Analytic Number Theory

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Contents

The proof we present uses **??** and requires a few different preliminary results. Many of these results are somewhat disconnected, so we will prove them seperately and then prove the prime number theorem. However, we will outline the overall idea. Start with the **Chebychef functions**:

$$\theta(x) = \sum_{p \le x} \log(p) \quad \text{and} \quad \psi(x) = \sum_{n \le x} \Lambda(n),$$

defined for a real x, where $m \ge 1$ is an integer, and where $\Lambda(n)$ is the von Mangoldt function. Since $\frac{\log(p^m)}{\log(p)} = m$ and $\frac{\log(x)}{\log(p)}$ is continuous, for $x > 0$ we may write

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{p^m \le x} \log(p) = \sum_{p \le x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p). \tag{1}$$

This is often a more useful representation. We will first reduce the asymptotis of $\pi(x)$ to that of the Chebychef functions, in particular, $\psi(x)$. We will then show $\psi(x) = O(x)$ which is a weaker statement than the prime number theorem. After, we introduce a technical result that will be needed in the proof of the prime number theorem. Once all of this is done we will be ready to prove the theorem itself. This will be acomplished by relating $\zeta(s)$ to $\psi(x)$ and using the technical theorem to deduce asymptotics for $\psi(x)$ which will complete the proof. Our first result, as we have mentioned, relates the asymptotics of $\pi(x)$, $\theta(x)$, and $\psi(x)$. Actually, it is an equivalence:

Lemma 0.0.1. *The following are equivalent:*

(i) $\pi(x) \sim \frac{x}{\log(x)}$.

(ii) $\theta(x) \sim x$.

(iii) $\psi(x) \sim x$.

Proof. Let $x > 0$. Then

$$\theta(x) = \sum_{p \le x} \log(p) \le \sum_{p \le x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p) \le \sum_{p \le x} \frac{\log(x)}{\log(p)} \log(p) \le \sum_{p \le x} \log(x) = \pi(x)\log(x).$$

This chain of inequalities and Equation (1) together imply

$$\frac{\theta(x)}{x} \le \frac{\psi(x)}{x} \le \frac{x\log(x)}{x}.$$

Therefore we have

$$\lim_{x\to\infty} \frac{\theta(x)}{x} \le \lim_{x\to\infty} \frac{\psi(x)}{x} \le \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x}. \tag{2}$$

Now fix an α with $0 < \alpha < 1$ and let $x > 1$. Then

$$\theta(x) = \sum_{p \le x} \log(p) \ge \sum_{x^\alpha < p \le x} \log(p) \ge \sum_{x^\alpha < p \le x} \alpha \log(x) = \alpha \log(x)(\pi(x) - \pi(x^\alpha)) > \alpha \log(x)(\pi(x) - x^\alpha),$$

where the last inequality follows because $\pi(x) < x$ provided $x > 0$. This chain of inequalities implies

$$\frac{\theta(x)}{x} \ge \alpha \frac{\pi(x)\log(x)}{x} - \alpha x^{\alpha-1}\log(x).$$

Note that $x^{\alpha-1}\log(x) \to 0$ as $x \to \infty$ because $0 < \alpha < 1$. Then

$$\lim_{x\to\infty} \frac{\theta(x)}{x} \ge \alpha \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x},$$

and letting $\alpha \to 1$ we conclude

$$\lim_{x\to\infty} \frac{\theta(x)}{x} \ge \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x}. \tag{3}$$

So Equations (2) and (3) together give

$$\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} \le \lim_{x\to\infty} \frac{\theta(x)}{x} \le \lim_{x\to\infty} \frac{\psi(x)}{x} \le \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x}.$$

This completes the proof. □

We now prove the weaker asymptotic $\psi(x) = O(x)$:

Proposition 0.0.1.

$$\psi(x) = O(x).$$

Proof. Let $m > 1$ be an integer and fix an $x > 0$ such that $2^m < x \le 2^{m+1}$. By Equation (1)

$$\psi(x) = \sum_{p \le x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p).$$

Then by our choice of m,

$$\begin{aligned}
\psi(x) &= \psi(x) + \psi(2^m) - \psi(2^m) \\
&\le \psi(2^m) + \psi(2^{m+1}) - \psi(2^m) \\
&= \sum_{p \le 2^m} \left\lfloor \frac{\log(2^m)}{\log(p)} \right\rfloor \log(p) + \sum_{2^m < p \le 2^{m+1}} \left\lfloor \frac{\log(2^{m+1})}{\log(p)} \right\rfloor \log(p).
\end{aligned} \tag{4}$$

We will now discuss two general estimates and then return to the two sums in Equation (4). For the first estimate, if $n \ge 1$ is an integer and p is a prime such that $n < p \le 2n$, then p divides $\frac{(2n)!}{n!} = n!\binom{2n}{n}$. Since p does not divide $n!$ it must divide $\binom{2n}{n}$ so that

$$\prod_{n < p \le 2n} p \le \binom{2n}{n} < (1+1)^{2n} = 2^{2n},$$

where the last inequality follows by the binomial theorem. In particular,

$$\sum_{n<p\leq 2n} \log(p) = \log\left(\prod_{n<p\leq 2n} p\right) < \log(2^{2n}) = 2n\log(2).$$

Therefore

$$\sum_{p\leq 2^m} \log(p) = \sum_{1\leq k\leq m}\left(\sum_{2^{k-1}<p\leq 2^k} \log(p)\right) < \sum_{1\leq k\leq m} 2^k\log(2) < 2^{m+1}\log(2). \tag{5}$$

For our second estimate, if $p \leq x$ is a prime such that $\left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor > 1$ then $\left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \geq 2$ so that $x \geq p^2$ and hence $\sqrt{x} \geq p$. So

$$\sum_{p\leq\sqrt{x}} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p) \leq \sum_{p\leq\sqrt{x}} \frac{\log(x)}{\log(p)}\log(p) = \log(x)\sum_{p\leq\sqrt{x}} 1 = \pi(\sqrt{x})\log(x). \tag{6}$$

Returning to the first of our two sums in Equation (4) and recalling that $2^m < x \leq 2^{m+1}$, Equations (5) and (6) imply

$$\begin{aligned}
\sum_{p\leq 2^m} \left\lfloor \frac{\log(2^m)}{\log(p)} \right\rfloor \log(p) &= \sum_{p\leq\sqrt{2^m}} \left\lfloor \frac{\log(2^m)}{\log(p)} \right\rfloor \log(p) + \sum_{\sqrt{2^m}<p\leq 2^m} \log(p) \\
&\leq \sum_{p\leq\sqrt{2^m}} \left\lfloor \frac{\log(2^m)}{\log(p)} \right\rfloor \log(p) + \sum_{p\leq 2^m} \log(p) \qquad (7)\\
&< \pi(\sqrt{2^m})\log(2^m) + 2^{m+1}\log(2) \\
&= \pi(\sqrt{x})\log(x) + 2^{m+1}\log(2).
\end{aligned}$$

As for the second sum in Equation (4), $p > 2^m$ implies $p > \sqrt{2^{m+1}}$ because $m > 1$. Therefore $\left\lfloor \frac{\log(2^{m+1})}{\log(p)} \right\rfloor = 1$ so from Equation (5)

$$\sum_{2^m<p\leq 2^{m+1}} \left\lfloor \frac{\log(2^{m+1})}{\log(p)} \right\rfloor \log(p) = \sum_{2^m<p\leq 2^{m+1}} \log(p) < 2^{m+1}\log(2). \tag{8}$$

Altogether, Equations (4), (7) and (8) give the first inequality in the following chain:

$$\begin{aligned}
\psi(x) &< \pi(\sqrt{x})\log(x) + 2^{m+1}\log(2) + 2^{m+1}\log(2) \\
&= \pi(\sqrt{x})\log(x) + 4(2^m)\log(2) \\
&< \pi(\sqrt{x})\log(x) + 4x\log(2) \\
&< \sqrt{x}\log(x) + 4x\log(2) \\
&= \left(\frac{1}{\sqrt{x}}\log(x) + 4\log(2)\right)x.
\end{aligned}$$

Since $\frac{1}{\sqrt{x}}\log(x) \to 0$ as $x \to \infty$, there is a positive M such that $\left|\frac{1}{\sqrt{x}}\log(x)\right| < M$ for all $x \geq 0$. Hence

$$\psi(x) < (M + 4\log(2))x,$$

for all $x \geq 0$. But this is to say that $\psi(x) = O(x)$. □

We now discuss our technical result. A **Tauberian theorem** is a theorem which gives conditions for when a series or integral converges at some part of the boundary of its domain of definition. Our technical theorem is of this kind and is due to Newman (see [**?**] for a proof):

Theorem 0.0.1. *Let $f(x)$ be bounded and locally integrable function on $[1, \infty)$. Moreover, suppose*

$$g(s) = \int_1^\infty f(x) x^{-(s+1)}\, dx,$$

defines an analytic function for $\mathrm{Re}(s) > 0$ and admits analytic continuation to a neighborhood of $\mathrm{Re}(s) = 0$. Then $\int_1^\infty \frac{f(x)}{x}\, dx$ exists and

$$\int_1^\infty \frac{f(x)}{x}\, dx = g(0).$$

Theorem 0.0.1 is interesting because the analytic continuation of $g(s)$ is not necessarily given by its defining integral, but this theorem guarantees that it is at $s = 0$. We need to make one last technical remark. Since $\psi(x)$ is discontinuous when x is a prime power, we need to work with a slightly modified version in order to apply the Mellin inversion formula. Define $\psi_0(x)$ by

$$\psi_0(x) = \begin{cases} \psi(x) & \text{if } x \text{ is not a prime power,} \\ \psi(x) - \frac{1}{2}\Lambda(x) & \text{if } x \text{ is a prime power.} \end{cases}$$

Equivalently, $\psi_0(x)$ is $\psi(x)$ except that its value is halfway between the limit values when x is a prime power. Stated another way, if x is a prime power the last term in the sum for $\psi_0(x)$ is multiplied by $\frac{1}{2}$. We are now ready to prove the prime number theorem