathbb{C} : \zeta(s) = 0, \mathrm{Re}(s) \geq \frac{1}{2} + \varepsilon, T \leq \mathrm{Im}(s) \leq T + 1 \right\}.$$

Then

$$|Z_{\varepsilon,T}| = o(\log(T)).$$

If the Riemann hypothesis is true, then Proposition 6.1.2 is immediate because $Z_{\varepsilon,T}$ would be empty for any $\varepsilon > 0$ and all T. Therefore the Riemann hypothesis implies the Lindelöf hypothesis. There are more interesting implications and consequences but we will not discuss them here. Instead, we would like to make an interesting comment about Proposition 5.3.3. The original proof (see [HL16]) is actually independent of the truth of the Riemann hypothesis. Assuming the Riemann hypothesis is true, one deduces a contradiction if $\pi - \mathrm{Li}(x)$ changes sign finitely many times. Then assuming the Riemann hypothesis is false, one again deduces a contradiction if $\pi - \mathrm{Li}(x)$ changes sign finitely many times.
Lastly, there is also the **Selberg class Riemann hypothesis** which is an analogous conjecture for Selberg class L-functions:

Conjecture 6.1.1 (Selberg class Riemann hypothesis). *For any Selberg class L-function $L(s)$, all of the nontrivial zeros of $L(s)$ lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

The Selberg class Riemann hypothesis also implies other interesting results, but we will not discuss them here.

6.2 The Lindelöf Hypothesis & Convexity Arguments

A slightly weaker conjectured result than the Riemann hypothesis is the Lindelöf hypothesis. In 1908 Lindelöf made a conjecture which is about the rate of growth of the zeta function on the critical line (see [Lin08]). This is now known as the **classical Lindelöf hypothesis**:

Conjecture 6.2.1 (Classical Lindelöf hypothesis). *For any $\varepsilon > 0$,*

$$\zeta\left(\frac{1}{2} + it\right) \ll_\varepsilon t^\varepsilon.$$

This conjecture still remains wide open today, but over the years there have been some advances toward proving this conjecture. These advances go by the name of subconvexity arguments. This motivates the question: what is a convexity argument anways? A **convexity argument** is one where estimates about the growth of an L-function on the critical line is dervied from **trivial bounds**, that is bounds given by

absolute convergence, via the functional equation. Usually this is achieved by methods of complex analysis and Sirling's formula.
We will demonstrate a standard convexity argument, also referred to as the **Lindelöf convexity argument** for the Riemann zeta function. Let $s = \sigma + it$. The first step is to guarantee the Phragmén-Lindelöf convexity principle in a region containing the critical strip. The zeta function being order 1 implies this immediately (see Appendix B.6). Therefore, we are reduced to estimating the growth of $\zeta(s,f)$ for σ to the left of 0 and to the right of 1. That is, just outside the edges of the critical strip. The right edge is easy. Letting $\varepsilon > 0$ and setting $\sigma = 1+\varepsilon$, we have that the zeta function is absolutely convergent in this region giving the trivial bound

$$\zeta((1+\varepsilon)+it, f) \ll_\varepsilon 1. \tag{6.3}$$

The left edge is only slightly more difficult. The functional equation implies

$$|\zeta(s,f)| \leq \left|\frac{\gamma(1-s,f)}{\gamma(s,f)}\right| |\zeta(1-s,f)|. \tag{6.4}$$

We now require an estimate for the ratio of the gamma factors. We will get this estimate from Equation (1.6). Since $s \sim_\sigma t$, we replace s with t, except for the exponenets, provided we let the implicit constant depend upon σ. For the exponents, we can reaplce s with σ because any complex number raised to a purely imaginary power has absolute value 1. So, our simplified estimate is

$$\Gamma(s) = \sqrt{2\pi} t^{\sigma-\frac{1}{2}} e^{-\sigma}(1+O_\sigma(1)),$$

which is equivalent to

$$\frac{1}{\Gamma(s)} = \frac{1}{\sqrt{2\pi}} t^{\frac{1}{2}-\sigma} e^{\sigma}(1+O_\sigma(1)),$$

In terms of Vinogradov's symbol the above estimates imply

$$\Gamma(s) \ll_\sigma t^{\sigma-\frac{1}{2}} e^{-\sigma} \ll_\sigma t^{\sigma-\frac{1}{2}} \quad \text{and} \quad \frac{1}{\Gamma(s)} \ll_\sigma t^{\frac{1}{2}-\sigma} e^{\sigma} \ll_\sigma t^{\frac{1}{2}-\sigma}.$$

We can then estimate the ratio of gamma factors as

$$\frac{\Gamma(1-s)}{\Gamma(s)} \ll_\sigma t^{1-2\sigma}. \tag{6.5}$$

From Equation (6.5) it follows that

$$\frac{\gamma(1-s,f)}{\gamma(s,f)} \ll_\sigma t^{\frac{1-2\sigma}{2}}.$$

Let $\varepsilon > 0$ and set $\sigma = -\varepsilon$. Then plugging our estimate above into Equation (6.4) and using the trivial bound $|\zeta((1+\varepsilon)+it)| \ll_\varepsilon 1$ gives

$$\zeta(-\varepsilon+it) \ll_\varepsilon t^{\frac{1+2\varepsilon}{2}}. \tag{6.6}$$

By the Phragmén-Lindelöf convexity principle, Equations (6.3) and (6.6) imply the **convexity bound**

$$\zeta(s) \ll_{\sigma,\varepsilon} t^{\frac{1+\varepsilon-\sigma}{2}},$$

for $-\varepsilon \leq \sigma \leq 1+\varepsilon$. At the critical line, the convexity bound becomes

$$\zeta\left(\frac{1}{2}+it\right) \ll_\varepsilon t^{\frac{1}{4}+\varepsilon}. \tag{6.7}$$

So the classical Lindelöf hypothesis says that the exponenet $\frac{1}{4}+\varepsilon$ can be improved to be ε. For a general Selberg class L-function, the Lindelöf convexity argument is the following:

Method 6.2.1 (Lindelöf convexity argument). Suppose we are given a general L-function $L(s,f)$ with degree d Euler product (not necessarily of the Selberg class) and such that the following hold:

(i) $L(s,f)$ is absolutely convergent for $\mathrm{Re}(s) > 1$.

(ii) $L(s,f)$ has a functional equation of shape $s \to 1-s$ with conductor $q(f)$.

(iii) $L(s,f)$ is of finite order.

(iv) $L(s,f)$ has analytic continuation to the critical strip.

Then the **convexity bound**

$$L(s,f) \ll_{\sigma,\varepsilon} q(f)^{\frac{1+\varepsilon-\sigma}{2}} t^{\frac{d(1+\varepsilon-\sigma)}{2}},$$

can be obtained inside the critical strip. Indeed, if $L(s)$ has poles inside the critical strip, remove them by multiplying by the corresponding polar divisors. We can then divide out by these factors after obtaining the estimate. Since $L(s)$ is finite order, $L(s)$ will satisfy the Phragmén-Lindelöf convexity principle in a region containing the critical strip. Now use the absolute convergence of $L(s)$ to bound the right edge by a constant. Then use the functional equation to isolate the L-function at the left edge and apply Stirling's formula to the ratio of gamma factors coming from the functional equation to deduce a polynomial bound of shape $q(f)^{\frac{1+2\varepsilon}{2}} t^{\frac{d(1+2\varepsilon)}{2}}$. The Phragmén-Lindelöf convexity principle will then give the result.

Notice that the convexity bound depends on the degree d of the Euler product and the conductor $q(f)$ (for the Riemann zeta function the conductor is 1). In particular, at the critical line the Lindelöf convexity argument gives a polynomials bound of $\frac{d}{4}+\varepsilon$ in t and $\frac{1}{4}+\varepsilon$ in $q(f)$. The polynomial bound in t depends upon d because there is one gamma function in the gamma factor for each degree of the Euler product. For any such L-function, any improvment upon either exponent at the critical line is called **breaking convexity** and any argument used to do so is called a **subconvexity argument**. Often, we refer to the **t-aspect** (or **height aspect**) to mean the polynomial bound in t. Similarly, we refer to the **$q(f)$-aspect** (or **conductor aspect**) to mean the polynomial bound in $q(f)$. Lastly, we refer to the **hybrid aspect** to mean the combined polynomial bound in t and $q(f)$. For example, from the Lindelöf convexity argument we have the convexity bound

$$L\left(\frac{1}{2}+it,\chi\right) \ll_{t,\varepsilon} q^{\frac{1}{4}+\varepsilon},$$

in the q-aspect, for any primitive Dirichlet L-function of conductor $q > 1$. The Lindelöf hypothesis in the q-aspect for Dirichlet L-functions is that this exponent can be improved to be ε:

Conjecture 6.2.2 (Lindelöf hypothesis for Dirichlet L-functions, q-aspect). *For any primitive Dirichlet character of conductor $q > 1$ and any $\varepsilon > 0$,*

$$L\left(\frac{1}{2}+it,\chi\right) \ll_{t,\varepsilon} q^{\varepsilon}.$$

The **Selberg class Lindelöf hypothesis** is the analog to the classical Lindelöf hypothesis for any Selberg class L-function:

Conjecture 6.2.3 (Selberg class Lindelöf hypothesis). *For any Selberg class L-function $L(s,f)$ of conductor $q(f)$ and any $\varepsilon > 0$,*

$$L\left(\frac{1}{2}+it\right) \ll_\varepsilon (q(f)t)^\varepsilon.$$

Chapter 7

Zero-free Regions & Zero Counting of L-functions

In this chapter, we expand our discussion about the zeros of L-functions. We prove the classical zero-free regions for the Riemann zeta function and Dirichlet L-functions. In the latter case, we introduce the infamous Siegel zeros. After the zero-free regions, we discuss zero counting up the critical strip. The main result we prove is the Riemann–von Mangoldt formula and its analog for Dirichlet L-functions.

7.1 The Explicit Formula for Logarithmic Derivatives

Information about the distribution of primes is ultimatiely contained in the information about the zeros of the L-functions $\zeta(s)$ and $L(s,\chi)$. In order to extract this information we will require more flexiable tools than the L-functions alone. These tools are explicit formulas for logarithmic derivatives of L-functions. Essentially, explicit formulas are partial fraction decomposition analogs to analytic functions.

The Explicit Formula for the Logarithmic Derivative of $\zeta(s)$

Define the **Riemann xi function** $\xi(s)$ by

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Note that $\xi(s)$ is holomorphic on $\mathbb{C}$ since at $s=1$ the pole of $\zeta(s)$ is canceled by the zero from the $s(s-1)$ term and the poles coming from the gamma function are canceled by the trivial zeros of $\zeta(s)$ or by s in the case $s=0$. By the functional equation for $\zeta(s)$,

$$\xi(s) = \xi(1-s),$$

which is the functional equation for the Riemann xi function. Moreover, $\xi(s)$ is of order 1. To see this, by the functional equation for the Riemann xi function it suffices to prove $\xi(s)$ is of order 1 for $\mathrm{Re}(s) \geq \frac{1}{2}$. Now $\zeta(s)$ is of order 1, being a Selberg class L-function, and $\Gamma\left(\frac{s}{2}\right)$ is of order 1 for $\mathrm{Re}(s) > 0$. By the Hadamard factorization theorem (see Appendix B.5),

$$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}, \tag{7.1}$$

where A and B are constants and ρ runs over the nontrivial zeros of $\zeta(s)$, counted with multiplicity, ordered with respect to the size of the ordinate. That is, if $\rho = \beta + i\gamma$, the product is ordered by $|\gamma|$. Equation (7.1) is an infinite product expression for $\xi(s)$ and we will use it to obtain the explicit formula for logarithmic derivative of $\zeta(s)$. To do this we tak the logarithmic derivative of $\xi(s)$ in two ways. First, observe that we can write $\xi(s) = (s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)\zeta(s)$. The logarithm derivative of this expression gives

$$\frac{\xi'}{\xi}(s) = \frac{1}{s-1} + \frac{1}{2}\log(\pi) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \frac{\zeta'}{\zeta}(s). \tag{7.2}$$

On the other hand, we can take the logarithmic derivative Equation (7.1) to obtain

$$\frac{\xi'}{\xi}(s) = B + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{7.3}$$

Upon combining Equations (7.2) and (7.3) yields the **explicit formula** for $\frac{\zeta'}{\zeta}(s)$:

$$\frac{\zeta'}{\zeta}(s) = B - \frac{1}{s-1} - \frac{1}{2}\log(\pi) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

The explicit formula for $\frac{\zeta'}{\zeta}(s)$ is quite important for analytic investigations. Notice that the second term contains information about the pole of $\zeta(s)$ while the last term contains all the information about the nontrivial zeros. The digamma function term contains the information about the trivial zeros as can be seen from Corollary 1.5.1.

The constants A and B in Equation (7.1) can be computed. To compute A, Equation (7.1) implies $A = \log(\xi(0))$. Now $\zeta(s)$ has residue 1 at $s = 1$ so that $\lim_{s\to 1}(s-1)\zeta(s) = 1$. This fact along with $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ together imply $\xi(1) = \frac{1}{2}$. By the functional equation for $\xi(s)$, we find that

$$A = \log\left(\frac{1}{2}\right).$$

The constant B requires slightly more work. We first claim that $\sum_{\rho}\frac{1}{\rho}$ converges provided we group the terms ρ and $\overline{\rho}$ together. This is in accordance with the order of ρ in the Hadamard factorization for $\xi(s)$. Letting $\rho = \beta + i\gamma$, we have

$$\sum_{\rho}\frac{1}{\rho} = \sum_{\rho}{}^{*}\left(\frac{1}{\rho} + \frac{1}{\overline{\rho}}\right) = \sum_{|\gamma|\geq 0}\frac{2\beta}{\beta^2+\gamma^2}, \tag{7.4}$$

where the $*$ indicates that we are summing over ρ and not $\overline{\rho}$. The last sum in Equation (7.4) is at most $2\sum_{\rho}\frac{1}{|\rho|^2}$ and this sum converges by the Hadamard factorizaton of $\xi(s)$ since the rank is 1. Now Equation (7.3) and the functional equation for $\xi(s)$ together give

$$B + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = -B - \sum_{\rho}\left(\frac{1}{1-s-\rho} + \frac{1}{\rho}\right). \tag{7.5}$$

The terms corresponding to $s-\rho$ and $1-s-\rho$ cancel because if ρ is a nontrivial zero so is $1-\rho$. Indeed, add $\sum_{p}\frac{1}{1-s-\rho}$ to both sides of Equation (7.5) and notice that $\frac{1}{1-s-\rho} = -\frac{1}{s-(1-\rho)}$. Using Equation (7.4) we can rewrite the result as

$$B = -\sum_{\rho}\frac{1}{\rho} = -\sum_{|\gamma|\geq 0}\frac{2\beta}{\beta^2+\gamma^2}.$$

The Explicit Formula for the Logarithmic Derivative of $L(s,\chi)$

Let χ be a primitive Dirichlet character with conductor $q>1$. We will derive an analogous explicit formula for the logarithmic derivative of $L(s,\chi)$. To do this, define the **Dirichlet xi function** $\xi(s,\chi)$ by

$$\xi(s,\chi)=\left(\frac{q}{\pi}\right)^{\frac{s+\mathfrak{a}}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi).$$

We claim $\xi(s,\chi)$ is holomorphic on $\mathbb{C}$. Indeed, $L(s,\chi)$ is holomorphic on $\mathbb{C}$ and the poles coming from the gamma factor are canceled by the trivial zeros of $L(s,\chi)$. Moreover, the functional equation for $L(s,\chi)$ implies

$$\xi(s,\chi)=\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}\xi(1-s,\overline{\chi}),$$

which is the functional equation for the Dirichlet xi function. It is also not hard to show that $\xi(s,\chi)$ is of order 1. Indeed, by the functional equation for the Dirichlet xi function it suffices to prove $\xi(s,\chi)$ is of order 1 for $\mathrm{Re}(s)\geq\frac{1}{2}$. Since $L(s,\chi)$ is a Selberg class L-function it is of order 1 and $\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)$ is of order 1 for $\mathrm{Re}(s)>0$. By the Hadamard factorization theorem (see Appendix B.5),

$$\xi(s,\chi)=e^{A(\chi)+B(\chi)s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}},\tag{7.6}$$

where $A(\chi)$ and $B(\chi)$ are constants and ρ runs over the nontrivial zeros of $L(s,\chi)$, counted with multiplicity, and ordered with respect to the size of the ordinate. In other words, if $\rho=\beta+i\gamma$, the product is ordered by $|\gamma|$. Just like for $\zeta(s)$, we can use Equation (7.6) to prove the explicit formula for $\frac{L'}{L}(s,\chi)$ by taking the logarithmic derivative of $\xi(s,\chi)$ in two ways. First, taking the logarithmic derivative of $\xi(s,\chi)$ directly gives

$$\frac{\xi'}{\xi}(s,\chi)=\frac{1}{2}\log\left(\frac{q}{\pi}\right)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)+\frac{L'}{L}(s,\chi).\tag{7.7}$$

On the other hand, we can take the logarithmic derivative using Equation (7.6) to obtain

$$\frac{\xi'}{\xi}(s,\chi)=B(\chi)+\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).\tag{7.8}$$

Combining Equations (7.7) and (7.8) we obtain the **explicit formula** for $\frac{L'}{L}(s,\chi)$:

$$\frac{L'}{L}(s,\chi)=B(\chi)-\frac{1}{2}\log\left(\frac{q}{\pi}\right)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)+\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

The constants $A(\chi)$ can be computed. Indeed, Equation (7.6) implies $A(\chi)=\log(\xi(0,\chi))$. Now $\xi(1,\overline{\chi})=\left(\frac{q}{\pi}\right)^{\frac{1+\mathfrak{a}}{2}}\Gamma\left(\frac{1+\mathfrak{a}}{2}\right)L(1,\overline{\chi})$ so the functional equation for $\xi(s,\chi)$ implies

$$A(\chi)=\log\left(\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}\left(\frac{q}{\pi}\right)^{\frac{1+\mathfrak{a}}{2}}\Gamma\left(\frac{1+\mathfrak{a}}{2}\right)L(1,\overline{\chi})\right).$$

The constant $B(\chi)$ is significantly more difficult to compute and there are currently no good estimates in terms of the conductor q alone. However, we do have a useful expression for the real part of $B(\chi)$ in terms of the real parts of the nontrivial zeros. To obtain this expression, note that Equation (7.4) holds for the nontrivial zeros of $L(s,\chi)$ and the sum converges because $\xi(1,\chi)$ has rank 1. Taking Equation (7.8) at $s=0$ gives $B(\chi) = \frac{\xi'}{\xi}(0,\chi)$. The functional equation for $\xi(s,\chi)$ implies $\frac{\xi'}{\xi}(0,\chi) = -\frac{\xi'}{\xi}(1,\overline{\chi})$ so using Equation (7.8) at $s=1$, we have

$$B(\chi) = -B(\overline{\chi}) - \sum_{\overline{\rho}} \left(\frac{1}{1-\overline{\rho}} + \frac{1}{\overline{\rho}} \right). \tag{7.9}$$

We claim $B(\overline{\chi}) = \overline{B(\chi)}$. To see this, since $\overline{L(s,\chi)} = L(\overline{s},\overline{\chi})$ we have $\overline{\xi(s,\chi)} = \xi(\overline{s},\overline{\chi})$. But then

$$\overline{B(\chi)} = \overline{\frac{\xi'}{\xi}(0,\chi)} = \frac{\xi'}{\xi}(0,\overline{\chi}) = B(\overline{\chi}).$$

Then the fact that $B(\overline{\chi}) = \overline{B(\chi)}$ and Equation (7.9) together imply

$$\mathrm{Re}(B(\chi)) = -\frac{1}{2}\sum_{\rho} \mathrm{Re}\left(\frac{1}{1-\overline{\rho}} + \frac{1}{\overline{\rho}} \right). \tag{7.10}$$

Now if $\rho = \beta + i\gamma$, we have $\mathrm{Re}\left(\frac{1}{1-\overline{\rho}}\right) = \frac{1-\beta}{(1-\beta)^2+\gamma^2} > 0$ and $\mathrm{Re}\left(\frac{1}{\rho}\right) = \frac{\beta}{\beta^2+\gamma^2} > 0$. Recalling that $1-\overline{\rho}$ is a nontrivial zero if ρ is, we may reaplace $1-\overline{\rho}$ with ρ in Equation (7.10) since the terms correspond to these zeros are positive as we have just shown. We can rewrite the result as follows:

$$B(\chi) = -\frac{1}{2}\sum_{\rho} \mathrm{Re}\left(\frac{1}{\rho} + \frac{1}{\overline{\rho}} \right) = -\sum_{\rho} \mathrm{Re}\left(\frac{1}{\rho}\right). \tag{7.11}$$

7.2 Zero-free Regions, Siegel Zeros & Siegel's Theorem

While the Riemann hypothesis, and its generalization, are currently out of reach, some progress has still be made regarding the location of the zeros of L-functions. In particular, we want to establish regions inside the critical strip where L-functions do not vanish. These regions are commonly known as **zero-free regions**.

The Classical Zero-free Region for $\zeta(s)$

The classical zero-free region for the Riemann zeta function is due to de la Vallée Poussin. It describes a region slightly to the left of the line $\mathrm{Re}(s) = 1$ inside of the critical strip where $\zeta(s)$ cannot exhibit a zero. The key step in proving the classical zero-free region for $\zeta(s)$ is to leverage $\eta(s)$:

Theorem 7.2.1 (The classical zero-free region for $\zeta(s)$). *There exists a positive constant c such that $\zeta(s)$ contains no zeros in the region*

$$\left\{ s = \sigma + it \in \mathbb{C} : \sigma \geq 1 - \frac{c}{\log(|t|+2)} \right\}.$$

Proof. Let $s = \sigma + it$ and suppose $1 < \sigma \leq 2$ and $|t| > 2$. We will derive a bound for real part of a nontrivial zero of $\zeta(s)$. This bound will be obtained by finding upper bounds for all of the individal zeta terms in Lemma 5.2.1. For the first term, since $\zeta(s)$ has a simple pole at $s=1$, $-\frac{\zeta'}{\zeta}(s)$ has a simple pole there too. In particular,

$$-\frac{\zeta'}{\zeta}(\sigma) < A + \frac{1}{\sigma - 1}, \tag{7.12}$$

for some positive constant A. For the last term, since $|t| > 2$, s is bounded away from zero. In particular, $\frac{1}{s-1}$ is bounded. Moreover, $\frac{s}{2}+1 \sim |t|$ so from Proposition 1.5.3 so we also deduce $\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) = \log|t| + O(1) = O(\log|t|)$. These two estimates along with the explicit formula for $\frac{\zeta'}{\zeta}(s)$ imply

$$-\mathrm{Re}\left(\frac{\zeta'}{\zeta}(s)\right) < A\log|t| - \sum_{\rho}\mathrm{Re}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right), \tag{7.13}$$

where we take A larger (to satisfy the O-estimate), if necessary. Letting $\rho = \beta + i\gamma$, observe that $\mathrm{Re}\left(\frac{1}{s-\rho}\right) = \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2} \geq 0$ and $\mathrm{Re}\left(\frac{1}{\rho}\right) = \frac{\beta}{\beta^2+\gamma^2} > 0$. Therefore the sum in Equation (7.13) is a sum of nonnegative terms so that we may discard it. Discarding the sum and letting $s = \sigma + 2it$, we obtain

$$-\mathrm{Re}\left(\frac{\zeta'}{\zeta}(\sigma+2it)\right) < A\log|t|, \tag{7.14}$$

for a possibly larger constant A. For the middle term, we finally assume that $t = \gamma$ is the ordinate of the zero $\rho = \beta + i\gamma$ and repeat the argument for the middle term at $s = \sigma + it$ except keeping the term $\frac{1}{s-\rho}$ correspoding to ρ. So Equation (7.13) gives the bound

$$-\mathrm{Re}\left(\frac{\zeta'}{\zeta}(\sigma+it)\right) < A\log|t| - \frac{1}{\sigma-\beta}. \tag{7.15}$$

Upon combining Equation (7.12), Equation (7.14), and Equation (7.15) with Lemma 5.2.1, we obtain

$$0 < 3A + \frac{3}{\sigma-1} + 5A\log|t| - \frac{4}{\sigma-\beta}.$$

which, taking A to be larger if necessary, implies the estimate

$$0 < \frac{3}{\sigma-1} + 5A\log|t| - \frac{4}{\sigma-\beta}. \tag{7.16}$$

Letting δ be the positive constant such that $\sigma = 1 + \frac{\delta}{\log|t|}$, we can write Equation (7.16) as

$$0 < \left(5A + \frac{3}{\delta}\right)\log|t| - \frac{4}{(1-\beta)+\frac{\delta}{\log|t|}},$$

and solving for β yields

$$\beta < 1 + \frac{\delta}{\log|t|} - \frac{1}{\left(5A+\frac{3}{\delta}\right)\log|t|}.$$

Choosing δ such that $5A\delta + 3\delta^2 < 1$ (δ is free to choose because σ is), we find that

$$\beta < 1 - \frac{c}{\log|t|},$$

for some positive constant c. It follows that there cannot be a zero in the region

$$\sigma \geq 1 - \frac{c}{\log|t|},$$

for all σ and $|t| > 2$. By Theorem 5.2.1, $\zeta(s)$ cannot have a zero arbitrarily near $s = 1 + it$ with $|t| \leq 2$. As $\log(|t|+2) > \log|t|$, enlarging our choice of c if necessary, there cannot be a zero in the region

$$\sigma \geq 1 - \frac{c}{\log(|t|+2)},$$

for all σ and t. $\square$

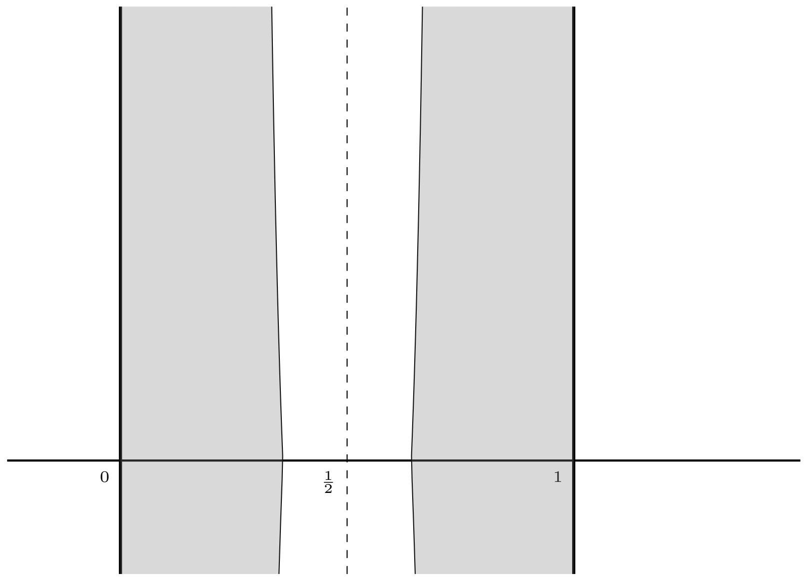

Figure 7.1: The classical zero-free region for $\zeta(s)$ (symmetrized).

A couple of comments are in order. The classical zero-free region describes a zero-free region for the Riemann zeta function in the right-half of the critical strip. As the zeros are symmetric about the critical line, we immediately obtain a zero-free region symmetric about the critical line (see Figure 7.1). This region has been improved many times. In 1922, the breadth of the zero-free region was enlarged to

$$\frac{c\log\log(|t|+2)}{\log(|t|+2)},$$

by Littlewood, and in 1958 Vinogradov and Korobov independently enlarged the zero-free region to

$$\frac{c}{(\log(|t|+2))^{\alpha}},$$

for any $\alpha > \frac{2}{3}$ and where c depends upon α (see [Dav80] for a more detailed historical account). These improvements depend on upper bounds for $\zeta(s)$ is a region just to the left of $\text{Re}(s) = 1$ and these upper bounds are deduced from delicate estimates of exponential sums. It is important to mention that the constant c is in general different for all the zero-free regions. This means c depends implicitly upon the shape of the zero-free region. Many current problems revolve around making improvements upon zero-free region results. Also, the constant c is effective and there are constant efforts to compute it to high degrees of precision.

It is also an interesting question to know the height of the first zero (with nonnegative imaginary part) the Riemann zeta function. It is not difficult to show that this height must necessarily be positive. In other words, $\zeta(s) \neq 0$ for $0 < s < 1$ (we know $\zeta(0) = -\frac{1}{2}$ and that there is a pole at $s = 1$). One way to see this is to consider the **Dirichlet eta function** $\eta(s)$ defined by

$$\eta(s) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n^s}.$$

Note that $\eta(s)$ converges for $\text{Re}(s) > 0$ by Proposition 3.1.1. Now for real s with $0 < s < 1$ and even n, $\frac{1}{n^s} - \frac{1}{(n+1)^s} > 0$ so that $\eta(s) > 0$. But for $\text{Re}(s) > 0$, we have

$$(1-2^{1-s})\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} - 2\sum_{n\geq 1}\frac{1}{(2n)^s} = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n^s} = \eta(s).$$

Therefore $\zeta(s)$ cannot admit a zero for $0 < s < 1$ because then $\eta(s)$ would be zero too. Actually, since $1 - 2^{1-s} < 0$ in this segment we also know $\zeta(s) < 0$ in this segment as well. As for the height of the first zero, it occurs on the critical line (as predicted by the Riemann hypothesis) at height $t \approx 14.134$. In particular, this gives a slight improvement to the classical zero-free region in Figure 7.1. Actually, the first 15 zeros were computed by Gram in 1903 (see [Gra03]). Since then, billions of zeros have been computed and have all been verified to lie on the critical line.

The Classical Zero-free Region for $L(s, \chi)$ & Siegel Zeros

For a fixed Dirichlet character χ of conductor q, it is not difficult to obtain a zero-free region result for $L(s, \chi)$ analogous to the classical zero-free region for $\zeta(s)$. However, it is more useful to have zero-free regions in terms of q alone so that the character χ can vary. Unfortunately, an additional obstruction arises in the case that the character is quadratic. This is due to the possible existance of an additional real zero. Throughout, we will only be interested in non-principal characters. For convenience we use the standard notation

$$\mathscr{L} = \log(q(|t| + 2)),$$

because this quantity appears all too often. The classical result, analogous to the case for the Riemann zeta function, is the following:

Theorem 7.2.2 (The classical zero-free region for $L(s, \chi)$). *There exists a positive constant c such that for any non-principal Dirichlet character χ of conductor $q > 1$, $L(s, \chi)$ contains no zeros in the region*

$$\left\{ s = \sigma + it \in \mathbb{C} : \sigma > 1 - \frac{c}{\mathscr{L}} \right\},$$

unless χ is quadratic in which case $L(s, \chi)$ has at most one, necessarily real, zero in this region.

Proof of the classical zero-free region for $L(s, \chi)$. We will eventually break the argument depending on whether χ is quadratic or complex, but for the moment we only assume χ is primitive. Let $s = \sigma + it$ and suppose $1 < \sigma \leq 2$. We will derive bounds for each of the terms in Lemma 5.2.2. For the first term, since $L(s, \chi_{q,0})$ has a simple pole at $s = 1$, $-\frac{L'}{L}(s, \chi_{q,0})$ has a simple pole there too. Therefore

$$-\frac{L'}{L}(\sigma, \chi_{q,0}) < A + \frac{1}{\sigma - 1}, \tag{7.17}$$

for some positive constant A. For the middle term, $\frac{s+\mathfrak{a}}{2} \sim |t| + 2$ so by Proposition 1.5.3, $\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) = \log(|t| + 2) + O(1) = O(\log(|t| + 2))$. This estimate and explicit formula for $\frac{L'}{L}(s, \chi)$ together give

$$-\mathrm{Re}\left(\frac{L'}{L}(s, \chi)\right) < A\mathscr{L} - \mathrm{Re}(B(\chi)) - \sum_{\rho} \mathrm{Re}\left(\frac{1}{s - \rho} + \frac{1}{\rho}\right), \tag{7.18}$$

where we take A larger (to satisfy the O-estimate) if necessary. Upon substituting Equation (7.11) into Equation (7.18) we obtain the simpler expression

$$-\mathrm{Re}\left(\frac{L'}{L}(s, \chi)\right) < A\mathscr{L} - \sum_{\rho} \mathrm{Re}\left(\frac{1}{s - \rho}\right). \tag{7.19}$$

If $\rho = \beta + i\gamma$, $\mathrm{Re}\left(\frac{1}{s-\rho}\right) = \frac{\sigma - \beta}{(\sigma-\beta)^2 + (t+\gamma)^2} \geq 0$ which implies the sum in Equation (7.19) is a sum of nonnegative terms and hence any part of it may be discarded. Now suppose χ is primitive and complex. We claim that Equation (7.19) holds for $L(s,\chi^2)$ too for a possibly larger constant A. This is certainly true if χ^2 is primitive, but when χ^2 is not primitive we have a minor complication. To get around this complication, let $\widetilde{\chi}$ be the primitive character inducing χ^2. Taking the logarithmic derivative of Equation (3.16) with χ^2 in place of χ, we find that

$$\left|\frac{L'}{L}(s,\chi^2)-\frac{L'}{L}(s,\widetilde{\chi})\right|=\left|\sum_{p|q}\frac{\widetilde{\chi}(p)\log(p)p^{-s}}{1-\widetilde{\chi}(p)p^{-s}}\right|\le\sum_{p|q}\frac{\log(p)p^{-\sigma}}{1-p^{-\sigma}}\le\sum_{p|q}\log(p)\le\log(q). \tag{7.20}$$

Now Equation (7.20) implies the weaker inequality

$$-\mathrm{Re}\left(\frac{L'}{L}(s,\chi^2)\right)\le\log(q)-\mathrm{Re}\left(\frac{L'}{L}(s,\widetilde{\chi})\right), \tag{7.21}$$

and combining Equations (7.19) and (7.21) with $\widetilde{\chi}$ in place of χ yields

$$-\mathrm{Re}\left(\frac{L'}{L}(s,\chi^2)\right)<A\mathscr{L}-\sum_{\rho}\mathrm{Re}\left(\frac{1}{s-\rho}\right), \tag{7.22}$$

for a possibly larger constant A. We can now obtain the bound on the last term. Taking $s=\sigma+2it$ and discarding the entire sum, Equation (7.22) gives

$$-\mathrm{Re}\left(\frac{L'}{L}(s+2it,\chi)\right)<A\mathscr{L}, \tag{7.23}$$

for a possibly larger constant A. For the middle term, assume $t=\gamma$ is the ordinate of the zero $\rho=\beta+i\gamma$. Letting $s=\sigma+it$ and retaining only the term in the sum corresponding to ρ, Equation (7.19) implies

$$-\mathrm{Re}\left(\frac{L'}{L}(\sigma+it,\chi^2)\right)<A\mathscr{L}-\frac{1}{\sigma-\beta}. \tag{7.24}$$

Upon combining Equations (7.17), (7.23) and (7.24) with Lemma 5.2.2 we obtain

$$0<3A+\frac{3}{\sigma-1}+5A\mathscr{L}-\frac{4}{\sigma-\beta},$$

which, taking A to be larger if necessary, gives

$$0<\frac{3}{\sigma-1}+5A\mathscr{L}-\frac{4}{\sigma-\beta}. \tag{7.25}$$

Letting δ be such that $\sigma=1+\frac{\delta}{\mathscr{L}}$, Equation (7.25) becomes

$$0<\left(5A+\frac{3}{\delta}\right)\mathscr{L}-\frac{4}{(1-\beta)+\frac{\delta}{\mathscr{L}}}.$$

Arguing exactly as in the proof of the classical zero-free region for $\zeta(s)$, we obtain a bound of the form

$$\beta<1-\frac{c}{\mathscr{L}}. \tag{7.26}$$

It follows that there can be no zero in the region

$$\sigma\ge1-\frac{c}{\mathscr{L}}, \tag{7.27}$$

for all σ and t. This finishes the proof in the case that χ is primitive and complex. Actually, the assumption that χ is primitive can be dropped since if $\widetilde{\chi}$ induces χ then Equation (3.16) implies that any zero of $L(s,\chi)$ which is not a zero of $L(s,\widetilde{\chi})$ is a zero of one of the factors $(1-\widetilde{\chi}(p)p^{-s})$ and hence lies on the line $\mathrm{Re}(s)=0$ which is a boundary line of the critial strip. Therefore our zero-free region holds for all complex χ. We now suppose χ is primitive and quadratic. The only part of the argument that needs adjustment is upper bound for the last term. Equations (7.20) and (7.21) still hold but since $\chi^2=\chi_{q,0}$, the primitive character inducing χ^2 is the trivial character so $L(s,\widetilde{\chi})=\zeta(s)$ and hence Equation (7.19) with $\widetilde{\chi}$ in place of χ no longer applies. Instead, we use the explicit formula for $\frac{\zeta'}{\zeta}(s)$. Since $\frac{s}{2}+1\sim|t|+2$, Proposition 1.5.3 implies $\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)=\log(|t|+2)+O(1)=O(\log(|t|+2))$. This estimate, the explicit formula for $\frac{\zeta'}{\zeta}(s)$, and discarding the sum over nontrivial zeros (because this is a sum of nonpositive terms), together give

$$-\mathrm{Re}\left(\frac{\zeta'}{\zeta}(s)\right)<\mathrm{Re}\left(\frac{1}{s-1}\right)+A\log(|t|+2), \tag{7.28}$$

for some possibly larger constant A. Observe that the term corresponding to $\frac{1}{s-1}$ in the explicit formula for $\frac{\zeta'}{\zeta}(s)$ is present in Equation (7.28) because t may very well be small and so this term is not necessarily bounded from above. Letting $s=\sigma+2it$, and combining Equations (7.21) and (7.28) (recall $L(s,\widetilde{\chi})=\zeta(s)$) yields

$$-\mathrm{Re}\left(\frac{L'}{L}(\sigma+2it,\chi^2)\right)<\mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right)+A\mathscr{L}, \tag{7.29}$$

for some possibly larger constant A. Combining Equations (7.17), (7.23) and (7.29) with Lemma 5.2.2 gives

$$0<3A+\frac{3}{\sigma-1}+5A\mathscr{L}+\mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right)-\frac{4}{\sigma-\beta},$$

and taking A to be larger if necessary, we have

$$0<\frac{3}{\sigma-1}+5A\mathscr{L}+\mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right)-\frac{4}{\sigma-\beta}. \tag{7.30}$$

Let δ be such that $\sigma=1+\frac{\delta}{\mathscr{L}}$. If we also assume $|t|\geq\frac{\delta}{\mathscr{L}}$, then $\mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right)=\frac{\sigma-1}{(\sigma-1)^2+2t^2}\leq\frac{1}{5\delta}\mathscr{L}$ and Equation (7.30) becomes

$$0<\left(5A+\frac{3}{\delta}+\frac{1}{5\delta}\right)\mathscr{L}-\frac{4}{(1-\beta)+\frac{\delta}{\log|t|}}.$$

The additional estimate $|t|\geq\frac{\delta}{\mathscr{L}}$ is essential, because if t is close to 0 and σ is close to 1, then $\mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right)$ is not bounded from above. Arguing again as in the proof of the classical zero-free region for $\zeta(s)$, we arrive at the bound Equation (7.26) for some possibly larger constant c. Therefore Equation (7.27) holds for primitive quadratic χ. Just as for complex characters, we can remove the primitive assumption by appealing Equation (3.16). So altogether, we have the desired zero-free region for all complex characters for all t and all quadratic characters with $|t|\geq\frac{\delta}{\mathscr{L}}$ for some δ. It remains to show that in the quadratic case, that there is at most one zero with $|t|<\frac{\delta}{\mathscr{L}}$ and that this zero is real. Actually it suffices to prove that there is at most one zero because if ρ is a zero then so is $\overline{\rho}$ since χ is quadratic. But both ρ and $\overline{\rho}$ have the same imaginary part in absolute value (so they both belong to this region). Again, we may assume χ is primitive. Now suppose there were two complex conjugate zeros $\rho=\beta+i\gamma$ and $\overline{\rho}=\beta-i\gamma$ belonging to the region $|t|<\frac{\delta}{\mathscr{L}}$. Taking $s=\sigma>1$ in Equation (7.19) and retaining only the terms in the sum corresponding to these zeros, we have

$$-\mathrm{Re}\left(\frac{L'}{L}(\sigma,\chi)\right)<A\mathscr{L}-\frac{2(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2}, \tag{7.31}$$

where we have used $\mathrm{Re}\left(\frac{1}{\sigma-\rho}\right) = \frac{\sigma-\beta}{(\sigma-\beta)^2+\gamma^2}$. Appealing to Equations (5.8), (5.12) and (7.12) we have the following crude lower bound:

$$-\mathrm{Re}\left(\frac{L'}{L}(\sigma,\chi)\right) \geq \mathrm{Re}\left(\sum_{n\geq 1}\frac{\chi(n)\Lambda(n)}{n^\sigma}\right) \geq \sum_{n\geq 1}\frac{\Lambda(n)}{n^\sigma} = \frac{\zeta'}{\zeta}(\sigma) > -\frac{1}{\sigma-1} - A, \tag{7.32}$$

for a possibly larger constant A. Taking A larger, if necessary, Equations (7.31) and (7.32) together yield

$$-\frac{1}{\sigma-1} < A\mathscr{L} - \frac{2(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2}. \tag{7.33}$$

Now let δ (possibly a different δ than before) be such that $\sigma = 1 + \frac{\delta}{\mathscr{L}}$. Since $|\gamma| < \frac{\delta}{\mathscr{L}}$ by assumption, we have

$$|\gamma| < \frac{\sigma-1}{2} < \frac{\sigma-\beta}{2}.$$

The estimate above implies $\gamma^2 < \frac{(\sigma-\beta)^2}{4}$ and inserting this into Equation (7.33) gives

$$-\frac{1}{\sigma-1} < A\mathscr{L} - \frac{5}{8(\sigma-\beta)}.$$

Upon solving for β and substiuting $\sigma = 1 + \frac{\delta}{\mathscr{L}}$, we find that

$$\beta < 1 + \frac{2\delta - \frac{5}{8}\left(\frac{1}{2\delta} + A\right)}{\mathscr{L}}.$$

Choosing δ such that $2\delta - \frac{5}{8}\left(\frac{1}{2\delta} + A\right) < -1$ (δ is free to choose because σ is) results in the bound

$$\beta < 1 - \frac{\delta}{\mathscr{L}},$$

which contradicts the zero-free region just establish because $|\gamma| < \frac{\delta}{\mathscr{L}}$. Hence there cannot be two complex conjugate zeros. In the case of two real zeros $\rho = \beta$ and $\rho' = \beta'$, without loss of generality we may assume ρ is such that $\mathrm{Re}\left(\frac{1}{s-\rho}\right) \leq \mathrm{Re}\left(\frac{1}{s-\rho'}\right)$. Then taking $s = \sigma > 1$ in Equation (7.19) and only retaining the terms in the sum correspond to these zeros but bounding both of them by the term corresponding to ρ gives Equation (7.31) and we may repeat the argument. In the case of a real zero $\rho = \beta$ with multiplicity at least two, we perform the same procedure but only retain two terms in the sum corresponding to ρ (because we at least have a double zero) and obtain Equation (7.31) with $\gamma = 0$. The same argument now works but is even simpler because we do not need to estimate γ. All of this shows that if there is a zero it must be a single real zero. □

The classical zero-free region for Dirichlet L-functions is almost an exact analog to the classical zero-free region for the Riemann zeta function. In particular, the constant c is effective. The strength of this zero-free region result rests in the fact that it only depends upon the conductor q of the character. Unfortunately, the drawback is that when χ is quadratic there may very well be a single real zero. Of course, the Selberg class Riemann hypothesis implies that such zeros do not exist. These hypothetical zeros are referred to as **Siegel zeros** or **exceptional zeros**. So far, no Siegel zero has been shown to exist or not exist. Now the effective constant c is absolute and hence is independent of every quadratic character χ. So for any fixed modulus, we can choose a smaller constant to make the region zero-free for all characters of that modulus. Also, some progress has been made towards showing that they are exceptionally rare (see [MV06]):

Proposition 7.2.1. *For every integer $q > 1$, there is at most one primitive and quadratic Dirichlet character χ of conductor q such that $L(s, \chi)$ has a Siegel zero.*

A natural question to ask after Proposition 7.2.1 is the following: how small can the distance between the moduli of two quadratic characters whose Dirichlet L-functions posess Siegel zeros be? This has been answered (see [MV06] for a proof):

Proposition 7.2.2. *For any positive $A > 0$ there exists a positive constant c_A such that if $(q_i)_{1\le i\le n}$ is a strictly increasing sequence of positive integers with $q_1 > 1$ and such that there exist primitive quadratic Dirichlet characters χ_i of conductor q_i whose Dirichlet L-function $L(s, \chi_{q_i})$ posses Siegel zeros β_i such that*

$$\beta_i > 1 - \frac{c_A}{\log(q_i)},$$

for $1 \le i \le n$, then

$$q_{i+1} > q_i^A,$$

for $1 \le i \le n-1$.

Siegel's Theorem

Siegel zeros present an unfortunate obstruction to zero-free region results for Dirichlet L-functions when the character χ is quadratic. However, if we no longer require the constant c in the zero-free region to be effective, we can obtain a much better result for how close the Siegel zero can be to 1. Ultimately, this improved bound results from a lower bound for $L(1, \chi)$ (recall that this is nonzero from our discussion of non-vanishing on $\mathrm{Re}(s) = 1$). **Siegel's theorem** refers to either this lower bound or to the improved zero-free region. In its first version, Siegel's theorem is the following:

Theorem 7.2.3 (Siegel's theorem, first version). *Let χ be a primitive quadratic Dirichlet character modulo q. Then for any $\varepsilon > 0$ there exists a positive constant $c_1(\varepsilon)$ such that*

$$L(1, \chi) \ge c_1(\varepsilon) q^{-\varepsilon}.$$

In its second version, the statement is about how close a Siegel zero can be to 1:

Theorem 7.2.4 (Siegel's theorem, second version). *Let χ be a primitive quadratic Dirichlet character modulo q. Then for any $\varepsilon > 0$ there exists a positive constant $c_2(\varepsilon)$ such that $L(s, \chi)$ has no real zero β with*

$$1 - \beta \le c_2(\varepsilon) q^{-\varepsilon}.$$

The largest defect of Siegel's theorem, in either version, is that the implicit constants $c_1(\varepsilon)$ and $c_2(\varepsilon)$ are ineffective (and not necessarily equal). Actually, the first version is slighly stronger as it implies the second. We will prove the second version (given the first) first and then we will prove the first version. Before we begin, however, we need two small lemmas about the size of $L'(\sigma, \chi)$ and $L(\sigma, \chi)$ for real s close to 1:

Lemma 7.2.1. *Let χ be a Dirichlet character modulo $m > 1$. Then $L'(\sigma, \chi) = O(\log^2(m))$ for any σ such that $0 \le 1 - \sigma \le \frac{1}{\log(m)}$.*

Proof. Setting $A(X) = \sum_{n \le X} \chi(n)$ we have $A(X) \ll 1$ by Corollary 1.2.1 (i) and that χ is periodic. Therefore $\sigma_c \le 0$ by Proposition 3.1.1. Hence for σ in the prescribed region, $L(\sigma, \chi)$ is holomorphic and its derivative is given by

$$L'(\sigma, \chi) = \sum_{n \ge 1} \frac{\chi(n)\log(n)}{n^\sigma} = \sum_{n<m} \frac{\chi(n)\log(n)}{n^\sigma} + \sum_{n \ge m} \frac{\chi(n)\log(n)}{n^\sigma}.$$

We will show that the last two sums are both $O(\log^2(m))$. For the first sum, if $n < m$, we have

$$\left|\frac{\chi(n)\log(n)}{n^\sigma}\right| \le \frac{1}{n^\sigma}\log(n) = \frac{n^{1-\sigma}}{n}\log(n) < \frac{m^{1-\sigma}}{n}\log(n) < \frac{e}{n}\log(m),$$

where the last inequality follows because $1 - \sigma \le \frac{1}{\log(m)}$. Then

$$\left|\sum_{n \le m} \frac{\chi(n)\log(n)}{n^s}\right| < e\log(m)\sum_{n<m}\frac{1}{n} < e\log(m)\int_1^m \frac{1}{n}\,dn \ll \log^2(m).$$

For the second sum, $A(Y) \ll 1$ so that $A(Y)\log(Y)Y^{-\sigma} \to 0$ as $Y \to \infty$. Then Abel's summation formula (see Appendix B.3) gives

$$\sum_{n \ge m} \frac{\chi(n)\log(n)}{n^\sigma} = -A(m)\log(m)m^{-\sigma} - \int_m^\infty A(u)(1 - \sigma\log(u))u^{-(\sigma+1)}\,du. \tag{7.34}$$

Since $0 \le 1-\sigma \le \frac{1}{\log(m)}$, we have $1-\sigma\log(u) \le \frac{\log(u)}{\log(m)}$. Also, we have the more precise estimate $|A(X)| \le m$ because χ is m-periodic and $|\chi(n)|$ is at most 1. With these estimates and Equation (7.34) we make the following computation:

$$\begin{aligned}
\left|\sum_{n \ge m} \frac{\chi(n)\log(n)}{n^\sigma}\right| &\le |A(m)|\log(m)m^{-\sigma} + \int_m^\infty |A(u)|(1 - \sigma\log(u))u^{-(\sigma+1)}\,du \\
&\le |A(m)|\log(m)m^{-\sigma} + \log(m)\int_m^\infty |A(u)|\log(u)u^{-(\sigma+1)}\,du \\
&\le m^{1-\sigma}\log(m) + m\int_m^\infty \log(u)u^{-(\sigma+1)}\,du \\
&= m^{1-\sigma}\log(m) + m\left(-\log(u)\frac{u^{-\sigma}}{\sigma}\bigg|_m^\infty + \int_m^\infty \frac{u^{-(\sigma+1)}}{s}\,du\right) \\
&= m^{1-\sigma}\log(m) + m\left(-\log(u)\frac{u^{-\sigma}}{\sigma} - \frac{u^{-\sigma}}{\sigma^2}\right)\bigg|_m^\infty \\
&= m^{1-\sigma}\log(m) + m\left(\log(m)\frac{m^{-\sigma}}{\sigma} + \frac{m^{-\sigma}}{\sigma^2}\right) \\
&\ll m^{1-\sigma}\log(m) \\
&\ll e\log(m),
\end{aligned}$$

where in the fourth line we have used integration by parts and the last line holds because $1 - \sigma \le \frac{1}{\log(m)}$. But $e\log(m) = O(\log^2(m))$ so the second sum is also $O(\log^2(m))$. Therefore we have shown $L'(\sigma, \chi) = O(\log^2(m))$ finishing the proof. $\square$

The second lemma is even easier and is proved in exactly the same way:

Lemma 7.2.2. *Let χ be a Dirichlet character modulo $m > 1$. Then $L(\sigma, \chi) = O(\log(m))$ for any σ such that $0 \leq 1 - \sigma \leq \frac{1}{\log(m)}$.*

Proof. Note that

$$L(\sigma, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^\sigma} = \sum_{n < m} \frac{\chi(n)}{n^\sigma} + \sum_{n \geq m} \frac{\chi(n)}{n^\sigma}.$$

It suffices to show that the last two sums are both $O(\log^2(m))$. For the first sum, since $n < m$, we have

$$\left| \frac{\chi(n)}{n^\sigma} \right| \leq \frac{1}{n^\sigma} = \frac{n^{1-\sigma}}{n} < \frac{m^{1-\sigma}}{n} < \frac{e}{n},$$

where the last inequality follows because $1 - \sigma \leq \frac{1}{\log(m)}$. Therefore

$$\left| \sum_{n \leq m} \frac{\chi(n)}{n^s} \right| < e \sum_{n < m} \frac{1}{n} < e \log(m) \int_1^m \frac{1}{n}\, dn \ll \log(m).$$

As for the second sum, setting $A(Y) = \sum_{n \leq Y} \chi(n)$ we have $A(Y) \ll 1$ so that $A(Y)Y^{-\sigma} \to 0$ as $Y \to \infty$. Then Abel's summation formula (see Appendix B.3) gives

$$\sum_{n \geq m} \frac{\chi(n)}{n^\sigma} = -A(m)m^{-\sigma} - \int_m^\infty A(u)u^{-(\sigma+1)}\, du. \tag{7.35}$$

Using the more precise estimate $|A(X)| \leq m$ because χ is m-periodic and $|\chi(n)|$ is at most 1 and Equation (7.35), we make the following computation:

$$\begin{aligned}
\left| \sum_{n \geq m} \frac{\chi(n)\log(n)}{n^\sigma} \right| &\leq |A(m)|m^{-\sigma} + \int_m^\infty |A(u)|u^{-(\sigma+1)}\, du \\
&\leq |A(m)|m^{-\sigma} + \int_m^\infty |A(u)|\log(u)u^{-(\sigma+1)}\, du \\
&\leq m^{1-\sigma} + m \int_m^\infty \log(u)u^{-(\sigma+1)}\, du \\
&= m^{1-\sigma} + m \left(-\log(u)\frac{u^{-\sigma}}{\sigma} \Big|_m^\infty + \int_m^\infty \frac{u^{-(\sigma+1)}}{s}\, du \right) \\
&= m^{1-\sigma} + m \left(-\log(u)\frac{u^{-\sigma}}{\sigma} - \frac{u^{-\sigma}}{\sigma^2} \right) \Big|_m^\infty \\
&= m^{1-\sigma} + m \left(\log(m)\frac{m^{-\sigma}}{\sigma} + \frac{m^{-\sigma}}{\sigma^2} \right) \\
&\ll m^{1-\sigma} \\
&\ll e,
\end{aligned}$$

where in the fourth line we have used integration by parts and the last line holds because $1 - \sigma \leq \frac{1}{\log(m)}$. But $e = O(\log^2(m))$ so the second sum is also $O(\log^2(m))$. Therefore we have shown $L(\sigma, \chi) = O(\log(m))$ which completes the proof. ∎

We will now prove the second version of Sigel's theorem, assuming the first, and using Lemma 7.2.1:

Proof of Siegel's theorem, second version. We we will prove the theorem by contradiction. Let $\varepsilon > 0$. Clearly the result holds for a single q, and notice that the result also holds provided we bound q from above by taking the maximum of all the $c_2(\varepsilon)$. Therefore we may suppose q is arbitrarily large. In this case, if there was a real zero β with $1-\beta \le c_2(\varepsilon)q^{-\varepsilon}$ then for large enough q we have $0 \le 1-\beta \le \frac{1}{\log(q)}$ so that $L'(\sigma,\chi) = O(\log^2(q))$ for $\beta \le \sigma \le 1$ by Lemma 7.2.1. These two estimates and the mean value theorem together give

$$L(1,\chi) = L(1,\chi) - L(\beta,\chi) = L'(\sigma,\chi)(1-\beta) \ll \log^2(q)q^{-\varepsilon}.$$

Upon taking $\frac{\varepsilon}{2}$ in the first version of Siegel's theorem, we obtain

$$q^{-\frac{\varepsilon}{2}} \ll L(1,\chi) \ll \log^2(q)q^{-\varepsilon},$$

which is a contradiction for large q. □

It remains to prove the first version of Siegel's theorem. The idea is to combine two Dirichlet L-functions attached to distinct characters with distinct moduli and use this new L-function to derivative a lower bound for a single Dirichlet L-function at $s = 1$:

Proof of Siegel's theorem, first version. Let χ_1 and χ_2 be two primitive quadratic and non-principal characters modulo q_1 and q_2 repsectively. Set

$$F(s) = \zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2).$$

The key ingredient in the proof is a lower bound for $F(s)$ relative to the modulus q_1q_2 in a small interval on the real axis close to 1. We will first deduce this estiamte from which the rest of the proof follows easily. Observe that $F(s)$ is holomorphic on $\mathbb{C}$ except for a simple pole at $s = 1$. Let λ be the residue at this pole so that

$$\lambda = L(1,\chi_1)L(1,\chi_2)L(1,\chi_1\chi_2).$$

Taking the logarithim of $F(s)$ with $\mathrm{Re}(s) > 1$ and using Equation (5.5) gives

$$\log F(s) = \sum_p \sum_{k\ge1} \frac{1+\chi_1(p^k)+\chi_2(p^k)+(\chi_1\chi_2)(p^k)}{kp^{ks}} = \sum_p \sum_{k\ge1} \frac{(1+\chi_1(p^k))(1+\chi_2(p^k))}{kp^{ks}}.$$

Since $(1+\chi_1(p^k))(1+\chi_2(p^k))$ is nonnegative, the coefficients of $\log F(s)$ are nonnegative and therefore the coefficients of $F(s)$, call them $a(n)$, are nonnegative as well. Moreover, $a(0) = 1$ which can be seen by expanding out the Dirichlet series defining $F(s)$. Now $F(s)$ is represented as an absolutely convergent series for $\mathrm{Re}(s) > 1$ so that it has a power series expansion about $s = 2$ with radius 1:

$$F(s) = \sum_{m\ge0} \frac{F^{(m)}(2)}{m!}(s-2)^m,$$

for $|s-2| < 1$. We can compute $F^{(m)}(2)$ using the Dirichlet series by differentiating termwise:

$$F^{(m)}(2) = \frac{d^m}{ds^m}\left(\sum_{n\ge1}\frac{a(n)}{n^s}\right)\Bigg|_{s=2} = (-1)^m \sum_{n\ge1}\frac{a(n)\log^m(n)}{n^s}\Bigg|_{s=2} = (-1)^m \sum_{n\ge1}\frac{a(n)\log^m(n)}{n^2}. \tag{7.36}$$

Since the $a(n)$ are nonnegative, as just mentioned, it follows that $F^{(m)}(2)$ is nonnegative and therefore we may write

$$F(s) = \sum_{m \geq 0} b(m)(2-s)^m,$$

for $|s-2| < 1$ and with $b(m)$ nonnegative. Also, Equation (7.36) and the fact that the $a(n)$ are nonnegative with $a(0) = 1$ together imply that $b(0) > 1$. Then

$$F(s) - \frac{\lambda}{s-1} = F(s) - \lambda \sum_{m \geq 0} (2-s)^m = \sum_{m \geq 0} (b(m) - \lambda)(2-s)^m, \tag{7.37}$$

and the last series must be absolutely convergent for say $|s-2| < 2$ because $F(s) - \frac{\lambda}{s-1}$ is entire as we have removed the pole at $s = 1$. We now wish to estiamte $F(s)$ and $\frac{\lambda}{s-1}$ on the circle $|s-2| = \frac{3}{2}$. Let χ be a Dirichlet character modulo q and let $A(X) = \sum_{n \leq X} \chi(n)$. Then Abel's summation formula and that $A(X) \ll 1$ (just as in the proof of Proposition 3.1.1) together imply

$$L(s, \chi) = s \int_1^\infty A(u) u^{-(s+1)} \, du,$$

for $\mathrm{Re}(s) > 0$. Restrict to $\mathrm{Re}(s) \geq \frac{1}{2}$ and let $s = \sigma + it$. As $|A(X)| \leq q$, we obtain

$$|L(s, \chi)| \leq q|s| \int_1^\infty u^{-(\sigma+1)} \, du = -q|s| \frac{u^{-\sigma}}{\sigma} \bigg|_1^\infty = \frac{q|s|}{\sigma} \leq 2q|s|.$$

In particular, on the disk $|s-2| \leq \frac{3}{2}$ we have the estimates

$$|L(s, \chi_1)| \ll q_1, \quad |L(s, \chi_2)| \ll q_2, \quad \text{and} \quad |L(s, \chi_1\chi_2)| \ll q_1 q_2.$$

Since $\zeta(s)$ is bounded on the circle $|s-2| = \frac{3}{2}$ (it's a compact set) and $\lambda = L(1, \chi_1)L(1, \chi_2)L(1, \chi_1\chi_2)$, we obtain the bounds

$$|F(s)| \ll q_1^2 q_2^2 \quad \text{and} \quad \left| \frac{\lambda}{s-1} \right| \ll q_1^2 q_2^2,$$

on this circle as well. Cauchy's inequality for the size of coefficients of a power series applied to Equation (7.37) on the circle $|s-2| = \frac{3}{2}$ gives

$$|b(m) - \lambda| \ll q_1^2 q_2^2 \left(\frac{2}{3} \right)^m. \tag{7.38}$$

Let M be a positive integer. For real s with $\frac{7}{8} < s < 1$ we have $2 - s < \frac{9}{8}$ and using Equations (7.37) and (7.38) together we can upper bound the tail of $F(s) - \frac{\lambda}{s-1}$:

$$\begin{aligned}
\left| \sum_{m \geq M} (b(m) - \lambda)(2-s)^m \right| &\leq \sum_{m \geq M} |b(m) - \lambda|(2-s)^m \\
&\ll q_1^2 q_2^2 \sum_{m \geq M} \left(\frac{2}{3}(2-s) \right)^m \\
&\ll q_1^2 q_2^2 \sum_{m \geq M} \left(\frac{3}{4} \right)^m \\
&\ll q_1^2 q_2^2 \left(\frac{3}{4} \right)^M \\
&\ll q_1^2 q_2^2 e^{-\frac{M}{4}},
\end{aligned}$$

where the last estimate follows because $(\frac{3}{4})^M < e^{-\frac{M}{4}}$ (which is equivalent to $\log\left(\frac{3}{4}\right) < -\frac{1}{4}$). Let c be the implict constant. Using that the $b(m)$ are nonnegative, $b(0) > 1$, and the previous estimate for the tail, we can estimate $F(s) - \frac{\lambda}{s-1}$ from below for $\frac{7}{8} < s < 1$. Ineed, throwing away the $b(m)$ terms for $1 \le m \le M$, bounding the constant term below by 1, and use the tail estimate gives

$$F(s) - \frac{\lambda}{s-1} \ge 1 - \lambda \sum_{0 \le m \le M-1} (2-s)^m - cq_1^2 q_2^2 e^{-\frac{M}{4}} = 1 - \lambda \frac{(2-s)^M - 1}{1-s} - cq_1^2 q_2^2 e^{-\frac{M}{4}}, \tag{7.39}$$

which is valid for any positive integer M. Now chose M to be a positive integer such that

$$\frac{1}{2} e^{-\frac{1}{4}} \le cq_1^2 q_2^2 e^{-\frac{M}{4}} < \frac{1}{2}. \tag{7.40}$$

Upon isolating $F(s)$ in Equation (7.39) and using the second estimate in Equation (7.40), we get

$$F(s) \ge \frac{1}{2} - \lambda \frac{(2-s)^M}{1-s}. \tag{7.41}$$

Taking the logarithm of the first esimate in Equation (7.40) and isolating M, we obtain

$$M \le 8\log(q_1 q_2) + c, \tag{7.42}$$

for some different constant c. It follows that

$$(2-s)^M = e^{M \log(2-s)} < e^{M(1-s)} \le c(q_1 q_2)^{8(1-s)}, \tag{7.43}$$

for some different constant c, where in the first estimate we have used the Taylor series of the logarithm truncated at the first term and in the second estimate we have used Equation (7.42). Since $1-s$ is positive for $\frac{7}{8} < s < 1$, we can combine Equations (7.41) and (7.43) which gives

$$F(s) \ge \frac{1}{2} - \lambda \frac{c}{1-s} (q_1 q_2)^{8(1-s)}. \tag{7.44}$$

This is our desired lower bound for $F(s)$. Now let $\varepsilon > 0$ be given. We will choose the character χ_1 depending upon ε. If there exists a Siegel zero β_1 with $1 - \frac{\varepsilon}{16} < \beta_1 < 1$, let χ_1 be such a character correspoding to the Dirichlet L-function that admits this Siegel zero. Then $F(\beta_1) = 0$ independent of the choice of χ_2. If there is no such Siegel zero, choose χ_1 to be any quadratic primitive character and β_1 to be any number such that $1 - \frac{\varepsilon}{16} < \beta_1 < 1$. Then $F(\beta_1) < 0$ independent of the choice of χ_2. Indeed, $\zeta(s)$ is negative in this segment (actually for $0 \le s < 1$) and each of the Dirichlet L-function defining $F(s)$ is positive at $s = 1$ (the Euler product implies Dirichlet L-function are positive for real $s > 1$ and they are in fact nonzero for $s = 1$ by Theorem 5.1.3) and do not admit a zero for $\beta_1 \le s \le 1$ by our choice of β_1. In either case, $F(\beta_1) \le 0$ so isolating λ and disregarding the constants in Equation (7.44) with $s = \beta_1$ gives the weaker estimate

$$\lambda \gg (1-\beta_1)(q_1 q_2)^{-8(1-\beta_1)}. \tag{7.45}$$

We will now choose $\chi_2 = \chi$ and hence $q_2 = q$ as in the statement of the theorem. Notice that, independent of any work we have done, the theorem holds for a single q. Moreover, the theorem holds provided we bound q from above by taking the minimum of the $c_1(\varepsilon)$. Therefore we may assume q is arbitrarily large and in particular that $q > q_1$. Using Lemma 7.2.2 with $\sigma = 1$ applied to $L(s, \chi_1)$ and $L(s, \chi_1\chi)$ and that Dirichlet L-function take positive real values at $s = 1$, we obtain

$$\lambda \ll \log(q_1) \log(q_1 q) L(1, \chi). \tag{7.46}$$

Combining Equations (7.45) and (7.46) yields

$$(1-\beta_1)(q_1q)^{-8(1-\beta_1)} \ll \log(q_1)\log(q_1q)L(1,\chi).$$

As β_1 and q_1 are fixed and $\log(q_1q) = O(\log(q))$, isolating $L(1,\chi)$ gives the weaker estimate

$$L(1,\chi) \gg q^{-8(1-\beta_1)}(\log(q))^{-1}. \tag{7.47}$$

But $1-\frac{\varepsilon}{16} < \beta < 1$ so that $0 < 8(1-\beta) < \frac{\varepsilon}{2}$ which combined with Equation (7.47) yields

$$L(1,\chi) \gg q^{-\frac{\varepsilon}{2}}(\log(q))^{-1} \gg q^{-\varepsilon},$$

where the last estimate follows because $\log(q) \ll q^{\frac{\varepsilon}{2}}$ and thus $(\log(q))^{-1} \gg q^{-\frac{\varepsilon}{2}}$ for sufficiently large q. $\square$

The part of the proof of the first version of Siegel's theorem which makes $c_1(\varepsilon)$ (and hence $c_2(\varepsilon)$) ineffective is the value of β_1. The choice of β_1 depends upon the existance of a Siegel zero near 1 and relative to the given $\varepsilon > 0$ and since we don't know if Siegel zeros exist, this makes estimating β_1 relative to ε ineffective. Many results in analytic number theory make use of Siegel's theorem and hence are also ineffective. Many important problems investiage methods to get around using Siegele's theroem in favor of a weaker result that is effective.

7.3 Zero Counting & Riemann-von Mangoldt Formulas

Although the Riemann hypothesis is currently out of reach, it is an interesting question to know how many nontrivial zeros there are up to height T or around a specific height. This can be acomplished with some tools from complex analysis, explicit formulas for logarithimic derivatives, and lots of estimation.

The Riemann–von Mangoldt formula

Let $Z(\zeta)$ be the set of nontrivial zeros for $\zeta(s)$, counted with multiplicity, and for any $T > 0$ set

$$N(T) = \#\{\rho = \beta + i\gamma \in Z(\zeta) : \gamma \in (0,T)\}.$$

We will be interested in estimating $N(T)$. In order to acomplish this, we will require a small lemma:

Lemma 7.3.1. *Let $\rho = \beta + i\gamma$ be a nontrivial zero of $\zeta(s)$. Then for $\mathscr{L}$,*

$$\sum_{|\gamma|\geq 0} \frac{1}{1+(t-\gamma)^2} \ll \log(t).$$

Proof. Let $s = \sigma + it$ with $1 < \sigma \leq 2$ and $\mathscr{L}$. By Equation (5.8), $\mathrm{Re}\left(\frac{\zeta'}{\zeta}(s)\right)$ is bounded so Equation (7.13) implies

$$\sum_{\rho} \mathrm{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \ll \log(t),$$

As we have already mentioned (see the proof of the classical zero-free region for $\zeta(s)$) the terms in the sum are positive. Noting that $\mathrm{Re}\left(\frac{1}{s-\rho}\right) = \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}$, discarding the terms corresponding to ρ in the sum gives

$$\sum_{|\gamma|\geq 0} \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2} \ll \log(t). \tag{7.48}$$

If we take $\sigma = 2$ in Equation (7.48), then

$$\sum_{|\gamma| \geq 0} \frac{1}{4 + (t - \gamma)^2} \ll \log(t),$$

because $1 < (\sigma - \beta) < 2$. As $4 + (t - \gamma)^2 \ll 1 + (t - \gamma)^2$, the desired estimate follows. $\square$

Actually, Lemma 7.3.1 already gives us some information about the number of nontrivial zeros around a fixed height. Specifically, we have the following corollary:

Corollary 7.3.1. *Let $\mathscr{L}$.*

(i) The number of nontrivial zeros of $\zeta(s)$ with ordinate in $[t-1, t+1]$ is $O(\log(t))$.

(ii)

$$\sum_{|\gamma - t| > 1} \frac{1}{(t - \gamma)^2} = O(\log(t)).$$

.

Proof. For (i), γ is bounded so $1 + (t - \gamma)^2 \ll 1$. The statement follows by discarding all the terms in the sum in Lemma 7.3.1 with $|\gamma - t| > 1$ and using the previous bound. To prove (ii), $|\gamma - t| > 1$ implies $1 + (t - \gamma)^2 \ll (t - \gamma)^2$ (if $|\gamma - t|$ was not bounded below the implicit constant would depend upon γ and t). This last estimate with Lemma 7.3.1 gives (ii). $\square$

With Corollary 7.3.1 we can prove the most well-known result which estimates $N(T)$. The result was conjectured by Riemann in [Rie59] but was proved by von Mangoldt in 1905 (see [Man05]). Consequently, it is known as the **Riemann–von Mangoldt formula**:

Theorem 7.3.1 (Riemann–von Mangoldt formula).

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log(T)).$$

Proof. Let $\mathscr{L}$. We will prove this estimate $N(T)$ by an application of the argument principle to $\xi(s)$. Recall that $\xi(s)$ is holomorphic on $\mathbb{C}$ and its zeros are exactly the nontrivial zeros of $\zeta(s)$. Then for any region Ω, the argument principle implies

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{\xi'}{\xi}(s)\, ds = |Z(\zeta) \cap \Omega|.$$

Let Ω be the region enclosed by the contours $\eta_1, \ldots, \eta_6$ in Figure 7.2 and let $\eta = \sum_{1 \leq i \leq 6} \eta_i$ so that $\eta = \partial\Omega$. It will also be useful to let $\eta_L = \eta_2 + \eta_3$ and $\eta_R = \eta_4 + \eta_5$.
Since $\overline{\rho}$ is a nontrivial zero if ρ is, the number of zeros enclosed by η is $N(T)$. Therefore $|Z(\zeta) \cap \Omega| = N(T)$ and hence

$$\frac{1}{2\pi i} \int_{\eta} \frac{\xi'}{\xi}(s)\, ds = N(T). \tag{7.49}$$

We now examine the left-hand side of Equation (7.49). The facts $\frac{\xi'}{\xi}(s) = \frac{d}{ds} \log(\xi(s))$ and $\log(s) = \log|s| + \arg(s)$ together imply

$$\frac{1}{2\pi i} \int_{\eta} \frac{\xi'}{\xi}(s)\, ds = \frac{1}{2\pi i} \int_{\eta} \frac{d}{ds} \log|\xi(s)|\, ds + \frac{1}{2\pi} \int_{\eta} \frac{d}{ds} \arg(\xi(s))\, ds = \frac{1}{2\pi} \Delta_{\eta} \arg(\xi(s)), \tag{7.50}$$

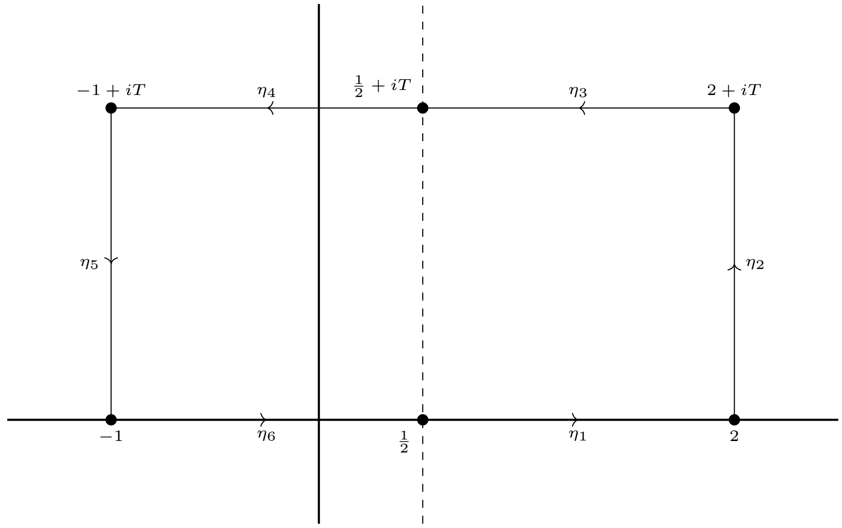

Figure 7.2: Contour for the Riemann–von Mangoldt formula

where the last equality follows because η is closed. Since $\xi(s)$ is real for real $s > 0$, the functional equation implies $\xi(s)$ is real for all real s. Then $\overline{\xi(s)} = \xi(\overline{s})$ holds for all s by the identity theorem. This implies that there is no change in the argument along η_1 and η_6. Moreover, the functional equation for $\xi(s)$ and that $-\arg(s) = \arg(\overline{s})$ together imply

$$\begin{aligned}\Delta_{\eta_L} \arg(\xi(s)) &= \Delta_{\eta_L} \arg(\xi(1-s)) \\ &= \Delta_{-\overline{\eta_R}} \arg(\xi(s)) \\ &= -\Delta_{\overline{\eta_R}} \arg(\xi(s)) \\ &= \Delta_{\overline{\eta_R}} \arg(\overline{\xi(s)}) \\ &= \Delta_{\overline{\eta_R}} \arg(\xi(\overline{s})) \\ &= \Delta_{\eta_R} \arg(\xi(s)).\end{aligned}$$

So the change in the argument of $\xi(s)$ along η_L is the same as the change in the argument along η_R. So this discussion together with Equations (7.49) and (7.50) tell us that

$$N(T) = \frac{1}{\pi}\Delta_{\eta_L} \arg(\xi(s)). \tag{7.51}$$

Since the argument function is additive, $\xi(s) = (s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)\zeta(s)$ implies

$$\Delta_{\eta_L} \arg(\xi(s)) = \Delta_{\eta_L} \arg(s-1) + \Delta_{\eta_L} \arg(\pi^{-\frac{s}{2}}) + \Delta_{\eta_L}\left(\Gamma\left(\frac{s}{2}+1\right)\right) + \Delta_{\eta_L}(\zeta(s)). \tag{7.52}$$

We now estimate the terms on the right-hand side of Equation (7.52). For the first term, we compute

$$\Delta_{\eta_L} \arg(s-1) = \arg(s-1)\Big|_{2}^{\frac{1}{2}+iT} = \arg\left(-\frac{1}{2}+iT\right) = \frac{\pi}{2} + O\left(\frac{1}{T}\right), \tag{7.53}$$

where in the last equality we have used the estimate $\arg(s) = \arctan\left(\frac{t}{\sigma}\right) = \frac{\pi}{2} + O_\sigma\left(\frac{1}{t}\right)$, with $s = \sigma + it$, which holds by the Laurent series of arctangent near ∞. For the second term, we perform an analogous computation:

$$\Delta_{\eta_L} \arg(\pi^{-\frac{s}{2}}) = \arg(\pi^{-\frac{s}{2}})\Big|_{2}^{\frac{1}{2}+iT} = \arg\left(\pi^{-\frac{\frac{1}{2}+iT}{2}}\right) = \operatorname{Im}\left(\log\left(\pi^{-\frac{\frac{1}{2}+iT}{2}}\right)\right) = -\frac{T}{2}\log(\pi). \tag{7.54}$$

To compute the gamma term we apply Stirling's formula to the log gamma function (see Equation (1.7)):

$$\begin{aligned}
\Delta_{\eta_L} \arg\left(\Gamma\left(\frac{s}{2}+1\right)\right) &= \arg\left(\Gamma\left(\frac{s}{2}+1\right)\right)\Bigg|_{2}^{\frac{1}{2}+iT} \\
&= \arg\left(\Gamma\left(\frac{5}{4}+i\frac{T}{2}\right)\right) \\
&= \mathrm{Im}\left(\log\Gamma\left(\frac{5}{4}+i\frac{T}{2}\right)\right) \\
&= \mathrm{Im}\left(\frac{1}{2}\log(2\pi)+\left(\frac{3}{4}+i\frac{T}{2}\right)\log\left(\frac{5}{4}+i\frac{T}{2}\right)-\left(\frac{5}{4}+i\frac{T}{2}\right)\right)+O(1) \\
&= \frac{3}{4}\arg\left(\frac{5}{4}+i\frac{T}{2}\right)+\frac{T}{2}\log\left|\frac{5}{4}+i\frac{T}{2}\right|-\frac{T}{2}+O(1) \\
&= \frac{3\pi}{8}+\frac{T}{2}\log\left(\frac{T}{2}\right)-\frac{T}{2}+O(1),
\end{aligned} \tag{7.55}$$

where in the last line we have used the estimate $\arg(s) = \frac{\pi}{2}+O_\sigma\left(\frac{1}{t}\right)$ again and that $\frac{5}{4}+i\frac{T}{2} \sim \frac{T}{2}$. Substituting Equations (7.53) to (7.55) into Equation (7.52) yields

$$\Delta_{\eta_L} \arg(\xi(s)) = \frac{T}{2}\log\left(\frac{T}{2\pi}\right)-\frac{T}{2}+\frac{7\pi}{8}+\Delta_{\eta_L}(\zeta(s))+O(1).$$

By Equation (7.51), we conclude

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right)-\frac{T}{2\pi}+\frac{7}{8}+S(T)+O(1), \tag{7.56}$$

where we set $S(T) = \frac{1}{\pi}\Delta_{\eta_L}(\zeta(s))$. We now show $S(T) = O(\log(T))$. To prove this estimate, we first rewrite $S(T)$ in terms of an integral:

$$\begin{aligned}
S(T) &= \frac{1}{\pi}\Delta_{\eta_L}(\zeta(s)) \\
&= \frac{1}{\pi}\arg(\zeta(s))\Big|_{2}^{\frac{1}{2}+iT} \\
&= \frac{1}{\pi}\mathrm{Im}(\log\zeta(s))\Big|_{2}^{\frac{1}{2}+iT} \\
&= \frac{1}{\pi}\mathrm{Im}\left(\log\zeta(s)\Big|_{2}^{\frac{1}{2}+iT}\right) \\
&= \frac{1}{\pi}\mathrm{Im}\left(\int_{\eta_L}\frac{d}{ds}\log\zeta(s)\,ds\right) \\
&= \frac{1}{\pi}\mathrm{Im}\left(\int_{\eta_L}\frac{\zeta'}{\zeta}(s)\,ds\right) \\
&= \frac{1}{\pi}\int_{\eta_L}\mathrm{Im}\left(\frac{\zeta'}{\zeta}(s)\right)\,ds \\
&= \frac{1}{\pi}\int_{2}^{2+iT}\mathrm{Im}\left(\frac{\zeta'}{\zeta}(s)\right)\,ds+\frac{1}{\pi}\int_{2+iT}^{\frac{1}{2}+iT}\mathrm{Im}\left(\frac{\zeta'}{\zeta}(s)\right)\,ds \\
&= \frac{1}{\pi}\int_{2+iT}^{\frac{1}{2}+iT}\mathrm{Im}\left(\frac{\zeta'}{\zeta}(s)\right)\,ds+O(1),
\end{aligned} \tag{7.57}$$

where the last line follows because $\frac{\zeta'}{\zeta}(s)$ is bounded for $\mathrm{Re}(s) > 1$ by Equation (5.8). It's now necessary to estimate $\frac{\zeta'}{\zeta}(s)$. The explict formula for $\frac{\zeta'}{\zeta}(s)$ at $s = 2+it$ with $t > 0$ implies

$$0 = -\frac{\zeta'}{\zeta}(2+it) + B - \frac{1}{1+it} - \frac{1}{2}\log(\pi) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(2+i\frac{t}{2}\right) + \sum_{\rho}\left(\frac{1}{2+it-\rho} + \frac{1}{\rho}\right). \tag{7.58}$$

Subtracting Equation (7.58) from the explicit formula for $\frac{\zeta'}{\zeta}(s)$ (s is now arbitrary) gives

$$\frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(2+it) + \frac{1}{1+it} - \frac{1}{s-1} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(2+i\frac{t}{2}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right).$$

Let $s = \sigma+it$ with $-1 \le s \le 2$, $\mathscr{L}$, and not coinciding with the ordinate of a nontrivial zero. In particular, $s \sim t$. We will estimate all of the terms except for the sum. For the first term, Equation (5.8) implies that it is bounded. For the second and third terms, $\frac{1}{1+it} \sim \frac{1}{t} = O(\log(t))$ and $\frac{1}{s-1} \sim \frac{1}{t} = O(\log(t))$. For the first gamma term, $2+i\frac{t}{2} \sim t$ and Proposition 1.5.3 imply $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(2+i\frac{t}{2}\right) = \log(t) + O(1) = O(\log(t))$. The second gamma term is handled in exactly the same way since $\frac{s}{2}+1 \sim t$. All of these estimates together give the weaker asymptotic

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) + O(\log(t)). \tag{7.59}$$

Now we estimate part of the the sum in Equation (7.59). Let $\rho = \beta+i\gamma$. For those terms in the sum with $|\gamma - t| \le 1$, $|2+it-\rho| \ge |\mathrm{Im}(2+it-\rho)| = |t-\gamma| \ge 1$ so that those terms corresponding to $2+it-\rho$ are at most 1 in absolute value and there are at most $O(\log(t))$ of them by Corollary 7.3.1 (i). Hence

$$\sum_{|\gamma-t|\le 1}\left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) = \sum_{|\gamma-t|\le 1}\frac{1}{s-\rho} + O(\log(t)), \tag{7.60}$$

For those terms with $|\gamma - t| > 1$, since $t-\gamma \ll s-\rho$ and $t-\gamma \ll 2+it-\rho$ we have

$$\frac{1}{s-\rho} - \frac{1}{2+it-\rho} = \frac{2-\sigma}{(s-\rho)(2+it-\rho)} \ll \frac{1}{(t-\gamma)^2}.$$

Therefore Corollary 7.3.1 (ii) gives

$$\sum_{|\gamma-t|>1}\left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) \ll \sum_{|\gamma-t|>1}\frac{1}{(t-\gamma)^2} = O(\log(t)). \tag{7.61}$$

Interting Equations (7.60) and (7.61) into Equation (7.59) yields the estimate

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma-t|\le 1}\frac{1}{s-\rho} + O(\log(t)). \tag{7.62}$$

We are now ready to estimate $S(T)$. Equations (7.57) and (7.62) give the first line in the following chain:

$$\begin{aligned}
S(T) &= \frac{1}{\pi}\int_{2+iT}^{\frac{1}{2}+iT} \operatorname{Im}\left(\sum_{|\gamma-T|\le 1}\frac{1}{s-\rho}\right) ds + O(\log(T)) \\
&= \frac{1}{\pi}\sum_{|\gamma-T|\le 1}\int_{2+iT}^{\frac{1}{2}+iT} \operatorname{Im}\left(\frac{1}{s-\rho}\right) ds + O(\log(T)) \\
&= \frac{1}{\pi}\sum_{|\gamma-T|\le 1}\operatorname{Im}\left(\int_{2+iT}^{\frac{1}{2}+iT} \frac{1}{s-\rho}\, ds\right) + O(\log(T)) \\
&= \frac{1}{\pi}\sum_{|\gamma-T|\le 1}\operatorname{Im}\left(\int_{2+iT}^{\frac{1}{2}+iT} \frac{d}{ds}\log(s-\rho)\, ds\right) + O(\log(T)) \\
&= \frac{1}{\pi}\sum_{|\gamma-T|\le 1}\arg(s-\rho)\Big|_{2+iT}^{\frac{1}{2}+iT} + O(\log(T)) \\
&= \sum_{|\gamma-T|\le 1} 2 + O(\log(T)) \\
&= O(\log(T)),
\end{aligned}$$

where the second to last line follows because the change in argument is at most 2π in absolute value, and the last line follows by Corollary 7.3.1 (i). So we have shown that $S(T) = O(\log(T))$. Then from Equation (7.56) we obtain

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + O(\log(T)),$$

which implies the weaker estimate

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log(T)),$$

as desired. □

We can obtain good information about the distribution of the nontrivial zeros follows from Riemann–von Mangoldt formula. For example, following corollary computes the density of the nontrivial zeros at height T:

Corollary 7.3.2.

$$\frac{N(T)}{T} \sim \frac{1}{2\pi}\log\left(\frac{T}{2\pi}\right).$$

Proof. From the Riemann-von Mangoldt formula,

$$\frac{N(T)}{T} = \frac{1}{2\pi}\log\left(\frac{T}{2\pi}\right) + O\left(\frac{\log(T)}{T}\right).$$

Since $\log(T) = o(T)$, the asymptotic follows. □

Corollary 7.3.2 can be interpreted as saying that for large T the density of $N(T)$ is approximately $\frac{1}{2\pi}\log\left(\frac{T}{2\pi}\right)$. Since this grows with T, we see that the nontrivial zeros tend to accumulate farther up the critial strip with logarithmic growth in the density.

The Riemann–von Mangoldt formula for Dirichlet L-functions

We can prove a result analogous to the Riemann-von Mangoldt formula for Dirichlet L-functions. Let χ be a primitive Dirichlet character of conductor $q > 1$ and let $Z(\chi)$ be the set of nontrivial zeros for $L(s, \chi)$, counted with multiplicity. For any $T > 0$ set

$$N(T, \chi) = \#\{\rho = \beta + i\gamma \in Z(\chi) : 0 < |\gamma| < T\}.$$

We need to use the condition $0 < |\gamma| < T$ instead of $\gamma \in (0, T)$ because the zeros of $L(s, \chi)$ need not be symmetric with respect to the real axis ($\overline{\rho}$ is not necessarily a zero if ρ is when χ is complex). We will compensate for this by introducing a factor of $\frac{1}{2}$ in the statement of the theorem. Just as for the Riemann-von Mangoldt formula, we will require a small lemma:

Lemma 7.3.2. *Let χ be a primitive Dirichlet character of conductor $q > 1$. Let $\rho = \beta + i\gamma$ be a nontrivial zero of $L(s, \chi)$. Then*

$$\sum_{|\gamma| \geq 0} \frac{1}{1 + (t - \gamma)^2} \ll \log(\mathscr{L}).$$

Proof. Let $s = \sigma + it$ with $1 < \sigma \leq 2$. By Equation (5.12), $\mathrm{Re}\left(\frac{L'}{L}(s, \chi)\right)$ is bounded so Equation (7.19) implies

$$\sum_{\rho} \mathrm{Re}\left(\frac{1}{s - \rho}\right) \ll \log(\mathscr{L}),$$

As we have already mentioned (see the proof of the classical zero-free region for $L(s, \chi)$) the terms in the sum are positive. Since $\mathrm{Re}\left(\frac{1}{s-\rho}\right) = \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2}$, we immeditely have

$$\sum_{|\gamma| \geq 0} \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} \ll \log(\mathscr{L}). \tag{7.63}$$

Taking $\sigma = 2$ in Equation (7.63) gives

$$\sum_{|\gamma| \geq 0} \frac{1}{4 + (t - \gamma)^2} \ll \log(\mathscr{L}),$$

because $1 < (\sigma - \beta) < 2$. Since $4 + (t - \gamma)^2 \ll 1 + (t - \gamma)^2$, the desired estimate follows. ☐

Just as for Lemma 7.3.1, Lemma 7.3.2 tells us some information about the number of nontrivial zeros around a fixed height:

Corollary 7.3.3. *Let χ be a primitive Dirichlet character of conductor $q > 1$. Then*

(i) The number of nontrivial zeros of $L(s, \chi)$ with ordinate in $[t - 1, t + 1]$ is $O(\mathscr{L})$.

(ii)

$$\sum_{|\gamma - t| > 1} \frac{1}{(t - \gamma)^2} = O(\mathscr{L}).$$

.

Proof. To prove (i), γ is bounded so $1+(t-\gamma)^2 \ll 1$. The claim follows by discarding all the terms in the sum in Lemma 7.3.2 with $|\gamma - t| > 1$ and using the previous bound. As for (ii), $|\gamma - t| > 1$ implies $1+(t-\gamma)^2 \ll (t-\gamma)^2$ (as we mentioned in Corollary 7.3.1). This estimate with Lemma 7.3.2 gives (ii).□

We are not ready to prove the **Riemann–von Mangoldt formula for Dirichlet L-functions**:

Theorem 7.3.2 (Riemann–von Mangoldt formula for Dirichlet L-functions). *Let χ be a primitive Dirichlet character of conductor $q > 1$. Then*

$$\frac{1}{2}N(T,\chi) = \frac{T}{2\pi}\log\left(\frac{qT}{2\pi}\right) - \frac{T}{2\pi} + O(\log(qT)).$$

Proof. The idea is the same as the proof of the Riemann–von Mangoldt formula with some slight modifications. To begin, for any region Ω, the argument principle implies

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\xi'}{\xi}(s,\chi)\,ds = |Z(\chi)\cap\Omega|.$$

Let Ω be the region enclosed by the contours $\eta_1,\ldots,\eta_6$ in Figure 7.3 and set $\eta = \sum_{1\le i\le 6}\eta_i$ so we have $\eta = \partial\Omega$. It will be helpful to let $\eta_L = \eta_1+\eta_2+\eta_3$ and $\eta_R = \eta_3+\eta_4+\eta_5$.

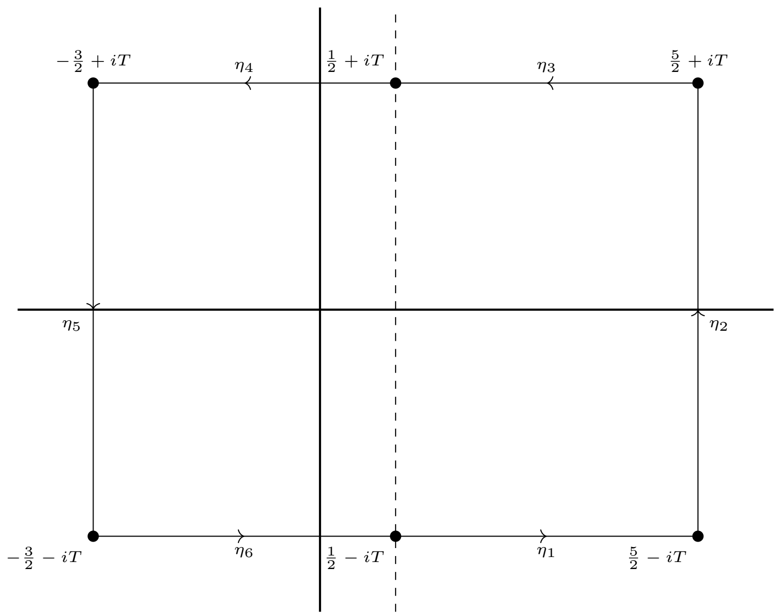

Figure 7.3: Contour for the Riemann–von Mangoldt formula for Dirichlet L-functions

We have chosen this region so that the boundary avoids the potential zero at $s=-1$ if χ is odd (we need to enlarge the region to make sure we contain the width of the entire critical strip). Then $|Z(\chi)\cap\Omega| = N(T,\chi)$ and so

$$\frac{1}{2\pi i}\int_{\eta}\frac{\xi'}{\xi}(s,\chi)\,ds = N(T,\chi). \tag{7.64}$$

We now manipulate left-hand side of Equation (7.64). Since $\frac{\xi'}{\xi}(s,\chi) = \frac{d}{ds}\log(\xi(s,\chi))$ and $\log(s) = \log|s| + \arg(s)$, we have

$$\frac{1}{2\pi i}\int_{\eta}\frac{\xi'}{\xi}(s,\chi)\,ds = \frac{1}{2\pi i}\int_{\eta}\frac{d}{ds}\log|\xi(s,\chi)|\,ds + \frac{1}{2\pi}\int_{\eta}\frac{d}{ds}\arg(\xi(s,\chi))\,ds = \frac{1}{2\pi}\Delta_\eta\arg(\xi(s,\chi)), \tag{7.65}$$

where the last equality holds because η is closed. Recall that we have already shown $\overline{(\xi(s,\chi))} = \xi(\overline{s},\overline{\chi})$. This fact along with the functional equation for $\xi(s,\chi)$ and that $-\arg(s) = \arg(\overline{s})$ together imply

$$\begin{aligned}
\Delta_{\eta_L} \arg(\xi(s,\chi)) &= \Delta_{\eta_L} \arg\left(\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}\xi(1-s,\overline{\chi})\right) \\
&= \Delta_{-\overline{\eta_R}} \arg\left(\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}\xi(s,\overline{\chi})\right) \\
&= -\Delta_{\overline{\eta_R}} \arg\left(\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}\xi(s,\overline{\chi})\right) \\
&= \Delta_{\overline{\eta_R}} \arg\left(\overline{\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}\xi(s,\overline{\chi})}\right) \\
&= \Delta_{\overline{\eta_R}} \arg\left(\frac{\overline{\varepsilon_\chi}}{i^{\mathfrak{a}}}\xi(\overline{s},\chi)\right) \\
&= \Delta_{\eta_R} \arg\left(\frac{\overline{\varepsilon_\chi}}{i^{\mathfrak{a}}}\xi(s,\chi)\right) \\
&= \Delta_{\eta_R} \arg\left(\frac{\overline{\varepsilon_\chi}}{i^{\mathfrak{a}}}\right) + \Delta_{\eta_R} \arg(\xi(s,\chi)) \\
&= \Delta_{\eta_R} \arg(\xi(s,\chi)),
\end{aligned}$$

where the second to last line follows because the argument function is additive. Hence the change in the argument of $\xi(s,\chi)$ along η_L is the same as the change in the argument along η_R. This fact along with Equations (7.64) and (7.65) together imply

$$N(T,\chi) = \frac{1}{\pi}\Delta_{\eta_L} \arg(\xi(s,\chi)). \tag{7.66}$$

By additivity of the argument function and that $\xi(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s+\mathfrak{a}}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi)$, we have

$$\Delta_{\eta_L} \arg(\xi(s,\chi)) = \Delta_{\eta_L} \arg\left(\left(\frac{q}{\pi}\right)^{\frac{s+\mathfrak{a}}{2}}\right) + \Delta_{\eta_L}\left(\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)\right) + \Delta_{\eta_L}(L(s,\chi)). \tag{7.67}$$

We will estimate the terms on the right-hand side of Equation (7.67). The first term is a direct computation:

$$\begin{aligned}
\Delta_{\eta_L} \arg\left(\left(\frac{q}{\pi}\right)^{\frac{s+\mathfrak{a}}{2}}\right) &= \arg\left(\left(\frac{q}{\pi}\right)^{\frac{s+\mathfrak{a}}{2}}\right)\Bigg|_{\frac{5}{2}-iT}^{-\frac{3}{2}+iT} \\
&= \arg\left(\left(\frac{q}{\pi}\right)^{\frac{-\frac{3}{2}+iT+\mathfrak{a}}{2}}\right) - \arg\left(\left(\frac{q}{\pi}\right)^{\frac{\frac{5}{2}-iT+\mathfrak{a}}{2}}\right) \\
&= \operatorname{Im}\left(\log\left(\left(\frac{q}{\pi}\right)^{\frac{-\frac{3}{2}+iT+\mathfrak{a}}{2}}\right)\right) - \operatorname{Im}\left(\log\left(\left(\frac{q}{\pi}\right)^{\frac{\frac{5}{2}-iT+\mathfrak{a}}{2}}\right)\right) \\
&= T\log\left(\frac{q}{\pi}\right).
\end{aligned} \tag{7.68}$$

For the gamma term we use Stirling's formula applied to the log gamma function (see Equation (1.7)):

$$\begin{aligned}
\Delta_{\eta_L} \arg\left(\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)\right) &= \arg\left(\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)\right)\Bigg|_{\frac{5}{2}-iT}^{-\frac{3}{2}+iT} \\
&= \arg\left(\Gamma\left(\frac{-3+2\mathfrak{a}}{4}+i\frac{T}{2}\right)\right) - \arg\left(\Gamma\left(\frac{5+2\mathfrak{a}}{4}-i\frac{T}{2}\right)\right) \\
&= \operatorname{Im}\left(\log\Gamma\left(\frac{-3+2\mathfrak{a}}{4}+i\frac{T}{2}\right)\right) - \operatorname{Im}\left(\log\Gamma\left(\frac{5+2\mathfrak{a}}{4}-i\frac{T}{2}\right)\right) \\
&= \operatorname{Im}\Bigg(\frac{1}{2}\log(2\pi) + \left(\frac{-5+2\mathfrak{a}}{4}+i\frac{T}{2}\right)\log\left(\frac{-3+2\mathfrak{a}}{4}+i\frac{T}{2}\right) \\
&\quad - \left(\frac{-3+2\mathfrak{a}}{4}+i\frac{T}{2}\right)\Bigg) - \operatorname{Im}\Bigg(\frac{1}{2}\log(2\pi) + \left(\frac{3+2\mathfrak{a}}{4}-i\frac{T}{2}\right) \qquad (7.69) \\
&\quad \cdot\log\left(\frac{5+2\mathfrak{a}}{4}-i\frac{T}{2}\right) - \left(\frac{5+2\mathfrak{a}}{4}-i\frac{T}{2}\right)\Bigg) + O(1) \\
&= -\frac{5+2\mathfrak{a}}{4}\arg\left(\frac{-3+2\mathfrak{a}}{4}+i\frac{T}{2}\right) + \frac{T}{2}\log\left|\frac{-3+2\mathfrak{a}}{4}+i\frac{T}{2}\right| - \frac{T}{2} \\
&\quad - \frac{3+2\mathfrak{a}}{4}\arg\left(\frac{5+2\mathfrak{a}}{4}-i\frac{T}{2}\right) + \frac{T}{2}\log\left|\frac{5+2\mathfrak{a}}{4}+i\frac{T}{2}\right| - \frac{T}{2} + O(1) \\
&= -\pi\left(1+\frac{\mathfrak{a}}{2}\right) + T\log\left(\frac{T}{2}\right) - T + O(1),
\end{aligned}$$

where in the last line we have used the estimate $\arg(s) = \frac{\pi}{2} + O_\sigma\left(\frac{1}{t}\right)$ and that asymptotics $\frac{-3+2\mathfrak{a}}{4} + i\frac{T}{2} \sim \frac{T}{2}$ and $\frac{5+2\mathfrak{a}}{4} - i\frac{T}{2} \sim \frac{T}{2}$. Substituting Equations (7.68) and (7.69) into Equation (7.67) yields

$$\Delta_{\eta_L}\arg(\xi(s,\chi)) = T\log\left(\frac{T}{2\pi}\right) - T + -\pi\left(1+\frac{\mathfrak{a}}{2}\right) + \Delta_{\eta_L}(L(s,\chi)) + O(1).$$

Using Equation (7.66), we get

$$N(T,\chi) = \frac{T}{\pi}\log\left(\frac{qT}{2\pi}\right) - \frac{T}{\pi} - \left(1+\frac{\mathfrak{a}}{2}\right) + S(T,\chi) + O(1), \qquad (7.70)$$

where we set $S(T,\chi) = \frac{1}{\pi}\Delta_{\eta_L}(L(s,\chi))$. We will now show $S(T,\chi) = O(\log(qT))$. To acomplish this we first rewrite $S(T,\chi)$ in terms of an an integral:

$$\begin{aligned}
S(T,\chi) &= \frac{1}{\pi}\Delta_{\eta_L}(L(s,\chi)) \\
&= \frac{1}{\pi}\arg(L(s,\chi))\Big|_{\frac{5}{2}-iT}^{-\frac{3}{2}+iT} \\
&= \frac{1}{\pi}\mathrm{Im}(\log L(s,\chi))\Big|_{\frac{5}{2}-iT}^{-\frac{3}{2}+iT} \\
&= \frac{1}{\pi}\mathrm{Im}\left(\log L(s,\chi)\Big|_{\frac{5}{2}-iT}^{-\frac{3}{2}+iT}\right) \\
&= \frac{1}{\pi}\mathrm{Im}\left(\int_{\eta_L}\frac{d}{ds}\log L(s,\chi)\,ds\right) \\
&= \frac{1}{\pi}\mathrm{Im}\left(\int_{\eta_L}\frac{L'}{L}(s,\chi)\,ds\right) \\
&= \frac{1}{\pi}\int_{\eta_L}\mathrm{Im}\left(\frac{L'}{L}(s,\chi)\right)\,ds \\
&= \frac{1}{\pi}\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}\mathrm{Im}\left(\frac{L'}{L}(s,\chi)\right)\,ds+\frac{1}{\pi}\int_{\frac{5}{2}+iT}^{-\frac{3}{2}+iT}\mathrm{Im}\left(\frac{L'}{L}(s,\chi)\right)\,ds \\
&= \frac{1}{\pi}\int_{\frac{5}{2}+iT}^{-\frac{3}{2}+iT}\mathrm{Im}\left(\frac{L'}{L}(s,\chi)\right)\,ds+O(1),
\end{aligned} \tag{7.71}$$

where the last line holds because $\frac{L'}{L}(s,\chi)$ is bounded for $\mathrm{Re}(s)>1$ by Equation (5.12). Now we estimate $\frac{L'}{L}(s,\chi)$. The explicit formula for $\frac{L'}{L}(s,\chi)$ at $s=2+it$ with $t>0$ gives

$$0=-\frac{L'}{L}(2+it,\chi)+B(\chi)-\frac{1}{2}\log\left(\frac{q}{\pi}\right)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1+\frac{\mathfrak{a}}{2}+i\frac{t}{2}\right)+\sum_{\rho}\left(\frac{1}{2+it-\rho}+\frac{1}{\rho}\right). \tag{7.72}$$

Subtracting Equation (7.72) from the explicit formula for $\frac{L'}{L}(s,\chi)$ (s is now arbitrary) gives

$$\frac{L'}{L}(s,\chi)=\frac{L'}{L}(2+it)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1+\frac{\mathfrak{a}}{2}+i\frac{t}{2}\right)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)+\sum_{\rho}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right).$$

Let $s=\sigma+it$ with $-\frac{3}{2}\le\sigma\le\frac{5}{2}$ and not coinciding with the ordinate of a nontrivial zero. Note that $s\sim t$. We will estimate all of the terms save for the sum. The first term is bounded by Equation (5.12). For the first gamma term, $1+\frac{\mathfrak{a}}{2}+i\frac{t}{2}\sim t$ and Proposition 1.5.3 imply $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1+\frac{\mathfrak{a}}{2}+i\frac{t}{2}\right)=\log(t)+O(1)=O(\log(t))$. The second gamma term is handled similarly since $\frac{s+\mathfrak{a}}{2}\sim t$. All of these estimates together give the weaker asymptotic

$$\frac{L'}{L}(s,\chi)=\sum_{\rho}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right)+O(\log(t)). \tag{7.73}$$

Now we are ready to estimate part of the the sum in Equation (7.73). Let $\rho=\beta+i\gamma$. For the terms in the sum with $|\gamma-t|\le 1$, $|2+it-\rho|\ge|\mathrm{Im}(2+it-\rho)|=|t-\gamma|\ge 1$ so that the terms corresponding to $2+it-\rho$ are at most 1 in absolute value and there are at most $O(\mathscr{L})$ of them by Corollary 7.3.3 (i).

Therefore

$$\sum_{|\gamma-t|\le 1}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right)=\sum_{|\gamma-t|\le 1}\frac{1}{s-\rho}+O(\mathscr{L}), \tag{7.74}$$

As for the terms with $|\gamma-t|>1$, we have $t-\gamma \ll s-\rho$ and $t-\gamma \ll 2+it-\rho$ so that

$$\frac{1}{s-\rho}-\frac{1}{2+it-\rho}=\frac{2-\sigma}{(s-\rho)(2+it-\rho)} \ll \frac{1}{(t-\gamma)^2}.$$

Hence Corollary 7.3.1 (ii) implies

$$\sum_{|\gamma-t|>1}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right) \ll \sum_{|\gamma-t|>1}\frac{1}{(t-\gamma)^2}=O(\mathscr{L}). \tag{7.75}$$

Interting Equations (7.74) and (7.75) into Equation (7.73) yields the estimate

$$\frac{L'}{L}(s)=\sum_{|\gamma-t|\le 1}\frac{1}{s-\rho}+O(\mathscr{L}). \tag{7.76}$$

We can now estimate $S(T,\chi)$. Equations (7.71) and (7.76) give the first line in the following chain:

$$\begin{aligned}
S(T,\chi) &= \frac{1}{\pi}\int_{2+iT}^{\frac{1}{2}+iT}\mathrm{Im}\left(\sum_{|\gamma-T|\le 1}\frac{1}{s-\rho}\right)ds+O(\mathscr{L})\\
&= \frac{1}{\pi}\sum_{|\gamma-T|\le 1}\int_{2+iT}^{\frac{1}{2}+iT}\mathrm{Im}\left(\frac{1}{s-\rho}\right)ds+O(\mathscr{L})\\
&= \frac{1}{\pi}\sum_{|\gamma-T|\le 1}\mathrm{Im}\left(\int_{2+iT}^{\frac{1}{2}+iT}\frac{1}{s-\rho}\,ds\right)+O(\mathscr{L})\\
&= \frac{1}{\pi}\sum_{|\gamma-T|\le 1}\mathrm{Im}\left(\int_{2+iT}^{\frac{1}{2}+iT}\frac{d}{ds}\log(s-\rho)\,ds\right)+O(\mathscr{L})\\
&= \frac{1}{\pi}\sum_{|\gamma-T|\le 1}\arg(s-\rho)\Big|_{2+iT}^{\frac{1}{2}+iT}+O(\mathscr{L})\\
&= \sum_{|\gamma-T|\le 1}2+O(\mathscr{L})\\
&= O(\mathscr{L}),
\end{aligned}$$

where the second to last line follows because the change in argument is at most 2π in absolute value, and the last line follows by Corollary 7.3.3 (i). But $\mathscr{L}=O(\log(qT))$ so that $S(T)=O(\log(qT))$. By Equation (7.70) we get

$$N(T,\chi)=\frac{T}{\pi}\log\left(\frac{qT}{2\pi}\right)-\frac{T}{\pi}-\left(1+\frac{\mathfrak{a}}{2}\right)+O(\log(qT)),$$

and this implies the weaker estimate

$$N(T,\chi)=\frac{T}{\pi}\log\left(\frac{qT}{2\pi}\right)-\frac{T}{\pi}+O(\log(qT)),$$

which is equivalent to the claim. $\square$

Chapter 8

Additional Results

8.1 The Value of Dirichlet L-functions at $s = 1$

Let χ be a primitive Dirichlet character with conductor $q > 1$. We know from Theorem 5.1.3 that $L(1, \chi) \neq 0$. It is interesting to know whether or not this value is computable in general. Indeed it is. The computation is fairly straightforward and only requires some basic properties of Gauss sums that we have already devloped. The idea is to rewrite the character values $\chi(n)$ so that we can collapse the infinite series into a Taylor series. Our result is the following:

Theorem 8.1.1. *Let χ be a primitive Dirichlet character with conductor $q > 1$. Then*

$$L(1, \chi) = -\frac{\chi(-1)\tau(\chi)}{q} \sideset{}{'}\sum_{a \pmod q} \chi(a) \log\left(\sin\left(\frac{\pi a}{q}\right)\right) \quad \text{or} \quad L(1, \chi) = -\frac{\chi(-1)\tau(\chi)\pi i}{q^2} \sideset{}{'}\sum_{a \pmod q} \chi(a)a,$$

according to whether χ is even or odd.

Proof. Make the following computation:

$$\begin{aligned}
\chi(n) &= \frac{1}{\overline{\tau(\chi)}}\overline{\tau(n, \chi)} && \text{Corollary 1.3.1} \\
&= \frac{1}{\tau(\overline{\chi})}\tau(n, \chi) && \text{Proposition 1.3.2 (i)} \\
&= \frac{\tau(\chi)}{\tau(\chi)\tau(\overline{\chi})}\tau(n, \chi) && \\
&= \frac{\chi(-1)\tau(\chi)}{q}\tau(n, \chi) && \text{Theorem 1.3.1 and Proposition 1.3.3} \\
&= \frac{\chi(-1)\tau(\chi)}{q} \sideset{}{'}\sum_{a \pmod q} \chi(a)e^{\frac{2\pi i a n}{q}}. &&
\end{aligned}$$

Substituting this result into the definition of $L(1,\chi)$, we find that

$$\begin{aligned} L(1,\chi) &= \sum_{n\geq 1}\frac{1}{n}\left(\frac{\chi(-1)\tau(\chi)}{q}\sideset{}{'}\sum_{a \pmod q}\chi(a)e^{\frac{2\pi i a n}{q}}\right) \\ &= \frac{\chi(-1)\tau(\chi)}{q}\sideset{}{'}\sum_{a \pmod q}\chi(a)\sum_{n\geq 1}\frac{e^{\frac{2\pi i a n}{q}}}{n} \\ &= \frac{\chi(-1)\tau(\chi)}{q}\sideset{}{'}\sum_{a \pmod q}\chi(a)\log\left(\left(1-e^{\frac{2\pi i a}{q}}\right)^{-1}\right), \end{aligned} \tag{8.1}$$

where in the last line we have used the Taylor series of the logarithm (notice $a \neq q$ so that $e^{\frac{2\pi i a}{q}} \neq 1$ and hence the logarithm is defined). We have now expressed $L(1,\chi)$ as a finite sum. In order to simplify the last expression in Equation (8.1), we deal with the logarithm. Since $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, we have

$$1-e^{\frac{2\pi i a}{q}} = -2ie^{\frac{\pi i a}{q}}\left(\frac{e^{\frac{\pi i a}{q}}-e^{-\frac{\pi i a}{q}}}{2i}\right) = -2ie^{\frac{\pi i a}{q}}\sin\left(\frac{\pi a}{q}\right).$$

Therefore the last expression in Equation (8.1) becomes

$$\frac{\chi(-1)\tau(\chi)}{q}\sideset{}{'}\sum_{a \pmod q}\chi(a)\log\left(\left(-2ie^{\frac{\pi i a}{q}}\sin\left(\frac{\pi a}{q}\right)\right)^{-1}\right).$$

As $0 < a < q$, we have $0 < \frac{\pi a}{q} < \pi$ so that $\sin\left(\frac{\pi a}{q}\right)$ is never negative. Therefore we can split up the logarithm term and obtain

$$-\frac{\chi(-1)\tau(\chi)}{q}\left(\log(-2i)\sideset{}{'}\sum_{a \pmod q}\chi(a)+\frac{\pi i}{q}\sideset{}{'}\sum_{a \pmod q}\chi(a)a+\sideset{}{'}\sum_{a \pmod q}\chi(a)\log\left(\sin\left(\frac{\pi a}{q}\right)\right)\right).$$

By the orthogonality relations (Corollary 1.2.1 (i)), the first sum above vanishes. Therefore

$$L(1,\chi) = -\frac{\chi(-1)\tau(\chi)}{q}\left(\frac{\pi i}{q}\sideset{}{'}\sum_{a \pmod q}\chi(a)a+\sideset{}{'}\sum_{a \pmod q}\chi(a)\log\left(\sin\left(\frac{\pi a}{q}\right)\right)\right). \tag{8.2}$$

Equation (8.2) simplifies in that one of the two sums vanish depending on if χ is even or odd. For the first sum in Equation (8.2), observe that

$$\frac{\pi i}{q}\sideset{}{'}\sum_{a \pmod q}\chi(a)a = -\frac{\chi(-1)\pi i}{q}\sideset{}{'}\sum_{a \pmod q}\chi(-a)(-a),$$

which vanishes if χ is even. For the second sum in Equation (8.2), we have an analogous relation of the form

$$\sideset{}{'}\sum_{a \pmod q}\chi(a)\log\left(\sin\left(\frac{\pi a}{q}\right)\right) = \chi(-1)\sideset{}{'}\sum_{a \pmod q}\chi(-a)\log\left(\sin\left(\frac{\pi a}{q}\right)\right),$$

which vanishes if χ is odd. This finishes the proof. $\square$

Theorem 8.1.1 encodes some interesting identities. For example, if χ is the non-principal Dirichlet character modulo 4, then χ is uniquely defined by $\chi(1) = 1$ and $\chi(3) = \chi(-1) = -1$. In particular, χ is odd and its conductor is 4. Now

$$\tau(\chi) = \sum_{a \pmod 4}{}' \chi(a) e^{\frac{2\pi i a}{4}} = e^{\frac{2\pi i}{4}} - e^{\frac{6\pi i}{4}} = i - (-i) = 2i,$$

so by Theorem 8.1.1 we get

$$L(1, \chi) = -\frac{\chi(-1)\tau(\chi)\pi i}{16}(1 - 3) = \frac{\pi}{4}.$$

Expanding out $L(1, \chi)$ gives

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4},$$

which is the famous **Madhava–Leibniz formula** for π. Alternatively, it's the Taylor series expansion for $\arctan(x)$ centered at $x = 0$ and evaulated at $x = 1$ (a boundary point of the disk of absolute convergence).

Appendix A

Number Theory

A.1 Arithmetic Functions

An arithmetic function f is a function $f : \mathbb{N} \to \mathbb{C}$. That is, it takes the positive integers into the complex numbers. We say that f is **additive** if $f(nm) = f(n) + f(m)$ for all positive integers n and m such that $(n, m) = 1$. If this condition simply holds for all n and m then we say f is **completely additive**. Similarly, we say that f is **multiplicative** if $f(nm) = f(n)f(m)$ for all positive integers n and m such that $(n, m) = 1$. If this condition simply holds for all n and m then we say f is **completely multiplicative**. Many important arithmetic functions are either additive, completely additive, multiplicative, or completely multiplicative. Note that if a f is additive or multiplicative then f is uniquely determined by its values on prime powers and if f is completely additive or completely multiplicative then it is uniquely determined by its values on primes. Moreover, if f is additive or completely additive then $f(1) = 0$ and if f is multiplicative or completely multiplicative then $f(1) = 1$.

Below is a list defining the most important arithmetic functions. Some of these functions are restrictions of common functions but we define them here as arithmetic functions because their domain being $\mathbb{N}$ is important.

(i) The **constant function**: The function $\mathbf{1}(n)$ defined by $\mathbf{1}(n) = 1$ for all $n \geq 1$. This function is neither additive or multiplicative.

(ii) The **unit function**: The function $e(n)$ defined by

$$e(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

This function is completely multiplicative.

(iii) The **identity function**: The identity function $\mathrm{id}(n)$ restricted to all $n \geq 1$. This function is completely multiplicative.

(iv) The **logarithm**: The function $\log(n)$ restricted to all $n \geq 1$. This function is completely additive.

(v) The **Möbius function**: The function $\mu(n)$ defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free with an even number of prime factors,} \\ -1 & \text{if } n \text{ is square-free with an odd number of prime factors,} \\ 0 & \text{if } n \text{ is not square-free,} \end{cases}$$

for all $n \geq 1$. This function is multiplicative.

(vi) The **characteristic function of square-free integers**: The square of the Möbius function $\mu^2(n)$ or $|\mu(n)|$ for all $n \geq 1$. This function is multiplicative.

(vii) **Liouville's function**: The function $\lambda(n)$ defined by

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is composed of } k \text{ not necessarily distinct prime factors,} \end{cases}$$

for all $n \geq 1$. This function is completely multiplicative.

(viii) **Euler's totient function**: The function $\phi(n)$ defined by

$$\phi(n) = \sideset{}{'}\sum_{m \pmod{n}} 1,$$

for all $n \geq 1$. This function is multiplicative.

(ix) The **divisor function**: The function $\sigma_0(n)$ defined by

$$\sigma_0(n) = \sum_{d|n} 1,$$

for all $n \geq 1$. This function is multiplicative.

(x) The **sum of divisors function**: The function $\sigma_1(n)$ defined by

$$\sigma_1(n) = \sum_{d|n} d,$$

for all $n \geq 1$. This function is multiplicative.

(xi) The **generalized sum of divisors function**: The function $\sigma_s(n)$ defined by

$$\sigma_s(n) = \sum_{d|n} d^s,$$

for all $n \geq 1$ and any complex number s. This function is multiplicative.

(xii) The **number of distinct prime factors function**: The function $\omega(n)$ defined by

$$\omega(n) = \sum_{p|n} 1,$$

for all $n \geq 1$. This function is additive.

(xiii) The **total number of prime divisors function**: The function $\Omega(n)$ defined by

$$\Omega(n) = \sum_{p^m|n} 1,$$

for all $n \geq 1$ and where $m \geq 1$. This function is completely additive.

(xiv) The **von Mangoldt function**: The function $\Lambda(n)$ defined by

$$\Lambda(n) = \begin{cases} 0 & \text{if } n \text{ is not a prime power,} \\ \log(p) & \text{if } n = p^m \text{ for some prime } p \text{ and integer } m \geq 1, \end{cases}$$

for all $n \geq 1$. This function is neither additive or multiplicative.

A.2 The Möbius Function

Recall that the Möbius function is the arithmetic function μ defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free with an even number of prime factors,} \\ -1 & \text{if } n \text{ is square-free with an odd number of prime factors,} \\ 0 & \text{if } n \text{ is not square-free,} \end{cases}$$

and it is multiplicative. It also satisfies an important summation property:

Proposition A.2.1.

$$\sum_{d|n} \mu(d) = \delta_{n,1}.$$

From this property, the important **Möbius inversion formula** can be derived:

Theorem A.2.1 (Möbius inversion formula). *Suppose f and g are arithmetic functions. Then*

$$g(n) = \sum_{d|n} f(d),$$

for all $n \geq 1$, if and only if

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$$

for all $n \geq 1$.

Using Möbius inversion, the following useful formula can also be derived:

Proposition A.2.2. *For* $\mathrm{Re}(s) > 1$,

$$\sum_{n \geq 1} \frac{\mu(n)}{n^s} = \zeta(s)^{-1} = \prod_p (1 - p^{-s}).$$

There is also an important similar statement to the Möbius inversion formula that we will need:

Theorem A.2.2. *Let f be an arithmetic function and B be the completely multiplicitive function defined on primes p by*

$$B(p) = f(p)^2 - f(p^2).$$

Then

$$f(n)f(m) = \sum_{d|(n,m)} B(d) f\left(\frac{nm}{d^2}\right),$$

for all $n, m \geq 1$, if and only if

$$f(nm) = \sum_{d|(n,m)} \mu(d) B(d) f\left(\frac{n}{d}\right) f\left(\frac{m}{d}\right),$$

for all $n, m \geq 1$.

Any function f satisfying the conditions of Theorem A.2.2 is said to be **specially multiplicative**.

A.3 The Generalized Sum of Divisors Function

Recall that the generalized sum of divisor function $\sigma_s(n)$ defined by

$$\sigma_s(n) = \sum_{d|n} d^s.$$

As we always take divisors to by positive, the sign of n is irrelevent. So

$$\sigma_s(n) = \sigma_s(-n) = \sigma_s(|n|).$$

More importantly, $\sigma_s(n)$ has a remarkable property: it can be written as a product. Recalling that $\text{ord}_p(n)$ is the positive integer satisfying $p^{\text{ord}_p(n)} \parallel n$, we have the following statement:

Proposition A.3.1. *For $s \neq 0$,*

$$\sigma_s(n) = \prod_{p|n} \frac{p^{(\text{ord}_p(n)+1)s} - 1}{p^s - 1}.$$

A.4 Quadratic Reciprocity

Let p be an odd prime. We are ofted interested in when the equation $x^2 = a \pmod p$ is solvable for some $a \in \mathbb{Z}$. The **Legendre symbol** $\left(\frac{a}{p}\right)$ keeps track of this:

$$\left(\frac{a}{q}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a \pmod p \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv a \pmod p \text{ is not solvable,} \\ 0 & \text{if } a \equiv 0 \pmod p. \end{cases}$$

Euler's criterion gives an alernative expression for Legendre symbol when a is coprime to p:

Proposition A.4.1 (Euler's criterion). *Let p be an odd prime and suppose $(a, p) = 1$. Then*

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} \pmod p.$$

From the definition and Euler's criterion is it not difficult to show that the Legendre symbol satisfies the following properties:

Proposition A.4.2. *Let p be an odd prime and let $a, b \in \mathbb{Z}$. Then the following hold:*

(i) If $a \equiv b \pmod p$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

(ii) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

From Proposition A.4.2, to compute the Legendre symbol in general it suffices to know how to compute $\left(\frac{-1}{p}\right)$, $\left(\frac{2}{p}\right)$, and $\left(\frac{q}{p}\right)$ where q is another odd prime. The **supplemental laws of quadratic reciprocity** are formulas for the first two symbols:

Proposition A.4.3 (Supplemental laws of quadratic reciprocity). *Let p be an odd prime.*

(i)

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \textit{if } p \equiv 1 \pmod 4, \\ -1 & \textit{if } p \equiv 3 \pmod 4. \end{cases}$$

(ii)

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \textit{if } p \equiv 1,7 \pmod 8, \\ -1 & \textit{if } p \equiv 3,5 \pmod 8. \end{cases}$$

The **law of quadratic reciprocity** handles the last symbol by relating $\left(\frac{q}{p}\right)$ to $\left(\frac{p}{q}\right)$:

Theorem A.4.1 (Law of quadratic reciprocity). *Let p and q be distinct odd primes. Then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} = \begin{cases} 1 & \textit{if } p \equiv 1 \pmod 4 \textit{ or } q \equiv 1 \pmod 4, \\ -1 & \textit{if } p \equiv 3 \pmod 4 \textit{ and } q \equiv 3 \pmod 4. \end{cases}$$

Let n be a positive odd integer and let $a \in \mathbb{Z}$. The **Jacobi symbol** $\left(\frac{a}{n}\right)$ is defined by

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{r_1}\left(\frac{a}{p_2}\right)^{r_2}\cdots\left(\frac{a}{p_k}\right)^{r_k},$$

where $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the prime factorization of n. When $n = p$ is prime, the Jacobi symbol reduces to the Legendre symbol, and the Jacobi symbol is precisely the unique multiplicative extension of the Legendre symbol to all positive odd integers. Accordingly, the Jacobi symbol has the following properties:

Proposition A.4.4. *Let n and m be positive odd integers and let $a, b \in \mathbb{Z}$. Then the following hold:*

(i) If $a \equiv b \pmod p$, then $\left(\frac{a}{n}\right) = \left(\frac{b}{n}\right)$.

(ii) $\left(\frac{a}{mn}\right) = \left(\frac{a}{m}\right)\left(\frac{a}{n}\right)$.

(iii) $\left(\frac{ab}{n}\right) = \left(\frac{a}{n}\right)\left(\frac{b}{n}\right)$.

There is also an associated reciprocity law:

Proposition A.4.5. *Let n and m be distinct positive odd integers. Then*

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (-1)^{\frac{m-1}{2}\frac{n-1}{2}} = \begin{cases} 1 & \textit{if } m \equiv 1 \pmod 4 \textit{ or } n \equiv 1 \pmod 4, \\ -1 & \textit{if } m \equiv 3 \pmod 4 \textit{ and } n \equiv 3 \pmod 4. \end{cases}$$

Appendix B

Analysis

B.1 Local Uniform Absolute Convergence

Often, we are interested in some series

$$\sum_{n\geq 1} f_n(z),$$

where the $f_n(z)$ are analytic functions on some region Ω. We say that the series above is **locally absolutely uniformly convergent** if

$$\sum_{n\geq 1} |f_n(z)|,$$

convergs uniformly on compact subsets of Ω. This mode of convergence is very useful because it is enough to guarantee the series is analytic on Ω:

Theorem B.1.1. *Suppose $(f_n(z))_{n\geq 1}$ is a sequence of analytic functions on a region Ω. Then if*

$$\sum_{n\geq 1} f_n(z),$$

is locally absolutely uniformly convergent, it is analytic on Ω.

We can also apply this idea in the case of integrals. Suppose we have an integral

$$\int_D f(z,x)\,dx,$$

where $f(z,x)$ is an analytic function on some region $\Omega \times D$. The integral is a function of z, and we say that the integral is **locally absolutely uniformly bounded** if

$$\int_D |f(z,x)|\,dx,$$

is uniformly bounded on compact subsets of Ω. Similar to the series case, this mode of convergence is very useful because it guarantees the integral is analytic on Ω:

Theorem B.1.2. *Suppose $f(z,x)$ is an analytic function on a region $\Omega \times D$. Then if*

$$\int_D f(z,x)\,dx,$$

is locally absolutely uniformly bounded, it is holomorphic on Ω.

B.2 Interchange of Integrals, Sums & Derivatives

Often, we would like to interchange a limit and a integral. This process is not always allowed, but in many instances it is. The **dominated convergence theorem** (DCT) covers the most well-known sufficient condition:

Theorem B.2.1 (Dominated convergence theorem). *Let $(f_n(z))_{n\geq 1}$ be a sequence of continuous real or complex integrable functions on some region Ω. Suppose that that the sequence converges pointwise to a function f, and that there is some integrable function g on Ω such that*

$$|f_n(z)| \leq g(z)$$

for all $n \geq 1$ and all $z \in \Omega$. Then f is integrable on Ω and

$$\lim_{n\to\infty} \int_\Omega f_n(z)\, dz = \int_\Omega f(z)\, dz.$$

This theorem is often employed when the underlying sequence is a sequence of partial sums of an absolutely convergent series ($g(s)$ will be the absolute series). In this case we have the following result:

Corollary B.2.1. *Suppose $\sum_{n\geq 1} f_n(z)$ is an absolutely convergent series of real or complex continuous functions that are integrable on some region Ω and that either $\int_\Omega \sum_{n\geq 1} |f_n(z)|\, dz$ or $\sum_{n\geq 1} \int_\Omega |f_n(z)|\, dz$ is finite. Then*

$$\sum_{n\geq 1} \int_\Omega f_n(z)\, dz = \int_\Omega \sum_{n\geq 1} f_n(z)\, dz.$$

Of course, we can apply Corollary B.2.1 repeatedly to interchange a sum with multiple integrals provided that the partial sums are absolutely convergent in each variable.
Other times we would also like to interchange a derivative and an integral. The **Leibniz integral rule** tells us when this is allowed:

Theorem B.2.2 (Leibniz integral rule). *Suppose $f(\mathbf{x}, t)$ is a function such that both $f(\mathbf{x}, t)$ and its partial derivative $\frac{\partial}{\partial x_i} f(\mathbf{x}, t)$ are continuous in $\mathbf{x}$ and t in some region including $\Omega \times [a(\mathbf{x}), b(\mathbf{x})]$ for some real-valued functions $a(\mathbf{x})$ and $b(\mathbf{x})$ and region Ω. Also suppose that $a(\mathbf{x})$ and $b(\mathbf{x})$ are continuous with continuous partial derivatives $\frac{\partial}{\partial x_i} a(\mathbf{x})$ and $\frac{\partial}{\partial x_i} b(\mathbf{x})$ for $\mathbf{x} \in \Omega$. Then for $\mathbf{x} \in \Omega$, we have*

$$\frac{\partial}{\partial x_i}\left(\int_{a(\mathbf{x})}^{b(\mathbf{x})} f(\mathbf{x}, t)\, dt\right) = f(\mathbf{x}, b(\mathbf{x}))\frac{\partial}{\partial x_i} b(\mathbf{x}) - f(\mathbf{x}, a(\mathbf{x}))\frac{\partial}{\partial x_i} a(\mathbf{x}) + \int_{a(\mathbf{x})}^{b(\mathbf{x})} \frac{\partial}{\partial x_i} f(\mathbf{x}, t)\, dt.$$

The Leibniz integral rule is sometimes applied in the case when $a(\mathbf{x}) = a$ and $b(\mathbf{x}) = b$ are constant. In this case, we get the following corollary:

Corollary B.2.2. *Suppose $f(\mathbf{x}, t)$ is a function such that both $f(\mathbf{x}, t)$ and its partial derivative $\frac{\partial}{\partial x_i} f(\mathbf{x}, t)$ are continuous in $\mathbf{x}$ and t in some region including $\Omega \times [a, b]$ for some reals a and b and region Ω. Then for $\mathbf{x} \in \Omega$, we have*

$$\frac{\partial}{\partial x_i}\left(\int_a^b f(\mathbf{x}, t)\, dt\right) = \int_a^b \frac{\partial}{\partial x_i} f(\mathbf{x}, t)\, dt.$$

B.3 Summation Formulas

The most well-known summation formula is **partial summation**:

Theorem B.3.1 (Partial summation). *Let $(a_n)_{n\geq1}$ and $(b_n)_{n\geq1}$ be two seqeunces of complex numbers. Then for any positive integers N and M with $1 \leq M < N$ we have*

$$\sum_{M\leq k\leq N} a_k(b_{k+1}-b_k) = (a_N b_{N+1} - a_M b_M) - \sum_{M+1\leq k\leq N} b_k(a_k - a_{k-1}).$$

There is a more useful summation formula for analytic number theory as it lets one estimate discrete sums by integrals. For this we need some notation. If $(a_n)_{n\geq1}$ is a sequence of complex numbers, for every real X set

$$A(X) = \sum_{n\leq X} a_n.$$

Then **Abel's summation formula** is the following:

Theorem B.3.2 (Abel's summation formula). *Let $(a_n)_{n\geq1}$ be a sequence of complex numbers. For every real X and Y with $X < Y$ and continuously differentiable function $\phi : [X, Y] \to \mathbb{C}$, we have*

$$\sum_{X\leq n\leq Y} a_n\phi(n) = A(Y)\phi(Y) - A(X)\phi(X) - \int_X^Y A(u)\phi'(u)\,du.$$

There are also some useful corollaries. For example, if we take the limit as $Y \to \infty$ we obtain:

Corollary B.3.1. *Let $(a_n)_{n\geq1}$ be a sequence of complex numbers. For every real X and continuously differentiable function $\phi : \mathbb{R}_{\geq X} \to \mathbb{C}$, we have*

$$\sum_{n\geq X} a_n\phi(n) = \lim_{Y\to\infty} A(Y)\phi(Y) - A(X)\phi(X) - \int_X^\infty A(u)\phi'(u)\,du.$$

We can take this corollary further by letting $X = 0$ so that $A(X) = 0$ to get the following:

Corollary B.3.2. *Let $(a_n)_{n\geq1}$ be a sequence of complex numbers. For every continuously differentiable function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{C}$, we have*

$$\sum_{n\geq1} a_n\phi(n) = \lim_{Y\to\infty} A(Y)\phi(Y) - \int_1^\infty A(u)\phi'(u)\,du.$$

B.4 Fourier Series

Let $N \geq 1$ be an integer. If $f : \mathbb{R} \to \mathbb{C}$ is N-periodic and integrable on $[0, N]$, then we define the n-th **Fourier coefficient** $\hat{f}(n)$ of f to be

$$\hat{f}(n) = \int_0^N f(x)e^{-\frac{2\pi inx}{N}}\,dx.$$

The **Fourier series** of f is defined by the series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{\frac{2\pi inx}{N}}.$$

There is the question of whether the Fourier series of f converges at all and if so does it even converge to f itself. Under reasonable conditions the answer is yes as is seen in the following proposition:

Proposition B.4.1. *If f is smooth and N-periodic it converges uniformly to its Fourier series eveywhere.*

In particular, all holomorphic N-periodic functions f converge uniformly to their Fourier series everywhere because for fixed x or y in $z = x + iy$, f restricts to an N-periodic smooth function on $\mathbb{R}$ that's integrable on $[0, N]$. Actually we can do a little better. If f is N-periodic and meromorphic on $\mathbb{C}$, then after clearing polar divisors we will have a holomorphic function on $\mathbb{C}$ and hence it will have a Fourier series converging uniformly everywhere. Therefore f will have such a Fourier series with meromorphic Fourier coefficients. In either situation, the case $N = 1$ is the most commonly seen. So for meromorphic (or holomorphic) 1-periodic functions f,

$$f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inz},$$

uniformly everywhere.

B.5 Factorizations, Order & Rank

The **elementary factors**, also referred to as **primary factors**, are the entire functions $E_n(z)$ defined by

$$E_n(z) = \begin{cases} 1-z & \text{if } n = 0, \\ (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^n}{n}} & \text{if } n \neq 0. \end{cases}$$

If f is an entire function, then it admits a factorization in terms of its zeros and the elementary factors. This is called the **Weierstrass factorization** of f:

Theorem B.5.1 (Weierstrass factorization). *Let f be an entire function with $\{a_n\}_{n\geq 1}$ the nonzero zeros of f counted with multiplicity. Also suppose that f has a zero of order m at $z = 0$ where it is understood that if $m = 0$ we mean $f(0) \neq 0$ and if $m < 0$ we mean f has a pole of order $|m|$ at $z = 0$. Then there exists an entire function g and sequence of nonnegative integers $(p_n)_{n\geq 1}$ such that*

$$f(z) = z^m e^{g(z)} \prod_{n\geq 1} E_{p_n}\left(\frac{z}{a_n}\right).$$

The Weierstrass factorization of f can be strengthened if f does not grow too fast. We say f is of **finite order** if there exists a $\rho_0 > 0$ such that

$$f(z) \ll e^{|z|^{\rho_0}},$$

for all $z \in \mathbb{C}$. The **order** ρ of f is the infimum of the ρ_0. Let $q = \lfloor \rho \rfloor$. If there is no such ρ_0, f is said to be of **infinite order** and we set $\rho = q = \infty$. Let $\{a_n\}_{n\geq 1}$ be the nonzero zeros of f that are not zero and ordered such that $a_n \to \infty$ as $n \to \infty$ if there are infinitely many zeros. Then we define the **rank** of f to be the smallest positive integer p such that the series

$$\sum_{n\geq 1} \frac{1}{|a_n|^{p+1}},$$

converges. If there is no such integer we set $p = \infty$ and if there are finitely many zeros we set $p = 0$. We set $g = max\{p, q\}$ and call g the **genus** of f. We can now state the **Hadamard factorization** of f:

Theorem B.5.2 (Hadamard factorization). *Let f be an entire function of finite order ρ. If p is the rank and g is the genus, then $g \le \rho$. Moreover, let $\{a_n\}_{n\ge1}$ be the nonzero zeros of f counted with multiplicity and suppose that f has a zero of order m at $z=0$ where it is understood that if $m=0$ we mean $f(0)\neq 0$ and if $m<0$ we mean f has a pole of order $|m|$ at $z=0$. Then there exists a polynomial $Q(z)$ of degree at most q such that*

$$f(z) = z^m e^{Q(z)} \prod_{n\ge1} E_p\left(\frac{z}{a_n}\right).$$

B.6 The Phragmen-Lindelöf Convexity Principle for a Strip

The **Phragmen-Lindelöf convexity principle** is a generic name for extending the maximum modulus principle to unbounded regions. The **Phragmen-Lindelöf Convexity principle for a strip** is the case when the unbounded region is the vertical strip $a < \mathrm{Re}(z) < b$:

Theorem B.6.1 (Phragmen-Lindelöf Convexity principle for a strip). *Suppose f is a holomorphic functions on an open neighborhood of a strip $a < \mathrm{Re}(z) < b$ such that $f(z) \ll e^{|z|^A}$ for some $A \ge 0$. Set $z = \sigma + it$. Then the following hold:*

(i) If $|f(z)| \le M$ for $\sigma = a, b$, that is on the boundary edges of the strip, then $|f(z)| \le M$ for all z in the strip.

(ii) Assume that

$$f(a+it) \ll_a t^{\alpha} \quad \text{and} \quad f(b+it) \ll_b t^{\beta},$$

for all $t \in \mathbb{R}$. Then

$$f(\sigma+it) \ll_\sigma t^{\alpha\ell(\sigma)+\beta(1-\ell(\sigma))},$$

where ℓ is the linear function such that $\ell(a)=1$ and $\ell(b)=0$.

B.7 Bessel Functions

For any $\nu \in \mathbb{C}$, the **Bessel equation** is the ODE

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-\nu^2)y = 0.$$

There are two linearly independent solutions to this equation. One solution is the **Bessel function of the first kind** $J_\nu(x)$ defined by

$$J_\nu(x) = \sum_{n\ge0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu}.$$

For integers n, $J_n(x)$ is entire and we have

$$J_n(x) = (-1)^n J_{-n}(x).$$

Otherwise, $J_\nu(x)$ has a pole at $x=0$ and $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent solutions to the Bessel equation. The other solution is the **Bessel function of the second kind** $Y_\nu(x)$ defined by

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)},$$

for non-integers ν, and for integers n is

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x).$$

For any integer n, we also have

$$Y_n(x) = (-1)^n Y_{-n}(x).$$

For the J-Bessel function there is also an important integral representation called the **Schläflin integral representation**:

Proposition B.7.1 (Schläflin integral representation for the J-Bessel function). *For any $\nu \in \mathbb{C}$ and $\mathrm{Re}(z) > 0$,*

$$J_\nu(z) = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^\nu \int_{-\infty}^{(0^+)} t^{-(\nu+1)} e^{t - \frac{z^2}{4t}}\, dt.$$

The **modified Bessel equation** is the ODE

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \nu^2)y = 0.$$

Like the Bessel equation, there are two linearly independent solutions. One solution is the **modified Bessel function of the first kind** $I_\nu(x)$ given by

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = \sum_{n \geq 0} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}.$$

For integers n, this solution is symmetric in n. That is,

$$I_n(x) = I_{-n}(x).$$

We also have a useful integral repersentation in a half-plane:

Proposition B.7.2. *For any $\nu \in \mathbb{C}$ and $\mathrm{Re}(x) > 0$,*

$$I_\nu(x) = \frac{1}{\pi} \int_0^\pi e^{x\cos(t)} \cos(\nu t)\, dt - \frac{\sin(\nu\pi)}{\pi} \int_0^\infty e^{-x\cosh(t) - \nu t}\, dt,$$

where the integrals are understood to be complex integrals.

From this integral representation we can show that $K_\nu(x)$ has at most exponential grwoth in this half-plane:

Lemma B.7.1. *For any $\nu \in \mathbb{C}$ and $\mathrm{Re}(x) > 0$,*

$$I_\nu(x) = O_\nu(e^x).$$

The other solution is the **modified Bessel function of the second kind** $K_\nu(x)$ defined by

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)},$$

for non-integers ν, and for integers n is

$$K_n(x) = \lim_{\nu \to n} K_\nu(x).$$

This is one of the more important types of Bessel functions as they appear in the Fourier coefficients of certian Eisenstein series. This function is symmetric in ν even when ν is an integer. That is,

$$K_\nu(x) = K_{-\nu}(x),$$

for all ν. We also have a very useful integral representation in a half-plane:

Proposition B.7.3. *For any* $\nu \in \mathbb{C}$ *and* $\mathrm{Re}(x) > 0$,

$$K_\nu(x) = \int_0^\infty e^{-x\cosh(t)} \cosh(\nu t)\, dt,$$

where the integral is understood to be a complex integral.

From this integral representation it does not take much to show that $K_\nu(x)$ has better than exponential decay in this half-plane:

Lemma B.7.2. *For any* $\nu \in \mathbb{C}$ *and* $\mathrm{Re}(x) > 0$,

$$K_\nu(x) = o_\nu(e^{-x}).$$

B.8 Sums Over Lattices

Let $\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d$ and let $||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_d^2}$ be the usual norm. We are often interested in series that obtained by summing over the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. In particular, we have the following general result:

Theorem B.8.1. *Let* $d \geq 1$ *be an integer. Then*

$$\sum_{\mathbf{a} \in \mathbb{Z}^d - \{\mathbf{0}\}} \frac{1}{||a||^s},$$

is locally absolutely uniformly convergent in the region $\mathrm{Re}(s) > d$.

In a practical setting, we usually restrict to the case $d = 2$. In this setting, with a little more work can show a more useful result:

Proposition B.8.1. *Let* $z \in \mathbb{H}$. *Then*

$$\sum_{(n,m) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \frac{1}{|nz + m|^s},$$

is locally absolutely uniformly convergent in the region $\mathrm{Re}(s) > 2$. *In addition, it is locally absolutely uniformly convergent as a function of* z *provided* $s > 2$.

Appendix C

Algebra

C.1 Character Groups

For any finite abelian group G, a **character** φ is a homomorphism $\varphi : G \to \mathbb{C}$. They form a group, denoted $\widehat{G}$, under multiplication called the **character group** of G. If G is an additive group, we say that any $\varphi \in \Gamma$ is a **additive character**. Similarly, if G is a multiplicative group, we say that any $\varphi \in \Gamma$ is a **multiplicative character**. In any case, if $|G| = n$ then $\varphi(g)^n = \varphi(g^n) = 1$ so that φ takes values in the n-th roots of unity. Moreover, to every character φ there is its **conjugate character** $\overline{\varphi}$ defined by $\overline{\varphi}(g) = \overline{\varphi(a)}$. Clearly the conjugate character is also a character. Since φ takes its value in the roots of unity, $\overline{\varphi(a)} = \varphi(a)^{-1}$ so that $\overline{\varphi} = \varphi^{-1}$. One of the central theorems about characters is that the character group of G is isomorphic to G:

Proposition C.1.1. *Any finite abelian group G is isomorphic to its character group. That is,*

$$G \cong \widehat{G}.$$

The characters also satisfy certain **orthogonality relations**:

Proposition C.1.2 (Orthogonality relations). *Let G be a finite abelian group.*

(i) For any two characters χ and ψ of G,

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi}(g) = \delta_{\chi,\psi}.$$

(ii) For any $g, h \in G$,

$$\frac{1}{|G|}\sum_{\chi\in \widehat{G}}\chi(g)\overline{\chi}(h) = \delta_{g,h}.$$

C.2 Representation Theory

Let G be a group and V be a vector space over a field $\mathbb{F}$. A **representation** (ρ, V), or just ρ if the underlying vector space V is clear, of G on V is a map

$$\rho : G \times V \to V \qquad (g, v) \mapsto \rho(g, v) = g \cdot v,$$

such that the following properties are satisfied:

1. For any $g \in G$, the map
$$\rho(g) : V \to V \qquad v \mapsto g \cdot v,$$
is linear.

2. For any $g, h \in G$ and $v \in V$,
$$1 \cdot v = v \quad \text{and} \quad g \cdot (h \cdot v) = (gh) \cdot v.$$

Therefore ρ defines an action of G on V. An equivalent defintion of a representation of G on V is a homomorphism from G into $\mathrm{Aut}(V)$. By abuse of notation, we also denote this homomorphism by ρ. If the dimension of V is n, then (ρ, V) is sai to be an **n-dimensional**. We say that (ρ, W) is a **subrepresentation** of (V, ρ) if $W \subseteq V$ is a G-invariant subspace. In particular, (ρ, W) is a representation itself. Lastly, if (ρ_1, V_1) and (ρ_2, V_2) are two representations, we can form the **direct sum representation** $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$ where $\rho_1 \oplus \rho_2$ acts diagionally on $V_1 \oplus V_2$. A natural question to ask is how representation can be decomposed as a direct sum of other representations. We say (ρ, V) is **irreducible** if it contains no proper G-invariant subspaces and is **completely irreducible** if it decomposes as a direct sum of irreducible subrepresentations.
We will only need one very useful theorem about representations when G is a finite abelian group and V is a vector space over $\mathbb{C}$. In this case G has a group of characters $\widehat{G}$, and the underlying vector space V is completely reducible with respect to the characters of G:

Theorem C.2.1. *Let V be a vector space over $\mathbb{C}$ and let Φ be a representation of a group G on V. If G is a finite abelian group, then*
$$V = \bigoplus_{\chi \in \widehat{G}} V_\chi,$$
where
$$V_\chi = \{v \in V : g \cdot v = \chi(g)v \text{ for all } g \in G\}.$$
In particular, V is completely reducible and every irreducible subrepresentation is 1-dimensional.

Appendix D

Miscellaneous

D.1 Special Integrals

Below is a table of well-known integrals that are used throughout the text:

Reference	Assumptions	Integral
Gaussian		$\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$
[Gol06]	$n \in \mathbb{Z}$, $s \in \mathbb{C}$, $y > 0$	$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^s} e^{-2\pi i n x y}\,dx = \begin{cases} \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} & \text{if } n = 0, \\ \frac{2\pi^s \lvert n\rvert^{s-\frac{1}{2}} y^{s-\frac{1}{2}}}{\Gamma(s)} K_{s-\frac{1}{2}}(2\pi\lvert n\rvert y) & \text{if } n \neq 0. \end{cases}$
[Dav80]	$c > 0$	$\frac{1}{2\pi i}\int_{(c)} y^s \,\frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$

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