

# A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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ABSTRACT. We construct a quadratic double Dirichlet series  $Z(s, w)$  built from single variable quadratic Dirichlet  $L$ -functions  $L(s, \chi)$  over  $\mathbb{Q}$ . We prove that  $Z(s, w)$  admits meromorphic continuation to the  $(s, w)$ -plane and satisfies a group of functional equations.

## 1. PRELIMINARIES

We present an overview of quadratic Dirichlet  $L$ -functions over  $\mathbb{Q}$ . We begin with the Riemann zeta-function. The zeta function  $\zeta(s)$  is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m \geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for  $\operatorname{Re}(s) > 1$ . The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Now we recall characters on  $\mathbb{Z}$ . They are multiplicative functions  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ . The two flavors of characters of interest to use are:

- Dirichlet characters: multiplicative functions  $\chi_d : \mathbb{Z} \rightarrow \mathbb{C}$  modulo  $d \geq 1$  (in that they are  $d$ -periodic) and such that  $\chi_d(m) = 0$  if  $(m, d) > 1$ .
- Hilbert symbols: Dirichlet characters modulo 1.

In either case, the image always lands in the roots of unity. If  $\chi$  is a Dirichlet character then its conjugate  $\bar{\chi}$  is also a Dirichlet character. Moreover,  $\bar{\chi}$  is the multiplicative inverse to  $\chi$  and the Dirichlet characters modulo  $m$  form a group under multiplication. This group is always finite and its order is  $\phi(d) = |(\mathbb{Z}/d\mathbb{Z})^*|$ . Dirichlet characters also satisfy orthogonality relations:

**Theorem 1.1** (Orthogonality relations).

(i) For any two Dirichlet characters  $\chi$  and  $\psi$  modulo  $d$ ,

$$\frac{1}{\phi(d)} \sum'_{a \pmod{d}} \chi(a) \bar{\psi}(a) = \delta_{\chi, \psi}.$$

(ii) For any  $a, b \in (\mathbb{Z}/d\mathbb{Z})^*$ ,

$$\frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \chi(a) \bar{\chi}(b) = \delta_{a, b}.$$

The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on  $\mathbb{Z}$ . First let us recall this symbol. For any odd prime  $p$  and any  $m \geq 1$ , we define the quadratic residue

symbol  $\left(\frac{m}{p}\right)$  by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod{p} = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod{p} \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod{p} \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod{p}. \end{cases}$$

This symbol only depends upon  $m$  modulo  $p$  and is multiplicative in  $m$ . We can extend the quadratic residue symbol multiplicatively in the denominator. If  $d = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  is the prime factorization of  $d$ , then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \leq i \leq k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any odd  $d \geq 1$ . We can extend this symbol further and allow  $d \geq 1$  to be even. To this end, we define

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases}$$

and extend  $\left(\frac{m}{d}\right)$  multiplicatively in  $d$  when  $d$  is even. Now the quadratic residue symbol makes sense for any  $m, d \geq 1$ . Moreover, it is multiplicative in both  $m$  and  $d$  but no longer depends upon only  $m$  modulo  $d$  (it also depends upon  $m$  modulo 8). In particular,

$$\left(\frac{-1}{d}\right) = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases}$$

and if  $d \not\equiv 0 \pmod{2}$ , we can compactly write

$$\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}} \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = (-1)^{\frac{d^2-1}{8}} = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}. \end{cases}$$

The quadratic residue symbol also admits the following reciprocity law:

**Theorem 1.2** (Quadratic reciprocity). *If  $d, m \geq 1$  are relatively prime, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2} \frac{m^{(2)}-1}{2}} \left(\frac{m}{d}\right),$$

where  $d^{(2)}$  and  $m^{(2)}$  are the parts of  $d$  and  $m$  relatively prime to 2 respectively.

We can now define the quadratic Dirichlet characters. For any odd square-free  $d \geq 1$ , define the quadratic Dirichlet character  $\chi_d$  by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod{4}, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod{4}. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension  $\mathbb{Q}(\sqrt{d})$ . We extend  $\chi_d$  multiplicatively in the denominator so that  $\chi_d$  makes sense for any odd  $d \geq 1$ . In particular,  $\chi_d(m) = \pm 1$  provided  $d$  and  $m$  are relatively prime and  $\chi_d(m) = 0$  if  $(m, d) > 1$ . Quadratic reciprocity implies that  $\chi_d$  is a Dirichlet character modulo  $d$  if  $d \equiv 1 \pmod{4}$  and is a Dirichlet character modulo  $4d$  if  $d \equiv 2, 3 \pmod{4}$ . Indeed, if  $d \equiv 1 \pmod{4}$  then  $d^{(2)} = d$  and the sign is always 1. If  $d \equiv 3 \pmod{4}$ , then  $d^{(2)} = d$  and the sign is  $\left(\frac{-1}{m}\right)$  which is a character modulo 4. If  $d \equiv 2 \pmod{4}$ , then  $d^{(2)} \equiv 1, 3 \pmod{4}$  and we are reduced to one of the previous two cases. To compactify these statements, we use the Dirichlet characters modulo 8:

Todo: [XXX]

This is a Hilbert symbol as we will now see. The only Hilbert symbols we will need are those given by the quadratic residue symbol. There are only two of them: one nontrivial and one trivial. The nontrivial Hilbert symbol is  $\chi_\theta$  where  $\theta \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$ :

$$\chi_\theta(m) = (-1)^{\deg(m)}.$$

Note that  $\overline{\chi_\theta} = \chi_\theta$ . The other Hilbert symbol is the trivial Dirichlet character  $\chi_\theta^2 = \chi_{\theta\theta} = \chi_1$ . In general, we denote a Hilbert symbol by  $\chi_a$  where  $a \in \{1, \theta\}$ . The Hilbert symbols are necessary because they control the possible sign change in the statement of quadratic reciprocity coming from the  $\text{sgn}(d)^{\deg(m)}$  and  $\text{sgn}(m)^{-\deg(d)}$  factors. Technically speaking, we would also require Hilbert symbols to keep track of the  $(-1)^{\frac{q-1}{2}\deg(d)\deg(m)}$  factor but as we will assume  $q \equiv 1 \pmod{4}$ , we will not discuss this additional difficulty.

Todo: [XXXXXX]

With the Dirichlet characters and Hilbert symbols introduced, we are ready to discuss the  $L$ -functions associated to quadratic Dirichlet characters. We define the  $L$ -function  $L(s, \chi_d)$  attached to  $\chi_d$  by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \text{ monic}} \frac{\chi_d(m)}{|m|^s} = \prod_{P \text{ monic irr}} \left(1 - \frac{\chi_d(P)}{|P|^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character,  $L(s, \chi_d) \ll \zeta(s)$  for  $\text{Re}(s) > 1$  so that  $L(s, \chi_d)$  is locally absolutely uniformly convergent in this region.  $L(s, \chi_d)$  also admits meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  if  $d$  is a perfect square and is analytic otherwise (see [?] for a proof). Moreover,  $L(s, \chi_d)$  is a polynomial in  $q^{-s}$  of degree at most  $\deg(d) - 1$ . The completed  $L$ -function is defined as follows:

$$L^*(s, \chi_d) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_d) & \text{if } \deg(d) \text{ is even,} \\ L(s, \chi_d) & \text{if } \deg(d) \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_d) = \begin{cases} q^{2s-1} |d|^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \deg(d) \text{ is even,} \\ q^{2s-1} (q|d|)^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \deg(d) \text{ is odd.} \end{cases}$$

Note that in the case  $\deg(d)$  is even, the conductor is  $|d|$  and in the case  $\deg(d)$  is odd, the conductor is  $q|d|$ . In other words, the gamma factors depend upon the degree of  $d$ . This will cause a small but important technical issue later when we want to derive functional equations for the quadratic double Dirichlet series.

## THE QUADRATIC DOUBLE DIRICHLET SERIES

### THE INTERCHANGE

### WEIGHTING TERMS

### FUNCTIONAL EQUATIONS

### MEROMORPHIC CONTINUATION

### POLES AND RESIDUES

### REFERENCES

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