COMMENTS ON COEFFICIENTS OF MAASS FORMS AND THE SIEGEL ZERO & AN EFFECTIVE ZERO-FREE REGION

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Abstract. We present a condensed exposition on [1] and its appendix [2]. All of the notation will be as in [1].

1. Main Theorems

Let f be a Maass form that is a newform for $\Gamma_0(N)$, with eigenvalue λ, and character χ, analytically normalized so that $\langle f, f\rangle = 1$ where $\langle \cdot, \cdot \rangle$ is the Petersson inner product. Denote the Fourier coefficients of f by $\rho(n)$ and the Hecke eigenvalues by $a(n)$. The primary aim of [1] is to establish an upper bound for $\rho(1)$ in terms of λ and N. An upper bound for $\rho(1)$ induces an upper bound for $\rho(n)$ because

$$\rho(n) = \pm\rho(-n) \quad \text{and} \quad \rho(n) = a(n)\rho(1),$$

for all $n \geq 1$ since f is a newform. The general stragety to estimate $\rho(1)$ is as follows: consider the Rankin-Selberg convolution

$$L(s, f \times f) = \zeta(2s)\sum_{n\geq 1}\frac{|a(n)|^2}{n^s}.$$

Then $L(s, f\times f)$ admits an Euler product and meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$. The residue is given by

$$\operatorname*{Res}_{s=1} L(s, f \times f) = \frac{2\pi}{3}|\rho(1)|^{-2}. \tag{1}$$

On the other hand, to any newform f, there is an associated GL(3) form F called the **adjoint square lift** with Fourier coefficients $a(m, n)$ and L-function

$$L(s, F) = \sum_{n\geq 1}\frac{a(1, n)}{n^s}.$$

The existance of F and its properties are establish in [3]. Now $\zeta(s)L(s, F)$ admits an Euler product and the local p factors of $\zeta(s)L(s, F)$ and $L(s, f \times f)$ agree provided $p \nmid N$. Therefore

$$L(s, f \times f) = \zeta(s)L_N(s)L(s, F), \tag{2}$$

where $L_N(s)$ is a Dirichlet polynomial supported on the primes dividing N. Moreover, $L(s, F)$ is entire and $L(1, F) \neq 0$. It follows from Equation (2) that

$$\operatorname*{Res}_{s=1} L(s, f \times f) = L_N(1)L(1, F). \tag{3}$$

Now one can deduce from [3] that the growth of $L_N(1)$ is minor in the sense that

$$N^{-\varepsilon} \ll_\varepsilon L_N(1) \ll_\varepsilon N^\varepsilon, \tag{4}$$

for small $\varepsilon > 0$. Combining Equations (1), (3) and (4) yields

$$N^{-\varepsilon}L(1, F)^{-1} \ll_\varepsilon |\rho(1)|^2 \ll_\varepsilon N^\varepsilon L(1, F)^{-1}. \tag{5}$$

We see from Equation (5) that finding effective upper bounds for $|\rho(1)|$ follows from effective lower bounds for $L(s, F)$ at the special value $s = 1$.

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This situation largely mimics the classical class number problem for quadratic number fields of discriminant $D < 0$. Indeed, let $h(D)$ be the ideal class number of $\mathbb{Q}(\sqrt{D})$ and let χ_D be the aossciated quadratic character. Then it is known that

$$L(1, \chi_D) = \frac{2\pi}{\omega_D \sqrt{|D|}} h(D),$$

where ω_D is the number of roots of unity in $\mathbb{Q}(\sqrt{D})$. It follows that estimates for $L(1, \chi_D)$ induce corresponding estimates for $h(D)$. Via Siegel's theorem, estimates for $L(1, \chi_D)$ are intimately related to existance of Siegel zeros for $L(1, \chi_D)$. That is, how close real zeros of $L(s, \chi_D)$ can be to $s = 1$. In our setting, $\rho(1)$ is playing the role of $h(D)$ and $L(s, F)$ is playing the role of $L(s, \chi_D)$. Indeed, [1] shows that lower bounds for $L(1, F)$ are closely related to the existance or non-existance of **Siegel zeros** for $L(s, F)$. That is, real zeros of $L(s, F)$ close to $s = 1$. The main results of [1] are the following:

Theorem 1.1. *Suppose there exists a constant c such that $L(s, F)$ has no real zeros in the range*

$$1 - \frac{c}{\log(\lambda N + 1)} < s < 1.$$

Then there are effective constants c_1 and c_2, depending only on c, such that

$$L(1, F) \geq \frac{c_1}{\log(\lambda N + 1)},$$

and

$$|\rho(1)|^2 \leq c_2 \log(\lambda N + 1).$$

Theorem 1.1 gives an upper bound for $\rho(1)$ in the case that the Siegel zero of $L(s, F)$ does not exist. The following result gives an estimate that is unconditional on the existance of a Siegel zero:

Theorem 1.2. *For any $\varepsilon > 0$, there exsits an effective constant $c(\varepsilon)$ so that the inequality*

$$L(1, F) \geq c(\varepsilon)(\lambda N)^{\varepsilon},$$

holds for all F with at most one exception.

The second statement in Theorem 1.2 shows that unconditionally the existance of Siegel zeros are rare. In particular, Theorem 1.2 implies $L(1, F) \gg (\lambda N)^{\varepsilon}$ with an inneffective constant. Combining with Equation (5) gives the following corollary:

Corollary 1.1. *Let f be a newform for $\Gamma_0(N)$ with eigenvalue λ and analytically normalized so that $\langle f, f \rangle = 1$. Then for any $\varepsilon > 0$,*

$$\rho(1) \ll_{\varepsilon} (\lambda N)^{\varepsilon}.$$

The Appendix

About a year after [1] was circulated, an appendix (see [2]) was written. This occured because through some discussions it became apparent how eliminate the existance of Siegel zeros of $L(s, F)$ for many F. This boils down to an additional factorization of $L(s, F \times F)$. In particular, Theorem 1.1 is true unconditionally as long as f is not a lift from GL(1). Even if these forms are included, the result still holds in the λ-aspect but either the constant must be weakened in the N-aspect. Explicitely:

Theorem 1.3. *Let f be a Maass form that is a newform for $\Gamma_0(N)$, with eigenvalue λ, and character χ, analytically normalized so that $\langle f, f \rangle = 1$. Let $\rho(1)$ denote the first Fourier coefficient of f and let F be the adjoint square lift of f to* GL(3). *The following are true* *(i) If f is not a lift from* $\mathrm{GL}(1)$*, then there exists effective constants c_1 and c_2 such that*

$$L(1,F) \geq \frac{c_1}{\log(\lambda N+1)},$$

and

$$|\rho(1)|^2 \leq c_2 \log(\lambda N+1).$$

(ii) If f is a lift from $\mathrm{GL}(1)$*, then there exists effective constants c_3 and c_4 such that*

$$L(1,F) \geq c_3 \min\left(\frac{1}{\sqrt{N}}, \frac{1}{\log(\lambda N+1)}\right),$$

and

$$|\rho(1)|^2 \leq c_4 \max\left(\sqrt{N}, \log(\lambda N+1)\right).$$

Moreover, $\sqrt{N}$ can be replaced by N^{ε}, for any $\varepsilon > 0$, at the cost of making c_3 and c_4 inneffective depending on ε.

The proof of Theorem 1.3 naturally breaks into two cases (i) and (ii). In the first case, one can eliminate the existance of Siegel zeros for $L(s,F)$. In the second case, if f is a lift from $\mathrm{GL}(1)$, the L-function $L(s,F)$ is divisible by a quadratic Dirichlet L-series which may exhibit a Siegel zero. Hence, $L(s,F)$ may have a Siegel zero induced from one for a quadratic Dirichlet L-function.

Remark 1.1. *In many instances there are no forms f which are lifts from* $\mathrm{GL}(1)$*. For example, on* $\mathrm{PSL}_2(\mathbb{Z})$ *or* $\Gamma_0(N)$ *for prime N and trivial character.*

We being with a lemma:

Lemma 1.1. *Let $\varphi(s)$ be a Dirichlet series with non-negative coefficients and absolutely convergent for* $\mathrm{Re}(s) > 1$*. Also suppose that $\varphi(s)$ admits an Euller product so that $\varphi(s) \neq 0$ for* $\mathrm{Re}(s) > 1$*, and the logarthmic derivative of $\varphi(s)$ is negative for real $s > 1$. Let $\varphi(s)$ have a pole of order m at $s = 1$ and set*

$$\Lambda(s) = s^m(1-s)^m G(s)\varphi(s),$$

is entire of order 1 satisfying the functional equation

$$\Lambda(s) = \Lambda(1-s),$$

where

$$G(s) = D^s \prod_{1\leq i\leq l} \Gamma\left(\frac{s+c_l}{2}\right),$$

for some constants c_l and an integer $D > 1$. Then there exists an effective constant c, depending only on l and m, such that $\varphi(s)$ has at most m real zeros in the range

$$1 - \frac{c}{\log(M)} < s < 1,$$

where $M = 1 + D\max_{1\leq i\leq l}\{|c_i|\}$.

Proof. Since $\Lambda(s)$ is entire and of order 1, it admits the Hadamard factorization

$$\Lambda(s) = e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}},$$

where the product is taken over the zeros ρ of $\Lambda(s)$. Taking the logarithmic derivative of this expression gives

$$\frac{\Lambda'}{\Lambda}(s) = B + \sum_{\rho} \frac{1}{s-\rho} + \frac{1}{\rho},$$

and using the functional equation (exactly as in the case for $\zeta(s)$) we see that

$$B = -\sum_{\rho} \frac{1}{\rho}.$$

On the other hand, the defintion of $\Lambda(s)$ gives

$$\frac{\Lambda'}{\Lambda}(s) = \frac{m}{s} + \frac{m}{s-1} + \frac{G'}{G}(s) + \frac{\varphi'}{\varphi}(s).$$

Combing our work results in the identity

$$\sum_{\rho} \frac{1}{s-\rho} = \frac{m}{s} + \frac{m}{s-1} + \frac{G'}{G}(s) + \frac{\varphi'}{\varphi}(s).$$

By assumption $\frac{\varphi'}{\varphi}(s) < 0$ if s is real and $s > 1$. Moreover, if we write

$$\sum_{\rho} \frac{1}{s-\rho} = \sum_{\rho}^{*} \left(\frac{1}{s-\rho} + \frac{1}{s-\overline{\rho}} \right),$$

where the $*$ indicates that we are summing over ρ and not $\overline{\rho}$, then each term in the latter sum is real and positive. So upon removing $\frac{\varphi'}{\varphi}(s)$ and all of the terms ρ except for those where $\rho = \beta$ for some positive real root $\beta \geq 1 - \frac{c}{\log(M)}$, there exists an effective constant c_1 such that

$$\sum_{\beta} \frac{1}{s-\beta} \leq \frac{m}{s-1} + c_1 \log(M).$$

Setting $s = 1 + \frac{\delta}{\log(M)}$ with $\delta < \frac{1}{c_1}$ and taking c small enough compared to δ, we obtain a contradiction if there are least $m+1$ roots β in the sum. □

Now we can begin the work to prove Theorem 1.3. Similar to the proof of Siegel's theroem, we will prove the claim from estimates of a Dirichlet series $\varphi(s)$ that is a multiple of two L-functions $L(s, F_1)$ and $L(s, F_2)$ corresponding to two Maass forms f_1 and f_2 and has non-negative coefficeints. As presented in [1], this Dirichlet series is

$$\varphi(s) = \zeta(s) L(s, F_1) L(s, F_2) L(s, F_1 \times F_2).$$

Analgous to Siegel's theorem, $L(s, F_1)$ plays the role of a quadratic Dirichlet L-series with a possible Siegel zero, $L(s, F_2)$ plays the role of quadratic Dirichlet L-series with another character, and $L(s, F_1 \times F_2)$ plays the role of the Dirichlet L-series formed with the product of the two characters. It was proved in [1] that if $F_1 \neq F_2$, then $\varphi(s)$ has non-negative coefficients, admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$, posses a functional equation of shape $s \to 1 - s$, and has polynomial growth in bounded vertical strips. By Lemma 1.1 it follows that $\varphi(s)$ has at most a single real zero close to 1.

In [2], they take $F_1 = F_2 = F$ so that

$$\varphi(s) = \zeta(s) L(s, F)^2 L(s, F \times F).$$

It turns out that $\varphi(s)$ retains all of the properties in the generic case when $F_1 \neq F_2$ except for the order of the pole. To compute the order of the pole, it follows from inspecting local factors that $L(s, F \times F)$ decomposes as

$$L(s, F \times F) = L(s, F) L(s, F, \vee^2),$$

where $L(s, F, \vee^2)$ is the symmetric square L-function of F. Therefore,

$$\varphi(s) = \zeta(s) L(s, F)^3 L(s, F, \vee^2).$$

The remainder of the argument breaks into cases depending on if f is a lift from GL(1) or not:

(i) Suppose f is not a lift from GL(1). It was shown in [4] that $L(s, F, \vee^2)$ has a simple pole at $s = 1$ and is analytic elsewhere. As $L(1, F) \neq 0$, it follows that $\varphi(s)$ has a double pole at $s = 1$. Moreover, any zero of $F(s, F)$ will be a zero of order at least 3 for $\varphi(s)$. Applying Lemma 1.1 (where $m = 2$ and $M = \lambda N + 1$) it follows that $\varphi(s)$ cannot have a triple zero within $\frac{c}{\log(M)}$ of 1 because it only has a double pole. In short, the lemme implies the existance of an effective constant c such that $L(s, F)$ has no real zeros in the interval

$$1 - \frac{c}{\log(\lambda N + 1)} < s < 1.$$

This eliminates the existance of Siegel zeros for such F and the result then follows from Theorem 1.1.

(ii) Now suppose f is a lift from GL(1). This means that $L(s, f) = L(s, \psi, K)$ where K is a quadratic number field and ψ is a Hecke character defined over K. Moreover, $L(s, F)$ factors as

$$L(s, F) = L(s, \psi_K)L(s, \psi^2(\psi^{-1} \circ N_{K/\mathbb{Q}}), K),$$

where $L(s, \psi_K)$ is the quadratic Dirichlet L-function associated to K. It follows that an Siegel zero for $L(s, \psi_K)$ or $L(s, \psi^2(\psi^{-1} \circ N_{K/\mathbb{Q}}), K)$ induces a Siegel zero for $L(s, F)$. Actally, if ψ^2 is not trivial, then $L(s, \psi^2(\psi^{-1} \circ N_{K/\mathbb{Q}}), K)$ cannot have a Siegel zero. In any case, the level of $L(s, F)$ is bounded above by N and then one can prove the result using the effective bound $1 - \beta \gg \frac{1}{\sqrt{N}}$ or the inneffective bound $1 - \beta \gg N^{-\varepsilon}$ coming from Siegel's theorem.

References

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