A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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Abstract. We construct a quadratic double Dirichlet series $Z(s,w)$ built from single variable quadratic Dirichlet L-functions $L(s,\chi)$ over $\mathbb{Q}$. We prove that $Z(s,w)$ admits meromorphic continuation to the (s,w)-plane and satisfies a group of functional equations.

1. Preliminaries

We present an overview of quadratic Dirichlet L-functions over $\mathbb{Q}$. We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m\geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Now we recall characters on $\mathbb{Z}$. They are multiplicative functions $\chi : \mathbb{Z} \to \mathbb{C}$ and form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions $\chi_d : \mathbb{Z} \to \mathbb{C}$ modulo $d \geq 1$ (in that they are d-periodic) and such that $\chi_d(m) = 0$ if $(m, d) > 1$.
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. Moreover, $\overline{\chi}$ is the multiplicative inverse to χ and the Dirichlet characters modulo d form a subgroup under multiplication. This group is always finite and its order is $\phi(d) = |(\mathbb{F}_q[t]/d\mathbb{F}_q[t])^\times|$. The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{Z}$. First let us recall this symbol. For any odd prime p and any $d \in \mathbb{Z}$, we define the quadratic residue symbol $\left(\frac{d}{p}\right)$ by

$$\left(\frac{d}{p}\right) \equiv d^{\frac{p-1}{2}} \pmod{p} = \begin{cases} 1 & \text{if } x^2 \equiv d \pmod{p} \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv d \pmod{p} \text{ is not solvable,} \\ 0 & \text{if } d \equiv 0 \pmod{p}. \end{cases}$$

This symbol only depends upon d modulo p and is multiplicative in d. We can extend the quadratic residue symbol multiplicatively in the denominator. First we define

$$\left(\frac{d}{-1}\right) = \begin{cases} 1 & \text{if } d \geq 0, \\ -1 & \text{if } d < 0, \end{cases} \quad \text{and} \quad \left(\frac{d}{2}\right) = \begin{cases} 1 & \text{if } d \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } d \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } d \equiv 0 \pmod{2}. \end{cases}$$

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If $m = up_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ is the prime factorization of m (with $u = \pm 1$), then we define

$$\left(\frac{d}{m}\right) = \left(\frac{d}{u}\right)\prod_{1\le i\le k}\left(\frac{d}{p_i}\right)^{e_i}.$$

The quadratic residue symbol now makes sense for any $m \in \mathbb{Z}$ and is multiplicative in both d and m. The quadratic residue symbol also admits the following reciprocity law:

Theorem 1.1 (Quadratic reciprocity)**.** *If $d, m \in \mathbb{Z}$, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2}\frac{m^{(2)}-1}{2}}\left(\frac{m}{|d|}\right),$$

where $d^{(2)}$ and $m^{(2)}$ are the parts of d and m relatively prime to 2 respectively.

Moreover, we have the additional relations

$$\left(\frac{-1}{m}\right) = (-1)^{\frac{m^{(2)}-1}{2}} \quad \text{and} \quad \left(\frac{2}{m}\right) = (-1)^{\frac{m^2-1}{8}},$$

and if $m \not\equiv 0 \pmod 2$, we can write

$$\left(\frac{-1}{m}\right) = (-1)^{\frac{m-1}{2}} = \begin{cases} 1 & m \equiv 1 \pmod 4, \\ -1 & m \equiv 3 \pmod 4, \end{cases} \quad \text{and} \quad \left(\frac{2}{m}\right) = (-1)^{\frac{m^2-1}{8}} = \begin{cases} 1 & m \equiv 1, 7 \pmod 8, \\ -1 & m \equiv 3, 5 \pmod 8. \end{cases}$$

We can now define the quadratic Dirichlet characters. For any square-free $d \in \mathbb{Z}$, define the quadratic Dirichlet character χ_d by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod 4, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension $\mathbb{Q}(\sqrt{d})$. We extend χ_d multiplicatively in the denominator so that χ_d makes sense for any odd d. In particular, $\chi_d(m) = \pm 1$ provided d and m are relatively prime and $\chi_d(m) = 0$ if $(m, d) > 1$. Quadratic reciprocity implies that χ_d is a Dirichlet character modulo $|d|$ if $d \equiv 1 \pmod 4$ and is a Dirichlet character modulo $|4d|$ if $d \equiv 2, 3 \pmod 4$. Indeed, if $d \equiv 1 \pmod 4$ then $d^{(2)} = d$ and the sign is always 1. If $d \equiv 3 \pmod 4$, then $d^{(2)} = d$ and the sign is $\left(\frac{-1}{m}\right)$ which is a character modulo 4. If $d \equiv 2 \pmod 4$, then $d^{(2)} \equiv 1, 3 \pmod 4$ and we are reduced to one of the previous two cases. We will also set

$$q(d) = \begin{cases} |d| & \text{if } d \equiv 1 \pmod 4, \\ |4d| & \text{if } d \equiv 2, 3 \pmod 4, \end{cases} \quad \text{and} \quad \varepsilon_{\chi_d} = \frac{\tau(\chi_d)}{\sqrt{q(d)}} = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ \text{Todo : [xxx]} & \end{cases},$$

where $\tau(\chi_d)$ is the Gauss sum attached to χ_d. We will also require an associated character. For each χ_m (here we are purposely interchanging the roles of d and m to keep consistency with the notation when discussing the quadratic double Dirichlet series later), we define $\widetilde{\chi}_m$ by

$$\widetilde{\chi}_m(d) = (-1)^{\frac{m^{(2)}-1}{2}\frac{d^{(2)}-1}{2}}\chi_m(d).$$

By quadratic reciprocity, $\widetilde{\chi}_m$ is a quadratic Dirichlet character of the same modulus as χ_m and is multiplicative in m. We now discuss the Hilbert characters. We will only need four of them: the quadratic

Dirichlet characters modulo 8. They are given as follows:

$$\chi_1(m)=\begin{cases}1 & \text{if } m\not\equiv 0 \pmod 2,\\ 0 & \text{if } m\equiv 0 \pmod 2,\end{cases}\qquad \chi_{-1}(m)=\begin{cases}1 & \text{if } m\equiv 1 \pmod 4,\\ -1 & \text{if } m\equiv 3 \pmod 4,\\ 0 & \text{if } m\equiv 0 \pmod 2,\end{cases}$$

$$\chi_2(m)=\begin{cases}1 & \text{if } m\equiv 1,7 \pmod 8,\\ -1 & \text{if } m\equiv 3,5 \pmod 8,\\ 0 & \text{if } m\equiv 0 \pmod 2,\end{cases}\qquad \chi_{-2}(m)=\begin{cases}1 & \text{if } m\equiv 1,3 \pmod 8,\\ -1 & \text{if } m\equiv 5,7 \pmod 8,\\ 0 & \text{if } m\equiv 0 \pmod 2.\end{cases}$$

In general, we will denote a Hilbert character by χ_a with $a\in\{\pm1,\pm2\}$. The Hilbert characters also satisfy an important orthogonality property:

Theorem 1.2 (Orthogonality of Hilbert characters)**.** *If $d,m\in\mathbb{Z}$ are odd, then*

$$\frac{1}{4}\sum_{a\in\{\pm1,\pm2\}}\chi_a(dm)=\begin{cases}1 & \text{if } d\equiv m \pmod 8,\\ 0 & \text{if } d\not\equiv m \pmod 8.\end{cases}$$

Also, we have the identities

$$\widetilde{\chi}_a(m)=\chi_a(m),\quad \chi_{-1}(m)=\left(\frac{-1}{m}\right),\quad\text{and}\quad \chi_2(m)=\left(\frac{2}{m}\right),$$

and the relations

$$\chi_{-2}(m)=\chi_{-1}(m)\chi_2(m),\quad \chi_1(m)=\chi_{-1}(m)\chi_{-1}(m),\quad\text{and}\quad \chi_{-1}(m)=\chi_2(m)\chi_{-2}(m).$$

We now return to χ_d for square-free d. If $d\equiv 1,2,5 \pmod 8$, then $d^{(2)}\equiv 1 \pmod 4$ so that the sign in the statement of quadratic reciprocity is 1. If $d\equiv 3,6,7 \pmod 8$, then $d^{(2)}\equiv 3 \pmod 4$ and the sign is $(-1)^{\frac{m^{(2)}-1}{2}}$. This fact together with the relations for the quadratic characters modulo 8 imply

$$\chi_d(m)=\begin{cases}\chi_m(d) & \text{if } d\equiv 1 \pmod 4,\\ \chi_{-1}(m)\chi_m(d) & \text{if } d\equiv 3 \pmod 4,\\ \chi_2(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d\equiv 2 \pmod 8,\\ \chi_{-2}(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d\equiv 6 \pmod 8.\end{cases}$$

With the Dirichlet and Hilbert characters introduced, we are ready to discuss the L-functions associated to quadratic Dirichlet characters. We define the L-function $L(s,\chi_d)$ attached to χ_d for square-free d, by a Dirichlet series or Euler product:

$$L(s,\chi_d)=\sum_{m\ge1}\frac{\chi_d(m)}{m^s}=\prod_{p\text{ prime}}\left(1-\frac{\chi_d(p)}{p^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character, $L(s,\chi_d)\ll\zeta(s)$ for $\mathrm{Re}(s)>1$ so that $L(s,\chi_d)$ is locally absolutely uniformly convergent in this region. $L(s,\chi_d)$ also admits analytic continuation to $\mathbb{C}$. The completed L-function $L^*(s,\chi_d)$ is defined as

$$L^*(s,\chi_d)=\begin{cases}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi_d) & \text{if } d>0,\\ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi_d) & \text{if } d<0,\end{cases}$$

and satisfies the functional equation

$$L^*(s,\chi_d)=\begin{cases}\varepsilon_{\chi_d}q(d)^{\frac{1}{2}-s}L^*(1-s,\chi_d) & \text{if } d>0,\\ -\varepsilon_{\chi_d}q(d)^{\frac{1}{2}-s}L^*(1-s,\chi_d) & \text{if } d<0.\end{cases}$$

Note that the gamma factors depend upon the partiy of χ_d. This the root cause of an important technical issue later when deriving functional equations for the quadratic double Dirichlet series. Analogously, the Dirichlet L-function $L(w,\widetilde{\chi}_m)$ attached to $\widetilde{\chi}_m$ for square-free m is defined by a Dirichlet series or Euler product:

$$L(w,\widetilde{\chi}_m)=\sum_{d\geq 1}\frac{\widetilde{\chi}_m(d)}{d^w}=\prod_{p \text{ prime}}\left(1-\frac{\widetilde{\chi}_m(p)}{p^w}\right)^{-1}.$$

As for $L(s,\chi_d)$, $L(w,\widetilde{\chi}_m)\ll\zeta(w)$ for $\mathrm{Re}(w)>1$ so that $L(w,\widetilde{\chi}_m)$ is locally absolutely uniformly convergent in this region. Moreover, $L(w,\widetilde{\chi}_m)$ admits analytic continuation to $\mathbb{C}$ and the completed L-function $L^*(w,\widetilde{\chi}_m)$ is defined as

$$L^*(w,\widetilde{\chi}_m)=\begin{cases}\pi^{-\frac{w}{2}}\Gamma\left(\frac{w}{2}\right)L(w,\widetilde{\chi}_m) & \text{if } m\equiv 1,2,5 \pmod 8,\\ \pi^{-\frac{w}{2}}\Gamma\left(\frac{w+1}{2}\right)L(w,\widetilde{\chi}_m) & \text{if } m\equiv 3,6,7 \pmod 8,\end{cases}$$

and satisfies the functional equation

$$L^*(w,\widetilde{\chi}_m)=\begin{cases}\varepsilon_{\widetilde{\chi}_m}q(m)^{\frac{1}{2}-w}L^*(1-w,\widetilde{\chi}_m) & \text{if } m\equiv 1,2,5 \pmod 8,\\ -\varepsilon_{\widetilde{\chi}_m}q(m)^{\frac{1}{2}-w}L^*(1-w,\widetilde{\chi}_m) & \text{if } m\equiv 3,6,7 \pmod 8.\end{cases}$$

Remark 1.1. *The definitions for $L(s,\chi_d)$, $L^*(s,\chi_d)$, $L(w,\widetilde{\chi}_m)$, and $L^*(w,\widetilde{\chi}_m)$ work perfectly well even when d and m are not square-free (however the functional equations do not hold). We purposely do not define these L-functions, yet, for d and m not necessarily square-free.*

The Quadratic Double Dirichlet Series

We will now define the quadratic double Dirichlet series $Z(s,w)$. For any integer $d\geq 1$, write $d=d_0d_1^2$ where d_0 is square-free. Equivalently, d_0 is the square-free part of d and $\frac{d}{d_0}$ is a perfect square. The **quadratic double Dirichlet series** $Z(s,w)$ is defined as

$$Z(s,w)=\sum_{d \text{ odd}}\frac{L^{(2)}(s,\chi_{d_0})Q_{d_0d_1^2}(s)}{d^w},$$

where $Q_{d_0d_1^2}(s)$ is the **correction polynomial** defined by

$$Q_{d_0d_1^2}(s)=\sum_{e_1e_2\mid d_1}\mu(e_1)\chi_{d_0}(e_1)e_1^{-s}e_2^{1-s}=\sum_{e_1e_2e_3=d_1}\mu(e_1)\chi_{d_0}(e_1)e_1^{-s}e_2^{1-s},$$

and μ is the usual Möbius function. For $\mathrm{Re}(s)>1$, there is the trivial estimate

$$Q_{d_0d_1^2}(s)\ll\sum_{e_1e_2\mid d_1}1\ll\sigma_0(d_1)^2\ll_\varepsilon d_1^{2\varepsilon}\ll_\varepsilon d^\varepsilon,$$

for any $\varepsilon>0$. As $L(s,\chi_{d_0})\ll 1$ for $\mathrm{Re}(s)>1$, $Z(s,w)$ is locally absolutely uniformly convergent in the region $\Lambda=\{(s,w)\in\mathbb{C}^2:\mathrm{Re}(s)>1,\mathrm{Re}(w)>1\}$. It will also be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters χ_{a_1} and χ_{a_2}. The **quadratic double Dirichlet series** $Z_{a_1,a_2}(s,w)$ twisted by χ_{a_1} and χ_{a_2} is defined as

$$Z_{a_1,a_2}(s,w)=\sum_{d \text{ odd}}\frac{L^{(2)}(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{d^w},$$

where $Q_{d_0d_1^2}(s,\chi_{a_1})$ is the **correction polynomial** twisted by χ_{a_1} defined by

$$Q_{d_0d_1^2}(s,\chi_{a_1})=\sum_{e_1e_2\mid d_1}\mu(e_1)\chi_{a_1d_0}(e_1)e_1^{-s}e_2^{1-2s}=\sum_{e_1e_2e_3=d_1}\mu(e_1)\chi_{a_1d_0}(e_1)e_1^{-s}e_2^{1-2s},$$

and μ is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound $Q_{d_0d_1^2}(s,\chi_{a_1}) \ll d_\varepsilon$ so that $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly in the same region as $Z(s,w)$ does. In particular, $Z(s,w) = Z_{1,1}(s,w)$. We will also require quadratic double Dirichlet series correpsonding to the characters $\widetilde{\chi}_m$. Analogously writing $m = m_0m_1^2$, the **quadratic double Dirichlet series** $\widetilde{Z}(s,w)$ is defined as

$$\widetilde{Z}(w,s) = \sum_{m \text{ odd}} \frac{L^{(2)}(w,\widetilde{\chi}_{m_0})Q_{m_0m_1^2}(w)}{m^s},$$

where $Q_{d_0d_1^2}(w)$ is the **correction polynomial** defined by

$$Q_{m_0m_1^2}(w) = \sum_{e_1e_2|m_1} \mu(e_1)\chi_{m_0}(e_1)e_1^{-w}e_2^{1-w} = \sum_{e_1e_2e_3=m_1} \mu(e_1)\chi_{m_0}(e_1)e_1^{-w}e_2^{1-w},$$

and μ is the usual Möbius function. We have the analogous estimate $Q_{m_0m_1^2}(w) \ll_\varepsilon m^\varepsilon$ and as $L(w,\widetilde{\chi}_{m_0}) \ll 1$ for $\mathrm{Re}(s) > 1$, $\widetilde{Z}(w,s)$ is locally absolutely uniformly convergent in the same region as $Z(s,w)$. We also need to consider twists by a pair of Hilbert characters $\widetilde{\chi}_{a_2}$ and $\widetilde{\chi}_{a_1}$. The **quadratic double Dirichlet series** $\widetilde{Z}_{a_2,a_1}(w,s)$ twisted by $\widetilde{\chi}_{a_2}$ and $\widetilde{\chi}_{a_1}$ is defined as

$$\widetilde{Z}_{a_2,a_1}(w,s) = \sum_{m \text{ odd}} \frac{L^{(2)}(w,\widetilde{\chi}_{a_2}m_0)\widetilde{\chi}_{a_1}(m)Q^2_{m_0m_1}(w,\widetilde{\chi}_{a_2})}{m^s},$$

where $Q_{m_0m_1^2}(w,\widetilde{\chi}_{a_2})$ is the **correction polynomial** twisted by $\widetilde{\chi}_{a_2}$ defined by

$$Q_{m_0m_1^2}(w,\widetilde{\chi}_{a_2}) = \sum_{e_1e_2|m_1} \mu(e_1)\widetilde{\chi}_{a_2m_0}(e_1)e_1^{-w}e_2^{1-2w} = \sum_{e_1e_2e_3=m_1} \mu(e_1)\widetilde{\chi}_{a_2m_0}(e_1)e_1^{-w}e_2^{1-2w}.$$

and μ is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound $Q_{m_0m_1^2}(w,\widetilde{\chi}_{a_2}) \ll_\varepsilon m^\varepsilon$ so that $\widetilde{Z}_{a_2,a_1}(w,s)$ converges locally absolutely uniformly in the same region as $\widetilde{Z}(w,s)$ does. In particular, $\widetilde{Z}(w,s) = \widetilde{Z}_{1,1}(w,s)$.

The Interchange

As defined, $Z_{a_1,a_2}(s,w)$ is a sum of L-functions, and hence Euler products, in s. We will prove an interchange formula for $Z_{a_1,a_2}(s,w)$ which will show that it can be expressed as a sum of L-functions in w. That is, we want the variables s and w to change places. Precisely:

Theorem 1.3 (Interchange)**.** *Wherever $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly,*

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{d^w} = \sum_{m \text{ odd}} \frac{L^{(2)}(w,\widetilde{\chi}_{a_2m_0})\widetilde{\chi}_{a_1}(m)Q_{m_0m_1^2}(w,\widetilde{\chi}_{a_2})}{m^s}.$$

Proof. Only the second equality needs to be proved. To do this, first expand the L-function $L^{(2)}(s,\chi_{a_1d_0})$ and polynomial $Q_{d_0d_1^2}(s,\chi_{a_1})$ to get

$$\begin{aligned} Z(s,w) &= \sum_{d \text{ odd}} \frac{L^{(2)}(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{d^w} \\ &= \sum_{d \text{ odd}} \left(\sum_{m \text{ odd}} \chi_{a_1d_0}(m)m^{-s}\right)\left(\sum_{e_1e_2|d_1} \mu(e_1)\chi_{a_1d_0}(e_1)e_1^{-s}e_2^{1-2s}\right)\chi_{a_2}(d)d^{-w} \\ &= \sum_{m,d \text{ odd}} \sum_{e_1e_2|d_1} \mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(me_1)e_1^{-s}e_2^{1-2s}m^{-s}d^{-w}. \end{aligned}$$

Now $\chi_{a_1 d_0}(me_1) = 0$ unless $(d_0, me_1) = 1$. We make this restriction on the sum giving

$$\sum_{m,\, d \text{ odd}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, me_1)=1}} \mu(e_1)\chi_{a_2}(d)\chi_{a_1 d_0}(me_1) e_1^{-s} e_2^{1-2s} m^{-s} d^{-w}.$$

Making the change of variables $me_1 \to m$ yields

$$\sum_{d \text{ odd}} \sum_{\substack{m \text{ odd} \\ e_1 | m}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, m)=1}} \mu(e_1)\chi_{a_2}(d)\chi_{a_1 d_0}(m) e_2^{1-2s} m^{-s} d^{-w}.$$

For fixed $d = d_0 d_1^2$ and e_2, the subsum over m and e_1 is

$$\sum_{\substack{m \text{ odd} \\ e_1 | m}} \sum_{\substack{e_1 | \frac{d_1}{e_2} \\ (d_0, m)=1}} \mu(e_1)\chi_{a_1 d_0}(m) m^{-s} = \sum_{\substack{m \text{ odd} \\ (d_0, m)=1}} \chi_{a_1 d_0}(m) m^{-s} \left(\sum_{e_1 | \left(\frac{d_1}{e_2}, m\right)} \mu(e_1) \right).$$

The inner sum over e_1 of the Möbius function vanishes unless $\left(\frac{d_1}{e_2}, m\right) = 1$ in which case it is 1. Therefore the triple sum above becomes

$$\sum_{m,\, d \text{ odd}} \sum_{\substack{e_2 | d_1 \\ \left(\frac{d_0 d_1}{e_2}, m\right)=1}} \chi_{a_2}(d)\chi_{a_1 d_0}(m) e_2^{1-2s} m^{-s} d^{-w}.$$

Making the change of variables $d \to de_2^2$, the condition $\left(\frac{d_0 d_1}{e_2}, m\right) = 1$ becomes $(d_0 d_1, m) = 1$ which is equivalent to $(d, m) = 1$. Moreover, $\chi_{a_2}(de_2^2) = \chi_{a_2}(d)$. Altogether, we obtain

$$\sum_{\substack{m,\, d \text{ odd} \\ (d,m)=1}} \sum_{e_2} \chi_{a_2}(d)\chi_{a_1 d_0}(m) e_2^{1-2s-2w} m^{-s} d^{-w}.$$

Writing $m = m_0 m_1^2$ analogously as for d, quadratic reciprocity implies $\chi_{d_0}(m) = \widetilde{\chi}_m(d_0) = \widetilde{\chi}_{m_0}(d)$ where the last equality holds because $(d, m) = 1$ and both d_0 and m_0 differ from d and m respectively by perfect squares. As $\chi_{a_1}(m) = \widetilde{\chi}_{a_1}(m)$ and $\chi_{a_2}(d) = \widetilde{\chi}_{a_2}(d)$, the previous fact implies $\chi_{a_2}(d)\chi_{a_1 d_0}(m) = \widetilde{\chi}_{a_1}(m)\widetilde{\chi}_{a_2 m_0}(d)$ and so our expression becomes

$$\sum_{\substack{m,\, d \text{ odd} \\ (d,m)=1}} \sum_{e_2} \widetilde{\chi}_{a_1}(m)\widetilde{\chi}_{a_2 m_0}(d) e_2^{1-2s-2w} m^{-s} d^{-w}.$$

But now we can reverse the argument with the roles of d, m, χ_{a_1}, and χ_{a_2} interchanged respectively, but with $\widetilde{\chi}_{a_1}$ and $\widetilde{\chi}_{a_2}$, to obtain

$$Z(s, w) = \sum_{m \text{ odd}} \frac{L^{(2)}(w, \widetilde{\chi}_{a_2 m_0})\widetilde{\chi}_{a_1}(m) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})}{m^s}.$$

$\square$

Note that the interchange is not completely symmetric because of the characters $\widetilde{\chi}_{a_2 m_0}$, $\widetilde{\chi}_{a_1}$, and $\widetilde{\chi}_{a_2}$ in the second expression for $Z_{a_1, a_2}(s, w)$. This is due to the fact that reciprocity is not perfect. In even more general settings the correction polynomials in w need not be equal to those in s.

Remark 1.2. *When $a_1 = a_2 = 1$, the interchange implies*

$$Z(s, w) = \widetilde{Z}(w, s).$$

More generally, the interchange implies the following relations for twisted quadratic double Dirichlet series:

$$Z_{a_1,a_2}(s,w) = \widetilde{Z}_{a_2,a_1}(w,s),$$

for $a_1, a_2 \in \{\pm 1, \pm 2\}$.

Weighting Terms

We will now study the coefficients of $Z_{a_1,a_2}(s,w)$ expanded in s and w. Expanding $L^{(2)}(s,\chi_{a_1 d_0}) Q_{d_0 d_1^2}(s,\chi_{a_1})$ in the numerator of $Z_{a_1,a_2}(s,w)$, we can write

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s,\chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s,\chi_{a_1})}{d^w} = \sum_{m,\, d \text{ odd}} \frac{\chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) a(m,d)}{m^s d^w},$$

where $\widehat{m}$ is the part of m relatively prime to d_0 and the **weighting coefficient** $a(m,d)$ is given by

$$a(m,d) = \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2.$$

To see this, the coefficient of $m^{-s} d^{-w}$ in the definition of $Z_{a_1,a_2}(s,w)$ is

$$\begin{aligned}
\chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1}} \mu(e_1)\chi_{a_1 d_0}(e_1 e_3) e_2 &= \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1)\chi_{a_1 d_0}(e_1 e_3) e_2 \\
&= \chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2 \\
&= \chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) a(m,d),
\end{aligned}$$

where the first equality holds because $\chi_{d_0}(e_1 e_3) = 0$ unless $(d_0, e_1 e_3) = 1$ and the second equality holds because if $(d_0, e_1 e_3) = 1$, $\widehat{m}$ differs from $e_1 e_3$ by a perfect square (the divisors of which belong to (d_0, e_2)) and so $\chi_{d_0}(e_1 e_3) = \chi_{d_0}(\widehat{m})$. For completeness, we extend the definition of $a(m,d)$ to all $m, d \geq 1$. In particular, $a(m,d)$ makes sense when m or d may be even.

Remark 1.3. *Also,* $a(m,d) = 0$ *unless* $m = e_1 e_2^2 e_3$ *with* $(d_0, e_1 e_3) = 1$ *and* $e_1 e_2^2 \mid d_1$.

We will define $L(s,\chi_{a_1 d})$ to be the Dirichlet series given by

$$L(s,\chi_{a_1 d}) = L(s,\chi_{a_1 d_0}) Q_{d_0 d_1^2}(s,\chi_{a_1}) = \sum_{m \geq 1} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m,d)}{m^s}.$$

In particular, $L(s,\chi_d)$ now makes sense for d not necessarily square-free and this definition agrees with the former when d is square-free. Moreover, we have the representation

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ odd}} \frac{\chi_{a_2}(d) L^{(2)}(s,\chi_{a_1 d})}{d^w}.$$

If we perform the same procedure but with the interchange, we get

$$\widetilde{Z}_{a_2,a_1}(w,s) = \sum_{m \text{ odd}} \frac{L^{(2)}(w,\widetilde{\chi}_{a_2 m_0})\widetilde{\chi}_{a_1}(m) Q_{m_0 m_1^2}(w,\widetilde{\chi}_{a_2})}{m^s} = \sum_{m,\, d \text{ odd}} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d})\widetilde{\chi}_{a_1}(m) a(d,m)}{m^s d^w},$$

where $\widehat{d}$ is the part of d relatively prime to m_0. Analogously, we define $L(w, \widetilde{\chi}_{a_2 m})$ to be the Dirichlet series given by

$$L(w, \widetilde{\chi}_{a_2 m}) = L(w, \widetilde{\chi}_{a_2 m_0}) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2}) = \sum_{d \geq 1} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d}) a(d, m)}{d^w},$$

so that

$$\widetilde{Z}_{a_2, a_1}(w, s) = \sum_{m \text{ odd}} \frac{\widetilde{\chi}_{a_1}(m) L^{(2)}(w, \widetilde{\chi}_{a_2 m})}{m^s}.$$

We now investigate the structure of the weighting coefficients $a(m, d)$. Their structure controls the majority of the information about both the quadratic double Dirichlet series and the correction polynomials. We first show that the weighting coefficients posess a multiplicativity property:

Proposition 1.1. *We have* $a(m, 1) = a(1, d) = 1$ *and*

$$a(m, d) = \prod_{\substack{p^\alpha || m \\ p^\beta || d}} a(p^\alpha, p^\beta).$$

Proof. From the definition of the weighting coefficients, $a(m, 1) = a(1, d) = 1$. We will prove multiplicativity in m and then in d. Letting $m = m' p^\alpha$, we must show

$$a(m, d) = a(m', d) a(p^\alpha, d).$$

To acomplish this, for $e_1 e_2^2 e_3 = m$, let $e_1 = c_1 d_1$, $e_2 = c_2 d_2$, and $e_3 = c_3 d_3$ with $c_1, c_2, c_3 \mid m'$ and $d_1, d_2, d_3 \mid p^\alpha$. Because $(m', p^\alpha) = 1$, as $e_1 e_2^2 e_3$ runs over decompositions of m, $c_1 c_2^2 c_3$ and $d_1 d_2^2 d_3$ run over decompositions of m' and p^α respectively. Moreover, as $e_1 e_2$ runs over the divisors of d_1 so does $c_1 d_1 c_2 d_2$. These facts combined with multiplicativity of the Möbius function gives

$$\begin{aligned}
a(m, d) &= \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2 \\
&= \sum_{\substack{c_1 c_2^2 c_3 = m' \\ d_1 d_2^2 d_3 = p^\beta \\ c_1 d_1 c_2 d_2 | d_1 \\ (d_0, c_1 d_1 c_3 d_3) = 1}} \mu(c_1)(d_1) |c_2| d_2 \\
&= \left(\sum_{\substack{c_1 c_2^2 c_3 = m' \\ c_1 c_2 | d_1 \\ (d_0, c_1 c_3) = 1}} \mu(c_1) |c_2| \right) \left(\sum_{\substack{d_1 d_2^2 d_3 = p^\alpha \\ d_1 d_2 | d_1 \\ (d_0, d_1 d_3) = 1}} \mu(d_1) d_2 \right) \\
&= a(m', d) a(p^\alpha, d),
\end{aligned}$$

as desired. Now we prove multiplicativity in d. Since we have already proven multiplicativity in m, we may assume $m = p^\alpha$. Letting $d = d' p^\beta$, we must show

$$a(p^\alpha, d) = a(p^\alpha, p^\beta).$$

As $e_1 e_2^2 e_3 = p^\alpha$, the e_i are powers of p for $1 \leq i \leq 3$. It follows that $e_1 e_2 \mid d_1$ is equivalent to $e_1 e_2 \mid p^\beta$. Moreover, $(d_0, e_1 e_2) = 1$ is equivalent to $(1, e_1 e_2) = 1$ or $(p, e_1 e_2) = 1$ depending on of β is even or odd. These facts imply the desired identity. $\square$

The correction polynomials $Q_{d_0d_1^2}(s,\chi_{a_1})$ are tightly connected to the weighting coefficients $a(m,d)$. In particular, $Q_{d_0d_1^2}(s,\chi_{a_1})$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients. We first prove this relationship when d is an odd prime power:

Lemma 1.1. *For any prime p and $\alpha \geq 1$, we have*

$$Q_{p^{2\alpha+1}}(s) = \sum_{k\leq 2\alpha} \frac{a(p^k,p^{2\alpha+1})}{p^{ks}}.$$

Moreover, the same holds for $Q_{p^{2\alpha+1}}(w)$.

Proof. Expanding the correction polynomial in p^{-s} yields

$$Q_{p^{2\alpha+1}}(s) = \sum_{e_1e_2|p^\alpha} \mu(e_1)\chi_p(e_1)e_1^{-s}e_2^{1-2s} = \sum_{k\leq 2\alpha} \frac{b(p^k,p^{2\alpha+1})}{p^{ks}}.$$

where

$$b(p^k,p^{2\alpha+1}) = \sum_{e_1e_2^2=p^k} \mu(e_1)\chi_p(e_1)e_2.$$

The proof will be finished if we can show $b(p^k,p^{2\alpha+1}) = a(p^k,p^{2\alpha+1})$. To see this, first observe $\mu(e_1)\chi_p(e_1) = 0$ unless $e_1 = 1$ in which case it is 1. So $b(p^k,p^{2\alpha+1}) = 0$ if k is odd and $p^{\frac{k}{2}}$ if k is even. Compactly stated,

$$b(p^k,p^{2\alpha+1}) = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

On the other hand, $k \leq \alpha$ so that

$$a(p^k,p^{2\alpha+1}) = \sum_{\substack{e_1e_2^2e_3=p^k \\ e_1e_2|p^\alpha \\ (p,e_1e_3)=1}} \mu(e_1)e_2 = \sum_{\substack{e_1e_2^2|p^k \\ (p,e_1e_3)=1}} \mu(e_1)e_2 = \sum_{e_2^2=p^k} e_2 = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

This finishes the proof. Clearly the same holds for $Q_{p^{2\alpha+1}}(w)$. $\square$

There is an analogous statement when d is an even prime power up to a square-free factor and relatively prime factor:

Lemma 1.2. *For any square-free integer $d_0 \geq 1$, $a_1 \in \{\pm 1, \pm 2\}$, prime p not dividing d_0, and $\beta \geq 1$, we have*

$$Q_{d_0p^{2\beta}}(s,\chi_{a_1}) = (1-\chi_{a_1d_0}(p)p^{-s}) \sum_{k\leq 2\beta} \frac{\chi_{a_1d_0}(p^k)a(p^k,p^{2\beta})}{p^{ks}}.$$

Moreover, the same holds for $Q_{m_0p^{2\beta}}(w,\widetilde{\chi}_{a_2})$.

Proof. Expand the correction polynomial in p^{-s} to get

$$Q_{d_0p^{2\beta}}(s,\chi_{a_1}) = \sum_{e_1e_2|p^\alpha} \mu(e_1)\chi_{a_1d_0}(e_1)e_1^{-s}e_2^{1-2s} = \sum_{k\leq 2\beta} \frac{b(p^k,p^{2\beta})}{p^{ks}}.$$

where

$$b(p^k,p^{2\beta}) = \sum_{e_1e_2^2=p^k} \mu(e_1)\chi_{a_1d_0}(e_1)e_2.$$

It suffices to show $b(p^k, p^{2\beta}) = \chi_{a_1 d_0}(p^k)\left(a(p^k, p^{2\beta}) - a(p^{k-1}, p^{2\beta})\right)$. On the one hand, $\mu(e_1) = 0$ unless $e_1 = 1, p$ in which case $\mu(e_1) = \pm 1$ accordingly. So

$$b(p^k, p^{2\beta}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1)\chi_{a_1 d_0}(e_1)e_2 = \begin{cases} \chi_{a_1 d_0}(p^k)p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(p^k)p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

where we have used the identity $\chi_{a_1 d_0}(e_1) = \chi_{a_1 d_0}(p^k)$ which holds because this quadratic Dirichlet character only depends upon the parity of k. On the other hand, as in the proof of Lemma 1.1

$$a(p^k, p^{2\beta}) = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

But then

$$\chi_{a_1 d_0}(p^k)\left(a(p^k, p^{2\beta}) - a(p^{k-1}, p^{2\beta})\right) = \begin{cases} \chi_{a_1 d_0}(p^k)p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(p^k)p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

which completes the proof. Clearly the same holds for $Q_{m_0 p^{2\beta}}(w, \widetilde{\chi}_{a_2})$. □

Lemmas 1.1 and 1.2 together show that $Q_{d_0 d_1^2}(s, \chi_{a_1})$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients $a(m, d)$ when d is an prime power. The proof of these lemmas also give the value of $a(p^k, p^l)$ and we collect this as a corollary:

Corollary 1.1. *For any prime p,*

$$a(p^k, p^l) = \begin{cases} \min\left(p^{\frac{k}{2}}, p^{\frac{l}{2}}\right) & \textit{if } \min(k,l) \textit{ is even,} \\ 0 & \textit{otherwise.} \end{cases}$$

If we combine Proposition 1.1 and Corollary 1.1 we can compute $a(m, d)$ in general:

Corollary 1.2. *For any integers $m, d \geq 1$,*

$$a(m, d) = \begin{cases} (m,d)^{\frac{1}{2}} & \textit{if } (m,d) \textit{ is a perfect square,} \\ 0 & \textit{otherwise.} \end{cases}$$

As an immediate consquence of Corollary 1.2, $a(m, d)$ is symmetric in m and d. As the weighting coefficients are multiplicative, $Q_{d_0 d_1^2}(s, \chi_{a_1})$ will posess an Euler product. To state the Euler product explicitely, we write $d = d_0 d_1^2 d_2^2$ with d_0 square-free and, d_2 relatively prime to $d_0 d_1$, and such that every prime divisor of d_1 divides d_0. In other words, d_0 is the square-free part of d, d_1 is the square part of d whose prime factors divide d to odd power, and d_2 is the square part of d whose prime factors divide d to even power. We have the following Euler product:

Theorem 1.4. *Let $d = d_0 d_1^2 d_2^2$ be the square decomposition of d stratified by even and odd powers. Then for any $a_1 \in \{\pm 1, \pm 2\}$,*

$$Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = \prod_{p^\alpha || d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta || d_2} Q_{d_0 p^{2\beta}}(s, \chi_{a_1}).$$

Moreover, the same holds for $Q_{m_0 m_1^2 m_2^2}(w, \widetilde{\chi}_{a_2})$.

Proof. Recall that

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = \sum_{m \geq 1} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m, d)}{m^s}.$$

We will now derive an alterate expression for $L(s,\chi_{a_1d})$. By Proposition 1.1, the coefficients of $L(s,\chi_{a_1d})$ are multiplicative. Therefore $L(s,\chi_{a_1d})$ admits the Euler product

$$L(s,\chi_{a_1d})=\prod_{p\text{ prime}}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(\widehat{p^k})a(p^k,d)}{p^{ks}}\right).$$

Decomposing the product according to primes dividing $d=d_0d_1^2d_2^2$, we get

$$\begin{aligned}
&L(s,\chi_{a_1d})\\
&=\prod_{p\text{ prime}}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(\widehat{p^k})a(p^k,d)}{p^{ks}}\right)\\
&=\prod_{p\nmid d}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(\widehat{p^k})a(p^k,1)}{p^{ks}}\right)\prod_{p^\alpha||d_1}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(\widehat{p^k})a(p^k,p^{2\alpha+1})}{p^{ks}}\right)\cdot\prod_{p^\beta||d_2}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(\widehat{p^k})a(p^k,p^{\beta})}{p^{ks}}\right)\\
&=\prod_{p\nmid d}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(\widehat{p^k})}{p^{ks}}\right)\prod_{p^\alpha||d_1}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(\widehat{p^k})a(p^k,p^{2\alpha+1})}{p^{ks}}\right)\cdot\prod_{p^\beta||d_2}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(\widehat{p^k})a(p^k,p^{\beta})}{p^{ks}}\right)\\
&=\prod_{p\nmid d}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(\widehat{p^k})}{p^{ks}}\right)\prod_{p^\alpha||d_1}\left(\sum_{k\geq 0}\frac{a(p^k,p^{2\alpha+1})}{p^{ks}}\right)\cdot\prod_{p^\beta||d_2}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(p^k)a(p^k,p^{\beta})}{p^{ks}}\right).
\end{aligned}$$

Including the factors corresponding to primes $p\mid d_2$ into the first product, we must multiply the last factor by the inverse of $\sum_{k\geq 0}\chi_{a_1d_0}(p)p^{-ks}=(1-\chi_{a_1d_0}(p)p^{-s})^{-1}$ obtaining

$$\prod_{p\nmid d_0}\left(\sum_{k\geq 0}\frac{\chi_{a_1d_0}(\widehat{p^k})}{p^{ks}}\right)\prod_{p^\alpha||d_1}\left(\sum_{k\geq 0}\frac{a(p^k,p^{2\alpha+1})}{p^{ks}}\right)\cdot\prod_{p^\beta||d_2}\left((1-\chi_{a_1d_0}(p)p^{-s})\sum_{k\geq 0}\frac{\chi_{a_1d_0}(p^k)a(p^k,p^{\beta})}{p^{ks}}\right),$$

as every prime divisor of d_1 divdes d_0. The first product is $L(s,\chi_{a_1d_0})$. For the second and third products, Remark 1.3 implies that the sums run up to $k\leq 2\alpha$ and $k\leq 2\beta$ respectively. Therefore they are $Q_{p^{2\alpha+1}}(s)$ and $Q_{d_0p^{2\beta}}(s,\chi_{a_1})$ respectively. It follows that

$$L(s,\chi_{a_1d})=L(s,\chi_{a_1d_0})\cdot\prod_{p^\alpha||d_1}Q_{p^{2\alpha+1}}(s)\cdot\prod_{p^\beta||d_2}Q_{d_0p^{2\beta}}(s,\chi_{a_1}).$$

This is our alternate expression for $L(s,\chi_{a_1d})$ and equating the two results in

$$L(s,\chi_{a_1d_0})Q_{d_0d_1^2d_2^2}(s,\chi_{a_1})=L(s,\chi_{a_1d_0})\cdot\prod_{p^\alpha||d_1}Q_{p^{2\alpha+1}}(s)\cdot\prod_{p^\beta||d_2}Q_{d_0p^{2\beta}}(s,\chi_{a_1}),$$

from which the proof is complete since $L(s,\chi_{a_1d_0})\neq 0$ for $\mathrm{Re}(s)>1$ (so that we may divide by $L(s,\chi_{a_1d_0})$). Clearly the same holds for $Q_{m_0m_1^2m_2^2}(w,\widetilde{\chi}_{a_2})$. □

Observe that for $d=d_0d_1^2d_2^2$, the prime factors that divide d_1d_2 are exactly those factors that divide d to power larger than 1. Thus, from Theorem 1.4 the Euler product for $Q_{d_0d_1^2d_2^2}(s,\chi_{a_1})$ is supported on exactly the primes dividing d to order larger than 1 and also depends upon the character $\chi_{a_1d_0}$.

Functional Equations

We can now derive functional equations for $Z_{a_1,a_2}(s,w)$. These functional equations will be induced from the functional equations for $L(s,\chi_{a_1d})$ and $L(s,\widetilde{\chi}_{a_2m})$. To prove these latter functional equations, we require a functional equation for the correction polynomials:

Theorem 1.5. *$Q_{d_0d_1^2}(s,\chi_{a_1})$ admits the functional equation.*

$$Q_{d_0d_1^2}(s,\chi_{a_1}) = d_1^{1-2s}Q_{d_0d_1^2}(1-s,\chi_{a_1}).$$

Moreover, the same holds for $Q_{m_0m_1^2}(w,\widetilde{\chi}_{a_2})$.

Proof. The stragety is to interchange e_2 and e_3 in the sum defining $Q_{d_0d_1^2}(s,\chi_{a_1})$:

$$\begin{aligned}
d_1^{1-2s}Q_{d_0d_1^2}(1-s) &= d_1^{1-2s}\sum_{e_1e_2e_3=d_1}\mu(e_1)\chi_{a_1d_0}(e_1)e_1^{s-1}e_2^{2s-1} \\
&= \sum_{e_1e_2e_3=d_1}\mu(e_1)\chi_{a_1d_0}(e_1)e_1^{s-1}\left(\frac{d_1}{e_2}\right)^{1-2s} \\
&= \sum_{e_1e_2e_3=d_1}\mu(e_1)\chi_{a_1d_0}(e_1)e_1^{s-1}(e_1e_3)^{1-2s} \\
&= \sum_{e_1e_2e_3=d_1}\mu(e_1)\chi_{a_1d_0}(e_1)e_1^{-s}e_3^{1-2s} \\
&= Q_{d_0d_1^2}(s,\chi_{a_1}).
\end{aligned}$$

Clearly the same holds for $Q_{m_0m_1^2}(w,\widetilde{\chi}_{a_2})$. □

We will define the completed L-function $L^*(s,\chi_{a_1d})$ by

$$L^*(s,\chi_{a_1d}) = L^*(s,\chi_{a_1d_0})Q_{d_0d_1^2}(s,\chi_{a_1}).$$

In particular, $L^*(s,\chi_d)$ makes sense even when d is not square-free and agrees with the previous definition when d is square-free. Combining Theorem 1.5, the functional equation for $L^*(s,\chi_{a_1d_0})$, and that $d\equiv d_0 \pmod 4$, we obtain a functional equation for $L^*(s,\chi_{a_1d})$:

$$L^*(s,\chi_{a_1d}) = \begin{cases} \varepsilon_{\chi_{a_1d_0}}q(d)^{\frac{1}{2}-s}L^*(1-s,\chi_{a_1d}) & \text{if } a_1d>0, \\ -\varepsilon_{\chi_{a_1d_0}}q(d)^{\frac{1}{2}-s}L^*(1-s,\chi_{a_1d}) & \text{if } a_1d<0. \end{cases}$$

Analogously, define the completed L-function $L^*(w,\widetilde{\chi}_{a_2m})$ by

$$L^*(w,\widetilde{\chi}_{a_2m}) = L^*(w,\widetilde{\chi}_{a_2m_0})Q_{m_0m_1^2}(w,\widetilde{\chi}_{a_2}).$$

Then, as before, we have a functional equation for $L^*(w,\widetilde{\chi}_{a_2m})$:

$$L^*(w,\widetilde{\chi}_{a_2m}) = \begin{cases} \varepsilon_{\widetilde{\chi}_{a_2m_0}}q(m)^{\frac{1}{2}-w}L^*(1-w,\widetilde{\chi}_{a_2m}) & \text{if } a_2m\equiv 1,2,5 \pmod 8, \\ -\varepsilon_{\widetilde{\chi}_{a_2m_0}}q(m)^{\frac{1}{2}-w}L^*(1-w,\widetilde{\chi}_{a_2m}) & \text{if } a_2m\equiv 3,6,7 \pmod 8. \end{cases}$$

The functional equations for $L^*(s,\chi_{a_1d})$ and $L^*(w,\widetilde{\chi}_{a_2m})$ will induce functional equations for $Z_{a_1,a_2}(s,w)$. However, there is an obstruction caused by the gamma factors. Indeed, the gamma factors for $L^*(s,\chi_{a_1d})$ and $L^*(w,\widetilde{\chi}_{a_2m})$ depend a_1d and a_2m modulo 8 respectively. To induce functional equations we need these gamma factors to be constant. Orthogonality of the Hilbert characters will allow us to get past this issue. For $b\in\{1,3,5,7\}$, define $Z^b_{a_1,a_2}(s,w)$ and $\widetilde{Z}^b_{a_2,a_1}(w,s)$ by

$$Z^b_{a_1,a_2}(s,w) = \frac{1}{4}\sum_{a\in\{\pm1,\pm2\}}\chi_a(b)Z_{a_1,aa_2}(s,w) \quad\text{and}\quad \widetilde{Z}^b_{a_2,a_1}(w,s) = \frac{1}{4}\sum_{a\in\{\pm1,\pm2\}}\widetilde{\chi}_a(b)\widetilde{Z}_{a_2,aa_1}(w,s).$$

In terms of the representations

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ odd}}\frac{\chi_{a_2}(d)L^{(2)}(s,\chi_{a_1d})}{d^w} \quad\text{and}\quad \widetilde{Z}_{a_2,a_1}(w,s) = \sum_{m \text{ odd}}\frac{\widetilde{\chi}_{a_1}(m)L^{(2)}(w,\widetilde{\chi}_{a_2m})}{m^s},$$

and orthogonality of the Hilbert characters, $Z^b_{a_1,a_2}(s,w)$ and $\widetilde{Z}^b_{a_2,a_1}(w,s)$ are the subseries containing only those d and m equivalent to b modulo 8 respectively. Then $Z^b_{a_1,a_2}(s,w)$ and $\widetilde{Z}^b_{a_2,a_1}(w,s)$ are sums of L-functions with a fixed gamma factor and so we can obtain functional equations. The fact that $Z_{a_1,a_2}(s,w)$ and $\widetilde{Z}_{a_2,a_1}(w,s)$ are linear combinations of these series will induce function equations. Precisely, we have the following statement:

Theorem 1.6. *$Z_{a_1,a_2}(s,w)$ and $\widetilde{Z}_{a_2,a_1}(w,s)$ admit the functional equations*

$$Z_{a_1,a_2}(s,w) = \chi_{a_1}(-1)\varepsilon_{\chi_{a_1d_0}}\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{(1-s)+\delta_{a_1}}{2}\right)}{\Gamma\left(\frac{s+\delta_{a_1}}{2}\right)}\frac{(1-\chi_{a_1d}(2)2^{-s})}{(1-\chi_{a_1d}(2)2^{-(1-s)})}Z_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right),$$

where $\delta_{\alpha_1} = \frac{\chi_{a_1}(1)-\chi_{a_1}(-1)}{2}$, and

$$\begin{aligned}
&\widetilde{Z}_{a_2,a_1}(w,s)\\
&= \frac{1}{4}\left(\varepsilon_{\chi_{a_2m_0}}\pi^{w-\frac{1}{2}}\frac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)}\frac{(1-\chi_{a_2m}(2)2^{-w})}{(1-\chi_{a_2m}(2)2^{-(1-w)})}\right)\sum_{\substack{a\in\{\pm1,\pm2\}\\ a_2b\equiv 1,2,5 \ (mod\ 8)}}\widetilde{\chi}_a(b)\widetilde{Z}_{a_2,aa_1}\left(1-w,s+w-\frac{1}{2}\right)\\
&-\frac{1}{4}\left(\varepsilon_{\chi_{a_2m_0}}\pi^{w-\frac{1}{2}}\frac{\Gamma\left(\frac{(1-w)+1}{2}\right)}{\Gamma\left(\frac{w+1}{2}\right)}\frac{(1-\chi_{a_2m}(2)2^{-w})}{(1-\chi_{a_2m}(2)2^{-(1-w)})}\right)\sum_{\substack{a\in\{\pm1,\pm2\}\\ a_2b\equiv 3,6,7 \ (mod\ 8)}}\widetilde{\chi}_a(b)\widetilde{Z}_{a_2,aa_1}\left(1-w,s+w-\frac{1}{2}\right).
\end{aligned}$$

Proof. The functional equation for $Z_{a_1,a_2}(s,w)$ follows immeditely from the functional equation for the L-function attached to a quadratic Dirichlet character and that $a_1d>0$ or $a_1d<0$ according to if $a_1>0$ or $a_1<0$. For the functional equation for $\widetilde{Z}_{a_2,a_1}(w,s)$, the functional equation for the L-function attached to a quadratic Dirichlet character implies

$$\widetilde{Z}^b_{a_2,a_1}(w,s) = \varepsilon_{\chi_{a_2m_0}}\pi^{w-\frac{1}{2}}\frac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)}\frac{(1-\chi_{a_2m}(2)2^{-w})}{(1-\chi_{a_2m}(2)2^{-(1-w)})}\widetilde{Z}^b_{a_2,a_1}\left(1-w,s+w-\frac{1}{2}\right),$$

if $a_2b\equiv 1,2,5 \pmod 8$, and

$$\widetilde{Z}^b_{a_2,a_1}(w,s) = -\varepsilon_{\chi_{a_2m_0}}\pi^{w-\frac{1}{2}}\frac{\Gamma\left(\frac{(1-w)+1}{2}\right)}{\Gamma\left(\frac{w+1}{2}\right)}\frac{(1-\chi_{a_2m}(2)2^{-w})}{(1-\chi_{a_2m}(2)2^{-(1-w)})}\widetilde{Z}^b_{a_2,a_1}\left(1-w,s+w-\frac{1}{2}\right),$$

if $a_2b\equiv 3,6,7 \pmod 8$. Since $\widetilde{Z}_{a_2,a_1}(w,s) = \sum_{b\in\{1,3,5,7\}}\widetilde{Z}^b_{a_2,a_1}(w,s)$, the functional equations just stated imply

$$\begin{aligned}
&\widetilde{Z}_{a_2,a_1}(w,s)\\
&= \left(\varepsilon_{\chi_{a_2m_0}}\pi^{w-\frac{1}{2}}\frac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)}\frac{(1-\chi_{a_2m}(2)2^{-w})}{(1-\chi_{a_2m}(2)2^{-(1-w)})}\right)\sum_{a_2b\equiv 1,2,5 \pmod 8}\widetilde{Z}^b_{a_2,a_1}\left(1-w,s+w-\frac{1}{2}\right)\\
&-\left(\varepsilon_{\chi_{a_2m_0}}\pi^{w-\frac{1}{2}}\frac{\Gamma\left(\frac{(1-w)+1}{2}\right)}{\Gamma\left(\frac{w+1}{2}\right)}\frac{(1-\chi_{a_2m}(2)2^{-w})}{(1-\chi_{a_2m}(2)2^{-(1-w)})}\right)\sum_{a_2b\equiv 3,6,7 \pmod 8}\widetilde{Z}^b_{a_2,a_1}\left(1-w,s+w-\frac{1}{2}\right).
\end{aligned}$$

The desired functional equation for $\widetilde{Z}_{a_2,a_1}(w,s)$ follows by expressing $\widetilde{Z}^b_{a_2,a_1}(w,s)$ in terms of $\widetilde{Z}_{a_2,aa_1}(w,s)$ for $a\in\{\pm1,\pm2\}$. $\square$

These functional equations are quite unruly and it is often far more simple to compactify them in terms of vectors. Define $\mathbf{Z}(s,w)$ and $\widetilde{\mathbf{Z}}(w,s)$ by

$$\mathbf{Z}(s,w) = (Z_{a_1,a_2}(s,w))_{a_1,a_2\in\{\pm1,\pm2\}} \quad \text{and} \quad \widetilde{\mathbf{Z}}(w,s) = (\widetilde{Z}_{a_2,a_1}(w,s))_{a_1,a_2\in\{\pm1,\pm2\}},$$

with lexicographical ordering. Also set

$$\gamma^{a_1}(s) = \chi_{a_1}(-1)\varepsilon_{\chi_{a_1}},$$

and

$$\gamma^{a_2 b}(w) = \text{Todo : [xxx]}.$$

Meromorphic Continuation

Poles and Residues

References

[1] Rosen, M. (2002). Number theory in function fields (Vol. 210). Springer Science & Business Media.
[2] Hormander, L. (1973). An introduction to complex analysis in several variables. Elsevier.
[3] Chinta, G., & Gunnells, P. E. (2007). Weyl group multiple Dirichlet series constructed from quadratic characters. Inventiones mathematicae, 167, 327-353.
[4] Stanley, R. (2023). Enumerative Combinatorics: Volume 2. Cambridge University Press.