0.1 Todo: [The Kuznetsov Trace Formula]

The Kuznetsov trace formula is an analog of the Petersson trace formula for weight zero Maass forms. From ??, $\mathcal{L}(N,\chi)$ admits an orthonormal basis of Maass forms for the point spectrum (these forms are generally not Hecke-Maass eigenforms because they need not be Hecke normalized or even cuspidal in the case of the discrete spectrum). However, by ?? and ?? we make take this orthonormal basis to consist of Hecke-Maass eigenforms and the constant function. Denote this basis by $\{u_j\}_{j\geq 0}$ with $u_0(z)=1$ and let u_j be of type ν_j for $j\geq 1$. In particular, $\{u_j\}_{j\geq 1}$ is an orthonormal basis of Hecke-Maass eigenforms and each such form admits a Fourier series at the $\mathfrak{a}$ cusp given by

$$(u_j|\sigma_{\mathfrak{a}})(z)=\sum_{n\neq 0}a_{j,\mathfrak{a}}(n)\sqrt{y}K_{\nu_j}(2\pi ny)e^{2\pi inx}.$$

The Kuznetsov trace formula is an equation relating the Fourier coefficients $a_{j,\mathfrak{a}}(n)$ and $a_{j,\mathfrak{b}}(n)$ of the basis $\{u_j\}_{j\geq 1}$ for two cusps $\mathfrak{a}$ and $\mathfrak{b}$ of $\Gamma_0(N)\backslash\mathbb{H}$ to a sum of integral transforms involving test functions and Salié sums. Similar to the Petersson trace formula, we will compute the inner product of two Poincaré series $P_{n,\chi,\mathfrak{a}}(z,\psi)(z)$ and $P_{m,\chi,\mathfrak{b}}(z,\varphi)(z)$ in two different ways. The first will be geometric in nature while the second will be spectral. We first need to compute the Fourier series of such a Poincaré series. Although we will not need it explicitly, we will work over any congruence subgroup:

Proposition 0.1.1. *Let $m\geq 1$, χ be Dirichlet character with conductor dividing the level, $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$, and $\psi(y)$ be a smooth function such that $\psi(y)\ll_\varepsilon y^{1+\varepsilon}$ as $y\to 0$. The Fourier series of $P_{m,\chi,\mathfrak{a}}(z,\psi)$ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{b}$ cusp is given by*

$$(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z,\psi)=\sum_{t\in\mathbb{Z}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\mathrm{Im}(z))+\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,m,t,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c)\right)e^{2\pi itz},$$

where $\psi(y,m,t,c)$ is the integral transform given by

$$\psi(y,m,t,c)=\int_{\mathrm{Im}(z)=y}\psi\left(\frac{y}{|cz|^2}\right)e^{-\frac{2\pi im}{c^2z}-2\pi itz}\,dz.$$

Proof. From the cocycle condition and ??, we have

$$(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z,\psi)=\delta_{\mathfrak{a},\mathfrak{b}}\psi(\mathrm{Im}(z))e^{2\pi imz}+\sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}},d\in\mathbb{Z}\\ d \pmod c\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\overline{\chi}(d)\psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)e^{2\pi im\left(\frac{a}{c}-\frac{1}{c^2z+cd}\right)},$$

where a and b are chosen such that $\det\left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\right)=1$ and we have used the fact that

$$\frac{a}{c}-\frac{1}{c^2z+cd}=\frac{az+b}{cz+d}.$$

Summing over all pairs (c,d) with $c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $d\in\mathbb{Z}$, and $d\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is the same as summing over all triples (c,ℓ,r) with $c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $\ell\in\mathbb{Z}$, and r taken modulo c with $r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$. Indeed, this is seen by writing $d=c\ell+r$. Moreover, since $ad-bc=1$ we have $a(c\ell+r)-bc=1$ which further implies that

$ar \equiv 1 \pmod{c}$. So we may take a to be the inverse for r modulo c. Then

$$\begin{aligned}
\sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}},d\in\mathbb{Z}\\ d \pmod{c}\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(d)\psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+cd}\right)} &= \sum_{(c,\ell,r)} \overline{\chi}(c\ell+r)\psi\left(\frac{\mathrm{Im}(z)}{|cz+c\ell+r|^2}\right)e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}\\
&= \sum_{(c,\ell,r)} \overline{\chi}(r)\psi\left(\frac{\mathrm{Im}(z)}{|cz+c\ell+r|^2}\right)e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}\\
&= \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\sum_{\ell\in\mathbb{Z}} \overline{\chi}(r)\psi\left(\frac{\mathrm{Im}(z)}{|cz+c\ell+r|^2}\right)e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}\\
&= \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(r)\sum_{\ell\in\mathbb{Z}}\psi\left(\frac{\mathrm{Im}(z)}{|cz+c\ell+r|^2}\right)e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)},
\end{aligned}$$

where on the right-hand side it is understood that we are summing over all triples (c,ℓ,r) with the prescribed properties and the second line holds since χ has conductor diving the level and $d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is determined modulo c. Now let

$$I_{c,r}(z,\psi) = \sum_{\ell\in\mathbb{Z}}\psi\left(\frac{\mathrm{Im}(z)}{|cz+c\ell+r|^2}\right)e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}.$$

We apply the Poisson summation formula to $I_{c,r}(z,\psi)$. This is allowed since the summands are absolutely integrable by **??**, as they exhibit polynomial decay of order $\sigma > 1$ because $\psi(y) \ll_\varepsilon y^{1+\varepsilon}$ as $y \to 0$, and $I_{c,r}(z,\psi)$ is holomorphic because $(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z,\psi)$ is. By the identity theorem it suffices to apply the Poisson summation formula for $z = iy$ with $y > 0$. So let $f(x)$ be given by

$$f(x) = \psi\left(\frac{y}{|cx+r+icy|^2}\right)e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2x+cr+ic^2y}\right)}.$$

As we have just noted, $f(x)$ is absolutely integrable on $\mathbb{R}$. We compute the Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i t x}\,dx = \int_{-\infty}^{\infty}\psi\left(\frac{y}{|cx+r+icy|^2}\right)e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2x+cr+ic^2y}\right)}e^{-2\pi i t x}\,dx.$$

Complexify the integral to get

$$\int_{\mathrm{Im}(z)=0}\psi\left(\frac{y}{|cz+r+icy|^2}\right)e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+cr+ic^2y}\right)}e^{-2\pi i t z}\,dz.$$

Now make the change of variables $z \to z - \frac{r}{c} - iy$ to obtain

$$e^{2\pi i m\frac{a}{c}+2\pi i t\frac{r}{c}-2\pi t y}\int_{\mathrm{Im}(z)=y}\psi\left(\frac{y}{|cz|^2}\right)e^{-\frac{2\pi i m}{c^2z}-2\pi i t z}\,dz.$$

As the remaining integral is $\psi(y,m,t,c)$, it follows that

$$\hat{f}(t) = \psi(y,m,t,c)e^{2\pi i m\frac{a}{c}+2\pi i t\frac{r}{c}-2\pi t y}.$$

By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z,\psi) = \sum_{t\in\mathbb{Z}}(\psi(y,m,t,c)e^{2\pi i m\frac{a}{c}+2\pi i t\frac{r}{c}})e^{2\pi i t z},$$

for all $z \in \mathbb{H}$. Substituting this back into the Eisenstein series gives a form of the Fourier series:

$$\begin{aligned}
(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z,\psi) &= \delta_{\mathfrak{a},\mathfrak{b}}\psi(\mathrm{Im}(z))e^{2\pi imz} + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r)\sum_{t\in\mathbb{Z}}\psi(y,m,t,c)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}e^{2\pi itz} \\
&= \sum_{t\in\mathbb{Z}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\mathrm{Im}(z)) + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}}\overline{\chi}(r)\psi(y,m,t,c)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right)e^{2\pi itz} \\
&= \sum_{t\in\mathbb{Z}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\mathrm{Im}(z)) + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,m,t,c)\sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\overline{\chi}(r)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right)e^{2\pi itz}.
\end{aligned}$$

We will simplify the innermost sum. Since a is the inverse for r modulo c, the innermost sum above becomes

$$\sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\overline{\chi}(r)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}} = \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\overline{\chi}(\overline{a})e^{2\pi im\frac{a}{c}+2\pi it\frac{\overline{a}}{c}} = \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\chi(a)e^{\frac{2\pi i(am+\overline{a}t)}{c}} = S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c).$$

So at last, we obtain our desired Fourier series:

$$(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t\in\mathbb{Z}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t}\psi(\mathrm{Im}(z)) + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,m,t,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c)\right)e^{2\pi itz}.$$

□

We can now derive the first half of the Kuznetsov trace formula by computing the inner product between $P_{n,\chi,\mathfrak{a}}(z,\psi)$ and $P_{m,\chi,\mathfrak{b}}(z,\varphi)$:

$$\begin{aligned}
\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), P_{m,\chi,\mathfrak{b}}(\cdot,\varphi)\rangle &= \frac{1}{V_{\Gamma_0(N)}}\int_{\mathcal{F}_{\Gamma_0(N)}} P_{n,\chi,\mathfrak{a}}(z,\psi)\overline{P_{m,\chi,\mathfrak{b}}(z,\varphi)}\,d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}}\int_{\mathcal{F}_{\Gamma_0(N)}}\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma_0(N)}\chi(\gamma)P_{n,\chi,\mathfrak{a}}(z,\psi)\overline{\varphi(\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}\gamma z))}e^{-2\pi im\overline{\sigma_{\mathfrak{b}}^{-1}\gamma z}}\,d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}}\int_{\mathcal{F}_{\Gamma_0(N)}}\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma_0(N)}P_{n,\chi,\mathfrak{a}}(\gamma z,\psi)\overline{\varphi(\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}\gamma z))}e^{-2\pi im\overline{\sigma_{\mathfrak{b}}^{-1}\gamma z}}\,d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}}\int_{\mathcal{F}_{\sigma_{\mathfrak{b}}^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}}}\sum_{\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma_0(N)}P_{n,\chi,\mathfrak{a}}(\gamma\sigma_{\mathfrak{b}}z,\psi)\overline{\varphi(\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}\gamma\sigma_{\mathfrak{b}}z))}e^{-2\pi im\overline{\sigma_{\mathfrak{b}}^{-1}\gamma\sigma_{\mathfrak{b}}z}}\,d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}}\int_{\mathcal{F}_{\sigma_{\mathfrak{b}}^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}}}\sum_{\gamma\in\Gamma_\infty\backslash\sigma_{\mathfrak{b}}^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}^{-1}}P_{n,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}}\gamma z,\psi)\overline{\varphi(\mathrm{Im}(\gamma z))}e^{-2\pi im\overline{\gamma z}}\,d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}}\int_{\Gamma_\infty\backslash\mathbb{H}}(P_{n,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z,\psi)\overline{\varphi(\mathrm{Im}(z))}e^{-2\pi im\overline{z}}\,d\mu,
\end{aligned}$$

where in the third line we have used the automorphy of $P_{n,\chi,\mathfrak{a}}(z,\psi)$, in the forth and fifth lines we have made the change of variables $z \to \sigma_{\mathfrak{b}}z$ and $\gamma \to \sigma_{\mathfrak{b}}\gamma\sigma_{\mathfrak{b}}^{-1}$ respectively, and in the sixth line we have unfolded. Now substitute in the Fourier series of $P_{n,\chi,\mathfrak{a}}(z,\psi)$ at the $\mathfrak{b}$ cusp to obtain

$$\frac{1}{V_{\Gamma_0(N)}}\int_{\Gamma_\infty\backslash\mathbb{H}}\sum_{t\in\mathbb{Z}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,t}\psi(\mathrm{Im}(z)) + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,n,t,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(n,t,c)\right)\overline{\varphi(\mathrm{Im}(z))}e^{2\pi itz-2\pi im\overline{z}}\,d\mu,$$

which is equivalent to

$$\frac{1}{V_{\Gamma_0(N)}}\int_0^\infty\int_0^1\sum_{t\geq 1}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,t}\psi(y)+\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,n,t,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(n,t,c)\right)\overline{\varphi(y)}e^{2\pi i(t-m)x}e^{-2\pi(t+m)y}\,\frac{dx\,dy}{y^2}.$$

By the dominated convergence theorem, we can interchange the sum and the two integrals. Then **??** implies that the inner integral cuts off all of the terms except the diagonal $t=m$. This leaves

$$\frac{1}{V_{\Gamma_0(N)}}\int_0^\infty\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m}\psi(y)+\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\psi(y,n,m,c)S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c)\right)\overline{\varphi(y)}e^{-4\pi my}\,\frac{dy}{y^2}.$$

Interchanging the integral and the remaining sum by the dominated convergence theorem again, we arrive at

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),P_{m,\chi,\mathfrak{b}}(\cdot,\varphi)\rangle=\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m}(\psi,\varphi)_{n,m}+\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c)V(n,m,c,\psi,\varphi),$$

where we have set

$$(\psi,\varphi)_{n,m}=\frac{1}{V_{\Gamma_0(N)}}\int_0^\infty\psi(y)\overline{\varphi(y)}e^{-2\pi(n+m)y}\,\frac{dy}{y^2},$$

and

$$V(n,m,c;\psi,\varphi)=\frac{1}{V_{\Gamma_0(N)}}\int_0^\infty\int_{\mathrm{Im}(z)=y}\psi\left(\frac{y}{|cz|^2}\right)\overline{\varphi(y)}e^{-\frac{2\pi im}{c^2z}-2\pi inz-4\pi my}\,\frac{dz\,dy}{y^2}.$$

This is the first half of the Kuznetsov trace formula. For the second half, **??** gives

$$P_{n,\chi,\mathfrak{a}}(\cdot,\psi)=\sum_{j\geq 0}\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),u_j\rangle u_j(z)+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^\infty\left\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right)\right\rangle E_{\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)\,dr,$$

and

$$P_{m,\chi,\mathfrak{a}}(\cdot,\varphi)=\sum_{j\geq 0}\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi),u_j\rangle u_j(z)+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^\infty\left\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi),E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right)\right\rangle E_{\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)\,dr.$$

By orthonormality, it follows that

$$\begin{aligned}\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),P_{m,\chi,\mathfrak{a}}(\cdot,\varphi)\rangle&=\sum_j\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),u_j\rangle\overline{\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi),u_j\rangle}\\&\quad+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^\infty\left\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right)\right\rangle\overline{\left\langle P_{m,\chi,\mathfrak{a}}(\cdot,\varphi),E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right)\right\rangle}\,dr.\end{aligned}$$

Now we must simplify the remaining inner products. Let $f\in\mathcal{L}(N,\chi)$ with Fourier series

$$f(z)=a^+(0)y^{\frac{1}{2}+\nu}+a^-(0)y^{\frac{1}{2}-\nu}+\sum_{n\neq 0}a(n)\sqrt{y}K_\nu(2\pi|n|y)e^{2\pi inx}.$$

By unfolding the integral in the Petersson inner product and cutting off everything except the diagonal using **??** exactly as in the case for $\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),P_{m,\chi,\mathfrak{a}}(\cdot,\varphi)\rangle$, we see that

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi),f\rangle=\frac{1}{V_\Gamma}\int_0^\infty\overline{a(n)\sqrt{y}K_\nu(2\pi ny)}\psi(y)e^{-4\pi my}\frac{dy}{y^2}.$$

Now set
$$\omega_\nu(n,\psi) = \frac{1}{V_\Gamma}\int_0^\infty \sqrt{y}K_\nu(2\pi|n|y)\overline{\psi(y)}e^{-4\pi my}\frac{dy}{y^2}.$$
Then it follows from the Fourier series of cusp forms and Eisenstein series that
$$\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), u_j\rangle = \overline{a_j(n)\omega_{\nu_j}(n,\psi)},$$
for $j \ge 1$ and
$$\left\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+ir\right)\right\rangle = \overline{\tau_{\mathfrak{a}}\left(n,\frac{1}{2}+ir\right)\omega_{ir}(n,\psi)}.$$
In particular, $\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), u_0\rangle = 0$. So we obtain
$$\begin{aligned}\langle P_{n,\chi,\mathfrak{a}}(\cdot,\psi), P_{m,\chi,\mathfrak{a}}(\cdot,\varphi)\rangle &= \sum_{j\ge1}\overline{a_j(n)}a_j(m)\overline{\omega(n,\psi)}\omega(m,\varphi)\\ &\quad+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^\infty \overline{\tau_{\mathfrak{a}}\left(n,\frac{1}{2}+ir\right)}\tau_{\mathfrak{a}}\left(m,\frac{1}{2}+ir\right)\overline{\omega(n,\psi)}\omega(m,\varphi)\,dr.\end{aligned}$$

This is the second half of the Kuznetsov trace formula. Equating the first and second halves we get the **Kuznetsov trace formula**:
$$\begin{aligned}\delta_{n,m}(\psi,\varphi)+\sum_{\substack{c\ge1\\ c\equiv 0 \pmod N}}\frac{1}{c}S_\chi(n,m,c)V(n,m,c,\psi,\varphi) &= \sum_{j\ge1}\overline{a_j(n)}a_j(m)\overline{\omega(n,\psi)}\omega(m,\varphi)\\ &\quad+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^\infty \overline{\tau_{\mathfrak{a}}\left(n,\frac{1}{2}+ir\right)}\tau_{\mathfrak{a}}\left(m,\frac{1}{2}+ir\right)\overline{\omega(n,\psi)}\omega(m,\varphi)\,dr.\end{aligned}$$

The left-hand side is called the **geometric side** and the right-hand side is called the **spectral side**. We collect our work as a theorem:

Theorem 0.1.1 (Kuznetsov trace formula). *Let $\{u_j\}_{j\ge1}$ be an orthonormal basis of Hecke-Maass eigenforms for $\mathcal{L}(N,\chi)$ of types ν_j with Fourier coefficients $a_j(n)$. Then for any positive integers $n, m \ge 1$, we have*
$$\begin{aligned}\delta_{n,m}(\psi,\varphi)+\sum_{\substack{c\ge1\\ c\equiv 0 \pmod N}}\frac{1}{c}S_\chi(n,m,c)V(n,m,c,\psi,\varphi) &= \sum_{j\ge1}\overline{a_j(n)}a_j(m)\overline{\omega(n,\psi)}\omega(m,\varphi)\\ &\quad+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^\infty \overline{\tau_{\mathfrak{a}}\left(n,\frac{1}{2}+ir\right)}\tau_{\mathfrak{a}}\left(m,\frac{1}{2}+ir\right)\overline{\omega(n,\psi)}\omega(m,\varphi)\,dr.\end{aligned}$$

0.2 The Ideal Norm

Let us now prove some properties about the ideal norm. We first show that it respects localization:

Proposition 0.2.1. *Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of separable extension L/K and let $D \subseteq \mathcal{o}-\{0\}$ be a multiplicative subset. Then for any fractional ideal $\mathfrak{F}$ of $\mathcal{O}$, we have*
$$\mathrm{N}_{\mathcal{O}D^{-1}/\mathcal{o}D^{-1}}(\mathfrak{F}D^{-1}) = \mathrm{N}_{\mathcal{O}/\mathcal{o}}(\mathfrak{F})D^{-1}.$$

Proof. Since the ideal norm is multiplicative, it suffices to prove the claim in the case of a prime $\mathfrak{P}$ of $\mathcal{O}$. Then we must show

$$\mathrm{N}_{\mathcal{O}D^{-1}/\mathcal{o}D^{-1}}(\mathfrak{P}D^{-1}) = \mathrm{N}_{\mathcal{O}/\mathcal{o}}(\mathfrak{P})D^{-1}.$$

This is immediate from **??** and the definition of the ideal norm. $\square$

The ideal norm is also compatible with the field trace:

Proposition 0.2.2. *Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of degree n separable extension L/K. Then for any $\lambda \in \mathcal{O}$, we have*

$$\mathrm{N}_{\mathcal{O}/\mathcal{o}}(\lambda\mathcal{O}) = \mathrm{N}_{L/K}(\lambda)\mathcal{o}.$$

Proof. In light of Proposition 0.2.1, it suffices to assume $\mathcal{O}/\mathcal{o}$ is a local Dedekind extension. Therefore $\mathcal{o}$ is a discrete valuation ring, $\mathcal{O}$ is a principal ideal domain, and $\mathcal{O}/\mathcal{o}$ admits an integral basis $\alpha_1, \ldots, \alpha_n$ making $\mathcal{O}$ a free $\mathcal{o}$-module of rank n. Let $\mathfrak{p}$ be the unique prime of $\mathcal{o}$ and π be a uniformizer so that $\mathfrak{p} = \pi\mathcal{o}$. Since the ideal norm and the field norm are both multiplicative and $\mathcal{O}$ and $\mathcal{o}$ are both unique factorization domains, we may assume that λ is prime. Then $\lambda\mathcal{O} = \mathfrak{P}$ for some prime $\mathfrak{P}$ of $\mathcal{O}$. So on the one hand,

$$\mathrm{N}_{\mathcal{O}/\mathcal{o}}(\lambda\mathcal{O}) = \mathfrak{p}^{f_{\mathfrak{p}}(\mathfrak{P})}.$$

As $\mathcal{o}$ is a discrete valuation ring, we have the prime factorization $\mathrm{N}_{L/K}(\lambda) = \mu\pi^f$. So on the other hand,

$$\mathrm{N}_{L/K}(\lambda)\mathcal{o} = \mathfrak{p}^f.$$

It now suffices to show that $f = f_{\mathfrak{p}}(\mathfrak{P})$. Todo: [xxx] $\square$

The different and discriminant and related to each other via the ideal norm. In particular, the ideal norm of the different is the discriminant:

Proposition 0.2.3. *Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of a degree n separable extension L/K. Then*

$$\mathfrak{d}_{\mathcal{O}/\mathcal{o}} = \mathrm{N}_{\mathcal{O}/\mathcal{o}}(\mathfrak{D}_{\mathcal{O}}/\mathcal{o}).$$

Proof. In view of **??**, we may assume $\mathcal{O}/\mathcal{o}$ is a local Dedekind extension. Therefore $\mathcal{o}$ is a discrete valuation ring, $\mathcal{O}$ is a principal ideal domain, and $\mathcal{O}/\mathcal{o}$ admits an integral basis $\alpha_1, \ldots, \alpha_n$ making $\mathcal{O}$ a free $\mathcal{o}$-module of rank n. Then $\mathfrak{d}_{\mathcal{O}/\mathcal{o}}$ is a principal integral ideal where

$$\mathfrak{d}_{\mathcal{O}/\mathcal{o}} = d_{\mathcal{o}}(\mathcal{O})\mathcal{o}.$$

As $\mathcal{O}$ is a principal ideal domain, every fractional ideal is also principal. So on the one hand, $\mathfrak{C}_{\mathcal{O}/\mathcal{o}} = \lambda\mathcal{O}$ for some nonzero $\lambda \in L$ and $\lambda\alpha_1, \ldots, \lambda\alpha_n$ is a basis of L/K contained in $\mathfrak{C}_{\mathcal{O}/\mathcal{o}}$. Moreover,

$$d_{L/K}(\lambda\alpha_1, \ldots, \lambda\alpha_n) = \mathrm{N}_{L/K}(\lambda)^2 d_{L/K}(\alpha_1, \ldots, \alpha_n),$$

by **??** and that base change matrix from $\alpha_1, \ldots, \alpha_n$ to $\lambda\alpha_1, \ldots, \lambda\alpha_n$ is the multiplication by λ map. Todo: [xxx] $\square$

0.3 Misc.

(ii) The gamma factor $\gamma(s, f \otimes g)$ takes the form

$$\gamma(s, f \otimes g) = \pi^{-\frac{d_{f\otimes g} s}{2}} \prod_{\substack{1\le j\le d_f \\ 1\le \ell\le d_g}} \Gamma\left(\frac{s+\mu_{j,\ell}}{2}\right),$$

with the local roots at infinity satisfying the additional bounds $\mathrm{Re}(\mu_{j,\ell}) \le \mathrm{Re}(\kappa_j) + \mathrm{Re}(\nu_\ell)$ and $|\mu_{j,\ell}| \le |\kappa_j| + |\nu_\ell|$.

(iii) The root number $q(f\otimes g)$ satisfies $q(f\otimes g) \mid q(f)^{d_f} q(g)^{d_g}$. If $q(f\otimes g)$ is a proper divisor of $q(f)^{d_f} q(g)^{d_g}$, we say that $L(s, f\otimes g)$ exhibits **conductor dropping**.

(v) $L(s, f\otimes g)$ has a pole of order $r_{f\otimes g} \ge 1$ at $s = 1$ if $f = g$.

0.4 Cyclotomic Dedekind Zeta Functions

Similarly to quadratic number fields, it is a simple matter to determine cyclotomic Dedekind zeta function explicitly:

Theorem 0.4.1. *Let $\mathbb{Q}(\omega)$ be the cyclotomic number field generated by a primitive d-th root of unity ω. Then*

$$\zeta_{\mathbb{Q}(\omega)}(s) = \prod_{\chi \pmod d} L(s, \widetilde{\chi}),$$

where $\widetilde{\chi}$ is the primitive character inducing χ.

Proof. By the identity theorem it suffices to prove this for $\sigma > 1$. Using **??** and letting ω_p be a primitive f_p-th root of unity, the Euler product of $\zeta_{\mathbb{Q}(\omega)}(s)$ can be expressed as

$$\zeta_{\mathbb{Q}(\omega)}(s) = \prod_p \prod_{1\le k\le f_p} (1 - \omega_p^k p^{-s})^{-r_p}.$$

Now the subgroup of $\left(\mathbb{Z}/\frac{n}{p^{e_p}}\mathbb{Z}\right)^*$ generated by p is of order f_p by definition of f_p. Moreover, the characters of this must be of the form

$$\chi(np) = \omega_p^{nk},$$

for $0 \le k \le f_p - 1$, since there are f_p many such characters. These are clearly primitive Dirichlet characters modulo f_p. Now the fundamental equality gives $\varphi(d) = r_p \varphi(p^{e_p}) f_p$ from which it follows that $\varphi\left(\frac{n}{p^{e_p}}\right) = r_p f_p$. Therefore each χ has r_p many lifts to a Dirichlet character modulo $\frac{n}{p^{e_p}}$. Todo: [xxx] □

We also deduce a useful expression for $a_{\mathbb{Q}(\omega)}(n)$:

Proposition 0.4.1. *Let $\mathbb{Q}(\omega)$ be the cyclotomic number field generated by a primitive d-th root of unity ω and let $\widetilde{\chi}_1, \ldots, \widetilde{\chi}_{\varphi(d)}$ be the primitive Dirichlet characters inducing those modulo d. Then*

$$a_{\mathbb{Q}(\omega)}(n) = \sum_{n = n_1\cdots n_{\varphi(d)}} \widetilde{\chi}_1(n_1)\cdots\widetilde{\chi}_{\varphi(d)}(n_{\varphi(d)}).$$

Proof. This follows from **??** and Theorem 0.4.1 since the coefficients of the L-series of $\prod_{\chi \pmod d} L(s, \widetilde{\chi})$ are $\sum_{n=n_1\cdots n_{\varphi(d)}} \widetilde{\chi}_1(n_1)\cdots\widetilde{\chi}_{\varphi(d)}(n_{\varphi(d)})$. □