

A DOUBLE DIRICHLET SERIES OVER FUNCTION FIELDS

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ABSTRACT. We construct a double Dirichlet series $Z(s, w)$ built from single variable Dirichlet L -functions $L(s, \chi)$ attached to the function field $\mathbb{F}_q(t)$. We prove that $Z(s, w)$ admits meromorphic continuation to the (s, w) -plane and satisfies a group of functional equations. This is the simplest construction of a Weyl group multiple Dirichlet series over a global field.

1. PRELIMINARIES

We will give an overview of the zeta function and Dirichlet L -functions attached to $\mathbb{F}_q(t)$. For proofs of these facts and a more detailed analysis see [1]. Let q be a power of an odd prime and let $\mathbb{F}_q[t]$ be the polynomial ring in t with coefficients in the finite field \mathbb{F}_q . This is a principal ideal domain. Moreover, the non-zero prime ideals in $\mathbb{F}_q[t]$ are generated by irreducible polynomials. Let $\mathbb{F}_q(t)$ denote the quotient field. Define the norm function $N(f)$ by

$$N(f) = |f| = q^{\deg(f)},$$

for any $f \in \mathbb{F}_q[t]$. The zeta function $\zeta(s)$ on $\mathbb{F}_q[t]$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s} = \prod_{P \text{ monic irr}} \left(1 - \frac{1}{|P|^s}\right)^{-1},$$

where the second equality holds since $\mathbb{F}_q[t]$ is a unique factorization domain. As for questions of convergence, there are q^n monic polynomials of degree n so, provided $\operatorname{Re}(s) > 1$, we can sum up the Dirichlet series according to degree and obtain an explicit expression:

$$\zeta(s) = \sum_{n \geq 0} \frac{\# \text{ of monic poly of deg } n}{q^{ns}} = \sum_{n \geq 1} \frac{1}{q^{n(1-s)}} = \frac{1}{1 - q^{1-s}}.$$

The latter expression is meromorphic on \mathbb{C} with a simple pole at $s = 1$ of residue $\frac{1}{\log(q)}$. Therefore $\zeta(s)$ admits meromorphic continuation to \mathbb{C} . The zeta function also satisfies a functional equation. Define the completed zeta function (this is also the zeta function attached to $\mathbb{F}_q(t)$) by

$$\zeta^*(s) = \frac{1}{1 - q^{-s}} \zeta(s).$$

Then

$$\zeta^*(s) = q^{2s-1} \zeta^*(1-s).$$

Recall that characters on $\mathbb{F}_q[t]$ are multiplicative functions to the complex numbers. The two flavors we care about are:

- Dirichlet characters: multiplicative functions to the complex numbers χ_f that are f -periodic for some $f \in \mathbb{F}_q[t]$ and such that $\chi_f(g) = 0$ if $(f, g) > 1$.
- Hilbert symbols: Dirichlet characters modulo 1.

The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{F}_q[t]$. If $f \in \mathbb{F}_q[t]$ is a monic non-constant irreducible, define the quadratic residue symbol χ_f by

$$\chi_f(g) = \left(\frac{f}{g} \right) = g^{\frac{|f|-1}{2}} \pmod{f},$$

for any $g \in \mathbb{F}_q[t]$. Then $\chi_f(g) \in \{\pm 1\}$ provided f and g are relatively prime and $\chi_f(g) = 0$ if $(f, g) > 1$. If $b \in \mathbb{F}^\times$, then we define the quadratic residue symbol χ_b by

$$\chi_b(g) = \left(\frac{b}{m} \right) = \text{sgn}(b)^{\deg(f)},$$

where $\text{sgn}(b) = \pm 1$ depending on if $b \in (\mathbb{F}^\times)^2$ or not. Moreover, if $d \in \mathbb{F}_q[t]$ then we set $\text{sgn}(d) = \text{sgn}(b_n)$ if $d(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_0$ (with $b_n \neq 0$). Extending χ_f multiplicativity in f , χ_f is defined for any f not necessarily monic. The quadratic residue symbol also has the following reciprocity property:

Theorem 1.1 (Quadratic reciprocity). *If $f, g \in \mathbb{F}_q[t]$ are monic, square-free, and relatively prime, then*

$$\left(\frac{f}{g} \right) = (-1)^{\frac{q-1}{2} \deg(f) \deg(g)} \left(\frac{g}{f} \right).$$

Note that if $q \equiv 1 \pmod{4}$, the sign in the statement of quadratic reciprocity is always 1 so that the reciprocity is perfect. We now describe the Hilbert symbols on $\mathbb{F}_q[t]$. In fact, there is only one non-trivial character ψ defined by

$$\psi(f) = (-1)^{\deg(f)}.$$

The other Hilbert symbol is the trivial character $\psi^2 = 1$. To see that ψ is given by a quadratic residue symbol, just notice that for $\theta \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$ we have $\chi_\theta(f) = (-1)^{\deg(f)}$. Moreover, the trivial character is then given by χ_1 . We will find it more useful to denote the non-trivial Hilbert symbol by χ_θ . In general, we denote a Hilbert symbol by χ_a where $a \in \{1, \theta\}$.

We can now define the L -functions attached to the symbol χ_f for not necessarily monic f . We define the L -series $L(s, \chi_f)$ attached to χ_f by a Dirichlet series or Euler product:

$$L(s, \chi_f) = \sum_{g \text{ monic}} \frac{\chi_f(g)}{|g|^s} = \prod_{P \text{ monic irr}} \left(1 - \frac{\chi_f(P)}{|P|^s} \right)^{-1}.$$

By definition of the quadratic residue symbol, $L(s, \chi_f) \ll \zeta(s)$ for $\text{Re}(s) > 1$ so that $L(s, \chi_f)$ is absolutely uniformly convergent on compacta in this region. $L(s, \chi_f)$ also admits meromorphic continuation to \mathbb{C} with a simple pole at $s = 1$ if f is square-free and is analytic otherwise (see [1] for a proof). Moreover, $L(s, \chi_f)$ is a polynomial in q^{-s} of degree at most $\deg(f) - 1$. The completed L -function is defined as follows:

$$L^*(s, \chi_f) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_f) & \text{if } \deg(f) \text{ is even,} \\ L(s, \chi_f) & \text{if } \deg(f) \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_f) = \begin{cases} q^{2s-1} |f|^{\frac{1}{2}-s} L^*(1-s, \chi_f) & \text{if } \deg(f) \text{ is even,} \\ q^{2s-1} (q|f|)^{\frac{1}{2}-s} L^*(1-s, \chi_f) & \text{if } \deg(f) \text{ is odd.} \end{cases}$$

Note that in the case $\deg(f)$ is even, the conductor is $|f|$ and in the case $\deg(f)$ is odd, the conductor is $q|f|$. In other words, the gamma factors depend upon the degree of f . This will cause a small but important technical issue later when we want to derive functional equations for the double Dirichlet series.

2. THE DOUBLE DIRICHLET SERIES $Z(z, w)$

From now on we assume $q \equiv 1 \pmod{4}$. This assumption is not strictly necessary to build the double Dirichlet series but it does allow for some technical simplifications as the statement of quadratic reciprocity is perfect. We are ready to define the double Dirichlet series $Z(s, w)$. For any monic $f \in \mathbb{F}_q[t]$, write $f = f_0 f_1^2$ where f_0 is square-free. In other words, f_0 is the square-free part of f so that $\frac{f}{f_0}$ is a perfect square. The **double Dirichlet series** $Z(s, w)$ is defined as

$$Z(s, w) = \sum_{f \text{ monic}} \frac{L(s, \chi_{f_0}) Q_{f_0 f_1^2}(s)}{|f|^w},$$

where $Q_{f_0 f_1^2}(s)$ is the **correction polynomial** defined by

$$Q_{f_0 f_1^2}(s) = \sum_{e_1 e_2 | f_1} \mu(e_1) \chi_{f_0}(e_1) |e_1|^{-s} |e_2|^{1-2s} = \sum_{e_1 e_2 e_3 = f_1} \mu(e_1) \chi_{f_0}(e_1) |e_1|^{-s} |e_2|^{1-2s},$$

where μ is the usual Möbius function on $\mathbb{F}_q[t]$. For $\operatorname{Re}(s) > 1$, we have the trivial bound

$$Q_{f_0 f_1^2}(s) \ll \sum_{e_1 e_2 | f_1} 1 \ll \sigma_0(f_1)^2 \ll_\varepsilon |f_1|^\varepsilon \ll_\varepsilon |f|^\varepsilon,$$

for some potentially large $\varepsilon > 0$. Combining this estimate with the bound $L(s, \chi_{f_0}) \ll 1$ for $\operatorname{Re}(s) > 1$, we see that $Z(s, w)$ is absolutely uniformly convergent on compacta in the region $\Lambda = \{(s, w) \in \mathbb{C}^2 : \operatorname{Re}(s) > 1, \operatorname{Re}(w) > 1\}$.

While $Z(s, w)$ is the double Dirichlet series we are after, it will be necessary to consider double Dirichlet series twisted by a pair of Hilbert symbols χ_{a_1} and χ_{a_2} . The **double Dirichlet series** $Z_{a_1, a_2}(s, z)$ twisted by χ_{a_1} and χ_{a_2} is defined by

$$Z_{a_1, a_2}(s, w) = \sum_{f \text{ monic}} \frac{L(s, \chi_{a_1 f_0}) \chi_{a_2}(f) Q_{a_1 f_0 f_1^2}(s)}{|f|^w},$$

where $Q_{a_1 f_0 f_1^2}(s)$ is the **correction polynomial** twisted by χ_{a_1} defined by

$$Q_{a_1 f_0 f_1^2}(s) = \sum_{e_1 e_2 | f_1} \mu(e_1) \chi_{a_1 f_0}(e_1) |e_1|^{-s} |e_2|^{1-2s} = \sum_{e_1 e_2 e_3 = f_1} \mu(e_1) \chi_{a_1 f_0}(e_1) |e_1|^{-s} |e_2|^{1-2s},$$

where μ is the usual Möbius function on $\mathbb{F}_q[t]$. As the Hilbert symbols are given by quadratic residue symbols, we have the analogous bound $Q_{a_1 f_0 f_1^2}(s) \ll |f|^\varepsilon$ so that $Z_{a_1, a_2}(s, w)$ converges absolutely uniformly on compacta in the same region as $Z(s, w)$ does. In this generalized setup, $Z_{1, 1}(s, w) = Z(s, w)$. The only genuinely twisted double Dirichlet series we will need is $Z_{1, \theta}$ but we state our results in full generality.

3. THE INTERCHANGE

Since L -functions attached to quadratic residue symbols admit Euler products, $Z_{a_1, a_2}(s, w)$ is a sum of Euler products in s . We will now argue that $Z_{a_1, a_2}(s, w)$ can be written as a sum of Euler products in w . In effect, we want the variable s to appear in the denominator of $Z_{a_1, a_2}(s, w)$ and the L -functions in the numerator to be in the variable w . To be precise, we will prove the following:

Theorem 3.1 (The interchange). *Wherever $Z_{a_1, a_2}(s, w)$ converges absolutely uniformly on compacta,*

$$Z_{a_1, a_2}(s, w) = \sum_{f \text{ monic}} \frac{L(s, \chi_{a_1 f_0}) \chi_{a_2}(f) Q_{a_1 f_0 f_1^2}(s)}{|f|^w} = \sum_{g \text{ monic}} \frac{L(w, \chi_{g_0 a_2}) \chi_{a_1}(g) Q_{a_2 g_0 g_1^2}(w)}{|g|^s},$$

Proof. Only the second equality needs to be justified since the first is the definition of $Z(s, w)$. Expanding the L -function $L(s, \chi_{a_1 f_0})$ and polynomial $Q_{a_1 f_0 f_1^2}(s)$ gives

$$\begin{aligned} Z(s, w) &= \sum_{f \text{ monic}} \frac{L(s, \chi_{a_1 f_0}) \chi_{a_2}(f) Q_{a_1 f_0 f_1^2}(s)}{|f|^w} \\ &= \sum_{f \text{ monic}} \left(\sum_{g \text{ monic}} \chi_{a_1 f_0}(g) |g|^{-s} \right) \left(\sum_{e_1 e_2 | f_1} \mu(e_1) \chi_{a_1 f_0}(e_1) |e_1|^{-s} |e_2|^{1-2s} \right) \chi_{a_2}(f) |f|^{-w} \\ &= \sum_{f, g \text{ monic}} \sum_{e_1 e_2 | f_1} \mu(e_1) \chi_{a_2}(f) \chi_{a_1 f_0}(g e_1) |e_1|^{-s} |e_2|^{1-2s} |g|^{-s} |f|^{-w}. \end{aligned}$$

Now $\chi_{a_1 f_0}(g e_1) = 0$ unless $(f_0, g e_1) = 1$. So we may make this restriction on the sum giving

$$\sum_{f, g \text{ monic}} \sum_{\substack{e_1 e_2 | f_1 \\ (f_0, g e_1) = 1}} \mu(e_1) \chi_{a_2}(f) \chi_{f_0}(g e_1) |e_1|^{-s} |e_2|^{1-2s} |g|^{-s} |f|^{-w}.$$

Making the change of variables $g e_1 \rightarrow g$ yields

$$\sum_{f \text{ monic}} \sum_{g \text{ monic}} \sum_{\substack{e_1 e_2 | f_1 \\ e_1 | g \\ (f_0, g) = 1}} \mu(e_1) \chi_{a_2}(f) \chi_{a_1 f_0}(g) |e_2|^{1-2s} |g|^{-s} |f|^{-w}.$$

Now for fixed $f = f_0 f_1^2$ and e_2 , the resulting subsum over g and e_1 is

$$\sum_{\substack{g \text{ monic} \\ e_1 | g \\ (f_0, g) = 1}} \sum_{\substack{e_1 | \frac{f_1}{e_2} \\ (f_0, g) = 1}} \mu(e_1) \chi_{a_1 f_0}(g) |g|^{-s} = \sum_{\substack{g \text{ monic} \\ (f_0, g) = 1}} \chi_{a_1 f_0}(g) |g|^{-s} \left(\sum_{e_1 | \left(\frac{f_1}{e_2}, g \right)} \mu(e_1) \right).$$

The inner sum over e_1 of the Möbius function vanishes unless $\left(\frac{f_1}{e_2}, g \right) = 1$ in which case it is 1. Therefore the triple sum above becomes

$$\sum_{f, g \text{ monic}} \sum_{\substack{e_2 | f_1 \\ \left(\frac{f_0 f_1}{e_2}, g \right) = 1}} \chi_{a_2}(f) \chi_{a_1 f_0}(g) |e_2|^{1-2s} |g|^{-s} |f|^{-w}.$$

Making the change of variables $f \rightarrow f e_2^2$, the condition $\left(\frac{f_0 f_1}{e_2}, g \right) = 1$ becomes $(f_0 f_1, g) = 1$ which is equivalent to $(f, g) = 1$. Moreover, $\chi_{a_2}(f e_2^2) = \chi_{a_2}(f)$. Altogether

$$\sum_{f, g \text{ monic}} \sum_{\substack{e_2 | f_1 \\ \left(\frac{f_0 f_1}{e_2}, g \right) = 1}} \chi_{a_2}(f) \chi_{a_1 f_0}(g) |e_2|^{1-2s} |g|^{-s} |f|^{-w} = \sum_{\substack{f, g \text{ monic} \\ (f, g) = 1}} \sum_{e_2} \chi_{a_2}(f) \chi_{a_1 f_0}(g) |e_2|^{1-2s-2w} |g|^{-s} |f|^{-w}.$$

Writing $g = g_0 g_1^2$ analogously as for f , quadratic reciprocity implies $\chi_{f_0}(g) = \chi_g(f_0) = \chi_{g_0}(f)$ where the last equality holds because $(f, g) = 1$ and both f_0 and g_0 differ from f and g respectively by perfect squares. Therefore $\chi_{a_2}(f) \chi_{a_1 f_0}(g) = \chi_{a_1}(g) \chi_{a_2 g_0}(f)$ and our expression becomes

$$\sum_{\substack{f, g \text{ monic} \\ (f, g) = 1}} \sum_{e_2} \chi_{a_1}(g) \chi_{a_2 g_0}(f) |e_2|^{1-2s-2w} |g|^{-s} |f|^{-w}.$$

But now we can reverse the argument with the roles of f and g interchangeed to obtain

$$Z(s, w) = \sum_{g \text{ monic}} \frac{L(w, \chi_{a_2 g_0}) Q_{a_1 f_0 f_1^2}(w)}{|g|^w}.$$

□

The symmetry in the interchange is not typical of a larger reality. In more general setting the argument is more complicated than the proof presented. One does not usually arrive at such a nice symmetric expression because the correction polynomials in w need not be equal to those in s . In other words, when $Z(s, w)$ is represented as a sum of L -functions in s the correction polynomials are $Q_{a_1 f_0 f_1^2}(s)$, but when $Z(s, w)$ is represented as a sum of L -functions in w the correction polynomials take the form $P_{a_2 g_0 g_1^2}(w)$. In our setting, it is a special case that $P_{g_0 g_1^2}(w) = Q_{g_0 g_1^2}(w)$.

Remark 3.1. When $a_1 = a_2 = 1$, the interchange implies that $Z(s, w)$ is symmetric in s and w . That is,

$$Z(s, w) = Z(w, s).$$

4. WEIGHTING TERMS

We will begin investigating the coefficients of $Z_{a_1, a_2}(s, w)$ further. Expanding $L(s, \chi_{a_1 f_0}) Q_{a_1 f_0 f_1^2}(s)$ in the numerator of $Z_{a_1, a_2}(s, w)$ we obtain

$$Z_{a_1, a_2}(s, w) = \sum_{f \text{ monic}} \frac{L(s, \chi_{a_1 f_0}) \chi_{a_2}(f) Q_{a_1 f_0 f_1^2}(s)}{|f|^w} = \sum_{f, g \text{ monic}} \frac{\chi_{a_1 f_0}(\widehat{g}) \chi_{a_2}(f) a(f, g)}{|f|^w |g|^s},$$

where \widehat{g} is the part of g relatively prime to f_0 and the **weighting coefficient** $a(f, g,)$ is given by

$$a(f, g) = \sum_{\substack{e_1 e_2^2 e_3 = g \\ e_1 e_2 | f_1 \\ (f_0, e_1 e_3) = 1}} \mu(e_1) |e_2|.$$

Indeed, the coefficient of $|f|^{-w} |g|^{-s}$ in the definition of $Z_{a_1, a_2}(s, w)$ is

$$\begin{aligned} \chi_{a_2}(f) \sum_{\substack{e_1 e_2^2 e_3 = g \\ e_1 e_2 | f_1}} \mu(e_1) \chi_{a_1 f_0}(e_1 e_3) |e_2| &= \chi_{a_2}(f) \sum_{\substack{e_1 e_2^2 e_3 = g \\ e_1 e_2 | f_1 \\ (f_0, e_1 e_3) = 1}} \mu(e_1) \chi_{a_1 f_0}(e_1 e_3) |e_2| \\ &= \chi_{a_1 f_0}(\widehat{g}) \chi_{a_2}(f) \sum_{\substack{e_1 e_2^2 e_3 = g \\ e_1 e_2 | f_1 \\ (f_0, e_1 e_3) = 1}} \mu(e_1) |e_2| \\ &= \chi_{a_1 f_0}(\widehat{g}) \chi_{a_2}(f) a(f, g), \end{aligned}$$

where the first equality holds because $\chi_{f_0}(e_1 e_3) = 0$ unless $(f_0, e_1 e_3) = 1$ and the second equality holds because when $(f_0, e_1 e_3) = 1$ we know that \widehat{g} differs from $e_1 e_3$ by a perfect square (they differ by divisors of (f_0, e_2)) and so $\chi_{f_0}(e_1 e_3) = \chi_{f_0}(\widehat{g})$.

Remark 4.1. Note that $a(f, g) = 0$ unless there is a decomposition $g = e_1 e_2^2 e_3$ of g such that $(f_0, e_1 e_3) = 1$ and $e_1 e_2^2 | f_1$. In the case of an irreducible P ,

It will be useful to let $L(s, \widehat{\chi}_{a_1 f})$ be the Dirichlet series defined by

$$L(s, \widehat{\chi}_{a_1 f}) = L(s, \chi_{a_1 f_0}) Q_{a_1 f_0 f_1^2}(s) = \sum_{g \text{ monic}} \frac{\chi_{a_1 f_0}(\widehat{g}) a(f, g)}{|g|^s},$$

so that

$$Z_{a_1, a_2}(s, w) = \sum_{f \text{ monic}} \frac{\chi_{a_2}(f) L(s, \widehat{\chi}_{a_1} f)}{|f|^w}.$$

Now on the other hand, performing the same procedure to the interchange we see that

$$Z_{a_1, a_2}(s, w) = \sum_{g \text{ monic}} \frac{L(w, \chi_{a_2 g_0}) \chi_{a_1}(g) Q_{a_2 g_0 g_1^2}(w)}{|g|^s} = \sum_{f, g \text{ monic}} \frac{\chi_{a_2 g_0}(\widehat{f}) \chi_{a_1}(g) a(g, f)}{|f|^w |g|^s},$$

where \widehat{f} is the part of f relatively prime to g_0 . Analogously, we define $L(w, \widehat{\chi}_{a_2} g)$ by

$$L(w, \widehat{\chi}_{a_2} g) = L(w, \chi_{a_2 g_0}) Q_{a_2 g_0 g_1^2}(w) = \sum_{f \text{ monic}} \frac{\chi_{g_0}(\widehat{f}) a(g, f)}{|f|^w},$$

so that

$$Z_{a_1, a_2}(s, w) = \sum_{g \text{ monic}} \frac{\chi_{a_1}(g) L(s, \widehat{\chi}_{a_2} g)}{|g|^w}.$$

We can now derive an important relationship for the weighting coefficients. Equating the coefficients of the two expansions for $Z_{a_1, a_2}(s, w)$ gives

$$\chi_{a_1 f_0}(\widehat{g}) \chi_{a_2}(f) a(f, g) = \chi_{a_2 g_0}(\widehat{f}) \chi_{a_1}(g) a(g, f)$$

By applying quadratic reciprocity twice we have

$$\chi_{f_0}(\widehat{g}) = \chi_{\widehat{g}}(f_0) = \chi_{\widehat{g_0}}(f_0) = \chi_{g_0}(\widehat{f_0}) = \chi_{g_0}(\widehat{f}),$$

so upon setting $a_1 = a_2 = 1$ we have

$$a(f, g) = a(g, f).$$

In other words, the weighting coefficients are symmetric. The weighting coefficients also possess a multiplicativity property:

Proposition 4.1. *We have $a(f, 1) = a(1, g) = 1$ and*

$$a(f, g) = \prod_{\substack{P^\alpha \parallel f \\ P^\beta \parallel g}} a(P^\alpha, P^\beta).$$

Proof. It is clear that $a(f, 1) = a(1, g) = 1$ from the definition of the weighting coefficients. It suffices to prove the second statement at irreducibles. As $a(f, g)$ is symmetric, we only need to prove multiplicativity in g . So letting $g = g' P^\beta$, we have to show

$$a(f, g) = a(f, g') a(f, P^\beta).$$

Indeed, for $e_1 e_2^2 e_3 = g$, let $e_1 = c_1 d_1$, $e_2 = c_2 d_2$, and $e_3 = c_3 d_3$ with $c_1, c_2, c_3 \mid g'$ and $d_1, d_2, d_3 \mid P^\beta$. Because $(g', P^\beta) = 1$, as $e_1 e_2^2 e_3$ runs over decompositions of g , $c_1 c_2^2 c_3$ and $d_1 d_2^2 d_3$ run over decompositions of g' and P^β respectively. Moreover, as $e_1 e_2$ runs over the divisors of f_1 so does $c_1 d_1 c_2 d_2$. Then using multiplicativity

of the Möbius function, we have

$$\begin{aligned}
a(f, g) &= \sum_{\substack{e_1 e_2^2 e_3 = g \\ e_1 e_2 | f_1 \\ (f_0, e_1 e_3) = 1}} \mu(e_1) |e_2| \\
&= \sum_{\substack{c_1 c_2^2 c_3 = g' \\ d_1 d_2^2 d_3 = P^\beta \\ c_1 d_1 c_2 d_2 | f_1 \\ (f_0, c_1 d_1 c_3 d_3) = 1}} \mu(c_1) (d_1) |c_2| |d_2| \\
&= \left(\sum_{\substack{c_1 c_2^2 c_3 = g' \\ c_1 c_2 | f_1 \\ (f_0, c_1 c_3) = 1}} \mu(c_1) |c_2| \right) \left(\sum_{\substack{d_1 d_2^2 d_3 = P^\beta \\ d_1 d_2 | f_1 \\ (f_0, d_1 d_3) = 1}} \mu(d_1) |d_2| \right) \\
&= a(f, g') a(f, P^\beta),
\end{aligned}$$

as desired. \square

There is a connection between the correction polynomials $Q_{a_1 f_0 f_1^2}(s)$ and weighting coefficients $a(f, g)$. It will turn out that $Q_{a_1 f_0 f_1^2}(s)$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients. We first establish this relationship when f is an odd irreducible power:

Lemma 4.1. *For any irreducible P and $\alpha \geq 1$, we have*

$$Q_{P^{2\alpha+1}}(s) = \sum_{k \leq 2\alpha} \frac{a(P^{2\alpha+1}, P^k)}{|P|^{ks}}.$$

Proof. Expanding the correction polynomial in $|P|^{-s}$ yields

$$Q_{P^{2\alpha+1}}(s) = \sum_{e_1 e_2 | P^\alpha} \mu(e_1) \chi_P(e_1) |e_1|^{-s} |e_2|^{1-2s} = \sum_{k \leq 2\alpha} \frac{b(P^{2\alpha+1}, P^k)}{|P|^{ks}}.$$

where

$$b(P^{2\alpha+1}, P^k) = \sum_{e_1 e_2^2 = P^k} \mu(e_1) \chi_P(e_1) |e_2|.$$

The proof will be complete if we can show $b(P^{2\alpha+1}, P^k) = a(P^{2\alpha+1}, P^k)$. To see this, first observe $\mu(e_1) \chi_P(e_1) = 0$ unless $e_1 = 1$ in which case it is 1. So $b(P^{2\alpha+1}, P^k) = 0$ if k is odd and $|P|^{\frac{k}{2}}$ if k is even. Compactly stated,

$$b(P^{2\alpha+1}, P^k) = \begin{cases} |P|^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

On the other hand, $k \leq \alpha$ so that

$$a(P^{2\alpha+1}, P^k) = \sum_{\substack{e_1 e_2^2 e_3 = P^k \\ e_1 e_2 | P^\alpha \\ (P, e_1 e_3) = 1}} \mu(e_1) |e_2| = \sum_{\substack{e_1 e_2^2 | P^k \\ (P, e_1 e_3) = 1}} \mu(e_1) |e_2| = \sum_{e_2^2 = P^k} |e_2| = \begin{cases} |P|^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

This completes the proof. \square

We have a similar statement if f is an even power of an irreducible up to a square-free factor and relatively prime factor:

Lemma 4.2. *For any square-free monic f_0 , $a_1 \in \{1, \theta\}$, irreducible P not dividing f_0 , and $\beta \geq 1$, we have*

$$Q_{a_1 f_0 P^{2\beta}}(s) = (1 - \chi_{a_1 f_0}(P)|P|^{-s}) \sum_{k \leq 2\beta} \frac{\chi_{a_1 f_0}(P^k) a(P^{2\beta}, P^k)}{|P|^{ks}}.$$

Proof. Expand the correction polynomial in $|P|^{-s}$ to get

$$Q_{a_1 f_0 P^{2\beta}}(s) = \sum_{e_1 e_2 | P^{2\beta}} \mu(e_1) \chi_{a_1 f_0}(e_1) |e_1|^{-s} |e_2|^{1-2s} = \sum_{k \leq 2\beta} \frac{b(P^{2\beta}, P^k)}{|P|^{ks}}.$$

where

$$b(P^{2\beta}, P^k) = \sum_{e_1 e_2^2 = P^k} \mu(e_1) \chi_{a_1 f_0}(e_1) |e_2|.$$

It now suffices to show $b(P^{2\beta}, P^k) = \chi_{a_1 f_0}(P^k) (a(P^{2\beta}, P^k) - a(P^{2\beta}, P^{k-1}))$. On the one hand, $\mu(e_1) = 0$ unless $e_1 = 1, P$ in which case it is ± 1 . So

$$b(P^{2\beta}, P^k) = \sum_{e_1 e_2^2 = P^k} \mu(e_1) \chi_{a_1 f_0}(e_1) |e_2| = \begin{cases} \chi_{a_1 f_0}(P^k) |P|^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 f_0}(P^k) |P|^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

where we have used the identity $\chi_{a_1 f_0}(e_1) = \chi_{a_1 f_0}(P^k)$ which holds because this quadratic residue symbol only depends upon the parity of k . On the other hand, as in the proof of Lemma 4.1

$$a(P^{2\beta}, P^k) = \begin{cases} |P|^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

But then

$$\chi_{a_1 f_0}(P^k) (a(P^{2\beta}, P^k) - a(P^{2\beta}, P^{k-1})) = \begin{cases} \chi_{a_1 f_0}(P^k) |P|^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 f_0}(P^k) |P|^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

which completes the proof. \square

From Lemmas 4.1 and 4.2 we see that $Q_{a_1 f}(s)$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients $a(f, g)$ when f is an irreducible power. From the proofs of these two lemmas, we have evaluated $a(P^k, P^l)$ and we collect this as a corollary:

Corollary 4.1. *For any irreducible P ,*

$$a(P^k, P^l) = \begin{cases} \min(|P|^{\frac{k}{2}}, |P|^{\frac{l}{2}}) & \text{if } \min(k, l) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Using Proposition 4.1 and Corollary 4.1 together lets us evaluate $a(f, g)$ in general:

Corollary 4.2. *For any monics f and g ,*

$$a(f, g) = \begin{cases} |(f, g)|^{\frac{1}{2}} & \text{if } (f, g) \text{ is a perfect square,} \\ 0 & \text{otherwise.} \end{cases}$$

Since the weighting coefficients are multiplicative, it will follow that $Q_{a_1 f}(s)$ admits an Euler product. To describe this Euler product explicitly, let $f = f_0 f_1^2 f_2^2$ with f_0 square-free and monic, f_2 relatively prime to $f_0 f_1$, and such that every irreducible divisor of f_1 divides f_0 . In other words, f_0 is the square-free part of f , f_1 is the square part of f whose irreducible factors divide f to odd power, and f_2 is the square part of f whose irreducible factors divide f to even power. Then we have the following Euler product:

Theorem 4.1. *Let $f = f_0 f_1^2 f_2^2$ be the square decomposition of f stratified by even and odd powers. Then for any $a_1 \in \{1, \theta\}$,*

$$Q_{a_1 f_0 f_1^2 f_2^2}(s) = \prod_{P^\alpha \parallel f_1} Q_{P^{2\alpha+1}}(s) \cdot \prod_{P^\beta \parallel f_2} Q_{a_1 f_0 P^{2\beta}}(s),$$

Proof. The coefficient of $|f|^{-w}$ in

$$Z_{a_1, a_2}(s, w) = \sum_{f, g \text{ monic}} \frac{\chi_{a_1 f_0}(\widehat{g}) \chi_{a_2}(f) a(f, g)}{|f|^w |g|^s},$$

is the Dirichlet series

$$\chi_{a_2}(f) L(s, \widehat{\chi}_f) = \chi_{a_2}(f) L(s, \chi_{a_1 f_0}) Q_{a_1 f_0 f_1^2 f_2^2}(s) = \chi_{a_2}(f) \sum_{g \text{ monic}} \frac{\chi_{a_1 f_0}(\widehat{g}) a(f, g)}{|g|^s}.$$

By Proposition 4.1 the coefficients of $L(s, \widehat{\chi}_f)$ are multiplicative. Therefore $L(s, \widehat{\chi}_f)$ admits the Euler product

$$L(s, \widehat{\chi}_f) = \prod_P \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(\widehat{P}^k) a(f, P^k)}{|P|^{ks}} \right).$$

Decomposing the product according to irreducibles dividing $f = f_0 f_1^2 f_2^2$, we see that

$$\begin{aligned} L(s, \widehat{\chi}_f) &= \prod_P \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(\widehat{P}^k) a(f, P^k)}{|P|^{ks}} \right) \\ &= \prod_{P \nmid f} \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(\widehat{P}^k) a(1, P^k)}{|P|^{ks}} \right) \prod_{P^\alpha \parallel f_1} \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(\widehat{P}^k) a(P^{2\alpha+1}, P^k)}{|P|^{ks}} \right) \cdot \prod_{P^\beta \parallel f_2} \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(\widehat{P}^k) a(P^\beta, P^k)}{|P|^{ks}} \right) \\ &= \prod_{P \nmid f} \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(\widehat{P}^k)}{|P|^{ks}} \right) \prod_{P^\alpha \parallel f_1} \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(\widehat{P}^k) a(P^{2\alpha+1}, P^k)}{|P|^{ks}} \right) \cdot \prod_{P^\beta \parallel f_2} \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(\widehat{P}^k) a(P^\beta, P^k)}{|P|^{ks}} \right) \\ &= \prod_{P \nmid f} \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(\widehat{P}^k)}{|P|^{ks}} \right) \prod_{P^\alpha \parallel f_1} \left(\sum_{k \geq 0} \frac{a(P^{2\alpha+1}, P^k)}{|P|^{ks}} \right) \cdot \prod_{P^\beta \parallel f_2} \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(P^k) a(P^\beta, P^k)}{|P|^{ks}} \right). \end{aligned}$$

Including the factors corresponding to primes $P \mid f_2$ into the first product, we must multiply the last factor by the inverse of $\sum_{k \geq 0} \chi_{a_1 f_0}(P) |P|^{-ks} = (1 - \chi_{a_1 f_0}(P) |P|^{-s})^{-1}$ obtaining

$$\prod_{P \nmid f_0} \left(\sum_{k \geq 0} \frac{\chi_{a_1 f_0}(\widehat{P}^k)}{|P|^{ks}} \right) \prod_{P^\alpha \parallel f_1} \left(\sum_{k \geq 0} \frac{a(P^{2\alpha+1}, P^k)}{|P|^{ks}} \right) \cdot \prod_{P^\beta \parallel f_2} \left((1 - \chi_{a_1 f_0}(P) |P|^{-s}) \sum_{k \geq 0} \frac{\chi_{a_1 f_0}(P^k) a(P^\beta, P^k)}{|P|^{ks}} \right),$$

since every irreducible divisor of f_1 divides f_0 . We recognize the first product as $L(s, \chi_{a_1 f_0})$. For the second and third products, Remark 4.1 implies that the sums run up to $k \leq 2\alpha$ and $k \leq 2\beta$ respectively. We then recognize the sums as $Q_{P^{2\alpha+1}}(s)$ and $Q_{a_1 f_0 P^{2\beta}}(s)$. It follows that the entire Euler product is

$$L(s, \chi_{a_1 f_0}) \cdot \prod_{P^\alpha \parallel f_1} Q_{P^{2\alpha+1}}(s) \cdot \prod_{P^\beta \parallel f_2} Q_{a_1 f_0 P^{2\beta}}(s).$$

But then upon canceling the $\chi_{a_2}(f)$ term, we have

$$L(s, \widehat{\chi}_f) = L(s, \chi_{a_1 f_0}) Q_{a_1 f_0 f_1^2 f_2^2}(s) = L(s, \chi_{a_1 f_0}) \cdot \prod_{P^\alpha \parallel f_1} Q_{P^{2\alpha+1}}(s) \cdot \prod_{P^\beta \parallel f_2} Q_{a_1 f_0 P^{2\beta}}(s),$$

and the proof is complete. \square

It is worth noting that for $f = f_0 f_1^2 f_2^2$, the irreducible factors that divide $f_1 f_2$ are exactly those factors that divide f to power larger than 1. In particular, from Theorem 4.1 we see that the Euler product for $Q_{a_1 f_0 f_1^2 f_2^2}(s)$ is supported on exactly the primes dividing f to order larger than 1 but also depends upon the character $\chi_{a_1 f_0}$. Also, Theorem 4.1 is typical of a larger reality. In more general settings when the correction polynomials $P_{a_2 g_0 g_1^2}(w)$ on the other side of the interchange are not of the form $Q_{a_1 f_0 f_1^2}(s)$, the correction polynomials $P_{a_2 g_0 g_1^2}(w)$ still possess Euler products.

5. FUNCTIONAL EQUATIONS

Returning to the double Dirichlet series $Z(s, w)$, we are ready to derive functional equations. In order to do this we require a functional equation for $L(s, \widehat{\chi}_f)$ which amounts to having a functional equation for the correction polynomials $Q_{f_0 f_1^2}(s)$:

Theorem 5.1. $Q_{f_0 f_1^2}(s)$ admits the functional equation.

$$Q_{f_0 f_1^2}(s) = |f_1|^{1-2s} Q_{f_0 f_1^2}(1-s).$$

Proof. The idea is that e_2 and e_3 are interchanged in the sum. The computation is simple:

$$\begin{aligned} |f_1|^{1-2s} Q_{f_0 f_1^2}(1-s) &= |f_1|^{1-2s} \sum_{e_1 e_2 e_3 = f_1} \mu(e_1) \chi_{f_0}(e_1) |e_1|^{s-1} |e_2|^{2s-1} \\ &= \sum_{e_1 e_2 e_3 = f_1} \mu(e_1) \chi_{f_0}(e_1) |e_1|^{s-1} \left| \frac{f_1}{e_2} \right|^{1-2s} \\ &= \sum_{e_1 e_2 e_3 = f_1} \mu(e_1) \chi_{f_0}(e_1) |e_1|^{s-1} |e_1 e_3|^{1-2s} \\ &= \sum_{e_1 e_2 e_3 = f_1} \mu(e_1) \chi_{f_0}(e_1) |e_1|^{-s} |e_3|^{1-2s} \\ &= Q_{f_0 f_1^2}(s). \end{aligned}$$

□

Define the completed L -function

$$L^*(s, \widehat{\chi}_f) = L^*(s, \chi_{f_0}) Q_{f_0 f_1^2}(s).$$

Combining Theorem 5.1, the functional equation for $L^*(s, \chi_{f_0})$, and noting that $\deg(f) \equiv \deg(f_0) \pmod{2}$, we obtain a functional equation for $L^*(s, \widehat{\chi}_f)$:

$$L^*(s, \widehat{\chi}_f) = \begin{cases} q^{2s-1} |f|^{\frac{1}{2}-s} L^*(1-s, \widehat{\chi}_f) & \text{if } \deg(f) \text{ is even,} \\ q^{s-\frac{1}{2}} |f|^{\frac{1}{2}-s} L^*(1-s, \widehat{\chi}_f) & \text{if } \deg(f) \text{ is odd.} \end{cases}$$

Now we return to $Z(s, w)$. We would like the functional equation for $L^*(s, \widehat{\chi}_f)$ to induce a functional equation for $Z(s, w)$ but there is one technical issue. The gamma factors for $L^*(s, \widehat{\chi}_f)$ depend upon the degree of f and in order to obtain a functional equation we need the gamma factors to be constant. We will be able to move past this obstruction by using the twisted double Dirichlet series $Z_{1,\theta}(s, w)$. Define $Z_{\text{even}}(s, w)$ and $Z_{\text{odd}}(z, w)$ by

$$Z_{\text{even}}(s, w) = \frac{Z(s, w) + Z_{1,\theta}(z, w)}{2} \quad \text{and} \quad Z_{\text{odd}}(s, w) = \frac{Z(s, w) - Z_{1,\theta}(z, w)}{2}.$$

In terms of the representations

$$Z(s, w) = \sum_{f \text{ monic}} \frac{L(s, \widehat{\chi}_f)}{|f|^w} = \sum_{g \text{ monic}} \frac{L(w, \widehat{\chi}_g)}{|g|^s},$$

given by the interchange we see that $Z_{\text{even}}(s, w)$ is the subseries containing only those f (or g) with even degree and $Z_{\text{odd}}(s, w)$ is the subseries containing only those f (or g) with odd degree. As $Z_{\text{even}}(s, w)$ and $Z_{\text{odd}}(s, w)$ are sums of L -functions with a fixed gamma factor we can obtain functional equations for them. The fact that $Z(s, w)$ is a linear combination of these series will induce a function equation for $Z(s, w)$ in terms of $Z(s, w)$ and $Z_{1,\theta}(s, w)$:

Theorem 5.2. $Z(s, w)$ admits the functional equations

$$Z(s, w) = \frac{1}{2} \left(\frac{q^{2s-1}(1-q^{-s})}{1-q^{s-1}} + q^{s-\frac{1}{2}} \right) Z \left(1-s, s+w-\frac{1}{2} \right) \\ + \frac{1}{2} \left(\frac{q^{2s-1}(1-q^{-s})}{1-q^{s-1}} - q^{s-\frac{1}{2}} \right) Z_{1,\theta} \left(1-s, s+w-\frac{1}{2} \right),$$

and

$$Z(s, w) = \frac{1}{2} \left(\frac{q^{2w-1}(1-q^{-w})}{1-q^{w-1}} + q^{w-\frac{1}{2}} \right) Z \left(s+w-\frac{1}{2}, 1-w \right) \\ + \frac{1}{2} \left(\frac{q^{2w-1}(1-q^{-w})}{1-q^{w-1}} - q^{w-\frac{1}{2}} \right) Z_{1,\theta} \left(s+w-\frac{1}{2}, 1-w \right).$$

Proof. From the interchange, the correction polynomials when $Z(s, w)$ is represented a sum of L -functions is s are the same as those when $Z(s, w)$ is represented as a sum of L -functions in w . Therefore it only suffices to prove the first functional equation as the second one will follow by symmetry. The functional equation for the L -function attached to a quadratic residue symbol induces the functional equations

$$Z_{\text{even}}(s, w) = \frac{q^{2s-1}(1-q^{-s})}{1-q^{s-1}} Z_{\text{even}} \left(1-s, s+w-\frac{1}{2} \right) \quad \text{and} \quad Z_{\text{odd}}(s, w) = q^{s-\frac{1}{2}} Z_{\text{odd}} \left(1-s, s+w-\frac{1}{2} \right).$$

As $Z(s, w) = Z_{\text{even}}(s, w) + Z_{\text{odd}}(s, w)$, the functional equations just stated give

$$Z(s, w) = \frac{q^{2s-1}(1-q^{-s})}{1-q^{s-1}} Z_{\text{even}} \left(1-s, s+w-\frac{1}{2} \right) + q^{s-\frac{1}{2}} Z_{\text{odd}} \left(1-s, s+w-\frac{1}{2} \right).$$

The desired functional equation for $Z(s, w)$ follows by expressing the even and odd double Dirichlet series in terms of $Z(s, w)$ and $Z_{1,\theta}(s, w)$. \square

In more general settings, the functional equations for $Z(s, w)$ need not be so symmetric. Indeed, when the correction polynomials $P_{g_0 g_1^2}(w)$ are not of the form $Q_{f_0 f_1^2}(s)$, we require functional equations for both of these correction polynomials. We then have to use each functional equation for each correction polynomial to deduce functional equations for $Z(s, w)$ of shape $s \rightarrow 1-s$ and $w \rightarrow 1-w$. From Theorem 5.2 we have two functional equations of shapes

$$\sigma_1 : (s, w) \rightarrow \left(1-s, s+w-\frac{1}{2} \right) \quad \text{and} \quad \sigma_2 : (s, w) \rightarrow \left(s+w-\frac{1}{2}, 1-w \right).$$

These transformations also naturally act on the (s, w) -plane. Moreover, we have the relation

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 : (s, w) \rightarrow (1-w, 1-s) \quad \text{or equivalently} \quad (\sigma_1 \sigma_2)^3 = 1 : (s, w) \rightarrow (s, w).$$

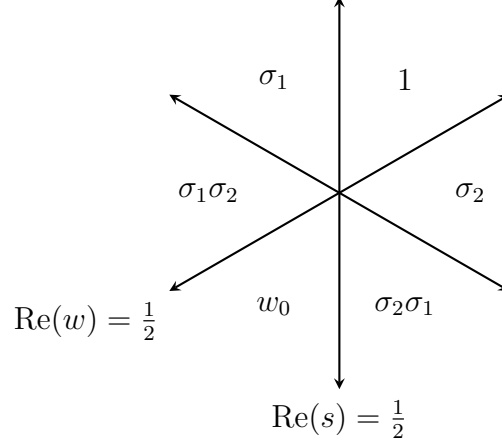
As $\sigma_1^2 = \sigma_2^2 = 1$, σ_1 and σ_2 generate the group

$$W = \langle \sigma_1, \sigma_2 : \sigma_1^2 = \sigma_2^2 = (\sigma_1 \sigma_2)^3 = 1 \rangle \cong D_6 \cong S_3.$$

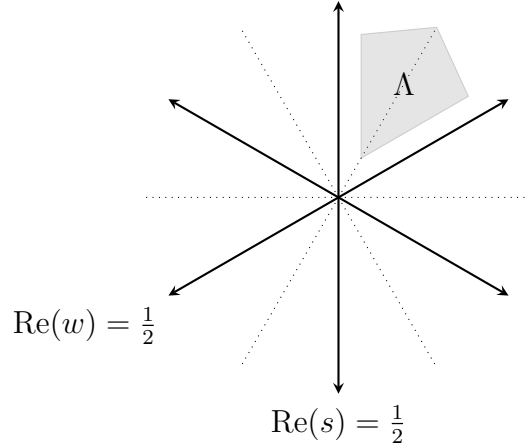
For convenience we will set $w_0 = \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$. It follows that $Z(s, w)$ doesn't just possess two functional equations, it has a group of 6 functional equations. Actually, if we include the functional equation $(s, w) \rightarrow (w, s)$ coming from the fact that $Z(s, w)$ is symmetric in s and w , there are 12 functional equations and the group is isomorphic to D_{12} . We can use these functional equations to meromorphically continue $Z(s, w)$ to the entire (s, w) -plane.

6. MEROMORPHIC CONTINUATION

Now it is time to establish the meromorphic continuation of $Z(s, w)$ to the entire (s, w) -plane. We will start by describing the action of W on the (s, w) -plane geometrically. First, it is easy to see from the definition of the actions σ_1 and σ_2 that there is a unique W -invariant point $(\frac{1}{2}, \frac{1}{2})$. Representing the point (s, w) by $(\text{Re}(s), \text{Re}(w))$ we have a geometric visualization of the action of W on the (s, w) -plane:



In the diagram, we have shifted the (s, w) -plane so that the origin lies at $(\frac{1}{2}, \frac{1}{2})$ and we have tiled the (s, w) -axes so that they are no longer perpendicular. The effect of these adjustments is that σ_1 and σ_2 act by rigid motions sending the region enclosing 1 (corresponding to the identity) to either of the adjacent triangles. The other regions are obtained by acting by the corresponding element of W . The initial region Λ that $Z(s, w)$ is defined on is displayed in the figure below:



To meromorphically continue $Z(s, w)$ to all of the (s, w) -plane, we first need to analytically continue the double Dirichlet series $Z(s, w)$ and $Z_{1,\theta}(s, w)$ to a slightly larger region than Λ . This continuation will be achieved by the Phragmén-Lindelöf convexity principal. Fix some small $\varepsilon > 0$. The functional equations for $L^*(s, \widehat{\chi}_{f_0})$ and $L^*(w, \widehat{\chi}_{g_0})$ provide the estimates

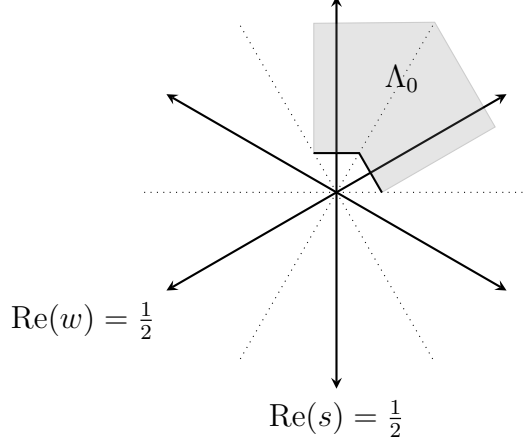
$$L(-\varepsilon, \widehat{\chi}_{f_0}) \ll |f|^{\frac{1}{2}+\varepsilon} \quad \text{and} \quad L(-\varepsilon, \widehat{\chi}_{g_0}) \ll |g|^{\frac{1}{2}+\varepsilon},$$

because $L(1+\varepsilon, \widehat{\chi}_{f_0}) \ll 1$ and $L(1+\varepsilon, \widehat{\chi}_{g_0}) \ll 1$. Since both of these L -functions at most have a simple pole at $s = 1$ and $w = 1$ respectively, the Phragmén-Lindelöf convexity principal implies the weak estimates

$$(s-1)L(s, \widehat{\chi}_{f_0}) \ll |f|^{\frac{1}{2}+\varepsilon} \quad \text{and} \quad (w-1)L(w, \widehat{\chi}_{g_0}) \ll |g|^{\frac{1}{2}+\varepsilon},$$

for $\operatorname{Re}(s) > -\varepsilon$ and $\operatorname{Re}(w) > -\varepsilon$. But then from the interchange we see that $(s-1)(w-1)Z(s, w)$ and $(s-1)(w-1)Z_{1,\theta}(s, w)$ are absolutely uniformly convergent on compacta on the region

$$\Lambda_0 = \Lambda \cup \left\{ (s, w) \in \mathbb{C}^2 : \operatorname{Re}(s) > 0, \operatorname{Re}(w) > \frac{3}{2} \right\} \cup \left\{ (s, w) \in \mathbb{C}^2 : \operatorname{Re}(s) > \frac{3}{2}, \operatorname{Re}(w) > 0 \right\}.$$



In particular, $Z(s, w)$ and $Z_{1,\theta}(s, w)$ are meromorphic on this region with at most polar lines at $s = 1$ and $w = 1$. The advantage of the region Λ_0 over the initial region Λ is that Λ_0 intersects the hyperplanes $s = \frac{1}{2}$ and $w = \frac{1}{2}$ so that the union of the reflections $w\Lambda_0$ for $w \in \Omega$ is connected. The idea now is to reflect Λ_0 via the functional equations and then apply a theorem of Bochner. To state this theorem we need a small definition. We say that a domain $\Omega \subset \mathbb{C}^n$ is a **tube domain** if there is an open set $\omega \subset \mathbb{R}^n$ such that

$$\Omega = \{(s_1, \dots, s_n) \in \mathbb{C}^n : \operatorname{Re}((s_1, \dots, s_n)) \in \omega\}.$$

Tube domains are generalizations of vertical strips in the complex plane. Now we can state the theorem of Bochner (see [2] for a proof):

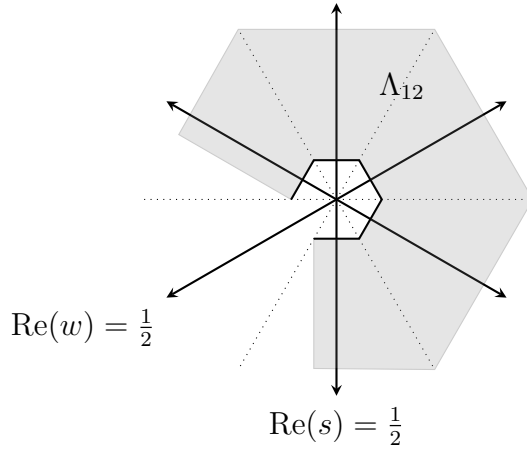
Theorem 6.1 (Bochner's continuation theorem). *If Ω is a connected tube domain, then any holomorphic function on Ω can be extended to a holomorphic function on the convex hull $\widehat{\Omega}$.*

By clearing polar divisors, Bochner's continuation theorem implies that any meromorphic function on a connected tube domain possessing a finite amount of hyperplane polar divisors can be extended to a meromorphic function on the convex hull. This is exactly the situation for $Z(s, w)$. Indeed, it is clear that a union of tube domains is a tube domain and so, in particular, Λ_0 is a tube domain. But on Λ_0 there are at most polar lines at $s = 1$ and $w = 1$. Reflecting these hyperplanes via W we obtain the finite set of possible polar divisors:

$$\left\{ s = 1, w = 1, s = 0, w = 0, s + w = \frac{1}{2}, s + w = \frac{3}{2} \right\}.$$

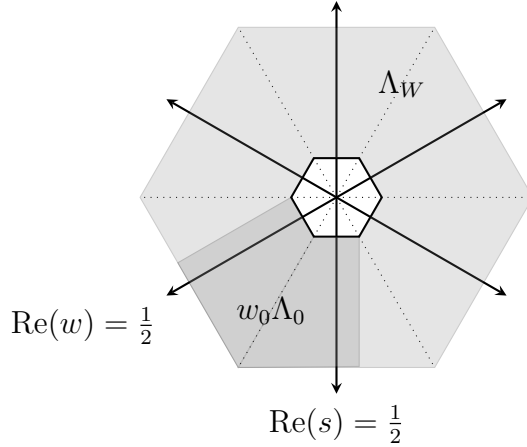
Therefore we are reduced to extending $Z(s, w)$ meromorphically. By applying the functional equations corresponding to σ_1 , σ_2 , and $\sigma_1\sigma_2$, $Z(s, w)$ admits meromorphic continuation to the region

$$\Lambda_{12} = \Lambda_0 \cup \sigma_1\Lambda_0 \cup \sigma_2\Lambda_0 \cup \sigma_1\sigma_2\Lambda_0.$$



Because the functional equation for $Z(s, w)$ involves both $Z(s, w)$ and $Z_{1,\psi}(s, w)$, it was necessarily to analytically continue both of these double Dirichlet series to Λ_0 before we applied the functional equation. Now Λ_{12} is a connected tube domain whose convex hull is \mathbb{C}^2 . So by applying Bochner's continuation theorem (or rather our comment for meromorphic functions) we see that $Z(s, w)$ admits meromorphic continuation to the (s, w) -plane with at most a finite set of polar divisors. This method is better than repeatedly applying the functional equations corresponding to every $w \in W$. Indeed, if we did we would obtain meromorphic continuation to the region

$$\Lambda_W = \bigcup_{w \in W} w\Lambda_0.$$



There are two issues here. The first is that $Z(s, w)$ has two meromorphic continuations to the region $w_0\Lambda_0$ given by the functional equations corresponding to $w_0 = \sigma_1\sigma_2\sigma_1$ and $w_0 = \sigma_2\sigma_1\sigma_2$ and we would need to show that these agree. The second is that we have not obtained meromorphic continuation to $\mathbb{C}^2 - \Lambda_W$ which is a compact hexagon about the origin. By using Bochner's theorem after meromorphically continuing to Λ_{12} , we have avoided these issues and as a consequence shown that the meromorphic continuations given by $w_0 = \sigma_1\sigma_2\sigma_1$ and $w_0 = \sigma_2\sigma_1\sigma_2$ agree.

7. POLES AND RESIDUES

We will inspect the polar divisors of $Z(s, w)$ more carefully. It turns out that the set of polar divisors is smaller than

$$\left\{ s = 1, w = 1, s = 0, w = 0, s + w = \frac{1}{2}, s + w = \frac{3}{2} \right\}.$$

Indeed, there are no poles on the hyperplanes $s = 0$, $w = 0$, and $s + w = \frac{3}{2}$. To see this, first note that by our earlier application of the Phragmén-Lindelöf convexity principal we actually obtained continuation

to an open set containing Λ_0 (because our estimates held for $\operatorname{Re}(s) > -\varepsilon$ and $\operatorname{Re}(w) > -\varepsilon$). We did not need this larger region for the meromorphic continuation but we do need it to study the poles. Now consider the possible polar divisor $s = 0$. We know $(s-1)(w-1)Z(s, w)$ and $(s-1)(w-1)Z_{1,\theta}(s, w)$ are holomorphic on an open set containing Λ_0 which contains half of the hyperplane defined by $s = 0$ outside of the hexagon $\mathbb{C}^2 - \Lambda_W$. As $(s-1)(w-1)$ is holomorphic on this region it follows that $Z(s, w)$ and $Z_{1,\theta}(s, w)$ do not have a polar divisor at $s = 0$ on an open set containing Λ_0 . Now note that an open set containing $\sigma_1\sigma_2\Lambda_0$ contains the other half of the hyperplane defined by $s = 0$ outside of the hexagon $\mathbb{C}^2 - \Lambda_W$. Upon applying the functional equation corresponding to $\sigma_1\sigma_2$, Theorem 5.2 implies that the gamma factors in the corresponding functional equation have a simple pole when $s + w = \frac{3}{2}$ (the gamma factors in the functional equation for σ_1 have a simple pole at $s = 1$ and $s - 1 \rightarrow s + w - \frac{3}{2}$ under σ_2). Therefore $Z(s, w)$ and $Z_{1,\theta}(s, w)$ do not have polar divisors at $s = 0$ on an open set containing $\sigma_1\sigma_2\Lambda_0$ away from $s + w = \frac{3}{2}$. In particular, $Z(s, w)$ does not have a polar divisor at $s = 0$ on Λ_W and away from the other polar divisors. By Bochner's continuation theorem (after clearing all of the other possible polar divisors), we see that $Z(s, w)$ does not have a polar divisors at $s = 0$ on all of \mathbb{C}^2 and away from the other polar divisors. An identical argument holds for the case $w = 0$ with the regions Λ_0 and $\sigma_2\sigma_1\Lambda_0$. For the polar divisor $s + w = \frac{1}{2}$, we argue in the same way with the regions $\sigma_2\sigma_1\Lambda_0$, $\sigma_1\sigma_2\Lambda_0$, and $w_0\Lambda_0$. The only difference is that for these regions the gamma factors in the corresponding functional equations are different. For the first two regions $\sigma_2\sigma_1\Lambda_0$ and $\sigma_1\sigma_2\Lambda_0$ the gamma factors have a simple pole when $s + w = \frac{3}{2}$. For the third region $w_0\Lambda_0$ the gamma factors have simple poles at $s = 1$ and $w = 1$ which is seen by using both representations $w_0 = \sigma_1\sigma_2\sigma_1$ and $w_0 = \sigma_2\sigma_1\sigma_2$. So in conclusion, there are no poles on the hyperplanes $s = 0$, $w = 0$, and $s + w = \frac{1}{2}$ and away from the other polar divisors. As for the hyperplanes $s = 1$, $w = 1$, and $s + w = \frac{3}{2}$, there are clearly genuine poles for $s = 1$ and $w = 1$ coming from $L(s, \chi_{f_0})$ and $L(w, \chi_{g_0})$ when f and g are perfect squares (so that $f_0 = g_0 = 1$). For $s + w = \frac{3}{2}$, we have a pole coming from the gamma factors corresponding to the functional equations for $\sigma_2\sigma_1$ and $\sigma_1\sigma_2$. We collect all of our work as a theorem:

Theorem 7.1. *$Z(s, w)$ admits meromorphic continuation to \mathbb{C}^2 with polar divisors $s = 1$, $w = 1$, and $s + w = \frac{3}{2}$.*

We can now look at the residue of $Z(s, w)$ at these poles. Since all of the poles are obtained from each other by applying the functional equations of $Z(s, w)$, we begin by looking at the pole at $w = 1$. To compute the residue we use the representation

$$Z(s, w) = \sum_{g \text{ monic}} \frac{L(w, \chi_{g_0}) Q_{g_0 g_1^2}(w)}{|g|^s},$$

coming from the interchange. For a fixed g , the numerator $L(w, \chi_{g_0}) Q_{g_0 g_1^2}(w)$ in the summand corresponding to g has a pole at $w = 1$ if and only if g_0 is square-free, that is $g_0 = 1$, or equivalently $g = g_1^2$ itself is a perfect square. In this case, $L(w, \chi_{g_0}) = \zeta(w)$ so that

$$\operatorname{Res}_{w=1} L(w, \chi_{g_0}) Q_{g_0 g_1^2}(w) = \frac{1}{\log(q)} Q_{g_1^2}(1).$$

But from Lemma 4.2 and Theorem 4.1 we see that $Q_{g_1^2}(1) = 1$, and so

$$\operatorname{Res}_{w=1} Z(s, w) = \frac{1}{\log(q)} \sum_{g \text{ monic perfect square}} \frac{Q_{g_1^2}(1)}{|g|^s} = \frac{1}{\log(q)} \sum_{g \text{ monic}} \frac{1}{|g|^{2s}} = \frac{1}{\log(q)} \zeta(2s).$$

Notice that this expression has a simple pole at $s = \frac{1}{2}$ which is exactly when the polar lines $w = 1$ and $s + w = \frac{3}{2}$ intersect. The residue of $Z(s, w)$ at $s = 1$ is computed in the same way. Indeed, by applying

the interchange, the exact same argument holds with the roles of s and w interchanged so that

$$\operatorname{Res}_{s=1} Z(s, w) = \frac{1}{\log(q)} \zeta(2w).$$

Again, this expression has a simple pole at $w = \frac{1}{2}$ which is when the polar lines $s = 1$ and $s + w = \frac{3}{2}$ intersect. The other residues of the simple poles can be computed by apply the functional equations for $Z(s, w)$ and using the residues at $s = 1$ and $w = 1$. Now consider the point where the polar lines $w = 1$ and $s + w = \frac{3}{2}$ intersect. Setting $s = \frac{1}{2}$, we see that $Z(\frac{1}{2}, w)$ has a pole at $w = 1$ and we would like to understand this pole better. To accomplish this, the Mittag-Leffler theorem applied to $Z(s, w)$ (in w) implies that

$$Z(s, w) = \frac{R_1(s)}{w - 1} + \frac{R_2(s)}{s + w - \frac{3}{2}} + Y(s, w),$$

in some neighborhood of $(\frac{1}{2}, 1)$, where $Y(s, w)$ is holomorphic, and we have set

$$R_1(s) = \operatorname{Res}_{w=1} Z(s, w) \quad \text{and} \quad R_2(s) = \operatorname{Res}_{w=\frac{3}{2}-s} Z(s, w).$$

From our residue computations, $R_1(s) = \frac{1}{\log(q)} \zeta(2s)$ which implies that it has a simple pole at $s = \frac{1}{2}$. The residue is given by $A = \frac{1}{2\log(q)}$. On the other hand, $Z(\frac{1}{2}, w)$ is holomorphic for $\operatorname{Re}(w) > 1$. These two facts together imply that $R_2(s)$ must have a simple pole at $s = \frac{1}{2}$ which cancels the simple pole coming from $R_1(s)$. So by Mittag-Leffler again, we may write

$$R_1(s) = \frac{A}{s - \frac{1}{2}} + R_3(s) \quad \text{and} \quad R_2(s) = -\frac{A}{s - \frac{1}{2}} + R_4(s),$$

in a neighborhood of $s = \frac{1}{2}$ and where $R_3(s)$ and $R_4(s)$ are holomorphic. It follows that

$$\begin{aligned} Z(s, w) &= \frac{R_1(s)}{w - 1} + \frac{R_2(s)}{s + w - \frac{3}{2}} + Y(s, w) \\ &= \frac{A}{(w - 1)(s - \frac{1}{2})} + \frac{R_3(s)}{w - 1} - \frac{A}{(s + w - \frac{3}{2})(s - \frac{1}{2})} + \frac{R_4(s)}{s + w - \frac{3}{2}} + Y(s, w) \\ &= \frac{A}{(w - 1)(s + w - \frac{3}{2})} + \frac{R_3(s)}{w - 1} + \frac{R_4(s)}{s + w - \frac{3}{2}} + Y(s, w). \end{aligned}$$

We can now set $s = \frac{1}{2}$ and let $B = R_3(\frac{1}{2}) + R_4(\frac{1}{2})$ so that

$$Z\left(\frac{1}{2}, w\right) = \frac{A}{(w - 1)^2} + \frac{B}{w - 1} + O(1).$$

It follows that $Z(\frac{1}{2}, w)$ has a double pole at $w = 1$. This can be thought of as follows: the polar lines $w = 1$ and $s + w = \frac{3}{2}$ correspond to simple poles of $Z(s, w)$ except in the case when they intersect and the order of the poles combine to give $Z(\frac{1}{2}, w)$ a double pole at $w = 1$. Applying the interchange, the exact same argument holds to show that $Z(s, \frac{1}{2})$ has a double pole at $s = 1$.

$Z(s, w)$ AS A RATIONAL FUNCTION

Recall that $L(s, \chi_f)$ is a polynomial in q^{-s} of degree at most $\deg(f) - 1$. A similar situation happens for the double Dirichlet series $Z(s, w)$, it will be a rational function in the variables $x = q^{-s}$ and $y = q^{-w}$. Since this property is a special case of Dirichlet series over function fields, we present the argument but

supress the more detailed computations. Before we begin, we recall some properties of Hadamard products of power series. The details can be found in [4]. For any two power series

$$R_1(x, y) = \sum_{n, m \geq 0} r_1(n, m) x^n y^m \quad \text{and} \quad R_2(x, y) = \sum_{n, m \geq 0} r_2(n, m) x^n y^m,$$

or more generally generating series, their **Hadamard product** $(R_1 * R_2)(x, y)$ is defined by

$$(R_1 * R_2)(x, y) = \sum_{n, m \geq 0} r_1(n, m) r_2(n, m) x^n y^m.$$

If we assume $R_1(x, y)$ and $R_2(x, y)$ are regular around the origin $x = y = 0$, then the Hadamard product can be expressed as two contour integrals around the origin:

$$(R_1 * R_2)(x, y) = \frac{1}{(2\pi i)^2} \int_{|z|=\rho} \int_{|w|=\rho} R_1(z, w) R_2\left(\frac{x}{z}, \frac{y}{w}\right) \frac{dz}{z} \frac{dw}{w},$$

for sufficiently small $\rho > 0$. By the residue theorem,

$$(R_1 * R_2)(x, y) = \sum_{s=s(x, y)} \text{Res}_s \left(R_1(z, w) R_2\left(\frac{x}{z}, \frac{y}{w}\right) \frac{1}{zw} \right),$$

where the sum is over all poles $s = s(x, y)$ of the integrand such that $\lim_{x, y \rightarrow 0} s(x, y) = 0$. This formula can be used to compute the Hadamard product of rational functions and in the following arguemnt where computing a Hadamard product is necessary, this method can be used.

Now we show that $Z(s, w)$ is a rational function in $x = q^{-s}$ and $y = q^{-w}$. Throughout, we work in the region of absolute uniform convergence on compacta for $Z(s, w)$. Consider the representation

$$Z(s, w) = \sum_{f \text{ monic}} \frac{L(s, \widehat{\chi}_{f_0})}{|f|^w} = \sum_{f, g \text{ monic}} \frac{\chi_{f_0}(\widehat{g}) a(f, g)}{|f|^w |g|^s}.$$

Since $L(s, \chi_f)$ is a polynomial in q^{-s} of degree at most $\deg(f) - 1$ and correction polynomials $Q_{f_0 f_1^2}(s)$ are Dirichlet polynomials, one of the two following cases occur:

- If f is a perfect square, then $L(s, \chi_{f_0}) = \zeta(s)$ and

$$L(s, \widehat{\chi}_{f_0}) = \zeta(s) Q_{f_1^2}(s),$$

where $Q_{f_1^2}(s)$ is a polynomial in q^{-s} of degree $\deg(f)$.

- If f is not a perfect square, then

$$L(s, \widehat{\chi}_{f_0}) = L(s, \chi_{f_0}) Q_{f_0 f_1^2}(s),$$

is a polynomial in q^{-s} of degree at most $\deg(f) - 1$.

So if f is not a perfect square, then

$$L(s, \widehat{\chi}_{f_0}) = \sum_{\deg(g) \leq \deg(f) - 1} \frac{\chi_{f_0}(\widehat{g}) a(f, g)}{|g|^s},$$

which is to say that

$$\sum_{\deg(g)=m} \chi_{f_0}(\widehat{g}) a(f, g) = 0,$$

for every $m \geq \deg(f)$ provided f is not a perfect square. To exploit this fact, we will decompose $Z(s, w)$ according to whether $\deg(f) \geq \deg(g)$ or $\deg(f) \leq \deg(g)$. So we write

$$Z(s, w) = Z_0(s, w) + Z_0(w, s) - Z_1(s, w),$$

where

$$Z_0(s, w) = \sum_{0 \leq n \leq m} \frac{1}{q^{nw} q^{ms}} \sum_{\substack{f, g \text{ monic} \\ \deg(f)=n \\ \deg(g)=m}} \chi_{f_0}(\widehat{g}) a(f, g) \quad \text{and} \quad Z_1(s, w) = \sum_{n \geq 0} \frac{1}{q^{nw} q^{ns}} \sum_{\substack{f, g \text{ monic} \\ \deg(f)=n \\ \deg(g)=n}} \chi_{f_0}(\widehat{g}) a(f, g).$$

We will now show that $Z_0(s, w)$ and $Z_1(s, w)$ are rational functions in q^{-s} and q^{-w} via a convolution procedure. For $Z_0(s, w)$, our remarks about $L(s, \widehat{\chi}_{f_0})$ imply the term in the inner sum of $Z_0(s, w)$ vanishes unless f is a perfect square and in this case $\chi_{f_0}(\widehat{g}) = 1$. Therefore

$$Z_0(s, w) = \sum_{0 \leq n \leq m} \frac{1}{q^{nw} q^{ms}} \sum_{\substack{f, g \text{ monic} \\ f \text{ a perfect square} \\ \deg(f)=n \\ \deg(g)=m}} a(f, g).$$

Now consider

$$Y(s, w) = \sum_{\substack{f, g \text{ monic} \\ f \text{ monic perfect square}}} \frac{a(f, g)}{|f|^s |g|^w} \quad \text{and} \quad K_0(s, w) = \sum_{0 \leq n \leq m} \frac{1}{q^{nw} q^{ms}}.$$

We can express both of these as rational functions in q^{-s} and q^{-w} . Indeed, for $Y(s, w)$, Proposition 4.1 we that $Y(s, w)$ possesses a Euler product. Using Corollary 4.1, we compute

$$Y(s, w) = \frac{1 - q^{1-s-2w}}{(1 - q^{1-2w})(1 - q^{1-s})(1 - q^{2-2s-2w})},$$

which can also be seen by comparing coefficients of the power series expansion of the right-hand side. $K_0(s, w)$ is even easier because it is a geometric series:

$$K_0(s, w) = \sum_{0 \leq n \leq m} \frac{1}{q^{nw} q^{ms}} = \frac{1}{(1 - q^{-s})(1 - q^{-s-w})}.$$

Then $Z_0(s, w)$ is expressed as a Hadamard product of power series in q^{-s} and q^{-w} :

$$Z_0(s, w) = Y(s, w) * K_0(s, w).$$

Then using the countour integral representation of the Hadamard product, we compute

$$Z_0(s, w) = \frac{1}{(1 - q^{1-s})(1 - q^{3-2s-2w})}.$$

For $Z_1(s, w)$, our remarks about $L(s, \widehat{\chi}_{f_0})$ similarly imply

$$Z_1(s, w) = \sum_{n \geq 0} \frac{1}{q^{ns} q^{nw}} \sum_{\substack{f, g \text{ monic} \\ f \text{ a perfect square} \\ \deg(f)=n \\ \deg(g)=n}} a(f, g).$$

But then we may repeat the same argument as for $Z_0(s, w)$ with

$$Y(s, w) = \sum_{\substack{f, g \text{ monic} \\ f \text{ monic perfect square}}} \frac{a(f, g)}{|f|^s |g|^w} \quad \text{and} \quad K_1(s, w) = \sum_{n \geq 0} \frac{1}{q^{ns} q^{nw}},$$

and arrive at

$$Z_1(s, w) = \frac{1}{(1 - q^{3-2s-2w})}.$$

Combining these representations for $Z_0(s, w)$ and $Z_1(s, w)$ with our decomposition of $Z(s, w)$ yields

$$Z(s, w) = \frac{1 - q^{2-s-w}}{(1 - q^{1-s})(1 - q^{1-w})(1 - q^{3-2s-2w})}.$$

Setting $x = q^{-s}$ and $y = q^{-w}$ gives

$$Z(x, y) = \frac{1 - q^2xy}{(1 - qx)(1 - qy)(1 - q^3x^2y^2)},$$

which is a rational function in x and y .

Remark 7.1. *Some authors, especially those using the Chinta-Gunnells construction of building multiple Dirichlet series, prefer the W -invariant point to be at the origin. For this, we make the change of variables $(s + \frac{1}{2}, w + \frac{1}{2}) \rightarrow (s, w)$ which gives*

$$Z(s, w) = \frac{1 - q^{1-s-w}}{(1 - q^{\frac{1}{2}-s})(1 - q^{\frac{1}{2}-w})(2 - q^{3-2s-2w})},$$

and upon setting $x = q^{-s}$ and $y = q^{-w}$, we have

$$Z(x, y) = \frac{1 - qxy}{(1 - q^{\frac{1}{2}}x)(1 - q^{\frac{1}{2}}y)(1 - q^2x^2y^2)}.$$

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