An Introduction to Number Theory

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Part I

Foundations

Chapter 1

Preliminaries

A good selection of topics that need some discussion before studying number theory are the following:

- Asymptotics,
- Dirichlet Characters,
- Exponential Sums,
- Integral Lattices,
- The Gamma Function,
- Locally Compact Groups,
- Integration Techniques and Transforms.

In the interest of keeping this text almost completely self-contained, this chapter is dedicated to the basics of these topics as they are essential tools that we will require in our investigations. The concepts presented in this chapter are tools in a toolbox rather than pure number theory. In order to improve the readability of the remainder of the text we will use the results presented here without reference unless it is a matter of clarity. As for standard knowledge, we assume familiarity with basic number theory, complex analysis, real analysis, functional analysis, topology, and algebra. We have also outsourced specific subtopics to the appendix and we will reference them when necessary.

1.1 Notational Conventions

Here we make some notational conventions that will be used throughout the rest of the text unless specified otherwise:

- The symbols $\subset$ and $\supset$ denote strict containment.
- Any ring is understood to be a commutative ring with 1 and a subring is understood to contain 1.
- The finite field with p elements $\mathbb{F}_p$ stands for $\mathbb{Z}/p\mathbb{Z}$.
- Primes and divisors of integers are taken to be positive.
- The symbol ε denotes a small positive constant ($\varepsilon > 0$) that is not necessarily the same from line to line.

- If $a \in (\mathbb{Z}/m\mathbb{Z})^*$, we will always let $\overline{a}$ denote the multiplicative inverse. That is, $a\overline{a} \equiv 1 \pmod{m}$.
- By analytic we mean real analytic or complex analytic accordingly.
- For the complex variables z, s, and u, we write

$$z = x + iy, \quad s = \sigma + it, \quad \text{and} \quad u = \tau + ir,$$

for the real and imaginary parts of these variables respectively unless specified otherwise. Moreover, in certain expressions we often write $\mathrm{Im}(z)$ for clarity.

- For $\mathbf{z} \in \mathbb{C}^n$, we write

$$\mathbf{z} = \mathbf{x} + i\mathbf{y},$$

for the real and imaginary parts of $\mathbf{z}$ respectively unless specified otherwise. Moreover,

$$\mathbf{z}^{-1} = (z_1^{-1}, \dots, z_n^{-1}) = \frac{1}{\mathbf{z}},$$

provided $z_i \neq 0$ for $1 \leq i \leq n$. Also,

$$\alpha\mathbf{z} = (\alpha z_1, \dots, \alpha z_n) \quad \text{and} \quad \mathbf{z}^\alpha = (z_1^\alpha, \dots, z_n^\alpha),$$

for all $\alpha, \beta \in \mathbb{C}$. Lastly,

$$\mathbf{z}^{\mathbf{w}} = z_1^{w_1} \cdots z_n^{w_n}, \quad \text{and} \quad \mathbf{z}\mathbf{w} = (z_1 w_1, \dots, z_n w_n),$$

for all $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{C}^n$.

- If $r \in \mathbb{Z}$ denotes the order of a possible pole of a complex function, $r \geq 0$ if it is a pole and $r \leq 0$ if it is a zero.
- The nontrivial zeros of an L-function will be denoted by $\rho = \beta + i\gamma$ unless specified otherwise.
- We will always take the principal branch of the logarithm.
- For a sum $\sum$ over integers satisfying a congruence condition, $\sum^{'}$ will denote the sum restricted to relatively prime integers satisfying the same congruence.
- All integrals are taken with positive orientation.
- We will write $\int_{(a)}$ for the complex integral over the line whose real part is a and with positive orientation.
- $\delta_{a,b}$ will denote the indicator function for $a = b$. That is, $\delta_{a,b} = 1, 0$ according to if $a = b$ or not.

1.2 Asymptotics

Throughout we assume $f, g : \mathbb{R}^n \to \mathbb{C}$. Much of the language of number theory is given in terms of asymptotics (or estimates or bounds) as they allows us to discuss approximate growth and dispense with superfluous constants. For this reason, asymptotics will be the first material that we will present. The asymptotics that we will cover are listed in the following table:

Asymptotics	Notation
Big O	$f(\mathbf{x}) = O(g(\mathbf{x}))$
Vinogradov's symbol	$f(\mathbf{x}) \ll g(\mathbf{x})$
Order of magnitude symbol	$f(\mathbf{x}) \asymp g(\mathbf{x})$
Little o	$f(\mathbf{x}) = o(g(\mathbf{x}))$
Asymptotic equivalence	$f(\mathbf{x}) \sim g(\mathbf{x})$
Omega symbol	$f(\mathbf{x}) = \Omega(g(\mathbf{x}))$

Implicit in all of these asymptotics is some limiting process $\mathbf{x} \to \mathbf{x}_0$ where $\mathbf{x}_0$ is finite or ∞. If $\mathbf{x}_0$ is finite then it is understood that the asymptotic is assumed to hold for all $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$ in norm. If $\mathbf{x}_0$ is infinite then the asymptotic is assumed to hold for sufficiently large $\mathbf{x}$. If the limiting process is not explicitly mentioned, it is assumed to be as $\mathbf{x} \to \infty$. Often times, asymptotics will hold for all admissible values of $\mathbf{x}$ and this will be clear from context although we still might suppress the specific limiting process.

Remark 1.2.1. *If $f, g : \mathbb{C}^n \to \mathbb{C}$ then the following theory still holds by identifying $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Moreover, if $f, g : \mathbb{Z}_+^n \to \mathbb{C}$ then by extending $f(\mathbf{n})$ and $g(\mathbf{n})$ to $\mathbb{R}^n$ by making them piecewise linear, so that they are piecewise continuous, the following theory still holds with $\mathbf{n}$ in place of $\mathbf{x}$. In particular, we may take $f(\mathbf{x})$ or $g(\mathbf{x})$ to be a constant function.*

Implicit in some asymptotics will be a constant (such constants are in general not unique and any sufficiently large constant will do). Any such constant is called the **implicit constant** of the asymptotic. The implicit constant may depend on one or more parameters, ε, σ, etc. If we wish to make these dependencies known, we use subscripts. If it is possible to choose the implicit constant independent of a certain parameter then we say that the asymptotic is **uniform** with respect to that parameter. Moreover, we say that an implicit constant is **effective** if the constant is numerically computable and **ineffective** otherwise. Moreover, if we are interested in the dependence of an asymptotic on a certain parameter, say p, we will refer to the **p-aspect** to mean the part of the asymptotic that is dependent upon p.

O-estimates and Symbols

We say $f(\mathbf{x})$ **is of order** $g(\mathbf{x})$ or $f(\mathbf{x})$ is $O(g(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{x}_0$ and write $f(\mathbf{x}) = O(g(\mathbf{x}))$ if there is some positive constant c such that

$$|f(\mathbf{x})| \leq c|g(\mathbf{x})|,$$

holds as $\mathbf{x} \to \mathbf{x}_0$. We call this an **$O$-estimate** and say that $f(\mathbf{x})$ has **growth at most** $g(\mathbf{x})$. The O-estimate says that for $\mathbf{x}$ close to $\mathbf{x}_0$, the size of $f(\mathbf{x})$ grows like $g(\mathbf{x})$.

Remark 1.2.2. *Many authors assume that $g(\mathbf{x})$ is a nonnegative function so that the absolute value on $g(\mathbf{x})$ can be dropped. As we require asymptotics that will be used more generally, we do not make this assumption since one could very well replace $O(g(\mathbf{x}))$ with $O(|g(\mathbf{x})|)$. In practice this deviation causes no issue.*

As a symbol, let $O(g(\mathbf{x}))$ stand for a function $f(\mathbf{x})$ that is $O(g(\mathbf{x}))$. Then we may use the O-estimates in algebraic equations and inequalities. Note that this extends the definition of the symbol because $f(\mathbf{x}) = O(g(\mathbf{x}))$ means $f(\mathbf{x})$ is $O(g(\mathbf{x}))$. Moreover, in algebraic equations and inequalities involving O-estimates it is often customary to refer to such an O-estimate as an **error term**. The symbol $\ll$ is known as

Vinogradov's symbol and it is an alternative way to express O-estimates. We write $f(\mathbf{x}) \ll g(\mathbf{x})$ as $\mathbf{x} \to \mathbf{x}_0$ if $f(\mathbf{x}) = O(g(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{x}_0$. We also write $f(\mathbf{x}) \gg g(\mathbf{x})$ as $\mathbf{x} \to \mathbf{x}_0$ to mean $g(\mathbf{x}) \ll f(\mathbf{x})$ as $\mathbf{x} \to \mathbf{x}_0$. If there is a dependence of the implicit constant on parameters, we use subscripts to denote dependence on these parameters. If both $f(\mathbf{x}) \ll g(\mathbf{x})$ and $g(\mathbf{x}) \ll f(\mathbf{x})$ as $\mathbf{x} \to \mathbf{x}_0$ then we say $f(\mathbf{x})$ and $g(\mathbf{x})$ have the **same order of magnitude** and write $f(\mathbf{x}) \asymp g(\mathbf{x})$ as $\mathbf{x} \to \mathbf{x}_0$. We also say $f(\mathbf{x})$ has **growth** $g(\mathbf{x})$. If there is a dependence of the implicit constant on parameters, we use subscripts to denote dependence on these parameters. From the definition of the O-estimate, this is equivalent to the existence of positive constants c_1 and c_2 such that

$$c_1|g(\mathbf{x})| \leq |f(\mathbf{x})| \leq c_2|g(\mathbf{x})|.$$

Equivalently, we can interchange $f(\mathbf{x})$ and $g(\mathbf{x})$ in the above equation.

o-estimates and Symbols

We say $f(\mathbf{x})$ **is of smaller order than** $g(\mathbf{x})$ or $f(\mathbf{x})$ is $o(g(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{x}_0$ and write $f(\mathbf{x}) = o(g(\mathbf{x}))$ if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \left|\frac{f(\mathbf{x})}{g(\mathbf{x})}\right| = 0,$$

provided $g(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$ in norm. We call this an o**-estimate** and say that $f(\mathbf{x})$ has **growth less than** $g(\mathbf{x})$. The o-estimate says that for $\mathbf{x}$ close to $\mathbf{x}_0$, $g(\mathbf{x})$ dominates $f(\mathbf{x})$. If $f(\mathbf{x}) = o(g(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{x}_0$ then $f(\mathbf{x}) = O(g(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{x}_0$ where the implicit constant can be taken arbitrarily small by definition of the o-estimate. Therefore, o-estimates are stronger than O-estimates. As a symbol, let $o(g(\mathbf{x}))$ stand for a function $f(\mathbf{x})$ that is $o(g(\mathbf{x}))$. Then we may use the o-estimates in algebraic equations and inequalities. Note that this extends the definition of the symbol because $f(\mathbf{x}) = o(g(\mathbf{x}))$ means $f(\mathbf{x})$ is $o(g(\mathbf{x}))$. We say $f(\mathbf{x})$ **is asymptotic to** $g(\mathbf{x})$ as $\mathbf{x} \to \mathbf{x}_0$ and write $f(\mathbf{x}) \sim g(\mathbf{x})$ if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x})}{g(\mathbf{x})} = 1,$$

provided $g(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$ in norm. We call this an **asymptotic** and say that $f(\mathbf{x})$ and $g(\mathbf{x})$ are **asymptotically equivalent**. It is useful to think of asymptotic equivalence as $f(\mathbf{x})$ and $g(\mathbf{x})$ being the same size in the limit as $\mathbf{x} \to \mathbf{x}_0$. Immediately from the definition, we see that this is an equivalence relation on functions. In particular, if $f(\mathbf{x}) \sim g(\mathbf{x})$ and $g(\mathbf{x}) \sim h(\mathbf{x})$ then $f(\mathbf{x}) \sim h(\mathbf{x})$. Also, if $f(\mathbf{x}) \sim g(\mathbf{x})$ as $\mathbf{x} \to \mathbf{x}_0$ then $f(\mathbf{x}) \asymp g(\mathbf{x})$ as $\mathbf{x} \to \mathbf{x}_0$ with $c_1 \leq 1 \leq c_2$. So asymptotic equivalence is stronger than being of the same order of magnitude. Also note that $f(\mathbf{x}) \sim g(\mathbf{x})$ is equivalent to $f(\mathbf{x}) = g(\mathbf{x})(1 + o(1))$ and hence implies $f(\mathbf{x}) = g(\mathbf{x})(1 + O(1))$. We say $f(\mathbf{x})$ **is of larger order than** $g(\mathbf{x})$ or $f(\mathbf{x})$ is $\Omega(g(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{x}_0$ and write $f(\mathbf{x}) = \Omega(g(\mathbf{x}))$ if

$$\limsup_{\mathbf{x}\to\mathbf{x}_0} \left|\frac{f(\mathbf{x})}{g(\mathbf{x})}\right| > 0.$$

provided $g(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$ in norm. We call this an Ω**-estimate** and say that $f(\mathbf{x})$ has **growth at least** $g(\mathbf{x})$. Observe that $f(\mathbf{x}) = \Omega(g(\mathbf{x}))$ is precisely the negation of $f(\mathbf{x}) = o(g(\mathbf{x}))$, so that $f(\mathbf{x}) = \Omega(g(\mathbf{x}))$ means $f(\mathbf{x}) = o(g(\mathbf{x}))$ is false. This is weaker than $f(\mathbf{x}) \gg g(\mathbf{x})$ because $f(\mathbf{x}) = \Omega(g(\mathbf{x}))$ means $|f(\mathbf{x})| \geq c|g(\mathbf{x})|$ for some values of $\mathbf{x}$ arbitrarily close to $\mathbf{x}_0$ whereas $f(\mathbf{x}) \gg g(\mathbf{x})$ means $|f(\mathbf{x})| \geq c|g(\mathbf{x})|$ for all values of $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$ in norm.

Algebraic Manipulation for O-estimates and o-estimates

Asymptotics become increasingly more useful when we can use them in equations to represent approximations. We catalogue some of the most useful algebraic manipulations for O-estimates and o-estimates. Most importantly, if an algebraic equation involves a O-estimate or o-estimate then it is understood that the equation is not symmetric and is interpreted to be read from left to right. That is, any function of the form satisfying the estimate on the left-hand side also satisfies the estimate on the right-hand side too. We begin with O-estimates. The trivial algebraic manipulations are collected in the proposition below:

Proposition 1.2.1. *The following O-estimates hold as $\mathbf{x} \to \mathbf{x}_0$:*

(i) If $f(\mathbf{x}) = O(g(\mathbf{x}))$ and $g(\mathbf{x}) = O(h(\mathbf{x}))$ then $f(\mathbf{x}) = O(h(\mathbf{x}))$. Equivalently, $O(O(h(\mathbf{x}))) = O(h(\mathbf{x}))$.

(ii) If $f_i(\mathbf{x}) = O(g_i(\mathbf{x}))$ for $i = 1, 2$ then $f_1(\mathbf{x})f_2(\mathbf{x}) = O(g_1(\mathbf{x})g_2(\mathbf{x}))$.

(iii) If $f(\mathbf{x}) = O(g(\mathbf{x})h(\mathbf{x}))$ then $f(\mathbf{x}) = g(\mathbf{x})O(h(\mathbf{x}))$.

(iv) If $f_i(\mathbf{x}) = O(g_i(\mathbf{x}))$ for $i = 1, 2, \ldots, n$ then $\sum_{1 \le i \le n} f_i(\mathbf{x}) = O\left(\sum_{1 \le i \le n} |g_i(\mathbf{x})|\right)$.

(v) If $f_n(\mathbf{x}) = O(g_n(\mathbf{x}))$ for $n \ge 1$ then $\sum_{n \ge 1} f_n(\mathbf{x}) = O\left(\sum_{n \ge 1} |g_n(\mathbf{x})|\right)$ provided both $\sum_{n \ge 1} f_n(\mathbf{x})$ and $\sum_{n \ge 1} |g_n(\mathbf{x})|$ converge.

(vi) If $f(\mathbf{x}) = O(g(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{x}_0$ and $T(\mathbf{x})$ is such that $T(\mathbf{x}) \to \mathbf{x}_0$ as $\mathbf{x} \to \mathbf{x}_0$ then $(f \circ T)(\mathbf{x}) = O((g \circ T)(\mathbf{x}))$.

(vii) If $f(\mathbf{x}) = O(g(\mathbf{x}))$ then $\mathrm{Re}(f(\mathbf{x})) = O(g(\mathbf{x}))$ and $\mathrm{Im}(f(\mathbf{x})) = O(g(\mathbf{x}))$.

Proof. Statements (i)-(iii) and (vi) follow immediately from the definition of the O-estimate. Statement (iv) follows from the definition and the triangle inequality. Statement (v) follows in the same way as (iv) given that both sums converge. Statement (vii) follows from the definition the O-estimate and the bounds $|x| \le |z|$ and $|y| \le |z|$ for any complex z. □

The most common application of Proposition 1.2.1 (vi) will be in the single variable case when $z \asymp w$ or $|z| \sim |w|$ (the latter case implying the former) where w is a function of z (usually one that is more simple than x itself). Taking $h(z) = w$, Proposition 1.2.1 (vi) says that if $f(z) = O(g(z))$ then $f(w) = O(g(w))$. In terms of Vinogradov's symbol, $f(z) \ll g(z)$ implies $f(w) \ll g(w)$. O-estimates also behave well with respect to integrals provided the functions are continuous and we are integrating over a compact region:

Proposition 1.2.2. *Suppose $f(\mathbf{x}) = O(g(\mathbf{x}))$ as $\mathbf{x} \to \infty$ and $f(\mathbf{x})$ and $g(\mathbf{x})$ are continuous on a compact region D where this estimate holds. Then*

$$\int_D f(\mathbf{x})\, d\mathbf{x} = O\left(\int_D |g(\mathbf{x})|\, d\mathbf{x}\right).$$

Proof. This follows immediately from the definition of the O-estimate and that continuous functions are bounded on compact regions. □

The next proposition is a collection of some useful expressions for simplifying equations involving O-estimates:

Proposition 1.2.3. *Let $f(\mathbf{x})$ be a function such that $f(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{x}_0$. The following O-estimates hold as $\mathbf{x} \to \mathbf{x}_0$:*

(i) $\frac{1}{1+O(f(\mathbf{x}))} = 1 + O(f(\mathbf{x}))$.

(ii) $(1 + O(f(\mathbf{x})))^w = 1 + O(f(\mathbf{x}))$ *for any complex* w.

(iii) $\log(1 + O(f(\mathbf{x}))) = O(f(\mathbf{x}))$.

(iv) $e^{1+O(f(\mathbf{x}))} = 1 + O(f(\mathbf{x}))$.

Proof. Taking the Taylor series truncated after the first term and applying Taylor's theorem gives the following O-estimates as $z \to 0$:

(i) $\frac{1}{1+z} = 1 + O(z)$.

(ii) $(1 + z)^z = 1 + O(z)$.

(iii) $\log(1 + z) = O(z)$.

(iv) $e^z = 1 + O(z)$.

Now apply Proposition 1.2.1 (v) to each of these O-estimates, and use Proposition 1.2.1 (i). □

For o-estimates, the following properties are useful:

Proposition 1.2.4. *The following o-estimates hold as* $\mathbf{x} \to \mathbf{x}_0$*:*

(i) If $f(\mathbf{x}) = o(g(\mathbf{x}))$ *and* $g(\mathbf{x}) = o(h(\mathbf{x}))$ *then* $f(\mathbf{x}) = o(h(\mathbf{x}))$. *Equivalently,* $o(o(h(\mathbf{x}))) = o(h(\mathbf{x}))$.

(ii) If $f_i(\mathbf{x}) = o(g_i(\mathbf{x}))$ *for* $i = 1, 2$ *then* $f_1(\mathbf{x})f_2(\mathbf{x}) = o(g_1(\mathbf{x})g_2(\mathbf{x}))$.

(iii) If $f(\mathbf{x}) = o(g(\mathbf{x})h(\mathbf{x}))$ *then* $f(\mathbf{x}) = g(\mathbf{x})o(h(\mathbf{x}))$.

(iv) If $f_i(\mathbf{x}) = o(g_i(\mathbf{x}))$ *for* $i = 1, 2, \ldots, n$ *then* $\sum_{1\le i\le n} f_i(\mathbf{x}) = o\left(\sum_{1\le i\le n} |g_i(\mathbf{x})|\right)$.

(v) If $f(\mathbf{x}) = o(g(\mathbf{x}))$ *as* $\mathbf{x} \to \mathbf{x}_0$ *and* $h(\mathbf{x})$ *is such that* $h(\mathbf{x}) \to \mathbf{x}_0$ *as* $\mathbf{x} \to \mathbf{x}_0$ *then* $(f\circ h)(\mathbf{x}) = o((g\circ h)(\mathbf{x}))$.

Proof. Statements (i)-(iii) and (v) follow immediately from the definition of the o-estimate. Statement (iv) follows from the definition and that $\sum_{1\le i\le n} |g_i(\mathbf{x})| \ge |g_i(\mathbf{x})|$. □

Growth and Decay of Functions

We will also be interested in the growth rate of functions. There are many types of growth rates, but we will only recall the ones that are standard. Throughout let $c \ge 1$. First suppose $\mathbf{x} \to \mathbf{x}_0$. If $f(\mathbf{x}) \asymp \log^c ||\mathbf{x}||_\infty$, we say that $f(\mathbf{x})$ is of **logarithmic growth**. If $f(\mathbf{x}) \asymp ||\mathbf{x}||_\infty^c$, we say that $f(\mathbf{x})$ is of **polynomial growth**. If $f(\mathbf{x}) \asymp e^{c||\mathbf{x}||_\infty}$, we say that $f(\mathbf{x})$ is of **exponential growth**. Now suppose $\mathbf{x} \to \infty$. If $f(\mathbf{x}) \asymp \log^{-c} ||\mathbf{x}||_\infty$, we say that $f(\mathbf{x})$ is of **logarithmic decay**. If $f(\mathbf{x}) \asymp ||\mathbf{x}||_\infty^{-c}$ for some $c \ge 1$, we say that $f(\mathbf{x})$ is of **polynomial decay**. If $f(\mathbf{x}) \asymp e^{-c||\mathbf{x}||_\infty}$, we say that $f(\mathbf{x})$ is of **exponential decay**. In all of these cases, we refer to the constant c as the **order** of growth or decay respectively. If $f(\mathbf{x}) = \Omega(||\mathbf{x}||_\infty^n)$ for all $n \ge 0$ then we say $f(\mathbf{x})$ is of **rapid growth**. Alternatively, if $f(\mathbf{x}) = o(||\mathbf{x}||_\infty^{-n})$ for all $n \ge 0$ then we say $f(\mathbf{x})$ is of **rapid decay**.

1.3 Dirichlet Characters

The most important multiplicative periodic functions for an analytic number theorist are the Dirichlet characters. A **Dirichlet character** χ modulo $m \geq 1$ is an m-periodic homomorphism $\chi : \mathbb{Z} \to \mathbb{C}$ such that $\chi(a) = 0$ if and only if $(a, m) > 1$. Note that any Dirichlet character is necessarily a completely multiplicative arithmetic function when restricted $\mathbb{N}$. We call m the **modulus** of χ. Sometimes we will also write χ_m to denote a Dirichlet character modulo m if we need to express the dependence upon the modulus. For any $m \geq 1$, there is always the **principal Dirichlet character** modulo m which we denote by $\chi_{m,0}$ (sometimes also seen as $\chi_{0,m}$ or the ever more confusing χ_0) and is defined by

$$\chi_{m,0}(a) = \begin{cases} 1 & (a, m) = 1, \\ 0 & (a, m) > 1. \end{cases}$$

When $m = 1$, the principal Dirichlet character is identically 1 and we call this the **trivial Dirichlet character**. This is also the only Dirichlet character modulo 1, so $\chi_1 = \chi_{1,0}$. In general, we say a Dirichlet character χ is **principal** if it only takes values 0 or 1. We now discuss some basic facts of Dirichlet characters. Since $a^{\varphi(m)} \equiv 1 \pmod{m}$ by Euler's little theorem, where φ is Euler's totient function, the multiplicativity of χ implies $\chi(a)^{\varphi(m)} = 1$. Therefore the nonzero values of χ_m are $\varphi(m)$-th roots of unity. In particular, there are only finitely many Dirichlet characters of any fixed modulus m. Given two Dirichlet character χ and ψ modulo m, we define $\chi\psi$ by $\chi\psi(a) = \chi(a)\psi(a)$. This is also a Dirichlet character modulo m, so the Dirichlet characters modulo m form an abelian group denoted by X_m. If we have a Dirichlet character χ modulo m then $\overline{\chi}$ defined by $\overline{\chi}(a) = \overline{\chi(a)}$ is also a Dirichlet character modulo m and is called the **conjugate Dirichlet character** of χ. Since the nonzero values of χ are roots of unity, if $(a, m) = 1$ then $\overline{\chi}(a) = \chi(a)^{-1}$. So $\overline{\chi}$ is the inverse of χ. This is all strikingly similar to characters on $(\mathbb{Z}/m\mathbb{Z})^*$ (see Appendix C.3) and there is indeed a connection. By the periodicity of χ, the nonzero values are uniquely determined by $(\mathbb{Z}/m\mathbb{Z})^*$. As χ is multiplicative, it descends to a character χ of $(\mathbb{Z}/m\mathbb{Z})^*$. Conversely, if we are given a character χ of $(\mathbb{Z}/m\mathbb{Z})^*$ we can extend it to a Dirichlet character by defining it to be m-periodic and declaring $\chi(a) = 0$ if $(a, m) > 1$. We call this extension the **zero extension**. So in other words, Dirichlet characters modulo m are the zero extensions of the character group of $(\mathbb{Z}/m\mathbb{Z})^*$. As groups are isomorphic to their character groups (see Proposition C.3.1), we deduce that the group of Dirichlet characters modulo m is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^*$. That is, $X_m \cong \widehat{(\mathbb{Z}/m\mathbb{Z})^*} \cong (\mathbb{Z}/m\mathbb{Z})^*$. In particular, there are $\varphi(m)$ Dirichlet characters modulo m and we identify them with the character group of $(\mathbb{Z}/m\mathbb{Z})^*$. We now state two very useful relations called **Dirichlet orthogonality relations** for Dirichlet characters (this follows from the more general orthogonality relations in Appendix C.3 but we wish to give a direct proof):

Proposition (Dirichlet orthogonality relations).

(i) For any two Dirichlet characters χ and ψ modulo m,

$$\frac{1}{\varphi(m)} \sideset{}{'}\sum_{a \pmod{m}} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

(ii) For any $a, b \in (\mathbb{Z}/m\mathbb{Z})^*$,

$$\frac{1}{\varphi(m)} \sum_{\chi \pmod{m}} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

Proof. We will prove the statements separately.

(i) Denote the left-hand side by S and let b be such that $(b,m)=1$. Then

$$\begin{aligned}\chi(b)\overline{\psi}(b)S &= \frac{\chi(b)\overline{\psi}(b)}{\varphi(m)}\sideset{}{'}\sum_{a \pmod m}\chi(a)\overline{\psi}(a)\\ &= \frac{1}{\varphi(m)}\sideset{}{'}\sum_{a \pmod m}\chi(ab)\overline{\psi}(ab)\\ &= \frac{1}{\varphi(m)}\sideset{}{'}\sum_{a \pmod m}\chi(a)\overline{\psi}(a) && a \mapsto a\bar{b}\\ &= S.\end{aligned}$$

Consequently $S=0$ unless $\chi(b)\overline{\psi}(b)=1$ for all b such that $(b,m)=1$. This happens if and only if $\psi=\chi$ in which case $S=1$ proving (i).

(ii) Denote the left-hand side by S. Let ψ be any Dirichlet character modulo m. Then

$$\begin{aligned}\psi(a)\overline{\psi}(b)S &= \frac{\psi(a)\overline{\psi}(b)}{\varphi(m)}\sum_{\chi \pmod m}\chi(a)\overline{\chi}(b)\\ &= \frac{1}{\varphi(m)}\sum_{\chi \pmod m}\chi\psi(a)\overline{\chi\psi}(b)\\ &= \frac{1}{\varphi(m)}\sum_{\chi \pmod m}\chi(a)\overline{\chi}(b) && \chi \mapsto \chi\overline{\psi}\\ &= S.\end{aligned}$$

Thus $S=0$ unless $\psi(a)\overline{\psi}(b)=\psi(a\bar{b})=1$ for all Dirichlet characters ψ modulo m. If this happens then $a\bar{b}\equiv 1 \pmod m$. To see this, let $m=p_1^{r_1}\cdots p_k^{r_k}$ be the prime factorization of m. By the structure theorem for finite abelian groups, we have

$$(\mathbb{Z}/m\mathbb{Z})^* \cong (\mathbb{Z}/p_1^{r_1}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_k^{r_k}\mathbb{Z})^*.$$

Under this isomorphism, any n taken modulo m with $(n,m)=1$ may be written uniquely as $n=n_1\cdots n_k$ where n_i is taken modulo $p_i^{r_i}$ with $(n_i,p_i^{r_i})=1$ for $1\le i\le k$. Let ω_i be a primitive $p_i^{r_i}$-th root of unity for all i and set

$$\psi(n)=\omega_1^{n_1}\cdots\omega_k^{n_k}.$$

Clearly ψ is a character of $(\mathbb{Z}/m\mathbb{Z})^*$ and is therefore a Dirichlet character modulo m. Writing $a=e_1\cdots e_k$ and $b=f_1\cdots f_k$ under this isomorphism, it follows that

$$1=\omega_1^{e_1\overline{f_1}}\cdots\omega_k^{e_k\overline{f_k}},$$

since $\psi(a\bar{b})=1$. As ω_i has order $p_i^{r_i}$ and $1\le e_i,f_i\le p_i^{r_i}-1$ for all i, the only way the above identity holds is if $e_i\equiv f_i \pmod{p_i^{r_i}}$ for all i. This implies $a\bar{b}\equiv 1 \pmod m$. But then $S=1$ and (ii) follows. □

In many practical settings, the Dirichlet orthogonality relations are often used in the following form:

Corollary 1.3.1.

(i) For any Dirichlet character χ modulo m,

$$\frac{1}{\varphi(m)} \sideset{}{'}\sum_{a \pmod m} \chi(a) = \delta_{\chi,\chi_{m,0}}.$$

(ii) For any $a \in (\mathbb{Z}/m\mathbb{Z})^*$,

$$\frac{1}{\varphi(m)} \sum_{\chi \pmod m} \chi(a) = \delta_{a,1}.$$

Proof. For (i), take $\psi = \chi_{m,0}$ in the Dirichlet orthogonality relations (namely (i)). For (ii), take $b \equiv 1 \pmod m$ in the Dirichlet orthogonality relations (namely (ii)). □

We will now describe how Dirichlet characters of a fixed modulus arise from Dirichlet characters of a smaller modulus. Let χ_{m_1} and χ_{m_2} be Dirichlet characters modulo m_1 and m_2. If $m_1 \mid m_2$ then $(a, m_2) = 1$ implies $(a, m_1) = 1$. Accordingly, we say χ_{m_2} is **induced** from χ_{m_1} (or that χ_{m_1} **lifts** to χ_{m_2}) if

$$\chi_{m_2}(a) = \begin{cases} \chi_{m_1}(a) & \text{if } (a, m_2) = 1, \\ 0 & \text{if } (a, m_2) > 1. \end{cases}$$

All this means is that χ_{m_2} is a Dirichlet character modulo m_2 whose values are given by those of χ_{m_1}. Clearly every Dirichlet character is induced from itself. On the other hand, if there is a prime p dividing m_2 and not m_1 (so m_2 is a larger modulus), χ_{m_2} will be different from χ_{m_1} since $\chi_{m_2}(p) = 0$ but $\chi_{m_1}(p) \neq 0$. In general, we say a Dirichlet character is **primitive** if it is not induced by any character other than itself and **imprimitive** otherwise. Notice that the principal Dirichlet characters are precisely those Dirichlet characters induced from the trivial Dirichlet character, and the only primitive one is the trivial Dirichlet character. It is not a hard matter to determine when Dirichlet characters are induced:

Proposition 1.3.1. *A Dirichlet character χ_{m_2} is induced from a Dirichlet character χ_{m_1} if and only if χ_{m_2} is constant on the residue classes in $(\mathbb{Z}/m_2\mathbb{Z})^*$ that are congruent modulo m_1. When this happens, χ_{m_1} is uniquely determined.*

Proof. For the forward implication, if χ_{m_2} is induced from χ_{m_1} then χ_{m_2} is constant on the residue classes in $(\mathbb{Z}/m_2\mathbb{Z})^*$ that are congruent modulo m_1 because χ_{m_1} is. For the reverse implication, first note that the surjective homomorphism $\mathbb{Z}/m_2\mathbb{Z} \to \mathbb{Z}/m_1\mathbb{Z}$ given by reduction modulo m_1 induces a surjective homomorphism $(\mathbb{Z}/m_2\mathbb{Z})^* \to (\mathbb{Z}/m_1\mathbb{Z})^*$ (because reduction modulo m_1 preserve inverses). Now suppose χ_{m_2} is constant on the residue classes in $(\mathbb{Z}/m_2\mathbb{Z})^*$ that are congruent modulo m_1. Surjectivity of the previously mentioned map implies that χ_{m_2} induces a unique character on $(\mathbb{Z}/m_1\mathbb{Z})^*$ and hence a unique Dirichlet character modulo m_1. By construction χ_{m_2} is induced from χ_{m_1}. □

We are interested in primitive Dirichlet characters because they are the building blocks for all Dirichlet characters:

Theorem 1.3.1. *Every Dirichlet character χ is induced from a primitive Dirichlet character $\widetilde{\chi}$ that is uniquely determined by χ.*

Proof. Let the modulus of χ be m. Define a partial ordering on the set of Dirichlet characters where $\psi \leq \chi$ if χ is induced from ψ. This ordering is clearly reflexive, and it is transitive by Proposition 1.3.1. Set

$$X = \left\{ \psi \in \bigcup_{d \mid m} X_d : \psi \leq \chi \right\}$$

This set is nonempty and finite by Proposition 1.3.1. Now suppose $\chi_{m_1}, \chi_{m_2} \in X$. Set $m_3 = (m_1, m_2)$. Also from Proposition 1.3.1, χ is constant on the residue classes of $(\mathbb{Z}/m\mathbb{Z})^*$ that are congruent modulo m_1 or m_2 and hence also m_3. Therefore Proposition 1.3.1 implies there is a unique Dirichlet character χ_{m_3} modulo m_3 that lifts to χ_{m_1} and χ_{m_2}. We have now shown that every pair $\chi_{m_1}, \chi_{m_2} \in X$ has a lower bound χ_{m_3}. Hence X contains a primitive Dirichlet character $\widetilde{\chi}$ that is minimal with respect to this partial ordering. There is only one such element. Indeed, since $m_3 \leq m_1, m_2$ the partial ordering is compatible with the total ordering by modulus. Thus $\widetilde{\chi}$ is unique. □

In light of Theorem 1.3.1, we define **conductor** q of a Dirichlet character χ modulo m to be the modulus of the unique primitive character $\widetilde{\chi}$ that induces χ. This is the most important data of a Dirichlet character since it tells us how χ is built. Note that χ is primitive if and only if its conductor and modulus are equal. Also observe that if χ has conductor q then χ is actually q-periodic (necessarily $q \mid m$), and the nonzero values of χ are all q-th roots of unity because those are the nonzero values of $\widetilde{\chi}$. Moreover, $\chi = \widetilde{\chi}\chi_{\frac{m}{q},0}$ by the definition of induced Dirichlet characters. Moreover, primitive characters behave well with respect to multiplication if the conductors are relatively prime as the following proposition shows:

Proposition 1.3.2. *Suppose χ_1 and χ_2 are Dirichlet characters modulo q_1 and q_2 respectively with q_1 and q_2 relatively prime. Set $\chi = \chi_1\chi_2$ so that χ is a Dirichlet character modulo q_1q_2. Then χ is a primitive if and only if χ_1 and χ_2 are both primitive.*

Proof. First suppose χ is primitive of conductor q. If d_1 and d_2 are the conductors of χ_1 and χ_2 respectively then χ is d_1d_2-periodic and primitivity further implies that $q \mid d_1d_2$. But as $d_1 \mid q_1$, $d_2 \mid q_2$, and $q = q_1q_2$, we must have $q = d_1d_2$ and hence $d_1 = q_1$ and $d_2 = q_2$. It follows that χ_1 and χ_2 are both primitive. Conversely, suppose χ_1 and χ_2 are both primitive. If d is the conductor of χ, set $d_1 = (d, q_1)$ and $d_2 = (d, q_2)$. As $(q_1, q_2) = 1$ and $q = q_1q_2$, we must have $(d_1, d_2) = 1$ and $d_1d_2 = q$. But then $d_1 = q_1$ and $d_2 = q_2$. Hence $d = q_1q_2$ which implies that χ is primitive. □

We would now like to distinguish Dirichlet characters whose nonzero values are either real or imaginary. We say χ is **real** if it is real-valued. Hence the nonzero values of χ are 1 or -1 since they must be roots of unity. We say χ is an **complex** if it is not real. More commonly, we distinguish Dirichlet characters modulo m by their order as an element of $(\mathbb{Z}/m\mathbb{Z})^*$. If χ is of order 2, 3, etc. in $(\mathbb{Z}/m\mathbb{Z})^*$ then we say it is **quadratic**, **cubic**, etc. In particular, a Dirichlet character is quadratic if and only if it is real. For any Dirichlet character χ, $\chi(-1) = \pm 1$ because $\chi(-1)^2 = 1$. We would like to distinguish this parity. Accordingly, we say χ is **even** if $\chi(-1) = 1$ and **odd** if $\chi(-1) = -1$. Clearly even Dirichlet characters are even functions and odd Dirichlet characters are odd functions. Moreover, χ and $\overline{\chi}$ have the same parity and any lift of χ has the same parity as χ. Also note that

$$\frac{\chi(1) - \chi(-1)}{2} = \begin{cases} 0 & \text{if } \chi \text{ is even,} \\ 1 & \text{if } \chi \text{ is odd.} \end{cases}$$

Lastly, we would like to discuss quadratic Dirichlet characters. We can construct quadratic Dirichlet characters using Jacobi symbols. If $m \geq 1$ is odd, consider

$$\chi_m(n) = \left(\tfrac{n}{m}\right).$$

Clearly χ_m a quadratic Dirichlet character modulo m because the Jacobi symbol is multiplicative, nonzero if and only if $(n, m) = 1$, and determined modulo m. However, quadratic Dirichlet characters given

by Jacobi symbols do not exhaust all possible quadratic Dirichlet characters. For this, we need to use Kronecker symbols. We say that $D \in \mathbb{Z}$ is a **fundamental discriminant** if D is of the form

$$D = \begin{cases} d & \text{if } D \equiv 1 \pmod 4, \\ 4d & \text{if } \frac{D}{4} \equiv 2, 3 \pmod 4, \end{cases}$$

for some square-free $d \in \mathbb{Z}$. Necessarily $d \equiv 1 \pmod 4$ or $d \equiv 2, 3 \pmod 4$ respectively and thus nonzero. We define the **quadratic Dirichlet character** χ_D associated to the fundamental discriminant D by

$$\chi_D(m) = \left(\frac{D}{m}\right).$$

It turns out that χ_D defines a primitive quadratic Dirichlet character, and exhausts all primitive quadratic Dirichlet characters, as the following theorem shows:

Theorem 1.3.2. *If D is a fundamental discriminant and $D \neq 1$ then χ_D is a primitive quadratic Dirichlet character of conductor $|D|$. Moreover, all primitive quadratic Dirichlet characters are of this form.*

Proof. We first show that χ_D is a quadratic Dirichlet character of conductor $|D|$. If $D \equiv 1 \pmod 4$, the sign in quadratic reciprocity is always 1 so that

$$\chi_D(m) = \left(\frac{m}{|D|}\right),$$

and hence is a quadratic Dirichlet character modulo $|D|$ because it is given by the Jacobi symbol. If $\frac{D}{4} \equiv 3 \pmod 4$, the sign in quadratic reciprocity is $\left(\frac{-1}{m}\right)$ which is the primitive quadratic Dirichlet character modulo 4 (there are only two Dirichlet characters modulo 4 since $\varphi(4) = 2$ and clearly $\left(\frac{-1}{m}\right)$ is not principal) so that

$$\chi_D(m) = \left(\frac{-1}{m}\right)\left(\frac{m}{\left|\frac{D}{4}\right|}\right),$$

and hence is a Dirichlet character modulo $|D|$. If $\frac{D}{4} \equiv 2 \pmod{16}$, first observe that $\left(\frac{D}{m}\right) = \left(\frac{8}{m}\right)\left(\frac{\frac{D}{8}}{m}\right)$ where $\left(\frac{8}{m}\right)$ is one of the two primitive quadratic Dirichlet character modulo 8 (the other is $\left(\frac{-8}{m}\right)$ as there are four Dirichlet character modulo 8 because $\varphi(8) = 4$ and the other two are the principal Dirichlet character and the Dirichlet character induced from $\left(\frac{-1}{m}\right)$ as mentioned previously). As $\frac{D}{8} \equiv 1, 3 \pmod 4$, the sign in quadratic reciprocity is either 1 or $\left(\frac{-1}{m}\right)$ according to these two cases. Thus

$$\chi_D(m) = \left(\frac{8}{m}\right)\left(\frac{m}{\left|\frac{D}{8}\right|}\right) \quad \text{or} \quad \chi_D(m) = \left(\frac{-8}{m}\right)\left(\frac{m}{\left|\frac{D}{8}\right|}\right),$$

according to if $\frac{D}{8} \equiv 1, 3 \pmod 4$ respectively, and hence is a quadratic Dirichlet character modulo $|D|$. We can compactly express all of these cases as follows:

$$\chi_D(m) = \begin{cases} \left(\frac{m}{|D|}\right) & \text{if } D \equiv 1 \pmod 4, \\ \left(\frac{-1}{m}\right)\left(\frac{m}{\left|\frac{D}{4}\right|}\right) & \text{if } \frac{D}{4} \equiv 3 \pmod 4, \\ \left(\frac{8}{m}\right)\left(\frac{m}{\left|\frac{D}{8}\right|}\right) & \text{if } \frac{D}{8} \equiv 1 \pmod 4, \\ \left(\frac{-8}{m}\right)\left(\frac{m}{\left|\frac{D}{8}\right|}\right) & \text{if } \frac{D}{8} \equiv 3 \pmod 4. \end{cases}$$

This shows that χ_D is a quadratic Dirichlet characters modulo $|D|$. It easily follows from the above that χ_D is primitive. Indeed, we have already mentioned that the characters $\left(\frac{-1}{m}\right)$, $\left(\frac{8}{m}\right)$, and $\left(\frac{-8}{m}\right)$ are all primitive. Therefore, since D, $\frac{D}{4}$, and $\frac{D}{8}$ are square-free according to their equivalences modulo 4 as given above, and $D \neq 1$, it suffices to show by Proposition 1.3.2 that χ_p is primitive for all primes p with $p \neq 2$. This is immediate since p is prime and clearly χ_p is not principal. We now show that every primitive quadratic Dirichlet character is of the form χ_D for some fundamental discriminant D. By Proposition 1.3.2, it suffices to consider primitive quadratic Dirichlet character modulo $q = p^m$ for some prime p and $m \geq 1$. First suppose that $p \neq 2$. Then $(\mathbb{Z}/q\mathbb{Z})^*$ is cyclic and so every $n \in (\mathbb{Z}/p^m\mathbb{Z})^*$ is of the form $n = v^\nu$ for some $\nu \in (\mathbb{Z}/\varphi(p^m)\mathbb{Z})$ and where v is a generator of $(\mathbb{Z}/p^m\mathbb{Z})^*$. It follows that every Dirichlet character χ modulo p^m is of the form

$$\chi(n) = e^{\frac{2\pi i k\nu}{\varphi(p^m)}},$$

where $0 \leq k \leq \varphi(q) - 1$. Indeed, this is a unique Dirichlet character for every such k and there are $\varphi(p^m)$ Dirichlet characters modulo p^m which is the same number of choices for k. Moreover, χ is primitive if and only if $p \nmid k$ for otherwise χ is a Dirichlet character modulo p^{m-1}. Similarly, χ is quadratic if and only if $\frac{k}{\varphi(p^m)}$ has at most 2 in its denominator which is equivalent to $k \equiv \frac{\varphi(p^m)}{2} \pmod{\varphi(p^m)}$ and hence such a k exists and is unique because $p \neq 2$. We also see that if χ is quadratic, it is imprimitive unless $m = 1$ for then $\varphi(p) = p - 1$ is not a multiple of p. All of this is to say that there is a unique quadratic Dirichlet character modulo q and it is primitive if and only if $q = p$. Necessarily, this unique primitive quadratic Dirichlet character modulo p is given by χ_D for the fundamental discriminant $D = p$ if $p \equiv 1 \pmod 4$ and $D = -p$ if $p \equiv 3 \pmod 4$. Now suppose $p = 2$ so that $q = 2^m$ for some $m \geq 1$. If $m = 1$, $\varphi(2) = 1$ and there are no primitive quadratic Dirichlet characters as the only Dirichlet character is principal. If $m = 2$, $\varphi(4) = 2$ so that there are two Dirichlet characters. They are both quadratic but only one is primitive, namely the principal Dirichlet character as well as the aforementioned primitive quadratic Dirichlet character $\left(\frac{-1}{m}\right)$. For $m \geq 3$, $(\mathbb{Z}/2^m\mathbb{Z})^* \cong C_2 \times C_{2^{m-2}}$ where C_2 and $C_{2^{m-2}}$ are the cyclic groups of order 2 and 2^{m-2} respectively. Therefore every $n \in (\mathbb{Z}/2^m\mathbb{Z})^*$ is of the form $n = (-1)^\mu 5^\nu$ for $\mu \in \mathbb{Z}/2\mathbb{Z}$ and $\nu \in \mathbb{Z}/2^{m-2}\mathbb{Z}$ (because the orders of -1 and 5 modulo 2^m are 2 and 2^{m-2} respectively, with the latter case following by induction for $m \geq 3$, and that $\langle -1\rangle \cap \langle 5\rangle = \{1\}$). Then every Dirichlet character χ modulo 2^m, for $m \geq 3$, is of the form

$$\chi(n) = e^{\frac{2\pi i j\mu}{2}} e^{\frac{2\pi i k\nu}{2^{m-2}}},$$

where $0 \leq j \leq 1$ and $0 \leq k \leq 2^{m-2} - 1$. Indeed, this is a unique Dirichlet character for every such choice of j and k and there are 2^{m-1} Dirichlet characters modulo 2^m which is the same number of choices for j and k. Similarly to the case for $p \neq 2$, χ is primitive if and only if $2^{m-2} \nmid k$, or equivalently, k is odd. Moreover, χ is quadratic if and only if $\frac{k}{2^{m-2}}$ has at most 2 in its denominator which is to say that $2^{m-3} \mid k$. Therefore for a primitive quadratic Dirichlet to exist we must have k odd and $2^{m-3} \mid k$ which can happen if and only if $m = 3$. Then $\phi(8) = 4$, so that there are four Dirichlet characters. They are all quadratic but only two are primitive, namely the principal Dirichlet character, the Dirichlet character induced from $\left(\frac{-1}{m}\right)$, and the two aforementioned primitive quadratic Dirichlet characters given by $\left(\frac{8}{m}\right)$ and $\left(\frac{-8}{m}\right)$. These three primitive quadratic Dirichlet characters are given by χ_D for the fundamental discriminants $D = -4$, $D = 8$, and $D = -8$ respectively. We have now shown that all primitive quadratic Dirichlet characters of prime power modulus are given by χ_D for some fundamental discriminant D and thus the same follows for all primitive quadratic Dirichlet characters by Proposition 1.3.2. This completes the proof. □

It follows from Theorem 1.3.2 that all quadratic Dirichlet characters are induced from some χ_D (including $D = 1$ since this corresponds to the trivial Dirichlet character). In particular, so too are the quadratic Dirichlet characters given by Jacobi symbols.

1.4 Exponential Sums

Number theory comes with its class of exponential sums that appear naturally. They play the role of discrete counterparts to continuous objects (there is a rich underpinning here). Without a sufficient understanding of these sums, they would cause a discrete obstruction to an analytic problem that we wish to solve.

Ramanujan and Gauss Sums

Let's begin with the Ramanujan sum. For $m \geq 1$ and $b \in \mathbb{Z}$, the **Ramanujan sum** $r(b, m)$ is defined by

$$r(b,m) = \sideset{}{'}\sum_{a \pmod m} e^{\frac{2\pi i a b}{m}}.$$

Note that the Ramanujan sum is a finite sum of m-th roots of unity on the unit circle. Clearly we have $r(0, m) = \varphi(m)$. Ramanujan sums can be computed explicitly by means of the Möbius function (see Appendix A.1):

Proposition 1.4.1. *For any $m \geq 1$ and any nonzero $b \in \mathbb{Z}$,*

$$r(b,m) = \sum_{\ell \mid (b,m)} \ell \mu\left(\frac{m}{\ell}\right).$$

Proof. This is a computation:

$$\begin{aligned}
r(b,m) &= \sideset{}{'}\sum_{a \pmod m} e^{\frac{2\pi i a b}{m}} \\
&= \sum_{a \pmod m} e^{\frac{2\pi i a b}{m}} \sum_{d \mid (a,m)} \mu(d) && \text{Proposition A.2.1} \\
&= \sum_{d \mid m} \mu(d) \sum_{\substack{a \pmod m \\ d \mid a}} e^{\frac{2\pi i a b}{m}} \\
&= \sum_{d \mid m} \mu(d) \sum_{k d \pmod m} e^{\frac{2\pi i k d b}{m}} && a \mapsto kd \\
&= \sum_{d \mid m} \mu(d) \sum_{k \pmod{\frac{m}{d}}} e^{\frac{2\pi i k b}{\frac{m}{d}}}.
\end{aligned}$$

Now if $\frac{m}{d} \mid b$ the inner sum is $\frac{m}{d}$, and otherwise it is zero because the change of variables $k \mapsto k\bar{b}$ shows that the sum is over all $\left(\frac{m}{d}\right)$-th roots of unity. So the double sum above reduces to

$$\sum_{\substack{\frac{m}{d} \mid b \\ d \mid m}} \frac{m}{d} \mu(d) = \sum_{\ell \mid (b,m)} \ell \mu\left(\frac{m}{\ell}\right),$$

upon performing the change of variables $\frac{m}{d} \mapsto \ell$. This completes the proof. □

We can also define a Ramanujan sum associated to Dirichlet characters. Let χ be a Dirichlet character modulo m. For any $b \in \mathbb{Z}$, the **Ramanujan sum** $\tau(b, \chi)$ associated to χ is given by

$$\tau(b,\chi) = \sum_{a \pmod m} \chi(a) e^{\frac{2\pi i a b}{m}}.$$

If $b = 1$ we will write $\tau(\chi)$ instead. That is, $\tau(\chi) = \tau(1, \chi)$. We call $\tau(\chi)$ the **Gauss sum** associated to χ. Observe that if $m = 1$ then χ is the trivial character and $\tau(b, \chi) = 1$. So the interesting cases are when $m \geq 2$. There are some basic properties of these sums which are very useful:

Proposition 1.4.2. *Let χ and ψ be nontrivial Dirichlet characters modulo m and n respectively and let $b \in \mathbb{Z}$. Then the following hold:*

(i) $\overline{\tau(b, \overline{\chi})} = \chi(-1)\tau(b, \chi)$.

(ii) If $(b, m) = 1$ *then* $\tau(b, \chi) = \overline{\chi}(b)\tau(\chi)$.

(iii) If $(b, m) > 1$ *and* χ *is primitive then* $\tau(b, \chi) = 0$.

(iv) If $(m, n) = 1$ *then* $\tau(b, \chi\psi) = \chi(n)\psi(m)\tau(b, \chi)\tau(b, \psi)$.

(v) Let q be the conductor of χ and let $\widetilde{\chi}$ be the primitive Dirichlet character that lifts to χ. Then

$$\tau(\chi) = \mu\left(\frac{m}{q}\right)\widetilde{\chi}\left(\frac{m}{q}\right)\tau(\widetilde{\chi}).$$

Proof. We will prove the statements separately.

(i) We compute

$$\begin{aligned}
\overline{\tau(b, \overline{\chi})} &= \overline{\sum_{a \pmod m} \overline{\chi}(a) e^{\frac{2\pi i a b}{m}}} \\
&= \sum_{a \pmod m} \chi(a) e^{-\frac{2\pi i a b}{m}} \\
&= \sum_{a \pmod m} \chi(-a) e^{\frac{2\pi i a b}{m}} && a \mapsto -a \\
&= \chi(-1) \sum_{a \pmod m} \chi(a) e^{\frac{2\pi i a b}{m}} \\
&= \chi(-1)\tau(b, \chi).
\end{aligned}$$

This proves (i).

(ii) We compute

$$\begin{aligned}
\tau(b, \chi) &= \sum_{a \pmod m} \chi(a) e^{\frac{2\pi i a b}{m}} \\
&= \sum_{a \pmod m} \chi(a\overline{b}) e^{\frac{2\pi i a}{m}} && a \mapsto a\overline{b} \\
&= \overline{\chi}(b) \sum_{a \pmod m} \chi(a) e^{\frac{2\pi i a}{m}} \\
&= \overline{\chi}(b)\tau(\chi).
\end{aligned}$$

This proves (ii).

(iii) Suppose d is a proper divisor of m and c is an integer c such that $c \equiv 1 \pmod{m}$. Then necessarily $(c, m) = 1$. Also note that as $d \mid m$, $c \equiv 1 \pmod{d}$ and $(c, d) = 1$. Moreover, there is such a c with the additional property that $\chi(c) \neq 1$. For if not, χ is induced from $\chi_{d,0}$ which contradicts χ being primitive. Now take $d = \frac{m}{(b,m)}$ and choose c as above. Then

$$\chi(c)\tau(b,\chi) = \sum_{a \pmod{m}} \chi(ac) e^{\frac{2\pi i a b}{m}} = \sum_{a \pmod{m}} \chi(a) e^{\frac{2\pi i a b \overline{c}}{m}} = \tau(b,\chi)$$

upon making the change of variables $a \mapsto a\overline{c}$ and where the last equality holds because $\overline{c} \equiv 1 \pmod{d}$ and $e^{\frac{2\pi i b}{m}}$ is a d-th root of unity. So altogether $\chi(c)\tau(b,\chi) = \tau(b,\chi)$. Since $\chi(c) \neq 1$, we conclude $\tau(b,\chi) = 0$ and (iii) follows.

(iv) Since $(m, n) = 1$, the Chinese remainder theorem implies that we have an isomorphism

$$(\mathbb{Z}/m\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/mn\mathbb{Z}) \qquad a \oplus a' \mapsto an + a'm.$$

Under this isomorphism, we make the following computation:

$$\begin{aligned}
\tau(b, \chi\psi) &= \sum_{an+a'm \pmod{mn}} \chi\psi(an + a'm) e^{\frac{2\pi i (an+a'm) b}{mn}} \\
&= \sum_{a \pmod{m}} \sum_{a' \pmod{n}} \chi\psi(an + a'm) e^{\frac{2\pi i (an+a'm) b}{mn}} \\
&= \sum_{a \pmod{m}} \sum_{a' \pmod{n}} \chi(an)\psi(a'm) e^{\frac{2\pi i a b}{m}} e^{\frac{2\pi i a' b}{n}} \\
&= \chi(n)\psi(m) \sum_{a \pmod{m}} \chi(a) e^{\frac{2\pi i a b}{m}} \sum_{a' \pmod{n}} \psi(a') e^{\frac{2\pi i a' b}{n}} \\
&= \chi(n)\psi(m)\tau(b,\chi)\tau(b,\psi).
\end{aligned}$$

This proves (iv).

(v) If $\left(\frac{m}{q}, q\right) > 1$ then $\widetilde{\chi}\left(\frac{m}{q}\right) = 0$ so we need to show $\tau(\chi) = 0$. As $\left(\frac{m}{q}, q\right) > 1$, there exists a prime p such that $p \mid \frac{m}{q}$ and $p \mid q$. By Euclidean division we may write any a modulo m in the form $a = a'\frac{m}{p} + a''$ with a' taken modulo p and a'' taken modulo $\frac{m}{p}$. Then

$$\tau(\chi) = \sum_{a \pmod{m}} \chi(a) e^{\frac{2\pi i a}{m}} = \sum_{\substack{a' \pmod{p} \\ a'' \pmod{\frac{m}{p}}}} \chi\left(a'\frac{m}{p} + a''\right) e^{\frac{2\pi i \left(a'\frac{m}{p} + a''\right)}{m}}. \tag{1.1}$$

Since $p \mid \left(\frac{m}{q}, q\right)$, we have $p^2 \mid m$. Therefore $\left(a'\frac{m}{p} + a'', m\right) = 1$ if and only if $\left(a'\frac{m}{p} + a'', \frac{m}{p}\right) = 1$ and this latter condition is equivalent to $\left(a'', \frac{m}{p}\right) = 1$. Thus the last sum in Equation (1.1) is

$$\sum_{\substack{a' \pmod{p} \\ a'' \pmod{\frac{m}{p}} \\ \left(a'', \frac{m}{p}\right) = 1}} \chi\left(a'\frac{m}{p} + a''\right) e^{\frac{2\pi i \left(a'\frac{m}{p} + a''\right)}{m}}.$$

As $p \mid \frac{m}{q}$, we know $q \mid \frac{m}{p}$ so that $a'\frac{m}{p} + a'' \equiv a'' \pmod q$. Then Proposition 1.3.1 implies $\chi\left(a'\frac{m}{p} + a''\right) = \widetilde{\chi}(a'')$ and this sum is further reduced to

$$\sum_{a'' \pmod{\frac{m}{p}}}{}' \widetilde{\chi}(a'')e^{\frac{2\pi i a''}{m}} \sum_{a' \pmod p} e^{\frac{2\pi i a'}{p}}. \tag{1.2}$$

The inner sum in Equation (1.2) vanishes since it is the sum over all p-th roots of unity and thus $\tau(\chi) = 0$. Now suppose $\left(\frac{m}{q}, q\right) = 1$. Then (iv) implies

$$\tau(\chi) = \tau(\widetilde{\chi}\chi_{\frac{m}{q},0}) = \widetilde{\chi}\left(\frac{m}{q}\right)\chi_{\frac{m}{q},0}(q)\tau(\widetilde{\chi})\tau(\chi_{\frac{m}{q},0}) = \tau(\chi_{\frac{m}{q},0})\widetilde{\chi}\left(\frac{m}{q}\right)\tau(\widetilde{\chi}).$$

Now observe that $\tau(\chi_{\frac{m}{q},0}) = r\left(1, \frac{m}{q}\right)$. By Proposition 1.4.1 we see that $r\left(1, \frac{m}{q}\right) = \mu\left(\frac{m}{q}\right)$ and

$$\tau(\chi) = \mu\left(\frac{m}{q}\right)\widetilde{\chi}\left(\frac{m}{q}\right)\tau(\widetilde{\chi}),$$

as claimed. This proves (v). □

Notice that Proposition 1.4.2 reduces the evaluation of the Ramanujan sum $\tau(b, \chi)$ to that of the Gauss sum $\tau(\chi)$ at least when χ is primitive. When χ is imprimitive and $(b, m) > 1$ we need to appeal to evaluating $\tau(b, \chi)$ by more direct means. Evaluating $\tau(\chi)$ for general characters χ turns out to be a very difficult problem and is still open. However, it is not difficult to determine the modulus of $\tau(\chi)$ when χ is primitive:

Theorem 1.4.1. *Let χ be a primitive Dirichlet character of conductor q. Then*

$$|\tau(\chi)| = \sqrt{q}.$$

Proof. If χ is the trivial character this is obvious since $\tau(\chi) = 1$. So we may assume χ is nontrivial. Now this is just a computation:

$$\begin{aligned}
|\tau(\chi)|^2 &= \tau(\chi)\overline{\tau(\chi)} \\
&= \sum_{a \pmod q} \tau(\chi)\overline{\chi}(a)e^{-\frac{2\pi i a}{q}} \\
&= \sum_{a \pmod q} \tau(a, \chi)e^{-\frac{2\pi i a}{q}} && \text{Proposition 1.4.2 (ii)} \\
&= \sum_{a \pmod q} \left(\sum_{a' \pmod q} \chi(a')e^{\frac{2\pi i a a'}{q}}\right) e^{-\frac{2\pi i a}{q}} \\
&= \sum_{a, a' \pmod q} \chi(a')e^{\frac{2\pi i a(a'-1)}{q}} \\
&= \sum_{a' \pmod q} \chi(a') \left(\sum_{a \pmod q} e^{\frac{2\pi i a(a'-1)}{q}}\right).
\end{aligned}$$

Let $S(a')$ denote the inner sum. For the a' such that $a' - 1 \equiv 0 \pmod q$, we have $S(a') = q$. Otherwise, the change of variables $a \mapsto a\overline{(a'-1)}$ shows that $S(a') = 0$ because it is the sum of all q-th roots of unity. It follows that the double sum is $\chi(1)q = q$. So altogether $|\tau(\chi)|^2 = q$ and hence $|\tau(\chi)| = \sqrt{q}$. □

As an almost immediate corollary to Theorem 1.4.1, we deduce a useful expression for primitive Dirichlet characters of conductor q:

Corollary 1.4.1. *Let χ be a primitive Dirichlet character of conductor q. Then*

$$\tau(n,\chi) = \overline{\chi}(n)\tau(\chi),$$

for all $n \in \mathbb{Z}$. In particular,

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{a \pmod q} \overline{\chi}(a) e^{\frac{2\pi i a n}{q}},$$

for all $n \in \mathbb{Z}$.

Proof. If χ is the trivial character this is obvious since $\tau(n,\chi) = 1$. So assume χ is nontrivial. If $(n,q) = 1$ then the first identity is Proposition 1.4.2 (ii). If $(n,q) > 1$ then the first identity follows from Proposition 1.4.2 (iii) and that $\overline{\chi}(n) = 0$. This proves the first identity in full. For the second identity, first note that $\tau(\chi) \neq 0$ by Theorem 1.4.1. Replacing χ with $\overline{\chi}$, dividing the first identity by $\tau(\chi)$, and expanding the Ramanujan sum, gives the second identity. □

In light of Theorem 1.4.1 we define the **epsilon factor** ε_χ for a Dirichlet character χ modulo m by

$$\varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{m}}.$$

Theorem 1.4.1 says that this value lies on the unit circle when χ is primitive and not the trivial character. The question of the evaluation of Gauss sums boils down to determining what value the epsilon factor is. This is the real difficultly as the epsilon factor is quite difficult to calculate and its value is not known for general Dirichlet characters. However, when χ is primitive there is a simple relationship between ε_χ and $\varepsilon_{\overline{\chi}}$:

Proposition 1.4.3. *Let χ be a primitive Dirichlet character of conductor q. Then*

$$\varepsilon_\chi \varepsilon_{\overline{\chi}} = \chi(-1).$$

Proof. If χ is trivial this is obvious since $\varepsilon_\chi = \varepsilon_{\overline{\chi}} = 1$. So assume χ is nontrivial. By Proposition 1.4.2 (i) and that ε_χ lies on the unit circle, we have

$$\varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{q}} = \chi(-1)\frac{\overline{\tau(\chi)}}{\sqrt{q}} = \chi(-1)\varepsilon_{\overline{\chi}}^{-1},$$

from whence the statement follows. □

Quadratic Gauss Sums

Another important sum is the quadratic Gauss sum. For any $m \geq 1$ and any $b \in \mathbb{Z}$, the **quadratic Gauss sum** $g(b,m)$ is defined by

$$g(b,m) = \sum_{a \pmod m} e^{\frac{2\pi i a^2 b}{m}}.$$

If $b = 1$ we write $g(m)$ instead. That is, $g(m) = g(1,m)$. It turns out that if χ_m is the quadratic Dirichlet character given by the Jacobi symbol then $\tau(b,\chi_m) = g(b,m)$ provided m is square-free. This will take a little work to prove. We first reduce to the case when $(b,m) = 1$:

Proposition 1.4.4. *Let $m \geq 1$ be odd and let $b \in \mathbb{Z}$. Then*

$$g(b,m) = (b,m) g\left(\frac{b}{(b,m)}, \frac{m}{(b,m)}\right).$$

Proof. By Euclidean division write any a modulo m in the form $a = a' \frac{m}{(b,m)} + a''$ with a' take modulo (b,m) and a'' take modulo $\frac{m}{(b,m)}$. Then

$$\begin{aligned}
g(b,m) &= \sum_{a \pmod m} e^{\frac{2\pi i a^2 b}{m}} \\
&= \sum_{\substack{a' \pmod{(b,m)} \\ a'' \pmod{\frac{m}{(b,m)}}}} e^{\frac{2\pi i \left(a' \frac{m}{(b,m)} + a''\right)^2 b}{m}} \\
&= \sum_{a'' \pmod{\frac{m}{(b,m)}}} e^{\frac{2\pi i (a'')^2 b}{m}} \sum_{a' \pmod{(b,m)}} e^{\frac{2\pi i \left(2a''a' \frac{m}{(b,m)} + \left(a' \frac{m}{(b,m)}\right)^2\right) b}{m}} \\
&= \sum_{a'' \pmod{\frac{m}{(b,m)}}} e^{\frac{2\pi i (a'')^2 \frac{b}{(b,m)}}{\frac{m}{(b,m)}}} \sum_{a' \pmod{(b,m)}} e^{\frac{2\pi i \left(2a''a' \frac{m}{(b,m)} + \left(a' \frac{m}{(b,m)}\right)^2\right) \frac{b}{(b,m)}}{\frac{m}{(b,m)}}} \\
&= (b,m) \sum_{a'' \pmod{\frac{m}{(b,m)}}} e^{\frac{2\pi i (a'')^2 \frac{b}{(b,m)}}{\frac{m}{(b,m)}}},
\end{aligned}$$

where the last line follows because $\left(2a''a' \frac{m}{(b,m)} + \left(a' \frac{m}{(b,m)}\right)^2\right) \equiv 0 \pmod{\frac{m}{(b,m)}}$ and thus the inner sum is (b,m). The remaining sum is $g\left(\frac{b}{(b,m)}, \frac{m}{(b,m)}\right)$ which finishes the proof. □

As a consequence of Proposition 1.4.4, we may always assume $(b,m) = 1$. Now we give an equivalent formulation of the Ramanujan sum associated to quadratic Dirichlet characters given by Jacobi symbols and show that in the case $m = p$ an odd prime, the Ramanujan and quadratic Gauss sums agree:

Proposition 1.4.5. *Let $m \geq 1$ and $b \in \mathbb{Z}$ be such that $(b,m) = 1$. Also let χ_m be the quadratic Dirichlet character given by the Jacobi symbol. Then*

$$\tau(b, \chi_m) = \sum_{a \pmod m} \left(1 + \left(\frac{a}{m}\right)\right) e^{\frac{2\pi i a b}{m}}.$$

Moreover, when $m = p$ is prime,

$$\tau(b, \chi_p) = g(b,p).$$

Proof. If $m = 1$ the claim is obvious since $\tau(b, \chi_1) = 1$ so assume $m > 1$. To prove the first statement, observe that

$$\sum_{a \pmod m} \left(1 + \left(\frac{a}{m}\right)\right) e^{\frac{2\pi i a b}{m}} = \sum_{a \pmod m} e^{\frac{2\pi i a b}{m}} + \sum_{a \pmod m} \left(\frac{a}{m}\right) e^{\frac{2\pi i a b}{m}}.$$

The first sum on the right-hand side is zero as it is the sum over all m-th roots of unity since $(b,m) = 1$. This proves the first claim. Now let $m = p$ be an odd prime. From the definition of the Jacobi symbol we

see that $1+\left(\frac{a}{p}\right)=2,0$ depending on if a is a quadratic residue modulo p or not provided $a \not\equiv 0 \pmod{p}$. If $a \equiv 0 \pmod{p}$ then $1+\left(\frac{a}{p}\right)=1$. Moreover, if a is a quadratic residue modulo p then $a \equiv (a')^2 \pmod{p}$ for some a'. So one the one hand,

$$\tau(b,\chi_p)=\sum_{a \pmod{p}}\left(1+\left(\frac{a}{p}\right)\right)e^{\frac{2\pi iab}{p}}=1+2\sum_{\substack{a \pmod{p}\\ a\equiv(a')^2 \pmod{p}\\ a\not\equiv 0 \pmod{p}}} e^{\frac{2\pi i(a')^2b}{p}}.$$

On the other hand,

$$g(b,p)=1+\sum_{\substack{a \pmod{p}\\ a\not\equiv 0 \pmod{p}}} e^{\frac{2\pi ia^2b}{p}},$$

but this last sum counts every quadratic residue twice because $(-a)^2=a^2$. Hence the previous two sums are equal completing the proof. □

We would like to generalize the second statement in Proposition 1.4.5 to when m is square-free. In this direction, a series of reduction properties will be helpful:

Proposition 1.4.6. *Let $m,n \geq 1$, p be an odd prime, and $b \in \mathbb{Z}$. Then the following hold:*

(i) If $(b,p)=1$ then $g(b,p^r)=pg(b,p^{r-2})$ for all $r \in \mathbb{Z}$ with $r \geq 2$.

(ii) If $(m,n)=1$ and $(b,mn)=1$ then $g(b,mn)=g(bn,m)g(bm,n)$.

(iii) If m is odd and $(b,m)=1$ then $g(b,m)=\left(\frac{b}{m}\right)g(m)$ where $\left(\frac{b}{m}\right)$ is the Jacobi symbol.

Proof. We will prove the statements separately.

(i) First notice that

$$g(b,p^r)=\sum_{a \pmod{p^r}} e^{\frac{2\pi ia^2b}{p^r}}=\sum_{a \pmod{p^r}}{}' e^{\frac{2\pi ia^2b}{p^r}}+\sum_{a \pmod{p^{r-1}}} e^{\frac{2\pi ia^2b}{p^{r-2}}},$$

since every a modulo p satisfies $(a,p)=1$ or not. By Euclidean division every element a modulo p^{r-1} is of the form $a=a'p^{r-2}+a''$ with a' taken modulo p and a'' taken modulo p^{r-2}. Since $(a'p^{r-2}+a'') \equiv a'' \pmod{p^{r-2}}$, every a'' is counted p times modulo p^{r-2}. Along with the fact that $(a'p^{r-2}+a'')^2 \equiv (a'')^2 \pmod{p^{r-2}}$, we have

$$\sum_{a \pmod{p^{r-1}}} e^{\frac{2\pi ia^2b}{p^{r-2}}}=\sum_{\substack{a' \pmod{p}\\ a'' \pmod{p^{r-2}}}} e^{\frac{2\pi i\left(a'p^{r-2}+a''\right)^2b}{p^{r-2}}}=p\sum_{a'' \pmod{p}} e^{\frac{2\pi i(a'')^2b}{p^{r-2}}}=pg(b,p^{r-2}).$$

It remains to show that the sum

$$\sum_{a \pmod{p^r}}{}' e^{\frac{2\pi ia^2b}{p^r}},$$

is zero. As this sum is exactly $r(b,p^r)$, Proposition 1.4.1 implies

$$\sum_{a \pmod{p^r}}{}' e^{\frac{2\pi ia^2b}{p^r}}=\mu(p^r)=0,$$

because $(b,p)=1$ and $r \geq 2$. This proves (i).

(ii) Observe that

$$g(bn,m)g(bm,n) = \left(\sum_{a \pmod m} e^{\frac{2\pi i a^2 bn}{m}}\right)\left(\sum_{a' \pmod n} e^{\frac{2\pi i (a')^2 bm}{n}}\right) = \sum_{\substack{a \pmod m \\ a' \pmod n}} e^{\frac{2\pi i\left((an)^2+(a'm)^2\right)b}{mn}}.$$

Since $(m,n)=1$, the Chinese remainder theorem gives an isomorphism

$$(\mathbb{Z}/m\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/mn\mathbb{Z}) \qquad a \oplus a' \mapsto an + a'm.$$

Set $a'' = an + a'm$ so that $(a'')^2 \equiv (an)^2 + (a'm)^2 \pmod{mn}$. Under this isomorphism, the last sum above is then equal to

$$\sum_{a'' \pmod{mn}} e^{\frac{2\pi i (a'')^2 b}{mn}},$$

which is precisely $g(b,mn)$. This proves (ii).

(iii) The claim is obvious if $m=1$ because $g(b,1)=1$ so assume $m>1$. If $m=p$ then Proposition 1.4.5, Proposition 1.4.2 (ii), and that quadratic Dirichlet characters are their own conjugate altogether imply the claim. Now let $r \geq 1$ and assume by induction that the claim holds when $m = p^{r'}$ for all positive integers r' such that $r' < r$. Then by (i), we have

$$g(b,p^r) = pg(b,p^{r-2}) = \left(\frac{b}{p^{r-2}}\right) pg(p^{r-2}) = \left(\frac{b}{p^{r-2}}\right) g(p^r) = \left(\frac{b}{p^r}\right) g(p^r). \tag{1.3}$$

It now suffices to prove the claim when $m = p^r q^s$ where q is another odd prime and $s \geq 1$. Then by (ii) and Equation (1.3), we compute

$$\begin{aligned} g(b,p^r q^s) &= g(bq^s,p^r)g(bp^r,q^s) \\ &= \left(\frac{bq^s}{p^r}\right)\left(\frac{bp^r}{q^s}\right) g(p^r)g(q^s) \\ &= \left(\frac{b}{p^r q^s}\right)\left(\frac{q^s}{p^r}\right)\left(\frac{p^r}{q^s}\right) g(p^r)g(q^s) \\ &= \left(\frac{b}{p^r q^s}\right) g(q^s,p^r)g(p^r,q^s) \\ &= \left(\frac{b}{p^r q^s}\right) g(p^r q^s). \end{aligned}$$

This proves (iii). □

At last we can prove that our Ramanujan and quadratic Gauss sums agree for square-free m:

Theorem 1.4.2. *Suppose $m \geq 1$ be square-free and odd and let χ_m be the quadratic Dirichlet character given by the Jacobi symbol. Let $b \in \mathbb{Z}$ such that $(b,m)=1$. Then*

$$\tau(b,\chi_m) = g(b,m).$$

Proof. The claim is obvious if $m = 1$ because $\tau(b, \chi_1) = 1$ and $g(b, 1) = 1$ so assume $m > 1$. Since χ_m is quadratic, it suffices to assume $b = 1$ by Proposition 1.4.2 (ii) and Proposition 1.4.6 (iii). Now let $m = p_1 p_2 \cdots p_k$ be the prime decomposition of m. Repeated application of Proposition 1.4.2 (iv) gives the first equality in the following chain:

$$\begin{aligned}\tau(\chi) &= \prod_{1 \leq i < j \leq k} \chi_{p_i}(p_j)\chi_{p_j}(p_i)\tau(\chi_{p_i})\tau(\chi_{p_j}) \\ &= \prod_{1 \leq i < j \leq k} \chi_{p_i}(p_j)\chi_{p_j}(p_i)g(p_i)g(p_j) \\ &= \prod_{1 \leq i < j \leq k} g(p_j, p_i)g(p_i, p_j) \\ &= g(q).\end{aligned}$$

This completes the proof. □

Now let's turn to Proposition 1.4.6 and the evaluation of the quadratic Gauss sum. Proposition 1.4.6 (ii) and (iii) reduce the evaluation of $g(b, m)$ for odd m and $(b, m) = 1$ to computing $g(p)$ for p an odd prime. As with the Gauss sum, it is not difficult to compute the modulus of the quadratic Gauss sum:

Theorem 1.4.3. *Let $m \geq 1$ be odd. Then*

$$|g(m)| = \sqrt{m}.$$

Proof. By Proposition 1.4.6 (ii), it suffices to assume $m = p^r$ is a power of an odd prime. By Euclidean division write $r = 2n + r'$ for some positive integer n and with $r' = 0, 1$ depending on if r is even or odd respectively. Then Proposition 1.4.6 (i) implies

$$|g(p^r)|^2 = p^{2n}|g(p^{r'})|^2.$$

If $r' = 0$ then $2n = r$ so that $p^{2n} = p^r$. Thus $|g(p^r)| = \sqrt{p^r}$. If $r' = 1$ then Theorem 1.4.1 and Proposition 1.4.5 together imply $|g(p^{r'})|^2 = p$ so that the right-hand side above is $p^{2n+1} = p^r$ and again we have $|g(p^r)| = \sqrt{p^r}$. □

Accordingly, we define the **epsilon factor** ε_m for any $m \geq 1$ by

$$\varepsilon_m = \frac{g(m)}{\sqrt{m}}.$$

Theorem 1.4.3 says that this value lies on the unit circle when m is odd. Thus the question of the evaluation of quadratic Gauss sums reduces to determining what the epsilon factor is. This was completely resolved and the original proof is due to Gauss in 1808 (see [Gau08]) while more modern proofs uses analytic techniques (see [Lan94]). The precise statement is the following:

Theorem 1.4.4. *Let $m \geq 1$. Then*

$$\varepsilon_m = \begin{cases} (1+i) & \text{if } m \equiv 0 \pmod 4, \\ 1 & \text{if } m \equiv 1 \pmod 4, \\ 0 & \text{if } m \equiv 2 \pmod 4, \\ i & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

As an immediate corollary, this implies the evaluation of the epsilon factor ε_{χ_p} where χ_p is the quadratic Dirichlet character given by the Jacobi symbol for an odd prime p:

Corollary 1.4.2. *Let p be an odd prime and χ_p be the quadratic Dirichlet character given by the Jacobi symbol. Then*

$$\varepsilon_{\chi_p} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ i & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. The statement follows immediately from Theorem 1.4.4 and Proposition 1.4.5. □

Kloosterman and Salié Sums

Our last class of sums generalize both types of Ramanujan sums. For any $c \geq 1$ and $n, m \in \mathbb{Z}$, the **Kloosterman sum** $K(n, m, c)$ is defined by

$$K(n,m,c) = \sum_{\substack{a \pmod c \\ (a,c)=1}} e^{\frac{2\pi i(an+\overline{a}m)}{c}} = \sum_{a \pmod c}{}' e^{\frac{2\pi i(an+\overline{a}m)}{c}}.$$

Notice that if either $n = 0$ or $m = 0$ then the Kloosterman sum reduces to a Ramanujan sum. Kloosterman sums have similar properties to those of Ramanujan sums, but we will not need them. The only result we will need is a famous bound, often called the **Weil bound** for Kloosterman sums, proved by Weil (see [Wei48] for a proof):

Theorem (Weil bound). *Let $c \geq 1$ and $n, m \in \mathbb{Z}$. Then*

$$|K(n,m,c)| \leq \sigma_0(c)\sqrt{(n,m,c)c}.$$

Lastly, Salié sums are Kloosterman sums with Dirichlet characters. To be precise, for any $c \geq 1$, $n, m \in \mathbb{Z}$, and a Dirichlet character χ with conductor $q \mid c$, the **Salié sum** $S_\chi(n, m, c)$ is defined by

$$S_\chi(n,m,c) = \sum_{\substack{a \pmod c \\ (a,c)=1}} \chi(a) e^{\frac{2\pi i(an+\overline{a}m)}{c}} = \sum_{a \pmod c}{}' \chi(a) e^{\frac{2\pi i(an+\overline{a}m)}{c}}.$$

If either $n = 0$ or $m = 0$ then the Salié sum reduces to the Ramanujan sum associated to χ.

1.5 Integral Lattices

Integral Lattices are ubiquitous in number theory because they provide a way to study discrete points by geometric methods. Let F be a characteristic zero field and V be an n-dimensional F-vector space with nondegenerate symmetric inner product $\langle \cdot, \cdot \rangle$. We say that a subset Λ of V is a **integral lattice** if Λ is a free abelian group. In particular, any integral lattice Λ is of the form

$$\Lambda = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_m,$$

for some linearly independent vectors $v_1, \ldots, v_m$ of V with $m \leq n$. We say Λ is **complete** if $n = m$. Necessarily, $v_1, \ldots, v_n$ is a basis of V. If $e_1, \ldots, e_n$ is an orthonormal basis for V, write

$$v_i = \sum_{1 \leq i \leq n} v_{i,j} e_j,$$

with $v_{i,j} \in F$ for $1 \leq i, j \leq n$, and define the associated **generator matrix** P by

$$P = \begin{pmatrix} v_{1,1} & \cdots & v_{n,1} \\ \vdots & & \vdots \\ v_{1,n} & \cdots & v_{n,n} \end{pmatrix}.$$

Then P is the base change matrix from $e_1, \ldots, e_n$ to $v_1, \ldots, v_n$. The **covolume** V_Λ of a complete integral lattice Λ is defined to be

$$V_\Lambda = |\det(P)|.$$

By Proposition C.1.1 and that the base change matrix between any two orthonormal basses is an orthogonal matrix and hence has determinant ± 1, the covolume is independent of the choice of bases. In particular, if V is a real or complex vector space we will take $e_1, \ldots, e_n$ to be the standard basis. We now introduce the notion of dual integral lattices. If Λ is a complete integral lattice in V then the **dual** $\Lambda^\vee$ of Λ is defined by

$$\Lambda^\vee = \{v \in V : \langle \lambda, v \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

In other words, $\Lambda^\vee$ consists of all of the vectors in V whose inner product with elements of Λ are integers. The dual integral lattice is indeed a complete integral lattice as the following proposition shows:

Proposition 1.5.1. *If $v_1, \ldots, v_n$ is a basis for the complete integral lattice Λ then the dual basis $v_1^\vee, \ldots, v_n^\vee$ is a basis for $\Lambda^\vee$. In particular, $\Lambda^\vee$ is a complete integral lattice.*

Proof. Let $v \in V$ and write

$$v = \sum_{1 \leq j \leq n} a_i v_j^\vee,$$

with $v_j^\vee \in \mathbb{R}$ for all j. Since $\langle v_i, v_j^\vee \rangle = \delta_{i,j}$ for $1 \leq i, j \leq n$, it follows that $v \in \Lambda^\vee$ if and only if $v_j^\vee \in \mathbb{Z}$ for all j. This means that $v_1^\vee, \ldots, v_n^\vee$ is a free abelian group of rank n. Hence $\Lambda^\vee$ is a complete integral lattice. □

Since the dual of the dual basis is the original basis, $(\Lambda^\vee)^\vee = \Lambda$. We say that Λ is **self-dual** if $\Lambda^\vee = \Lambda$. For example, the integral lattice $\mathbb{Z}^n$ is self-dual because the standard basis is self-dual. It also turns out that the covolume of the dual integral lattice is the inverse of the volume of the original integral lattice:

Proposition 1.5.2. *Let Λ be a complete integral lattice in V and let $\Lambda^\vee$ be its dual. Then*

$$V_{\Lambda^\vee} = \frac{1}{V_\Lambda}.$$

Proof. Let $e_1, \ldots, e_n$ be the standard basis for V and let $v_1, \ldots, v_n$ be a basis for Λ. By Proposition 1.5.1, $v_1^\vee, \ldots, v_n^\vee$ is a basis for $\Lambda^\vee$. If P is the associated generator matrix for Λ then P is the base change matrix from $e_1, \ldots, e_n$ to $v_1, \ldots, v_n$. Hence $(P^{-1})^t$ is the base change matrix from $e_1, \ldots, e_n$ to $v_1^\vee, \ldots, v_n^\vee$. The claim follows by Proposition C.1.1. □

We now turn to the case when V is an n-dimensional real inner product space with inner product $\langle \cdot, \cdot \rangle$. We let $d\lambda$ be the associated Lebesgue measure on V. If $v_1, \ldots, v_n$ is a basis of V so that any $v \in V$ has the form

$$v = t_1 v_1 + \cdots + t_n v_n,$$

for some $t_i \in \mathbb{R}$ for $1 \leq i \leq n$, the Lebesgue measure $d\lambda$ is $dt_1 \cdots dt_n$. Then

$$\mathrm{Vol}(X) = \int_X d\lambda = \int_X dt_1 \cdots dt_n,$$

is the volume of any measurable subset $X \subseteq V$. Note that Λ acts on V by automorphisms given by translation. That is, we have a group action

$$\Lambda \times V \to V \qquad (\lambda, v) \mapsto \lambda + v.$$

Moreover, Λ acts properly discontinuously on V (see Appendix D.1). To see this, let $v \in V$ and let δ_v be such that $0 < \delta_v < \min_{1 \leq i \leq n}(v - v_i)$. Then taking U_v to be the ball of radius δ_v about v, the intersection $\lambda + U_v \cap U_v$ is empty unless $\lambda = 0$. As Λ is also discrete, it follows by Proposition D.1.1 that V/Λ is also connected Hausdorff (recall that V is connected Hausdorff). In particular, V/Λ admits a fundamental domain

$$\mathcal{M} = \{t_1 v_1 + \cdots + t_n v_n \in V : 0 \leq t_i \leq 1 \text{ for } 1 \leq i \leq n\}.$$

Indeed, since Λ acts by translations, it is obvious that $\mathcal{M}$ is a fundamental domain for V/Λ. Moreover, any translation of $\mathcal{M}$ by an element of Λ is also a fundamental domain. As we might expect, the covolume of Λ is equal to the volume of $\mathcal{M}$:

Proposition 1.5.3. *Let Λ be a complete integral lattice in a finite dimensional real inner product space V and let $\mathcal{M}$ be a fundamental domain for Λ. Then*

$$V_\Lambda = \mathrm{Vol}(\mathcal{M}).$$

Proof. Let n be the dimension of V, $e_1, \ldots, e_n$ be an orthonormal basis of V, and $v_1, \ldots, v_n$ be a basis for Λ. The change of variables

$$x_1 e_1 + \cdots + x_n e_n \mapsto x_1 v_1 + \cdots + x_n v_n,$$

has the generator matrix P for $v_1, \ldots, v_n$ as its Jacobian matrix since P is the base change matrix from $e_1, \ldots, e_n$ to $v_1, \ldots, v_n$. Then

$$\mathrm{Vol}(\mathcal{M}) = \int_{\mathcal{M}} d\mathbf{x} = |\det(P)| \int_{[0,1]^n} d\mathbf{x} = |\det(P)| = V_\Lambda,$$

where the last equality follows by Proposition C.1.1. □

From Proposition 1.5.3 we see that the covolume of a complete integral lattice in a real inner product space is a measure of the density of the integral lattice. The smaller the covolume the smaller the fundamental domain and the more dense the integral lattice is. It turns out that for V, being a integral lattice is equivalent to being a discrete subgroup:

Proposition 1.5.4. *Let Λ be a subset of a finite dimensional real inner product space V. Then Λ is a integral lattice if and only if it is a discrete subgroup.*

Proof. It is clear that if Λ is a integral lattice then it is a discrete subgroup. So suppose Λ is a discrete subgroup. Then Λ is closed (since V is a metric space). Let V' be the subspace spanned by Λ and let its dimension be m. Choosing a basis $v'_1, \ldots, v'_m$ of V', set

$$\Lambda' = \mathbb{Z}v'_1 + \cdots + \mathbb{Z}v'_m.$$

Then Λ' is a complete integral lattice in V' and $\Lambda' \subseteq \Lambda \subset V$. We claim that Λ/Λ' is finite. To see this, let λ run over a complete set of representatives for Λ/Λ' and let $\mathcal{M}'$ be a fundamental domain for Λ' in V'. As $\mathcal{M}'$ is a fundamental domain, there exists unique $w' \in \mathcal{M}'$ and $\lambda' \in L$ such that $\lambda = w' + \lambda'$ for every λ. But then $w' = \lambda - \lambda' \in \mathcal{M}' \cap \Lambda$ and since $\mathcal{M}' \cap \Lambda$ is closed, discrete, and compact, it must be finite ($\mathcal{M}'$ is compact and Λ is closed and discrete). Hence there are finitely many w' and thus finitely many λ. Letting $|\Lambda/\Lambda'| = q$, we have $q\Lambda \subseteq \Lambda'$ and therefore

$$\Lambda \subseteq \Lambda' = \mathbb{Z}\frac{1}{q}v_1' + \cdots + \mathbb{Z}\frac{1}{q}v_m'.$$

In particular, Λ is a subgroup of a free abelian group and therefore is free abelian. This means Λ is a integral lattice. □

As for complete integral lattices in V, they are equivalent to the existence of a bounded set whose translates cover V:

Proposition 1.5.5. *Let Λ be a integral lattice in a finite dimensional real inner product space V. Then Λ complete if and only if there exists a bounded subset M of V whose translates by Λ cover V.*

Proof. First suppose Λ is complete. Then we may take $M = \mathcal{M}$ to be the fundamental domain of Λ which is bounded and whose translates by Λ cover V. Now suppose Λ is a integral lattice and there exists a bounded subset M of V whose translates by Λ cover V. Let W be the subspace of V spanned by Λ. Then W is closed. Moreover, Λ is complete if and only if $V = W$ and this is what we will show. So let $v \in V$. Since the translates of M by Λ cover V, for every $n \geq 1$ we may write

$$nv = w_n + \lambda_n,$$

with $v_n \in M$ and $\lambda_n \in L \subset W$. As M is bounded, $\lim_{n\to\infty} \frac{1}{n}v_n = 0$. Moreover, $\frac{1}{n}\lambda_n \in W$ for every n and since W is closed we must have that $\lim_{n\to\infty} \frac{1}{n}\lambda_n$ exists. These facts, and that v is independent of n, together imply

$$v = \lim_{n\to\infty} \frac{1}{n}w_n + \lim_{n\to\infty} \frac{1}{n}\lambda_n = \lim_{n\to\infty} \frac{1}{n}\lambda_n,$$

is an element of W. Since v was arbitrary, $V = W$ and hence Λ is complete. □

The most important result we will require about integral lattices is **Minkowski's lattice point theorem** which states that, under some mild conditions, a set of sufficiently large volume in V contains a nonzero point of a complete integral lattice:

Theorem (Minkowski's lattice point theorem). *Let Λ be a integral lattice in an n-dimensional real inner product space V. Suppose $X \subset V$ is a compact convex symmetric set. If*

$$\mathrm{Vol}(X) \geq 2^n V_\Lambda,$$

then there exists a nonzero $\lambda \in L$ with $\lambda \in X$

Proof. We will prove the claim depending on if the inequality is strict or not. First suppose $\mathrm{Vol}(X) > 2^n V_\Lambda$. Consider the linear map

$$\phi : \frac{1}{2}X \to V/\Lambda \qquad \frac{1}{2}x \mapsto \frac{1}{2}x \pmod{\Lambda}.$$

If ϕ were injective then

$$\mathrm{Vol}\left(\frac{1}{2}X\right) = \frac{1}{2^n}\mathrm{Vol}(X) \leq V_\Lambda,$$

so that $\text{Vol}(X) \leq 2^n V_\Lambda$. This is a contradiction, so ϕ cannot be injective. Hence there exists distinct $x_1, x_2 \in \frac{1}{2}X$ such that $\phi(x_1) = \phi(x_2)$. Thus $2x_1, 2x_2 \in X$. In particular, since X is symmetric we must have $-2x_2 \in X$. But then the fact X is convex implies that

$$\left(1 - \frac{1}{2}\right) 2x_1 + \frac{1}{2}(-2x_2) = x_1 - x_2,$$

is an element of X. Note that $x_1 - x_2 \in \Lambda$ because $\phi(x_1) = \phi(x_2)$ and ϕ is linear. Then $\lambda = x_1 - x_2$ is a nonzero element of Λ with $\lambda \in X$. Now suppose $\text{Vol}(X) = 2^n V_\Lambda$. Then

$$\text{Vol}((1+\varepsilon)X) = (1+\varepsilon)^n \,\text{Vol}(X) = (1+\varepsilon)^n 2^n V_\Lambda > 2^n V_\Lambda.$$

What we have just proved shows that there exists a nonzero $\lambda_\varepsilon \in \Lambda$ with $\lambda_\varepsilon \in (1+\varepsilon)X$. In particular, if $\varepsilon \leq 1$ then $\lambda_\varepsilon \in 2X \cap \Lambda$. The set $2X \cap \Lambda$ is compact and discrete, because X is compact and Λ is discrete, and therefore is finite. But as this holds for all $\varepsilon \leq 1$, the sequence $(\lambda_{\frac{1}{n}})_{n \geq 1}$ belongs to the finite set $2X \cap \Lambda$ and so must converge to a point λ. Since Λ is discrete and the $\lambda_{\frac{1}{n}}$ are nonzero so too is λ. As λ is an element of

$$\bigcap_{n \geq 1} \left(1 + \frac{1}{n}\right) X,$$

and X is closed, $\lambda \in X$ as well. Thus we have found a nonzero $\lambda \in L$ with $\lambda \in X$ and we are done. □

1.6 Integration Techniques and Transforms

Integration Techniques

Complex integrals are a core backbone of many number theory techniques. An extremely useful one is called **shifting the line of integration**:

Theorem (Shifting the line of integration). *Suppose we are given an integral*

$$\int_{\text{Re}(z)=a} f(z)\,dz \quad \text{or} \quad \int_{\text{Im}(z)=a} f(z)\,dz,$$

and some real b with $b < a$ in the first case and $b > a$ in the second case. Suppose $f(z)$ is meromorphic on a strip bounded by the lines $\text{Re}(z) = a, b$ or $\text{Im}(z) = a, b$ and is holomorphic about the lines $\text{Re}(z) = a, b$ or $\text{Im}(z) = a, b$ respectively. Moreover, suppose $f(z) \to 0$ as $y \to \infty$ or $x \to \infty$ respectively. Then

$$\int_{(a)} f(z)\,dz = \int_{(b)} f(z)\,dz + 2\pi i \sum_{\rho \in P} \operatorname*{Res}_{z=\rho} f(z),$$

where P is the set of poles inside of the strip bounded by the lines $\text{Re}(z) = a, b$ or $\text{Im}(z) = a, b$ respectively.

Proof. To collect these cases, let (a) stand for the line $\text{Re}(z) = a$ or $\text{Im}(z) = a$ respectively with positive orientation. Let R_T be a positively oriented rectangle of height or width T and with its edges on (a) or (b) respectively. Consider the limit

$$\lim_{T \to \infty} \int_{R_T} f(z)\,dz.$$

On the one hand, the residue theorem implies the integral is a sum of a $2\pi i$ multiple of the residues r_i in the rectangle R_T and hence the limit is a $2\pi i$ multiple of the sum of the residues in the strip bounded by

(a) and (b). On the other hand, the integral can be decomposed into a sum of four integrals along the edges of R_T and by taking the limit, the edges other than (a) and (b) will tend to zero because $f(z) \to 0$ as $y \to \infty$ or $x \to \infty$ respectively. What remains in the limit is the difference between the integral along (a) and (b). So in total,

$$\int_{(a)} f(z)\,dz = \int_{(b)} f(z)\,dz + 2\pi i \sum_{\rho \in P} \operatorname*{Res}_{z=\rho} f(z).$$

□

A particular application of interest is when the integral in question is real and over the entire real line, the integrand is entire as a complex function, and one is trying to shift the line of integration of the complexified integral to $\operatorname{Im}(z) = a$. In this case, shifting the line of integration amounts to making the change of variables $x \mapsto x - ia$ without affecting the initial line of integration. The second integral technique we will use is when we are summing integrals over a group and is called the **unfolding/folding method**:

Theorem (Unfolding/folding method). *Suppose $f(z)$ is holomorphic on some region Ω. Moreover, suppose G is a countable group acting by automorphisms on Ω and let D and F be regions such that*

$$D = \bigcup_{g \in G} gF,$$

where the intersections $gF \cap hF$ are measure zero for all $g, h \in G$ with respect to some G-invariant measure $d\mu$. Then

$$\int_F \sum_{g \in G} f(gz)\,d\mu = \int_D f(z)\,d\mu,$$

provided either side is absolutely convergent.

Proof. First suppose $\int_F \sum_{g \in G} f(gz)\,d\mu$ converges absolutely. By the Fubini–Tonelli theorem, we may interchange the sum and integral. Upon making the change of variables $z \mapsto g^{-1}z$, the invariance of $d\mu$ implies that the integral takes the form

$$\sum_{g \in G} \int_{gF} f(z)\,d\mu.$$

As G is countable and the intersections $gF \cap hF$ are measure zero, the overlap in $\bigcup_{g \in G} gF$ is also measure zero. As $D = \bigcup_{g \in G} gF$, the result follows. Of course, there is equality everywhere so we can also run the procedure in reverse provided $\int_D f(z)\,d\mu$ converges absolutely. □

In the unfolding/folding method, we refer to the going from the left-hand side to right-hand as **unfolding** and going from the right-hand to left-hand side as **folding**.

The Fourier Transform

The first type of integral transform we will need is the Fourier transform. Suppose $f(\mathbf{x})$ is absolutely integrable on $\mathbb{R}^n$. The **Fourier transform** $(\mathcal{F}f)(\boldsymbol{\zeta})$ of $f(\mathbf{x})$ is defined by

$$(\mathcal{F}f)(\boldsymbol{\zeta}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \langle \boldsymbol{\zeta}, \mathbf{x} \rangle}\,d\mathbf{x},$$

for $\boldsymbol{\zeta} \in \mathbb{R}^n$. This integral is absolutely convergent precisely because $f(\mathbf{x})$ is absolutely integrable on $\mathbb{R}^n$. The Fourier transform is intimately related to periodic functions. If $f(\mathbf{x})$ is 1-periodic in each component and integrable on $[0,1]^n$ then we define the $\mathbf{n}$-th **Fourier coefficient** $\hat{f}(\mathbf{n})$ of $f(\mathbf{x})$ by

$$\hat{f}(\mathbf{n}) = \int_{[0,1]^n} f(\mathbf{x}) e^{-2\pi i \langle \mathbf{n}, \mathbf{x} \rangle}\,d\mathbf{x}.$$

The **Fourier series** of $f(\mathbf{x})$ is defined by the series

$$\sum_{\mathbf{n}\in\mathbb{Z}^n} \hat{f}(\mathbf{n})e^{2\pi i\langle \mathbf{n},\mathbf{x}\rangle}.$$

There is the question of whether the Fourier series of $f(\mathbf{x})$ converges at all and if so does it even converge to $f(\mathbf{x})$ itself. Under reasonable conditions this is possible as the following proposition shows (see [G+08] for a proof):

Proposition 1.6.1. *If $f(\mathbf{x})$ is smooth and 1-periodic in each component then it converges uniformly to its Fourier series.*

The link between the Fourier transform and Fourier series is given by the **Poisson summation formula**:

Theorem (Poisson summation formula). *Suppose Λ is a complete integral lattice in $\mathbb{R}^n$, $f(\mathbf{x})$ is absolutely integrable on $\mathbb{R}^n$, and the function*

$$F(\mathbf{x}) = \sum_{\lambda\in\Lambda} f(\mathbf{x}+\lambda),$$

is locally absolutely uniformly convergent and smooth. Then

$$\sum_{\lambda\in\Lambda} f(\mathbf{x}+\lambda) = \frac{1}{V_\Lambda}\sum_{\lambda^\vee\in\Lambda^\vee} (\mathcal{F}f)(\lambda^\vee)e^{2\pi i\langle\lambda^\vee,\mathbf{x}\rangle},$$

and

$$\sum_{\lambda\in\Lambda} f(\lambda) = \frac{1}{V_\Lambda}\sum_{\lambda^\vee\in\Lambda^\vee} (\mathcal{F}f)(\lambda^\vee).$$

Proof. Fix a basis $\lambda_1,\dots,\lambda_n$ for Λ and let P be the associated generator matrix. Then P is the base change matrix from the standard basis to $\lambda_1,\dots,\lambda_n$. In particular, P is an invertible linear transformation on $\mathbb{R}^n$ satisfying $\Lambda = P\mathbb{Z}^n$ and $\Lambda^\vee = (P^{-1})^t\mathbb{Z}^n$. Letting $F_P(\mathbf{x}) = F(P\mathbf{x})$, we may write

$$F_P(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^n} f(P\mathbf{x}+P\mathbf{n}).$$

Now $F_P(\mathbf{x})$ is smooth and 1-periodic in each component because $F(\mathbf{x})$ is smooth and invariant under translation by Λ. Therefore $F_P(\mathbf{x})$ admits a Fourier series. We compute the $\mathbf{t}$-th Fourier coefficient of

$F_P(\mathbf{x})$ as follows:

$$\begin{aligned}
\hat{F}_P(\mathbf{t}) &= \int_{[0,1]^n} F_P(\mathbf{x}) e^{-2\pi i\langle \mathbf{t},\mathbf{x}\rangle}\, d\mathbf{x} \\
&= \int_{[0,1]^n} \sum_{\mathbf{n}\in\mathbb{Z}^n} f(P\mathbf{x}+P\mathbf{n}) e^{-2\pi i\langle \mathbf{t},\mathbf{x}\rangle}\, d\mathbf{x} \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^n} \int_{[0,1]^n} f(P\mathbf{x}+P\mathbf{n}) e^{-2\pi i\langle \mathbf{t},\mathbf{x}\rangle}\, d\mathbf{x} && \text{FTT} \\
&= \frac{1}{V_\Lambda} \sum_{\mathbf{n}\in\mathbb{Z}^n} \int_{P[0,1]^n} f(\mathbf{x}+P\mathbf{n}) e^{-2\pi i\langle \mathbf{t},P^{-1}\mathbf{x}\rangle} && \mathbf{x}\mapsto P^{-1}\mathbf{x} \text{ and Proposition 1.5.2} \\
&= \frac{1}{V_\Lambda} \sum_{\mathbf{n}\in\mathbb{Z}^n} \int_{P[0,1]^n} f(\mathbf{x}+P\mathbf{n}) e^{-2\pi i\langle \mathbf{t},P^{-1}\mathbf{x}\rangle}\, d\mathbf{x} \\
&= \frac{1}{V_\Lambda} \sum_{\mathbf{n}\in\mathbb{Z}^n} \int_{P[0,1]^n} f(\mathbf{x}+P\mathbf{n}) e^{-2\pi i\left\langle (P^{-1})^t\mathbf{t},\mathbf{x}\right\rangle}\, d\mathbf{x} \\
&= \frac{1}{V_\Lambda} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i\left\langle (P^{-1})^t\mathbf{t},\mathbf{x}\right\rangle}\, d\mathbf{x} && \text{FTT} \\
&= \frac{1}{V_\Lambda} (\mathcal{F}f)\left((P^{-1})^t\mathbf{t}\right).
\end{aligned}$$

Therefore the definition of $F_P(\mathbf{x})$ and its representation as a Fourier series together give

$$\sum_{\lambda\in\Lambda} f(P\mathbf{x}+\lambda) = \frac{1}{V_\Lambda} \sum_{\mathbf{t}\in\mathbb{Z}^n} (\mathcal{F}f)\left((P^{-1})^t\mathbf{t}\right) e^{2\pi i\langle \mathbf{t},\mathbf{x}\rangle}.$$

Changing variables $\mathbf{x}\mapsto P^{-1}\mathbf{x}$ and using the fact $\Lambda^\vee = (P^{-1})^t\mathbb{Z}^n$ results in

$$\sum_{\lambda\in\Lambda} f(\mathbf{x}+\lambda) = \frac{1}{V_\Lambda} \sum_{\lambda^\vee\in\Lambda^\vee} (\mathcal{F}f)(\lambda^\vee) e^{2\pi i\langle \lambda^\vee,\mathbf{x}\rangle}.$$

This proves the first statement. Setting $\mathbf{x}=\mathbf{0}$ proves the second statement. □

For convenience, we state the Poisson summation formula in the simplified case for the self-dual integral lattice $\Lambda=\mathbb{Z}^n$ as a corollary since it is how the Poisson summation formula is usually applied:

Corollary 1.6.1. *Suppose $f(\mathbf{x})$ is absolutely integrable on $\mathbb{R}^n$, and the function*

$$F(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^n} f(\mathbf{x}+\mathbf{n}),$$

is locally absolutely uniformly convergent and smooth. Then

$$\sum_{\mathbf{n}\in\mathbb{Z}^n} f(\mathbf{x}+\mathbf{n}) = \sum_{\mathbf{t}\in\mathbb{Z}^n} (\mathcal{F}f)(\mathbf{t}) e^{2\pi i\langle \mathbf{t},\mathbf{x}\rangle},$$

and

$$\sum_{\mathbf{n}\in\mathbb{Z}^n} f(\mathbf{n}) = \sum_{\mathbf{t}\in\mathbb{Z}^n} (\mathcal{F}f)(\mathbf{t}).$$

Proof. This is the Poisson summation formula when $\Lambda = \mathbb{Z}^n$ since $\mathbb{Z}^n$ is self-dual. □

In practical settings, we need a class of functions $f(\mathbf{x})$ for which the assumptions of the Poisson summation formula hold. We say that $f(\mathbf{x})$ is of **Schwarz class** if $f \in C^\infty(\mathbb{R}^n)$ and $f(\mathbf{x})$ along with all of its partial derivatives have rapid decay. If $f(\mathbf{x})$ is of Schwarz class, the rapid decay implies that $f(\mathbf{x})$ and all of its derivatives are absolutely integrable over $\mathbb{R}^n$. Moreover, this also implies that $F(\mathbf{x})$ and all of its derivatives are locally absolutely uniformly convergent by the Weierstrass M-test. The uniform limit theorem then implies $F(\mathbf{x})$ is smooth and thus the conditions of the Poisson summation formula are satisfied. We will now derive some properties of the Fourier transform including a case specific to Schwarz class functions:

Proposition 1.6.2. *Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be absolutely integrable on $\mathbb{R}^n$. Then the following hold:*

(i) For any $\alpha, \beta \in \mathbb{R}$, we have

$$(\mathcal{F}(\alpha f + \beta g))(\boldsymbol{\zeta}) = \alpha(\mathcal{F}f)(\boldsymbol{\zeta}) + \beta(\mathcal{F}g)(\boldsymbol{\zeta}).$$

(ii) If $g(\mathbf{x}) = f(\mathbf{x} + \boldsymbol{\alpha})$ for any $\boldsymbol{\alpha} \in \mathbb{R}^n$ then

$$(\mathcal{F}g)(\boldsymbol{\zeta}) = e^{2\pi i\langle \boldsymbol{\alpha}, \boldsymbol{\zeta}\rangle}(\mathcal{F}f)(\boldsymbol{\zeta}).$$

(iii) If $g(\mathbf{x}) = f(A\mathbf{x})$ for any $A \in \mathrm{GL}_n(\mathbb{R})$ then

$$(\mathcal{F}g)(\boldsymbol{\zeta}) = \frac{1}{|\det(A)|}(\mathcal{F}f)((A^{-1})^t\boldsymbol{\zeta}).$$

(iv) If $f(\mathbf{x})$ is of Schwarz class and $g(\mathbf{x}) = \frac{\partial}{\partial \mathbf{x}}^{\mathbf{k}} f(\mathbf{x})$ for some $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$ then

$$(\mathcal{F}g)(\boldsymbol{\zeta}) = (2\pi i\boldsymbol{\zeta})^{\mathbf{k}}(\mathcal{F}f).$$

Proof. We will prove the statements separately:

(i) This is immediate from linearity of the integral.

(ii) Applying the change of variables $\mathbf{x} \mapsto \mathbf{x} - \boldsymbol{\alpha}$ to

$$(\mathcal{F}g)(\boldsymbol{\zeta}) = \int_{\mathbb{R}^n} f(\mathbf{x} + \boldsymbol{\alpha})e^{-2\pi i\langle \boldsymbol{\zeta}, \mathbf{x}\rangle}\, d\mathbf{x}$$

gives

$$\int_{\mathbb{R}^n} f(\mathbf{x})e^{-2\pi i\langle \boldsymbol{\zeta}, \mathbf{x}-\boldsymbol{\alpha}\rangle}\, d\mathbf{x} = e^{2\pi i\langle \boldsymbol{\zeta}, \boldsymbol{\alpha}\rangle}\int_{\mathbb{R}^n} f(\mathbf{x})e^{-2\pi i\langle \boldsymbol{\zeta}, \mathbf{x}\rangle}\, d\mathbf{x} = e^{2\pi i\langle \boldsymbol{\alpha}, \boldsymbol{\zeta}\rangle}(\mathcal{F}f)(\boldsymbol{\zeta}).$$

(iii) Applying the change of variables $\mathbf{x} \mapsto A^{-1}\mathbf{x}$ to

$$(\mathcal{F}g)(\boldsymbol{\zeta}) = \int_{\mathbb{R}^n} f(A\mathbf{x})e^{-2\pi i\langle \boldsymbol{\zeta}, \mathbf{x}\rangle}\, d\mathbf{x},$$

gives

$$\frac{1}{|\det(A)|}\int_{\mathbb{R}^n} f(\mathbf{x})e^{-2\pi i\langle \boldsymbol{\zeta}, A^{-1}\mathbf{x}\rangle}\, d\mathbf{x} = \frac{1}{|\det(A)|}\int_{\mathbb{R}^n} f(\mathbf{x})e^{-2\pi i\left\langle (A^{-1})^t\boldsymbol{\zeta}, \mathbf{x}\right\rangle}\, d\mathbf{x} = (\mathcal{F}f)((A^{-1})^t\boldsymbol{\zeta}),$$

since the Jacobian matrix is A.

(iv) We may assume $\mathbf{k} \neq \mathbf{0}$ for otherwise there is nothing to prove. Since $f(\mathbf{x})$ is of Schwarz class, so is $g(\mathbf{x})$. First suppose $n = 1$ so that $f(\mathbf{x}) = f(x)$, $g(\mathbf{x}) = g(x)$, and $\mathbf{k} = k \geq 1$. Applying integration by parts to

$$(\mathcal{F}g)(\zeta) = \int_{-\infty}^{\infty} \frac{d^k}{dx^k} f(x) e^{-2\pi i \zeta x}\, dx,$$

gives

$$\frac{d^{k-1}}{dx^{k-1}} f(x) e^{-2\pi i \zeta x}\Big|_{-\infty}^{\infty} + 2\pi i \zeta \int_{-\infty}^{\infty} \frac{d^{k-1}}{dx^{k-1}} f(x) e^{-2\pi i \zeta x}\, dx = 2\pi i \zeta \int_{-\infty}^{\infty} \frac{d^{k-1}}{dx^{k-1}} f(x) e^{-2\pi i \zeta x}\, dx,$$

where the first term vanishes because $f(x)$ is of Schwarz class. Repeating this procedure $k-1$ times results in

$$(2\pi i \zeta)^k \int_{-\infty}^{\infty} f(x) e^{-2\pi i \zeta x}\, dx = (2\pi i \zeta)^k (\mathcal{F})(\zeta),$$

proving the case when $n = 1$. The general case follows from what we have shown by applying repeated integration by parts in each variable to

$$(\mathcal{F}g)(\boldsymbol{\zeta}) = \int_{\mathbb{R}^n} \frac{\partial}{\partial \mathbf{x}}^{\mathbf{k}} f(\mathbf{x}) e^{-2\pi i \langle \boldsymbol{\zeta}, \mathbf{x} \rangle}\, d\mathbf{x}.$$

□

The properties in Proposition 1.6.2 are immensely useful when computing Fourier transforms. We now introduce examples of Schwarz class functions and compute their Fourier transforms. The classic example of a Schwarz class function is $e^{-2\pi x^2}$. This function is particularly important because it is essentially its own Fourier transform:

Proposition 1.6.3. *Let $\alpha > 0$ and set $f(x) = e^{-2\pi\alpha x^2}$. Then*

$$(\mathcal{F}f)(\zeta) = \frac{e^{-\frac{\pi\zeta^2}{2\alpha}}}{\sqrt{2\alpha}}.$$

In particular, $e^{-\pi x^2}$ is its own Fourier transform.

Proof. Note that $f(x)$ is absolutely integrable because it exhibits exponential decay. We compute its Fourier transform

$$(\mathcal{F}f)(\zeta) = \int_{-\infty}^{\infty} e^{-2\pi\alpha x^2} e^{-2\pi i \zeta x}\, dx = \int_{-\infty}^{\infty} e^{-2\pi\alpha(x^2 + i\zeta x)}\, dx.$$

By performing the change of variables $x \mapsto \frac{x}{\sqrt{\alpha}}$, the last integral becomes

$$\frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-2\pi\left(x^2 + \frac{i\zeta x}{\sqrt{\alpha}}\right)}\, dx.$$

Complete the square in the exponent by observing

$$x^2 + \frac{i\zeta x}{\sqrt{\alpha}} = \left(x + \frac{i\zeta}{2\sqrt{\alpha}}\right)^2 + \frac{\zeta^2}{4\alpha},$$

so that the previous integral is equal to

$$\frac{e^{-\frac{\pi\zeta^2}{2\alpha}}}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-2\pi\left(x + \frac{i\zeta}{2\sqrt{\alpha}}\right)^2}\, dx.$$

The change of variables $x \mapsto \frac{x}{\sqrt{2}} - \frac{i\zeta}{\sqrt{\alpha}}$ is permitted without affecting the line of integration by viewing the integral as a complex integral, noting that the integrand is entire as a complex function, and shifting the line of integration. Then the integral takes the form

$$\frac{e^{-\frac{\pi\zeta^2}{2\alpha}}}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = \frac{e^{-\frac{\pi\zeta^2}{2\alpha}}}{\sqrt{2\alpha}},$$

where the equality holds because the integral is 1 since it is the Gaussian integral (see Appendix E.1). This proves the first statement. The second statement follows by taking $\alpha = \frac{1}{2}$. □

The analog of $e^{-2\pi x^2}$ on $\mathbb{R}^n$ is $e^{-2\pi\langle \mathbf{x},\mathbf{x}\rangle}$ which is clearly Schwarz class because $e^{-2\pi x^2}$ is. We also obtain an analog of Proposition 1.6.3 for this Schwarz class function as a corollary:

Corollary 1.6.2. *Let $\alpha > 0$ and set $f(\mathbf{x}) = e^{-2\pi\alpha\langle \mathbf{x},\mathbf{x}\rangle}$. Then*

$$(\mathcal{F}f)(\boldsymbol{\zeta}) = \frac{e^{-\frac{\pi\langle \boldsymbol{\zeta},\boldsymbol{\zeta}\rangle}{2\alpha}}}{(2\alpha)^{\frac{n}{2}}}.$$

In particular, $e^{-\pi\langle \mathbf{x},\mathbf{x}\rangle}$ is its own Fourier transform.

Proof. Applying Proposition 1.6.3 to each variable separately proves the first statement. The second statement follows upon setting $\alpha = \frac{1}{2}$. □

The Mellin Transform

Like the Fourier transform, the Mellin transform is another type of integral transform. If $f(\mathbf{x})$ is a continuous function on $\mathbb{R}^n_+$ then the **Mellin transform** $(\mathcal{M}f)(\mathbf{s})$ of $f(\mathbf{x})$ is defined by

$$(\mathcal{M}f)(\mathbf{s}) = \int_{\mathbb{R}^n_+} f(\mathbf{x})\mathbf{x}^{\mathbf{s}}\, \frac{d\mathbf{x}}{\mathbf{x}},$$

for $\mathbf{s} \in \mathbb{C}^n$. However, this integral is not guaranteed to converge unless specific conditions upon $f(\mathbf{x})$ are imposed. For example, if $f(\mathbf{x})$ exhibits rapid decay and remains bounded as $\mathbf{x} \to 0$ then the integral is locally absolutely uniformly convergent for $\sigma > 0$. We will only be interested in the case when $\mathbf{s} = (s, \dots, s)$ so that the Mellin transform is a function on $\mathbb{C}$. Moreover, most of the time we will take $n = 1$ so that the initial function we are taking the Mellin transform of is defined on $\mathbb{R}_+$. There is also an inverse transform in this case. If $g(s)$ holomorphic and tends to zero as $t \to \infty$ in a vertical strip $a < \sigma < b$ then the inverse **inverse Mellin transform** $(\mathcal{M}^{-1}g)(x)$ of $g(s)$ is given by

$$(\mathcal{M}^{-1}g)(x) = \frac{1}{2\pi i} \int_{(c)} g(s)x^{-s}\, ds,$$

for any $a < c < b$. It is not immediately clear that this integral converges or is independent of c. The following theorem makes precise what properties $f(x)$ needs to satisfy so that the inverse Mellin transform recovers $f(x)$ (see [DB15] for a proof):

Theorem (Mellin inversion formula). *Let $a < b$ and suppose $g(s)$ is analytic in the strip vertical $a < \sigma < b$, tends to zero uniformly as $t \to \infty$ along any line $\sigma = c$ for $a < c < b$, and that the integral of $g(s)$ along this line is locally absolutely uniformly convergent. Then if*

$$f(x) = \frac{1}{2\pi i} \int_{(c)} g(s)x^{-s}\, ds,$$

this integral is independent of c and moreover $g(s) = (\mathcal{M}f)(s)$. Conversely, suppose $f(x)$ is piecewise continuous such that its value is halfway between the limit values at any jump discontinuity and

$$g(s) = \int_0^\infty f(x)x^s \frac{dx}{x},$$

is locally absolutely uniformly convergent in the vertical strip $a < \sigma < b$. Then $f(x) = (\mathcal{M}^{-1}g)(x)$.

1.7 The Gamma Function

The gamma function is ubiquitous in number theory and the better one understands this function the better one will be at seeing the forest for the trees. The **gamma function** $\Gamma(s)$ is defined to be the Mellin transform of e^{-x}:

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,dx,$$

for $\sigma > 0$. The integral is locally absolutely uniformly convergent in this region. Indeed, let K is a compact subset in this region and set $\alpha = \min_{s\in K}(\sigma)$. Then we have to show that $\Gamma(s)$ is absolutely uniformly convergent on K. Now split the integral by writing

$$\Gamma(s) = \int_0^1 e^{-x}x^{s-1}\,dx + \int_1^\infty e^{-x}x^{s-1}\,dx.$$

The second integral is absolutely uniformly convergent on K since the integrand exhibits exponential decay. As for the first integral, we have

$$\int_0^1 e^{-x}x^{s-1}\,dx \ll \int_0^1 x^{\sigma-1}\,dx \ll_\alpha 1,$$

so that this integral is absolutely uniformly convergent on K too. Thus $\Gamma(s)$ is absolutely uniformly convergent on K. Also note that $\Gamma(s)$ is real for $s > 0$. The most basic properties of $\Gamma(s)$ are the following:

Proposition 1.7.1. *$\Gamma(s)$ satisfies the following properties:*

(i) $\Gamma(1) = 1$.

(ii) $\Gamma(s+1) = s\Gamma(s)$.

(iii) $\Gamma(\overline{s}) = \overline{\Gamma(s)}$.

Proof. We obtain (i) by direct computation:

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1.$$

An application of integration by parts gives (ii):

$$\Gamma(s+1) = \int_0^\infty e^{-x}x^s\,dx = -e^{-x}x^s\Big|_0^\infty + s\int_0^\infty e^{-x}x^{s-1}\,dx = s\int_0^\infty e^{-x}x^{s-1}\,dx = s\Gamma(s).$$

For (iii), since $\Gamma(s)$ is real for $s > 0$ we have $\Gamma(\overline{s}) = \overline{\Gamma(s)}$ on this half-line and then the identity theorem implies that this holds everywhere. □

From Proposition 1.7.1 we see that for $s = n$ a positive integer, $\Gamma(n) = (n-1)!$. So $\Gamma(s)$ can be thought of as a holomorphic extension of the factorial function. We can use property (ii) of Proposition 1.7.1 to extended $\Gamma(s)$ to a meromorphic function on all of $\mathbb{C}$:

Theorem 1.7.1. *$\Gamma(s)$ admits meromorphic continuation to $\mathbb{C}$ with poles at $s = -n$ for $n \geq 0$. All of these poles are simple and with residue $\frac{(-1)^n}{n!}$ at $s = -n$.*

Proof. Using Proposition 1.7.1, (ii) repeatedly, for any integer $n \geq 0$ we have

$$\Gamma(s) = \frac{\Gamma(s+1+n)}{s(s+1)\cdots(s+n)}.$$

The right-hand side defines an meromorphic function in the region $\sigma > -n$ and away from the points $0, -1, \ldots, -n$. Letting n be arbitrary, we see that $\Gamma(s)$ has meromorphic continuation to $\mathbb{C}$ with poles at $0, -1, -2, \ldots$. We now compute the residue at $s = -n$. Around this point $\Gamma(s)$ admits meromorphic continuation with representation

$$\frac{\Gamma(s+1+n)}{s(s+1)\cdots(s+n)},$$

where all of the factors except for $s+n$ are holomorphic at $s = -n$. Thus the pole is simple, and

$$\operatorname*{Res}_{s=-n} \Gamma(s) = \lim_{z \to -n} \frac{\Gamma(s+1+n)(s+n)}{s(s+1)\cdots(s+n)} = \frac{\Gamma(1)}{(-n)(1-n)\cdots(-1)} = \frac{(-1)^n}{n!}.$$ □

In particular, Theorem 1.7.1 implies $\operatorname{Res}_{s=0} \Gamma(s) = 1$ and $\operatorname{Res}_{s=1} \Gamma(s) = -1$. There are a few other properties of the gamma function that are famous and which we will use frequently. The first of which is the **Legendre duplication formula** (see [Rem98] for a proof):

Theorem (Legendre duplication formula). *For any $s \in \mathbb{C} - \{0, -1, -2, \ldots\}$,*

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s).$$

As a first application, we can use this formula to compute $\Gamma\left(\frac{1}{2}\right)$. Letting $z = \frac{1}{2}$ in the Legendre duplication formula and recalling $\Gamma(1) = 1$, we see that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. There is also the important Hadamard factorization of the reciprocal of $\Gamma(s)$ (see [SSS03] for a proof):

Proposition 1.7.2. *For all $s \in \mathbb{C}$,*

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n \geq 1} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}},$$

where γ is the Euler-Mascheroni constant.

In particular, $\frac{1}{\Gamma(s)}$ is entire so that $\Gamma(s)$ is nowhere vanishing on $\mathbb{C}$. Also, $\frac{1}{\Gamma(s)}$ is of order 1 (see Appendix B.4). In particular, this means that $\Gamma(s)$ is also order 1 for $\sigma > 0$. We call $\frac{\Gamma'}{\Gamma}(s)$ the **digamma function**. Equivalently, the digamma function is the logarithmic derivative of the gamma function. Note that upon taking the logarithmic derivative of $\Gamma(s+1) = s\Gamma(s)$, we see that the digamma function satisfies the related formula

$$\frac{\Gamma'}{\Gamma}(s+1) = \frac{\Gamma'}{\Gamma}(s) + \frac{1}{s}.$$

If we take the logarithmic derivative of the Hadamard factorization for $\frac{1}{s\Gamma(s)}$, we obtain a useful expression for the digamma function:

Corollary 1.7.1. *For all $s \in \mathbb{C}$,*

$$\frac{\Gamma'}{\Gamma}(s+1) = -\gamma + \sum_{n \geq 1} \left(\frac{1}{n} - \frac{1}{s+n} \right),$$

where γ is the Euler-Mascheroni constant. In particular, the digamma function has simple poles of residue -1 at the poles of the gamma function.

Proof. By Proposition 1.7.1 (ii), $\frac{1}{\Gamma(s+1)} = \frac{1}{s\Gamma(s)}$. Taking the logarithmic derivative using Proposition 1.7.2 we obtain

$$-\frac{\Gamma'}{\Gamma}(s+1) = \gamma + \sum_{n \geq 1} \left(\frac{1}{s+n} - \frac{1}{n} \right),$$

provided s is distance ε away from the poles of $\Gamma(s)$. This is the desired formula and the statement regarding the poles follows immediately. □

We will also require a well-known approximation for the gamma function known as **Stirling's formula** (see [Rem98] for a proof):

Theorem (Stirling's formula).

$$\Gamma(s) \sim \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s},$$

provided $|\arg(s)| < \pi - \varepsilon$ and $|s| > \delta$ for some $\varepsilon, \delta > 0$.

If σ is bounded, Stirling's formula gives a useful asymptotic showing that $\Gamma(s)$ decays as $s \to \infty$:

Corollary 1.7.2. *Let $|\arg(s)| < \pi - \varepsilon$ and $|s| > \delta$ for some $\varepsilon, \delta > 0$. Then if σ is bounded, we have*

$$\Gamma(s) \sim \sqrt{2\pi} t^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|}.$$

Proof. Stirling's formula can be equivalently expressed as

$$\Gamma(s) \sim \sqrt{2\pi} (\sigma + it)^{\sigma-\frac{1}{2}+it} e^{-\sigma-it}.$$

Since σ is bounded, $e^{-\sigma-it} \ll 1$ and we obtain the simplified asymptotic

$$\Gamma(s) \sim \sqrt{2\pi} (it)^{\sigma-\frac{1}{2}+it}.$$

Similarly, x being bounded implies $i^{\sigma-\frac{1}{2}} \ll 1$ and we compute

$$(it)^{it} = e^{i|t|\log(i|t|)} = e^{i|t|(\log(i)+\log|t|)} = e^{-\frac{\pi}{2}|t|+i|t|\log|t|} \sim e^{-\frac{\pi}{2}|t|},$$

where we have used the fact that $\log(i) = i\frac{\pi}{2}$. Together, we obtain the further simplified asymptotic

$$\Gamma(s) \sim \sqrt{2\pi} t^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|},$$

which is the desired result □

Equivalent to Stirling's formula is the asymptotic

$$\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s}(1 + O_{\varepsilon,\delta}(1)), \tag{1.4}$$

provided $|\arg(s)| < \pi - \varepsilon$ and $|s| > \delta$ for some $\varepsilon, \delta > 0$. Taking the logarithm (since $|\arg(s)| < \pi - \varepsilon$ the logarithm is defined) of this asymptotic gives

$$\log \Gamma(s) = \frac{1}{2}\log(2\pi) + \left(s - \frac{1}{2}\right)\log(s) - s + O_{\varepsilon,\delta}(1), \tag{1.5}$$

which will be useful. In fact, from Equation (1.5) we can obtain another useful estimate formula for the digamma function:

Proposition 1.7.3.

$$\frac{\Gamma'}{\Gamma}(s) = \log(s) + O_{\varepsilon,\delta}(1),$$

provided $|\arg(s)| < \pi - \varepsilon$ *and* $|s| > \delta$ *for some* $\varepsilon, \delta > 0$.

Proof. Equation (1.5) give the simplified estimate

$$\log \Gamma(s) = \frac{1}{2}\log(2\pi) + s\log(s) - s + O_{\varepsilon,\delta}(1).$$

Set $g(s) = \frac{1}{2}\log(2\pi) + s\log(s) - s$ so that $\log\Gamma(s) = g(s) + O_{\varepsilon,\delta}(1)$. Then $\log\Gamma(s) - g(s) = O_{\varepsilon,\delta}(1)$, and by Cauchy's integral formula, we have

$$\begin{aligned}\frac{\Gamma'}{\Gamma}(s) &= \frac{d}{ds}\left(g(s) + O_{\varepsilon,\delta}(1)\right) \\ &= g'(s) + \frac{d}{ds}(\log\Gamma(s) - g(s)) \\ &= \log(s) + \frac{1}{2\pi i}\int_C \frac{\log\Gamma(u) - g(u)}{(u-s)^2}\,du,\end{aligned}$$

where C is the circle about s of sufficiently small radius η depending upon ε and δ. Therefore

$$\left|\frac{\Gamma'}{\Gamma}(s) - \log(s)\right| \le \frac{1}{2\pi}\int_C \frac{|\log\Gamma(u) - g(u)|}{\eta^2}\,|du| \ll_{\varepsilon,\delta} 1,$$

where the last estimate follows because $\log\Gamma(s) - g(s) = O_{\varepsilon,\delta}(1)$. □

Lastly, we introduce a useful function related to the Gamma function. We define the **beta function** $B(s,u)$ by

$$B(s,u) = \int_0^1 t^{s-1}(1-t)^{u-1}\,dt,$$

for $\sigma > 0$ and $\tau > 0$. The integral is locally absolutely uniformly convergent in this region. Indeed, let $K \times L$ be a compact subset in this region and set $\alpha = \min_{s\in K}(\sigma)$ and $\beta = \min_{u\in L}(\tau)$. Then we have to show that $B(s,u)$ is absolutely uniformly convergent on $K \times L$. Observe

$$\int_0^1 t^{s-1}(1-t)^{u-1}\,dt \ll \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \ll_{\alpha,\beta} 1,$$

so that this integral is absolutely uniformly convergent on $K \times L$. Thus $B(s,u)$ is absolutely uniformly convergent on $K \times L$. Most importantly, the beta function is related to the gamma function in this region (see [Rem98] for a proof):

Proposition 1.7.4. *For* $\mathrm{Re}(s) > 0$ *and* $\mathrm{Re}(u) > 0$,

$$B(s,u) = \frac{\Gamma(s)\Gamma(u)}{\Gamma(s+u)}.$$

In particular, Proposition 1.7.4 shows that $B(s,u)$ has meromorphic continuation to $\mathbb{C}^2$ since the gamma function does by Theorem 1.7.1.

Part II

Dirichlet Series and L-functions

Chapter 2

The Theory of Dirichlet Series

We start our discussion of number theory with Dirichlet series. Dirichlet series are essential tools because they are a way of analytically encoding arithmetic information. We discuss basic properties of Dirichlet series such as convergence properties and Euler products. Then we prove several variants of Perron's formula which connects coefficient sums to Dirichlet series via the inverse Mellin transform.

2.1 Convergence of Dirichlet Series

A **Dirichlet series** $D(s)$ is a sum of the form

$$D(s) = \sum_{n \geq 1} \frac{a(n)}{n^s},$$

with $a(n) \in \mathbb{C}$. We exclude the case $a(n) = 0$ for all $n \geq 1$ so that $D(s)$ is not identically zero. We would first like to understand where this series converges. It does not take much for $D(s)$ to converge uniformly in a sector:

Theorem 2.1.1. *Suppose $D(s)$ is a Dirichlet series that converges at $s_0 = \sigma_0 + it_0$. Then for any $H > 0$, $D(s)$ converges uniformly in the sector*

$$\{s \in \mathbb{C} : \sigma \geq \sigma_0 \text{ and } |t - t_0| \leq H(\sigma - \sigma_0)\}.$$

Proof. Set $R(u) = \sum_{n \geq u} \frac{a(n)}{n^{s_0}}$ so that $a(n) = (R(n) - R(n+1))n^{s_0}$. Then for positive integers N and M with $1 \leq M \leq N$, summation by parts (see Appendix B.3) implies

$$\sum_{M \leq n \leq N} \frac{a(n)}{n^s} = R(M)M^{s_0 - s} - R(N+1)(N+1)^{s_0 - s} - \sum_{M \leq n \leq N} R(n+1)(n^{s_0 - s} - (n+1)^{s_0 - s}). \tag{2.1}$$

We will now express the sum on the right-hand side of Equation (2.1) as an integral. To do this, observe that

$$n^{s_0 - s} - (n+1)^{s_0 - s} = -(s_0 - s) \int_n^{n+1} u^{s_0 - s - 1}\, du.$$

Therefore

$$\begin{aligned}\sum_{M\le n\le N} R(n+1)(n^{s_0-s}-(n+1)^{s_0-s}) &= -(s_0-s)\sum_{M+1\le n\le N} R(n+1)\int_n^{n+1} u^{s_0-s-1}\,du \\ &= -(s_0-s)\sum_{M\le n\le N}\int_n^{n+1} R(u)u^{s_0-s-1}\,du \qquad (2.2)\\ &= -(s_0-s)\int_M^{N+1} R(u)u^{s_0-s-1}\,du,\end{aligned}$$

where the second line follows because $R(u)$ is constant on the interval $(n, n+1]$. Combining Equations (2.1) and (2.2) gives

$$\sum_{M\le n\le N}\frac{a(n)}{n^s} = R(M)M^{s_0-s} - R(N+1)(N+1)^{s_0-s} + (s_0-s)\int_M^{N+1} R(u)u^{s_0-s-1}\,du. \qquad (2.3)$$

We can choose M such that $|R(u)| < \varepsilon$ for all $u \ge M$ because $D(s)$ converges at s_0. It follows that $|R(u)u^{s_0-s}| < \varepsilon$ for all $u \ge M$ because $\sigma \ge \sigma_0$. Moreover, for s in the prescribed sector, we have

$$|s-s_0| \le (\sigma-\sigma_0) + |t-t_0| \le (H+1)(\sigma-\sigma_0).$$

These estimates and Equation (2.3) together imply

$$\left|\sum_{M\le n\le N}\frac{a(n)}{n^s}\right| \le 2\varepsilon + \varepsilon|s-s_0|\int_M^{N+1} u^{\sigma_0-\sigma-1}\,du \le 2\varepsilon + \varepsilon(H+1)(\sigma-\sigma_0)\int_M^{N+1} u^{\sigma_0-\sigma-1}\,du.$$

Since the last integral is finite, $\sum_{M\le n\le N}\frac{a(n)}{n^s}$ can be made uniformly arbitrarily small for s in the desired sector. The claim follows by uniform Cauchy's criterion. □

By taking $H \to \infty$ in Theorem 2.1.1 we see that $D(s)$ converges in the region $\sigma > \sigma_0$. Let σ_c be the infimum of all σ for which $D(s)$ converges. We call σ_c the **abscissa of convergence** of $D(s)$. Similarly, let σ_a be the infimum of all σ for which $D(s)$ converges absolutely. Since the terms of $D(s)$ are holomorphic, the convergence is locally absolutely uniform (actually uniform in sectors) for $\sigma > \sigma_a$. It follows that $D(s)$ is holomorphic in the region $\sigma > \sigma_a$. We call σ_a the **abscissa of absolute convergence** of $D(s)$. One should think of σ_c and σ_a as the boundaries of convergence and absolute convergence respectively. Of course, anything can happen at $\sigma = \sigma_c$ and $\sigma = \sigma_a$, but to the right of these lines we have convergence and absolute convergence of $D(s)$ respectively. It turns out that σ_a is never far from σ_c provided σ_c is finite:

Theorem 2.1.2. *If $D(s)$ is a Dirichlet series with finite abscissa of convergence σ_c then*

$$\sigma_c \le \sigma_a \le \sigma_c + 1.$$

Proof. The first inequality is trivial since absolute convergence implies convergence. For the second inequality, since $D(s)$ converges at $\sigma_c + \varepsilon$, the terms $a(n)n^{-(\sigma_c+\varepsilon)}$ tend to zero as $n \to \infty$. Therefore $a(n) \ll_\varepsilon n^{\sigma_c+\varepsilon}$ where the implicit constant is independent of n. But then $a(n)n^{-(\sigma_c+\varepsilon)} \ll_\varepsilon 1$ which implies $\sum_{n\ge1} a(n)n^{-(\sigma_c+1+2\varepsilon)}$ is absolutely convergent by the comparison test with respect to $\sum_{n\ge1} n^{-(1+\varepsilon)}$. In terms of $D(s)$, this means $\sigma_a \le \sigma_c + 1 + 2\varepsilon$ and taking $\varepsilon \to 0$ gives the second inequality. □

It turns out that the coefficients of Dirichlet series are uniquely determined:

Proposition 2.1.1. *Suppose $D(s)$ is a Dirichlet series with finite abscissa of convergence σ_c such that*

$$D(s) = \sum_{n\geq 1} \frac{a(n)}{n^s} \quad \textit{and} \quad D(s) = \sum_{n\geq 1} \frac{b(n)}{n^s}.$$

Then $a(n) = b(n)$ for all $n \geq 1$.

Proof. If $\sigma \geq \sigma_c + 1$ then $D(s)$ converges absolutely by Theorem 2.1.2. Letting $\sigma \to \infty$, the dominated convergence theorem implies that

$$\lim_{\sigma\to\infty} D(\sigma) = a(1) \quad \text{and} \quad \lim_{\sigma\to\infty} D(\sigma) = b(1),$$

because every term other than the first tends to zero as $\sigma \to \infty$. Therefore $a(1) = b(1)$. Now assume by induction that the claim holds for $n \leq N$ and set $c(n) = a(n) - b(n)$ for all $n \geq 1$. Then by assumption and our induction hypothesis, we have

$$\sum_{n>N} \frac{c(n)}{n^s} = 0.$$

Moreover, the sum on the left-hand side has abscissa of convergence at most σ_c since $D(s)$ does. It follows that

$$c(N+1) = -\sum_{n>N} c(n) \left(\frac{N}{n}\right)^{\sigma},$$

where the sum on the right-hand side has abscissa of convergence σ_c as well. By Theorem 2.1.2 again, this sum converges absolutely for $\sigma \geq \sigma_c + 1$. Letting $\sigma \to \infty$, the dominated convergence theorem implies that

$$\lim_{\sigma\to\infty} \left(-\sum_{n>N} c(n) \left(\frac{N}{n}\right)^{\sigma}\right) = 0,$$

because every term tends to zero as $\sigma \to \infty$ since $n > N$. Hence $c(N+1) = 0$ as well which completes the proof. □

We will now introduce several convergence theorems for Dirichlet series. It will be useful to setup some notation first. If $D(s)$ is a Dirichlet series with coefficients $a(n)$ then for $X > 0$, we set

$$A(X) = \sum_{n\leq X} a(n) \quad \text{and} \quad |A|(X) = \sum_{n\leq X} |a(n)|.$$

These are the partial sums of the coefficients $a(n)$ and $|a(n)|$ up to X respectively. Our first convergence theorem relates boundedness of $A(X)$ to the value of σ_c:

Proposition 2.1.2. *Suppose $D(s)$ is a Dirichlet series and that $A(X) \ll 1$. Then $\sigma_c \leq 0$.*

Proof. Let s be such that $\sigma > 0$. Since $A(X) \ll 1$, $A(X)X^{-s} \to 0$ as $X \to \infty$. Abel's summation formula (see Corollary B.3.4) then implies

$$D(s) = s\int_1^{\infty} A(u)u^{-(s+1)}\,du.$$

But because $A(u) \ll 1$, we have

$$s\int_1^{\infty} A(u)u^{-(s+1)}\,du \ll s\int_1^{\infty} u^{-(\sigma+1)}\,du = -\frac{s}{\sigma}u^{-\sigma}\Big|_1^{\infty} = \frac{s}{\sigma},$$

and so the integral converges for $\sigma > 0$. Thus $D(s)$ converges for $\sigma > 0$ and therefore $\sigma_c \leq 0$. □

Our next theorem states that if the coefficients of $D(s)$ are of polynomial growth, we can obtain an upper bound for the abscissa of absolute convergence:

Proposition 2.1.3. *Suppose $D(s)$ is a Dirichlet series whose coefficients satisfy $a(n) \ll_\alpha n^\alpha$ for some real α. Then the abscissa of absolute convergence satisfies $\sigma_a \leq 1 + \alpha$.*

Proof. It suffices to show that $D(s)$ is absolutely convergent in the region $\sigma > 1+\alpha$. For s is in this region, the polynomial bound gives

$$D(s) \ll \sum_{n\geq 1} \left|\frac{a(n)}{n^s}\right| \ll_\alpha \sum_{n\geq 1} \frac{1}{n^{\sigma-\alpha}}.$$

The latter series converges by the integral test because $\sigma-\alpha > 1$. Therefore $D(s)$ is absolutely convergent.□

Obtaining polynomial bounds on coefficients of Dirichlet series are, in most cases, not hard to establish. So the assumption in Proposition 2.1.3 is mild. Actually, there is a partial converse to Proposition 2.1.3 which gives an approximate size to $A(X)$:

Proposition 2.1.4. *Suppose $D(s)$ is a Dirichlet series with finite and nonnegative abscissa of absolute convergence σ_a. Then*

$$A(X) \ll_\varepsilon X^{\sigma_a+\varepsilon}.$$

Proof. By Abel's summation formula (see Corollary B.3.2),

$$\sum_{n\leq X} \frac{a(n)}{n^{\sigma_a+\varepsilon}} = A(X)X^{-(\sigma_a+\varepsilon)} + (\sigma_a+\varepsilon)\int_0^X A(u)u^{-(\sigma_a+\varepsilon+1)}\,du. \tag{2.4}$$

If we set $R(u) = \sum_{n\geq u} \frac{a(n)}{n^{\sigma_a+\varepsilon}}$ then $a(n) = (R(n) - R(n+1))n^{\sigma_a+\varepsilon}$ and it follows that

$$A(u) = \sum_{n\leq u} (R(n) - R(n+1))n^{\sigma_a+\varepsilon}.$$

Substituting this into Equation (2.4), we obtain

$$\int_0^X \sum_{n\leq u} (R(n) - R(n+1))n^{\sigma_a+\varepsilon}u^{-(\sigma_a+\varepsilon+1)}\,du.$$

As $R(n)$ is constant on the interval $[n, n+1)$, linearity of the integral implies

$$\int_0^X \sum_{n\leq u} (R(n) - R(n+1))n^{\sigma_a+\varepsilon}u^{-(\sigma_a+\varepsilon+1)}\,du = \sum_{0\leq n\leq X} (R(n) - R(n+1))n^{\sigma_a+\varepsilon}\int_n^{n+1} u^{-(\sigma_a+\varepsilon+1)} + O_\varepsilon(1),$$

where the O-estimate is present since X may not be an integer. Now $R(n) \ll_\varepsilon 1$ since it is the tail of $D(\sigma_a+\varepsilon)$ and moreover,

$$\int_n^{n+1} u^{-(\sigma_a+\varepsilon+1)} = -\frac{u^{-(\sigma_a+\varepsilon)}}{\sigma_a+\varepsilon}\bigg|_n^{n+1} = \frac{n^{-(\sigma_a+\varepsilon)}}{\sigma_a+\varepsilon} - \frac{(n+1)^{-(\sigma_a+\varepsilon)}}{\sigma_a+\varepsilon} \ll_\varepsilon 1,$$

because $\sigma_a + \varepsilon > 0$. So

$$\int_0^X A(u)u^{-(\sigma_a+\varepsilon+1)}\,du = \int_0^X \sum_{n\leq u} (R(n) - R(n+1))n^{\sigma_a+\varepsilon}u^{-(\sigma_a+\varepsilon+1)}\,du \ll_\varepsilon 1.$$

Also, $\sum_{n\leq X} \frac{a(n)}{n^{\sigma_a+\varepsilon}} \ll_\varepsilon 1$ because $D(\sigma_a+\varepsilon)$ converges. We conclude

$$A(X)X^{-(\sigma_a+\varepsilon)} = \sum_{n\leq X} \frac{a(n)}{n^{\sigma_a+\varepsilon}} - (\sigma_a+\varepsilon)\int_0^X A(u)u^{-(\sigma_a+\varepsilon+1)}\,du. \ll_\varepsilon 1,$$

which is equivalent to the desired estimate. □

A way to think about Proposition 2.1.4 is that if the abscissa of absolute convergence is $\sigma_a \geq 0$ then the size of the coefficients $a(n)$ is at most $n^{\sigma_a+\varepsilon}$ on average. Of corse, if $a(n) \ll_\alpha n^\alpha$ then Proposition 2.1.3 implies that $\sigma_a \leq 1+\alpha$ and so Proposition 2.1.4 gives the significantly weaker estimate $A(X) \ll_\varepsilon X^{1+\alpha+\varepsilon}$. However, if we have a bound of the form $|A|(X) \ll_\alpha X^\alpha$ we can still obtain an upper estimate for the abscissa of absolute convergence:

Proposition 2.1.5. *Suppose $D(s)$ is a Dirichlet series such that $|A|(X) \ll_\alpha X^\alpha$ for some real $\alpha \geq 0$. Then the abscissa of absolute convergence satisfies $\sigma_a \leq \alpha$.*

Proof. It suffices to show that $D(s)$ is absolutely convergent in the region $\sigma > \alpha$. Let s be in this region. Then

$$D(s) \ll \sum_{n\geq 1}\left|\frac{a(n)}{n^s}\right| = \sum_{n\geq 1}\frac{|a(n)|}{n^\sigma}.$$

By Abel's summation formula (see Appendix B.3),

$$\sum_{n\leq N}\frac{|a(n)|}{n^\sigma} = |a(N)|N^{-\sigma} - |a(1)| + \sigma\int_1^N |A|(u)u^{-(\sigma+1)}\,du,$$

By assumption $|A|(u) \ll_\alpha u^\alpha$ and so $a(N) \ll_\alpha N^\alpha$. We then estimate as follows:

$$\sum_{n\leq N}\frac{|a(n)|}{n^\sigma} = |a(N)|N^{-\sigma} - |a(1)| + \sigma\int_1^N |A|(u)u^{-(\sigma+1)}\,du \ll_\alpha |a(N)|N^{-\sigma} + |a(1)| + \sigma\int_1^N u^{\alpha-(\sigma+1)}\,du.$$

As $N\to\infty$, the left-hand side tends towards $\sum_{n\geq 1}\frac{|a(n)|}{n^\sigma}$. As for the right-hand side, the first term tends to zero since $\sigma > \alpha$ and the second term remains bounded as they are independent of N. For the third term, we compute

$$\int_1^N u^{\alpha-(\sigma+1)}\,du = \frac{u^{\alpha-\sigma}}{\alpha-\sigma}\Big|_1^N = \frac{N^{\alpha-\sigma}}{\alpha-\sigma} - \frac{1}{\alpha-\sigma},$$

which is also bounded as $N\to\infty$. This finishes the proof. □

In general, Proposition 2.1.5 is weaker than Proposition 2.1.3. For example, from our comments following Proposition 2.1.4, if $D(s)$ is a Dirichlet series with coefficients $a(n)$ and we have the estimate $|A|(X) \ll_\beta X^\beta$ for some real β then Proposition 2.1.5 only says $\sigma_a \leq \beta$. This is a significantly worse upper bound for the abscissa of absolute convergence than what Proposition 2.1.3 would imply if $a(n) \ll_\alpha n^\alpha$ and α is very small compared to β. Actually, the question of sharp polynomial bounds for these coefficients can be very deep. However, if the coefficients $a(n)$ are always nonnegative then **Landau's theorem** provides a way of obtaining a lower bound for their growth as well as locating a singularity of $D(s)$:

Theorem (Landau's theorem). *Suppose $D(s)$ is a Dirichlet series with nonnegative coefficients $a(n)$ and finite abscissa of absolute convergence σ_a. Then σ_a is a singularity of $D(s)$.*

Proof. If we replace $a(n)$ by $a(n)n^{-\sigma_a}$ then we may assume $\sigma_a = 0$. Now suppose to the contrary that $D(s)$ was holomorphic at $s = 0$. Therefore for some $\delta > 0$, $D(s)$ is holomorphic in the domain

$$\mathcal{D} = \{s \in \mathbb{C} : \sigma_a > 0\} \cup \{s \in \mathbb{C} : |s| < \delta\}.$$

Let $P(s)$ be the power series expansion of $D(s)$ at $s = 1$ so that

$$P(s) = \sum_{k\geq 0} c_k(s-1)^k,$$

where

$$c_k = \frac{D^{(k)}(1)}{k!} = \frac{1}{k!}\sum_{n\geq 1}\frac{a(n)(-\log(n))^k}{n},$$

because $D(s)$ is holomorphic and so we can differentiate termwise. The radius of convergence of $P(s)$ is the distance from $s = 1$ to the nearest singularity of $P(s)$. Since $P(s)$ is holomorphic on $\mathcal{D}$, the closest points are $\pm i\delta$. Therefore, the radius of convergence is at least $|1 \pm \delta| = \sqrt{1+\delta^2}$. We can write $\sqrt{1+\delta^2} = 1 + \delta'$ for some $\delta' > 0$. Then for $|s-1| < 1 + \delta'$, write $P(s)$ as

$$P(s) = \sum_{k\geq 0}\frac{(s-1)^k}{k!}\sum_{n\geq 1}\frac{a(n)(-\log(n))^k}{n} = \sum_{k\geq 0}\frac{(1-s)^k}{k!}\sum_{n\geq 1}\frac{a(n)(\log(n))^k}{n}.$$

If s is real with $s < 1$ then this last double sum is a sum of positive terms because $a(n) \geq 0$. Moreover, since $P(s)$ is absolutely convergent the two sums can be interchanged by the Fubini–Tonelli theorem. Interchanging sums we see that

$$P(s) = \sum_{n\geq 1}\frac{a(n)}{n}\sum_{k\geq 0}\frac{(1-s)^k(\log(n))^k}{k!} = \sum_{n\geq 1}\frac{a(n)}{n}e^{(1-s)\log(n)} = \sum_{n\geq 1}\frac{a(n)}{n^s} = D(s),$$

for $-\delta' < s < 1$. As $\delta' > 0$, this implies that $D(s)$ converges absolutely (since $a(n)$ is nonnegative) for some $s < 0$ (say $s = -\frac{\delta'}{2}$) which contradicts $\sigma_a = 0$. □

2.2 Euler Products

Generally speaking, if the coefficients $a(n)$ are chosen at random, $D(s)$ will not possess any good properties outside of convergence in some region (it might not even possess that). However, many Dirichlet series of interest will have coefficients that exhibit polynomial growth and are multiplicative. These Dirichlet series admits infinite product expressions:

Proposition 2.2.1. *Suppose the coefficients $a(n)$ of a Dirichlet series $D(s)$ are multiplicative and satisfy $a(n) \ll_\alpha n^\alpha$ for some real $\alpha \geq 0$. Then $D(s)$ converges absolutely for $\sigma > 1 + \alpha$ and admits the infinite product expression*

$$D(s) = \prod_p\left(\sum_{k\geq 0}\frac{a(p^k)}{p^{ks}}\right),$$

in this region. Conversely, suppose that there are coefficients $a(n)$ such that

$$\prod_p\left(\sum_{k\geq 0}\left|\frac{a(p^k)}{p^{ks}}\right|\right),$$

converges for $\sigma > 1 + \alpha$. Then the Dirichlet series $D(s)$ converges absolutely in this region and satisfies the former infinite product expression. Moreover, if the coefficients $a(n)$ are completely multiplicative in either case then

$$D(s) = \prod_p (1 - a(p)p^{-s})^{-1},$$

for $\sigma > 1 + \alpha$.

Proof. Since $a(n) \ll_\alpha n^\alpha$, Proposition 2.1.3 implies that $D(s)$ converges absolutely for $\sigma > 1 + \alpha$. Let s be such that $\sigma > 1 + \alpha$. Since

$$\sum_{k\geq 0} \left| \frac{a(p^k)}{p^{ks}} \right| < \sum_{n\geq 1} \left| \frac{a(n)}{n^s} \right|,$$

the infinite series on the left converges because the right does by the absolute convergence of $D(s)$. Now let $N \geq 1$. By the fundamental theorem of arithmetic

$$\prod_{p\leq N} \left(\sum_{k\geq 0} \frac{a(p^k)}{p^{ks}} \right) = \sum_{n\leq N} \frac{a(n)}{n^s} + \sum_{n>N}^{*} \frac{a(n)}{n^s}, \tag{2.5}$$

where the $*$ denotes that we are summing over only those additional terms $\frac{a(n)}{n^s}$ that appear in the expanded product on the left-hand side with $n > N$. As $N \to \infty$, the first sum on the right-hand side tends to $D(s)$ and the second sum tends to zero because it is part of the tail of $D(s)$ (which tends to zero by convergence). This proves that the product converges, and is equal to $D(s)$. Equation (2.5) also holds absolutely in the sense that

$$\prod_{p\leq N} \left(\sum_{k\geq 0} \left| \frac{a(p^k)}{p^{ks}} \right| \right) = \sum_{n\leq N} \left| \frac{a(n)}{n^s} \right| + \sum_{n>N}^{*} \left| \frac{a(n)}{n^s} \right|, \tag{2.6}$$

since $D(s)$ converges absolutely. For the converse statement, since the product

$$\prod_p \left(\sum_{k\geq 0} \left| \frac{a(p^k)}{p^{ks}} \right| \right),$$

converges for $\sigma > 1 + \alpha$ each factor is necessarily finite. That is, for each prime p the series $\sum_{k\geq 0} \frac{a(p^k)}{p^{ks}}$ converges absolutely in this region. Now fix an $N \geq 1$. Then Equation (2.6) holds. Taking $N \to \infty$ in Equation (2.6), the left-hand side converges by assumption. Therefore the right-hand sides does too. But the first sum on the right-hand side tends to

$$\sum_{n\geq 1} \left| \frac{a(n)}{n^s} \right|,$$

and the second sum is part of its tail. So the first sum must convergence hence defining an absolutely convergent Dirichlet series in $\sigma > 1 + \alpha$, and the second sum must tend to zero. Lastly, if the $a(n)$ are completely multiplicative we have

$$\prod_p \left(\sum_{k\geq 0} \frac{a(p^k)}{p^{ks}} \right) = \prod_p \left(\sum_{k\geq 0} \left(\frac{a(p)}{p^s} \right)^k \right) = \prod_p (1 - a(p)p^{-s})^{-1}.$$

□

Now suppose $D(s)$ is a Dirichlet series that has a product expression

$$D(s) = \prod_p \prod_{1 \leq i \leq d} (1 - \alpha_i(p) p^{-s})^{-1},$$

for $\sigma > \sigma_a$ and some $\alpha_i(p) \in \mathbb{C}$ for all i and all primes p. We call this product the **Euler product** of $D(s)$, and it is said to be of **degree** d. In Proposition 2.2.1, complete multiplicativity of the coefficients is enough to guarantee that $D(s)$ has an Euler product of degree 1, but in general $D(s)$ will admit an Euler product of degree $d > 1$ if the coefficients are merely multiplicative but satisfy additional properties like a recurrence relation. If $D(s)$ has an Euler product then for any $N \geq 1$ we let $D_{(N)}(s)$ denote the Dirichlet series with the factors $p \mid N$ in the Euler product removed. Then

$$D_{(N)}(s) = D(s) \prod_{p \mid N} \prod_{1 \leq i \leq d} (1 - \alpha_i(p) p^{-s}).$$

Dually, for any $N \geq 1$ we let $D_N(s)$ denote the Dirichlet series only consisting of the factors $p \mid N$ in the Euler product. That is

$$D_N(s) = \prod_{p \mid N} \prod_{1 \leq i \leq d} (1 - \alpha_i(p) p^{-s})^{-1}.$$

With this notation, we have the relationship

$$D(s) = D_{(N)}(s) D_N(s),$$

for any $N \geq 1$.

2.3 Perron Formulas

With the Mellin inversion formula, it is not hard to prove a very useful integral expression for the sum of coefficients of a Dirichlet series. First, we setup some general notation. If $D(s)$ is a Dirichlet series with coefficients $a(n)$ then for $X > 0$, we set

$$A^*(X) = \sum_{n \leq X}^{*} a(n),$$

where the $*$ indicates that the last term is multiplied by $\frac{1}{2}$ if X is an integer. We would like to relate $A^*(X)$ to an integral involving the Dirichlet series $D(s)$ via an inverse Mellin transform. Such a formula is desirable because it allows for the examination of a sum of coefficients of a Dirichlet series, a discrete object, by means of a complex integral where analytic techniques are available. We will prove a few of these types of representations, the first being (classical) **Perron's formula** which is a consequence of Abel's summation formula and the Mellin inversion formula applied to Dirichlet series:

Theorem (Perron's formula, classical). *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for any $c > \sigma_a$,*

$$A^*(X) = \frac{1}{2\pi i} \int_{(c)} D(s) X^s \frac{ds}{s}.$$

Proof. Let s be such that $\sigma > \sigma_a$. By Abel's summation formula (see Corollary B.3.4),

$$\sum_{n\geq 1} \frac{a(n)}{n^s} = \lim_{Y\to\infty} A^*(Y)Y^{-s} + s\int_1^\infty A^*(u)u^{-(s+1)}\,du.$$

Now $A^*(Y) \leq A(Y)$ and $A(Y) \ll_\varepsilon Y^{\sigma_a+\varepsilon}$ by Proposition 2.1.4 so that $A^*(Y)Y^{-s} \ll_\varepsilon Y^{\sigma_a+\varepsilon-\sigma}$. Choosing $\varepsilon < \sigma - \sigma_a$, this latter term tends to zero as $Y \to \infty$, which implies that $A^*(Y)Y^{-s} \to 0$ as $Y \to \infty$. Therefore we can write the equation above as

$$\frac{D(s)}{s} = \int_1^\infty A^*(u)u^{-(s+1)}\,du = \int_0^\infty A^*(u)u^{-(s+1)}\,du, \tag{2.7}$$

where the second equality follows because $A(u) = 0$ in the interval $[0, 1)$. The Mellin inversion formula immediately gives the result. □

As a corollary, we obtain a useful integral representation for Dirichlet series:

Corollary 2.3.1. *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for $\sigma > \sigma_a$,*

$$D(s) = s\int_1^\infty A^*(u)u^{-(s+1)}\,du.$$

Proof. The identity follows from Equation (2.7). □

There is also a truncated variant of Perron's formula which is often more useful for more subtle estimates. To state it, we need to setup some notation and will require a lemma. For any $c > 0$, consider the discontinuous integral (see Appendix E.1)

$$\delta(y) = \frac{1}{2\pi i}\int_{(c)} y^s\,\frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$$

Also, for any $T > 0$, let

$$I(y, T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} y^s\,\frac{ds}{s},$$

be $\delta(y)$ truncated outside of height T. The lemma we require gives an approximation for how close $I(y, T)$ is to $\delta(y)$ (see [Dav80] for a proof):

Lemma 2.3.1. *For any $c > 0$, $y > 0$, and $T > 0$, we have*

$$I(y, T) - \delta(y) = \begin{cases} O\left(y^c \min\left(1, \frac{1}{T|\log(y)|}\right)\right) & \text{if } y \neq 1, \\ O\left(\frac{c}{T}\right) & \text{if } y = 1. \end{cases}$$

We can now state and prove (truncated) **Perron's formula**:

Theorem (Perron's formula, truncated). *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for any $c > \sigma_a$ and $T > 0$,*

$$A^*(X) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} D(s)X^s\,\frac{ds}{s} + O\left(X^c \sum_{\substack{n\geq 1 \\ n\neq X}} \frac{a(n)}{n^c}\min\left(1, \frac{1}{T\left|\log\left(\frac{X}{n}\right)\right|}\right) + \delta_X a(X)\frac{c}{T}\right),$$

where $\delta_X = 1, 0$ according to if X is an integer or not.

Proof. By Appendix E.1, we have

$$A^*(X) = \sum_{n\geq 1} a(n)\delta\left(\frac{X}{n}\right).$$

Using Lemma 2.3.1, we may replace $\delta\left(\frac{X}{n}\right)$ and obtain

$$A^*(X) = \sum_{n\geq 1} a(n)\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{X^s}{n^s}\frac{ds}{s} + \sum_{\substack{n\geq 1\\ n\neq X}} a(n)O\left(\frac{X^c}{n^c}\min\left(1, \frac{1}{T\left|\log\left(\frac{X}{n}\right)\right|}\right)\right) + \delta_X a(X)O\left(\frac{c}{T}\right).$$

Since $D(s)$ converges absolutely, the sum may be moved inside of the first O-estimate and then we may combine the resulting two O-estimates. By the Fubini–Tonelli theorem, we may interchange the sum and the integral and the statement of the lemma follows. □

There is a slightly weaker variant of truncated Perron's formula that follows as a corollary:

Corollary 2.3.2. *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for any $c > \sigma_a$ and $T > 0$,*

$$A^*(X) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} D(s)X^s\frac{ds}{s} + O\left(\frac{X^c}{T}\right),$$

Proof. For sufficiently large X, we have

$$\min\left(1, \frac{1}{T\left|\log\left(\frac{X}{n}\right)\right|}\right) \ll \frac{X^c}{T}.$$

The statement now follows from truncated Perron's formula. □

There is also a variant of Perron's formula where we add a smoothing function. For any $X > 0$, we set

$$A_\psi(X) = \sum_{n\geq 1} a(n)\psi\left(\frac{n}{X}\right),$$

where $\psi(y)$ is smooth function such that $\psi(y) \to 0$ as $y \to \infty$. This is most useful in two cases. The first is when we choose $\psi(y)$ to be a bump function. In this setting, the bump function can be chosen such that it assigns weight 1 or 0 to the coefficients $a(n)$ and we can estimate sums like

$$\sum_{\frac{X}{2}\leq n<X} a(n) \quad \text{or} \quad \sum_{X\leq n<X+Y} a(n),$$

for some X and Y with $Y < X$. Sums of this type are called **unweighted**. As an example of an unweighted sum, let $\psi(y)$ be a bump function that is identically 1 on $[0,1]$ and is compactly supported in $\left[1, \frac{X+1}{X}\right]$. For example,

$$\psi(y) = \begin{cases} 1 & \text{if } 0 \leq y \leq 1, \\ e^{\frac{1-y}{\frac{X+1}{X}-y}} & \text{if } 1 \leq y < \frac{X+1}{X}, \\ 0 & \text{if } y \geq \frac{X+1}{X}. \end{cases}$$

Then

$$A_\psi(X) = \sum_{n\geq 1} a(n)\psi\left(\frac{n}{X}\right) = \sum_{n\leq X} a(n).$$

In the second case, we want $\psi(y)$ to dampen the $a(n)$ with a weight other than 1 or 0. Sums of this type are called **weighted**. In any case, suppose the support of the smooth function $\psi(y)$ is contained in $[0, M]$. These conditions will force the Mellin transform $\Psi(s)$ of $\psi(y)$ to exist and have nice properties. To see that $\Psi(s)$ exists, let K be a compact set in $\mathbb{C}$ and let $\alpha = \max_{s\in K}(\sigma)$ and $\beta = \min_{s\in K}\{\sigma\}$. Note that $\psi(y)$ is bounded because it is compactly supported. Then for $s \in K$,

$$\Psi(s) = \int_0^\infty \psi(y) y^s \frac{dy}{y} \ll \int_0^M y^{\sigma-1}\, dy \ll_{\alpha,\beta} 1.$$

Therefore $\Psi(s)$ is locally absolutely uniformly convergent for $s \in \mathbb{C}$. In particular, the Mellin inversion formula implies that $\psi(y)$ is the Mellin inverse of $\Psi(s)$. As for nice properties, $\Psi(s)$ exhibit polynomial decay of arbitrarily large order in vertical strips:

Proposition 2.3.1. *Suppose $\psi(y)$ is a bump function and let $\Psi(s)$ denote its Mellin transform. Then for any $N \geq 1$,*

$$\Psi(s) \ll (|s| + 1)^{-N},$$

provided s is contained in the vertical strip $a < \sigma < b$ for any a and b with $0 < a < b$.

Proof. Fix a and b with $a < b$. Also, let the support of $\psi(y)$ be contained in $[0, M]$. Now consider

$$\Psi(s) = \int_0^\infty \psi(y) y^s \frac{dy}{y}.$$

Since $\psi(y)$ is compactly supported, integrating by parts yields

$$\Psi(s) = \frac{1}{s}\int_0^\infty \psi'(y) y^{s+1} \frac{dy}{y}.$$

Repeatedly integrating by parts $N \geq 1$ times, we arrive at

$$\Psi(s) = \frac{1}{s(s+1)\cdots(s+N-1)}\int_0^\infty \psi^{(N)}(y) y^{s+N} \frac{dy}{y}.$$

Therefore

$$\Psi(s) \ll (|s| + 1)^{-N}\int_0^\infty \psi^{(N)}(y) y^{\sigma+N} \frac{dy}{y}.$$

The claim will follow if we can show that the integral is bounded. Since $\psi(y)$ is compactly supported in $[0, M]$ so is $\psi^{(N)}(y)$. In particular, $\psi^{(N)}(y)$ is bounded. Therefore

$$\int_0^\infty \psi^{(N)}(y) y^{\sigma+N} \frac{dy}{y} \ll \int_0^M y^{\sigma+N} \frac{dy}{y} = \frac{y^{\sigma+N}}{\sigma+N}\bigg|_0^M = \frac{M^{\sigma+N}}{\sigma+N} \ll \frac{M^{b+N}}{N} \ll 1,$$

where the second to last estimate follows because $a < \sigma < b$ with $0 < a < b$. So the integral is bounded and the claim follows. □

The following theorem is (smoothed) **Perron's formula**:

Theorem (Perron's formula, smoothed). *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Let $\psi(y)$ be a bump function and denote its Mellin transform by $\Psi(s)$. Then for any $c > \sigma_a$,*

$$A_\psi(X) = \frac{1}{2\pi i}\int_{(c)} D(s)\Psi(s)X^s\, ds.$$

In particular,

$$\sum_{n\geq 1} a(n)\psi(n) = \frac{1}{2\pi i}\int_{(c)} D(s)\Psi(s)\, ds.$$

Proof. The first statement is just a computation:

$$\begin{aligned} A_\psi(X) &= \sum_{n\geq 1} a(n)\psi\left(\frac{n}{X}\right) \\ &= \sum_{n\geq 1} \frac{a(n)}{2\pi i}\int_{(c)} \Psi(s)\left(\frac{n}{X}\right)^{-s} ds && \text{Mellin inversion formula} \\ &= \frac{1}{2\pi i}\int_{(c)} \sum_{n\geq 1} a(n)\Psi(s)\left(\frac{n}{X}\right)^{-s} ds && \text{FTT} \\ &= \frac{1}{2\pi i}\int_{(c)} D(s)\Psi(s)X^s\, ds. \end{aligned}$$

This proves the first statement. For the second statement, take $X = 1$. □

Smoothed Perron's formula is useful because it improves the convergence of the integral (in light of Proposition 2.3.1) while compensating by weighting the sum of coefficients of the Dirichlet series.

Chapter 3

The Theory of L-functions

If a Dirichlet series possesses special properties we call it an L-function. From the analytic properties of L-functions we can extract number theoretic results. First we define L-functions and their associated data. The material following this consists of many important discussions about the analytic properties of L-functions: the approximate functional equation, the Riemann hypothesis and Lindelöf hypothesis, the central value, logarithmic derivatives, zero density, zero-free regions, and explicit formulas.

3.1 Analytic Data of L-functions

We are now ready to discuss L-functions in some generality. In the following, we will denote L-series and L-functions by $L(s, f)$, and for the moment, f will carry no formal meaning. It is only used to suggest that the L-function is attached to some interesting arithmetic object f. When we discuss specific L-functions, f will carry a formal meaning. An **L-series** $L(s, f)$ is a Dirichlet series

$$L(s, f) = \sum_{n \geq 1} \frac{a_f(n)}{n^s},$$

where the $a_f(n) \in \mathbb{C}$ are coefficients usually attached to some arithmetic object f. We call $L(s, f)$ an **L-function** if it satisfies the following properties:

(i) $L(s, f)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits a degree d_f Euler product:

$$L(s, f) = \sum_{n \geq 1} \frac{a_f(n)}{n^s} = \prod_p \prod_{1 \leq j \leq d_f} (1 - \alpha_j(p) p^{-s})^{-1},$$

with $a_f(n), \alpha_j(p) \in \mathbb{C}$, $a_f(1) = 1$, and $|\alpha_j(p)| \leq p$ for all j and primes p. We call

$$L_p(s, f) = \prod_{1 \leq j \leq d_f} (1 - \alpha_j(p) p^{-s})^{-1},$$

the **local factor** at p, and the $\alpha_j(p)$ are called the **local roots** (or **local parameters**) at p.

(ii) There exists a factor

$$\gamma(s, f) = \pi^{-\frac{d_f s}{2}} \prod_{1 \leq j \leq d_f} \Gamma\left(\frac{s + \kappa_j}{2}\right),$$

with $\kappa_j \in \mathbb{C}$ that are either real or appear in conjugate pairs. We also require $\mathrm{Re}(\kappa_j) > -1$. The κ_j are called the **local roots** (or **local parameters**) at infinity.

(iii) There exists an integer $q(f) \geq 1$ called the **conductor** such that $\alpha_j(p) \neq 0$ for all prime p such that $p \nmid q(f)$. If $p \mid q(f)$ then we say p is **ramified** and is **unramified** otherwise. The **analytic conductor** $\mathfrak{q}(s,f)$ is defined to be

$$\mathfrak{q}(s,f) = q(f)\mathfrak{q}_\infty(s,f),$$

where we set

$$\mathfrak{q}_\infty(s,f) = \prod_{1\leq j\leq d_f} (|s+\kappa_j|+3).$$

For simplicity, we will also suppress the dependence upon s if $s=0$. That is, we write

$$\mathfrak{q}(f) = \mathfrak{q}(0,f) \quad \text{and} \quad \mathfrak{q}_\infty(f) = \mathfrak{q}_\infty(0,f).$$

(iv) The **completed L-function**

$$\Lambda(s,f) = q(f)^{\frac{s}{2}}\gamma(s,f)L(s,f),$$

must admit meromorphic continuation to $\mathbb{C}$ with at most poles at $s=0$ and $s=1$. Moreover, it must satisfy the **functional equation** given by

$$\Lambda(s,f) = \varepsilon(f)\Lambda(1-s,\overline{f}),$$

where $\varepsilon(f)$ is a complex number with $|\varepsilon(f)|=1$ called the **root number** of $L(s,f)$, and $\overline{f}$ is an object associated to f called the **dual** of f such that $L(s,\overline{f})$ satisfies $a_{\overline{f}}(n) = \overline{a_f(n)}$, $\gamma(s,\overline{f}) = \gamma(s,f)$, $q(\overline{f}) = q(f)$, and $\varepsilon(\overline{f}) = \overline{\varepsilon(f)}$. We call $L(s,\overline{f})$ the **dual** of $L(s,f)$. If $\overline{f} = f$, we say $L(s,f)$ is **self-dual**.

(v) $L(s,f)$ admits meromorphic continuation to $\mathbb{C}$ with at most a pole at $s=1$ of order $r_f \in \mathbb{Z}$, and must be of order 1 (see Appendix B.4) after clearing the possible polar divisor.

Property (ii) ensures that $\gamma(s,f)$ is holomorphic and nonzero for $\sigma \geq 1$. Then as $\gamma(1,f)$ is nonzero, r_f is also the order of possible poles of $\Lambda(s,f)$ at $s=0$ and $s=1$ which are equal by the functional equation. It follows that $r_{\overline{f}} = -r_f$. As $L(s,f)$ admits meromorphic continuation by property (v), we denote the continuation by $L(s,f)$ as well. Also note that the product of L-functions is an L-function but the sum of L-functions need not be.

Remark 3.1.1. *The only L-function of degree 0 is $L(s,\mathbf{1})$ defined by setting $a_{\mathbf{1}}(1)=1$ and $a_{\mathbf{1}}(n)=0$ for all $n>1$ so that $L(s,\mathbf{1}) \equiv 1$. Moreover, $\gamma(s,\mathbf{1}) \equiv 1$, $q(\mathbf{1})=1$, $\varepsilon(\mathbf{1})=1$, and $L(s,\mathbf{1})$ is self-dual.*

We say that an L-function $L(s,f)$ belongs to the **Selberg class** if the additional adjustments to (i), (ii), and (v) hold:

(i) The local roots at p satisfy $|\alpha_j(p)|=1$ if $p \nmid q(f)$ and $|\alpha_j(p)| \leq 1$ if $p \mid q(f)$.

(ii) The local roots at infinity satisfy $\mathrm{Re}(\kappa_j) \geq 0$.

(v) $L(s,f)$ has at most a simple pole at $s=1$. That is, $r_f \leq 1$.

The adjustment for (i) is called the **generalized Ramanujan-Petersson conjecture** and it forces $a_f(n) \ll \sigma_0(n) \ll_\varepsilon n^\varepsilon$ (recall Proposition A.3.1). The adjustment for (ii) is called the **generalized Selberg conjecture** and it ensures that $\gamma(s,f)$ is holomorphic and nonzero for $\sigma > 0$. As for the adjustment for (v), it is expected that $L(s,f)$ is entire unless $\alpha_i(p) \geq 0$. The Selberg class constitutes a very special class of L-functions. Note that the product of two Selberg class L-functions is also Selberg class. Accordingly, we say that a Selberg class L-function is **primitive** if it cannot be factored into a product of two Selberg class L-functions of strictly smaller positive degree (positive degree is necessary so that one of the factors cannot be $L(s,\mathbf{1})$). Clearly, any Selberg class L-function of degree 1 is primitive.

Remark 3.1.2. *In general, it is a difficult problem to determine when an L-function $L(s,f)$ of degree $d_f \geq 2$ is primitive.*

Let $L(s,f)$ be an L-function with local roots $\alpha_j(p)$ at p and let $k \geq 1$. We say that an L-function $L(s, \mathrm{sym}^k f)$ of degree $\binom{d_f+k-1}{k}$ is the k-th **symmetric power** of $L(s,f)$ if it satisfies the following adjustment:

(i) The Euler product of $L(s, \mathrm{sym}^k f)$ takes the form

$$L(s, \mathrm{sym}^k f) = \prod_p \prod_{1 \leq j_1 \leq \cdots \leq j_k \leq d_f} (1 - \alpha_{j_1}(p) \cdots \alpha_{j_k}(p) p^{-s})^{-1}.$$

When k is 2, 3, etc. we simply refer to $L(s, \mathrm{sym}^k f)$ as the **symmetric square**, **symmetric cube**, etc. When $k \leq d_f$, we say that an L-function $L(s, \mathrm{ext}^k f)$ of degree $\binom{d_f}{k}$ is the k-th **exterior power** of $L(s,f)$ if it satisfies the following adjustment:

(i) The Euler product of $L(s, \mathrm{ext}^k f)$ takes the form

$$L(s, \mathrm{ext}^k f) = \prod_p \prod_{1 \leq j_1 < \cdots < j_k \leq d_f} (1 - \alpha_{j_1}(p) \cdots \alpha_{j_k}(p) p^{-s})^{-1}.$$

When k is 2, 3, etc. we simply refer to $L(s, \mathrm{ext}^k f)$ as the **exterior square**, **exterior cube**, etc.

Remark 3.1.3. *It is not an easy matter to determine if the k-th symmetric power and k-th exterior power L-functions of $L(s,f)$ exist.*

Now suppose we are given two L-functions $L(s,f)$ and $L(s,g)$ with local roots $\alpha_j(p)$ and $\beta_\ell(p)$ at p. We say that an L-function $L(s, f \otimes g)$ of degree $d_{f \otimes g} = d_f d_g$ is the **Rankin-Selberg convolution** of $L(s,f)$ and $L(s,g)$ (or **Rankin-Selberg square** if $f = g$) if it satisfies the following adjustment:

(i) The Euler product of $L(s, f \otimes g)$ takes the form

$$L(s, f \otimes g) = \prod_{p \nmid q(f)q(g)} \prod_{\substack{1 \leq j \leq d_f \\ 1 \leq \ell \leq d_g}} \left(1 - \alpha_j(p)\overline{\beta_\ell(p)} p^{-s}\right)^{-1} \prod_{p \mid q(f)q(g)} \prod_{1 \leq i \leq d_f d_g} \left(1 - \gamma_i(p) p^{-s}\right)^{-1},$$

with $\gamma_j(p) \in \mathbb{C}$, $\gamma_j(1) = 1$, and $|\gamma_j(p)| \leq p$ for all i and primes $p \mid q(f)q(g)$.

We now introduce some important concepts associated to L-functions. The **critical strip** is the vertical strip left invariant by the transformation $s \mapsto 1 - s$, that is, the region defined by

$$\left\{ s \in \mathbb{C} : \left| \sigma - \frac{1}{2} \right| \leq \frac{1}{2} \right\}.$$

Moreover, it is also the region where we cannot determine the value of an L-function from its representation as a Dirichlet series or using the functional equation. It turns out that most of the important information about L-functions is contained inside of the critical strip. The **critical line** is the line left invariant by the transformation $s \mapsto 1 - s$ which is the line defined by $\sigma = \frac{1}{2}$. It is also the line that bisects the critical strip vertically. The **central point** is the fixed point of the transformation $s \mapsto 1 - s$, in other words, the point $s = \frac{1}{2}$. Clearly the central point is also the center of the critical line. The critical strip, critical line, and central point are all displayed in Figure 3.1:

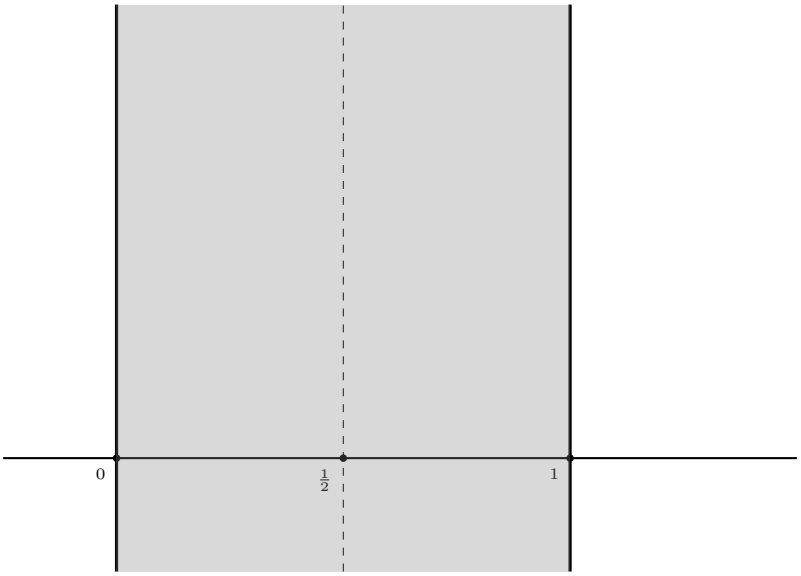


Figure 3.1: The critical strip, critical line, and central point.

Lastly, we provide some bounds about gamma functions, gamma factors, and their logarithmic derivatives that will be extremely useful in the study of L-functions. Suppose σ is bounded and $|t| > 1$. Then s is bounded away from zero and by Corollary 1.7.2 we have the asymptotic

$$\Gamma(s) \sim \sqrt{2\pi} t^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|}.$$

This gives the weaker estimates

$$\Gamma(s) \ll t^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|} \quad \text{and} \quad \frac{1}{\Gamma(s)} \ll t^{\frac{1}{2}-\sigma} e^{\frac{\pi}{2}|t|},$$

for $|t| > 1$. It is not hard to obtain estimates that holds in vertical strips. Suppose s is in the vertical strip $a \leq \sigma \leq b$ and is distance ε away from the poles of $\Gamma(s)$. Then $\Gamma(s)$ is bounded on the compact region $a \leq \sigma \leq b$ with $|t| \leq 1$ provided s is distance ε away from the poles of $\Gamma(s)$. It follows that the estimates

$$\Gamma(s) \ll_\varepsilon (|t|+1)^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|} \quad \text{and} \quad \frac{1}{\Gamma(s)} \ll (|t|+1)^{\frac{1}{2}-\sigma} e^{\frac{\pi}{2}|t|}, \tag{3.1}$$

are valid in the vertical strip $a \leq \sigma \leq b$ provided s is distance ε away from the poles of $\Gamma(s)$ in the former case. We immediately obtain the useful estimate

$$\frac{\Gamma(1-s)}{\Gamma(s)} \ll_\varepsilon (|t|+1)^{1-2\sigma},$$

valid in the vertical strip $a \leq \sigma \leq b$ provided s is distance ε away from the poles of $\Gamma(1-s)$. Moreover, it easily follows from the definition of $\gamma(s,f)$ that the estimates

$$\frac{\gamma(1-s,f)}{\gamma(s,f)} \ll_\varepsilon \mathfrak{q}_\infty(s,f)^{\frac{1}{2}-\sigma} \quad \text{and} \quad q(f)^{\frac{1}{2}-s} \frac{\gamma(1-s,f)}{\gamma(s,f)} \ll_\varepsilon \mathfrak{q}(s,f)^{\frac{1}{2}-\sigma}, \tag{3.2}$$

are valid in the vertical strip $a \leq \sigma \leq b$ provided s is distance ε away from the poles of $\gamma(1-s,f)$. There are analogous useful estimates for the digamma function as well. To derive them, suppose σ is bounded

and $|t| > 1$. Then s is bounded away from zero, so making use of the formula $\frac{\Gamma'}{\Gamma}(s+1) = \frac{\Gamma'}{\Gamma}(s) + \frac{1}{s}$ if $\sigma < 0$, Proposition 1.7.3 gives the estimate

$$\frac{\Gamma'}{\Gamma}(s) \ll \log(s).$$

We can also obtain estimates that holds in vertical strips. Suppose s is in the vertical strip $a \leq \sigma \leq b$ and is distance ε away from the poles of $\Gamma(s)$. Then $\frac{\Gamma'}{\Gamma}(s)$ is bounded on the compact region $a \leq \sigma \leq b$ with $|t| \leq 1$ provided s is distance ε away from the poles of $\Gamma(s)$. It follows that the estimate

$$\frac{\Gamma'}{\Gamma}(s) \ll_\varepsilon \log(|s|+1),$$

is valid in the vertical strip $a \leq \sigma \leq b$ provided s is distance ε away from the poles of $\Gamma(s)$. It follows from the definition of $\gamma(s,f)$ that the estimates

$$\frac{\gamma'}{\gamma}(s,f) \ll_\varepsilon \log \mathfrak{q}_\infty(s,f) \quad \text{and} \quad \log(q(f)) + \frac{\gamma'}{\gamma}(s,f) \ll_\varepsilon \log \mathfrak{q}(s,f), \tag{3.3}$$

are valid in the vertical strip $a \leq \sigma \leq b$ provided s is distance ε away from the poles of $\gamma(s,f)$. Also, our definition of $\mathfrak{q}_\infty(s,f)$ ensures

$$\log \mathfrak{q}(s,f) \geq \log \mathfrak{q}(f) \geq \log(3) \geq 1.$$

3.2 The Approximate Functional Equation

If $L(s,f)$ is an L-function then there is a formula which acts as a compromise between the functional equation for $L(s,f)$ and expressing $L(s,f)$ as a Dirichlet series. This formula is known as the approximate functional equation and it is important because it is valid inside of the critical strip and therefore can be used to obtain data about $L(s,f)$ in that region. First we use Equation (3.2) to show that $L(s,f)$ has polynomial growth in the t-aspect in vertical strips:

Proposition 3.2.1. *For any L-function $L(s,f)$ and $a < b$, $L(s,f)$ is of polynomial growth in the t-aspect in the vertical half-strip $a \leq \sigma \leq b$ with $|t| \geq 1$.*

Proof. On the one hand, for $\sigma > \max(1,b)$ we have $L(s,f) \ll 1$ on the line $\sigma = 1$ with $t \geq 1$. On the other hand, the functional equation and Equation (3.2) together imply

$$L(s,f) \ll \mathfrak{q}(s,f)^{\frac{1}{2}-\sigma} L(1-s,f).$$

Clearly $\mathfrak{q}(s,f)^{\frac{1}{2}-\sigma}$ is of polynomial growth in the t-aspect provided σ is bounded. As $|\sigma| < \max(a,b)$, we see that $L(s,f)$ is also of polynomial growth in the t-aspect for $|t| \geq 1$. Moreover, this estimate also implies $L(s,f)$ is bounded on the line $t = 1$ since σ is bounded. As $L(s,f)$ is holomorphic for $|t| \geq 1$ and of order 1, we can apply the Phragmén-Lindelöf convexity principle in this region (see Appendix B.5) so that $L(s,f)$ is of polynomial growth in the t-aspect in the vertical half-strip $a \leq \sigma \leq b$ with $|t| \geq 1$. □

We now prove the **approximate function equation**:

Theorem (Approximate functional equation). *Let $L(s,f)$ be an L-function, $\Phi(u)$ be an even holomorphic function bounded in the vertical strip $|\tau| < a+1$ for any $a > 1$ such that $\Phi(0) = 1$, and let $X > 0$. Then for s in the critical strip, we have*

$$L(s,f) = \sum_{n \geq 1} \frac{a_f(n)}{n^s} V_s\left(\frac{n}{\sqrt{q(f)}X}\right) + \varepsilon(s,f) \sum_{n \geq 1} \frac{\overline{a_f(n)}}{n^{1-s}} V_{1-s}\left(\frac{nX}{\sqrt{q(f)}}\right) + \frac{R}{q(f)^{\frac{s}{2}}\gamma(s,f)},$$

where $V_s(y)$ is the inverse Mellin transform defined by

$$V_s(y) = \frac{1}{2\pi i}\int_{(a)} \frac{\gamma(s+u,f)}{\gamma(s,f)}\Phi(u)y^{-u}\frac{du}{u},$$

and

$$\varepsilon(s,f) = \varepsilon(f)q(f)^{\frac{1}{2}-s}\frac{\gamma(1-s,f)}{\gamma(s,f)}.$$

Moreover, R is zero if $\Lambda(s,f)$ is entire, and otherwise

$$R = \operatorname*{Res}_{u=1-s} \frac{\Lambda(s+u,f)\Phi(u)X^u}{u} + \operatorname*{Res}_{u=-s} \frac{\Lambda(s+u,f)\Phi(u)X^u}{u}.$$

Proof. Let

$$I(X,s,f) = \frac{1}{2\pi i}\int_{(a)} \Lambda(s+u,f)\Phi(u)X^u\,\frac{du}{u}.$$

$L(s,f)$ has polynomial growth in the t-aspect by Proposition 3.2.1. From Equation (3.1) we see that $\gamma(s+u,f)$ exhibits exponential decay. Since $\Phi(u)$ is bounded, it follows that the integrand exhibits exponential decay in a vertical strip containing $|\tau| \leq a$. Therefore the integral is locally absolutely uniformly convergent. Moreover, we may shift the line of integration to $(-a)$. In doing so, we pass by a simple pole at $u=0$ and possible poles at $u = 1-s$ and $u=-s$, giving

$$I(X,s,f) = \frac{1}{2\pi i}\int_{(-a)} \Lambda(s+u,f)\Phi(u)X^u\,\frac{du}{u} + \Lambda(s,f) + R.$$

Applying the functional equation to $\Lambda(s+u,f)$ and performing the change of variables $u \mapsto -u$, we obtain

$$I(X,s,f) = -\varepsilon(f)I(X^{-1},1-s,\overline{f}) + \Lambda(s,f) - R,$$

since $\Phi(u)$ is even. This equation is equivalent to

$$\Lambda(s,f) = I(X,s,f) + \varepsilon(f)I(X^{-1},1-s,\overline{f}) + R.$$

Since $\operatorname{Re}(s+u) > 1$, we can expand the L-function $L(s,f)$ inside of $I(X,s,f)$ as a Dirichlet series:

$$\begin{aligned}
I(X,s,f) &= \frac{1}{2\pi i}\int_{(a)} \Lambda(s+u,f)\Phi(u)X^u\,\frac{du}{u} \\
&= \frac{1}{2\pi i}\int_{(a)} q(f)^{\frac{s+u}{2}}\gamma(s+u,f)L(s+u,f)\Phi(u)X^u\,\frac{du}{u} \\
&= \frac{1}{2\pi i}\int_{(a)} \sum_{n\geq 1}\frac{a_f(n)}{n^{s+u}}q(f)^{\frac{s+u}{2}}\gamma(s+u,f)\Phi(u)X^u\,\frac{du}{u} \\
&= \sum_{n\geq 1}\frac{1}{2\pi i}\int_{(a)} \frac{a_f(n)}{n^{s+u}}q(f)^{\frac{s+u}{2}}\gamma(s+u,f)\Phi(u)X^u\,\frac{du}{u} && \text{FTT} \\
&= q(f)^{\frac{s}{2}}\gamma(s,f)\sum_{n\geq 1}\frac{a_f(n)}{n^s}\frac{1}{2\pi i}\int_{(a)} \frac{\gamma(s+u,f)}{\gamma(s,f)}\Phi(u)\left(\frac{\sqrt{q(f)}X}{n}\right)^u\,\frac{du}{u} \\
&= q(f)^{\frac{s}{2}}\gamma(s,f)\sum_{n\geq 1}\frac{a_f(n)}{n^s}V_s\left(\frac{n}{\sqrt{q(f)}X}\right).
\end{aligned}$$

Performing the same computation for $I(X^{-1}, 1-s, \overline{f})$ and substituting in the results, we arrive at

$$\Lambda(s,f) = q(f)^{\frac{s}{2}}\gamma(s,f)\sum_{n\geq 1}\frac{a_f(n)}{n^s}V_s\left(\frac{n}{\sqrt{q(f)}X}\right) + \varepsilon(f)q(f)^{\frac{1-s}{2}}\gamma(1-s,f)\sum_{n\geq 1}\frac{a_f(n)}{n^{1-s}}V_{1-s}\left(\frac{nX}{\sqrt{q(f)}}\right) + R.$$

Diving by $q(f)^{\frac{s}{2}}\gamma(s,f)$ completes the proof. □

The approximate functional equation was first developed by Hardy and Littlewood in the series [HL21, HL23, HL29]. The function $V_s(y)$ has the effect of smoothing out the two sums on the right-hand side of the approximate functional equation. In most cases, we will take

$$\Phi(u) = \cos^{-4d_f M}\left(\frac{\pi u}{4M}\right),$$

for an integer $M \geq 1$. Clearly $\Phi(u)$ holomorphic in the vertical strip $|\tau| < (2M-2)+1$, even, and satisfies $\Phi(0) = 1$. To see that it is bounded in this vertical strip, using the formula $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$, we have

$$\cos^{-4d_f M}\left(\frac{\pi u}{4M}\right) = \left(\frac{e^{i\frac{\pi u}{4M}} + e^{-i\frac{\pi u}{4M}}}{2}\right)^{-4d_f M} \ll e^{-d_f\pi|r|}, \tag{3.4}$$

where in the estimate we have used the reverse triangle equality. It follows that $\Phi(u)$ exhibits exponential decay and is therefore bounded. With this choice of $\Phi(u)$, we can prove a useful bound for $V_s(y)$:

Proposition 3.2.2. *Let $L(s,f)$ be an L-function, set $\Phi(u) = \cos^{-4d_f M}\left(\frac{\pi u}{4M}\right)$ for some integer $M \geq 1$, and let $V_s(y)$ be the inverse Mellin transform defined by*

$$V_s(y) = \frac{1}{2\pi i}\int_{(2M-2)}\frac{\gamma(s+u,f)}{\gamma(s,f)}\Phi(u)y^{-u}\frac{du}{u}.$$

Then for s in the critical strip, $V_s(y)$ satisfies the estimate

$$V_s(y) \ll \left(1 + \frac{y}{\sqrt{\mathfrak{q}_\infty(s,f)}}\right)^{-M}.$$

Proof. Suppose $0 \leq \sigma \leq \frac{1}{2}$ so that $\sigma - \frac{1}{2} \leq 0$. Then from Equation (3.1) and the reverse triangle inequality, we deduce that

$$\frac{\Gamma(s+u)}{\Gamma(s)} \ll \frac{(|t+r|+1)^{\sigma+\tau-\frac{1}{2}}}{(|t|+1)^{\sigma-\frac{1}{2}}}e^{\frac{\pi}{2}(|t|-|t+r|)} \ll (|t+r|+1)^{\tau}e^{\frac{\pi}{2}|r|}.$$

From the definition of $\mathfrak{q}(s,f)$, the above estimate implies

$$\frac{\gamma(s+u,f)}{\gamma(s,f)} \ll \mathfrak{q}_\infty(s,f)^{\frac{\tau}{2}}e^{d_f\frac{\pi}{2}|r|},$$

for s in the critical strip. By Equation (3.4), the integral defining $V_s(y)$ is locally absolutely uniformly convergent and we may shift the line of integration to (M). In doing so, we do not pass by any poles and obtain

$$V_s(y) = \frac{1}{2\pi i}\int_{(M)}\frac{\gamma(s+u,f)}{\gamma(s,f)}\Phi(u)y^{-u}\frac{du}{u}$$

The fact that u is bounded away from zero, Equation (3.4), and the estimate for the ratio of gamma factors, gives the first estimate in the following chain:

$$\begin{aligned} V_s(y) &= \frac{1}{2\pi i}\int_{(M)} \frac{\gamma(s+u,f)}{\gamma(s,f)}\Phi(u)y^{-u}\frac{du}{u} \\ &\ll \int_{-\infty}^{\infty} \mathfrak{q}_\infty(s,f)^{\frac{M}{2}} e^{-d_f\frac{\pi}{2}|r|} y^{-M}\, dr \\ &\ll \int_{-\infty}^{\infty} \mathfrak{q}_\infty(s,f)^{\frac{M}{2}} e^{-d_f\frac{\pi}{2}|r|} (1+y)^{-M}\, dr \\ &\ll \int_{-\infty}^{\infty} e^{-\frac{\pi}{2}d_f|r|}\left(1+\frac{y}{\sqrt{\mathfrak{q}_\infty(s,f)}}\right)^{-M} dr \\ &\ll \left(1+\frac{y}{\sqrt{\mathfrak{q}_\infty(s,f)}}\right)^{-M}\int_{-\infty}^{\infty} e^{-\frac{\pi}{2}d_f|r|}\, dr \\ &\ll \left(1+\frac{y}{\sqrt{\mathfrak{q}_\infty(s,f)}}\right)^{-M}, \end{aligned}$$

where in the third and fourth lines we have used that $cy \ll (1+cy)$ and the last line holds since the integrand exhibits exponential decay. This completes the proof. □

From Proposition 3.2.2 we see that $V_s(y)$ is bounded for $y \ll_\varepsilon \mathfrak{q}_\infty(s,f)^{\frac{1}{2}+\varepsilon}$ and then starts to exhibit polynomial decay of arbitrarily large order. In a similar spirit to the approximate functional equation, a useful summation formula can be derived from the functional equation of each L-function:

Theorem 3.2.1. *Let $\psi(y)$ be a bump function and let $\Psi(s)$ denote its Mellin transform. Then for any L-function $L(s,f)$, we have*

$$\sum_{n\geq 1} a_f(n)\psi(n) = \frac{\varepsilon(f)}{\sqrt{q(f)}}\sum_{n\geq 1} a_{\overline{f}}(n)\phi(n) + R\Psi(1),$$

where $\phi(y)$ is the inverse Mellin transform defined by

$$\phi(y) = \frac{1}{2\pi i}\int_{(a)} q(f)^s \frac{\gamma(s,f)}{\gamma(1-s,f)} y^{-s}\Psi(1-s)\, ds,$$

for any $a > 1$. Moreover, R is zero if $L(s,f)$ is entire, and otherwise

$$R = \operatorname*{Res}_{s=1} L(s,f).$$

Proof. By smoothed Perron's formula,

$$\sum_{n\geq 1} a_f(n)\psi(n) = \frac{1}{2\pi i}\int_{(a)} L(s,f)\Psi(s)\, ds.$$

By Propositions 2.3.1 and 3.2.1, the integrand has polynomial decay of arbitrarily larger order and therefore is locally absolutely uniformly convergent. Shifting the line of integration to $(1-a)$, we pass by a potential pole at $s=1$ from $L(s,f)$ and obtain

$$\sum_{n\geq 1} a_f(n)\psi(n) = \frac{1}{2\pi i}\int_{(1-a)} L(s,f)\Psi(s)\, ds + R\Psi(1).$$

Applying the functional equation, we further have

$$\sum_{n\geq 1} a_f(n)\psi(n) = \frac{1}{2\pi i}\int_{(1-a)} \varepsilon(f)q(f)^{\frac{1}{2}-s}\frac{\gamma(1-s,f)}{\gamma(s,f)}L(1-s,\overline{f})\Psi(s)\,ds + R\Psi(1).$$

Performing the change of variables $s \mapsto 1-s$ in this latter integral gives

$$\sum_{n\geq 1} a_f(n)\psi(n) = \frac{1}{2\pi i}\int_{(a)} \varepsilon(f)q(f)^{s-\frac{1}{2}}\frac{\gamma(s,f)}{\gamma(1-s,f)}L(s,\overline{f})\Psi(1-s)\,ds + R\Psi(1).$$

The proof is complete upon interchanging the sum over the Dirichlet series and the integral by the Fubini–Tonelli theorem and factoring out $\frac{\varepsilon(f)}{\sqrt{q(f)}}$. □

3.3 The Riemann Hypothesis and Nontrivial Zeros

The zeros of an L-function $L(s,f)$ have interesting behavior. Recall that

$$L(s,f) = \prod_p \prod_{1\leq j\leq d_f} (1-\alpha_j(p)p^{-s})^{-1},$$

for $\sigma > 1$. This product vanishes if and only if one of its factors are zero. As $\sigma > 1$, this is impossible so that $L(s,f)$ has no zeros in this region. The functional equation will allow us to understand more about the zeros of $L(s,f)$. Rewrite the functional equation for $L(s,f)$ as

$$L(s,f) = \varepsilon(f)q(f)^{\frac{1}{2}-s}\frac{\gamma(1-s,f)}{\gamma(s,f)}L(1-s,\overline{f}). \tag{3.5}$$

If $\sigma < 0$ then $L(1-s,\overline{f})$ is nonzero by our previous comments. Moreover, $\gamma(1-s,f)$ is holomorphic and nonzero in this region because $\mathrm{Re}(\kappa_j) > -1$. We conclude that poles of $\gamma(s,f)$ are zeros of $L(s,f)$ for $\sigma < 0$. Such a zero is called a **trivial zero**. From the definition of $\gamma(s,f)$, they are all simple and of the form $s = -(\kappa_j + 2n)$ for some local root at infinity κ_j and some integer $n \geq 0$. Any other zero of $L(s,f)$ is called a **nontrivial zero** and it lies inside of the critical strip (it may also be a pole of $\gamma(s,f)$). Now let ρ be a nontrivial zero of $L(s,f)$. Note that $L(\overline{s},\overline{f}) = \overline{L(s,f)}$ for $\sigma > 1$ where $L(s,f)$ is defined by a Dirichlet series and thus for all s by the identity theorem. It follows that $\overline{\rho}$ is a nontrivial zero of $L(s,\overline{f})$ and hence, from the functional equation, $1-\overline{\rho}$ is a nontrivial zero of $L(s,f)$ too. In short, the nontrivial zeros occur in pairs:

$$\rho \quad \text{and} \quad 1-\overline{\rho}.$$

We can sometimes say more. If $L(s,f)$ takes real values for $s > 1$, the Schwarz reflection principle implies $L(\overline{s},f) = \overline{L(s,f)}$ and that $L(s,f)$ takes real values on the entire real axis save for the possible poles at $s = 0$ and $s = 1$. We find that $\overline{\rho}$ and $1-\overline{\rho}$ are nontrivial zeros too and therefore the nontrivial zeros of $L(s,f)$ come in sets of four and are displayed in Figure 3.2:

$$\rho, \quad \overline{\rho}, \quad 1-\rho, \quad \text{and} \quad 1-\overline{\rho}.$$

The **Riemann hypothesis** for $L(s,f)$ says that this symmetry should be as simple as possible:

Conjecture (Riemann hypothesis, $L(s,f)$). *For the L-function $L(s,f)$, all of the nontrivial zeros lie on the line $\sigma = \frac{1}{2}$.*

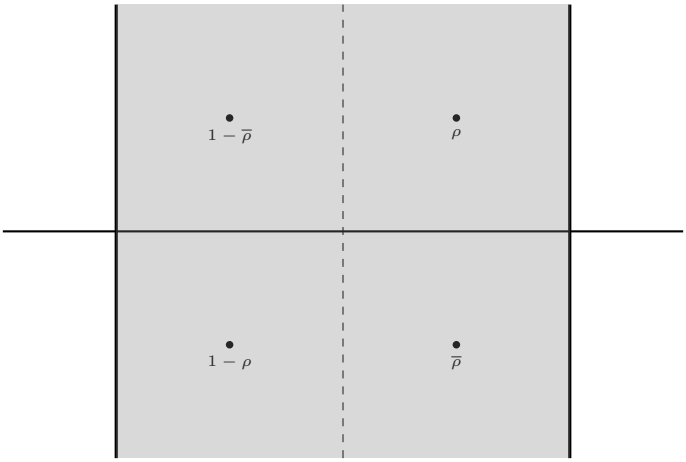


Figure 3.2: Symmetric nontrivial zeros.

Somewhat confusingly, do not expect the Riemann hypothesis to hold for just any L-function but we expect it to hold for many L-functions. In particular, the **grand Riemann hypothesis** says that this symmetry should hold for any L-function in the Selberg class:

Conjecture (Grand Riemann hypothesis). *For any Selberg class L-function $L(s,f)$, all of the nontrivial zeros lie on the line $\sigma = \frac{1}{2}$.*

So far, the Riemann hypothesis remains completely out of reach for any L-function and thus the grand Riemann hypothesis does as well.

3.4 The Lindelöf Hypothesis and Convexity Bounds

Instead of asking about the zeros of an L-function $L(s,f)$ on the critical line, we can ask about the growth of $L(s,f)$, and more generally its derivatives, on the critical line. More precisely, we want to derive an upper bound for $L(s,f)$, or one of its derivatives, on the critical line using the Phragmén-Lindelöf convexity principle. The argument we will describe for $L(s,f)$ is essentially a refinement of the proof of Proposition 3.2.1. Let

$$p_{r_f}(s) = \left(\frac{s-1}{s+1}\right)^{r_f}.$$

Note that $p_{r_f}(s) \sim 1$. The first step is to guarantee the Phragmén-Lindelöf convexity principle for $p_{r_f}(s)L(s,f)$ in a region containing the critical strip. As $L(s,f)$ is of order 1, this is assured (see Appendix B.5). Therefore, we are reduced to estimating the growth of $p_{r_f}(s)L(s,f)$ for σ to the left of 0 and to the right of 1. That is, just outside the edges of the critical strip. The right edge is easily estimated by setting $\sigma = 1+\varepsilon$ so that

$$p_{r_f}(1+\varepsilon+it)L(1+\varepsilon+it,f) \ll_\varepsilon 1,$$

which holds since $L(s,f)$ is defined by a locally absolutely uniformly convergent Dirichlet series for $\sigma > 1$. The left edge is only slightly more difficult. Upon isolating $L(s,f)$ in the functional equation, we have

$$L(s,f) = \varepsilon(f)q(f)^{\frac{1}{2}-s}\frac{\gamma(1-s,f)}{\gamma(s,f)}L(1-s,f).$$

Applying Equation (3.2) result in the bound

$$L(s,f) \ll_\varepsilon \mathfrak{q}(s,f)^{\frac{1}{2}-\sigma}L(1-s,f),$$

for s in any vertical strip with distance ε away from the poles of $\gamma(1-s,f)$. Multiplying both sides by $p_{r_f}(s)$ and taking $\sigma = -\varepsilon$, it follows that

$$p_{r_f}(-\varepsilon+it)L(-\varepsilon+it,f) \ll_\varepsilon \mathfrak{q}(s,f)^{\frac{1}{2}+\varepsilon},$$

which holds since $L(1-s,f)$ is defined by a locally absolutely uniformly convergent Dirichlet series for $\sigma < 0$. As $p_{r_f}(s)L(s,f)$ is holomorphic in a region containing the vertical strip $-\varepsilon \leq \sigma \leq 1+\varepsilon$ and $p_{r_f}(s) \sim 1$, the Phragmén-Lindelöf convexity principle gives

$$L(s,f) \ll_\varepsilon \mathfrak{q}(s,f)^{\frac{1-\sigma}{2}+\varepsilon}, \tag{3.6}$$

in the vertical strip $-\varepsilon \leq \sigma \leq 1+\varepsilon$ provided s is distance ε away from the pole of $L(s,f)$ at $s=1$ if it exists. At the critical line Equation (3.6) gives the following **convexity bound**:

$$L\left(\frac{1}{2}+it,f\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it,f\right)^{\frac{1}{4}+\varepsilon}.$$

The **Lindelöf hypothesis** for $L(s,f)$ says that the exponent can be reduced to ε:

Conjecture (Lindelöf hypothesis, $L(s,f)$). *The L-function $L(s,f)$ satisfies*

$$L\left(\frac{1}{2}+it,f\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it,f\right)^{\varepsilon}.$$

Just as for the Riemann hypothesis, we do not expect the Lindelöf hypothesis to hold for just any L-function. Accordingly, the **grand Lindelöf hypothesis** says that the exponent can be reduced to ε for any L-function in the Selberg class and we expect this to hold:

Conjecture (Grand Lindelöf hypothesis). *Any Selberg class L-function $L(s,f)$ satisfies*

$$L\left(\frac{1}{2}+it,f\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it,f\right)^{\varepsilon}.$$

Like the Riemann hypothesis, we have been unable to prove the Lindelöf hypothesis for any L-function. However, the Lindelöf hypothesis seems to be much more tractable. Generally speaking, any improvement upon the exponent in the convexity bound in any aspect of the analytic conductor is called a **subconvexity estimate** (or a **convexity breaking bound**). In other words, we would want a bound of the form

$$L\left(\frac{1}{2}+it,f\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it,f\right)^{\delta+\varepsilon},$$

for some $0 \leq \delta \leq \frac{1}{4}$. The convexity bound says that we may take $\delta = \frac{1}{4}$ while the Lindelöf hypothesis for $L(s,f)$ implies that we may take $\delta = 0$. Subconvexity estimates are viewed with great interest. This is primarily because of their connection to the associated Lindelöf hypothesis, but also because any improvement upon the convexity bound (or current best value of of δ) for an L-function has drastic consequences in applications.

Remark 3.4.1. *Some subconvexity bounds are deserving of names. The cases $\delta = \frac{3}{16}$ and $\delta = \frac{1}{6}$ are referred to as the **Burgess bound** and **Weyl bound** respectively.*

With a little more work, we can obtain a a similar bound for $L^{(k)}(s,f)$ for any $k \geq 1$. First observe that Equation (3.6) implies

$$L(s,f) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it,f\right)^{\frac{1}{4}+\varepsilon},$$

in the vertical strip $\left|\sigma - \frac{1}{2}\right| \leq \frac{\varepsilon}{2}$. This is just slightly stronger than the convexity bound. Now Cauchy's integral formula gives

$$L^{(k)}\left(\frac{1}{2}+it,f\right) = \frac{k!}{2\pi i}\int_C \frac{L(s,f)}{(s-\frac{1}{2}-it)^{k+1}}\,ds \ll_\varepsilon \int_C |L(s,f)|ds,$$

where C is the circle about $\frac{1}{2}+it$ of radius $\frac{\varepsilon}{2}$. These two estimates, and that the disk bounded by C is compact, together imply the following **convexity bound**:

$$L^{(k)}\left(\frac{1}{2}+it,f\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it,f\right)^{\frac{1}{4}+\varepsilon}.$$

3.5 Estimating the Central Value

The Lindelöf hypothesis is concerned with the growth of the L-function $L(s,f)$ along the critical line, but sometimes we are only concerned with the size of $L(s,f)$ at the central point. The value of $L(s,f)$ at the central point is called the **central value** of $L(s,f)$. Many important properties about $L(s,f)$ can be connected to its central value. Any argument used to estimate the central value of an L-function is called a **central value estimate**. We will prove central value estimate which gives a very useful upper bound in the $q(f)$-aspect. To state it, let $\psi(y)$ be a bump function with compact support in $\left[\frac{1}{2},2\right]$. For example, we may take

$$\psi(y) = \begin{cases} e^{-\frac{1}{9-(4y-5)^2}} & \text{if } |4y-5|<3, \\ 0 & \text{if } |4y-5|\geq 3. \end{cases}$$

The theorem is the following:

Theorem 3.5.1. *Let $L(s,f)$ be an L-function and let $\psi(y)$ be a bump function with compact support in $\left[\frac{1}{2},2\right]$. Then*

$$L\left(\frac{1}{2},f\right) \ll_\varepsilon \max_{X \ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \left(\left|\frac{A_\psi(X)}{q(f)^{\frac{1}{4}}}\right|\right) + \left|\frac{S}{q(f)^{\frac{1}{4}}}\right|,$$

where S is zero if $\Lambda(s,f)$ is entire, and otherwise

$$S = \operatorname*{Res}_{u=\frac{1}{2}} \Lambda\left(\frac{1}{2}+u,f\right) + \operatorname*{Res}_{u=-\frac{1}{2}} \Lambda\left(\frac{1}{2}+u,f\right).$$

Proof. Taking $s=\frac{1}{2}$, $X=1$, and $\Phi(u) = \cos^{-4dM}\left(\frac{\pi u}{4M}\right)$ with $M \gg 1$ in the approximate functional equation gives

$$L\left(\frac{1}{2},f\right) = \sum_{n\geq 1}\frac{a_f(n)}{\sqrt{n}}V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right) + \varepsilon(f)\sum_{n\geq 1}\frac{\overline{a_f(n)}}{\sqrt{n}}V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right) + \frac{R}{q(f)^{\frac{1}{4}}\gamma\left(\frac{1}{2},f\right)}.$$

This implies the bound

$$L\left(\frac{1}{2},f\right) \ll \left|\sum_{n\geq 1}\frac{a_f(n)}{\sqrt{n}}V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right)\right| + \left|\frac{S}{q(f)^{\frac{1}{4}}}\right|.$$

Now consider the set of functions $\left\{\psi\left(\frac{y}{2^k}\right)\right\}_{k\in\mathbb{Z}}$. Since $\psi\left(\frac{y}{2^k}\right)$ has support in $[2^{k-1},2^{k+1}]$, the sum $\sigma(y)=\sum_{k\in\mathbb{Z}}\psi\left(\frac{y}{2^k}\right)$, defined for $y>0$, is finite since at most finitely many terms are nonzero for every y. It is also bounded away from zero since for any $y>0$ there is some $k\in\mathbb{Z}$ for which $2^k\le y\le 3\cdot 2^{k-1}$ so that $\frac{y}{2^k}$ is at least distance $\frac{1}{2}$ from the endpoints of $\left[\frac{1}{2},2\right]$. Defining $\psi_k(y)=\psi\left(\frac{y}{2^k}\right)\sigma(y)^{-1}$, it follows that $\{\psi_k(y)\}_{k\in\mathbb{Z}}$ satisfies

$$\sum_{k\in\mathbb{Z}}\psi_k(y)=1,$$

for any $y>0$. Then we can write

$$V_s(y)=\sum_{k\in\mathbb{Z}}\psi_k(y)V_s(y).$$

It follows that

$$\begin{aligned}
\sum_{n\ge1}\frac{a_f(n)}{\sqrt{n}}V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right)&=\sum_{n\ge1}\frac{a_f(n)}{\sqrt{n}}\sum_{k\in\mathbb{Z}}\psi_k\left(\frac{n}{\sqrt{q(f)}}\right)V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right)\\
&=\sum_{k\in\mathbb{Z}}\sum_{n\ge1}\frac{a_f(n)}{\sqrt{n}}\psi_k\left(\frac{n}{\sqrt{q(f)}}\right)V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right) && \text{FTT}\\
&\ll_\varepsilon\sum_{k\in\mathbb{Z}}\left|\sum_{\substack{n\ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}\\ 2^{k-1}\sqrt{q(f)}\le n\le 2^{k+1}\sqrt{q(f)}}}\frac{a_f(n)}{\sqrt{n}}\psi_k\left(\frac{n}{\sqrt{q(f)}}\right)\right|\\
&\ll_\varepsilon\sum_{k\ll\log(q(f)^{\frac{\varepsilon}{2}})}\left|\sum_{\substack{n\ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}\\ 2^{k-1}\sqrt{q(f)}\le n\le 2^{k+1}\sqrt{q(f)}}}\frac{a_f(n)}{\sqrt{n}}\psi_k\left(\frac{n}{\sqrt{q(f)}}\right)\right|,
\end{aligned}$$

where in the second to last line we have used that $V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right)$ is bounded for $n\ll_\varepsilon \mathfrak{q}\left(\frac{1}{2},f\right)^{\frac{1}{2}+\varepsilon}\ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}$ (and then exhibits polynomial decay thereafter) by Proposition 3.2.2 and in the last line we have used that $\psi_k(y)$ has compact support in $[2^{k-1},2^{k+1}]$ (recall $k\in\mathbb{Z}$). Since $\sigma(y)$ is bounded away from zero and $\log(y)\ll y$, we obtain the crude bound

$$\sum_{k\ll\log(q(f)^{\frac{\varepsilon}{2}})}\left|\sum_{\substack{n\ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}\\ 2^{k-1}\sqrt{q(f)}\le n\le 2^{k+1}\sqrt{q(f)}}}\frac{a_f(n)}{\sqrt{n}}\psi_k\left(\frac{n}{\sqrt{q(f)}}\right)\right|\ll_\varepsilon q(f)^{\frac{\varepsilon}{2}}\max_{X\ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}}\left(\left|\sum_{\frac{X}{2}\le n\le 2X}\frac{a_f(n)}{\sqrt{n}}\psi\left(\frac{n}{X}\right)\right|\right).$$

We will estimate this latter sum. Abel's summation formula (see Appendix B.3) gives

$$\sum_{\frac{X}{2}\le n\le 2X}\frac{a_f(n)}{\sqrt{n}}\psi\left(\frac{n}{X}\right)=\frac{A_\psi(2X)}{\sqrt{2X}}-\frac{A_\psi\left(\frac{X}{2}\right)}{\sqrt{\frac{X}{2}}}+\frac{1}{2}\int_{\frac{X}{2}}^{2X}A_\psi(u)u^{-\frac{3}{2}}\,du.$$

But as

$$\frac{1}{2}\int_{\frac{X}{2}}^{2X}A_\psi(u)u^{-\frac{3}{2}}\,du\ll X\max_{\frac{X}{2}\le u\le 2X}\left(\left|A_\psi(u)u^{-\frac{3}{2}}\right|\right)\ll\max_{\frac{X}{2}\le u\le 2X}\left(\left|\frac{A_\psi(u)}{\sqrt{u}}\right|\right),$$

we obtain the bound

$$\max_{X \ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \left(\left| \sum_{\frac{X}{2} \leq n \leq 2X} \frac{a_f(n)}{\sqrt{n}} \psi\left(\frac{n}{X}\right) \right| \right) \ll \max_{X \ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \left(\left| \frac{A_\psi(X)}{\sqrt{X}} \right| \right).$$

Putting everything together gives

$$\sum_{n \geq 1} \frac{a_f(n)}{\sqrt{n}} V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right) \ll_\varepsilon q(f)^{\frac{\varepsilon}{2}} \max_{X \ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \left(\left| \frac{A_\psi(X)}{\sqrt{X}} \right| \right) \ll_\varepsilon \max_{X \ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \left(\left| \frac{A_\psi(X)}{q(f)^{\frac{1}{4}}} \right| \right),$$

where in the last estimate we may replace X with $q(f)^{\frac{1}{2}+\varepsilon}$ because $X \geq 1$ so that X is bounded away from zero. □

3.6 Logarithmic Derivatives

There is an incredibly useful formula for the logarithmic derivative of any L-function which is often the starting point for deeper analytic investigations. To deduce it, we will need a more complete understanding of $\Lambda(s, f)$. First observe that the zeros ρ of $\Lambda(s, f)$ are contained inside of the critical strip. Indeed, we have already remarked that $L(s, f)$ has no zeros for $\sigma > 0$ and clearly $\gamma(s, f)$ does not have zeros in this region as well. Therefore $\Lambda(s, f)$ is nonzero for $\sigma > 1$. By the functional equation, $\Lambda(s, f)$ is also nonzero for $\sigma < 0$ too. In other words, the zeros of $\Lambda(s, f)$ are the nontrivial zeros of $L(s, f)$. Before we state our result, we setup some notation. For an L-function $L(s, f)$ we define

$$\xi(s, f) = (s(1 - s))^{r_f} \Lambda(s, f).$$

Note that $\xi(s, f)$ is essentially just $\Lambda(s, f)$ with the potential poles at $s = 0$ and $s = 1$ removed. From the functional equation, we also have

$$\xi(s, f) = \varepsilon(f)\xi(1 - s, \overline{f}).$$

We now state our desired result:

Proposition 3.6.1. *For any L-function $L(s, f)$, there exist constants $A(f)$ and $B(f)$ such that*

$$\xi(s, f) = e^{A(f)+B(f)s} \prod_{\rho \neq 0,1} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

and hence the sum

$$\sum_{\rho \neq 0,1} \frac{1}{|\rho|^{1+\varepsilon}},$$

is convergent provided the product and sum are both counted with multiplicity and ordered with respect to the size of the ordinate. Moreover,

$$-\frac{L'}{L}(s, f) = \frac{r_f}{s} + \frac{r_f}{s - 1} + \frac{1}{2} \log q(f) + \frac{\gamma'}{\gamma}(s, f) - B(f) - \sum_{\rho \neq 0,1} \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right).$$

Proof. For the first statement, observe that $\xi(s, f)$ is entire since the only possible poles of $\Lambda(s, f)$ are at $s = 0$ and $s = 1$ and are of order r_f. We also claim that $\xi(s, f)$ is of order 1. By the functional equation,

it suffices to show this for $\sigma \geq \frac{1}{2}$. This follows from $L(s,f)$ being of order 1 and Equation (3.1) (within ε of the poles of $\gamma(s,f)$ we know $\xi(s,f)$ is bounded because it is entire). By the Hadamard factorization theorem (see Appendix B.4),

$$\xi(s,f) = e^{A(f)+B(f)s} \prod_{\rho \neq 0,1} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

for some constants $A(f)$ and $B(f)$ and the desired sum converges. This proves the first statement. For the second, taking the logarithmic derivative of the definition of $\xi(s,f)$ yields

$$\frac{\xi'}{\xi}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} + \frac{1}{2}\log q(f) + \frac{\gamma'}{\gamma}(s,f) + \frac{L'}{L}(s,f). \tag{3.7}$$

On the other hand, taking the logarithmic derivative of the Hadamard factorization gives

$$\frac{\xi'}{\xi}(s,f) = B(f) + \sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{3.8}$$

Equating Equations (3.7) and (3.8), we arrive at

$$B(f) + \sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = \frac{r_f}{s} + \frac{r_f}{s-1} + \frac{1}{2}\log q(f) + \frac{\gamma'}{\gamma}(s,f) + \frac{L'}{L}(s,f).$$

Isolating $-\frac{L'}{L}(s,f)$ completes the proof. □

We now need to make a few comments. Our first is regarding the constants $A(f)$ and $B(f)$. The explicit evaluation of these constants can be challenging and heavily depends upon the arithmetic object f. However, useful estimates are not too difficult to obtain. We also claim $A(\overline{f}) = \overline{A(f)}$ and $B(\overline{f}) = \overline{B(f)}$. To see this, recall that $L(\overline{s},\overline{f}) = \overline{L(s,f)}$. Then $\xi(\overline{s},\overline{f}) = \overline{\xi(s,f)}$ because $\gamma(s,\overline{f}) = \gamma(s,f)$, the κ_j are real or occur in conjugate pairs, and $\Gamma(\overline{s}) = \overline{\Gamma(s)}$. But then the functional equation and Proposition 3.6.1 together imply

$$e^{A(\overline{f})+B(\overline{f})s} = \frac{\xi'}{\xi}(0,\overline{f}) = \overline{\frac{\xi'}{\xi}(0,f)} = e^{\overline{A(f)}+\overline{B(f)}s},$$

and the claim follows. Our second comment concerns the negative logarithmic derivative of $L(s,f)$. In general, this function attracts much attention for analytic investigations. As $L(s,f)$ is holomorphic for $\sigma > 1$ and admits an Euler product there, we can take the logarithm of the Euler product (turning it into a sum) and differentiate termwise to obtain

$$-\frac{L'}{L}(s,f) = -\sum_p \sum_{1 \leq j \leq d_f} \frac{d}{ds}\log(1-\alpha_j(p)p^{-s}) = \sum_p \sum_{1 \leq j \leq d_f} \frac{\alpha_j(p)\log(p)}{(1-\alpha_j(p)p^{-s})p^s}. \tag{3.9}$$

From the Taylor series of $\frac{1}{1-s}$, it follows that $-\frac{L'}{L}(s,f)$ is a locally absolutely uniformly convergent Dirichlet series of the form

$$-\frac{L'}{L}(s,f) = \sum_{n \geq 1} \frac{\Lambda_f(n)}{n^s},$$

for $\sigma > 1$, where

$$\Lambda_f(n) = \begin{cases} \sum_{1 \leq j \leq d_f} \alpha_j(p)^k \log(p) & \text{if } n = p^k \text{ for some } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is worth noting that $\Lambda_{\overline{f}}(n) = \overline{\Lambda_f(n)}$.

3.7 Zero Density

The deepest subject of the theory of L-functions is arguably the distribution of the zeros of L-functions. Here we introduce a method of counting zeros of L-functions and gaining a very simple understanding of their density as a result. We first require an immensely useful lemma:

Lemma 3.7.1. *Let $L(s,f)$ be an L-function. The following statements hold:*

(i) The constant $B(f)$ satisfies

$$\mathrm{Re}(B(f)) = -\sum_{\rho\neq 0,1} \mathrm{Re}\left(\frac{1}{\rho}\right),$$

where the sum is counted with multiplicity and ordered with respect to the size of the ordinate.

(ii) For any $T \geq 0$, the number of nontrivial zeros $\rho = \beta + i\gamma$ with $\rho \neq 0, 1$ and such that $|T-\gamma| \leq 1$ is $O(\log \mathfrak{q}(iT, f))$.

(iii) For $\sigma > 1$, we have

$$\mathrm{Re}\left(\frac{1}{s-\rho}\right) > 0 \quad \textit{and} \quad \mathrm{Re}\left(\frac{1}{s+\kappa_j}\right) > 0.$$

(iv) We have

$$-\frac{L'}{L}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} - \sum_{|s+\kappa_j|<1} \frac{1}{s+\kappa_j} - \sum_{\substack{|s-\rho|<1 \\ \rho\neq 0,1}} \frac{1}{s-\rho} + O(\log \mathfrak{q}(s,f)),$$

for any s in the vertical strip $-\frac{1}{2} \leq \sigma \leq 2$. Moreover, in this vertical strip we also have

$$\mathrm{Re}\left(-\frac{L'}{L}(s,f)\right) \leq \mathrm{Re}\left(\frac{r_f}{s}\right) + \mathrm{Re}\left(\frac{r_f}{s-1}\right) - \sum_{|s+\kappa_j|<1} \mathrm{Re}\left(\frac{1}{s+\kappa_j}\right) - \sum_{\substack{|s-\rho|<1 \\ \rho\neq 0,1}} \mathrm{Re}\left(\frac{1}{s-\rho}\right) + O(\log \mathfrak{q}(s,f)),$$

and we may discard any term in either sum if $1 < \sigma \leq 2$.

Proof. We will prove each statement separately.

(i) To prove (i), first recall that $B(\overline{f}) = \overline{B(f)}$. Then Equation (3.8) and the functional equation together imply

$$2\mathrm{Re}(B(f)) = B(f) + B(\overline{f}) = -\sum_{\rho\neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{1-s-\overline{\rho}} + \frac{1}{\rho} + \frac{1}{\overline{\rho}}\right),$$

where we have made use of the fact that the nontrivial zeros occur is pairs ρ and $1-\overline{\rho}$ where the latter is also a nontrivial zero of $L(1-s,\overline{f})$. Now fix s such that it does not coincide with the ordinate of a nontrivial zero. Then s is bounded away from all of the nontrivial zeros and it follows that $\frac{1}{(s-\rho)} + \frac{1}{(1-s-\overline{\rho})} \ll \frac{1}{\rho^2}$ and $\frac{1}{\rho} + \frac{1}{\overline{\rho}} \ll \frac{1}{\rho^2}$. Therefore the sums

$$\sum_{\rho\neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{1-s-\overline{\rho}}\right) \quad \text{and} \quad \sum_{\rho\neq 0,1} \left(\frac{1}{\rho} + \frac{1}{\overline{\rho}}\right),$$

converge absolutely by Proposition 3.6.1 and so we can sum them separately. The first sum vanishes by again using the fact that the nontrivial zeros occur is pairs ρ and $1-\overline{\rho}$. Thus

$$2\mathrm{Re}(B(f)) = \sum_{\rho\neq 0,1} \left(\frac{1}{\rho} + \frac{1}{\overline{\rho}}\right) = -\sum_{\rho\neq 0,1} \mathrm{Re}\left(\frac{1}{\rho}\right),$$

which gives (i).

(ii) For (ii), we first bound two important quantities. For the first quantity, the definition of $\Lambda_f(n)$ and that $|\alpha_j(p)| \leq p$ together imply the weak bound $|\Lambda_f(n)| \leq d_f n \log(n)$. Then

$$\frac{L'}{L}(s,f) \ll d_f \zeta'(s-1) \ll \log \mathfrak{q}(f), \tag{3.10}$$

provided $\sigma > 2$. For the second quantity, Equation (3.3) implies

$$\frac{1}{2}\log q(f) + \frac{\gamma'}{\gamma}(s,f) \ll \log \mathfrak{q}(s,f), \tag{3.11}$$

for $\sigma > 0$. Now fix $T \geq 0$ and let $s = 3 + iT$. Taking the real part of the formula for the negative logarithmic derivative in Proposition 3.6.1 and combining Equations (3.10) and (3.11) with (i) results in

$$\sum_{\rho \neq 0,1} \mathrm{Re}\left(\frac{1}{s-\rho}\right) \ll \log \mathfrak{q}(iT, f).$$

But as

$$\frac{2}{9 + (T-\gamma)^2} \leq \mathrm{Re}\left(\frac{1}{s-\rho}\right) \leq \frac{3}{4 + (T-\gamma)^2},$$

we obtain

$$\sum_{\rho \neq 0,1} \frac{1}{1 + (T-\gamma)^2} \ll \log \mathfrak{q}(iT, f), \tag{3.12}$$

which is stronger than the first statement of (ii) since all of the terms in the sum are positive. The second statement is also clear.

(iii) For (iii), just observe that

$$\mathrm{Re}\left(\frac{1}{s-\rho}\right) = \frac{\sigma - \beta}{(\sigma-\beta)^2 + (t-g)^2} > 0 \quad \text{and} \quad \mathrm{Re}\left(\frac{1}{s+\kappa_j}\right) = \frac{\sigma + \mathrm{Re}(\kappa_j)}{(\sigma + \mathrm{Re}(\kappa_j))^2 + (t + \mathrm{Im}(\kappa_j))^2} > 0,$$

where the first bound holds because $\beta \leq 1$ and the second bound holds because $\mathrm{Re}(\kappa_j) > -1$.

(iv) To deduce (iv), let s be such that $-\frac{1}{2} \leq \sigma \leq 2$. Using Equation (3.10), we can write

$$-\frac{L'}{L}(s,f) = -\frac{L'}{L}(s,f) + \frac{L'}{L}(3+it, f) + O(\log \mathfrak{q}(s,f)).$$

Applying the formula for the negative logarithmic derivative in Proposition 3.6.1 to the two terms on the right-hand side and using Equation (3.3) for $\sigma > 0$, we get

$$-\frac{L'}{L}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} + \frac{\gamma'}{\gamma}(s,f) - \sum_{\rho \neq 0,1}\left(\frac{1}{s-\rho} - \frac{1}{3+it-\rho}\right) + O(\log \mathfrak{q}(s,f)).$$

We now estimate the remaining sum. Retain the first part of the terms for which $|s-\rho| < 1$. The contribution from the second part of these terms is $O(\log \mathfrak{q}(it,f))$ by (ii). For those terms with $|s-\rho| \geq 1$, we have

$$\left|\frac{1}{s-\rho} - \frac{1}{3+it-\rho}\right| \leq \frac{3-\sigma}{(3-\beta)^2 + (t-\gamma)^2} \leq \frac{3}{1+(t-\gamma)^2}.$$

Therefore from Equation (3.12), the contribution of these terms is $O(\log \mathfrak{q}(it, f))$ too. It follows that

$$-\frac{L'}{L}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} + \frac{\gamma'}{\gamma}(s,f) - \sum_{\substack{|s-\rho|<1 \\ \rho\neq 0,1}} \frac{1}{s-\rho} + O(\log \mathfrak{q}(s,f)).$$

Applying $\frac{\Gamma'}{\Gamma}(s+1) = \frac{\Gamma'}{\Gamma}(s) + \frac{1}{s}$ to $\frac{\gamma'}{\gamma}(s,f)$ and using Equation (3.3) gives

$$\frac{\gamma'}{\gamma}(s,f) = -\sum_{|s+\kappa_j|<1} \frac{1}{s+\kappa_j} + O(\log \mathfrak{q}_\infty(s,f)).$$

Then we obtain

$$-\frac{L'}{L}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} - \sum_{|s+\kappa_j|<1} \frac{1}{s+\kappa_j} - \sum_{\substack{|s-\rho|<1 \\ \rho\neq 0,1}} \frac{1}{s-\rho} + O(\log \mathfrak{q}(s,f)),$$

which is the first statement of (iv). For second statement, tak the real part of this estimate and write it as

$$\sum_{|s+\kappa_j|<1} \mathrm{Re}\left(\frac{1}{s+\kappa_j}\right) + \sum_{\substack{|s-\rho|<1 \\ \rho\neq 0,1}} \mathrm{Re}\left(\frac{1}{s-\rho}\right) \leq \mathrm{Re}\left(-\frac{L'}{L}(s,f)\right) + \mathrm{Re}\left(\frac{r_f}{s}\right) + \mathrm{Re}\left(\frac{r_f}{s-1}\right) + O(\log \mathfrak{q}(s,f)).$$

Now observe that we can discard any of the terms in either sum provided $1 < \sigma \leq 2$ by (iii). □

With Lemma 3.7.1 in hand, we can deduce a result which estimates the number of nontrivial zeros in a box. Accordingly, for any $T \geq 0$ we define

$$N(T,f) = |\{\rho = \beta + i\gamma \in \mathbb{C} : L(\rho,f) = 0 \text{ with } 0 \leq \beta \leq 1 \text{ and } |\gamma| \leq T\}|.$$

In other words, $N(T,f)$ is the number of nontrivial zeros of $L(s,f)$ with ordinate in $[-T,T]$. We will prove the following:

Theorem 3.7.1. *For any L-function $L(s,f)$ and $T \geq 1$,*

$$N(T,f) = \frac{T}{\pi}\log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right) + O(\log \mathfrak{q}(iT,f)).$$

Proof. Let $T \geq 1$ and set

$$N'(T,f) = |\{\rho = \beta + i\gamma \in \mathbb{C} : L(\rho,f) = 0 \text{ with } 0 \leq \beta \leq 1 \text{ and } 0 < \gamma \leq T\}|.$$

As the nontrivial zeros occur in pairs ρ and $1-\overline{\rho}$ where the latter is also a nontrivial zero of $L(s,\overline{f})$, it follows that

$$N(T,f) = N'(T,f) + N'(T,\overline{f}) + O(\log \mathfrak{q}(f)),$$

where $O(\log \mathfrak{q}(f))$ accounts for the possible real nontrivial zeros. There are finitely many such nontrivial zeros because the interval $0 \leq s \leq 1$ is compact. We will estimate $N'(T,f)$ and in doing so we may assume $L(s,f)$ does not vanish on the line $t = T$ by varying T by a sufficiently small constant, if necessary, and observing that $N(T,f)$ is modified by a quantity of size $O(\log \mathfrak{q}(iT,f))$ by Lemma 3.7.1 (i). Since

the nontrivial zeros are isolated, let $\delta > 0$ be small enough such that $\Lambda(s, f)$ has no nontrivial zeros for $-\delta \leq t < 0$. Then by our previous comments and the argument principle,

$$N'(T, f) = \frac{1}{2\pi i} \int_\eta \frac{\xi'}{\xi}(s, f)\, ds + O(\log \mathfrak{q}(iT, f)),$$

where $\eta = \sum_{1 \leq i \leq 6} \eta_i$ is the contour in Figure 3.3:
Since $\log(s) = \log|s| + i \arg(s)$, we have

$$\frac{1}{2\pi i} \int_\eta \frac{\xi'}{\xi}(s, f)\, ds = \frac{1}{2\pi i} \int_\eta \frac{d}{ds} \log|\xi(s, f)|\, ds + \frac{1}{2\pi} \int_\eta \frac{d}{ds} \arg \xi(s, f)\, ds = \frac{1}{2\pi} \Delta_\eta \arg(\xi(s, f)),$$

where the last equality holds by parameterizing the curve η and noting that η is closed so that the first integral vanishes. For convenience, set $\eta_L = \eta_1 + \eta_2 + \eta_3$ and $\eta_R = \eta_4 + \eta_5 + \eta_6$. Recall that we have already shown $\xi(\overline{s}, \overline{f}) = \overline{(\xi(s, f))}$. Using this fact along with the functional equation and that $-\arg(s) = \arg(\overline{s})$, we compute

$$\begin{aligned}
\Delta_{\eta_L} \arg(\xi(s, f)) &= \Delta_{\eta_L} \arg\left(\varepsilon(f)\xi(1 - s, \overline{f})\right) \\
&= -\Delta_{\overline{\eta_R}} \arg\left(\varepsilon(f)\xi(s, \overline{f})\right) \\
&= \Delta_{\overline{\eta_R}} \arg\left(\overline{\varepsilon(f)}\xi(\overline{s}, f)\right) \\
&= \Delta_{\eta_R} \arg\left(\overline{\varepsilon(f)}\xi(s, f)\right) \\
&= \Delta_{\eta_R} \arg\left(\overline{\varepsilon(f)}\right) + \Delta_{\eta_R} \arg(\xi(s, f)) \\
&= \Delta_{\eta_R} \arg(\xi(s, f)).
\end{aligned}$$

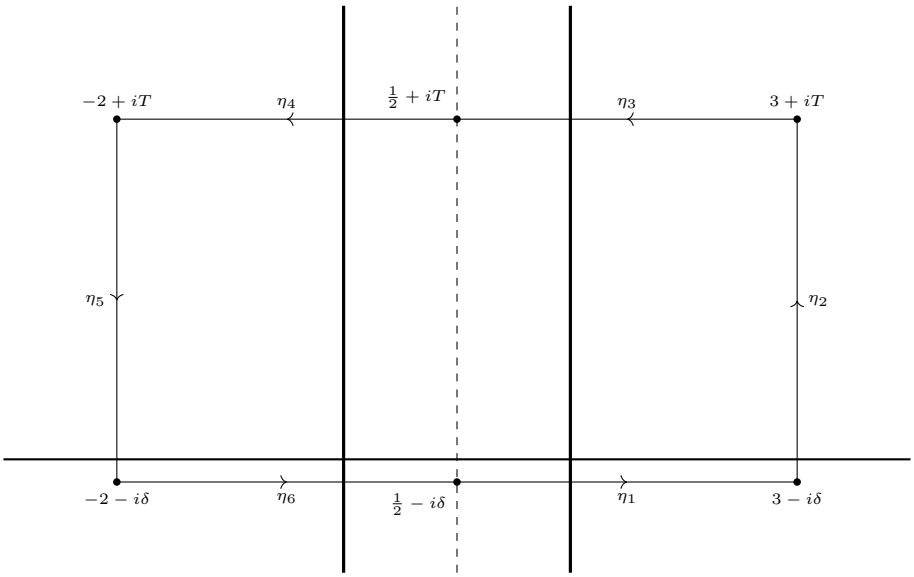


Figure 3.3: A zero counting contour.

In other words, the change in argument along η_L is equal to the change in argument along η_R and so

$$\frac{1}{2\pi i} \int_\eta \frac{\xi'}{\xi}(s, f)\, ds = \frac{1}{\pi} \Delta_{\eta_L} \arg \xi(s, f).$$

Thus to estimate the integral, we estimate the change in argument along η_L of each factor in

$$\xi(s,f) = (s(1-s))^{r_f} q(f)^{\frac{s}{2}} \pi^{-\frac{d_f s}{2}} \prod_{1\le j\le d_f} \Gamma\left(\frac{s+\kappa_j}{2}\right) L(s,f).$$

For the factor $(s(1-s))^{r_f}$, we first have

$$\Delta_{\eta_L} \arg(s) = \arg(s)\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} = \arg\left(\frac{1}{2}+iT\right) - \arg\left(\frac{1}{2}-i\delta\right) = O\left(\frac{1}{T}\right),$$

and

$$\Delta_{\eta_L} \arg(1-s) = \arg(1-s)\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} = \arg\left(\frac{1}{2}-iT\right) - \arg\left(\frac{1}{2}+i\delta\right) = O\left(\frac{1}{T}\right),$$

where in both computations we have used $\arg(s) = \tan^{-1}\left(\frac{t}{\sigma}\right) = \frac{\pi}{2} + O\left(\frac{1}{t}\right)$, provided $\sigma > 0$, which holds by the Laurent series of the inverse tangent. Combining these two bounds, we obtain

$$\Delta_{\eta_L} (s(1-s))^{r_f} = O\left(\frac{1}{T}\right). \tag{3.13}$$

For the factor $q(f)^{\frac{s}{2}}$, we use that $\arg(s) = \mathrm{Im}(\log(s))$ and compute

$$\Delta_{\eta_L} \arg q(f)^{\frac{s}{2}} = \mathrm{Im}(\log q(f)^{\frac{s}{2}})\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} = \log q(f)\left(\frac{T}{2}+\frac{\delta}{2}\right) = \frac{T}{2}\log q(f) + O(1). \tag{3.14}$$

For the factor $\pi^{-\frac{d_f s}{2}}$, we use that $\arg(s) = \mathrm{Im}(\log(s))$ and compute

$$\Delta_{\eta_L} \arg(\pi^{-\frac{d_f s}{2}}) = \mathrm{Im}(\log(\pi^{-\frac{d_f s}{2}}))\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} = \log\left(\frac{1}{\pi^{d_f}}\right)\left(\frac{T}{2}+\frac{\delta}{2}\right) = \frac{T}{2}\log\left(\frac{1}{\pi^{d_f}}\right) + O(1). \tag{3.15}$$

For the factor $\prod_{1\le j\le d_f} \Gamma\left(\frac{s+\kappa_j}{2}\right)$, we first use Equation (1.5) (valid since $T \ge 1$) and that $\arg(s) = \mathrm{Im}(\log(s))$ to obtain

$$\begin{aligned}
\Delta_{\eta_L} \arg\Gamma(s) &= \mathrm{Im}(\log\Gamma(s))\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} \\
&= T\log\left|\frac{1}{2}+iT\right| - T + \delta\log\left|\frac{1}{2}+i\delta\right| - \delta + O(1) \\
&= T\log(T) - T + O(1) \\
&= T\log\left(\frac{T}{e}\right) + O(1).
\end{aligned}$$

It follows that

$$\Delta_{\eta_L} \arg\left(\prod_{1\le j\le d_f} \Gamma\left(\frac{s+\kappa_j}{2}\right)\right) = \frac{T}{2}\log\left(\frac{T^{d_f}}{(2e)^{d_f}}\right) + O(\log \mathfrak{q}(f)). \tag{3.16}$$

For the factor $L(s,f)$, note that

$$\Delta_{\eta_L} \arg(L(s,f)) = \mathrm{Im}(\log L(s,f))\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} = \mathrm{Im}\left(\int_{\eta_L} \frac{L'}{L}(s,f)\,ds\right).$$

By Equation (3.10), the integral is $O(\log q(f))$ on η_2. On η_1 and η_3, Lemma 3.7.1 (ii) and (iv) together imply that the integral is $O(\log \mathfrak{q}(iT, f))$. It follows that

$$\Delta_{\eta_L} \arg(L(s, f)) = O(\log \mathfrak{q}(iT, f)). \tag{3.17}$$

Combining Equations (3.13) to (3.17) results in

$$\frac{1}{\pi}\Delta_{\eta_L} \arg \xi(s, f) = \frac{T}{2\pi} \log\left(\frac{T^{d_f}}{(2\pi e)^{d_f}}\right) + O(\log \mathfrak{q}(iT, f)),$$

and therefore

$$N'(T, f) = \frac{T}{2\pi} \log\left(\frac{T^{d_f}}{(2\pi e)^{d_f}}\right) + O(\log \mathfrak{q}(iT, f)),$$

The claim follows immediately from this estimate and that the same exact estimate holds for $N'(T, \overline{f})$ by using the L-function $L(s, \overline{f})$. □

It is worth noting that the main term in the proof of Theorem 3.7.1 comes from the change in argument of $q(f)^{\frac{s}{2}}\gamma(s, f)$ along η_3 (equivalently η_4). Moreover, the contribution from $L(s, f)$ is only to the error term. This is a good example of how analytic information of an L-function is intrinsically connected to its gamma factor. Also, with Theorem 3.7.1 we can now derive our zero density estimate:

Corollary 3.7.1. *For and L-function $L(s, f)$ and $T \geq 1$,*

$$\frac{N(T, f)}{T} \sim \frac{1}{\pi} \log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right).$$

Proof. From Theorem 3.7.1,

$$\frac{N(T, f)}{T} = \frac{1}{\pi} \log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right)\left(1 + O\left(\frac{\log \mathfrak{q}(iT, f)}{\log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right) T}\right)\right) = \frac{1}{\pi} \log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right)\left(1 + O\left(\frac{1}{T}\right)\right).$$

Since $O\left(\frac{1}{T}\right) = o(1)$, the result follows. □

Corollary 3.7.1 can be interpreted as saying that for large T the density of $N(T, f)$ is approximately $\frac{1}{\pi} \log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right)$. Since this grows as $T \to \infty$, we see that the nontrivial zeros tend to accumulate farther up the critical strip with logarithmic growth. We can dispense with this accumulation. If $\rho = \beta + i\gamma$ is a nontrivial zero of $L(s, f)$ then we call $\rho_{\text{unf}} = \beta + i\omega$ the **unfolded nontrivial zero** corresponding to ρ where

$$\omega = \frac{\gamma}{\pi} \log\left(\frac{q(f)|\gamma|^{d_f}}{(2\pi e)^{d_f}}\right).$$

Now for any $W \geq 0$, define

$$N_{\text{unf}}(W, f) = |\{\rho_{\text{unf}} = \beta + i\omega \in \mathbb{C} : L(\rho, f) = 0 \text{ with } 0 \leq \beta \leq 1 \text{ and } |\omega| \leq W\}|.$$

In other words, $N_{\text{unf}}(T, f)$ is the number of unfolded nontrivial zeros of $L(s, f)$ with ordinate in $[-W, W]$. We then have the following well-known result:

Proposition 3.7.1. *For any L-function $L(s,f)$ and $W \geq \frac{1}{\pi}\log\left(\frac{q(f)}{(2\pi e)^{d_f}}\right)$,*

$$\frac{N_{unf}(W,f)}{W} \sim 1.$$

Proof. Consider the function $f(t)$ defined by

$$f(t) = \frac{t}{\pi}\log\left(\frac{q(f)|t|^{d_f}}{(2\pi e)^{d_f}}\right),$$

for real t. Since $f(t)$ is a strictly increasing continuous function, it has an inverse $g(w)$ for real w. It follows that $|\omega| \leq W$ if and only if $|\rho| \leq g(W)$ and so $N_{\text{unf}}(W,f) = N(g(W),f)$. But by Corollary 3.7.1 and that $g(w)$ is the inverse of $f(t)$, we have $N(g(W),f) \sim W$. Then

$$N_{\text{unf}}(W,f) \sim W,$$

which is equivalent to the claim. □

We interpret Proposition 3.7.1 as saying that the unfolded nontrivial zeros are evenly spaced opposed to Corollary 3.7.1 which says that they tend to accumulate up the critical strip.

3.8 A Zero-free Region

Although the Riemann hypothesis remains out of reach, some progress has been made to understand regions inside of the critical strip for which L-functions are nonzero except for possibly one real exception. Such regions are known as **zero-free regions** and there is great interest in improving the breadth of such regions. We will derive a standard zero-free region for any L-function under some mild assumptions. First a useful lemma:

Lemma 3.8.1. *Let $L(s,f)$ be an L-function such that $\text{Re}(\Lambda_f(n)) \geq 0$ provided $(n,q(f)) = 1$. Also suppose that $|\alpha_j(p)| \leq \frac{p}{2}$ for ramified primes p. Then $L(1,f) \neq 0$ and hence $r_f \geq 0$. Moreover, there exists a constant $c > 0$ such that $L(s,f)$ has at most r_f real zeros in the region*

$$\sigma \geq 1 - \frac{c}{d_f(r_f+1)\log \mathfrak{q}(f)}.$$

Proof. Consider the real nontrivial zeros belonging to the interval $\frac{1}{2} \leq s \leq 1$. As this interval is compact, there are finitely many such zeros β_j (with the understanding that there may be no such zeros). Letting $1 < \sigma \leq 2$, and applying Lemma 3.7.1 (iv) while discarding all the terms except those corresponding to the nontrivial zeros β_j, we obtain the inequality

$$\sum_j \frac{1}{\sigma - \beta_j} < \frac{r_f}{\sigma - 1} + \text{Re}\left(\frac{L'}{L}(\sigma,f)\right) + O(\log \mathfrak{q}(f)).$$

To estimate $\text{Re}\left(\frac{L'}{L}(\sigma,f)\right)$, we first note that as $\text{Re}(\Lambda_f(n)) \geq 0$ provided $(n,q(f)) = 1$ by assumption, the Dirichlet series of $\frac{L'^{(q(f))}}{L^{(q(f))}}(s,f)$ shows that

$$\text{Re}\left(\frac{L^{(q(f))'}}{L^{(q(f))}}(\sigma,f)\right) \leq 0.$$

This gives an estimate for the contribution of the local factors of $L(s,f)$ corresponding to unramified primes. For the contribution of the local factors corresponding to ramified primes, we use Equation (3.9) to compute

$$\mathrm{Re}\left(\frac{L'_{q(f)}}{L_{q(f)}}(\sigma,f)\right) \leq \left|\frac{L'_{q(f)}}{L_{q(f)}}(\sigma,f)\right| = \left|\sum_{p|q(f)}\sum_{1\leq j\leq d_f}\frac{\alpha_j(p)\log(p)}{(1-\alpha_j(p)p^{-\sigma})p^{\sigma}}\right| \leq d_f\sum_{p|q(f)}\log(p) \leq d_f\log(q(f)),$$

where in the second inequality we have made use of the assumption $|\alpha_j(p)| \leq \frac{p}{2}$ for ramified primes p to conclude that $\left|\frac{\alpha_j(p)}{(1-\alpha_j(p)p^{-\sigma})p^{\sigma}}\right| \leq 1$. These estimates together imply

$$\sum_j \frac{1}{\sigma-\beta_j} < \frac{r_f}{\sigma-1} + O(\log d_f\mathfrak{q}(f)).$$

From this inequality we see that $\beta_j \neq 1$. For if some $\beta_j = 1$, $r_f < 0$ and the right-hand side is negative for σ sufficiently close to 1 contradicting the positivity of the left-hand side. Thus $L(1,f) \neq 0$ and hence $r_f \geq 0$. As there are finitely many β_j, there exists a $c > 0$ such that the β_j satisfy

$$\beta_j \geq 1 - \frac{c}{d_f(r_f+1)\log\mathfrak{q}(f)}.$$

Setting $\sigma = 1 + \frac{2c}{d_f\log\mathfrak{q}(f)}$ and choosing c smaller, if necessary, we guarantee $1 < \sigma \leq 2$. Then the two inequalities above together imply

$$\frac{nd_f\log\mathfrak{q}(f)}{2c+\frac{c}{(r_f+1)}} < \left(\frac{r_f}{2c}+O(1)\right)d_f\log\mathfrak{q}(f).$$

Isolating n, we see that

$$n < r_f + \frac{r_f}{2(r_f+1)} + O(c),$$

and taking c smaller, if necessary, we have $n \leq r_f$. As $L(s,f)$ is nonzero for $\sigma > 1$, it follows that there are at most r_f real zeros satisfying

$$\sigma \geq 1 - \frac{c}{d_f(r_f+1)\log\mathfrak{q}(f)}.$$

This completes the proof. □

We can now prove our zero-free region result:

Theorem 3.8.1. *Let $L(s,f)$ be an L-function with at most a simple pole at $s=1$, $\mathrm{Re}(\Lambda_f(n)) \geq 0$ provided $(n,q(f))=1$, and $|\alpha_j(p)| \leq \frac{p}{2}$ for ramified primes p. Then there exists a constant $c>0$ such that $L(s,f)$ has no zeros in the region*

$$\sigma \geq 1 - \frac{c}{d_f^2\log(\mathfrak{q}(f)(|t|+3))},$$

except for possibly one simple real zero β_f with $\beta_f < 1$ in the case $L(s,f)$ has a simple pole at $s=1$.

Proof. For real t, let $L(s,g)$ be the L-function defined by

$$L(s,g) = L^3(s,f)L^3(s,\overline{f})L^4(s+it,f)L^4(s+it,\overline{f})L(s+2it,f)L(s+2it,\overline{f}).$$

Then $d_g = 16d_f$ and $\mathfrak{q}(g)$ satisfies

$$\mathfrak{q}(g) \leq \mathfrak{q}(f)^6\mathfrak{q}(it,f)^8\mathfrak{q}(2it,f)^2 \leq \mathfrak{q}(f)^{16d_f}(|t|+3)^{10d_f} < (\mathfrak{q}(f)(|t|+3))^{16d_f}.$$

We claim that $\mathrm{Re}(\Lambda_g(n)) \geq 0$ for $(n, q(f)) = 1$. To see this, let p be an unramified prime. The local roots of $L(s,g)$ at p are $\alpha_j(p)$ and $\overline{\alpha_j(p)}$ both with multiplicity three, $\alpha_j(p)p^{-it}$ and $\overline{\alpha_j(p)}p^{-it}$ both with multiplicity four, and $\alpha_j(p)p^{-2it}$ and $\overline{\alpha_j(p)}p^{-2it}$ both with multiplicity one. So for any $k \geq 1$, the sum of k-th powers of these local roots is

$$\sum_{1\leq j\leq d_f} \left(6\mathrm{Re}(\alpha_j(p)^k) + 8\mathrm{Re}(\alpha_j(p)^k)p^{-kit} + 2\mathrm{Re}(\alpha_j(p)^k)p^{-2kit}\right).$$

The real part of this expression is

$$(6 + 8\cos\log(p^{kt}) + 2\cos\log(p^{2kt}))\mathrm{Re}(\Lambda_f(p^k)) = 4(1+\cos\log(p^{kt}))^2\mathrm{Re}(\Lambda_f(p^k)) \geq 0.$$

where we have made use of the identity $3+4\cos(x)+\cos(2x) = 2(1+\cos(x))^2$. It follows that $\mathrm{Re}(\Lambda_g(n)) \geq 0$ for $(n, q(f)) = 1$. Therefore the conditions of Lemma 3.8.1 are satisfied for $L(s,g)$. Now let $\rho = \beta + i\gamma$ be a complex nontrivial zero of $L(s,f)$. Setting $t = \gamma$, $L(s,g)$ has a real nontrivial zero at $s = \beta$ of order at least 8 and a pole at $s = 1$ of order 6. That is, $r_g = 6$. But Lemma 3.8.1 implies that $L(s,g)$ can have at most 6 real nontrivial zeros in the given region. Letting the constant for the region in Lemma 3.8.1 be c', it follows that β must satisfy

$$\beta < 1 - \frac{c'}{d_g(r_g+1)\log\mathfrak{q}(g)} < 1 - \frac{c'}{1792d_f^2\log(\mathfrak{q}(f)(|\gamma|+3))},$$

Take $c = \frac{c'}{1792}$. Now let β be a real nontrivial zero of $L(s,f)$. Since $\mathrm{Re}(\Lambda_f(n)) \geq 0$ and $L(s,f)$ has at most a simple pole at $s = 1$, Lemma 3.8.1 implies, upon shrinking c if necessary, that there is at most one simple real zero β_f in the desired region and it can only occur if $L(s,f)$ has a simple pole at $s = 1$. Note that if β_f exists then $\beta_f < 1$ because $L(s,f)$ is nonzero for $\sigma > 1$ and has a simple pole at $s = 1$. This completes the proof. □

Some comments are in order. First observe that the larger c can be taken in Theorem 3.8.1, the closer the curve bounding the zero-free region approaches the line $\sigma = \frac{1}{2}$. Since we only know $c > 0$ (and we do not have a nonzero lower bound), c could be quite small and so we are only guaranteed a zero-free region very close to the line $\sigma = 1$. This means that the possible simple real zero β_f could be very close to 1. Moreover, as the zeros of $L(s,f)$ occur in pairs ρ and $1-\overline{\rho}$, the zero-free region in Theorem 3.8.1 implies a symmetric zero-free region about the critical line with at most one real zero in each half as displayed in Figure 3.4.

In full generality, Theorem 3.8.1 is the best result that one can hope for. It is possible to obtain better zero-free regions in certain cases but this heavily depends upon the particular L-function of interest and hence the arithmetic object f attached to $L(s,f)$. In many cases it is possible to augment the proof of Theorem 3.8.1 to make the constant c effective and such results have important applications. The possible simple zero β_f in Theorem 3.8.1 satisfying

$$1 - \frac{c}{d_f^2\log(3\mathfrak{q}(f))} \leq \beta_f < 1$$

is referred to as a **Siegel zero** (or **exceptional zero**). Such zeros are conjectured to not exist and it is a very interesting question to see how the presence of a Siegel zero can affect the behavior of the associated

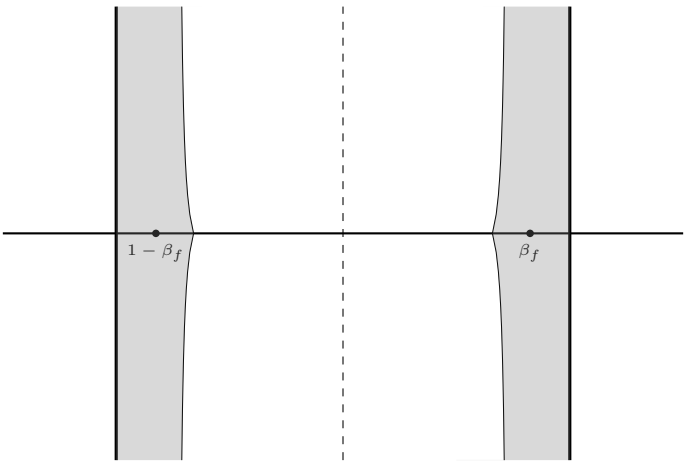


Figure 3.4: The symmetric zero-free region in Theorem 3.8.1.

L-function. If $L(s,f)$ is of Selberg class, the conclusion of Theorem 3.8.1 is expected to hold without the possibly of a Siegel zero since $L(s,f)$ is expected to satisfy the Riemann hypothesis. Nevertheless, we can often satisfy the assumptions of Theorem 3.8.1. Indeed, if $\alpha_i(p) \geq 0$ then the assumptions are satisfied immediately. If not, since $|\alpha_i(p)| \leq 1$ the assumptions will hold for $\zeta(s)L(s,f)$ provided $L(s,f)$ does not have a pole at $s=1$. This is often possible to prove in practice since $L(s,f)$ is expected to be entire unless $\alpha_i(p) \geq 0$.

3.9 The Explicit Formula

A formula somewhat analogous to the approximate functional equation can be derived for the negative logarithmic derivative of any L-function. This formula is called the **explicit formula**:

Theorem (Explict Formula). *Let $\psi(y)$ be a bump function with compact support and let $\Psi(s)$ denote its Mellin transform. Set $\phi(y) = y^{-1}\psi(y^{-1})$ so that its Mellin transform satisfies $\Phi(s) = \Psi(1-s)$. Then for any L-function $L(s,f)$, we have*

$$\sum_{n\geq 1}\big(\Lambda_f(n)\psi(n) + \Lambda_{\overline{f}}(n)\phi(n)\big) = \psi(1)\log q(f) + r_f\Psi(1) + \frac{1}{2\pi i}\int_{(\frac{1}{2})}\left(\frac{\gamma'}{\gamma}(s,f) + \frac{\gamma'}{\gamma}(1-s,\overline{f})\right)\Psi(s)\,ds - \sum_{\rho}\Psi(\rho),$$

where the sum is counted with multiplicity and ordered with respect to the size of the ordinate.

Proof. By smoothed Perron's formula, we have

$$\sum_{n\geq 1}\Lambda_f(n)\psi(n) = \frac{1}{2\pi i}\int_{(c)} -\frac{L'}{L}(s,f)\Phi(s)\,ds,$$

for any $c > 1$. Since the trivial zeros are isolated, let $\delta > 0$ be such that there are no trivial zeros in the vertical strip $-2\delta \leq \sigma < 0$. By Propositions 2.3.1 and 3.2.1, the integrand has polynomial decay of arbitrarily large order and therefore is locally absolutely uniformly convergent. Shifting the line of integration to $(-\delta)$, we pass by simple poles at $s=1$ and $s=\rho$ for every nontrivial zero ρ obtaining

$$\sum_{n\geq 1}\Lambda_f(n)\psi(n) = r_f\Psi(1) - \sum_{\rho}\Psi(\rho) + \frac{1}{2\pi i}\int_{(-\delta)} -\frac{L'}{L}(s,f)\Psi(s)\,ds.$$

For the latter integral, the functional equation and that $q(\overline{f}) = q(f)$ together imply

$$-\frac{L'}{L}(s,f) = \log q(f) + \frac{\gamma'}{\gamma}(s,f) + \frac{\gamma'}{\gamma}(1-s,\overline{f}) + \frac{L'}{L}(1-s,\overline{f}).$$

Thus

$$\frac{1}{2\pi i}\int_{(-\delta)} -\frac{L'}{L}(s,f)\Phi(s)\,ds = \frac{1}{2\pi i}\int_{(-\delta)} \left(\log q(f) + \frac{\gamma'}{\gamma}(s,f) + \frac{\gamma'}{\gamma}(1-s,\overline{f}) + \frac{L'}{L}(1-s,\overline{f})\right)\Psi(s)\,ds.$$

The integral over the first term on the right-hand side is $\psi(1)\log q(f)$ by the Mellin inversion formula. As for the last term, smoothed Perron's formula and that $\Phi(s) = \Psi(1-s)$ together give

$$\frac{1}{2\pi i}\int_{(-\delta)} -\frac{L'}{L}(1-s,\overline{f})\Psi(s)\,ds = \sum_{n\geq 1}\Lambda_{\overline{f}}(n)\phi(n).$$

So altogether, we have

$$\begin{aligned}\sum_{n\geq 1}\big(\Lambda_f(n)\psi(n) + \Lambda_{\overline{f}}(n)\phi(n)\big) &= \psi(1)\log q(f) + r_f\Psi(1) \\ &\quad + \frac{1}{2\pi i}\int_{(-\delta)}\left(\frac{\gamma'}{\gamma}(s,f) + \frac{\gamma'}{\gamma}(1-s,\overline{f})\right)\Psi(s)\,ds - \sum_{\rho}\Psi(\rho).\end{aligned}$$

Lastly, we note that by Propositions 1.7.3 and 2.3.1, the remaining integrand has rapid decay and therefore is locally absolutely uniformly convergent. Shifting the line of integration to $\left(\frac{1}{2}\right)$, we pass over no poles because the residues of $\frac{\gamma'}{\gamma}(1-s,\overline{f})$ are negative of those of $\frac{\gamma'}{\gamma}(s,f)$ for $\sigma \geq -\delta$ (the nontrivial zeros of $L(s,f)$ occur in pairs ρ and $1-\overline{\rho}$). This completes the proof. □

The explicit formula is a very useful tool for analytic investigations. Since $\Lambda_f(n)$ is essentially a weighted sum over prime powers, the explicit formula can be thought of as expressing a smoothed weighted sum over primes for f in terms of the zeros of $L(s,f)$.

Chapter 4

L-functions of Arithmetic Functions

We discuss the L-functions attached to arithmetic functions. Namely, we develop the theory of the Riemann zeta function and Dirichlet L-functions. Then we prove some classical applications of Dirichlet L-functions. Our first result is a gem of analytic number theory: Dirichlet's theorem on arithmetic progressions. This is a consequence of a non-vanishing result for Dirichlet L-series at $s = 1$. Next we discuss Siegel's theorem and the existence of Siegel zeros in the case of Dirichlet L-functions. After this we prove

4.1 The Riemann Zeta Function

The Definition and Euler Product

The **Riemann zeta function** $\zeta(s)$ is defined by the following Dirichlet series:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

This is the prototypical example of a Dirichlet series as all the coefficients are 1. We will see that $\zeta(s)$ is a Selberg class L-function. As the coefficients are uniformly bounded and completely multiplicative, $\zeta(s)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits the following degree 1 Euler product by Proposition 2.2.1:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

The local factor at p is

$$\zeta_p(s) = (1 - p^{-s})^{-1},$$

with local root 1.

The Integral Representation: Part I

We want to find an integral representation for $\zeta(s)$. To do this, consider the function

$$\omega(z) = \sum_{n \geq 1} e^{\pi i n^2 z},$$

defined for $z \in \mathbb{H}$. It is locally absolutely uniformly convergent in this region by Weierstrass M-test. Moreover, we have

$$\omega(z) = O\left(\sum_{n \geq 1} e^{-\pi n^2 y}\right) = O(e^{-\pi y}),$$

where the second equality holds because each term is of smaller order than the next so that the series is bounded by a constant times the first term. It follows that $\omega(z)$ exhibits exponential decay. Now consider the following Mellin transform:

$$\int_0^\infty \omega(iy) y^{\frac{s}{2}} \frac{dy}{y}.$$

By the exponential decay of $\omega(z)$, this integral is locally absolutely uniformly convergent for $\sigma > 1$ and hence defines an analytic function there. Then we compute

$$\begin{aligned}
\int_0^\infty \omega(iy) y^{\frac{s}{2}} \frac{dy}{y} &= \int_0^\infty \sum_{n\geq 1} e^{-\pi n^2 y} y^{\frac{s}{2}} \frac{dy}{y} \\
&= \sum_{n\geq 1} \int_0^\infty e^{-\pi n^2 y} y^{\frac{s}{2}} \frac{dy}{y} && \text{FTT} \\
&= \sum_{n\geq 1} \frac{1}{\pi^{\frac{s}{2}} n^s} \int_0^\infty e^{-y} y^{\frac{s}{2}} \frac{dy}{y} && y \mapsto \frac{y}{\pi n^2} \\
&= \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} \sum_{n\geq 1} \frac{1}{n^s} \\
&= \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} \zeta(s).
\end{aligned}$$

Therefore we have an integral representation

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty \omega(iy) y^{\frac{s}{2}} \frac{dy}{y}. \tag{4.1}$$

Unfortunately, we cannot proceed until we obtain a functional equation for $\omega(z)$. So we will make a detour and come back to the integral representation after.

The Jacobi Theta Function

The **Jacobi theta function** $\vartheta(z)$ is defined by

$$\vartheta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z},$$

for $z \in \mathbb{H}$. It is locally absolutely uniformly convergent in this region by the Weierstrass M-test. Moreover,

$$\vartheta(z) - 1 = O\left(\sum_{n\in\mathbb{Z}-\{0\}} e^{-2\pi n^2 y}\right) = O(e^{-2\pi y}),$$

where the second equality holds because each term is of smaller order than the $n = \pm 1$ terms so that the series is bounded by a constant times the order of these terms. In particular, $\vartheta(z) - 1$ exhibits exponential decay. The relationship to $\omega(z)$ is given by

$$\omega(z) = \frac{\vartheta\left(\frac{z}{2}\right) - 1}{2}.$$

The essential fact we will need is a functional equation for the Jacobi theta function:

Theorem 4.1.1. *For $z \in \mathbb{H}$,*

$$\vartheta(z) = \frac{1}{\sqrt{-2iz}}\vartheta\left(-\frac{1}{4z}\right).$$

Proof. We will apply the Poisson summation formula to

$$\vartheta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}.$$

To do this, we compute the Fourier transform of the summand and by the identity theorem it suffices to verify this for $z = iy$ with $y > 0$. So set

$$f(x) = e^{-2\pi x^2 y}.$$

Then $f(x)$ is of Schwarz class. By Proposition 1.6.3, we have

$$(\mathcal{F}f)(t) = \frac{e^{-\frac{\pi t^2}{2y}}}{\sqrt{2y}}.$$

By the Poisson summation formula and the identity theorem, we have

$$\vartheta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z} = \sum_{t\in\mathbb{Z}} \frac{e^{-\frac{\pi t^2}{-2iz}}}{\sqrt{-2iz}} = \frac{1}{\sqrt{-2iz}} \sum_{t\in\mathbb{Z}} e^{-\frac{\pi t^2}{-2iz}} = \frac{1}{\sqrt{-2iz}} \sum_{t\in\mathbb{Z}} e^{2\pi i t^2\left(-\frac{1}{4z}\right)} = \frac{1}{\sqrt{-2iz}}\vartheta\left(-\frac{1}{4z}\right). \qquad \square$$

We will use Theorem 4.1.1 to analytically continue $\zeta(s)$.

The Integral Representation: Part II

Returning to the Riemann zeta function, we split the integral in Equation (4.1) into two pieces by writing

$$\int_0^\infty \omega(iy) y^{\frac{s}{2}} \frac{dy}{y} = \int_0^1 \omega(iy) y^{\frac{s}{2}} \frac{dy}{y} + \int_1^\infty \omega(iy) y^{\frac{s}{2}} \frac{dy}{y}. \tag{4.2}$$

The idea now is to rewrite the first piece in the same form and symmetrize the result as much as possible. We being by performing a change of variables $y \mapsto \frac{1}{y}$ to the first piece to obtain

$$\int_1^\infty \omega\left(\frac{i}{y}\right) y^{-\frac{s}{2}} \frac{dy}{y}$$

Now we compute

$$\begin{aligned}
\omega\left(\frac{i}{y}\right) &= \omega\left(-\frac{1}{iy}\right) \\
&= \frac{\vartheta\left(-\frac{1}{2iy}\right) - 1}{2} \\
&= \frac{\sqrt{y}\vartheta\left(\frac{iy}{2}\right) - 1}{2} && \text{Theorem 4.1.1} \\
&= \sqrt{y}\omega(iy) + \frac{\sqrt{y}}{2} - \frac{1}{2}.
\end{aligned}$$

This chain implies that our first piece can be expressed as

$$\int_1^\infty \left(\sqrt{y}\omega(iy) + \frac{\sqrt{y}}{2} - \frac{1}{2}\right) y^{-\frac{s}{2}} \frac{dy}{y},$$

which is further equivalent to

$$\int_1^\infty \omega(iy) y^{\frac{1-s}{2}} \frac{dy}{y} - \frac{1}{s(1-s)},$$

because the integral over the last two pieces is $\frac{1}{1-s} - \frac{1}{s} = -\frac{1}{s(1-s)}$. Substituting this result back into Equation (4.2) and combining with Equation (4.1) yields the integral representation

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left[-\frac{1}{s(1-s)} + \int_1^\infty \omega(iy) y^{\frac{1-s}{2}} \frac{dy}{y} + \int_1^\infty \omega(iy) y^{\frac{s}{2}} \frac{dy}{y} \right]. \tag{4.3}$$

This integral representation will give analytic continuation. To see this, first observe that everything outside the brackets is entire. Moreover, the integrands exhibit exponential decay and therefore the integrals are locally absolutely uniformly convergent on $\mathbb{C}$. The fractional term is holomorphic except for simple poles at $s = 0$ and $s = 1$. The meromorphic continuation to $\mathbb{C}$ follows with possible simple poles at $s = 0$ and $s = 1$. There is no pole at $s = 0$. Indeed, $\Gamma\left(\frac{s}{2}\right)$ has a simple pole at $s = 0$ and so its reciprocal has a simple zero. This cancels the corresponding simple pole of $-\frac{1}{s(1-s)}$ so that $\zeta(s)$ has a removable singularity and thus is holomorphic at $s = 0$. At $s = 1$, $\Gamma\left(\frac{s}{2}\right)$ is nonzero and so $\zeta(s)$ has a simple pole. Therefore $\zeta(s)$ has meromorphic continuation to all of $\mathbb{C}$ with a simple pole at $s = 1$.

The Functional Equation

An immediate consequence of applying the symmetry $s \mapsto 1-s$ to Equation (4.3) is the following functional equation:

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} \zeta(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\pi^{\frac{1-s}{2}}} \zeta(1-s).$$

We identify the gamma factor as

$$\gamma(s, \zeta) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right),$$

with $\kappa = 0$ the only local root at infinity. Clearly it satisfies the required bounds. The conductor is $q(\zeta) = 1$ so no primes ramify. The completed Riemann zeta function is

$$\Lambda(s, \zeta) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

with functional equation

$$\Lambda(s, \zeta) = \Lambda(1-s, \zeta).$$

This is the functional equation of $\zeta(s)$ and in this case is just a reformulation of the previous functional equation. From it we find that the root number is $\varepsilon(\zeta) = 1$ and that $\zeta(s)$ is self-dual. We can now show that the order of $\zeta(s)$ is 1. Since there is only a simple pole at $s = 1$, multiply by $(s-1)$ to clear the polar divisor. As the integrals in Equation (4.3) are locally absolutely uniformly convergent, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s, \zeta)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

Thus the reciprocal of the gamma factor is also of order 1. It follows that

$$(s-1)\zeta(s) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

This shows $(s-1)\zeta(s)$ is of order 1 and thus $\zeta(s)$ is as well after removing the polar divisor. We now compute the residue of $\zeta(s)$ at $s=1$. First observe that the only term in Equation (4.3) contributing to the pole is $-\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\frac{1}{s(1-s)}$. Then

$$\operatorname*{Res}_{s=1}\zeta(s) = \operatorname*{Res}_{s=1}\left(-\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\frac{1}{s(1-s)}\right) = \lim_{s\to 1}\left(\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\frac{1}{s}\right) = 1,$$

where the second equality follows because $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. We summarize all of our work into the following theorem:

Theorem 4.1.2. *$\zeta(s)$ is a Selberg class L-function with degree 1 Euler product given by*

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}.$$

Moreover, it admits meromorphic continuation to $\mathbb{C}$, possesses the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Lambda(s,\zeta) = \Lambda(1-s,\zeta),$$

and has a simple pole at $s=1$ of residue 1.

Lastly, we note that by virtue of the functional equation we can also compute $\zeta(0)$. Indeed, since $\operatorname{Res}_{s=1}\zeta(s) = 1$, we have

$$\lim_{s\to 1}(s-1)\Lambda(s,\zeta) = \left(\operatorname*{Res}_{s=1}\zeta(s)\right)\left(\lim_{s\to 1}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right) = 1.$$

In other words, $\Lambda(s,\zeta)$ has a simple pole at $s=1$ of residue 1 too. It follows that $\Lambda(s,\zeta)$ also has a simple pole at $s=0$ of residue 1. Hence

$$1 = \lim_{s\to 1}(s-1)\Lambda(1-s,\zeta) = \left(\operatorname*{Res}_{s=1}\Gamma\left(\frac{1-s}{2}\right)\right)\left(\lim_{s\to 1}\pi^{-\frac{1-s}{2}}\zeta(1-s)\right) = -2\zeta(0),$$

because $\operatorname{Res}_{s=0}\Gamma(s) = 1$. Therefore $\zeta(0) = -\frac{1}{2}$.

4.2 Dirichlet L-functions

The Definition and Euler Product

To every Dirichlet character χ there is an associated L-function. Throughout we will let m denote the modulus and q the conductor of χ respectively. The **Dirichlet L-series** (respectively **Dirichlet L-function** if it is an L-function) $L(s,\chi)$ attached to the Dirichlet character χ is defined by the following Dirichlet series:

$$L(s,\chi) = \sum_{n\geq 1}\frac{\chi(n)}{n^s}.$$

Since $\chi(n) = 0$ if $(n, m) > 1$, the above sum can be restricted to all positive integers relatively prime to m. We will see that $L(s, \chi)$ is a Selberg class L-function if χ is primitive and of conductor $q > 1$ (in the case $q = 1$, $L(s, \chi) = \zeta(s)$). From now we make this assumption about χ. As the coefficients are uniformly bounded and completely multiplicative, $L(s, \chi)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits the following degree 1 Euler product by Proposition 2.2.1:

$$L(s, \chi) = \prod_{p} (1 - \chi(p)p^{-s})^{-1} = \prod_{p \nmid q} (1 - \chi(p)p^{-s})^{-1},$$

where the second equality holds because if $p \mid q$ we have $\chi(p) = 0$. The local factor at p is

$$L_p(s, \chi) = 1 \quad \text{or} \quad L_p(s, \chi) = (1 - \chi(p)p^{-s})^{-1},$$

with local root 0 or $\chi(p)$ respectively and according to if $p \mid q$ or not.

The Integral Representation: Part I

The integral representation for $L(s, \chi)$ is deduced in a similar way as for $\zeta(s)$. However, it will depend on if χ is even or odd. To handle both cases simultaneously let $\delta_\chi = 0, 1$ according to whether χ is even or odd. In other words,

$$\delta_\chi = \frac{\chi(1) - \chi(-1)}{2}.$$

We also have $\chi(-1) = (-1)^{\delta_\chi}$. Note that δ_χ takes the same value for both χ and $\overline{\chi}$. To find an integral representation for $L(s, \chi)$, consider the function

$$\omega_\chi(z) = \sum_{n \geq 1} \chi(n) n^{\delta_\chi} e^{\pi i n^2 z},$$

defined for $z \in \mathbb{H}$. It is locally absolutely uniformly convergent in this region by the Weierstrass M-test. Moreover, we have

$$\omega_\chi(z) = O\left(\sum_{n \geq 1} n e^{-\pi n^2 y} \right) = O(e^{-\pi y}),$$

where the second equality holds because each term is of smaller order than the next so that the series is bounded by a constant times the first term. Hence $\omega_\chi(z)$ exhibits exponential decay. Now consider the following Mellin transform:

$$\int_0^\infty \omega_\chi(iy) y^{\frac{s+\delta_\chi}{2}} \frac{dy}{y}.$$

By the exponential decay of w_χ, this integral is locally absolutely uniformly convergent for $\sigma > 1$ and hence defines an analytic function there. Then we compute

$$\begin{aligned}
\int_0^\infty \omega_\chi(iy) y^{\frac{s+\delta_\chi}{2}} \frac{dy}{y} &= \int_0^\infty \sum_{n\geq 1} \chi(n) n^{\delta_\chi} e^{-\pi n^2 y} y^{\frac{s+\delta_\chi}{2}} \frac{dy}{y} \\
&= \sum_{n\geq 1} \int_0^\infty \chi(n) n^{\delta_\chi} e^{-\pi n^2 y} y^{\frac{s+\delta_\chi}{2}} \frac{dy}{y} && \text{FTT} \\
&= \sum_{n\geq 1} \frac{\chi(n)}{\pi^{\frac{s+\delta_\chi}{2}} n^s} \int_0^\infty e^{-y} y^{\frac{s+\delta_\chi}{2}} \frac{dy}{y} && y \mapsto \frac{y}{\pi n^2} \\
&= \frac{\Gamma\left(\frac{s+\delta_\chi}{2}\right)}{\pi^{\frac{s+\delta_\chi}{2}}} \sum_{n\geq 1} \frac{\chi(n)}{n^s} \\
&= \frac{\Gamma\left(\frac{s+\delta_\chi}{2}\right)}{\pi^{\frac{s+\delta_\chi}{2}}} L(s,\chi).
\end{aligned}$$

Therefore we have an integral representation

$$L(s,\chi) = \frac{\pi^{\frac{s+\delta_\chi}{2}}}{\Gamma\left(\frac{s+\delta_\chi}{2}\right)} \int_0^\infty \omega_\chi(iy) y^{\frac{s+\delta_\chi}{2}} \frac{dy}{y}, \tag{4.4}$$

and just as for the Riemann zeta function, we need to find a functional equation for $\omega_\chi(z)$ before we can proceed.

The Dirichlet Theta Function

The **Dirichlet theta function** $\vartheta_\chi(z)$ attached to the character χ, is defined by

$$\vartheta_\chi(z) = \sum_{n\in\mathbb{Z}} \chi(n) n^{\delta_\chi} e^{2\pi i n^2 z},$$

for $z \in \mathbb{H}$. It is locally absolutely uniformly convergent in this region by the Weierstrass M-test. Moreover,

$$\vartheta_\chi(z) = O\left(\sum_{n\in\mathbb{Z}} n e^{-2\pi n^2 y}\right) = O(e^{-2\pi y}),$$

where the second equality holds because each term is of smaller order than the $n = \pm 1$ terms so that the series is bounded by a constant times the order of these terms. In particular, $\vartheta_\chi(z)$ exhibits exponential decay. The relationship to $\omega_\chi(z)$ is given by

$$\omega_\chi(z) = \frac{\vartheta_\chi\left(\frac{z}{2}\right)}{2}.$$

This is a slightly less complex relationship than the analog for the Jacobi theta function because assuming $q > 1$ means $\chi(0) = 0$. The essential fact we will need is a functional equation for the Dirichlet theta function:

Theorem 4.2.1. *Let χ be a primitive Dirichlet character of conductor $q > 1$. For $z \in \mathbb{H}$,*

$$\vartheta_\chi(z) = \frac{\varepsilon_\chi}{i^{\delta_\chi}(-2qiz)^{\frac{1}{2}+\delta_\chi}}\vartheta_{\overline{\chi}}\left(-\frac{1}{4q^2z}\right).$$

Proof. Since χ is q-periodic and $\vartheta_\chi(z)$ is absolutely convergent, we can write

$$\vartheta_\chi(z) = \sum_{a \pmod q} \chi(a) \sum_{m\in\mathbb{Z}} (mq+a)^{\delta_\chi} e^{2\pi i(mq+a)^2 z}.$$

Set

$$I_a(z) = \sum_{m\in\mathbb{Z}} (mq+a)^{\delta_\chi} e^{2\pi i(mq+a)^2 z}.$$

We will apply the Poisson summation formula to $I_a(z)$. To do this, we compute the Fourier transform of the summand and by the identity theorem it suffices to verify this for $z = iy$ with $y > 0$. So set

$$f(x) = (xq+a)^{\delta_\chi} e^{-2\pi(xq+a)^2 y}.$$

Then $f(x)$ is of Schwarz class. By Proposition 1.6.2 (i)-(iv) and Proposition 1.6.3, we have

$$(\mathcal{F}f)(t) = \left(\frac{-it}{2qy}\right)^{\delta_\chi} \frac{e^{\frac{2\pi iat}{q} - \frac{\pi t^2}{2q^2 y}}}{\sqrt{2q^2 y}}.$$

By the Poisson summation formula and the identity theorem, we have

$$I_a(z) = \sum_{m\in\mathbb{Z}} (mq+a)^{\delta_\chi} e^{2\pi i(mq+a)^2 z} = \sum_{t\in\mathbb{Z}} \left(\frac{-it}{2q(-iz)}\right)^{\delta_\chi} \frac{e^{\frac{2\pi iat}{q} - \frac{\pi t^2}{2q^2(-iz)}}}{\sqrt{2q^2(-iz)}}.$$

Plugging this back into the definition of $\vartheta_\chi(z)$ yields

$$\begin{aligned}
\vartheta_\chi(z) &= \sum_{a \pmod q} \chi(a) \sum_{t\in\mathbb{Z}} \left(\frac{-it}{2q(-iz)}\right)^{\delta_\chi} \frac{e^{\frac{2\pi iat}{q} - \frac{\pi t^2}{2q^2(-iz)}}}{\sqrt{2q^2(-iz)}} \\
&= \sum_{a \pmod q} \chi(a) \sum_{t\in\mathbb{Z}} \left(\frac{t}{i(-2qiz)}\right)^{\delta_\chi} \frac{e^{\frac{2\pi iat}{q} - \frac{\pi it^2}{2q^2 z}}}{\sqrt{q(-2qiz)}} \\
&= \frac{1}{i^{\delta_\chi}\sqrt{q}(-2qiz)^{\frac{1}{2}+\delta_\chi}} \sum_{t\in\mathbb{Z}} t^{\delta_\chi} e^{-\frac{\pi it^2}{2q^2 z}} \sum_{a \pmod q} \chi(a) e^{\frac{2\pi iat}{q}} \\
&= \frac{1}{i^{\delta_\chi}\sqrt{q}(-2qiz)^{\frac{1}{2}+\delta_\chi}} \sum_{t\in\mathbb{Z}} t^{\delta_\chi} e^{-\frac{\pi it^2}{2q^2 z}} \tau(t,\chi) && \text{definition of } \tau(t,\chi) \\
&= \frac{\tau(\chi)}{i^{\delta_\chi}\sqrt{q}(-2qiz)^{\frac{1}{2}+\delta_\chi}} \sum_{t\in\mathbb{Z}} \overline{\chi}(t) t^{\delta_\chi} e^{-\frac{\pi it^2}{2q^2 z}} && \text{Corollary 1.4.1} \\
&= \frac{\varepsilon_\chi}{i^{\delta_\chi}(-2qiz)^{\frac{1}{2}+\delta_\chi}} \sum_{t\in\mathbb{Z}} \overline{\chi}(t) t^{\delta_\chi} e^{-\frac{\pi it^2}{2q^2 z}} && \varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{q}} \\
&= \frac{\varepsilon_\chi}{i^{\delta_\chi}(-2qiz)^{\frac{1}{2}+\delta_\chi}} \sum_{t\in\mathbb{Z}} \overline{\chi}(t) t^{\delta_\chi} e^{2\pi it^2\left(-\frac{1}{4q^2 z}\right)} \\
&= \frac{\varepsilon_\chi}{i^{\delta_\chi}(-2qiz)^{\frac{1}{2}+\delta_\chi}} \vartheta_{\overline{\chi}}\left(-\frac{1}{4q^2 z}\right).
\end{aligned}$$

□

Notice that the functional equation relates $\vartheta_\chi(z)$ to $\vartheta_{\overline{\chi}}(z)$. Regardless, we will use Theorem 4.2.1 to analytically continue $L(s,\chi)$.

The Integral Representation: Part II

Returning to $L(s,\chi)$, split the integral in Equation (4.4) into two pieces by writing

$$\int_0^\infty \omega_\chi(iy) y^{\frac{s+\delta_\chi}{2}} \frac{dy}{y} = \int_0^{\frac{1}{q}} \omega_\chi(iy) y^{\frac{s+\delta_\chi}{2}} \frac{dy}{y} + \int_{\frac{1}{q}}^\infty \omega_\chi(iy) y^{\frac{s+\delta_\chi}{2}} \frac{dy}{y}. \tag{4.5}$$

We now rewrite the first piece in the same form and symmetrize the result as much as possible. Start by performing a change of variables $y \mapsto \frac{1}{q^2 y}$ to the first piece to obtain

$$\int_{\frac{1}{q}}^\infty \omega_\chi\left(\frac{i}{q^2 y}\right) (q^2 y)^{-\frac{s+\delta_\chi}{2}} \frac{dy}{y}.$$

Now we compute

$$\begin{aligned}
\omega_\chi\left(\frac{i}{q^2 y}\right) &= \omega_\chi\left(-\frac{1}{q^2 iy}\right) \\
&= \frac{\vartheta_\chi\left(-\frac{1}{2q^2 iy}\right)}{2} \\
&= \frac{i^{\delta_\chi}(qy)^{\frac{1}{2}+\delta_\chi}}{\varepsilon_{\overline{\chi}}} \frac{\vartheta_{\overline{\chi}}\left(\frac{iy}{2}\right)}{2} && \text{Theorem 4.2.1} \\
&= \varepsilon_\chi(-i)^{\delta_\chi}(qy)^{\frac{1}{2}+\delta_\chi} \frac{\vartheta_{\overline{\chi}}\left(\frac{iy}{2}\right)}{2} && \text{Proposition 1.4.3 and that } \chi(-1) = (-1)^{\delta_\chi} \\
&= \frac{\varepsilon_\chi(qy)^{\frac{1}{2}+\delta_\chi}}{i^{\delta_\chi}} \frac{\vartheta_{\overline{\chi}}\left(\frac{iy}{2}\right)}{2} \\
&= \frac{\varepsilon_\chi(qy)^{\frac{1}{2}+\delta_\chi}}{i^{\delta_\chi}} \omega_{\overline{\chi}}(iy).
\end{aligned}$$

This chain implies that our first piece can be expressed as

$$\frac{\varepsilon_\chi}{i^{\delta_\chi}} q^{\frac{1}{2}-s} \int_{\frac{1}{q}}^\infty \omega_{\overline{\chi}}(iy) y^{\frac{(1-s)+\delta_\chi}{2}} \frac{dy}{y}.$$

Substituting this expression back into Equation (4.5) and combining with Equation (4.4) gives the integral representation

$$L(s,\chi) = \frac{\pi^{\frac{s+\delta_\chi}{2}}}{\Gamma\left(\frac{s+\delta_\chi}{2}\right)} \left[\frac{\varepsilon_\chi}{i^{\delta_\chi}} q^{\frac{1}{2}-s} \int_{\frac{1}{q}}^\infty \omega_{\overline{\chi}}(iy) y^{\frac{(1-s)+\delta_\chi}{2}} \frac{dy}{y} + \int_{\frac{1}{q}}^\infty \omega_\chi(iy) y^{\frac{s+\delta_\chi}{2}} \frac{dy}{y} \right]. \tag{4.6}$$

This integral representation will give analytic continuation. Indeed, we know everything outside the brackets is entire. The integrands exhibit exponential decay and therefore the integrals are locally absolutely uniformly convergent on $\mathbb{C}$. This gives analytic continuation to all of $\mathbb{C}$. In particular, $L(s,\chi)$ has no poles.

The Functional Equation

An immediate consequence of applying the symmetry $s \mapsto 1-s$ to Equation (4.6) is the following functional equation:

$$q^{\frac{s}{2}} \frac{\Gamma\left(\frac{s+\delta_\chi}{2}\right)}{\pi^{\frac{s+\delta_\chi}{2}}} L(s,\chi) = \frac{\varepsilon_\chi}{i^{\delta_\chi}} q^{\frac{1-s}{2}} \frac{\Gamma\left(\frac{(1-s)+\delta_\chi}{2}\right)}{\pi^{\frac{(1-s)+\delta_\chi}{2}}} L(1-s,\overline{\chi}).$$

We identify the gamma factor as

$$\gamma(s,\chi) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+\delta_\chi}{2}\right),$$

with $\kappa = \delta_\chi$ the only local root at infinity. Clearly it satisfies the required bounds. The conductor is $q(\chi) = q$ and if p is unramified then the local root is $\chi(p) \neq 0$. The completed L-function is

$$\Lambda(s,\chi) = q^{\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+\delta_\chi}{2}\right) L(s,\chi),$$

with functional equation

$$\Lambda(s,\chi) = \frac{\varepsilon_\chi}{i^{\delta_\chi}} \Lambda(1-s,\overline{\chi}).$$

From it we see that the root number is $\varepsilon(\chi) = \frac{\varepsilon_\chi}{i^{\delta_\chi}}$ and that $L(s,\chi)$ has dual $L(s,\overline{\chi})$. We now show that $L(s,\chi)$ is of order 1. Since $L(s,\chi)$ has no poles, we do not need to clear any polar divisors. As the integrals in Equation (4.6) are locally absolutely uniformly convergent, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s,\chi)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So the reciprocal of the gamma factor is also of order 1. It follows that

$$L(s,\chi) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So $L(s,\chi)$ is of order 1. We summarize all of our work into the following theorem:

Theorem 4.2.2. *For any primitive Dirichlet character χ of conductor $q > 1$, $L(s,\chi)$ is a Selberg class L-function with degree 1 Euler product given by*

$$L(s,\chi) = \prod_p (1-\chi(p)p^{-s})^{-1}.$$

Moreover, it admits analytic continuation to $\mathbb{C}$ and possesses the functional equation

$$q^{\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+\delta_\chi}{2}\right) L(s,\chi) = \Lambda(s,\chi) = \frac{\varepsilon_\chi}{i^{\delta_\chi}} \Lambda(1-s,\overline{\chi}).$$

Beyond Primitivity

We can still obtain meromorphic continuation of the L-series $L(s,\chi)$ if χ is imprimitive. Indeed, if χ is induced by $\widetilde{\chi}$ then $\chi(p) = \widetilde{\chi}(p)$ if $p \nmid q$ and $\chi(p) = 0$ if $p \mid m$. Then just as for primitive characters, Proposition 2.2.1 implies that $L(s,\chi)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits the following degree 1 Euler product:

$$L(s,\chi) = \prod_{p \mid m} (1-\widetilde{\chi}(p)p^{-s})^{-1} = \prod_p (1-\widetilde{\chi}(p)p^{-s})^{-1} \prod_{p \mid m} (1-\widetilde{\chi}(p)p^{-s}) = L(s,\widetilde{\chi}) \prod_{p \mid m} (1-\widetilde{\chi}(p)p^{-s}). \quad (4.7)$$

From this relation, we can prove the following:

Theorem 4.2.3. *For any Dirichlet character χ modulo m, let $\widetilde{\chi}$ be the primitive character inducing χ. Then $L(s,\chi)$ is locally absolutely uniformly convergent for $\sigma > 1$ with degree 1 Euler product given by*

$$L(s,\chi) = \prod_{p \nmid m} (1 - \widetilde{\chi}(p)p^{-s})^{-1}.$$

Moreover, it admits meromorphic continuation to $\mathbb{C}$ and if χ is principal there is a simple pole at $s = 1$ of residue $\prod_{p|m}(1 - \widetilde{\chi}(p)p^{-1})$.

Proof. This follows from Theorems 4.1.2 and 4.2.2 and Equation (4.7). □

It is worth noting that for any non-principal Dirichlet character χ modulo m, the L-series $L(s,\chi)$ converges for $\sigma > 0$. Indeed, setting $A(X) = \sum_{n \leq X} \chi(n)$ we have $A(X) \leq m$ by Dirichlet orthogonality relations (namely (i)) and that χ is m-periodic, and then the claim follows from Proposition 2.1.2. While this fact is not too important from an analytic continuation standpoint, it is useful because it allows for the manipulation (without rearrangement since we do not have absolute convergence) of the L-series $L(s,\chi)$ in the vertical strip $0 < \sigma \leq 1$.

4.3 Dirichlet's Theorem on Arithmetic Progressions

One of the more well-known arithmetic results proved using L-series is **Dirichlet's theorem on arithmetic progressions**:

Theorem (Dirichlet's theorem on arithmetic progressions). *Let a and m be integers such that $m \geq 1$ and $(a,m) = 1$. Then the series*

$$\sum_{p \equiv a \pmod{m}} \frac{1}{p},$$

diverges. In particular, the arithmetic progression $\{a + km \mid k \in \mathbb{Z}_{\geq 0}\}$ contain infinitely many primes.

The proof can be broken into three steps. The first is to estimate the logarithm of a Dirichlet L-series $L(s,\chi)$ and show that almost all of the terms are uniformly bounded as $s \to 1$. The next step is to use the Dirichlet orthogonality relations of the characters to sieve out the correct sum. The last step is to show the non-vanishing result $L(1,\chi) \neq 0$ for all non-principal characters χ. This is the essential part of the proof as it is what assures us that the sum diverges. Fortunately, we have done all of the difficult work to prove this already:

Theorem 4.3.1. *Let χ be a non-principal Dirichlet character. Then $L(1,\chi)$ is finite and nonzero.*

Proof. This follows immediately by applying Lemma 3.8.1 to $\zeta(s)L(s,\chi)$ and noting that $L(s,\chi)$ is holomorphic. □

We now prove Dirichlet's theorem on arithmetic progressions:

Proof of Dirichlet's theorem on arithmetic progressions. Let χ be a Dirichlet character modulo m. Then for $\sigma > 1$, taking the logarithm of the Euler product of $L(s,\chi)$ gives

$$\log L(s,\chi) = -\sum_{p} \log(1 - \chi(p)p^{-s}).$$

The Taylor series of the logarithm implies

$$\log(1-\chi(p)p^{-s}) = \sum_{k\geq 1}(-1)^{k-1}\frac{(-\chi(p)p^{-s})^k}{k} = \sum_{k\geq 1}(-1)^{2k-1}\frac{\chi(p^k)}{kp^{ks}},$$

so that

$$\log L(s,\chi) = \sum_p \sum_{k\geq 1} \frac{\chi(p^k)}{kp^{ks}}. \tag{4.8}$$

The double sum restricted to $k \geq 2$ is uniformly bounded for $\sigma > 1$. Indeed, first observe

$$\left|\sum_{k\geq 2}\frac{\chi(p^k)}{kp^{ks}}\right| \ll \sum_{k\geq 2}\frac{1}{p^k} = \frac{1}{p^2}\sum_{k\geq 0}\frac{1}{p^k} = \frac{1}{p^2}(1-p^{-1})^{-1} \leq \frac{2}{p^2},$$

where the last inequality follows because $p > 2$. Then

$$\left|\sum_p\sum_{k\geq 2}\frac{\chi(p^k)}{kp^{ks}}\right| \leq 2\sum_p\frac{1}{p^2} < 2\sum_{n\geq 1}\frac{1}{n^2} = 2\zeta(2),$$

as desired. Now let a and m be integers such that $m \geq 1$ and $(a,m) = 1$. Using Equation (4.8), we may write

$$\sum_{\chi \pmod m}\overline{\chi}(a)\log L(s,\chi) = \sum_{\chi \pmod m}\sum_p \frac{\overline{\chi}(a)\chi(p)}{p^s} + \sum_{\chi \pmod m}\overline{\chi}(a)\sum_p\sum_{k\geq 2}\frac{\chi(p^k)}{kp^{ks}},$$

and by the Dirichlet orthogonality relations (namely (ii)), we have

$$\sum_{\chi \pmod m}\sum_p \frac{\overline{\chi}(a)\chi(p)}{p^s} = \sum_p \frac{1}{p^s}\sum_{\chi \pmod m}\overline{\chi}(a)\chi(p) = \varphi(m)\sum_{p\equiv a \pmod m}\frac{1}{p^s}.$$

Combining these two identities together gives

$$\sum_{\chi \pmod m}\overline{\chi}(a)\log L(s,\chi) - \sum_{\chi \pmod m}\overline{\chi}(a)\sum_p\sum_{k\geq 2}\frac{\chi(p^k)}{kp^{ks}} = \varphi(m)\sum_{p\equiv a \pmod m}\frac{1}{p^s}.$$

The triple sum in this identity is uniformly bounded for $\sigma > 1$ because the inner double sum is and there are finitely many Dirichlet characters modulo m. Therefore it suffices to show that the first sum on the left-hand side diverges as $s \to 1$. We have

$$L(s,\chi_{m,0}) = \zeta(s)\prod_{p\mid m}(1-p^{-s}),$$

for $\chi = \chi_{m,0}$, and so the corresponding term in the sum is

$$\overline{\chi_{m,0}}(a)\log L(s,\chi_{m,0}) = \log\zeta(s) + \sum_{p\mid m}\log(1-p^{-s}),$$

which diverges as $s \to 1$ because $\zeta(s)$ has a simple pole at $s = 1$. We will be done if $\log L(s,\chi)$ remains bounded as $s \to 1$ for all $\chi \neq \chi_{m,0}$. So assume χ is non-principal. Then we must show $L(1,\chi)$ is finite and nonzero. This follows from Theorem 4.3.1. □

4.4 Siegel's Theorem and Siegel Zeros

The discussion of Siegel zeros first arose during the study of zero-free regions for Dirichlet L-functions. Refining the argument used in Theorem 3.8.1, we can show that Siegel zeros only exist when the character χ is quadratic. But first we improve the zero-free region for the Riemann zeta function:

Theorem 4.4.1. *There exists a constant $c > 0$ such that $\zeta(s)$ has no zeros in the region*

$$\sigma \geq 1 - \frac{c}{\log(|t| + 3)}.$$

Proof. By Theorem 3.8.1 applied to $\zeta(s)$, it suffices to show that $\zeta(s)$ has no real nontrivial zeros. To see this, let $\eta(s)$ be defined by

$$\eta(s) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s}.$$

Note that $\eta(s)$ converges for $\sigma > 0$ by Proposition 2.1.2. Now for $0 < s < 1$ and even n, $\frac{1}{n^s} - \frac{1}{(n+1)^s} > 0$ so that $\eta(s) > 0$. But for $\sigma > 0$, we have

$$(1 - 2^{1-s})\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} - 2\sum_{n \geq 1} \frac{1}{(2n)^s} = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s} = \eta(s).$$

Therefore $\zeta(s)$ cannot admit a zero for $0 < s < 1$ because then $\eta(s)$ would be zero too. This completes the proof. □

Theorem 4.4.1 shows that $\zeta(s)$ has no Siegel zeros. Moreover, the proof shows that $\zeta(s)$ is negative for $0 < s < 1$. As for the height of the first zero, it occurs on the critical line (as predicted by the Riemann hypothesis) at height $t \approx 14.134$ (see [Dav80] for a further discussion). The first 15 zeros were computed by Gram in 1903 (see [Gra03]). Since then, billions of zeros have been computed and have all been verified to lie on the critical line. Our improved zero-free region for Dirichlet L-functions is only slightly different but causes increasing complexity in further study. The improvement is that if a Siegel zero exists for a Dirichlet L-function then the character is necessarily quadratic:

Theorem 4.4.2. *Let χ be a primitive Dirichlet character of conductor $q > 1$. Then there exists a constant $c > 0$ such that $L(s, \chi)$ has no zeros in the region*

$$\sigma \geq 1 - \frac{c}{\log(q(|t| + 3))},$$

except for possibly one simple real zero β_χ with $\beta_\chi < 1$ in the case χ is quadratic.

Proof. By Theorem 3.8.1 applied to $\zeta(s)L(s, \chi)$, and shrinking c if necessary, it remains to show that there not a simple real zero β_χ if χ is not quadratic. For this, let $L(s, f)$ be the L-series defined by

$$L(s, f) = L^3(s, \chi_{q,0})L^4(s, \chi)L(s, \chi^2).$$

Then $d_f = 8$ and $\mathfrak{q}(f)$ satisfies

$$\mathfrak{q}(f) \leq \mathfrak{q}(\chi_{q,0})^3\mathfrak{q}(\chi)^4\mathfrak{q}(\chi^2) \leq 3^8q^5 < (3q)^8.$$

Moreover, $\mathrm{Re}(\Lambda_f(n)) \geq 0$ for $(n,q) = 1$. To see this, suppose p is an unramified prime. The local roots of $L(s,f)$ at p are 1 with multiplicity three, $\chi(p)$ with multiplicity four, and $\chi^2(p)$ with multiplicity one. So for any $k \geq 1$, the sum of k-th powers of these local roots is

$$3 + 4\chi^k(p) + \chi^{2k}(p).$$

Writing $\chi(p) = e^{ix}$, the real part of this expression is

$$3 + 4\cos(x) + \cos(2x) = 2(1 + \cos(x))^2 \geq 0,$$

where we have also made use of the identity $3 + 4\cos(x) + \cos(2x) = 2(1 + \cos(x))^2$. Thus $\mathrm{Re}(\Lambda_f(n)) \geq 0$ for $(n,q) = 1$, and the conditions of Lemma 3.8.1 are satisfied for $L(s,f)$ (recall Equation (4.7) for the L-series $L(s,\chi_{q,0})$ and $L(s,\chi^2)$). On the one hand, if β be a real nontrivial zero of $L(s,\chi)$ then $L(s,f)$ has a real nontrivial zero at $s = \beta$ of order at least 4. On the other hand, using Equation (4.7) and that $\chi^2 \neq \chi_{q,0}$, $L(s,f)$ has a pole at $s = 1$ of order 3. Then, upon shrinking c if necessary, Lemma 3.8.1 gives a contradiction since $r_f = 3$. This completes the proof. □

Siegel zeros present an unfortunate obstruction to zero-free region results for Dirichlet L-functions when the primitive character χ is quadratic. However, if we no longer require the constant c in the zero-free region to be effective, we can obtain a much better result for how close the Siegel zero can be to 1. This result will follow from a lower bound for $L(1,\chi)$ known as **Siegel's theorem**:

Theorem (Siegel's theorem). *Let χ be a primitive quadratic Dirichlet character χ of conductor $q > 1$. Then there exists a positive constant $c_1(\varepsilon)$ such that*

$$L(1,\chi) \geq \frac{c_1(\varepsilon)}{q^\varepsilon}.$$

In particular, there exists a positive constant $c_2(\varepsilon)$ such that $L(s,\chi)$ has no real zeros in the interval

$$\sigma \geq 1 - \frac{c_2(\varepsilon)}{q^\varepsilon}.$$

The largest defect of Siegel's theorem is that the implicit constants $c_1(\varepsilon)$ and $c_2(\varepsilon)$ are ineffective (and not necessarily equal). To prove Siegel's theorem, we we will require two lemmas. The first is about the size of $L^{(k)}(\sigma,\chi)$ for σ close to 1:

Lemma 4.4.1. *Let χ be a non-principal Dirichlet character modulo $m > 1$. Then for any for any $k \geq 0$, we have $L^{(k)}(\sigma,\chi) = O(\log^{k+1}(m))$ provided σ is such that $0 \leq 1 - \sigma \leq \frac{1}{\log(m)}$.*

Proof. Recall that the Dirichlet L-series $L(s,\chi)$ converges for $\sigma > 0$ and thus for $0 \leq 1 - \sigma \leq \frac{1}{\log(m)}$. As $L(s,\chi)$ admits analytic continuation to $\mathbb{C}$, it is holomorphic for $0 \leq 1 - \sigma \leq \frac{1}{\log(m)}$ and so its k-th derivative is given by

$$L^{(k)}(\sigma,\chi) = \sum_{n\geq 1} \frac{\chi(n)\log^k(n)}{n^\sigma} = \sum_{n<m} \frac{\chi(n)\log^k(n)}{n^\sigma} + \sum_{n\geq m} \frac{\chi(n)\log^k(n)}{n^\sigma}.$$

We will show that the last two sums are both $O(\log^{k+1}(m))$. For the first sum, if $n < m$, we have

$$\left|\frac{\chi(n)\log^k(n)}{n^\sigma}\right| \leq \frac{1}{n^\sigma}\log^k(n) = \frac{n^{1-\sigma}}{n}\log^k(n) < \frac{m^{1-\sigma}}{n}\log^k(n) < \frac{e}{n}\log^k(m),$$

where the last inequality follows because $1-\sigma \leq \frac{1}{\log(m)}$. Then

$$\left|\sum_{n\leq m} \frac{\chi(n)\log^k(n)}{n^\sigma}\right| < e\log^k(m)\sum_{n<m}\frac{1}{n} < e\log^k(m)\int_1^m \frac{1}{n}\,dn \ll \log^{k+1}(m).$$

For the second sum, let $A(Y)=\sum_{n\leq Y}\chi(n)$. Then $|A(Y)|\leq m$ by the Dirichlet orthogonality relations (namely Corollary 1.3.1 (i)) and that χ is m-periodic. Hence $A(Y)\log(Y)Y^{-\sigma}\to 0$ as $Y\to\infty$ and Abel's summation formula (see Corollary B.3.3) gives

$$\sum_{n\geq m}\frac{\chi(n)\log^k(n)}{n^\sigma} = -A(m)\log^k(m)m^{-\sigma} - \int_m^\infty A(u)(k-\sigma\log(u))\log^{k-1}(u)u^{-(\sigma+1)}\,du. \tag{4.9}$$

Since $0\leq 1-\sigma\leq\frac{1}{\log(m)}$, we have $k-\sigma\log(u)\ll\log(u)$. With this estimate, that $|A(u)|\leq m$, and Equation (4.9), we make the following computation:

$$\begin{aligned}
\left|\sum_{n\geq m}\frac{\chi(n)\log^k(n)}{n^\sigma}\right| &\leq |A(m)|\log^k(m)m^{-\sigma} + \int_m^\infty |A(u)(k-\sigma\log(u))|\log^{k-1}(u)u^{-(\sigma+1)}\,du\\
&\ll m^{1-\sigma}\log^k(m) + m\int_m^\infty \log^k(u)u^{-(\sigma+1)}\,du\\
&= m^{1-\sigma}\log^k(m) + \frac{m}{\sigma}\left(\log^k(m)m^{-\sigma} + \int_m^\infty \log^{k-1}(u)u^{-(\sigma+1)}\,du\right)\\
&\ll m^{1-\sigma}\log^k(m) + \frac{m^{1-\sigma}}{\sigma}\log^k(m)\\
&\ll \log^{k+1}(m),
\end{aligned}$$

where in the third line we have used integration by parts, in the fourth we have used that the remaining integral is bounded by the former, and in the last line we have used that $1-\sigma\leq\frac{1}{\log(m)}$ (which implies that $\frac{1}{\sigma}\ll\log(m)$). Therefore the second sum is $O(\log^{k+1}(m))$ and hence $L^{(k)}(\sigma,\chi)=O(\log^{k+1}(m))$. □

Our second lemma is concerned with the positivity of the coefficients of a Dirichlet series formed by combining two Dirichlet L-series attached to distinct quadratic characters with distinct moduli:

Lemma 4.4.2. *Let χ_1 and χ_2 be quadratic Dirichlet characters and let $L(s,f)$ be the L-series defined by*

$$L(s,f)=\zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2).$$

Then $\Lambda_f(n)\geq 0$. In particular, $a_f(n)\geq 0$ and $a_f(0)=1$.

Proof. For any prime p, the local roots at p are 1 with multiplicity one, $\chi_1(p)$ with multiplicity one, $\chi_2(p)$ with multiplicity one, and $\chi_1\chi_2(p)$ with multiplicity one. So for any $k\geq 1$, the sum of k-th powers of these local roots is

$$(1+\chi_1^k(p))(1+\chi_2^k(p))\geq 0.$$

Thus $\Lambda_f(n)\geq 0$. It follows immediately from the Euler product of $L(s,f)$ that $a_f(n)\geq 0$ too. Also, it is clear from the Euler product of $L(s,f)$ that $a_f(0)=1$. □

We are now ready to prove Siegel's theorem and we follow a particularly simple proof due to Goldfeld (see [Gol74]):

Proof of Siegel's theorem. Observe that the first statement holds holds for a single q and hence for bounded q by taking the minimum of all the constants. Therefore we may assume q arbitrarily large throughout. We may also assume $\varepsilon \leq \frac{1}{4}$ because $\frac{1}{q^\varepsilon}$ is a decreasing function of ε. We now prove that the first statement implies the second by contradiction. For if there was a real zero β in the desired interval then for large enough q we have $0 \leq 1-\beta \leq \frac{1}{\log(q)}$ so that $L'(\sigma,\chi) = O(\log^2(q))$ for $\beta \leq \sigma \leq 1$ by Lemma 4.4.1. These two estimates and the mean value theorem together give

$$L(1,\chi) = L(1,\chi) - L(\beta,\chi) = L'(\sigma,\chi)(1-\beta) \ll \frac{\log^2(q)}{q^\varepsilon}.$$

Upon taking $\frac{\varepsilon}{2}$ in the lower bound, we obtain

$$\frac{1}{q^{\frac{\varepsilon}{2}}} \ll L(1,\chi) \ll \frac{\log^2(q)}{q^\varepsilon},$$

which is a contradiction. Therefore the second statement holds provided the first does. We will now prove the first statement which will complete the proof. Let χ_1 and χ_2 be distinct primitive quadratic characters of conductors $q_1 > 1$ and $q_2 > 1$ respectively. Consider the L-series $L(s,f)$ defined by

$$L(s,f) = \zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2).$$

Observe that $L(s,f)$ is holomorphic except for a simple pole at $s=1$. Let λ be the residue at this pole so that

$$\lambda = L(1,\chi_1)L(1,\chi_2)L(1,\chi_1\chi_2).$$

If there exists a Siegel zero β with $1-\varepsilon \leq \beta < 1$, let χ_1 be the character corresponding to the Dirichlet L-function that admits this Siegel zero. Then $L(\beta,f) = 0$ independent of the choice of χ_2. If there is no such Siegel zero, choose χ_1 to be any quadratic primitive character and β to be any number such that $1-\varepsilon \leq \beta < 1$. Then $L(\beta,f) < 0$ independent of the choice of χ_2. Indeed, $\zeta(s)$ is negative in this interval (actually $0 < s < 1$) while each of the Dirichlet L-series defining $L(s,f)$ is positive in this interval (the Dirichlet L-series converge for $\sigma > 0$, are positive for $\sigma > 1$ by their Euler product, are nonzero at $s=1$ by Theorem 4.3.1, and do not admit a real zero in this interval by our choice of β). So with our choice of χ_1 (depending on the existence of a Siegel zero or not) we see that $L(\beta,f) < 0$ for any choice of χ_2. We now take $\chi_2 = \chi$. Let $\psi(y)$ be a bump function that is identically 1 on $[0,1]$ and compactly supported. Let $\Psi(s)$ be the Mellin transform of $\psi(y)$. On the one hand, our choice of $\psi(y)$ and Lemma 4.4.2 together imply

$$\sum_{n\geq 1} \frac{a_f(n)}{n^\beta} \psi\left(\frac{n}{X}\right) \geq 1, \tag{4.10}$$

for any $X \geq 1$. On the other hand, smoothed Perron's formula gives

$$\sum_{n\geq 1} \frac{a_f(n)}{n^\beta} \psi\left(\frac{n}{X}\right) = \frac{1}{2\pi i}\int_{(c)} L(s+\beta,f)\Psi(s)X^s\,ds,$$

for any $c > 1-\beta$. Shifting the line of integration to $\left(\frac{1}{2}-\beta\right)$, we pass by a simple pole at $s = 1-\beta$ from $L(s+\beta,f)$ and obtain

$$\sum_{n\geq 1} \frac{a_f(n)}{n^\beta} \psi\left(\frac{n}{X}\right) = \frac{1}{2\pi i}\int_{\left(\frac{1}{2}-\beta\right)} L(s+\beta,f)\Psi(s)X^s\,ds + \lambda\Psi(1-\beta)X^{1-\beta}. \tag{4.11}$$

By the convexity bound and Proposition 2.3.1, the remaining integral satisfies the weak bound $O(qX^{\frac{1}{2}-\beta})$ and we have the estimate $\Psi(1-\beta) \ll (1-\beta)^{-1}$. These bounds together with Equations (4.10) and (4.11) give

$$1 \ll qX^{\frac{1}{2}-\beta} + X^{1-\beta}\frac{\lambda}{1-\beta}.$$

As $1-\varepsilon \leq \beta < 1$ and $\varepsilon \leq \frac{1}{4}$, we can take $X \asymp q^4$ to ensure that $qX^{\frac{1}{2}-\beta} \ll 1$. With this restriction on X, our estimate becomes

$$1 \ll q^{4(1-\beta)}\frac{\lambda}{1-\beta}.$$

By Lemma 4.4.1, $\lambda \ll \log^2(q)L(1,\chi)$ and we have

$$1 \ll q^{4(1-\beta)}\frac{\log^2(q)L(1,\chi)}{1-\beta}.$$

Isolating $L(1,\chi)$ results in

$$\frac{1-\beta}{q^{4(1-\beta)}\log^2(q)} \ll L(1,\chi).$$

But $1-\varepsilon \leq \beta < 1$ implies $0 < 1-\beta \leq \varepsilon$ and so

$$\frac{1}{q^{\varepsilon}} \ll_{\varepsilon} L(1,\chi),$$

where we have used that $\log(q) \ll_{\varepsilon} q^{\varepsilon}$. This is equivalent to the desired lower bound. □

The part of the proof in Siegel's theorem which makes $c_1(\varepsilon)$ and $c_2(\varepsilon)$ ineffective is the value of β. The choice of β depends upon the existence of a Siegel zero near 1 relative to the given ε. Since we don't know if Siegel zeros exist, this makes estimating β relative to ε ineffective and hence the constants $c_1(\varepsilon)$ and $c_2(\varepsilon)$ as well. Many results in analytic number theory make use of Siegel's theorem and hence also rely on ineffective constants. Moreover, some important problems investigate methods to get around using Siegel's theorem in favor of a weaker result that is effective. So far, no Siegel zero has been shown to exist or not exist for Dirichlet L-functions. But some progress has been made to showing that they are rare:

Proposition 4.4.1. *Let χ_1 and χ_2 be distinct quadratic Dirichlet characters of conductors q_1 and q_2. If $L(s,\chi_1)$ and $L(s,\chi_2)$ have Siegel zeros β_1 and β_2 respectively and $\chi_1\chi_2$ is not principal then there exists a positive constant c such that*

$$\min(\beta_1,\beta_2) < 1 - \frac{c}{\log(q_1q_2)}.$$

Proof. We may assume χ_1 and χ_2 are primitive since if $\widetilde{\chi}_i$ is the primitive character inducing χ_i, for $i = 1, 2$, the only difference in zeros between $L(s,\chi_i)$ and $L(s,\widetilde{\chi}_i)$ occur on the line $\sigma = 0$. Now let $\widetilde{\chi}$ be the primitive character of conductor q inducing $\chi_1\chi_2$. From Equation (4.7) with $\chi_1\chi_2$ in place of χ, we find that

$$\left|\frac{L'}{L}(s,\chi_1\chi_2) - \frac{L'}{L}(s,\widetilde{\chi})\right| = \left|\sum_{p|q_1q_2}\frac{\widetilde{\chi}(p)\log(p)p^{-s}}{1-\widetilde{\chi}(p)p^{-s}}\right| \leq \sum_{p|q_1q_2}\frac{\log(p)p^{-\sigma}}{1-p^{-\sigma}} \leq \sum_{p|q_1q_2}\log(p) \leq \log(q_1q_2). \tag{4.12}$$

Let $s = \sigma$ with $1 < \sigma \leq 2$. Using the reverse triangle inequality, we deduce from Equation (4.12) that

$$-\frac{L'}{L}(\sigma,\chi_1\chi_2) < c\log(q_1q_2), \tag{4.13}$$

for some positive constant c. By Lemma 3.7.1 (iv) applied to $\zeta(s)$ while discarding all of the terms in both sums, we have

$$-\frac{\zeta'}{\zeta}(\sigma) < A + \frac{1}{\sigma - 1}, \tag{4.14}$$

for some positive constant A. By Lemma 3.7.1 (iv) applied to $L(s, \chi_i)$ and only retaining the term corresponding to β_i, we have

$$-\frac{L'}{L}(\sigma, \chi_i) < A \log(q_i) + \frac{1}{\sigma - \beta_i}, \tag{4.15}$$

for $i = 1, 2$ and some possibly larger constant A. Now let $L(s, g)$ be the L-series defined by

$$L(s, g) = \zeta(s) L(s, \chi_1) L(s, \chi_2) L(s, \chi_1 \chi_2),$$

so that $-\frac{L'}{L}(\sigma, g) \geq 0$ by Lemma 4.4.2. Combining Equations (4.12) to (4.15) with this fact implies

$$0 < A + \frac{1}{\sigma - 1} + A \log(q_1) - \frac{1}{\sigma - \beta_1} + A \log(q_2) - \frac{1}{\sigma - \beta_2} + c \log(q_1 q_2).$$

Taking c larger, if necessary, we arrive at the simplified estimate

$$0 < \frac{1}{\sigma - 1} - \frac{1}{\sigma - \beta_1} - \frac{1}{\sigma - \beta_2} + c \log(q_1 q_2),$$

which we rewrite as

$$\frac{1}{\sigma - \beta_1} + \frac{1}{\sigma - \beta_2} < \frac{1}{\sigma - 1} + c \log(q_1 q_2),$$

Now let $\sigma = 1 + \frac{\delta}{\log(q_1 q_2)}$ for some $\delta > 0$. Upon substituting, we have

$$\frac{1}{\sigma - \beta_1} + \frac{1}{\sigma - \beta_2} < \left(c + \frac{1}{\delta} \right) \log(q_1 q_2).$$

If $\min(\beta_1, \beta_2) \geq 1 - \frac{c}{\log(q_1 q_2)}$, it follows that

$$2(\delta + c) < c + \frac{1}{\delta},$$

which is a contradiction if we take δ small enough to ensure $2\delta^2 + c\delta < 1$. □

From Proposition 4.4.1 we immediately see that for every modulus $m > 1$ there is at most one primitive quadratic Dirichlet character that can admit a Siegel zero:

Proposition 4.4.2. *For every integer $m > 1$, there is at most one Dirichlet character χ modulo m such that $L(s, \chi)$ has a Siegel zero. If this Siegel zero exists, χ is necessarily quadratic.*

Proof. Let $\widetilde{\chi}$ be the primitive character inducing χ. As the zeros of $L(s, \chi)$ and $L(s, \widetilde{\chi})$ differ only on the line $\sigma = 0$, Theorem 4.4.2 implies that $\widetilde{\chi}$, and hence χ, must be quadratic. So suppose χ_1 and χ_2 are two distinct characters modulo m, of conductors q_1 and q_2, admitting Siegel zeros β_1 and β_2. Then $\chi_1 \chi_2 \neq \chi_{m,0}$. Moreover, $\beta_1 \geq 1 - \frac{c_1}{\log(q_1)}$ and $\beta_2 \geq 1 - \frac{c_2}{\log(q_2)}$ for some positive constants c_1 and c_2. Taking c smaller, if necessary, we have $\min(\beta_1, \beta_2) \geq 1 - \frac{c}{\log(q_1 q_2)}$ which contradicts Proposition 4.4.1. □

4.5 The Prime Number Theorem

The function $\psi(x)$ is defined by

$$\psi(x) = \sum_{n \leq x} \Lambda(n),$$

for $x > 0$. We will obtain an explicit formula for $\psi(x)$ analogous to the explicit formula for the Riemann zeta function. The explicit formula for $\psi(x)$ will be obtained by applying truncated Perron's formula to the logarithmic derivative of $\zeta(s)$. Since $\psi(x)$ is discontinuous when x is a prime power, we need to work with a slightly modified function to apply the Mellin inversion formula. Define $\psi_0(x)$ by

$$\psi_0(x) = \begin{cases} \psi(x) & \text{if } x \text{ is not a prime power,} \\ \psi(x) - \frac{1}{2}\Lambda(x) & \text{if } x \text{ is a prime power.} \end{cases}$$

Equivalently, $\psi_0(x)$ is $\psi(x)$ except that its value is halfway between the limit values when x is a prime power. Stated another way, if x is a prime power the last term in the sum for $\psi_0(x)$ is multiplied by $\frac{1}{2}$. The **explicit formula** for $\psi(x)$ is the following:

Theorem (Explicit formula for $\psi(x)$). *For $x \geq 2$,*

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1 - x^{-2}),$$

where the sum is counted with multiplicity and ordered with respect to the size of the ordinate.

A few comments are in order before we prove the explicit formula for $\psi(x)$. First, since ρ is conjectured to be of the form $\rho = \frac{1}{2} + i\gamma$ via the Riemann hypothesis for the Riemann zeta function, x is conjectured to be the main term in the explicit formula. The constant $\frac{\zeta'}{\zeta}(0)$ can be shown to be $\log(2\pi)$ (see [Dav80] for a proof). Also, using the Taylor series of the logarithm, the last term can be expressed as

$$\frac{1}{2}\log(1 - x^{-2}) = \frac{1}{2}\sum_{m \geq 1} (-1)^{m-1} \frac{(-x^{-2})^m}{m} = \sum_{m \geq 1} (-1)^{2m-1} \frac{x^{-2m}}{2m} = \sum_{m \geq 1} \frac{x^{-2m}}{-2m} = \sum_{\omega} \frac{x^\omega}{\omega},$$

where ω runs over the trivial zeros of $\zeta(s)$. We will now prove the explicit formula for $\psi(x)$:

Proof of the explicit formula for $\psi(x)$. Applying truncated Perron's formula to $-\frac{\zeta'}{\zeta}(s)$ gives

$$\psi_0(x) - J(x,T) \ll x^c \sum_{\substack{n \geq 1 \\ n \neq x}} \frac{\Lambda(n)}{n^c} \min\left(1, \frac{1}{T\left|\log\left(\frac{x}{n}\right)\right|}\right) + \delta_x \Lambda(x)\frac{c}{T}, \tag{4.16}$$

where

$$J(x,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{\zeta'}{\zeta}(s) x^s \frac{ds}{s},$$

$c > 1$, and it is understood that $\delta_x = 0$ unless x is a prime power. Take $T > 2$ not coinciding with the ordinate of a nontrivial zero and let $c = 1 + \frac{1}{\log(x^2)}$ so that $x^c = \sqrt{e}x$ and $1 < c < 2$. The first step is to estimate the right-hand side of Equation (4.16). We deal with the terms corresponding to n such that n is

bounded away from x before anything else. So suppose $n \leq \frac{3}{4}x$ or $n \geq \frac{5}{4}x$. For these n, $\log\left(\frac{x}{n}\right)$ is bounded away from zero so that their contribution is

$$\ll \frac{x^c}{T}\sum_{n\geq 1}\frac{\Lambda(n)}{n^c} \ll \frac{x^c}{T}\left(-\frac{\zeta'}{\zeta}(c)\right) \ll \frac{x\log(x)}{T}, \tag{4.17}$$

where the last estimate follows from Lemma 3.7.1 (iv) applied to $\zeta(s)$ while discarding all of the terms in both sums and our choice of c (in particular $\log(c) \ll \log(x)$). Now we deal with the terms n close to x. Consider those n for which $\frac{3}{4}x < n < x$. Let x_1 be the largest prime power less than x. We may also suppose $\frac{3}{4}x < x_1 < x$ since otherwise $\Lambda(n) = 0$ and these terms do not contribute anything. Moreover, $\frac{x^c}{n^c} \ll 1$. For the term $n = x_1$, we have

$$\log\left(\frac{x}{n}\right) = -\log\left(1 - \frac{x - x_1}{x}\right) \geq \frac{x - x_1}{x},$$

where we have obtained the inequality by using Taylor series of the logarithm truncated after the first term. The contribution of this term is then

$$\ll \Lambda(x_1)\min\left(1, \frac{x}{T(x - x_1)}\right) \ll \log(x)\min\left(1, \frac{x}{T(x - x_1)}\right). \tag{4.18}$$

For the other such n, we can write $n = x_1 - v$, where v is an integer satisfying $0 < v < \frac{1}{4}x$, so that

$$\log\left(\frac{x}{n}\right) \geq \log\left(\frac{x_1}{n}\right) = -\log\left(1 - \frac{v}{x_1}\right) \geq \frac{v}{x_1},$$

where we have obtained the latter inequality by using Taylor series of the logarithm truncated after the first term. The contribution for these n is then

$$\ll \sum_{0<v<\frac{1}{4}x}\Lambda(x_1 - v)\frac{x_1}{Tv} \ll \frac{x}{T}\sum_{0<v<\frac{1}{4}x}\frac{\Lambda(x_1 - v)}{v} \ll \frac{x\log(x)}{T}\sum_{0<v<\frac{1}{4}x}\frac{1}{v} \ll \frac{x\log^2(x)}{T}. \tag{4.19}$$

The contribution for those n for which $x < n < \frac{5}{4}x$ is handled in exactly the same way with x_1 being the least prime power larger than x. Let $\langle x\rangle$ be the distance between x and the nearest prime power other than x if x itself is a prime power. Combining Equations (4.18) and (4.19) with our previous comment, the contribution for those n with $\frac{3}{4}x < n < \frac{5}{4}x$ is

$$\ll \frac{x\log^2(x)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right). \tag{4.20}$$

Putting Equations (4.17) and (4.20) together and noticing that the error term in Equation (4.17) is absorbed by the second error term in Equation (4.20), we obtain

$$\psi_0(x) - J(x, T) \ll \frac{x\log^2(x)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right). \tag{4.21}$$

This is the first part of the proof. Now we estimate $J(x, T)$ by appealing to the residue theorem. Let $U \geq 1$ be an odd integer. Let Ω be the region enclosed by the contours $\eta_1, \ldots, \eta_4$ in Figure 4.1 and set $\eta = \sum_{1\leq i\leq 4}\eta_i$ so that $\eta = \partial\Omega$.

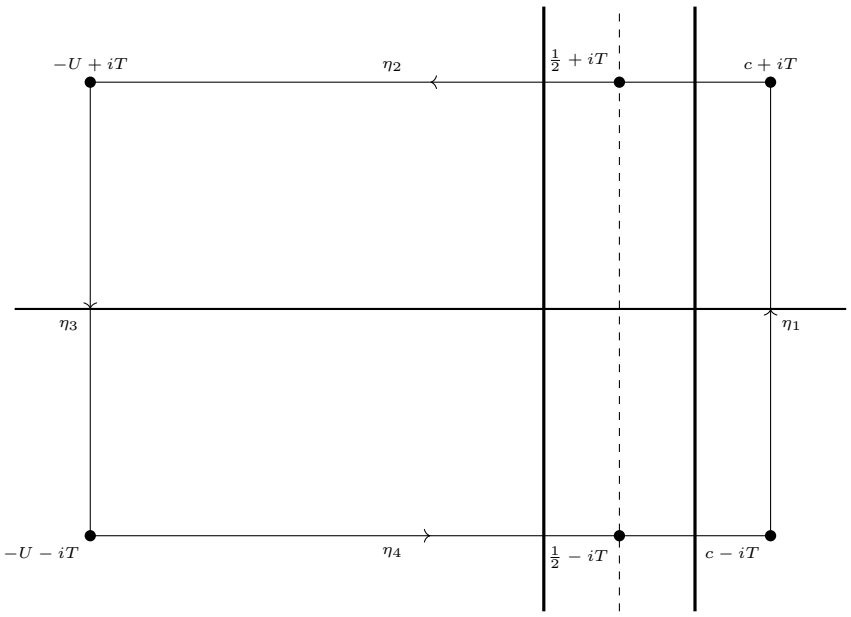


Figure 4.1: Contour for the explicit formula for $\psi(x)$

We may express $J(x,T)$ as

$$J(x,T) = \frac{1}{2\pi i}\int_{\eta_1} -\frac{\zeta'}{\zeta}(s)x^s\,\frac{ds}{s}.$$

The residue theorem together with the formula for the negative logarithmic derivative in Proposition 3.6.1 applied to $\zeta(s)$ and Corollary 1.7.1 imply

$$J(x,T) = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \sum_{0<2m<U}\frac{x^{-2m}}{-2m} + \frac{1}{2\pi i}\int_{\eta_2+\eta_3+\eta_4} -\frac{\zeta'}{\zeta}(s)x^s\,\frac{ds}{s}, \tag{4.22}$$

where $\rho = \beta + i\gamma$ is a nontrivial zero of ζ. We will estimate $J(x,T)$ by estimating the remaining integral. By Lemma 3.7.1 (ii) applied to $\zeta(s)$, the number of nontrivial zeros satisfying $|\gamma - T| < 1$ is $O(\log(T))$. Among the ordinates of these nontrivial zeros, there must be a gap of size larger than $O\left(\frac{1}{\log(T)}\right)$. Upon varying T by a bounded amount (we are varying in the interval $[T-1, T+1]$) so that it belongs to this gap, we can additionally ensure

$$\gamma - T \gg \frac{1}{\log(T)},$$

for all the nontrivial zeros of $\zeta(s)$. To estimate part of the horizontal integrals over η_2 and η_4, Lemma 3.7.1 (iv) applied to $\zeta(s)$ gives

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma-T|<1}\frac{1}{s-\rho} + O(\log(T)),$$

on the parts of these intervals with $-1 \le \sigma \le 2$. By our choice of T, $|s-\rho| \ge |\gamma - T| \gg \frac{1}{\log(T)}$ so that each term in the sum is $O(\log(T))$. There are at most $O(\log(T))$ such terms by Lemma 3.7.1 (ii) applied to $\zeta(s)$, so we find that

$$\frac{\zeta'}{\zeta}(s) = O(\log^2(T)),$$

on the parts of these intervals with $-1 \leq \sigma \leq 2$. It follows that the parts of the horizontal integrals over η_2 and η_4 with $-1 \leq \sigma \leq c$ (recall $c < 2$) contribute

$$\ll \frac{\log^2(T)}{T} \int_{-1}^{c} x^\sigma \, d\sigma \ll \frac{\log^2(T)}{T} \int_{-\infty}^{c} x^\sigma \, d\sigma \ll \frac{x \log^2(T)}{T \log(x)}, \tag{4.23}$$

where in the last estimate we have used the choice of c. To estimate the remainder of the horizontal integrals, we need a bound for $\frac{\zeta'}{\zeta}(s)$ when $\sigma < -1$ and away from the trivial zeros. To this end, write the functional equation for $\zeta(s)$ in the form

$$\zeta(s) = \pi^{s-1} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s),$$

and take the logarithmic derivative to get

$$\frac{\zeta'}{\zeta}(s) = \log(\pi) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{\zeta'}{\zeta}(1-s).$$

Let s be such that $\sigma < -1$ and suppose s is distance $\frac{1}{2}$ away from the trivial zeros. We will estimate every term on the right-hand side of the previous identity. The first term is constant and the last term is bounded since it is an absolutely convergent Dirichlet series. As for the digamma terms, since s is away from the trivial zeros, Proposition 1.7.3 implies $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) = O(\log|1-s|)$ and $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) = O(\log|s|)$. However, as $\sigma < -1$ and s is away from the trivial zeros, s and $1-s$ are bounded away from zero so that $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) = O(\log|s|)$. Putting these estimates together gives

$$\frac{\zeta'}{\zeta}(s) \ll \log|s|, \tag{4.24}$$

for $\sigma < -1$. Using Equation (4.24), the parts of the horizontal integrals over η_2 and η_4 with $-U \leq \sigma \leq -1$ contribute

$$\ll \frac{\log(T)}{T} \int_{-U}^{-1} x^\sigma \, d\sigma \ll \frac{\log(T)}{Tx\log(x)}. \tag{4.25}$$

Combining Equations (4.23) and (4.25) gives

$$\frac{1}{2\pi i}\int_{\eta_2+\eta_4} -\frac{\zeta'}{\zeta}(s) x^s \frac{ds}{s} \ll \frac{x\log^2(T)}{T\log(x)} + \frac{\log(T)}{Tx\log(x)} \ll \frac{x\log^2(T)}{T\log(x)}. \tag{4.26}$$

To estimate the vertical integral, we use Equation (4.24) again to conclude that

$$\frac{1}{2\pi i}\int_{\eta_3} -\frac{\zeta'}{\zeta}(s) x^s \frac{ds}{s} \ll \frac{\log(U)}{U}\int_{-T}^{T} x^{-U} \, dt \ll \frac{T\log(U)}{Ux^U}. \tag{4.27}$$

Combining Equations (4.22), (4.26) and (4.27) and taking the limit as $U \to \infty$, the error term in Equation (4.27) vanishes and the sum over m in Equation (4.22) evaluates to $\frac{1}{2}\log(1-x^{-2})$ (as we have already mentioned) giving

$$J(x,T) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1-x^{-2}) + \frac{x\log^2(T)}{T\log(x)}. \tag{4.28}$$

Substituting Equation (4.28) into Equation (4.21), we at last obtain

$$\psi_0(x) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1-x^{-2}) + \frac{x\log^2(xT)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right), \tag{4.29}$$

where the second to last term on the right-hand side is obtained by combining the error term in Equation (4.26) with the first error term in Equation (4.21). The theorem follows by taking the limit as $T \to \infty$. □

Note that the convergence of the right-hand side in the explicit formula for $\psi(x)$ is uniform in any interval not containing a prime power since $\psi(x)$ is continuous there. Moreover, we have an approximate formula for $\psi(x)$ as a corollary:

Corollary 4.5.1. *For $x \geq 2$ and $T > 2$,*

$$\psi_0(x) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} + R(x,T),$$

where ρ runs over the nontrivial zeros of $\zeta(s)$ counted with multiplicity and ordered with respect to the size of the ordinate, and

$$R(x,T) \ll \frac{x\log^2(xT)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right),$$

where $\langle x\rangle$ is the distance between x and the nearest prime power other than x if x itself is a prime power. Moreover, if x is an integer, we have the simplified estimate

$$R(x,T) \ll \frac{x\log^2(xT)}{T}.$$

Proof. This follows from Equation (4.29) since $\frac{\zeta'}{\zeta}(0)$ is constant and $\frac{1}{2}\log(1-x^2)$ is bounded for $x \geq 2$. If x is an integer then $\langle x\rangle \geq 1$ so that $\log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right) \leq \frac{x\log(x)}{T}$ and this term can be absorbed into $O\left(\frac{x\log^2(xT)}{T}\right)$. □

With our refined explicit formula in hand, we are ready to discuss and prove the prime number theorem. The **prime counting function** $\pi(x)$ is defined by

$$\pi(x) = \sum_{p\leq x} 1,$$

for $x > 0$. Equivalently, $\pi(x)$ counts the number of primes that no larger than x. That $\pi(x) \to \infty$ as $x \to \infty$ is equivalent to the existence of infinitely many primes. A more interesting question is to ask how the primes are distributed among the integers. The (classical) **prime number theorem** answers this question and the precise statement is the following:

Theorem (Prime number theorem, classical). *For $x \geq 2$,*

$$\pi(x) \sim \frac{x}{\log(x)}.$$

There is an equivalent statement to the classical prime number theorem. Define the **logarithmic integral** $\mathrm{Li}(x)$ by

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)},$$

for $x \geq 2$. Notice that $\mathrm{Li}(x) \sim \frac{x}{\log x}$ because

$$\lim_{x\to\infty} \left| \frac{\mathrm{Li}(x)}{\frac{x}{\log x}} \right| = \lim_{x\to\infty} \left| \frac{\int_2^x \frac{dt}{\log(t)}}{\frac{x}{\log x}} \right| = \lim_{x\to\infty} \left| \frac{\frac{1}{\log(x)}}{\frac{\log(x)-1}{\log^2(x)}} \right| = \lim_{x\to\infty} \left| \frac{\log(x)}{\log(x)-1} \right| = 1.$$

where in the second equality we have used L'Hôpital's rule. So an equivalent statement is the (logarithmic integral) **prime number theorem**:

Theorem (Prime number theorem, logarithmic integral). *For $x \geq 2$,*

$$\pi(x) \sim \mathrm{Li}(x).$$

For the moment, we will delay the proofs of these two variants of the prime number theorem and give some intuition and historical context to the results. Intuitively, the prime number theorem is a result about how dense the primes are in the integers. To see this, notice that the result is equivalent to the asymptotic

$$\frac{\pi(x)}{x} \sim \frac{1}{\log(x)}.$$

Letting $x \geq 2$, the left-hand side is the probability that a randomly chosen positive integer no larger than x is prime. Thus the asymptotic result says that for large enough x, the probability that a randomly chosen integer no larger than x is prime is approximately $\frac{1}{\log(x)}$. We can also interpret this as saying that the average gap between primes no larger than x is approximately $\frac{1}{\log(x)}$. As a consequence, a positive integer with at most $2n$ digits is half as likely to be prime than a positive integer with at most n digits for large n. Indeed, there are $10^n - 1$ numbers with at most n digits, $10^{2n} - 1$ with at most $2n$ digits, and $\log(10^{2n} - 1) \sim 2\log(10^n)$. Note that the prime number theorem says nothing about the exact error $\pi(x) - \frac{x}{\log(x)}$ as $x \to \infty$. It only says that the relative error tends to zero, which is to say

$$\lim_{x\to\infty} \frac{\pi(x) - \frac{x}{\log(x)}}{\frac{x}{\log(x)}} = 0.$$

Now for some historical context. While Gauss was not the first to put forth a conjectural form of the prime number theorem, he was known for compiling extensive tables of primes and he suspected that the density of the primes up to x was roughly $\frac{1}{\log(x)}$. How might one suspect this is the correct density? Well, let $d\delta_p$ be the weighted point measure that assigns $\frac{1}{p}$ at the prime p and zero everywhere else. Then

$$\sum_{p\leq x} \frac{1}{p} = \int_1^x d\delta_p(u).$$

We can interpret the integral as integrating the density $d\delta_p$ over $[1, x]$. We would like a more explicit description of $d\delta_p$. Euler (see [Eul44]), argued

$$\sum_{p\leq x} \frac{1}{p} \sim \log\log(x),$$

and notice that

$$\log\log(x) = \int_1^{\log(x)} \frac{du}{u} = \int_e^x \frac{1}{u}\frac{du}{\log u},$$

where in the second equality we have made the change of variables $u \mapsto \log(u)$. So altogether, we have

$$\sum_{p\leq x} \frac{1}{p} \sim \int_e^x \frac{1}{u}\frac{du}{\log u}.$$

This asymptotic gives a more explicit representation of the density $d\delta_p$. Notice that both sides of this asymptotic are essentially weighted the same with the left-hand side by $\frac{1}{p}$ and the right-hand side by $\frac{1}{u}$. Cancelling these weights (this is not strictly allowed), we might expect

$$\pi(x) = \sum_{p\leq x} 1 \sim \int_e^x \frac{du}{\log(u)},$$

which is simply the logarithmic integral prime number theorem. Legendre was the first to put forth a conjectural form of the prime number theorem. In 1798 (see [Leg98]) he claimed that $\pi(x)$ was of the form

$$\frac{x}{A\log(x) + B},$$

for some constants A and B. In 1808 (see [Leg08]) he refined his conjecture by claiming

$$\frac{x}{\log(x) + A(x)},$$

where $\lim_{x\to\infty} A(x) \approx 1.08366$. Riemann's 1859 manuscript (see [Rie59]) contains an outline for how to prove the prime number theorem, but it was not until 1896 that the prime number theorem was proved independently by Hadamard and de la Vallée Poussin (see [Had96] and [Pou97]). Their proofs, as well as every proof thereon out until 1949, used complex analytic methods in an essential way (there are now elementary proofs due to Erdös and Selberg). We are now ready to prove the prime number theorem. Strictly speaking, we will prove the (absolute error) **prime number theorem**, due to de la Vallée Poussin, which bounds the absolute error between $\pi(x)$ and $\mathrm{Li}(x)$:

Theorem (Prime number theorem, absolute error). *For $x \geq 2$, there exists a positive constant c such that*

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-c\sqrt{\log(x)}}\right).$$

Proof. It suffices to assume x is an integer, because $\pi(x)$ can only change value at integers and the other functions in the statement are increasing. We will first prove

$$\psi(x) = x + O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{4.30}$$

for some positive constant c. To achieve this, we estimate the sum over the nontrivial zeros of $\zeta(s)$ in Corollary 4.5.1. So let $T > 2$ not coinciding with the ordinate of a nontrivial zero, and suppose $\rho = \beta + i\gamma$ is a nontrivial zero with $|\gamma| < T$. By Theorem 4.4.1, we know $\beta < 1 - \frac{c}{\log(T)}$ for some positive constant c. It follows that

$$|x^\rho| = x^\beta < x^{1-\frac{c}{\log(T)}} = xe^{-c\frac{\log(x)}{\log(T)}}. \tag{4.31}$$

As $|\rho| > |\gamma|$, letting $\gamma_1 > 0$ (which is bounded away from zero since the zeros of $\zeta(s)$ are discrete and we know that there is no real nontrivial zero) be the ordinate of the first nontrivial zero, applying integration by parts gives

$$\sum_{|\gamma|<T} \frac{1}{\rho} \ll \sum_{\gamma_1 \leq |\gamma| < T} \frac{1}{\gamma} \ll \int_{\gamma_1}^{T} \frac{dN(t)}{t} = \frac{N(T)}{T} + \int_{\gamma_1}^{T} \frac{N(t)}{t^2}\, dt \ll \log^2(T), \tag{4.32}$$

where in the last estimate we have used that $N(t) \ll t\log(t)$ which follows from Corollary 3.7.1. Putting Equations (4.31) and (4.32) together gives

$$\sum_{|\gamma|<T} \frac{x^\rho}{\rho} \ll x\log^2(T) e^{-c\frac{\log(x)}{\log(T)}}. \tag{4.33}$$

As $\psi(x) \sim \psi_0(x)$ and x is an integer, Equation (4.33) with Corollary 4.5.1 together imply

$$\psi(x) - x \ll x\log^2(T) e^{-c\frac{\log(x)}{\log(T)}} + \frac{x\log^2(xT)}{T}. \tag{4.34}$$

We will now let T be determined by

$$\log^2(T) = \log(x),$$

or equivalently,

$$T = e^{\sqrt{\log(x)}}.$$

With this choice of T (note that if $x \geq 2$ then $T > 2$), we can estimate Equation (4.34) as follows:

$$\begin{aligned}
\psi(x) - x &\ll x\log(x) e^{-c\sqrt{\log(x)}} + x\left(\log^2(x) + \log(x)\right) e^{-\sqrt{\log(x)}} \\
&\ll x\log(x) e^{-c\sqrt{\log(x)}} + x\log^2(x) e^{-\sqrt{\log(x)}} \\
&\ll x\log^2(x) e^{-\min(1,c)\sqrt{\log(x)}}.
\end{aligned}$$

As $\log(x) = o\left(e^{-\varepsilon\sqrt{\log(x)}}\right)$, we conclude that

$$\psi(x) - x \ll x e^{-c\sqrt{\log(x)}},$$

for some smaller c with $c < 1$. This is equivalent to Equation (4.30). Now let

$$\pi_1(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\log(n)}.$$

We can write $\pi_1(x)$ in terms of $\psi(x)$ as follows:

$$\begin{aligned}
\pi_1(x) &= \sum_{n \leq x} \frac{\Lambda(n)}{\log(n)} \\
&= \sum_{n \leq x} \Lambda(n) \int_n^x \frac{dt}{t\log^2(t)} + \frac{1}{\log(x)} \sum_{n \leq x} \Lambda(n) \\
&= \int_2^x \sum_{n \leq t} \Lambda(n) \frac{dt}{t\log^2(t)} + \frac{1}{\log(x)} \sum_{n \leq x} \Lambda(n) \\
&= \int_2^x \frac{\psi(t)}{t\log^2(t)}\, dt + \frac{\psi(x)}{\log(x)}.
\end{aligned}$$

Applying Equation (4.30) to the last expression yields

$$\pi_1(x) = \int_2^x \frac{t}{t\log^2(t)}\,dt + \frac{x}{\log(x)} + O\left(\int_2^x \frac{e^{-c\sqrt{\log(t)}}}{\log^2(t)}\,dt + \frac{xe^{-c\sqrt{\log(x)}}}{\log(x)}\right). \tag{4.35}$$

Upon applying integrating by parts to the main term in Equation (4.35), we obtain

$$\int_2^x \frac{t}{t\log^2(t)}\,dt + \frac{x}{\log(x)} = \int_2^x \frac{dt}{\log(t)} + \frac{2}{\log(2)} = \mathrm{Li}(x) + \frac{2}{\log(2)}. \tag{4.36}$$

As for the error term in Equation (4.35), $\log^2(t)$ and $\log(x)$ are both bounded away from zero so that

$$\int_2^x \frac{e^{-c\sqrt{\log(t)}}}{\log^2(t)}\,dt + \frac{xe^{-c\sqrt{\log(x)}}}{\log(x)} \ll \int_2^x e^{-c\sqrt{\log(t)}}\,dt + xe^{-c\sqrt{\log(x)}}.$$

For $t \leq x^{\frac{1}{4}}$, we use the bound $e^{-c\sqrt{\log(t)}} < 1$ so that

$$\int_2^{x^{\frac{1}{4}}} e^{-c\sqrt{\log(t)}}\,dt < \int_2^{x^{\frac{1}{4}}}\,dt \ll x^{\frac{1}{4}}.$$

For $t > x^{\frac{1}{4}}$, $\sqrt{\log(t)} > \frac{1}{2}\sqrt{\log(x)}$ and thus

$$\int_2^{x^{\frac{1}{4}}} e^{-c\sqrt{\log(t)}}\,dt \leq e^{-c\frac{1}{2}\sqrt{\log(x)}} \int_2^{x^{\frac{1}{4}}}\,dt \ll x^{\frac{1}{4}} e^{-c\frac{1}{2}\sqrt{\log(x)}}.$$

All of these estimates together imply

$$\int_2^x \frac{e^{-c\sqrt{\log(t)}}}{\log^2(t)}\,dt + \frac{xe^{-c\sqrt{\log(x)}}}{\log(x)} \ll xe^{-c\sqrt{\log(x)}}, \tag{4.37}$$

for some smaller c. Combining Equations (4.35) to (4.37) yields

$$\pi_1(x) = \mathrm{Li}(x) + O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{4.38}$$

where the constant in Equation (4.36) has been absorbed into the error term. We now pass from $\pi_1(x)$ to $\pi(x)$. If p is a prime such that $p^m < x$, for some $m \geq 1$, then $p < x^{\frac{1}{2}} < x^{\frac{1}{3}} < \cdots < x^{\frac{1}{m}}$. Therefore

$$\pi_1(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\log(n)} = \sum_{p^m \leq x} \frac{\log(p)}{m\log(p)} = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \cdots. \tag{4.39}$$

Moreover, as $\pi(x^{\frac{1}{n}}) < x^{\frac{1}{n}}$ for any $n \geq 1$, we see that $\pi(x) - \pi_1(x) = O(x^{\frac{1}{2}})$. This estimate together with Equations (4.38) and (4.39) gives

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-c\sqrt{\log(x)}}\right),$$

because $x^{\frac{1}{2}} \ll xe^{-c\sqrt{\log(x)}}$. This completes the proof. □

The proof of the classical and logarithmic integral variants of the prime number theorem are immediate:

Proof of prime number theorem, classical and logarithmic integral. From the absolute error prime number theorem, we have

$$\pi(x) = \mathrm{Li}(x)\left(1 + O\left(\frac{xe^{-c\sqrt{\log(x)}}}{\mathrm{Li}(x)}\right)\right).$$

But we have shown $\mathrm{Li}(x) \sim \frac{x}{\log(x)}$ so that

$$\frac{xe^{-c\sqrt{\log(x)}}}{\mathrm{Li}(x)} \sim \log(x)e^{-c\sqrt{\log(x)}} = o(1),$$

where the equality holds since $\log(x) = o\left(e^{-\varepsilon\sqrt{\log(x)}}\right)$. The logarithmic integral prime number theorem follows. The classical prime number theorem also holds using the asymptotic $\mathrm{Li}(x) \sim \frac{x}{\log(x)}$ again. □

In the proof of the logarithmic integral and classical variants of the prime number theorem, we saw that $xe^{-c\sqrt{\log(x)}} < \frac{x}{\log(x)}$ for sufficiently large x. Therefore the exact error $\pi(x) - \mathrm{Li}(x)$ grows slower than $\pi(x) - \frac{x}{\log x}$ for sufficiently large x. This means that $\mathrm{Li}(x)$ is a better numerical approximation to $\pi(x)$ than $\frac{x}{\log(x)}$. There is also the following result due to Hardy and Littlewood (see [HL16]) which gives us more information:

Proposition 4.5.1. *$\pi(x) - \mathrm{Li}(x)$ changes sign infinitely often as $x \to \infty$.*

So in addition, Proposition 4.5.1 implies that $\mathrm{Li}(x)$ never underestimates or overestimates $\pi(x)$ continuously. On the other hand, the exact error $\pi(x) - \frac{x}{\log(x)}$ is positive provided $x \geq 17$ (see [RS62]). It is also worthwhile to note that in 1901 Koch showed that the Riemann hypothesis improves the error term in the absolute error prime number theorem (see [Koc01]):

Proposition 4.5.2. *For $x \geq 2$,*

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log(x)),$$

provided the Riemann hypothesis for the Riemann zeta function holds.

Proof. If ρ is a nontrivial zero of $\zeta(s)$, the Riemann hypothesis implies $|x^\rho| = \sqrt{x}$. Therefore as in the proof of the absolute error prime number theorem,

$$\sum_{|\gamma|<T} \frac{x^\rho}{\rho} \ll \sqrt{x}\log^2(T),$$

for $T > 2$ not coinciding with the ordinate of a nontrivial zero. Repeating the same argument with T determined by

$$T^2 = x,$$

gives

$$\psi(x) = x + O(\sqrt{x}\log^2(x)),$$

and then transferring to $\pi_1(x)$ and finally $\pi(x)$ gives

$$\pi(x) = x + O(\sqrt{x}\log(x)).$$

□

In fact, an improvement in the zero-free region for the Riemann zeta function will give an error term in between the absolute error prime number theorem and Proposition 4.5.2. So it is the strength of the zero-free region which controls the size of the error term.

4.6 The Siegel-Walfisz Theorem

Let χ be a Dirichlet character modulo $m > 1$. The function $\psi(x, \chi)$ is defined by

$$\psi(x, \chi) = \sum_{n \leq x} \chi(n)\Lambda(n),$$

for $x > 0$. This function plays the analogous role of $\psi(x)$ but for Dirichlet L-series. Accordingly, we will derive an explicit formula for $\psi(x, \chi)$ in a similar manner to that of $\psi(x)$. Because $\psi(x, \chi)$ is discontinuous when x is a prime power, we also introduce a slightly modified function. Define $\psi_0(x, \chi)$ by

$$\psi_0(x, \chi) = \begin{cases} \psi(x, \chi) & \text{if } x \text{ is not a prime power,} \\ \psi(x, \chi) - \frac{1}{2}\chi(x)\Lambda(x) & \text{if } x \text{ is a prime power.} \end{cases}$$

Thus $\psi_0(x, \chi)$ is $\psi(x, \chi)$ except that its value is halfway between the limit values when x is a prime power. We will also need to define a particular constant that will come up. For a character χ, define $b(\chi)$ to be $\frac{L'}{L}(0, \chi)$ if χ is odd and to be the constant term in the Laurent series of $\frac{L'}{L}(s, \chi)$ if χ is even (as in the even case $\frac{L'}{L}(s, \chi)$ has a pole at $s = 0$). The **explicit formula** for $\psi(x, \chi)$ is the following:

Theorem (Explicit formula for $\psi(x, \chi)$). *Let χ be a primitive Dirichlet character of conductor $q > 1$. Then for $x \geq 2$,*

$$\psi_0(x, \chi) = -\sum_{\rho} \frac{x^\rho}{\rho} - b(\chi) + \tanh^{-1}(x^{-1}),$$

if χ is odd, and

$$\psi_0(x, \chi) = -\sum_{\rho} \frac{x^\rho}{\rho} - \log(x) - b(\chi) - \frac{1}{2}\log(1 - x^{-2}),$$

if χ is even, and where in both expressions, ρ runs over the nontrivial zeros of $L(s, \chi)$ counted with multiplicity and ordered with respect to the size of the ordinate.

As for $\psi(x)$, a few comments are in order. Unlike the explicit formula for $\psi(x)$, there is no main term x in the explicit formula for $\psi(x, \chi)$. This is because $L(s, \chi)$ does not have a pole at $s = 1$. The constant $b(\chi)$ can be expressed in terms of $B(\chi)$ (see [Dav80] for a proof). Also, in the case χ is odd the Taylor series of the inverse hyperbolic tangent lets us write

$$\tanh^{-1}(x^{-1}) = \sum_{m \geq 1} \frac{x^{-(2m-1)}}{2m - 1} = -\sum_{m \geq 1} \frac{x^{-(2m-1)}}{-(2m - 1)} = -\sum_{\omega} \frac{x^\omega}{\omega},$$

where ω runs over the trivial zeros of $L(s, \chi)$. In the case χ is even, $\frac{1}{2}\log(1 - x^{-2})$ accounts for the contribution of the trivial zeros just as for $\zeta(s)$. We will now prove the explicit formula for $\psi(x, \chi)$:

Proof of the explicit formula for $\psi(x, \chi)$. By truncated Perron's formula applied to $-\frac{L'}{L}(s, \chi)$, we get

$$\psi_0(x, \chi) - J(x, T, \chi) \ll x^c \sum_{\substack{n \geq 1 \\ n \neq x}} \frac{\chi(n)\Lambda(n)}{n^c} \min\left(1, \frac{1}{T\left|\log\left(\frac{x}{n}\right)\right|}\right) + \delta_x \chi(x)\Lambda(x)\frac{c}{T}, \tag{4.40}$$

where

$$J(x, T, \chi) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{L'}{L}(s, \chi)x^s \frac{ds}{s},$$

$c > 1$, and it is understood that $\delta_x = 0$ unless x is a prime power. Take $T > 2$ not coinciding with the ordinate of a nontrivial zero and let $c = 1 + \frac{1}{\log(x^2)}$ so that $x^c = \sqrt{e}x$. We will estimate the right-hand side of Equation (4.40). First, we estimate the terms corresponding to n such that n is bounded away from x. So suppose $n \leq \frac{3}{4}x$ or $n \geq \frac{5}{4}x$. For these n, $\log\left(\frac{x}{n}\right)$ is bounded away from zero so that their contribution is

$$\ll \frac{x^c}{T} \sum_{n \geq 1} \frac{\chi(n)\Lambda(n)}{n^c} \ll \frac{x^c}{T} \left(-\frac{L'}{L}(c, \chi) \right) \ll \frac{x \log(x)}{T}, \tag{4.41}$$

where the last estimate follows from Lemma 3.7.1 (iv) applied to $L(s, \chi)$ while discarding all of the terms in both sums and our choice of c (in particular $\log(c) \ll \log(x)$). Now we estimate the terms n close to x. So consider those n for which $\frac{3}{4}x < n < x$ and let x_1 be the largest prime power less than x. We may also assume $\frac{3}{4}x < x_1 < x$ since otherwise $\Lambda(n) = 0$ and these terms do not contribute anything. Moreover, $\frac{x^c}{n^c} \ll 1$. For the term $n = x_1$, we have the estimate

$$\log\left(\frac{x}{n}\right) = -\log\left(1 - \frac{x - x_1}{x}\right) \geq \frac{x - x_1}{x},$$

where we have obtained the inequality by using Taylor series of the logarithm truncated after the first term. The contribution of this term is

$$\ll \chi(x_1)\Lambda(x_1) \min\left(1, \frac{x}{T(x - x_1)}\right) \ll \log(x) \min\left(1, \frac{x}{T(x - x_1)}\right). \tag{4.42}$$

For the other n, we write $n = x_1 - v$, where v is an integer satisfying $0 < v < \frac{1}{4}x$, so that

$$\log\left(\frac{x}{n}\right) \geq \log\left(\frac{x_1}{n}\right) = -\log\left(1 - \frac{v}{x_1}\right) \geq \frac{v}{x_1},$$

where we have obtained the latter inequality by using Taylor series of the logarithm truncated after the first term. The contribution for these n is

$$\ll \sum_{0 < v < \frac{1}{4}x} \chi(x_1 - v)\Lambda(x_1 - v)\frac{x_1}{Tv} \ll \frac{x}{T} \sum_{0 < v < \frac{1}{4}x} \frac{\Lambda(x_1 - v)}{v} \ll \frac{x \log(x)}{T} \sum_{0 < v < \frac{1}{4}x} \frac{1}{v} \ll \frac{x \log^2(x)}{T}. \tag{4.43}$$

The contribution for those n for which $x < n < \frac{5}{4}x$ is handled in exactly the same way with x_1 being the least prime power larger than x. Let $\langle x \rangle$ be the distance between x and the nearest prime power other than x if x itself is a prime power. Combining Equations (4.42) and (4.43) with our previous comment, the contribution for those n with $\frac{3}{4}x < n < \frac{5}{4}x$ is

$$\ll \frac{x \log^2(x)}{T} + \log(x) \min\left(1, \frac{x}{T\langle x \rangle}\right). \tag{4.44}$$

Putting Equations (4.41) and (4.44), the error term in Equation (4.41) is absorbed by the second error term in Equation (4.44) and we obtain

$$\psi_0(x, \chi) - J(x, T, \chi) \ll \frac{x \log^2(x)}{T} + \log(x) \min\left(1, \frac{x}{T\langle x \rangle}\right). \tag{4.45}$$

Now we estimate $J(x, T, \chi)$ by using the residue theorem. Let $U \geq 1$ be an integer with U is even if χ is odd and odd if χ is even. Let Ω be the region enclosed by the contours $\eta_1, \ldots, \eta_4$ in Figure 4.2 and set $\eta = \sum_{1 \leq i \leq 4} \eta_i$ so that $\eta = \partial\Omega$. We may write $J(x, T, \chi)$ as

$$J(x, T, \chi) = \frac{1}{2\pi i} \int_{\eta_1} -\frac{L'}{L}(s, \chi) x^s \frac{ds}{s}.$$

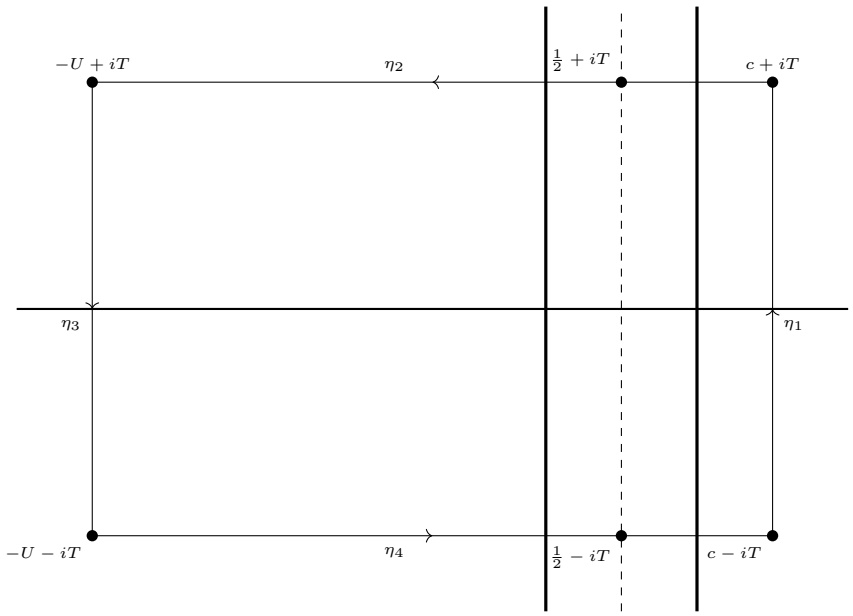


Figure 4.2: Contour for the explicit formula for $\psi(x,\chi)$

We now separate the cases that χ is even or odd. If χ is odd then the residue theorem, the formula for the negative logarithmic derivative in Proposition 3.6.1 applied to $L(s,\chi)$, and Corollary 1.7.1 together give

$$J(x,T,\chi) = -\sum_{|\gamma|<T} \frac{x^\rho}{\rho} - b(\chi) - \sum_{0<2m+1<U} \frac{x^{-(2m-1)}}{-(2m-1)} + \frac{1}{2\pi i}\int_{\eta_2+\eta_3+\eta_4} -\frac{L'}{L}(s,\chi)x^s\,\frac{ds}{s}, \tag{4.46}$$

where $\rho = \beta + i\gamma$ is a nontrivial zero of $L(s,\chi)$. If χ is even then there is a minor complication because $L(s,\chi)$ has a simple zero at $s=0$ and so the integrand has a double pole at $s=0$. To find the residue, the Laurent series are

$$\frac{L'}{L}(s,\chi) = \frac{1}{s} + b(\chi) + \cdots \quad \text{and} \quad \frac{x^s}{s} = \frac{1}{s} + \log(x) + \cdots,$$

and thus the residue of the integrand is $-(\log(x)+b(\chi))$. Now as before, the residue theorem, the formula for the negative logarithmic derivative in Proposition 3.6.1 applied to $L(s,\chi)$, and Corollary 1.7.1 together give

$$J(x,T,\chi) = -\sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \log(x) - b(\chi) - \sum_{0<2m<U} \frac{x^{-2m}}{-2m} + \frac{1}{2\pi i}\int_{\eta_2+\eta_3+\eta_4} -\frac{L'}{L}(s,\chi)x^s\,\frac{ds}{s}, \tag{4.47}$$

where $\rho = \beta+i\gamma$ is a nontrivial zero of $L(s,\chi)$. We now estimate the remaining integrals in Equations (4.46) and (4.47). For this estimate, the parity of χ does not matter so we make no such restriction. By Lemma 3.7.1 (ii) applied to $L(s,\chi)$, the number of nontrivial zeros satisfying $|\gamma - T| < 1$ is $O(\log(qT))$. Among the ordinates of these nontrivial zeros, there must be a gap of size larger than $O\left(\frac{1}{\log(qT)}\right)$. Upon varying T by a bounded amount (we are varying in the interval $[T-1, T+1]$) so that it belongs to this gap, we can additionally ensure

$$\gamma - T \gg \frac{1}{\log(qT)},$$

for all the nontrivial zeros of $L(s,\chi)$. To estimate part of the horizontal integrals over η_2 and η_4, Lemma 3.7.1 (iv) applied to $L(s,\chi)$ gives

$$\frac{L'}{L}(s,\chi) = \sum_{|\gamma - T|<1} \frac{1}{s-\rho} + O(\log(qT)),$$

on the parts of these intervals with $-1 \le \sigma \le 2$. Our choice of T implies $|s-\rho| \ge |\gamma - T| \gg \frac{1}{\log(qT)}$ so that each term in the sum is $O(\log(qT))$. As there are at most $O(\log(qT))$ such terms by Lemma 3.7.1 (ii) applied to $L(s,\chi)$, we have

$$\frac{L'}{L}(s,\chi) = O(\log^2(qT)),$$

on the parts of these intervals with $-1 \le \sigma \le 2$. It follows that the parts of the horizontal integrals over η_2 and η_4 with $-1 \le \sigma \le c$ (recall $c < 2$) contribute

$$\ll \frac{\log^2(qT)}{T} \int_{-1}^{c} x^\sigma \, d\sigma \ll \frac{\log^2(qT)}{T} \int_{-\infty}^{c} x^\sigma \, d\sigma \ll \frac{x \log^2(qT)}{T \log(x)}. \tag{4.48}$$

where in the last estimate we have used the choice of c. To estimate the remainder of the horizontal integrals, we require a bound for $\frac{L'}{L}(s,\chi)$ when $\sigma < -1$ and away from the trivial zeros. To find such a bound, write the functional equation for $L(s,\chi)$ in the form

$$L(s,\chi) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}} q^{\frac{1}{2}-s} \pi^{s-1} \frac{\Gamma\left(\frac{(1-s)+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)} L(1-s,\chi),$$

and take the logarithmic derivative to get

$$\frac{L'}{L}(s,\chi) = -\log(q) + \log(\pi) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{(1-s)+\mathfrak{a}}{2}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) + \frac{L'}{L}(1-s,\chi).$$

Now let s be such that $\sigma < -1$ and suppose s is distance $\frac{1}{2}$ away from the trivial zeros. We will estimate every term on the right-hand side of the identity above. The second term is constant and the last term is bounded since it is an absolutely convergent Dirichlet series. For the digamma terms, s is away from the trivial zeros so Proposition 1.7.3 implies $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{(1-s)+\mathfrak{a}}{2}\right) = O(\log|(1-s)+\mathfrak{a}|)$ and $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) = O(\log|s+\mathfrak{a}|)$. However, as $\sigma < -1$ and s is away from the trivial zeros, $s+\mathfrak{a}$ and $(1-s)+\mathfrak{a}$ are bounded away from zero so that $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{(1-s)+\mathfrak{a}}{2}\right) = O(\log|s|)$ and $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) = O(\log|s|)$. Putting these estimates together with the first term yields

$$\frac{L'}{L}(s,\chi) \ll \log(q|s|), \tag{4.49}$$

for $\sigma < -1$. Using Equation (4.49), the parts of the horizontal integrals over η_2 and η_4 with $-U \le \sigma \le -1$ contribute

$$\ll \frac{\log(qT)}{T} \int_{-U}^{-1} x^\sigma \, d\sigma \ll \frac{\log(qT)}{Tx\log(x)}. \tag{4.50}$$

Combining Equations (4.48) and (4.50) gives

$$\frac{1}{2\pi i}\int_{\eta_2+\eta_4} -\frac{L'}{L}(s) x^s \frac{ds}{s} \ll \frac{x\log^2(qT)}{T\log(x)} + \frac{\log(qT)}{Tx\log(x)} \ll \frac{x\log^2(qT)}{T\log(x)}. \tag{4.51}$$

To estimate the vertical integral, we use Equation (4.49) again to conclude that

$$\frac{1}{2\pi i}\int_{\eta_3} -\frac{L'}{L}(s,\chi)x^s\,\frac{ds}{s} \ll \frac{\log(qU)}{U}\int_{-T}^{T} x^{-U}\,dt \ll \frac{T\log(qU)}{Ux^U}. \tag{4.52}$$

Combining Equations (4.46), (4.51) and (4.52) and taking the limit as $U\to\infty$, the error term in Equation (4.52) vanishes and the sum over m in Equations (4.46) and (4.47) evaluates to $-\tanh^{-1}(x^{-1})$ or $\frac{1}{2}\log(1-x^{-2})$ respectively (as we have already mentioned) giving

$$J(x,T,\chi) = -\sum_{|\gamma|<T}\frac{x^\rho}{\rho} - b(\chi) + \tanh^{-1}(x^{-1}) + \frac{x\log^2(qT)}{T\log(x)}, \tag{4.53}$$

if χ is odd, and

$$J(x,T,\chi) = -\sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \log(x) - b(\chi) - \frac{1}{2}\log(1-x^{-2}) + \frac{x\log^2(qT)}{T\log(x)}, \tag{4.54}$$

if χ is even. Substituting Equations (4.53) and (4.54) into Equation (4.45) in the respective cases, we obtain

$$\psi_0(x,\chi) = -\sum_{|\gamma|<T}\frac{x^\rho}{\rho} - b(\chi) + \tanh^{-1}(x^{-1}) + \frac{x\log^2(xqT)}{T} + \log(x)\min\left(1,\frac{x}{T\langle x\rangle}\right), \tag{4.55}$$

if χ is odd, and

$$\psi_0(x,\chi) = -\sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \log(x) - b(\chi) - \frac{1}{2}\log(1-x^{-2}) + \frac{x\log^2(xqT)}{T} + \log(x)\min\left(1,\frac{x}{T\langle x\rangle}\right), \tag{4.56}$$

if χ is even, and where the second to last term on the right-hand side in both equations are obtained by combining the error term in Equation (4.51) with the first error term in Equation (4.45). The theorem follows by taking the limit as $T\to\infty$. □

As was the case for $\psi(x)$, the convergence of the right-hand side in the explicit formula for $\psi(x,\chi)$ is uniform in any interval not containing a prime power since $\psi(x,\chi)$ is continuous there. Moreover, there is an approximate formula for $\psi(x,\chi)$ as a corollary which holds for all Dirichlet characters:

Corollary 4.6.1. *Let χ be a Dirichlet character modulo $m>1$. Then for $2\le T\le x$,*

$$\psi_0(x,\chi) = -\frac{x^{\beta_\chi}}{\beta_\chi} - \sum_{|\gamma|<T}^{*}\frac{x^\rho}{\rho} + R(x,T,\chi),$$

where ρ runs over the nontrivial zeros of $L(s,\chi)$ counted with multiplicity and ordered with respect to the size of the ordinate, the $*$ in the sum indicates that we are excluding the terms corresponding to real zeros, the term corresponding to a Siegel zero β_χ is present only if $L(s,\chi)$ admits a Siegel zero, and

$$R(x,T,\chi) \ll \frac{x\log^2(xmT)}{T} + x^{1-\beta_\chi}\log(x) + \log(x)\min\left(1,\frac{x}{T\langle x\rangle}\right),$$

where $\langle x\rangle$ is the distance between x and the nearest prime power other than x if x itself is a prime power and again the term corresponding to a Siegel zero β_χ is present only if $L(s,\chi)$ admits a Siegel zero. Moreover, if x is an integer, we have the simplified estimate

$$R(x,T,\chi) \ll \frac{x\log^2(xmT)}{T} + x^{1-\beta_\chi}\log(x).$$

Proof. We first reduced to the case that χ is primitive. Let $\widetilde{\chi}$ be the primitive character inducing χ and denote its conductor by q. We estimate

$$\begin{aligned} |\psi_0(x,\chi) - \psi_0(x,\widetilde{\chi})| &\leq \sum_{\substack{n\leq x \\ (n,m)>1}} \Lambda(n) \\ &= \sum_{p|m} \sum_{\substack{v\geq 1 \\ p^v \leq x}} \log(p) \\ &\ll \log(x) \sum_{p|m} \log(p) \\ &\ll \log(x)\log(m) \\ &\ll \log^2(xm), \end{aligned} \tag{4.57}$$

where the third line holds because $p^v \leq x$ implies $v \leq \frac{\log(x)}{\log(p)}$ so that there are $O(\log(x))$ many terms in the inner sum and in the last line we have used the simple estimates $\log(x) \ll \log(xm)$ and $\log(m) \ll \log(xm)$. Therefore the difference between $\psi_0(x,\chi)$ and $\psi_0(x,\widetilde{\chi})$ is $O(\log^2(xm))$. Now for $2 \leq T \leq x$, we have $\log^2(xm) \ll \frac{x\log^2(xmT)}{T}$, which implies that the difference is absorbed into $O\left(\frac{x\log^2(xmT)}{T}\right)$ which is the first term in the error for $R(x,T,\chi)$. As $R(x,T,\widetilde{\chi}) \ll R(x,T,\chi)$ because $q \leq m$, and there are finitely many nontrivial zeros of $L(s,\chi)$ that are not nontrivial zeros of $L(s,\widetilde{\chi})$ (all occurring on the line $\sigma = 0$), it suffices to assume that χ is primitive. The claim will follow from estimating terms in Equations (4.55) and (4.56). We will estimate the constant $b(\chi)$ first. The formula for the negative logarithmic derivative in Proposition 3.6.1 applied to $L(s,\chi)$ at $s = 2$ implies

$$0 = -\frac{L'}{L}(2,\chi) - \frac{1}{2}\log(q) + \frac{1}{2}\log(\pi) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{2+\mathfrak{a}}{2}\right) + B(\chi) + \sum_{\rho}\left(\frac{1}{2-\rho} + \frac{1}{\rho}\right). \tag{4.58}$$

Adding Equation (4.58) to the formula for the negative logarithmic derivative in Proposition 3.6.1 applied to $L(s,\chi)$ gives

$$-\frac{L'}{L}(s,\chi) = -\frac{L'}{L}(2,\chi) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{2+\mathfrak{a}}{2}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) - \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2-\rho}\right).$$

As the first two terms are constant, we obtain a weaker estimate

$$-\frac{L'}{L}(s,\chi) = \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) - \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2-\rho}\right) + O(1).$$

If χ is odd, we set $s = 0$ since $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)$ does not have a pole there. If χ is even, we compare constant terms in the Laurent series using the Laurent series

$$\frac{L'}{L}(s,\chi) = \frac{1}{s} + b(\chi) + \cdots \quad \text{and} \quad \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) = \frac{1}{s} + b + \cdots,$$

for some constant b. In either case, our previous estimate gives

$$b(\chi) = -\sum_{\rho}\left(\frac{1}{\rho} + \frac{1}{2-\rho}\right) + O(1). \tag{4.59}$$

Let $\rho = \beta + i\gamma$. For all the terms with $|\gamma| > 1$, we estimate

$$\sum_{|\gamma|>1} \left(\frac{1}{\rho} + \frac{1}{2-\rho} \right) \leq \sum_{|\gamma|>1} \left| \frac{1}{\rho} + \frac{1}{2-\rho} \right| = \sum_{|\gamma|>1} \frac{2}{|\rho(2-\rho)|} \ll \sum_{|\gamma|>1} \frac{1}{|\rho|^2} \ll \log(q), \tag{4.60}$$

where the second to last estimate holds since $2 - \rho \gg \rho$ because β is bounded and the last estimate holds by the convergent sum in Proposition 3.6.1and Lemma 3.7.1 (ii) both applied to $L(s, \chi)$ (recall that the tail of a convergent series is bounded). For the terms corresponding to $2 - \rho$ with $|\gamma| \leq 1$, we have

$$\sum_{|\gamma|\leq 1} \frac{1}{2-\rho} \leq \sum_{|\gamma|\leq 1} \frac{1}{|2-\rho|} \ll \log(q), \tag{4.61}$$

where the last estimate holds by using Lemma 3.7.1 (ii) applied to $L(s, \chi)$ and because the nontrivial zeros are bounded away from 2. Combining Equations (4.59) to (4.61) yields

$$b(\chi) = -\sum_{|\gamma|\leq 1} \frac{1}{\rho} + O(\log(q)). \tag{4.62}$$

Inserting Equation (4.62) into Equations (4.55) and (4.56) and noting that $\tanh^{-1}(x^{-1})$ and $\frac{1}{2}\log(1 - x^{-2})$ are both bounded for $x \geq 2$ gives

$$\psi_0(x, \chi) = -\sum_{|\gamma|<T} \frac{x^\rho}{\rho} + \sum_{|\gamma|\leq 1} \frac{1}{\rho} + R'(x, T, \chi), \tag{4.63}$$

where

$$R'(x, T, \chi) \ll \frac{x \log^2(xqT)}{T} + \log(x) \min\left(1, \frac{x}{T\langle x \rangle}\right),$$

and we have absorbed the error in Equation (4.62) into $O\left(\frac{x\log^2(xqT)}{T}\right)$ because $2 \leq T \leq x$. Extracting the terms corresponding to the possible real zeros β_χ and $1 - \beta_\chi$ in Equation (4.63), we obtain

$$\psi_0(x, \chi) = -\sum_{|\gamma|<T}^{*} \frac{x^\rho}{\rho} + \sum_{|\gamma|\leq 1}^{*} \frac{1}{\rho} - \frac{x^{\beta_\chi} - 1}{\beta_\chi} - \frac{x^{1-\beta_\chi} - 1}{1 - \beta_\chi} + R'(x, T, \chi). \tag{4.64}$$

We now estimate some of the terms in Equation (4.64). For the second sum, we have $\rho \gg \frac{1}{\log(q)}$ since γ is bounded and $\beta < 1 - \frac{c}{\log(q|\gamma|)}$ for some positive constant c by Theorem 4.4.2. Thus

$$\sum_{|\gamma|\leq 1}^{*} \frac{1}{\rho} \ll \sum_{|\gamma|\leq 1}^{*} \log(q) \ll \log^2(q), \tag{4.65}$$

where the last estimate holds by Lemma 3.7.1 (ii) applied to $L(s, \chi)$. Similarly,

$$\frac{x^{1-\beta_\chi} - 1}{1 - \beta_\chi} \ll x^{1-\beta_\chi} \log(x), \tag{4.66}$$

because $\rho \gg \frac{1}{\log(q)}$ implies $1 - \beta_\chi \gg \frac{1}{\log(q)} \gg \frac{1}{\log(x)}$. Substituting Equations (4.65) and (4.66) into Equation (4.64) and noting that β_χ is bounded yields

$$\psi_0(x, \chi) = -\frac{x^{\beta_\chi}}{\beta_\chi} - \sum_{|\gamma|<T}^{*} \frac{x^\rho}{\rho} + R(x, T, \chi), \tag{4.67}$$

where

$$R(x,T,\chi) \ll \frac{x\log^2(xqT)}{T} + x^{1-\beta_\chi}\log(x) + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right)$$

and we have absorbed the error in Equation (4.65) into $O\left(\frac{x\log^2(xqT)}{T}\right)$ because because $2 \leq T \leq x$. If x is an integer then $\langle x\rangle \geq 1$ so that $\log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right) \leq \frac{x\log(x)}{T}$ and this term can be absorbed into $O\left(\frac{x\log^2(xqT)}{T}\right)$. □

We can now discuss the Siegel–Walfisz theorem. Let a and m be integers such that $m > 1$ and $(a,m) = 1$. The **prime counting function** $\pi(x : a, m)$ is defined by

$$\pi(x;a,m) = \sum_{\substack{p\leq x \\ p\equiv a \pmod m}} 1,$$

for $x > 0$. Equivalently, $\pi(x;a,m)$ counts the number of primes that no larger than x and are equivalent to a modulo m. This is the analog of $\pi(x)$ that is naturally associated to Dirichlet characters modulo m. Accordingly, there are asymptotics for $\pi(x;a,m)$ analogous to those for $\pi(x)$. To prove them, we will require an auxiliary function. The function $\psi(x;a,m)$ is defined by

$$\psi(x;a,m) = \sum_{\substack{n\leq x \\ n\equiv a \pmod m}} \Lambda(n),$$

for $x \geq 1$. This is just $\psi(x)$ restricted to only those terms equivalent to a modulo m. As for the asymptotic, the (classical) **Siegel–Walfisz theorem** is the first of them and the precise statement is the following:

Theorem (Siegel–Walfisz theorem, classical). *Let a and m be integers such that $m > 1$ and $(a,m) = 1$. For $x \geq 2$,*

$$\pi(x;a,m) \sim \frac{x}{\varphi(m)\log(x)}.$$

The (logarithmic integral) **Siegel–Walfisz theorem** is equivalent and sometimes more useful:

Theorem (Siegel–Walfisz theorem, logarithmic integral). *Let a and m be integers such that $m > 1$ and $(a,m) = 1$. For $x \geq 2$,*

$$\pi(x;a,m) \sim \frac{\mathrm{Li}(x)}{\varphi(m)}.$$

We will prove the (absolute error) **Siegel–Walfisz theorem** which is slightly stronger as it bounds the absolute error between $\pi(x;a,m)$ and $\frac{\mathrm{Li}(x)}{\varphi(m)}$:

Theorem (Siegel–Walfisz theorem, absolute error). *Let a and m be integers such that $m > 1$ and $(a,m) = 1$. For $N \geq 1$ and $x \geq 2$, there exists a positive constant c such that*

$$\pi(x;a,m) = \frac{\mathrm{Li}(x)}{\varphi(m)} + O\left(xe^{-c\sqrt{\log(x)}}\right),$$

provided $m \leq \log^N(x)$.

Proof. It suffices to assume x is an integer, because $\pi(x;a,m)$ can only change value at integers and the other functions in the statement are increasing. We being with the identity

$$\psi(x;a,m) = \frac{1}{\varphi(m)} \sum_{\chi \pmod m} \overline{\chi}(a)\psi(x,\chi), \tag{4.68}$$

which holds by the Dirichlet orthogonality relations (namely (ii)). Let $\widetilde{\chi}$ be the primitive character inducing χ. Then Equation (4.57) implies

$$\psi(x,\chi) = \psi(x,\widetilde{\chi}) + O(\log^2(xm)). \tag{4.69}$$

When $\chi = \chi_{m,0}$ we have $\psi(x,\widetilde{\chi}) = \psi(x)$, and as $\psi(x,\chi) \sim \psi_0(x,\chi)$, substituting Equation (4.30) into Equation (4.69) gives

$$\psi(x,\chi_{m,0}) = \psi(x) + O\left(xe^{-c\sqrt{\log(x)}} + \log^2(xm)\right), \tag{4.70}$$

for some positive constant c. Upon combining Equations (4.68) and (4.70), we obtain

$$\psi(x;a,m) = \frac{x}{\varphi(m)} + \frac{1}{\varphi(m)} \sum_{\substack{\chi \pmod m \\ \chi \neq \chi_{m,0}}} \overline{\chi}(a)\psi(x,\chi) + O\left(\frac{1}{\varphi(m)}\left(xe^{-c\sqrt{\log(x)}} + \log^2(xm)\right)\right). \tag{4.71}$$

We now prove

$$\psi(x,\chi) = -\frac{x^{\beta_\chi}}{\beta_\chi} + O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{4.72}$$

for some potentially different constant c, where $\chi \neq \chi_{m,0}$, and the term corresponding to β_χ appears if and only if $L(s,\chi)$ admits a Siegel zero. To accomplish this, we estimate the sum over the nontrivial zeros of $L(s,\chi)$ in Corollary 4.6.1. So fix a non-principal χ modulo m and let $\widetilde{\chi}$ be the primitive character inducing χ. Let $2 \leq T \leq x$ not coinciding with the ordinate of a nontrivial zero and let $\rho = \beta + i\gamma$ be a complex nontrivial zero of $L(s,\chi)$ with $|\gamma| < T$. By Theorem 4.4.2, all of the zeros ρ satisfy $\beta < 1 - \frac{c}{\log(mT)}$ for some possibly smaller c (recall that the nontrivial zeros of $L(s,\chi)$ that are not nontrivial zeros of $L(s,\widetilde{\chi})$ lie on the line $\sigma = 0$). It follows that

$$|x^\rho| = x^\beta < x^{1-\frac{c}{\log(mT)}} = xe^{-c\frac{\log(x)}{\log(mT)}}. \tag{4.73}$$

As $|\rho| > |\gamma|$, for those terms with $|\gamma| > 1$ (unlike the Riemann zeta function we do not have a positive lower bound for the first ordinate γ_1 of a nontrivial zero that is not real since Siegel zeros may exist), applying integration by parts gives

$$\sum_{1<|\gamma|<T} \frac{1}{\rho} \ll \sum_{1<|\gamma|<T} \frac{1}{\gamma} \ll \int_1^T \frac{dN(t,\widetilde{\chi})}{t} = \frac{N(T,\widetilde{\chi})}{T} + \int_1^T \frac{N(t,\widetilde{\chi})}{t^2}\,dt \ll \log^2(mT) \ll \log^2(xm), \tag{4.74}$$

where in the second to last estimate we have used that $N(t,\widetilde{\chi}) \ll t\log(qt) \ll t\log(mt)$ which follows from Corollary 3.7.1 applied to $L(s,\widetilde{\chi})$ and that $q \leq m$ and in the last estimate we have used the bound $T \leq x$. For the remaining terms with $|\gamma| \leq 1$ (that do not correspond to real nontrivial zeros), Equation (4.65) along with $q \leq m$ gives

$$\sum_{|\gamma|\leq 1}^{*} \frac{1}{\rho} \ll \log^2(m). \tag{4.75}$$

Combining Equations (4.72) to (4.75) yields

$$\sum_{|\gamma|<T}^{*} \frac{x^{\rho}}{\rho} \ll x \log^2(xm) e^{-c\frac{\log(x)}{\log(mT)}}, \tag{4.76}$$

where the $*$ in the sum indicates that we are excluding the terms corresponding to real zeros, and the error in Equation (4.75) has been absorbed by that in Equation (4.74). As $\psi(x,\chi) \sim \psi_0(x,\chi)$ and x is an integer, inserting Equation (4.76) into Corollary 4.6.1 results in

$$\psi(x,\chi) + \frac{x^{\beta_\chi}}{\beta_\chi} \ll x \log^2(xm) e^{-c\frac{\log(x)}{\log(mT)}} + \frac{x \log^2(xmT)}{T} + x^{1-\beta_\chi}\log(x), \tag{4.77}$$

where the terms corresponding to real zeros are present if and only if $L(s,\chi)$ admits a Siegel zero. We now let T be determined by $T = x$ for $2 \leq x < 3$ and

$$\log^2(T) = \log(x),$$

or equivalently,

$$T = e^{\sqrt{\log(x)}},$$

for $x \geq 3$. With this choice of T (note that if $x \geq 2$ then $2 \leq T \leq x$) and that $m \ll \log^N(x)$, we can estimate Equation (4.77) as follows:

$$\begin{aligned}
\psi(x) + \frac{x^{\beta_\chi}}{x} &\ll x(\log^2(x) + \log^2(m)) e^{-c\frac{\log(x)}{\log(m)+\sqrt{\log(x)}}} + x(\log^2(x) + \log^2(m) + \log(x)) e^{-\sqrt{\log(x)}} + x^{1-\beta_\chi}\log(x) \\
&\ll x(\log^2(x) + \log^2(m)) e^{-c\frac{\log(x)}{\log(m)+\sqrt{\log(x)}}} + x(\log^2(x) + \log^2(m) + \log(x)) e^{-\sqrt{\log(x)}} + x^{1-\beta_\chi}\log(x) \\
&\ll x(\log^2(x) + \log^2(m)) e^{-c\sqrt{\log(x)}} + x(\log^2(x) + \log^2(m)) e^{-\sqrt{\log(x)}} + x^{1-\beta_\chi}\log(x) \\
&\ll x\log^2(x) e^{-c\sqrt{\log(x)}} + x\log^2(x) e^{-\sqrt{\log(x)}} + x^{1-\beta_\chi}\log(x) \\
&\ll x\log^2(x) e^{-\min(1,c)\frac{\sqrt{\log(x)}}{\log(m)}},
\end{aligned}$$

where in the last estimate we have used that $x^{1-\beta_\chi} \leq x^{\frac{1}{2}}$ because $\beta_\chi \geq \frac{1}{2}$. As $\log(x) = o\left(e^{-\varepsilon\sqrt{\log(x)}}\right)$, we conclude that

$$\psi(x) + \frac{x^{\beta_\chi}}{x} \ll x e^{-c\sqrt{\log(x)}},$$

for some smaller c with $c < 1$. This is equivalent to Equation (4.72). Substituting Equation (4.72) into Equation (4.71) and noting that there is at most one Siegel zero for characters modulo m by Proposition 4.4.2, we arrive at

$$\psi(x;a,m) = \frac{x}{\varphi(m)} - \frac{\overline{\chi}_1(a)x^{\beta_\chi}}{\varphi(m)\beta_\chi} + O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{4.78}$$

where χ_1 is the single quadratic character modulo m such that $L(s,\chi_1)$ admits a Siegel zero if it exists and we have absorbed the second term in the error in Equation (4.71) into the first since $\log(x) = o\left(e^{-\varepsilon\sqrt{\log(x)}}\right)$ and $m \ll \log^N(x)$. Taking $\varepsilon = \frac{1}{2N}$ in Siegel's theorem, $m^{\frac{1}{2N}} \ll \sqrt{\log(x)}$ so that $\beta_\chi < 1 - \frac{c}{\sqrt{\log(x)}}$ for some potentially smaller constant c. It follows that m^{2N} Therefore

$$x^{\beta_\chi} < x^{1-\frac{c}{\sqrt{\log(x)}}} = xe^{-c\log(x)}. \tag{4.79}$$

Combining Equations (4.78) and (4.79) gives the simplified estimate

$$\psi(x;a,m) = \frac{x}{\varphi(m)} + O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{4.80}$$

for some potentially smaller constant c. Now let

$$\pi_1(x;a,m) = \sum_{\substack{n\leq x\\ n\equiv a \pmod m}} \frac{\Lambda(n)}{\log(n)}.$$

We can write $\pi_1(x;a,m)$ in terms of $\psi(x;a,m)$ as follows:

$$\begin{aligned}
\pi_1(x;a,m) &= \sum_{\substack{n\leq x\\ n\equiv a \pmod m}} \frac{\Lambda(n)}{\log(n)} \\
&= \sum_{\substack{n\leq x\\ n\equiv a \pmod m}} \Lambda(n)\int_n^x \frac{dt}{t\log^2(t)} + \frac{1}{\log(x)} \sum_{\substack{n\leq x\\ n\equiv a \pmod m}} \Lambda(n) \\
&= \int_2^x \sum_{\substack{n\leq x\\ n\equiv a \pmod m}} \Lambda(n)\frac{dt}{t\log^2(t)} + \frac{1}{\log(x)} \sum_{\substack{n\leq x\\ n\equiv a \pmod m}} \Lambda(n) \\
&= \int_2^x \frac{\psi(t;a,m)}{t\log^2(t)}\,dt + \frac{\psi(x;a,m)}{\log(x)}.
\end{aligned}$$

Applying Equation (4.80) to the last expression yields

$$\pi_1(x;a,m) = \int_2^x \frac{t}{\varphi(m)t\log^2(t)}\,dt + \frac{x}{\varphi(m)\log(x)} + O\left(\int_2^x \frac{e^{-c\sqrt{\log(t)}}}{\log^2(t)}\,dt + \frac{xe^{-c\sqrt{\log(x)}}}{\log(x)}\right). \tag{4.81}$$

Applying integrating by parts to the main term in Equation (4.81), we obtain

$$\int_2^x \frac{t}{\varphi(m)t\log^2(t)}\,dt + \frac{x}{\varphi(m)\log(x)} = \int_2^x \frac{dt}{\varphi(m)\log(t)} + \frac{2}{\varphi(m)\log(2)} = \frac{\mathrm{Li}(x)}{\varphi(m)} + \frac{2}{\varphi(m)\log(2)}. \tag{4.82}$$

As for the error term in Equation (4.81), we use Equation (4.37). Combining Equations (4.37), (4.81) and (4.82) yields

$$\pi_1(x;a,m) = \frac{\mathrm{Li}(x)}{\varphi(m)} + O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{4.83}$$

for some smaller c and where the constant in Equation (4.82) has been absorbed into the error term. As last we pass from $\pi_1(x;a,m)$ to $\pi(x;a,m)$. If p is a prime such that $p^m < x$, for some $m \geq 1$, then $p < x^{\frac{1}{2}} < x^{\frac{1}{3}} < \cdots < x^{\frac{1}{m}}$. Therefore

$$\pi_1(x;a,m) = \sum_{\substack{n\leq x\\ n\equiv a \pmod m}} \frac{\Lambda(n)}{\log(n)} = \sum_{\substack{p^m\leq x\\ p^m\equiv a \pmod m}} \frac{\log(p)}{m\log(p)} = \pi(x;a,m) + \frac{1}{2}\pi(x^{\frac{1}{2}};a,m) + \cdots. \tag{4.84}$$

Moreover, as $\pi(x^{\frac{1}{n}};a,m) < x^{\frac{1}{n}}$ for any $n \geq 1$, we see that $\pi(x;a,m) - \pi_1(x;a,m) = O(x^{\frac{1}{2}})$. This estimate together with Equations (4.83) and (4.84) gives

$$\pi(x) = \frac{\mathrm{Li}(x)}{\varphi(m)} + O\left(xe^{-c\sqrt{\log(x)}}\right),$$

because $x^{\frac{1}{2}} \ll xe^{-c\sqrt{\log(x)}}$. This completes the proof. □

It is interesting to note that the constant c in the Siegel-Walfisz theorem is ineffective because of the use of Siegel's theorem (it also depends upon N). This is unlike the prime number theorem, where the constant c can be made to be effective. The proof of the classical and logarithmic integral variants of the Siegel-Walfisz theorem are immediate:

Proof of Siegel-Walfisz theorem, classical and logarithmic integral. From the absolute error Siegel-Walfisz theorem, we have

$$\pi(x;a,m) = \frac{\mathrm{Li}(x)}{\varphi(m)}\left(1 + O\left(\frac{\varphi(m)xe^{-c\sqrt{\log(x)}}}{\mathrm{Li}(x)}\right)\right),$$

provided $m \ll \log^N(x)$ for some $N \geq 1$. As $\mathrm{Li}(x) \sim \frac{x}{\log(x)}$, we have

$$\frac{\varphi(m)xe^{-c\sqrt{\log(x)}}}{\mathrm{Li}(x)} \sim \varphi(m)\log(x)e^{-c\sqrt{\log(x)}} = o(1),$$

where the equality holds since $m \ll \log^N(x)$ and $\log(x) = o\left(e^{-\varepsilon\sqrt{\log(x)}}\right)$. The logarithmic integral Siegel-Walfisz theorem follows and then the classical Siegel-Walfisz theorem holds as well after using the asymptotic $\mathrm{Li}(x) \sim \frac{x}{\log(x)}$ again. □

Also, we have an optimal error term, in a much wider range of m, assuming the Riemann hypothesis for Dirichlet L-functions:

Proposition 4.6.1. *Let a and m be integers such that $m > 1$ and $(a,m) = 1$. For $x \geq 2$,*

$$\pi(x;a,m) = \frac{\mathrm{Li}(x)}{\varphi(m)} + O(\sqrt{x}\log(x)),$$

provided $m \leq x$ and the Riemann hypothesis for Dirichlet L-functions holds.

Proof. Let χ be a Dirichlet character modulo m. If ρ is a nontrivial zero of $L(s,\chi)$, the Riemann hypothesis for Dirichlet L-functions implies $|x^\rho| = \sqrt{x}$ and that Siegel zeros do not exist so we may merely assume $m \leq x$. Therefore as in the proof of the absolute error Siegel-Walfisz theorem,

$$\sum_{|\gamma|<T}\frac{x^\rho}{\rho} \ll \sqrt{x}\log^2(x),$$

for $2 \leq T \leq x$ not coinciding with the ordinate of a nontrivial zero. Repeating the same argument with T determined by $T = x$ for $2 \leq x < 3$ and

$$T^2 = x,$$

for $x \geq 3$ gives

$$\psi(x;a,m) = \frac{x}{\varphi(m)} + O(\sqrt{x}\log^2(x)),$$

and then transferring to $\pi_1(x)$ and finally $\pi(x)$ gives

$$\pi(x;a,m) = \frac{x}{\varphi(m)} + O(\sqrt{x}\log(x)).$$

□

Just like in the prime number theorem, an improvement in the zero-free region for Dirichlet L-functions will give an error term in between the absolute error Siegel-Walfisz theorem and Proposition 4.6.1. So again it is the strength of the zero-free region which controls the size of the error term.

Part III

Number Fields

Chapter 5

Algebraic Integers

Elementary number theory is done over $\mathbb{Q}$. The associated set of integers $\mathbb{Z}$ is a ring inside $\mathbb{Q}$. Moreover, the fundamental theorem of arithmetic tells us that prime factorization exists in $\mathbb{Z}$. That is, every integer is uniquely a product of primes (up to reordering of the factors). The study of number fields is concerned with finite extensions of $\mathbb{Q}$ where there might no longer be prime factorization. In the following, we introduce the primary tools in algebraic number theory. Namely, integrality, traces and norms, Dedekind domains, ramification, localization, and the discriminant and different.

5.1 Integrality

Let B/A be an extension of rings. We say that $b \in B$ is **integral** over A if β is the root of a monic polynomial $f(x) \in A[x]$. In other words, β satisfies

$$\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_0 = 0,$$

for some $n \geq 1$ and $\alpha_i \in A$ for $0 \leq i \leq n-1$. We say that B is **integral** over A if every element of B is integral over A. The following proposition shows that integral elements form a ring:

Proposition 5.1.1. *Let B/A be an extension of rings. Then the finitely many elements $\beta_1, \ldots, \beta_n \in B$ are all integral over A if and only if $A[\beta_1, \ldots, \beta_n]$ is a finitely generated A-module. In particular, the elements of B that are integral over A form a ring.*

Proof. First suppose $\beta \in B$ is integral over A. Then there exists a monic polynomial $f(x) \in A[x]$, of say degree $n \geq 1$, such that $f(\beta) = 0$. Now for any $g(x) \in A[x]$, Euclidean division implies

$$g(x) = q(x)f(x) + r(x),$$

with $q(x), r(x) \in A[x]$ and $\deg(r(x)) < n$. Letting

$$r(x) = \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + a_0,$$

with $\alpha_i \in \mathbb{Z}$ for $0 \leq i \leq n-1$, it follows that

$$g(\beta) = r(\beta) = \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0.$$

As $g(x)$ was arbitrary, we see that $1, \beta, \ldots, \beta^{n-1}$ generates $A[\beta]$ as an A-module. Now suppose $\beta_1, \ldots, \beta_n \in B$ are all integral over A. We will prove that $A[\beta_1, \ldots, \beta_n]$ is finitely generated as an A-module by induction.

Our previous work implies the base case. So assume by induction that $R = A[\beta_1, \ldots, \beta_{n-1}]$ is a finitely generated A-module. Then $R[\beta_n] = A[\beta_1, \ldots, \beta_n]$ is a finitely generated R-module and hence a finitely generated $\mathbb{Z}$-module as well by our induction hypothesis. This proves the forward implication of the first statement. For the reverse implication, suppose $A[\beta_1, \ldots, \beta_n]$ is a finitely generated A-module. Let $\omega_1, \ldots, \omega_r$ be a basis of $A[\beta_1, \ldots, \beta_n]$. Then for any $\beta \in A[\beta_1, \ldots, \beta_n]$, we have

$$\beta\omega_i = \sum_{1 \leq j \leq r} \alpha_{i,j}\omega_j,$$

with $\alpha_{i,j} \in A$ for $1 \leq i, j \leq r$. We can rewrite this as

$$(\beta - \alpha_{i,i})\omega_i - \sum_{\substack{1 \leq j \leq r \\ j \neq i}} \alpha_{i,j}\omega_j = 0,$$

for all i. These r equations are equivalent to the identity

$$\begin{pmatrix} \beta - \alpha_{1,1} & \alpha_{1,2} & \cdots & -\alpha_{1,r} \\ -\alpha_{2,1} & \beta - \alpha_{2,2} & & \\ \vdots & & \ddots & \\ -\alpha_{r,1} & & & \beta - \alpha_{r,r} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_r \end{pmatrix} = \mathbf{0}.$$

Thus the determinant of the matrix on the left-hand side must be zero. This shows that β is the root of the characteristic polynomial $\det(xI - (\alpha_{i,j})_{i,j})$ which is a monic polynomial with coefficients in A. Hence β is integral over A. As β was arbitrary, this shows that the elements $\beta_1, \ldots, \beta_n$ are all integral over A and that the sum and product of elements that are integral over A are also integral over A. This proves the reverse implication and the second statement completing the proof. □

Integrality is also transitive via the following statement:

Proposition 5.1.2. *Let $C/B/A$ be an extension of rings. If C is integral over B and B is integral over A then C is integral over A.*

Proof. Let $\gamma \in C$. Since C is integral over B, we have

$$\gamma^n + \beta_{n-1}\gamma^{n-1} + \cdots + \beta_0 = 0,$$

for some $n \geq 1$ and $\beta_i \in B$ for $0 \leq i \leq n-1$. Set $R = A[\beta_0, \ldots, \beta_{n-1}]$. Then $R[\gamma]$ is a finitely generated R-module and since B is integral over A, Proposition 5.1.1 implies that $R[\gamma]$ is also a finitely generated A-module. Thus γ is integral over A by Proposition 5.1.1 again. As γ was arbitrary, C is integral over A. □

In light of Proposition 5.1.2, we define the **integral closure** $\overline{A}$ of A in B by

$$\overline{A} = \{\beta \in B : \beta \text{ is integral over } A\}.$$

Clearly $A \subseteq \overline{A}$. Moreover, we say that A is **integrally closed** in B if $A = \overline{A}$. As $A \subseteq \overline{A} \subseteq \overline{\overline{A}}$, Proposition 5.1.2 implies that $\overline{A}$ is automatically integrally closed in B. Now suppose A is an integral domain with field of fractions K. Then we call the integral closure $\overline{A}$ of A in K the **normalization** of A and simply say that A is **integrally closed** if A is equal to its normalization. It turns out that every unique factorization domain is integrally closed:

Lemma 5.1.1. *Let A be a unique factorization domain with field of fractions K. Then A is integrally closed. In particular, every principal ideal domain is integrally closed.*

Proof. Since every principal ideal domain is a unique factorization domain, the second statement follows immediately from the first. To prove the first statement, let $\kappa \in K$ be such that

$$\kappa^n + \alpha_{n-1}\kappa^{n-1} + \cdots + \alpha_0 = 0,$$

for some $n \geq 1$ and $\alpha_i \in A$ for $0 \leq i \leq n-1$. Since A is a unique factorization domain, we may write $\kappa = \frac{\alpha}{\beta}$ for $\alpha, \beta \in A$ with β nonzero and $(\alpha, \beta) = 1$. Multiplying by β^n and isolating the leading term shows that

$$\alpha^n = -(\alpha_{n-1}\beta\alpha^{n-1} + \cdots + \alpha_0\beta^n).$$

As β divides the right-hand side it divides the left-hand side as well. But then $\beta \mid \alpha$ and hence β is a unit in A because $(\alpha, \beta) = 1$. This means $\kappa \in A$ and so A is integrally closed. □

Despite Lemma 5.1.1, we will often consider the following more general setting: A is an integrally closed integral domain with field of fractions K, L/K is a finite separable extension, and B is the integral closure of A in L. In this setting, the field of fractions of B has a simple description:

Proposition 5.1.3. *Let A be an integrally closed integral domain with field of fractions K, L/K be a finite separable extension, and B be the integral closure of A in L. Then every $\lambda \in L$ is of the form*

$$\lambda = \frac{\beta}{\alpha},$$

for some $\beta \in B$ and nonzero $\alpha \in A$. In particular, L is the field of fractions of B. Moreover, $\lambda \in L$ is integral over A if and only if the minimal polynomial $m_\lambda(x)$ of λ over K has coefficients in A.

Proof. As L/K is finite, it is necessarily algebraic so that any $\lambda \in L$ satisfies

$$\alpha\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0 = 0,$$

with $\alpha_i \in A$ for $0 \leq i \leq n-1$ and nonzero $\alpha \in A$. We claim that $\alpha\lambda$ is integral over A. Indeed, multiplying the previous identity by α^{n-1} yields

$$(\alpha\lambda)^n + \alpha'_{n-1}(\alpha\lambda)^{n-1} + \cdots + \alpha'_0 = 0,$$

where $\alpha'_i = \alpha_i\alpha^{n-1-i}$ for $0 \leq i \leq n-1$, and so $\alpha\lambda$ is the root of a monic polynomial with coefficients in A. Then $\alpha\lambda \in B$ and so $\alpha\lambda = \beta$ for some $\beta \in B$ which is equivalent to $\lambda = \frac{\beta}{\alpha}$. As $A \subseteq B$, this also implies that L is the field of fractions of B. For the last statement, suppose $\lambda \in L$. If the minimal polynomial $m_\lambda(x)$ of λ over K has coefficients in A then λ is automatically integral over A (since minimal polynomials are monic). So suppose λ is an integral over A so that λ is a root of a monic polynomial $f(x) \in A[x]$. Then $m_\lambda(x)$ divides $f(x)$ in $A[x]$ and thus all of the roots of $m_\lambda(x)$ are integral over A too. By Vieta's formulas, the coefficients of $m_\kappa(x)$ integral over A as well. But then $m_\kappa(x) \in A[x]$. This completes the proof. □

We are now ready to introduce number fields. A **number field** K is a finite extension of $\mathbb{Q}$. That is, K is a finite dimensional $\mathbb{Q}$-vector space. In particular, $K/\mathbb{Q}$ is a finite separable extension since $\mathbb{Q}$ is perfect so that the primitive element theorem applies. Moreover, $K/\mathbb{Q}$ is Galois if and only if it is normal. We say that the **degree** of K is $[K : \mathbb{Q}]$ which is simply the degree of K as a $\mathbb{Q}$-vector space. If K is of degree 2, 3, etc. then we say it is **quadratic**, **cubic**, etc. Any $\kappa \in K$ is called an **algebraic number**. We define the **ring of integers** $\mathcal{O}_K$ of K to be the integral closure of $\mathbb{Z}$ in K. In other words,

$$\mathcal{O}_K = \{\kappa \in K : \kappa \text{ is integral over } \mathbb{Z}\}.$$

Any $\alpha \in \mathcal{O}_K$ is called an **algebraic integer**. Then α is an algebraic integer if and only if it is the root of a monic polynomial $f(x) \in \mathbb{Z}[x]$.

Remark 5.1.1. *By Lemma 5.1.1, $\mathbb{Z}$ is integrally closed and therefore the ring of integers for the number field $\mathbb{Q}$ is exactly $\mathbb{Z}$.*

Let K be a number field of degree n. It follows from Proposition 5.1.3 that K is the field of fractions of $\mathcal{O}_K$ and every $\kappa \in K$ is of the form

$$\kappa = \frac{\alpha}{a},$$

for some $\alpha \in \mathcal{O}_K$ and nonzero $a \in \mathbb{Z}$. Moreover, $\kappa \in K$ is an algebraic integers if and only if the minimal polynomial $m_\kappa(x)$ of κ over $\mathbb{Q}$ has coefficients in $\mathbb{Z}$.

5.2 Traces and Norms

We will now introduce norms and traces of free modules. Let B/A be an extension of rings such that B is a free A-module of rank n. Then the **trace** and **norm** of B over A, denoted $\mathrm{Tr}_{B/A}$ and $\mathrm{N}_{B/A}$ respectively, are defined by

$$\mathrm{Tr}_{B/A}(\beta) = \mathrm{trace}(T_\beta) \quad \text{and} \quad \mathrm{N}_{B/A}(\beta) = \det(T_\beta),$$

for any $\beta \in B$, where $T_\beta : B \to B$ is the linear operator defined by

$$T_\beta(x) = \beta x,$$

for all $x \in B$. That is, T_β is the multiplication by β map. Letting $f_\beta(x)$ denote the characteristic polynomial of T_β, we have

$$f_\beta(x) = \det(xI - T_\beta) = x^n - \alpha_{n-1}x^{n-1} + \cdots + (-1)^n\alpha_0,$$

with $\alpha_i \in A$ for $0 \leq i \leq n-1$. Then the trace and the norm are given by

$$\mathrm{Tr}_{B/A}(\beta) = \alpha_{n-1} \quad \text{and} \quad \mathrm{N}_{B/A}(\beta) = \alpha_0, \tag{5.1}$$

and therefore take values in A. Moreover, we have

$$\mathrm{Tr}_{B/A}(\alpha\beta) = \alpha\,\mathrm{Tr}_{B/A}(\beta) \quad \text{and} \quad \mathrm{N}_{B/A}(\alpha\beta) = \alpha^n\,\mathrm{N}_{B/A}(\beta),$$

for all $\alpha \in A$ because $T_{\alpha\lambda} = \alpha T_\lambda$. Also note that $T_{\beta+\gamma} = T_\beta + T_\gamma$ and $T_{\beta\gamma} = T_\beta T_\gamma$ for all $\beta, \gamma \in B$. Moreover, recall that the trace and determinant of a linear map are additive and multiplicative respectively with respect to direct sums (since any matrix representation of the linear map becomes block upper triangular). Then if $B = B_1 \oplus B_2$ and $\beta = \beta_1 + \beta_2$ with $\beta_1 \in B_1$ and $\beta_2 \in B_2$, we have

$$\mathrm{Tr}_{B/A}(\beta) = \mathrm{Tr}_{B_1/A}(\beta_1) + \mathrm{Tr}_{B_2/A}(\beta_2) \quad \text{and} \quad \mathrm{N}_{B/A}(\beta) = \mathrm{N}_{B_1/A}(\beta_1)\,\mathrm{N}_{B_2/A}(\beta_2). \tag{5.2}$$

In the case of a degree n extension L/K, we call $\mathrm{Tr}_{L/K}$ and $\mathrm{N}_{L/K}$ the **trace** and **norm** of L/K. Moreover, $\mathrm{N}_{L/K}(\lambda) = 0$ if and only if $\lambda = 0$ because otherwise T_λ has inverse $T_{\lambda^{-1}}$ and hence nonzero determinant. Therefore we obtain homomorphisms

$$\mathrm{Tr}_{L/K} : L \to K \quad \text{and} \quad \mathrm{N}_{L/K} : L^* \to K^*.$$

When L/K is also separable, we can derive alternative descriptions of the trace and norm of L/K. This additional assumption is weak because we are mostly interested in finite extensions of $\mathbb{Q}$ and $\mathbb{F}_p$ which are always separable (because both $\mathbb{Q}$ and $\mathbb{F}_p$ are perfect). In any case, to do this we need to work in the algebraic closure $\overline{K}$ of K. As L/K is a degree n separable extension, there are exactly n distinct K-embeddings $\sigma_1, \ldots, \sigma_n$ of L into $\overline{K}$ (each given by letting θ be a primitive element for L/K so that $L = K(\theta)$ and sending θ to one of its conjugate roots in the minimal polynomial $m_\theta(x)$ of θ over K). In other words, there are n elements of $\mathrm{Hom}_K(L, \overline{K})$. Moreover, we prove the following proposition:

Proposition 5.2.1. *Let L/K be a degree n separable extension and let σ run over the elements of $\mathrm{Hom}_K(L, \overline{K})$. For any $\lambda \in L$, the characteristic polynomial $f_\lambda(x)$ of T_λ over K is a power of the minimal polynomial $m_\lambda(x)$ of λ over K and satisfies*

$$f_\lambda(x) = \prod_\sigma (x - \sigma(\lambda)).$$

We also have

$$\mathrm{Tr}_{L/K}(\lambda) = \sum_\sigma \sigma(\lambda) \quad \textit{and} \quad \mathrm{N}_{L/K}(\lambda) = \prod_\sigma \sigma(\lambda).$$

Moreover, if L/K is Galois and $\lambda_1, \ldots, \lambda_n$ are the conjugates of λ then

$$\mathrm{Tr}_{L/K}(\lambda) = \sum_{1 \le i \le n} \lambda_i \quad \textit{and} \quad \mathrm{N}_{L/K}(\lambda) = \prod_{1 \le i \le n} \lambda_i.$$

Proof. Write

$$m_\lambda(x) = x^m + \kappa_{m-1}x^{m-1} + \cdots + \kappa_0,$$

with $\kappa_i \in K$ for $0 \le i \le m-1$ (necessarily m is the degree of $K(\lambda)/K$) and let d be the degree of $L/K(\lambda)$. We first show that $f_\lambda(x)$ is a power of $m_\lambda(x)$. In particular, we claim

$$f_\lambda(x) = m_\lambda(x)^d.$$

To see this, recall that $1, \lambda, \ldots, \lambda^{n-1}$ is a basis of $K(\lambda)/K$. If $\alpha_1, \ldots, \alpha_d$ is a basis for $L/K(\lambda)$ then

$$\alpha_1, \alpha_1\lambda, \ldots, \alpha_1\lambda^{m-1}, \ldots, \alpha_d, \alpha_d\lambda, \ldots, \alpha_d\lambda^{m-1},$$

is a basis for L/K. Because $m_\lambda(x)$ gives the linear relation

$$\lambda^m = -\kappa_0 - k_1\lambda - \cdots - \kappa_{m-1}\lambda^{m-1},$$

the matrix of T_λ is block diagonal with d blocks each of the form

$$\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\kappa_0 & -\kappa_1 & \cdots & -\kappa_{m-1} \end{pmatrix}.$$

This is the companion matrix to $m_\lambda(x)$ and hence the characteristic polynomial is $m_\lambda(x)$ as well. Our claim follows since the matrix of T_λ is block diagonal. Since λ is algebraic over K of degree m, $K(\lambda)$ is the splitting field of $m_\lambda(x)$ and there are m elements of $\mathrm{Hom}_K(K(\lambda), \overline{K})$. Then the elements of $\mathrm{Hom}_K(L, \overline{K})$ are partitioned into m many equivalence classes each of size d (because $L/K(\lambda)$ is degree d) where σ and σ' are in the same class if and only if $\sigma(\lambda) = \sigma'(\lambda)$. In particular, a complete set of representatives is given by the τ. But then

$$f_\lambda(x) = m_\lambda(x)^d = \left(\prod_\tau (x - \tau(\lambda))\right)^d = \prod_\sigma (x - \sigma(\lambda)),$$

which proves the first statement. The formulas for the trace and norm follow from Vieta's formulas and Equation (5.1) which proves the second statement. Now suppose L/K is Galois. Then $\mathrm{Gal}(L/K) = \mathrm{Hom}_K(L, \overline{K})$. Therefore the conjugates of λ are exactly the images of λ under these K-embeddings and the last claim follows. □

As an application of Proposition 5.2.1, we can show how the field trace and field norm act when A is an integrally closed integral domain with field of fractions K, L/K is a finite separable extension, and B is the integral closure of A in L:

Proposition 5.2.2. *Let A be an integrally closed integral domain with field of fractions K, L/K be a finite separable extension, and B be the integral closure of A in L. If $\beta \in B$ then the trace and norm of β are in A.*

Proof. By Proposition 5.1.3, the minimal polynomial $m_\beta(x)$ of β over K has coefficients in A. By Proposition 5.2.1, the characteristic polynomial $f_\beta(x)$ is a power of $m_\beta(x)$ and hence $f_\beta(x)$ has coefficients in A too. From Equation (5.1) we conclude that the trace and norm of β are in A. □

We can also classify the units of B in terms of the units of A:

Proposition 5.2.3. *Let A be an integrally closed integral domain with field of fractions K, L/K be a finite separable extension, and B be the integral closure of A in L. Then $\beta \in B$ is a unit if and only if $\mathrm{N}_{L/K}(\beta) \in A$ is a unit*

Proof. First suppose $\beta \in B$ is a unit. Then $\frac{1}{\beta} \in B$ and so

$$\mathrm{N}_{L/K}(\beta)\,\mathrm{N}_{L/K}\left(\frac{1}{\beta}\right) = \mathrm{N}_{L/K}(1) = 1.$$

By Proposition 5.2.2, $\mathrm{N}_{L/K}(\beta), \mathrm{N}_{L/K}\left(\frac{1}{\beta}\right) \in A$ and hence $\mathrm{N}_{L/K}(\beta)$ is a unit. Now suppose $\mathrm{N}_{L/K}(\beta) \in A$ is a unit. By Proposition 5.1.3, the minimal polynomial $m_\beta(x)$ of β over K has coefficients in A. Moreover, Equation (5.1) and Proposition 5.2.1 together imply that the constant term is a unit because $\mathrm{N}_{L/K}(\beta)$ is. Letting the degree of $m_\beta(x)$ be m, we have shown that

$$m_\beta(x) = x^m + \alpha_{m-1}x^{m-1} + \cdots + \alpha,$$

with $\alpha_i \in A$ for $1 \leq i \leq m-1$ and $\alpha \in A$ a unit. Dividing $m_\beta(\beta)$ by β^m, we find that $\frac{1}{\beta}$ is a root of the polynomial

$$f(x) = \alpha x^m + \alpha_1 x^{m-1} + \cdots + 1.$$

Multiplying by $\frac{1}{\alpha}$, it follows that $\frac{1}{\beta}$ is a root of a monic polynomial with coefficients in A. Hence $\frac{1}{\beta} \in B$ and thus β is a unit. □

Having introduced traces and norms, we discuss discriminants of free modules. Let B/A be an extension of rings such that B is a free A-module of rank n. If $\beta_1, \ldots, \beta_n$ is a basis for B, we define its **trace matrix** $\mathrm{Tr}_{B/A}(\beta_1, \ldots, \beta_n)$ by

$$\mathrm{Tr}_{B/A}(\beta_1, \ldots, \beta_n) = \begin{pmatrix} \mathrm{Tr}_{B/A}(\beta_1\beta_1) & \cdots & \mathrm{Tr}_{B/A}(\beta_1\beta_n) \\ \vdots & & \vdots \\ \mathrm{Tr}_{B/A}(\beta_n\beta_1) & \cdots & \mathrm{Tr}_{B/A}(\beta_n\beta_n) \end{pmatrix}.$$

The trace matrix will play a prominent role in the theory. Most importantly is the **discriminant** $d_{B/A}(\beta_1, \ldots, \beta_n)$ of $\beta_1, \ldots, \beta_n$ defined by

$$d_{B/A}(\beta_1, \ldots, \beta_n) = \det(\mathrm{Tr}_{B/A}(\beta_1, \ldots, \beta_n)).$$

Equivalently, the discriminant is the determinant of the trace matrix. In particular, the discriminant $d_{B/A}(\beta_1, \ldots, \beta_n)$ is an element of A. It is also independent of the choice of basis up to elements of $(A^*)^2$. For if $\beta_1', \ldots, \beta_n'$ is another basis, we have

$$\beta_i' = \sum_{1 \leq j \leq m} \alpha_{i,j} \beta_j,$$

with $\alpha_{i,j} \in A$ for $1 \leq i, j \leq n$. Then $(\alpha_{i,j})_{i,j}$ is the base change matrix from $\beta_1, \ldots, \beta_n$ to $\beta_1', \ldots, \beta_n'$ and so has nonzero determinant. Thus $\det((\alpha_{i,j})_{i,j}) \in A^*$. Moreover,

$$\mathrm{Tr}_{B/A}(\beta_1', \ldots, \beta_n') = (\alpha_{i,j})_{i,j} \, \mathrm{Tr}_{B/A}(\beta_1, \ldots, \beta_n)(\alpha_{i,j})_{i,j}^t,$$

which, upon taking the determinant, shows that

$$d_{B/A}(\beta_1', \ldots, \beta_n') = \det((\alpha_{i,j})_{i,j})^2 d_{B/A}(\beta_1, \ldots, \beta_n), \tag{5.3}$$

as claimed. Accordingly, we define the **discriminant** $d_A(B)$ of B/A to be the class in $A/(A^*)^2$ represented by any discriminant $d_{B/A}(\beta_1, \ldots, \beta_n)$. In other words,

$$d_A(B) = d_{B/A}(\beta_1, \ldots, \beta_n)(A^*)^2.$$

This is well-defined by Equation (5.3). In particular, $d_A(B) = 0$ is independent of the choice of representative. The discriminant is also multiplicative with respect to direct sums:

Proposition 5.2.4. *Let B/A be an extension of rings such that B is a free A-module of rank n. Suppose we have a direct sum decomposition*

$$B = B_1 \oplus B_2,$$

for free A-modules B_1 and B_2 of ranks n_1 and n_2 respectively. Also let $\beta_{1,1}, \ldots, \beta_{n_1,1}$ and $\beta_{1,2}, \ldots, \beta_{n_2,2}$ be bases of B_1 and B_2 respectively. Then $\beta_{1,1}, \ldots, \beta_{n_1,1}, \beta_{1,2}, \ldots, \beta_{n_2,2}$ is a basis of B with

$$d_{B/A}(\beta_{1,1}, \ldots, \beta_{n_1,1}, \beta_{1,2}, \ldots, \beta_{n_2,2}) = d_{B_1/A}(\beta_{1,1}, \ldots, \beta_{n_1,1}) d_{B_2/A}(\beta_{1,2}, \ldots, \beta_{n_2,2}).$$

Proof. Clearly $\beta_{1,1}, \ldots, \beta_{n_1,1}, \beta_{1,2}, \ldots, \beta_{n_2,2}$ is a basis for B. We also have $\beta_{i,1}\beta_{j,2} = 0$ for all $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$. It follows that $d_{B/A}(\beta_{1,1}, \ldots, \beta_{n_1,1}, \beta_{1,2}, \ldots, \beta_{n_2,2})$ is the determinant of the block diagonal matrix

$$\begin{pmatrix} \mathrm{Tr}_{B/A}(\beta_{1,1}, \ldots, \beta_{n_1,1}) & \\ & \mathrm{Tr}_{B/A}(\beta_{1,2}, \ldots, \beta_{n_2,2}) \end{pmatrix}.$$

Moreover, Equation (5.2) implies

$$\mathrm{Tr}_{B/A}(\beta_1) = \mathrm{Tr}_{B_1/A}(\beta_1) \quad \text{and} \quad \mathrm{Tr}_{B/A}(\beta_2) = \mathrm{Tr}_{B_2/A}(\beta_2)$$

for any $\beta_1 \in B_1$ and $\beta_2 \in B_2$ since multiplication by β_1 and β_2 annihilate B_2 and B_1 respectively. But then the block diagonal matrix above is equal to

$$\begin{pmatrix} \mathrm{Tr}_{B_1/A}(\beta_{1,1}, \ldots, \beta_{n_1,1}) & \\ & \mathrm{Tr}_{B_2/A}(\beta_{1,2}, \ldots, \beta_{n_2,2}) \end{pmatrix}.$$

The determinant of this matrix is $d_{B_1/A}(\beta_{1,1}, \ldots, \beta_{n_1,1}) d_{B_2/A}(\beta_{1,2}, \ldots, \beta_{n_2,2})$ which completes the proof. □

We now specialize to the setting of a degree n separable extension L/K. In this case, it turns out that the discriminant of a basis is nonzero. To see this, we require a lemma:

Lemma 5.2.1. *Let L/K be a finite separable extension. Then the map*

$$\mathrm{Tr}_{L/K} : L \times L \to K \qquad (\lambda, \eta) \mapsto \mathrm{Tr}_{L/K}(\lambda\eta),$$

is a nondegenerate symmetric bilinear form.

Proof. From the definition of the trace, it is clear that the map is a symmetric bilinear form. To see that is is nondegenerate, suppose L/K is degree n. Then for any nonzero $\lambda \in L$, Proposition 5.2.1 implies that

$$\mathrm{Tr}_{L/K}(\lambda\lambda^{-1}) = \mathrm{Tr}_{L/K}(1) = n.$$

Hence the symmetric bilinear form is nondegenerate. □

We can now show that the discriminant of any basis for L/K never vanishes:

Proposition 5.2.5. *Let L/K be a degree n separable extension and let $\lambda_1, \ldots, \lambda_n$ be a basis for L/K. Then $d_{L/K}(\lambda_1, \ldots, \lambda_n) \neq 0$.*

Proof. Suppose by contradiction that $d_{L/K}(\lambda_1, \ldots, \lambda_n) = 0$. Then the matrix $\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n)$ is not invertible. Hence there exists $\kappa_i \in K$ for $1 \leq i \leq n$ such that

$$\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n) \begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_n \end{pmatrix} = \mathbf{0}.$$

This is equivalent to the n equations

$$\sum_{1 \leq j \leq n} \kappa_j \, \mathrm{Tr}_{L/K}(\lambda_i \lambda_j) = 0,$$

for all i. Setting

$$\lambda = \sum_{1 \leq j \leq n} \kappa_j \lambda_j,$$

linearity of the trace implies that these n equations are equivalent to the fact that $\mathrm{Tr}_{L/K}(\lambda\lambda_i) = 0$ for all i. As $\lambda_1, \ldots, \lambda_n$ is a basis for L/K, it follows that $\lambda \in L$ is a nonzero element for which $\mathrm{Tr}_{L/K}(\lambda\eta) = 0$ for all $\eta \in L$. This is impossible by Lemma 5.2.1. Hence $d_{L/K}(\lambda_1, \ldots, \lambda_n) \neq 0$ as desired. □

In addition to the discriminant $d_{L/K}(\lambda_1, \ldots, \lambda_n)$ never vanishing, we can also write it in an alternative form. To do this, we define the **embedding matrix** $M(\lambda_1, \ldots, \lambda_n)$ of the basis $\lambda_1, \ldots, \lambda_n$ by

$$M(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \sigma_1(\lambda_1) & \cdots & \sigma_1(\lambda_n) \\ \vdots & & \vdots \\ \sigma_n(\lambda_1) & \cdots & \sigma_n(\lambda_n) \end{pmatrix},$$

where $\sigma_1, \ldots, \sigma_n$ are the elements of $\mathrm{Hom}_K(L, \overline{K})$. Then we have the following result:

Proposition 5.2.6. *Let L/K be a degree n separable extension. Then for any basis $\lambda_1, \ldots, \lambda_n$ of L/K, we have*

$$d_{L/K}(\lambda_1, \ldots, \lambda_n) = \det(M(\lambda_1, \ldots, \lambda_n))^2.$$

Proof. Recalling that the (i,j)-entry of $M(\lambda_1,\dots,\lambda_n)^t M(\lambda_1,\dots,\lambda_n)$ is the dot product of the i-th and j-th columns of $M(\lambda_1,\dots,\lambda_n)$, we compute

$$\begin{aligned}
\det(M(\lambda_1,\dots,\lambda_n))^2 &= \det(M(\lambda_1,\dots,\lambda_n)^t M(\lambda_1,\dots,\lambda_n)) \\
&= \det\left(\left(\sum_{\sigma\in \mathrm{Hom}_K(L,\overline{K})} \sigma(\lambda_i)\sigma(\lambda_j)\right)_{i,j}\right) \\
&= \det\left(\left(\sum_{\sigma\in \mathrm{Hom}_K(L,\overline{K})} \sigma(\lambda_i\lambda_j)\right)_{i,j}\right) \\
&= \det(\mathrm{Tr}_{L/K}(\lambda_1,\dots,\lambda_n)) \\
&= d_{L/K}(\lambda_1,\dots,\lambda_n),
\end{aligned}$$

where the second to last equality follows by Proposition 5.2.1, as desired. □

When the degree n separable extension L/K admits a basis of the form $1,\lambda,\dots,\lambda^{n-1}$ (in fact such a basis always exists by choosing λ to be a primitive element for L/K) $d_{L/K}(1,\lambda,\dots,\lambda^{n-1})$ can be easily computed. Indeed, the embedding matrix becomes

$$M(1,\lambda,\dots,\lambda^{n-1}) = \begin{pmatrix} 1 & \sigma_1(\lambda) & \cdots & \sigma_1(\lambda)^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \sigma_n(\lambda) & \cdots & \sigma_n(\lambda)^{n-1} \end{pmatrix},$$

which is a Vandermonde matrix. Then

$$d_{L/K}(1,\lambda,\dots,\lambda^{n-1}) = \prod_{1\le i<j\le n} (\sigma_i(\lambda)-\sigma_j(\lambda))^2, \tag{5.4}$$

which is the square of the Vandermonde determinant of $M(1,\lambda,\dots,\lambda^{n-1})$, by Proposition 5.2.6. In any case, discriminants of bases are useful because of the following lemma:

Lemma 5.2.2. *Let A be an integrally closed integral domain with field of fractions K, L/K be a degree n separable extension, and B be the integral closure of A in L. If $\lambda_1,\dots,\lambda_n$ is a basis for L/K that is contained in B then*

$$d_{L/K}(\lambda_1,\dots,\lambda_n)B \subseteq A\lambda_1+\cdots+A\lambda_n.$$

Proof. Since $\lambda_1,\dots,\lambda_n$ is a basis, we may write $\beta=\kappa_1\lambda_1+\cdots+\kappa_n\lambda_n$ for any $\beta\in B$. Linearity of the trace implies

$$\sum_{1\le j\le n} \kappa_j \,\mathrm{Tr}_{L/K}(\lambda_i\lambda_j) = \mathrm{Tr}_{L/K}(\lambda_i\beta),$$

for $1\le i\le n$. These n equations are equivalent to the identity

$$\mathrm{Tr}_{L/K}(\lambda_1,\dots,\lambda_n)\begin{pmatrix}\kappa_1\\ \vdots \\ \kappa_n\end{pmatrix} = \begin{pmatrix}\mathrm{Tr}_{L/K}(\beta\lambda_1)\\ \vdots \\ \mathrm{Tr}_{L/K}(\beta\lambda_n)\end{pmatrix}.$$

Multiplying on the left by the adjugate $\mathrm{adj}(\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n))$ of $\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n)$ and recalling that a matrix times its adjugate is its determinant times the identity, we see that

$$d_{L/K}(\lambda_1, \ldots, \lambda_n) \begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_n \end{pmatrix} = \mathrm{adj}(\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n)) \begin{pmatrix} \mathrm{Tr}_{L/K}(\beta\lambda_1) \\ \vdots \\ \mathrm{Tr}_{L/K}(\beta\lambda_n) \end{pmatrix}.$$

Since $\lambda_1, \ldots, \lambda_n$ is a basis for L/K that is contained in B, the matrix $\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n)$ has entries in A by Proposition 5.2.2 and therefore $\mathrm{adj}(\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n))$ does too. Moreover, Proposition 5.2.2 again implies $\mathrm{Tr}_{L/K}(\beta\lambda_i) \in A$ for all i. So the right-hand side has entires in A and hence the left-hand side must as well. This means $d_{L/K}(\lambda_1, \ldots, \lambda_n)\kappa_i \in A$ for all i. But then

$$d_{L/K}(\lambda_1, \ldots, \lambda_n)B \subseteq A\lambda_1 + \cdots + A\lambda_n,$$

as desired. □

Again suppose A is an integrally closed integral domain with field of fractions K, L/K is a degree n separable extension, and B is the integral closure of A in L. We say that $\beta_1, \ldots, \beta_n$ is an **integral basis** for B/A if $\beta_1, \ldots, \beta_n$ is such that

$$B = A\beta_1 + \cdots + A\beta_n.$$

Equivalently, B is a free A-module of rank n. An integral basis is necessarily a basis for L/K by Proposition 5.1.3. However, an integral basis need not always exist. For if $\lambda_1, \ldots, \lambda_n$ is a basis of L/K, Proposition 5.1.3 implies that we can multiply by a nonzero element of A to ensure that this basis is contained in B. However, $\lambda_1, \ldots, \lambda_n$ need not also be a basis of B as an A-module. Nevertheless, if A is a principal ideal domain then we can ensure the existence of an integral basis:

Theorem 5.2.1. *Let A be a principal ideal domain with field of fractions K, let L/K be a degree n separable extension, and let B be the integral closure of A in L. Then B admits an integral basis over A. Moreover, every finitely generated nonzero B-submodule of L is a free A-module of rank n.*

Proof. A is integrally closed by Lemma 5.1.1. Let $\lambda_1, \ldots, \lambda_n$ be a basis for L/K. By Proposition 5.1.3 we may multiply by a nonzero element of A, if necessary, to ensure that this basis belongs to B. Then Lemma 5.2.2 implies

$$d_{L/K}(\lambda_1, \ldots, \lambda_n)B \subseteq A\lambda_1 + \cdots + A\lambda_n.$$

Since $A\lambda_1 + \cdots + A\lambda_n$ is a free A-module of rank n and A is a principal ideal domain, it follows from the structure theorem of finitely generated modules over principal ideal domains that B is also a free A-module of rank at most n. But any basis for B as an A-module must also be a basis for L/K by Proposition 5.1.3. Hence the rank is exactly n and B admits an integral basis over A which proves the first statement. Now suppose M is a nonzero B-submodule of L and let $\omega_1, \ldots, \omega_r$ be generators. By Proposition 5.1.3 again we may multiply by a nonzero element of A, if necessary, to ensure that these generators belong to B. But then

$$d_{L/K}(\omega_1, \ldots, \omega_n)M \subseteq d_{L/K}(\omega_1, \ldots, \omega_n)B.$$

By the structure theorem of finitely generated modules over principal ideal domains again, M is a free A-module of rank at most n. To see that the rank is at least n, let $\alpha \in M$ be nonzero and, as before, let $\lambda_1, \ldots, \lambda_n$ be a basis for L/K that is contained in B. Then $\alpha\lambda_1, \ldots, \alpha\lambda_n$ is a basis for L/K contained in M. Thus the rank of M is at least n, and in particular it must be n. This completes the proof. □

Recall that if L_1/K and L_2/K are finite separable extensions then the composite L of L_1 and L_2 is such that L/K is also a finite separable extension. Integral bases behave well with respect to composite fields provided the fields are linearly disjoint:

Proposition 5.2.7. *Let A be an integrally closed integral domain with field of fractions K, L_1/K and L_2/K be degree n_1 and n_2 separable extensions respectively, and B_1 and B_2 be the integral closures of A in L_1 and L_2 respectively. Suppose L_1 and L_2 are linearly disjoint over K in $\overline{K}$ and that B_1 and B_2 admit integral bases $\beta_{1,1}, \ldots, \beta_{n_1,1}$ and $\beta_{1,2}, \ldots, \beta_{n_2,2}$ over A with*

$$\alpha_1 d_{L_1/K}(\beta_{1,1}, \ldots, \beta_{n_1,1}) + \alpha_2 d_{L_2/K}(\beta_{1,2}, \ldots, \beta_{n_2,2}) = 1,$$

for some $\alpha_1, \alpha_2 \in A$. Let L be the composite of L_1 and L_2 and let B be the integral closure of A in L. Then $\beta_{1,1}\beta_{1,2}, \ldots, \beta_{n_1,1}\beta_{n_2,2}$ is an integral basis for B over A with

$$\delta_{L/K}(\beta_{1,1}\beta_{1,2}, \ldots, \beta_{n_1,1}\beta_{n_2,2}) = d_{L_1/K}(\beta_{1,1}, \ldots, \beta_{n_1,1})^{n_2} d_{L_2/K}(\beta_{1,2}, \ldots, \beta_{n_2,2})^{n_1}.$$

In particular,

$$B = B_1 B_2.$$

Proof. The last statement clearly follows from the first so we will start by proving that $\beta_{1,1}\beta_{1,2}, \ldots, \beta_{n_1,1}\beta_{n_2,2}$ is an integral basis for B over A. Since L_1 and L_2 are linearly disjoint over K in $\overline{K}$, it must be that $\beta_{1,1}\beta_{1,2}, \ldots, \beta_{n_1,1}\beta_{n_2,2}$ is a basis for L as a K-vector space. Therefore any $\lambda \in B$ is of the form

$$\lambda = \sum_{\substack{1 \le i \le n_1 \\ 1 \le j \le n_i}} \kappa_{i,j} \beta_{i,1} \beta_{j,2},$$

for some $\kappa_{i,j} \in K$ for $1 \le i \le n_1$ and $1 \le j \le n_2$. Then we need to show that $\kappa_{i,j} \in A$ for all i and j. Now let

$$\alpha_{i,2} = \sum_{1 \le j \le n_2} \kappa_{i,j} \beta_{j,2} \quad \text{and} \quad \alpha_{j,1} = \sum_{1 \le i \le n_1} \kappa_{i,j} \beta_{i,1},$$

for all i and j so that

$$\lambda = \sum_{1 \le i \le n_1} \alpha_{i,2} \beta_{i,1} \quad \text{and} \quad \lambda = \sum_{1 \le j \le n_2} \alpha_{j,1} \beta_{j,2}.$$

In particular, $\alpha_{i,2} \in L_2$ and $\alpha_{j,1} \in L_1$. By linear disjointness, we have

$$\mathrm{Hom}_K(L, \overline{K}) \cong \mathrm{Hom}_K(L_1, \overline{K}) \times \mathrm{Hom}_K(L_2, \overline{K}).$$

So letting $\sigma_{1,1}, \ldots, \sigma_{n_1,1}$ and $\sigma_{1,2}, \ldots, \sigma_{n_2,2}$ be the elements of $\mathrm{Hom}_K(L_1, \overline{K})$ and $\mathrm{Hom}_K(L_2, \overline{K})$ respectively, $\sigma_{1,1}\sigma_{1,2}, \ldots, \sigma_{n_1,1}\sigma_{n_2,2}$ are the elements of $\mathrm{Hom}_K(L, \overline{K})$. In particular, we may view $\sigma_{1,1}, \ldots, \sigma_{n_1,1}$ and $\sigma_{1,2}, \ldots, \sigma_{n_2,2}$ as elements of $\mathrm{Hom}_K(L, \overline{K})$ that act as the identity on L_2 and L_1 respectively. Then the n_1 and n_2 equations

$$\sum_{1 \le k \le n_1} \sigma_{i,1}(\beta_{k,1}) \alpha_{k,2} = \sigma_{i,1}(\lambda) \quad \text{and} \quad \sum_{1 \le k \le n_2} \sigma_{j,2}(\beta_{k,2}) \alpha_{k,1} = \sigma_{j,2}(\lambda),$$

respectively are equivalent to the identities

$$M(\beta_{1,1}, \ldots, \beta_{n_1,1}) \begin{pmatrix} \alpha_{1,2} \\ \vdots \\ \alpha_{n_1,2} \end{pmatrix} = \begin{pmatrix} \sigma_{1,1}(\lambda) \\ \vdots \\ \sigma_{n_1,1}(\lambda) \end{pmatrix} \quad \text{and} \quad M(\beta_{1,2}, \ldots, \beta_{n_2,2}) \begin{pmatrix} \alpha_{1,1} \\ \vdots \\ \alpha_{n_2,1} \end{pmatrix} = \begin{pmatrix} \sigma_{1,2}(\lambda) \\ \vdots \\ \sigma_{n_2,2}(\lambda) \end{pmatrix},$$

respectively. Multiplying on the left by the adjugates $\mathrm{adj}(M(\beta_{1,1},\dots,\beta_{n_1,1}))$ and $\mathrm{adj}(M(\beta_{1,2},\dots,\beta_{n_2,2}))$ of $M(\beta_{1,1},\dots,\beta_{n_1,1})$ and $M(\beta_{1,2},\dots,\beta_{n_2,2})$ respectively and recalling that a matrix times its adjugate is its determinant times the identity, we obtain

$$\det(M(\beta_{1,1},\dots,\beta_{n_1,1}))\begin{pmatrix}\alpha_{1,2}\\ \vdots \\ \alpha_{n_1,2}\end{pmatrix} = \mathrm{adj}(M(\beta_{1,1},\dots,\beta_{n_1,1}))\begin{pmatrix}\sigma_{1,1}(\lambda)\\ \vdots \\ \sigma_{n_1,1}(\lambda)\end{pmatrix},$$

and

$$\det(M(\beta_{1,2},\dots,\beta_{n_2,2}))\begin{pmatrix}\alpha_{1,1}\\ \vdots \\ \alpha_{n_2,1}\end{pmatrix} = \mathrm{adj}(M(\beta_{1,2},\dots,\beta_{n_2,2}))\begin{pmatrix}\sigma_{1,2}(\lambda)\\ \vdots \\ \sigma_{n_2,2}(\lambda)\end{pmatrix}.$$

Since $\lambda \in B$, we have $\sigma_{i,1}(\lambda) \in B$ and $\sigma_{j,2}(\beta) \in B$ for all i and j. Moreover, $M(\beta_{1,1},\dots,\beta_{n_1,1})$ and $M(\beta_{1,2},\dots,\beta_{n_2,2})$ have entries in B and thus $\mathrm{adj}(M(\beta_{1,1},\dots,\beta_{n_1,1}))$ and $\mathrm{adj}(M(\beta_{1,2},\dots,\beta_{n_2,2}))$ do too. This means the two right-hand sides have entries in B and so the two left-hand sides must as well. It follows from this fact and Proposition 5.2.6 that $d_{L_1/K}(\beta_{1,1},\dots,\beta_{n_1,1})\alpha_{i,2} \in B$ and $d_{L_2/K}(\beta_{1,2},\dots,\beta_{n_2,2})\alpha_{j,1} \in B$ for all i and j. But as $d_{L_1/K}(\beta_{1,1},\dots,\beta_{n_1,1})$ and $d_{L_2/K}(\beta_{1,2},\dots,\beta_{n_2,2})$ are in A, $\alpha_{i,2} \in L_2$, and $\alpha_{j,1} \in L_1$, we find that $d_{L_1/K}(\beta_{1,1},\dots,\beta_{n_1,1})\alpha_{i,2} \in B_2$ and $d_{L_2/K}(\beta_{1,2},\dots,\beta_{n_2,2})\alpha_{j,1} \in B_1$. Expanding $\alpha_{i,2}$ and $\alpha_{j,1}$ shows that $d_{L_1/K}(\beta_{1,1},\dots,\beta_{n_1,1})\kappa_{i,j} \in A$ and $d_{L_2/K}(\beta_{1,2},\dots,\beta_{n_2,2})\kappa_{i,j} \in A$. From the identity

$$k_{i,j} = k_{i,j}(\alpha_1 d_{L_1/K}(\beta_{1,1},\dots,\beta_{n_1,1}) + \alpha_2 d_{L_2/K}(\beta_{1,2},\dots,\beta_{n_2,2})),$$

we conclude $k_{i,j}$ is in A as desired. This proves $\beta_{1,1}\beta_{1,2},\dots,\beta_{n_1,1}\beta_{n_2,2}$ is an integral basis for B over A. We will now compute the discriminant of this basis. As $\sigma_{1,1}\sigma_{1,2},\dots,\sigma_{n_1,1}\sigma_{n_2,2}$ are the elements of $\mathrm{Hom}_K(L,\overline{K})$, we have

$$M(\beta_{1,1}\beta_{1,2},\dots,\beta_{n_1,1}\beta_{n_2,2}) = \begin{pmatrix}\sigma_{1,1}(\beta_{1,1})\sigma_{1,2}(\beta_{1,2}) & \cdots & \sigma_{1,1}(\beta_{n_1,1})\sigma_{1,2}(\beta_{n_2,2})\\ \vdots & & \vdots \\ \sigma_{n_1,1}(\beta_{1,1})\sigma_{n_2,2}(\beta_{1,2}) & \cdots & \sigma_{n_1,1}(\beta_{n_1,1})\sigma_{n_2,2}(\beta_{n_2,2})\end{pmatrix}.$$

This latter matrix admits the block factorization

$$\begin{pmatrix}M(\beta_{1,1},\dots,\beta_{n_1,1}) & & \\ & \ddots & \\ & & M(\beta_{1,1},\dots,\beta_{n_1,1})\end{pmatrix}\begin{pmatrix}I\sigma_{1,2}(\beta_{1,2}) & \cdots & I\sigma_{1,2}(\beta_{n_2,2})\\ \vdots & & \vdots \\ I\sigma_{n_2,2}(\beta_{1,2}) & \cdots & I\sigma_{n_2,2}(\beta_{n_2,2})\end{pmatrix},$$

where the second matrix is the Kronecker product $M(\beta_{1,2},\dots,\beta_{n_2,2}) \otimes I$. As $M(\beta_{1,2},\dots,\beta_{n_2,2}) \otimes I$ is $(I \otimes M(\beta_{1,2},\dots,\beta_{n_2,2}))P$ for some permutation matrix P, it takes the form

$$\begin{pmatrix}M(\beta_{1,2},\dots,\beta_{n_2,2}) & & \\ & \ddots & \\ & & M(\beta_{1,2},\dots,\beta_{n_2,2})\end{pmatrix}P,$$

with $\det(P) = \pm 1$. Putting these decompositions together and applying Proposition 5.2.6 shows

$$\delta_{L/K}(\beta_{1,1}\beta_{1,2},\dots,\beta_{n_1,1}\beta_{n_2,2}) = d_{L_1/K}(\beta_{1,1},\dots,\beta_{n_1,1})^{n_2} d_{L_2/K}(\beta_{1,2},\dots,\beta_{n_2,2})^{n_1}.$$

This completes the proof. □

It is generally very difficult to write down an integral basis explicitly. However, there is one instance in which this is possible. We say that B is **monogenic** over A if $B = A[\beta]$ for some $\beta \in B$. It follows immediately that $1, \beta, \ldots, \beta^{n-1}$ is an integral basis for B/A since B is of rank n. The discriminant of this basis is then given by Equation (5.4). In addition, the traces $\mathrm{Tr}_{L/K}$ and $\mathrm{Tr}_{B/A}$ and norms $\mathrm{N}_{L/K}$ and $\mathrm{N}_{B/A}$ agree:

Proposition 5.2.8. *Let A be an integrally closed integral domain with field of fractions K, L/K be a degree n separable extension, and B be the integral closure of A in L. If B admits an integral basis over A then*

$$\mathrm{Tr}_{L/K}(\beta) = \mathrm{Tr}_{B/A}(\beta) \quad \textit{and} \quad \mathrm{N}_{L/K}(\beta) = \mathrm{N}_{B/A}(\beta),$$

for all $\beta \in B$.

Proof. Let $\beta_1, \ldots, \beta_n$ be an integral basis for B/A. Then $\beta_1, \ldots, \beta_n$ is also a basis for L/K. It follows that the multiplication by β map in L has the same matrix representation as it does in B. Therefore

$$\mathrm{Tr}_{L/K}(\beta) = \mathrm{Tr}_{B/A}(\beta) \quad \text{and} \quad \mathrm{N}_{L/K}(\beta) = \mathrm{N}_{B/A}(\beta),$$

as desired. □

We now turn to the case of a number field K of degree n. We write $\mathrm{Tr}_K = \mathrm{Tr}_{K/\mathbb{Q}}$ and $\mathrm{N}_K = \mathrm{N}_{K/\mathbb{Q}}$ and call these the **field trace** and **field norm** of K respectively. Moreover, for any $\kappa \in K$ we call $\mathrm{Tr}_K(\kappa)$ and $\mathrm{N}_K(\kappa)$ the **trace** and **norm** of κ respectively. Then from Propositions 5.2.2 and 5.2.3 we see that the trace and norm of algebraic integers are themselves integers and $\kappa \in \mathcal{O}_K$ is a unit if and only if $\mathrm{N}_K(\kappa) = \pm 1$ (as these are the only units in $\mathbb{Z}$). Moreover, since $\mathbb{Z}$ is a principal ideal domain Theorem 5.2.1 implies that $\mathcal{O}_K$ admits an integral basis over $\mathbb{Z}$ of degree n. That is, $\mathcal{O}_K$ is a free abelian group of rank n. Accordingly, we say that $\alpha_1, \ldots, \alpha_n$ is an **integral basis** for K if it is an integral basis of $\mathcal{O}_K$ over $\mathbb{Z}$. Accordingly, we define the **discriminant** Δ_K of K to be the discriminant of $\mathcal{O}/\mathcal{o}$. As $(\mathbb{Z}^*)^2 = \{1\}$, Δ_K is a well-defined integer and satisfies

$$\Delta_K = d_{L/K}(\alpha_1, \ldots, \alpha_n),$$

for any integral basis $\alpha_1, \ldots, \alpha_n$ for K. Moreover, Δ_K is nonzero by Proposition 5.2.2 and Lemma 5.2.1 and may very well be negative. In light of Proposition 5.2.6, we also have

$$|\det(M(\alpha_1, \ldots, \alpha_n))| = \sqrt{|\Delta_K|}.$$

Lastly, we say K is **monogenic** if $\mathcal{O}_K$ is monogenic over $\mathbb{Z}$.

5.3 Dedekind Domains

Let $\mathcal{o}$ be a noetherian integral domain and denote its field of fractions by K. Any nonzero ideal $\mathfrak{a}$ of $\mathcal{o}$ is said to be an **integral ideal** of $\mathcal{o}$. We call any prime integral ideal $\mathfrak{p}$ of $\mathcal{o}$ a **prime** of $\mathcal{o}$ and if $\mathfrak{p}$ is principal with $\mathfrak{p} = \alpha\mathcal{o}$, with $\alpha \in K$ nonzero, we will simply refer to α as the prime instead of $\mathfrak{p}$. In any case, an integral ideal $\mathfrak{a}$ is just a $\mathcal{o}$-submodule of $\mathcal{o}$. Moreover, it is finitely generated since $\mathcal{o}$ is noetherian and is therefore a finitely generated $\mathcal{o}$-submodule of K. More generally, we say $\mathfrak{f}$ is a **fractional ideal** of $\mathcal{o}$ if $\mathfrak{f}$ a nonzero finitely generated $\mathcal{o}$-submodule of K. Moreover, we say that a fractional ideal is **principal** if it is generated by a single element. That is, if $\mathfrak{f} = \kappa\mathcal{o}$ for some nonzero $\kappa \in K$. In particular, all integral ideals are fractional ideals and all principal integral ideals are principal fractional ideals. Now let $\kappa_1, \ldots, \kappa_r \in K$ be generators for the fractional ideal $\mathfrak{f}$. Since K is the field of fractions of $\mathcal{o}$, $\kappa_i = \frac{\alpha_i}{\delta_i}$ with $\alpha_i, \delta_i \in \mathcal{o}$ and

where δ_i is nonzero for $1 \leq i \leq r$. Setting $\delta = \delta_1 \cdots \delta_r$, we have that $\delta\kappa_i \in \mathcal{O}$ for all i and hence $\delta\mathfrak{f}$ is an integral ideal. Conversely, if there exists some nonzero $\delta \in \mathcal{O}$ such that $\delta\mathfrak{f} = \mathfrak{a}$ is an integral ideal then $\mathfrak{f}$ is a fractional ideal because $\mathfrak{a}$ is a finitely generated $\mathcal{O}$-submodule of K and hence $\mathfrak{f}$ is too. Thus for any fractional ideal $\mathfrak{f}$, there exists a nonzero $\delta \in \mathcal{O}$ and integral ideal $\mathfrak{a}$ such that

$$\mathfrak{f} = \frac{1}{\delta}\mathfrak{a}.$$

Every fractional ideal is of this form, and integral ideals are precisely those for which $\delta = 1$, and fractional ideals need not be a subgroup of $\mathcal{O}$. We will be able to show that an fractional ideal of $\mathcal{O}$ factors into a product of primes provided that $\mathcal{O}$ satisfies a few more restrictive conditions. Accordingly, we say that a ring is a **Dedekind domain** if it satisfies the following properties:

(i) It is an integrally closed integral domain.

(ii) It is noetherian.

(iii) Every nonzero prime ideal is maximal.

In other words, a noetherian integral domain is a Dedekind domain if and only if it is integrally closed and every nonzero prime is maximal.

Remark 5.3.1. *$\mathbb{Z}$ is an integrally closed integral domain by Lemma 5.1.1. It is also noetherian since $\mathbb{Z}$ is a principal ideal domain. Every prime is of the form $p\mathbb{Z}$ for some prime p and therefore is maximal because $\mathbb{Z}/p\mathbb{Z}$ is a field. It follows that $\mathbb{Z}$ is a Dedekind domain and the primes of $\mathbb{Z}$ are exactly the primes p.*

With this setup, property (ii) can be rephrased as saying that every integral ideal is finitely generated while property (iii) is equivalent to the fact that every prime is maximal. One should think of Dedekind domains as generalizations of $\mathbb{Z}$. Note that we have not assumed $\mathcal{O}$ is a principal ideal domain. In fact, the most interesting Dedekind domains are not principal ideal domains. Our primary concern regarding Dedekind domains will be to show that, while there need not be prime factorization of elements, fractional ideals admit a factorization into a product of primes. We first show containment in one direction for integral ideals:

Lemma 5.3.1. *Let $\mathcal{O}$ be a Dedekind domain. For every integral ideal $\mathfrak{a}$, there exist primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ such that*

$$\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq \mathfrak{a}.$$

Proof. Let $\mathcal{S}$ be the set of integral ideals which do not contain a product of prime integral. Then it suffices to show $\mathcal{S}$ is empty. Assuming otherwise and ordering $\mathcal{S}$ by inclusion, the fact that $\mathcal{O}$ is noetherian implies that there exists a maximal integral ideal $\mathfrak{a} \in \mathcal{S}$. Moreover $\mathfrak{a}$ cannot be prime for otherwise $\mathfrak{a}$ contains a product of primes (namely itself). Since $\mathfrak{a}$ is not prime, there exist $\alpha_1, \alpha_2 \in \mathcal{O}$ with $\alpha_1\alpha_2 \in \mathfrak{a}$ and such that $\alpha_1, \alpha_2 \notin \mathfrak{a}$. Now define integral ideals

$$\mathfrak{b}_1 = \mathfrak{a} + \alpha_1\mathcal{O} \quad \text{and} \quad \mathfrak{b}_2 = \mathfrak{a} + \alpha_2\mathcal{O}.$$

Note that $\mathfrak{b}_1$ and $\mathfrak{b}_2$ strictly contain $\mathfrak{a}$ because $\alpha_1, \alpha_2 \notin \mathfrak{a}$. Moreover, $\mathfrak{b}_1\mathfrak{b}_2 \subseteq \mathfrak{a}$ because

$$\mathfrak{b}_1\mathfrak{b}_2 = (\mathfrak{a} + \alpha_1\mathcal{O})(\mathfrak{a} + \alpha_2\mathcal{O}) = \mathfrak{a}^2 + \alpha_1\mathcal{O} + \alpha_2\mathcal{O} + \alpha_1\alpha_2\mathcal{O},$$

and $\alpha_1\alpha_2 \in \mathfrak{a}$. Maximality of $\mathfrak{a}$ implies that there exist primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_\ell$ such that

$$\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq \mathfrak{b}_1 \quad \text{and} \quad \mathfrak{q}_1 \cdots \mathfrak{q}_\ell \subseteq \mathfrak{b}_2.$$

But then

$$\mathfrak{p}_1 \cdots \mathfrak{p}_k \mathfrak{q}_1 \cdots \mathfrak{q}_\ell \subseteq \beta_1\beta_2 \subseteq \mathfrak{a},$$

which contradicts the fact that $\mathfrak{a} \in \mathcal{S}$. Hence $\mathcal{S}$ is empty as desired. □

In order to obtain the reverse containment in Lemma 5.3.1, we need to do more work. Precisely, we want to show that every integral ideal factors into a product of primes. Let $\mathfrak{p}$ be a prime. We define $\mathfrak{p}^{-1}$ by

$$\mathfrak{p}^{-1} = \{\kappa \in K : \kappa\mathfrak{p} \subseteq \mathcal{O}\}.$$

It turns out that $\mathfrak{p}^{-1}$ is a fractional ideal. Indeed, since $\mathfrak{p}$ is an integral ideal there exists a nonzero $\alpha \in \mathfrak{p}$. By definition of $\mathfrak{p}^{-1}$, we have that $\alpha\mathfrak{p}^{-1} \subseteq \mathcal{O}$. Hence $\alpha\mathfrak{p}^{-1}$ is an integral ideal and therefore $\mathfrak{p}^{-1}$ is a fractional ideal. Unlike integral ideals, $1 \in \mathfrak{p}^{-1}$ so that $\mathfrak{p}^{-1}$ contains units. The following proposition proves a stronger version of this and more:

Lemma 5.3.2. *Let $\mathcal{O}$ be a Dedekind domain and $\mathfrak{p}$ be a prime. Then the following hold:*

(i)

$$\mathcal{O} \subset \mathfrak{p}^{-1}.$$

(ii)

$$\mathfrak{p}^{-1}\mathfrak{p} = \mathcal{O}.$$

Proof. We will prove the latter two statement separately:

(i) Clearly $\mathcal{O} \subseteq \mathfrak{p}^{-1}$ so it suffices to show that $\mathfrak{p}^{-1} - \mathcal{O}$ is nonempty. To this end, let $\alpha \in \mathfrak{p}$ be nonzero. By Lemma 5.3.1 let $k \geq 1$ be the minimal integer such that there exist primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ with

$$\mathfrak{p}_1 \cdots \mathfrak{p}_k \subseteq \alpha\mathcal{O}.$$

As $\alpha \in \mathfrak{p}$, we have $\alpha\mathcal{O} \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, there must be some i with $1 \leq i \leq k$ such that $\mathfrak{p}_i \subseteq \mathfrak{p}$. Without loss of generality, we may assume $\mathfrak{p}_1 \subseteq \mathfrak{p}$. As primes are maximal since $\mathcal{O}$ is noetherian, we conclude $\mathfrak{p}_1 = \mathfrak{p}$. Moreover, since k is minimal we must have

$$\mathfrak{p}_2 \cdots \mathfrak{p}_k \not\subseteq \alpha\mathcal{O}.$$

Hence there exists $\beta \in \mathfrak{p}_2 \cdots \mathfrak{p}_k$ with $\beta \notin \alpha\mathcal{O}$. We will now show that $\beta\alpha^{-1}$ is an element in $\mathfrak{p}^{-1} - \mathcal{O}$. Since $\mathfrak{p}_1 = \mathfrak{p}$, what we have previously shown implies $\beta\mathfrak{p} \subseteq \alpha\mathcal{O}$ and hence $\beta\alpha^{-1}\mathfrak{p} \in \mathcal{O}$ which means $\beta\alpha^{-1} \in \mathfrak{p}^{-1}$. But as $\beta \notin \alpha\mathcal{O}$, we also have $\beta\alpha^{-1} \notin \mathcal{O}$. Hence $\beta\alpha^{-1} \in \mathfrak{p}^{-1} - \mathcal{O}$ which proves (i).

(ii) By (i) and the definition of $\mathfrak{p}^{-1}$, we have $\mathfrak{p} \subseteq \mathfrak{p}^{-1}\mathfrak{p} \subseteq \mathcal{O}$. Since $\mathfrak{p}$ is maximal because $\mathcal{O}$ is noetherian, it follows that $\mathfrak{p}^{-1}\mathfrak{p}$ is either $\mathfrak{p}$ or $\mathcal{O}$. So it suffices to show that the first case cannot hold. Assume by contradiction that $\mathfrak{p}^{-1}\mathfrak{p} = \mathfrak{p}$. Let $\omega_1, \ldots, \omega_r$ generate $\mathfrak{p}$ and let $\alpha \in \mathfrak{p}^{-1} - \mathcal{O}$ which exists by (i). Then $\alpha\omega_i \in \mathfrak{p}^{-1}\mathfrak{p}$ for $1 \leq i \leq r$ and hence $\alpha\mathfrak{p} \subseteq \mathfrak{p}^{-1}\mathfrak{p}$. By our assumption, this further implies that $\alpha\mathfrak{p} \subseteq \mathfrak{p}$. But then

$$\alpha\omega_i = \sum_{1 \leq j \leq r} \alpha_{i,j}\omega_j,$$

with $\alpha_{i,j} \in \mathcal{O}$ for $1 \leq i, j \leq r$. We can rewrite this as,

$$(\alpha - \alpha_{i,i})\omega_i - \sum_{\substack{1 \leq j \leq r \\ j \neq i}} \alpha_{i,j}\omega_j = 0,$$

for all i. These r equations are equivalent to the identity

$$\begin{pmatrix} \alpha - \alpha_{1,1} & \alpha_{1,2} & \cdots & -\alpha_{1,r} \\ -\alpha_{2,1} & \alpha - \alpha_{2,2} & & \\ \vdots & & \ddots & \\ -\alpha_{r,1} & & & \alpha - \alpha_{r,r} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_r \end{pmatrix} = \mathbf{0}.$$

Thus the determinant of the matrix on the left-hand side must be zero. But this means α is a root of the characteristic polynomial $\det(xI - (\alpha_{i,j}))$ which is a monic polynomial with coefficients $\mathcal{O}$. As $\mathcal{O}$ is integrally closed, $\alpha \in \mathcal{O}$ which is a contraction. Thus $\mathfrak{p}^{-1}\mathfrak{p} = \mathcal{O}$ proving (ii). □

We can now show that every integral ideal factors uniquely into a product of primes (up to reordering of the factors):

Theorem 5.3.1. *Let $\mathcal{O}$ be a Dedekind domain. Then for every integral ideal $\mathfrak{a}$ there exist primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ such that $\mathfrak{a}$ factors as*

$$\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_k.$$

Moreover, this factorization is unique up to reordering of the factors.

Proof. We first prove existence and then uniqueness. For existence, let $\mathcal{S}$ be the set of integral ideals that are not a product of primes. We will show that $\mathcal{S}$ is empty. Assuming otherwise and ordering $\mathcal{S}$ by inclusion, the fact that $\mathcal{O}$ is noetherian implies that there exists a maximal integral ideal $\mathfrak{a} \in \mathcal{S}$. Necessarily $\mathfrak{a}$ is not prime and since primes are maximal in $\mathcal{O}$, there is some prime $\mathfrak{p}_1$ for which $\mathfrak{a} \subset \mathfrak{p}_1$. Then by Lemma 5.3.2 (ii), we have $\mathfrak{p}_1^{-1}\mathfrak{a} \subset \mathcal{O}$ so that $\mathfrak{p}_1^{-1}\mathfrak{a}$ is also an integral ideal. Also, Lemma 5.3.2 (i) implies that $\mathfrak{a} \subset \mathfrak{a}\mathfrak{p}_1^{-1}$. By maximality of $\mathfrak{a}$, $\mathfrak{a}\mathfrak{p}_1^{-1}$ factors into a product of primes. That is, there exist primes $\mathfrak{p}_2, \ldots, \mathfrak{p}_k$ such that

$$\mathfrak{a}\mathfrak{p}_1^{-1} = \mathfrak{p}_2, \ldots, \mathfrak{p}_k.$$

Hence

$$\mathfrak{a} = \mathfrak{p}_1, \ldots, \mathfrak{p}_k,$$

so that $\mathfrak{a}$ factors into a product of primes which contradicts the fact that $\mathfrak{a} \in \mathcal{S}$. Hence $\mathcal{S}$ is empty thus proving the existence of such a factorization. Now we prove uniqueness. Suppose that $\mathfrak{a}$ admits factorizations

$$\mathfrak{a} = \mathfrak{p}_1, \ldots, \mathfrak{p}_k \quad \text{and} \quad \mathfrak{a} = \mathfrak{q}_1, \ldots, \mathfrak{q}_\ell,$$

for primes $\mathfrak{p}_i$ and $\mathfrak{q}_j$ with $1 \leq i \leq k$ and $1 \leq j \leq \ell$. Since $\mathfrak{p}_1$ is prime, there is some j for which $\mathfrak{q}_j \subseteq \mathfrak{p}_1$. Without loss of generality, we may assume $\mathfrak{q}_1 \subseteq \mathfrak{p}_1$ and since primes are maximal in $\mathcal{O}$ we have $\mathfrak{q}_1 = \mathfrak{p}_1$. Then

$$\mathfrak{p}_2, \ldots, \mathfrak{p}_k = \mathfrak{q}_2, \ldots, \mathfrak{q}_\ell.$$

Repeating this process, we see that $k = \ell$ and $\mathfrak{q}_i = \mathfrak{p}_i$ for all i. This proves uniqueness of the factorization.□

As a near immediate corollary of Theorem 5.3.1, all fractional ideal admits a factorization into a product of primes and their inverses (up to reordering of the factors):

Corollary 5.3.1. *Let $\mathcal{O}$ be a Dedekind domain. Then for every fractional ideal $\mathfrak{f}$ there exist primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_\ell$ such that $\mathfrak{f}$ factors as*

$$\mathfrak{f} = \mathfrak{p}_1 \cdots \mathfrak{p}_k \mathfrak{q}_1^{-1}, \ldots, \mathfrak{q}_\ell^{-1}.$$

Moreover, this factorization is unique up to reordering of the factors.

Proof. If $\mathfrak{f}$ is a fractional ideal then there exists a nonzero $\delta \in \mathcal{O}$ and integral ideal $\mathfrak{a}$ such that $\mathfrak{f} = \frac{1}{\delta}\mathfrak{a}$. In particular, $\mathfrak{a}$ and $\delta\mathcal{O}$ are integral ideals such that $\delta\mathcal{O}\mathfrak{f} = \mathfrak{a}$. By Theorem 5.3.1, $\mathfrak{a}$ and $\delta\mathcal{O}$ admit unique factorizations

$$\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_k \quad \text{and} \quad \delta\mathcal{O} = \mathfrak{q}_1, \ldots, \mathfrak{q}_\ell,$$

for some primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_\ell$ up to reordering of the factors. Hence

$$\mathfrak{q}_1, \ldots, \mathfrak{q}_\ell \mathfrak{f} = \mathfrak{p}_1 \cdots \mathfrak{p}_k,$$

which is equivalent to the factorization for $\mathfrak{f}$. □

By Corollary 5.3.1, for any fractional ideal $\mathfrak{f}$ there exist distinct prime $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ such that $\mathfrak{f}$ admits a factorization

$$\mathfrak{f} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r},$$

with $e_i \in \mathbb{Z} - \{0\}$ for all i, called the **prime factorization** of $\mathfrak{f}$ with **prime factors** $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. In particular, the prime factorization of an integral ideal $\mathfrak{a}$ is of the form

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r},$$

with $e_i \geq 1$ for all i. Accordingly, for any two integral ideal $\mathfrak{a}$ and $\mathfrak{b}$ we say that $\mathfrak{a}$ **divides** $\mathfrak{b}$ and write $\mathfrak{a} \mid \mathfrak{b}$ if $\mathfrak{b} \subseteq \mathfrak{a}$. Sometimes this condition is expressed as *to contain is to divide* or *to divide is to contain*. By the prime factorization of fractional ideals, this is equivalent to the fact that every prime power factor of $\mathfrak{a}$ appears in the prime factorization of $\mathfrak{b}$. We also say that $\mathfrak{a}$ **exactly divides** $\mathfrak{b}$ and write $\mathfrak{a} \mid\mid \mathfrak{b}$ if $\mathfrak{a}$ divides $\mathfrak{b}$ but no power of $\mathfrak{a}$ divides $\mathfrak{b}$. This is equivalent to the fact that every prime power factor of $\mathfrak{a}$ appears in the prime factorization of $\mathfrak{b}$ but not for any power of $\mathfrak{a}$. In the case of a prime $\mathfrak{p}$ and an integral ideal $\mathfrak{a}$, $\mathfrak{p} \mid \mathfrak{a}$ if and only if $\mathfrak{p}$ is a prime factor of $\mathfrak{a}$ and $\mathfrak{p}^e \mid\mid \mathfrak{a}$, for some $e \geq 1$, if and only if $\mathfrak{p}^e$ is exactly the power of $\mathfrak{p}$ appearing in the prime factorization of $\mathfrak{a}$. Moreover, if $\mathfrak{a} \mid \mathfrak{p}$ then $\mathfrak{a} = \mathfrak{p}$. The **greatest common divisor** $(\mathfrak{a}, \mathfrak{b})$ of $\mathfrak{a}$ and $\mathfrak{b}$ is defined to be the integral ideal that all other common integral ideal divisors divide. Since to divide is to contain, $(\mathfrak{a}, \mathfrak{b})$ is the smallest ideal that contains both $\mathfrak{a}$ and $\mathfrak{b}$. This is $\mathfrak{a} + \mathfrak{b}$ and so $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}$. The **least common multiple** $[\mathfrak{a}, \mathfrak{b}]$ of $\mathfrak{a}$ and $\mathfrak{b}$ is defined to be the integral ideal that divides all other common multiples. Since to divide is to contain, $[\mathfrak{a}, \mathfrak{b}]$ is the largest integral ideal that is contained in both $\mathfrak{a}$ and $\mathfrak{b}$. This is $\mathfrak{a} \cap \mathfrak{b}$ and so $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{a} \cap \mathfrak{b}$. Moreover, we say that $\mathfrak{a}$ and $\mathfrak{b}$ are **relatively prime** if $(\mathfrak{a}, \mathfrak{b}) = \mathcal{O}$. In other words, we have

$$\mathfrak{a} + \mathfrak{b} = \mathcal{O}.$$

This is equivalent to the prime factorizations of $\mathfrak{a}$ and $\mathfrak{b}$ containing distinct primes. In particular, distinct primes and their powers are relatively prime. Moreover, if $\mathfrak{a}$ and $\mathfrak{b}$ and relatively prime then we also have

$$\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}.$$

The reverse containment is obvious. For the forward containment, since $\mathfrak{a} + \mathfrak{b} = \mathcal{O}$ there exists $\alpha \in \mathfrak{a}$ and $\beta \in \mathfrak{b}$ such that $\alpha + \beta = 1$. Now let $\gamma \in \mathfrak{a} \cap \mathfrak{b}$. Then $\gamma = \gamma\alpha + \gamma\beta$ and hence $\gamma \in \mathfrak{a}\mathfrak{b}$. The reverse containment follows which proves equality. Just as it is common to suppress the fundamental theorem of arithmetic and just state the prime factorization of an integer, we suppress referencing Theorem 5.3.1 and simply state the prime factorization of a fractional ideal. We can now show that for a Dedekind domain, being a principal ideal domain is equivalent to being a unique factorization domain:

Proposition 5.3.1. *Let $\mathcal{O}$ be a Dedekind domain. Then $\mathcal{O}$ is a principal ideal domain if and only if it is a unique factorization domain.*

Proof. The forward implication is trivial since every principal ideal domain is a unique factorization domain. For the reverse implication, suppose $\mathcal{O}$ is a unique factorization domain. Since every integral ideal factors into a product of primes by Theorem 5.3.1 and the product of principal integral ideals is principal, it suffices to show that primes are principal. So let $\mathfrak{p}$ be a prime. Then there exists nonzero $\alpha \in \mathfrak{p}$ and α is not a unit since $\mathfrak{p}$ is prime (primes are necessarily proper). Since $\mathcal{O}$ is a unique factorization domain, we may write

$$\alpha = \mu\rho_1^{e_1} \cdots \rho_r^{e_r},$$

for some unit μ and primes $\rho_i \in \mathcal{O}$ and integers $e_i \geq 1$ for $1 \leq i \leq r$ with $r \geq 1$. Since $\mathfrak{p}$ is prime, it follows that there is some i such that $\rho_i \in \mathfrak{p}$. Without loss of generality, we may assume $\rho_1 \in \mathfrak{p}$. Then the integral ideal $\rho_1\mathcal{O}$ satisfies $\rho_1\mathcal{O} \subseteq \mathfrak{p}$. As ρ_1 is prime and $\mathcal{O}$ is a unique factorization domain, $\rho_1\mathcal{O}$ is a prime and hence maximal since $\mathcal{O}$ is also a Dedekind domain. Thus $\rho_1\mathcal{O} = \mathfrak{p}$ and hence $\mathfrak{p}$ is principal completing the proof. □

With the prime factorization in hand, we will discuss the group structure of the fractional ideals of $\mathcal{O}$. Let $I_{\mathcal{O}}$ denote the set of fractional ideals of $\mathcal{O}$. We call $I_{\mathcal{O}}$ the **ideal group** of $\mathcal{O}$. The following theorem shows that $I_{\mathcal{O}}$ is indeed a group:

Theorem 5.3.2. *Let $\mathcal{O}$ be a Dedekind domain with field of fractions K. Then $I_{\mathcal{O}}$ is an abelian group with identity $\mathcal{O}$.*

Proof. It is clear that the product of fractional ideals is a fractional ideal. Associativity and commutativity of $I_{\mathcal{O}}$ are also obvious. The identity is $\mathcal{O}$ because every fractional ideal is a finitely generated $\mathcal{O}$-submodule of K. It follows that $\mathfrak{p}^{-1}$ is the inverse of any prime $\mathfrak{p}$ by Lemma 5.3.2 (ii). Therefore every prime is invertible. If $\mathfrak{a}$ is a integral ideal then it admits a prime factorization $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ and then $\mathfrak{b} = \mathfrak{p}_1^{-e_1} \cdots \mathfrak{p}_r^{-e_r}$ is its inverse. Hence every integral ideal is invertible. If $\mathfrak{f}$ is a fractional ideal, there exists a nonzero $\delta \in \mathcal{O}$ and integral ideal $\mathfrak{a}$ such that $\mathfrak{f} = \frac{1}{\delta}\mathfrak{a}$ and hence $\mathfrak{f}$ is invertible because δ and $\mathfrak{a}$ are. It follows that every fractional ideal is invertible which completes the proof. □

Now that we have proved that the ideal group $I_{\mathcal{O}}$ of K is indeed a group, we can also deduce the explicit form for the inverse $\mathfrak{f}^{-1}$ of any fractional ideal $\mathfrak{f}$:

Proposition 5.3.2. *Let $\mathcal{O}$ be a Dedekind domain with field of fractions K and let $\mathfrak{f}$ be a fractional ideal. Then*

$$\mathfrak{f}^{-1} = \{\kappa \in K : \kappa\mathfrak{f} \subseteq \mathcal{O}\}.$$

In particular, $\mathcal{O} \subseteq \mathfrak{f}$ if and only if $\mathfrak{f}^{-1}$ is an integral ideal.

Proof. Let $\mathfrak{f}$ be a fractional ideal. Then the inverse $\mathfrak{f}^{-1}$ exists by Theorem 5.3.2. In the case of an integral ideal $\mathfrak{a}$, we have

$$\mathfrak{a}^{-1} = \{\kappa \in K : \kappa\mathfrak{a} \subseteq \mathcal{O}\},$$

by the prime factorization of $\mathfrak{a}$ and the definition of the inverse of a prime. If $\mathfrak{f}$ is a fractional ideal, there exists a nonzero $\delta \in \mathcal{O}$ and integral ideal $\mathfrak{a}$ such that $\mathfrak{f} = \frac{1}{\delta}\mathfrak{a}$. But then $\delta\mathfrak{f} = \mathfrak{a}$ so that

$$\frac{1}{\delta}\mathfrak{f}^{-1} = \{\kappa \in K : \kappa\delta\mathfrak{f} \subseteq \mathcal{O}\},$$

which is equivalent to the first statement. For the second statement, if $\mathcal{O} \subseteq \mathfrak{f}$ then multiplying by $\mathfrak{f}^{-1}$ shows $\mathfrak{f}^{-1} \subseteq \mathcal{O}$ and hence $\mathfrak{f}^{-1}$ is an integral ideal. Running this argument backwards by multiplying by $\mathfrak{f}$ proves the converse. □

We will now discuss applications of the Chinese remainder theorem in the context of integral ideals. With it we can prove some interesting results. First, we recall a useful fact. Suppose $\mathfrak{a}$ is an integral ideal with prime factorization

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}.$$

As powers of distinct primes are relatively prime, the integral ideals $\mathfrak{p}_1^{e_1}, \ldots, \mathfrak{p}_r^{e_r}$ are pairwise relatively prime so that the Chinese remainder theorem gives an isomorphism

$$\mathcal{O}/\mathfrak{a} \cong \bigoplus_{1 \leq i \leq r} \mathcal{O}/\mathfrak{p}_i^{e_i}.$$

In particular, for any $\alpha_i \in \mathcal{O}$ for all i, there exists a unique $\alpha \in \mathcal{O}$ such that

$$\alpha \equiv \alpha_i \pmod{\mathfrak{p}_i^{e_i}},$$

for all i. We will use the Chinese remainder theorem to prove a few useful lemmas about Dedekind domains. For convenience, if $\mathcal{O}$ is a Dedekind domain with prime $\mathfrak{p}$ we will let $\mathbb{F}_\mathfrak{p} = \mathcal{O}/\mathfrak{p}$ be the residue class field of $\mathcal{O}$ by $\mathfrak{p}$ (recall $\mathfrak{p}$ is maximal). The first lemma is an isomorphism between quotients by primes (and powers thereof):

Lemma 5.3.3. *Let $\mathcal{O}$ be a Dedekind domain. Then for any prime $\mathfrak{p}$ and $n \geq 0$, we have an isomorphism*

$$\mathbb{F}_\mathfrak{p} \cong \mathfrak{p}^n/\mathfrak{p}^{n+1}.$$

Proof. By uniqueness of prime factorizations of fractional ideals, there exists $\beta \in \mathfrak{p}^n - \mathfrak{p}^{n+1}$. Now consider the homomorphism

$$\phi : \mathcal{O} \to \mathfrak{p}^n/\mathfrak{p}^{n+1} \qquad \alpha \mapsto \alpha\beta \pmod{\mathfrak{p}^{n+1}}.$$

By the first isomorphism theorem, it suffices to show $\ker\phi = \mathfrak{p}$ and that ϕ is surjective. Let us first show $\ker\phi = \mathfrak{p}$. As $\beta \in \mathfrak{p}^n$, it is obvious that $\mathfrak{p} \subseteq \ker\phi$. Conversely, suppose $\alpha \in \mathcal{O}$ is such that $\phi(\alpha) = 0$. Then $\alpha\beta \in \mathfrak{p}^{n+1}$ and as $\beta \in \mathfrak{p}^n - \mathfrak{p}^{n+1}$, we must have $\alpha \in \mathfrak{p}$. It follows that $\ker\phi = \mathfrak{p}$. We now show that ϕ is surjective. Let $\gamma \in \mathfrak{p}^n$ be a representative of a class in $\mathfrak{p}^n/\mathfrak{p}^{n+1}$. As $\beta \in \mathfrak{p}^n$, we have $\beta\mathcal{O} \subseteq \mathfrak{p}^n$. But since $\beta \notin \mathfrak{p}^{n+1}$, we see that $\beta\mathcal{O}\mathfrak{p}^{-n}$ is necessarily an integral ideal relatively prime to $\mathfrak{p}^{n+1}$. As $\mathfrak{p}^{n+1}$ and $\beta\mathcal{O}\mathfrak{p}^{-n}$ are relatively prime, the Chinese remainder theorem implies that we can find a unique $\alpha \in \mathcal{O}$ such that

$$\alpha \equiv \gamma \pmod{\mathfrak{p}^{n+1}} \quad \text{and} \quad \alpha \equiv 0 \pmod{\beta\mathcal{O}\mathfrak{p}^{-n}}.$$

The second condition forces $\alpha \in \beta\mathcal{O}\mathfrak{p}^{-n}$. As $\gamma \in \mathfrak{p}^n$ and α and γ differ by an element in $\mathfrak{p}^{n+1} \subset \mathfrak{p}^n$, we have that $\alpha \in \beta\mathcal{O}\mathfrak{p}^{-n} \cap \mathfrak{p}^n = \beta\mathcal{O}$ where the equality holds because the intersection of ideals is equal to their product provided the ideals are relatively prime. Thus $\alpha\beta^{-1} \in \mathcal{O}$ and hence

$$\phi(\alpha\beta^{-1}) = \alpha \equiv \gamma \pmod{\mathfrak{p}^{n+1}}.$$

This shows ϕ is surjective completing the proof. □

Our second lemma shows that given two integral ideals, we can multiply by a relatively prime integral ideal and produce a principal integral ideal:

Lemma 5.3.4. *Let $\mathcal{O}$ be a Dedekind domain and $\mathfrak{a}$ and $\mathfrak{b}$ be integral ideals. Then there exists an integral ideal $\mathfrak{c}$ relatively prime to $\mathfrak{b}$ such that $\mathfrak{ac}$ is principal.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the prime factors of both $\mathfrak{a}$ and $\mathfrak{b}$ so that

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} \quad \text{and} \quad \mathfrak{b} = \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_r^{f_r},$$

with $e_i \geq 0$ and $f_i \geq 0$ for all $1 \leq i \leq r$. By the prime factorization of fractional ideals, there exists $\alpha_i \in \mathfrak{p}_i^{e_i} - \mathfrak{p}_i^{e_i+1}$ for all i. Since $\mathfrak{p}_1^{e_1+1}, \ldots, \mathfrak{p}_r^{e_r+1}$ are pairwise relatively prime, there exists $\alpha \in \mathcal{O}$ such that $\alpha \equiv \alpha_i \pmod{\mathfrak{p}_i^{e_i+1}}$ for all i. But then

$$\alpha \equiv 0 \pmod{\mathfrak{p}_i^{e_i}} \quad \text{and} \quad \alpha \equiv \alpha_i \pmod{\mathfrak{p}_i^{e_i+1}},$$

since $\alpha_i \in \mathfrak{p}_i^{e_i}$ for all i. Hence $\alpha \in \mathfrak{p}_i^{e_i} - \mathfrak{p}_i^{e_i+1}$ and so $\mathfrak{p}_i^{e_i} \mid\mid \alpha\mathcal{O}$ for all i. It follows that $(\alpha\mathcal{O}, \mathfrak{ab}) = \mathfrak{a}$. Letting $\mathfrak{c}$ be the integral ideal such that $\alpha\mathcal{O} = \mathfrak{ac}$, we have $(\mathfrak{ac}, \mathfrak{ab}) = \mathfrak{a}$. Thus $(\mathfrak{c}, \mathfrak{b}) = \mathcal{O}$ which is to say that $\mathfrak{c}$ must be relatively prime to $\mathfrak{b}$. □

Our last lemma shows that multiplying by fractional ideals does not affect quotients:

Lemma 5.3.5. *Let $\mathcal{O}$ be a Dedekind domain and $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ be fractional ideals with $\mathfrak{g} \subseteq \mathfrak{f}$. Then we have an isomorphism*

$$\mathfrak{f}/\mathfrak{g} \cong \mathfrak{fh}/\mathfrak{gh}.$$

In particular,

$$\mathfrak{a}^{-1}/\mathcal{O} \cong \mathcal{O}/\mathfrak{a},$$

for any integral ideal $\mathfrak{a}$.

Proof. We have $\mathfrak{f} = \frac{\mathfrak{a}}{\alpha}$, $\mathfrak{g} = \frac{\mathfrak{b}}{\beta}$, and $\mathfrak{h} = \frac{\mathfrak{c}}{\gamma}$ for nonzero $\alpha, \beta, \gamma \in \mathcal{O}$ and integral ideals $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ with $\mathfrak{b} \subseteq \mathfrak{a}$. In view of the isomorphisms,

$$\mathfrak{f}/\mathfrak{g} \cong \beta\mathfrak{a}/\alpha\mathfrak{b} \quad \text{and} \quad \mathfrak{fh}/\mathfrak{gh} \cong \beta\mathfrak{ac}/\alpha\mathfrak{bc},$$

it suffices to show

$$\mathfrak{a}/\mathfrak{b} \cong \mathfrak{ac}/\mathfrak{bc}.$$

As to contain is to divide, we know $\mathfrak{a} \mid \mathfrak{b}$. Therefore $\mathfrak{b}\mathfrak{a}^{-1}$ is an integral ideal. By Lemma 5.3.4, there exists an integral ideal $\mathfrak{d}$ relatively prime to $\mathfrak{b}\mathfrak{a}^{-1}$ such that $\mathfrak{cd} = \delta\mathcal{O}$ for some $\delta \in \mathcal{O}$. But then $\mathfrak{d} + \mathfrak{b} = \mathcal{O}$ and hence

$$\delta\mathfrak{c}^{-1} + \mathfrak{b}\mathfrak{a}^{-1} = \mathcal{O}.$$

Multiplying by $\mathfrak{ac}$ yields

$$\delta\mathfrak{a} + \mathfrak{bc} = \mathfrak{ac}.$$

As $\mathfrak{cd} = \delta\mathcal{O}$, we have $\delta\mathcal{O} \subseteq \mathfrak{c}$ so that $\delta \in \mathfrak{c}$. Therefore we may consider the homomorphism

$$\phi : \mathfrak{a} \to \mathfrak{ac}/\mathfrak{bc} \qquad \alpha \mapsto \alpha\delta \pmod{\mathfrak{bc}}.$$

It is surjective by what we have just proved. Moreover, $\ker\phi = \mathfrak{a} \cap \delta^{-1}\mathfrak{bc}$. But observe that

$$\mathfrak{a} \cap \delta^{-1}\mathfrak{bc} = \delta^{-1}\mathfrak{ac}(\delta\mathfrak{c}^{-1} \cap \mathfrak{b}\mathfrak{a}^{-1}) = \delta^{-1}\mathfrak{ac}(\mathfrak{d} \cap \mathfrak{b}\mathfrak{a}^{-1}) = \mathfrak{b},$$

where the last equality follows since $\mathfrak{d}$ and $\mathfrak{b}\mathfrak{a}^{-1}$ are relatively prime so that their intersection is equal to their product. Hence $\ker\phi = \mathfrak{b}$ and the first statement now follows upon applying the first isomorphism theorem. For the second statement, take $\mathfrak{f} = \mathfrak{a}^{-1}$, $\mathfrak{g} = \mathcal{O}$, and $\mathfrak{h} = \mathfrak{a}$ (note that $\mathcal{O} \subseteq \mathfrak{a}^{-1}$ by Proposition 5.3.2). □

We now state two additional interesting facts about Dedekind domains. The first is that any Dedekind domains with only finitely many primes is a principal ideal domain:

Proposition 5.3.3. *Let $\mathcal{O}$ be a Dedekind domain. If there are only finitely many primes then $\mathcal{O}$ is a principal ideal domain.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the primes of $\mathcal{O}$. Then for any integral ideal $\mathfrak{a}$, the prime factorization is

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r},$$

with $e_i \geq 0$ for $1 \leq i \leq r$. By uniqueness of prime factorizations of fractional ideals, there exists $\alpha_i \in \mathfrak{p}_i^{e_i} - \mathfrak{p}_i^{e_i+1}$ for all i. Since $\mathfrak{p}_1^{e_1+1}, \ldots, \mathfrak{p}_r^{e_r+1}$ are pairwise relatively prime, the Chinese remainder theorem implies that there exists $\alpha \in \mathcal{O}$ with

$$\alpha \equiv \alpha_i \pmod{\mathfrak{p}_i^{e_i+1}},$$

for all i. As $\alpha_i \in \mathfrak{p}_i^{e_i}$ for all i, we have that $\alpha \in \mathfrak{a}$ and hence $\alpha\mathcal{O} \subseteq \mathfrak{a}$. But as $\alpha \notin \mathfrak{p}_i^{e_i+1}$ for all i, we see that $\alpha\mathcal{O}\mathfrak{a}^{-1}$ is necessarily an integral ideal relatively prime to all primes of $\mathcal{O}$. This means $\alpha\mathcal{O}\mathfrak{a}^{-1} = \mathcal{O}$ and hence $\mathfrak{a} = \alpha\mathcal{O}$ so that $\mathfrak{a}$ is principal. As $\mathfrak{a}$ was arbitrary, $\mathcal{O}$ is a principal ideal domain. □

The second fact is that any fractional ideal is generated by at most two elements:

Proposition 5.3.4. *Let $\mathcal{O}$ be a Dedekind domain. Then every fractional ideal is generated by at most two elements.*

Proof. We first prove the claim for an integral ideal $\mathfrak{a}$. Let $\alpha \in \mathfrak{a}$ be nonzero and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the prime factors of $\alpha\mathcal{O}$. As $\alpha\mathcal{O} \subseteq \mathfrak{a}$, the prime factorization of $\mathfrak{a}$ is

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r},$$

with $e_i \geq 0$ for $1 \leq i \leq r$. By uniqueness of prime factorizations of fractional ideals, there exists $\beta_i \in \mathfrak{p}_i^{e_i} - \mathfrak{p}_i^{e_i+1}$ for all i. Since $\mathfrak{p}_1^{e_1+1}, \ldots, \mathfrak{p}_r^{e_r+1}$ are pairwise relatively prime, the Chinese remainder theorem implies that there exists $\beta \in \mathcal{O}$ with

$$\beta \equiv \beta_i \pmod{\mathfrak{p}_i^{e_i+1}},$$

for all i. As $\beta_i \in \mathfrak{p}_i^{e_i}$ for all i, we have that $\beta \in \mathfrak{a}$ and hence $\beta\mathcal{O} \subseteq \mathfrak{a}$. But as $\beta \notin \mathfrak{p}_i^{e_i+1}$ for all i, we see that $\beta\mathcal{O}\mathfrak{a}^{-1}$ is necessarily an integral ideal relatively prime to $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ and hence to $\alpha\mathcal{O}$. This means

$$\beta\mathcal{O}\mathfrak{a}^{-1} + \alpha\mathcal{O} = \mathcal{O},$$

and hence

$$\beta\mathcal{O} + \alpha\mathfrak{a} = \mathfrak{a}.$$

But as $\alpha, \beta \in \mathfrak{a}$, we have $\beta\mathcal{O} + \alpha\mathfrak{a} \subseteq \beta\mathcal{O} + \alpha\mathcal{O} \subseteq \mathfrak{a}$ and so

$$\beta\mathcal{O} + \alpha\mathcal{O} = \mathfrak{a}.$$

This shows that $\mathfrak{a}$ is generated by at most two elements. Now suppose $\mathfrak{f}$ is a fractional ideal. Then there exists a nonzero $\delta \in \mathcal{O}$ and integral ideal $\mathfrak{a}$ such that $\mathfrak{f} = \frac{1}{\delta}\mathfrak{a}$. Since $\mathfrak{a}$ is generated by at most two elements, say α and β, we have

$$\mathfrak{f} = \frac{\alpha}{\delta}\mathcal{O} + \frac{\beta}{\delta}\mathcal{O},$$

and so $\mathfrak{f}$ is also generated by at most two elements as well. □

Proposition 5.3.4 shows that while a Dedekind domain $\mathcal{o}$ may not be a principal ideal domain, it is not far off from one since every integral ideal needs at most two generators. We can give a more refined interpretation of this using the ideal group $I_{\mathcal{o}}$. Let $P_{\mathcal{o}}$ denote the subgroup of principal fractional ideals of $I_{\mathcal{o}}$. Since $I_{\mathcal{o}}$ is abelian by Theorem 5.3.2, $P_{\mathcal{o}}$ is normal. The **ideal class group** $\mathrm{Cl}(\mathcal{o})$ of $\mathcal{o}$ is defined to be the quotient group

$$\mathrm{Cl}(\mathcal{o}) = I_{\mathcal{o}}/P_{\mathcal{o}},$$

We call an element of $\mathrm{Cl}(\mathcal{o})$ an **ideal class** of $\mathcal{o}$. As every fractional ideal $\mathfrak{f}$ can be expressed as $\mathfrak{f} = \frac{1}{\delta}\mathfrak{a}$ for some nonzero $\delta \in \mathcal{o}$ and integral ideal $\mathfrak{a}$, we have $\delta\mathfrak{f} = \mathfrak{a}$ and hence every ideal class can be represented by an integral ideal $\mathfrak{a}$. The **class number** $h_{\mathcal{o}}$ of $\mathcal{o}$ is defined by

$$h_{\mathcal{o}} = |\,\mathrm{Cl}(\mathcal{o})|.$$

The ideal class group is an object which encodes how much $\mathcal{o}$ fails to be a principal ideal domain (equivalently a unique factorization domain by Proposition 5.3.1) while the class number $h_{\mathcal{o}}$ is a measure of the degree of failure. For example, $\mathcal{o}$ is a principal ideal domain if and only if $h_{\mathcal{o}} = 1$. Indeed, if $\mathcal{o}$ is a principal ideal domain then every integral ideal is principal and hence every fractional ideal is too (because every fractional ideal is of the form $\frac{1}{\delta}\mathfrak{a}$ for some nonzero $\delta \in \mathcal{o}$ and integral ideal $\mathfrak{a}$). But then $\mathrm{Cl}(\mathcal{o})$ is the trivial group and hence $h_{\mathcal{o}} = 1$. Conversely, if $h_{\mathcal{o}} = 1$ then every integral ideal is principal so that $\mathcal{o}$ is a principal ideal domain.

Remark 5.3.2. *The class number $h_{\mathcal{o}}$ need not be finite for a general Dedekind domain $\mathcal{o}$.*

The **unit group** of $\mathcal{o}$ is defined to be $\mathcal{o}^*$. That is, the unit group is the group of units in $\mathcal{o}$. The ideal class group and unit group of $\mathcal{o}$ are related via the following exact sequence:

Proposition 5.3.5. *Let $\mathcal{o}$ be a Dedekind domain with field of fractions K. Then the sequence*

$$1 \longrightarrow \mathcal{o}^* \longrightarrow K^* \xrightarrow{\mathcal{o}} I_{\mathcal{o}} \longrightarrow \mathrm{Cl}(\mathcal{o}) \longrightarrow 1,$$

where the middle map takes any $\kappa \in K^*$ to its associated principal fractional ideal $\kappa\mathcal{o}$, is exact.

Proof. The sequence is exact at $\mathcal{o}^*$ because the second map is injective. For exactness at K^*, the image of the second map is contained in the kernel of the third map since $\mathcal{o}$ is the identity in $I_{\mathcal{o}}$ by Theorem 5.3.2. So suppose $\kappa \in K^*$ is contained in the kernel of the third map. Then $\kappa\mathcal{o} = \mathcal{o}$ which implies $\kappa \in \mathcal{o}^*$ because $1 \in \mathcal{o}$. This proves exactness at K^*. Exactness at $I_{\mathcal{o}}$ follows by the definition of $\mathrm{Cl}(\mathcal{o})$. Lastly, the sequence is exact at $\mathrm{Cl}(\mathcal{o})$ since the fourth map is surjective which completes the proof. □

Thinking of the third map in Proposition 5.3.5 as passing from numbers in K^* to fractional ideals in $I_{\mathcal{o}}$, exactness means that unit group is measuring the contraction (how many numbers are annihilated) taking place during this process while the class group is measuring the expansion (how many fractional ideal are created).

Remark 5.3.3. *The class number $h_{\mathcal{o}}$ and unit group $\mathcal{o}^*$ of $\mathcal{o}$ are generally two of the most difficult pieces of algebraic data of $\mathcal{o}$ to compute.*

We now turn to the case of a number field K for which our developments so far can be refined. However, in order to apply our results on Dedekind domains, we need to show that $\mathcal{O}_K$ is one:

Theorem 5.3.3. *Let K be a number field. Then $\mathcal{O}_K$ is a Dedekind domain.*

Proof. Since $\mathcal{O}_K$ is the integral closure of $\mathbb{Z}$ in K, $\mathcal{O}_K$ is automatically an integrally closed integral domain. As $\mathcal{O}_K$ is a free abelian group of finite rank, any ideal is a finitely generated $\mathbb{Z}$-module and hence a finitely generated $\mathcal{O}_K$-module. This proves that $\mathcal{O}_K$ is noetherian. It remains to prove that every prime of $\mathcal{O}_K$ is maximal. Letting $\mathfrak{p}$ be a prime, it suffices to show $\mathcal{O}_K/\mathfrak{p}$ is a field. To this end, consider the homomorphism

$$\phi : \mathbb{Z} \to \mathcal{O}_K/\mathfrak{p} \qquad a \mapsto a \pmod{\mathfrak{p}}.$$

Then $\ker \phi = \mathfrak{p} \cap \mathbb{Z}$ and we claim $\mathfrak{p} \cap \mathbb{Z}$ is a prime of $\mathbb{Z}$. It is obviously an ideal of $\mathbb{Z}$ and is prime because $\mathfrak{p}$ is. To see that it is nonzero, let $\alpha \in \mathfrak{p}$ be nonzero. As α is an algebraic integer, we have

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0,$$

for some $n \geq 1$ and $a_i \in \mathbb{Z}$ for $0 \leq i \leq n-1$. Taking n minimal, we have $a_0 \neq 0$. Isolating a_0 shows that $a_0 \in \mathfrak{p}$ and hence $a_0 \in \mathfrak{p} \cap \mathbb{Z}$. Therefore $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ for some prime p. Hence $\ker \phi = p\mathbb{Z}$ and by the first isomorphism theorem, ϕ induces an injection $\phi : \mathbb{F}_p \to \mathcal{O}_K/\mathfrak{p}$. As $\mathcal{O}_K$ is the integral closure of $\mathbb{Z}$ in K, it is obtained from $\mathbb{Z}$ by forming a polynomial ring with algebraic elements in K. The latter injection then shows that $\mathcal{O}_K/\mathfrak{p}$ is obtained from $\mathbb{F}_p$ by adjoining these algebraic elements reduced modulo $\mathfrak{p}$. Their reductions are seen to be algebraic over $\mathbb{F}_p$ by reducing their minimal polynomials over $\mathbb{Q}$, viewed as elements of $\mathcal{o}[x]$, modulo $\mathfrak{p}$ upon recalling that the coefficients of which are in $\mathbb{Z}$ by Proposition 5.1.3. Hence $\mathcal{O}_K/\mathfrak{p}$ is obtained from $\mathbb{F}_p$ by adjoining algebraic elements to $\mathbb{F}_p$ (since $\mathbb{F}_p[\overline{\alpha}] = \mathbb{F}_p(\overline{\alpha})$ if $\overline{\alpha}$ is algebraic over $\mathbb{F}_p$) and is therefore a field. □

In view of the fact that $\mathcal{O}_K$ is a Dedekind domain by Theorem 5.3.3, we simplify some terminology. An **integral ideal** of K is simply an integral ideal of $\mathcal{O}_K$, a **prime** of K is a prime of $\mathcal{O}_K$, and a **fractional ideal** of K is a fractional ideal of $\mathcal{O}_K$. The **ideal group** I_K of K is the ideal group of $\mathcal{O}_K$, we write P_K for the subgroup of principal fractional ideals of K, and the **ideal class group** $\mathrm{Cl}(K)$ of K is the ideal class group of $\mathcal{O}_K$. In particular,

$$\mathrm{Cl}(K) = I_K/P_K.$$

The **class number** h_K of K is the class number of $\mathcal{O}_K$ and so

$$h_K = |\,\mathrm{Cl}(K)|.$$

The **unit group** of K is the unit group of $\mathcal{O}_K$ and we call any element of $\mathcal{O}_K^*$ a **unit** of K (with the understanding that every element of K is invertible in K). It follows from Theorems 5.3.1 and 5.3.3 that integral ideals $\mathfrak{a}$ of K admit prime factorizations. One of our core investigations will be to understand how the principal integral ideal $p\mathcal{O}_K$ factors into a product of primes of K for any prime p. Moreover, we will be able to leverage geometric tools to show that the class number h_K is finite and completely describe the unit group $\mathcal{O}_K^*$.

5.4 Localization

Let $\mathcal{o}$ be a noetherian integral domain with field of fractions K. If $D \subseteq \mathcal{o} - \{0\}$ is a multiplicative subset (recall that necessarily $1 \in D$) then the **localization** $\mathcal{o}D^{-1}$ of $\mathcal{o}$ at D is defined by

$$\mathcal{o}D^{-1} = \left\{ \frac{\alpha}{\delta} \in K : \alpha \in \mathcal{o} \text{ and } \delta \in D \right\}.$$

Moreover, for any subset N of $\mathcal{o}$, we set

$$ND^{-1} = \left\{ \frac{\eta}{\delta} \in K : \eta \in N \text{ and } \delta \in D \right\}.$$

Now $\mathcal{O}D^{-1}$ is clearly a subring of K which is an integral domain and is it formed from $\mathcal{O}$ by making every element of D invertible. It is also noetherian for if $\mathfrak{A}$ is an ideal of $\mathcal{O}D^{-1}$, set $\mathfrak{a} = \mathfrak{A} \cap \mathcal{O}$. Then

$$\mathfrak{A} = \mathfrak{a}D^{-1}.$$

Indeed, the reverse containment is trivial. For the forward containment, if $\frac{\alpha}{\delta} \in \mathfrak{A}$ then $\alpha \in \mathfrak{a}$ because $\delta \in D$ and $\mathfrak{A}$ is an ideal of $\mathcal{O}D^{-1}$. Hence $\frac{\alpha}{\delta} \in \mathfrak{a}D^{-1}$ as desired. Since $\mathcal{O}$ is Dedekind, $\mathfrak{a}$ is a finitely generated $\mathcal{O}$-module and hence $\mathfrak{A}$ is a finitely generated $\mathcal{O}D^{-1}$-module by the identity we have just proved. Thus $\mathcal{O}D^{-1}$ is noetherian. It follows that if $\mathfrak{f}$ is a fractional ideal of $\mathcal{O}$ then $\mathfrak{f}D^{-1}$ is a fractional ideal of $\mathcal{O}D^{-1}$. We call $\mathfrak{f}D^{-1}$ the **localization** of $\mathfrak{f}$ at D. In the case of primes, we have an exact correspondence:

Proposition 5.4.1. *Let $\mathcal{O}$ be a noetherian integral domain and $D \subseteq \mathcal{O} - \{0\}$ be a multiplicative subset. Then the maps*

$$\mathfrak{q} \mapsto \mathfrak{q}D^{-1} \quad \textit{and} \quad \mathfrak{Q} \mapsto \mathfrak{Q} \cap \mathcal{O}.$$

are inverse inclusion-preserving bijections between the primes $\mathfrak{q}$ of $\mathcal{O}$ disjoint from D and the primes $\mathfrak{Q}$ of $\mathcal{O}D^{-1}$.

Proof. First suppose $\mathfrak{q}$ is a prime of $\mathcal{O}$ that is disjoint from D. We claim that the integral ideal $\mathfrak{q}D^{-1}$ of $\mathcal{O}D^{-1}$ is prime. Indeed, suppose $\frac{\alpha}{\delta}, \frac{\beta}{\gamma} \in \mathcal{O}D^{-1}$ are such that $\frac{\alpha\beta}{\delta\gamma} \in \mathfrak{q}D^{-1}$. Then $\alpha\beta \in \mathfrak{q}$ and thus $\alpha \in \mathfrak{q}$ or $\beta \in \mathfrak{q}$ because $\mathfrak{q}$ is prime. Hence $\frac{\alpha}{\delta} \in \mathfrak{q}D^{-1}$ or $\frac{\beta}{\gamma} \in \mathfrak{q}D^{-1}$ as desired. Also, the prime $\mathfrak{q}D^{-1}$ satisfies

$$\mathfrak{q} = \mathfrak{q}D^{-1} \cap \mathcal{O}.$$

The forward inclusion is obvious. For the reverse inclusion, if $\frac{\alpha}{\delta} \in \mathfrak{q}D^{-1} \cap \mathcal{O}$ then $\alpha \in \mathfrak{q}$ and hence $\frac{\alpha}{\delta} \in \mathfrak{q}$ because $\mathfrak{q}$ is prime and $\delta \notin \mathfrak{q}$ (as $\mathfrak{q}$ is disjoint from D). Now suppose $\mathfrak{Q}$ is a prime of $\mathcal{O}D^{-1}$. We claim that $\mathfrak{Q} \cap \mathcal{O}$ is a prime of $\mathcal{O}$ that is disjoint from D. It is clearly an integral ideal of $\mathcal{O}$ and is prime because $\mathfrak{Q}$ is. If $\mathfrak{Q} \cap \mathcal{O}$ is not disjoint from D then $\delta \in \mathfrak{Q} \cap \mathcal{O}$ with $\delta \in D$. Hence $1 \in \mathfrak{Q} \cap \mathcal{O}$ because $\frac{1}{\delta} \in \mathcal{O}D^{-1}$ and $\mathfrak{Q}$ is an integral ideal of $\mathcal{O}D^{-1}$. This is impossible since $\mathfrak{Q} \cap \mathcal{O}$ is prime (hence proper) and therefore must be disjoint from D. Moreover, the prime $\mathfrak{Q} \cap \mathcal{O}$ satisfies

$$\mathfrak{Q} = (\mathfrak{Q} \cap \mathcal{O})D^{-1}.$$

The reverse inclusion is obvious since $1 \in D$. For the forward inclusion, if $\frac{\alpha}{\delta} \in \mathfrak{Q}$ then $\alpha \in \mathfrak{Q} \cap \mathcal{O}$ and thus $\frac{\alpha}{\delta} \in (\mathfrak{Q} \cap \mathcal{O})D^{-1}$. All of this together shows that the mappings

$$\mathfrak{q} \mapsto \mathfrak{q}D^{-1} \quad \text{and} \quad \mathfrak{Q} \mapsto \mathfrak{Q} \cap \mathcal{O}.$$

are inverse bijections between the primes $\mathfrak{q}$ of $\mathcal{O}$ disjoint from D and the primes $\mathfrak{Q}$ of $\mathcal{O}D^{-1}$. They are clearly inclusion-preserving. □

From Proposition 5.4.1, the primes of $\mathcal{O}D^{-1}$ are of the form $\mathfrak{p}D^{-1}$ for primes $\mathfrak{p}$ of $\mathcal{O}$ disjoint from D.

Remark 5.4.1. *The bijections in Proposition 5.4.1 need not hold for all integral ideals.*

Localization is most useful when we localize at the compliment of a prime or collection of primes. If $\mathfrak{p}$ is a prime of $\mathcal{O}$ then $\mathcal{O} - \mathfrak{p} \subset \mathcal{O} - \{0\}$ is a multiplicative subset. Indeed, the inclusion is obvious, $1 \in \mathcal{O} - \mathfrak{p}$ because $\mathfrak{p}$ is proper and thus does not contain units, and if $\alpha, \beta \in \mathcal{O} - \mathfrak{p}$ we have $\alpha\beta \notin \mathfrak{p}$ because $\mathfrak{p}$ is prime so that $\alpha\beta \in \mathcal{O} - \mathfrak{p}$ and thus $\mathcal{O} - \mathfrak{p}$ is closed under multiplication. We define the **localization** $\mathcal{O}_{\mathfrak{p}}$ of $\mathcal{O}$ at $\mathfrak{p}$ by

$$\mathcal{O}_{\mathfrak{p}} = \mathcal{O}(\mathcal{O} - \mathfrak{p})^{-1}.$$

In other words,

$$\mathcal{o}_{\mathfrak{p}} = \left\{ \frac{\alpha}{\delta} \in K : \alpha, \delta \in \mathcal{o} \text{ with } \delta \not\equiv 0 \pmod{\mathfrak{p}} \right\}.$$

Essentially, localizing at a prime $\mathfrak{p}$ removes all of the algebraic information about $\mathcal{o}$ that has nothing to do with $\mathfrak{p}$. If $\mathfrak{f}$ is a fractional ideal of $\mathcal{o}$ then the **localization** $\mathfrak{f}_{\mathfrak{p}}$ of $\mathfrak{f}$ at $\mathfrak{p}$ is defined to be

$$\mathfrak{f}_{\mathfrak{p}} = \mathfrak{f}(\mathcal{o} - \mathfrak{p})^{-1}.$$

More generally, let X be a set of primes in $\mathcal{o}$ and consider

$$\mathcal{o} - \bigcup_{\mathfrak{p} \in X} \mathfrak{p}.$$

Then $\mathcal{o} - \bigcup_{\mathfrak{p} \in X} \mathfrak{p} \subseteq \mathcal{o} - \{0\}$ is a multiplicative subset because

$$\mathcal{o} - \bigcup_{\mathfrak{p} \in X} \mathfrak{p} = \bigcap_{\mathfrak{p} \in X} (\mathcal{o} - \mathfrak{p}),$$

and the $\mathcal{o} - \mathfrak{p}$ are. We define the **localization** $\mathcal{o}(X)$ of $\mathcal{o}$ at X by

$$\mathcal{o}(X) = \mathcal{o} \left(\mathcal{o} - \bigcup_{\mathfrak{p} \in X} \mathfrak{p} \right)^{-1}.$$

Equivalently,

$$\mathcal{o}(X) = \left\{ \frac{\alpha}{\delta} \in K : \alpha, \delta \in \mathcal{o} \text{ with } \delta \not\equiv 0 \pmod{\mathfrak{p}} \text{ for some } \mathfrak{p} \in X \right\}.$$

If $\mathfrak{f}$ is a fractional ideal of $\mathcal{o}$ then the **localization** $\mathfrak{f}(X)$ of $\mathfrak{f}$ at X is defined to be

$$\mathfrak{f}(X) = \mathfrak{f} \left(\mathcal{o} - \bigcup_{\mathfrak{p} \in X} \mathfrak{p} \right)^{-1}.$$

Note that if $X = \{\mathfrak{p}\}$ then $\mathcal{o}(X) = \mathcal{o}_{\mathfrak{p}}$ and $\mathfrak{f}(X) = \mathfrak{f}_{\mathfrak{p}}$. In any case, localizing is a useful tool in algebraic investigations and we quickly collect some useful properties. First, localization respects integral closure:

Proposition 5.4.2. *Let $\mathcal{o}$ be a noetherian integral domain with field of fractions K, L/K be a finite separable extension, and $\mathcal{O}$ be the integral closure of $\mathcal{o}$ in L. Then for any multiplicative set $D \subseteq \mathcal{o} - \{0\}$, $\mathcal{O}D^{-1}$ is the integral closure of $\mathcal{o}D^{-1}$ in L.*

Proof. We need to show that $\mathcal{O}D^{-1} = \overline{\mathcal{o}D^{-1}}$. For the forward inclusion, let $\frac{\alpha}{\delta} \in \mathcal{O}D^{-1}$. As $\mathcal{O}$ is the integral closure of $\mathcal{o}$ in L, α is integral over $\mathcal{o}$ so that

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0,$$

for some $n \geq 1$ and $a_i \in \mathcal{o}$ for $0 \leq i \leq n-1$. Diving by δ^n, we obtain

$$\left(\frac{\alpha}{\delta}\right)^n + \frac{a_{n-1}}{\delta}\left(\frac{\alpha}{\delta}\right)^{n-1} + \cdots + \frac{a_0}{\delta^n} = 0.$$

Thus $\frac{\alpha}{\delta}$ is the root of a monic polynomial with coefficients in $\mathcal{o}D^{-1}$. Therefore the forward inclusion holds. For the reverse inclusion, suppose $\lambda \in \overline{\mathcal{o}D^{-1}}$. Then

$$\lambda^n + \frac{a_{n-1}}{\delta_{n-1}}\lambda^{n-1} + \cdots + \frac{a_0}{\delta_0} = 0,$$

for some $n \geq 1$ and $\frac{a_i}{\delta_i} \in \mathfrak{o}D^{-1}$ for $0 \leq i \leq n-1$. Letting $\delta = \delta_0 \cdots \delta_{n-1}$ and multiplying by δ^n, we obtain

$$(\lambda\delta)^n + \frac{a_{n-1}\delta}{\delta_{n-1}}(\lambda\delta)^{n-1} + \cdots + \frac{a_0\delta^n}{\delta_0} = 0.$$

It follows that $\lambda\delta$ is the root of a monic polynomial with coefficients in $\mathfrak{o}$. As $\mathcal{O}$ is the integral closure of $\mathfrak{o}$ in L, we have $\lambda\delta \in \mathfrak{o}$ and thus $\lambda \in \mathfrak{o}D^{-1}$. This proves the reverse inclusion which means $\mathcal{O}D^{-1}$ is the integral closure of $\mathfrak{o}D^{-1}$ in L. □

In the case of Proposition 5.4.2, we setup some additional notation. Again suppose $\mathfrak{o}$ is a noetherian integral domain with field of fractions K, L/K be a finite separable extension, and $\mathcal{O}$ be the integral closure of $\mathfrak{o}$ in L. If $\mathfrak{p}$ is a prime of $\mathfrak{o}$ and $\mathfrak{F}$ is a fractional ideal of $\mathcal{O}$, the **localization** $\mathfrak{F}_\mathfrak{p}$ of $\mathfrak{F}$ at $\mathfrak{p}$ is defined by

$$\mathfrak{F}_\mathfrak{p} = \mathfrak{F}(\mathfrak{o} - \mathfrak{p})^{-1}.$$

More generally, let X be a set of primes of $\mathfrak{o}$. Then the **localization** $\mathfrak{F}(X)$ of $\mathfrak{F}$ at X is defined by

$$\mathfrak{F}(X) = \mathfrak{F}\left(\mathfrak{o} - \bigcup_{\mathfrak{p}\in X} \mathfrak{p}\right)^{-1}.$$

In particular, Proposition 5.4.2 shows that $\mathcal{O}_\mathfrak{p}$ and $\mathcal{O}(X)$ are the integral closures of $\mathfrak{o}_\mathfrak{p}$ and $\mathfrak{o}(X)$ respectively. Our second property says intersections of localizations behave with respect to fractional ideals and units:

Proposition 5.4.3. *Let $\mathfrak{o}$ be a noetherian integral domain. Then for every fractional ideal $\mathfrak{f}$ of $\mathfrak{o}$, we have*

$$\mathfrak{f} = \bigcap_{\mathfrak{p}\ prime} \mathfrak{f}_\mathfrak{p}.$$

In particular,

$$\mathfrak{o} = \bigcap_{\mathfrak{p}\ prime} \mathfrak{o}_\mathfrak{p} \quad and \quad \mathfrak{o}^* = \bigcap_{\mathfrak{p}\ prime} \mathfrak{o}_\mathfrak{p}^*.$$

Proof. The first identity follows from the second by multiplying by a fractional ideal $\mathfrak{f}$ of $\mathfrak{o}$. As for the second identity, the forward containment is obvious. For the reverse containment, suppose $\frac{\alpha}{\beta} \in \bigcap_{\mathfrak{p}\text{ prime}} \mathfrak{o}_\mathfrak{p}$ and set

$$\mathfrak{a} = \{\gamma \in \mathfrak{o} : \alpha\gamma \in \beta\mathfrak{o}\}.$$

Then $\mathfrak{a}$ is clearly an integral ideal of $\mathfrak{o}$ as $\beta \in \mathfrak{a}$. It also cannot be contained in any prime $\mathfrak{p}$. Indeed, since $\frac{\alpha}{\beta} \in \mathfrak{o}_\mathfrak{p}$ we have $\beta \notin \mathfrak{p}$ and so $\beta \in \mathfrak{a} - \mathfrak{p}$. As $\mathfrak{o}$ is noetherian (which we recall is equivalent to every nonempty collection of ideals having a maximal element), there exists a maximal integral ideal $\mathfrak{m}$ which is necessarily prime. As $\mathfrak{a}$ is not contained in any prime, $\mathfrak{a}$ is not contained in $\mathfrak{m}$ and thus $\mathfrak{a} = \mathfrak{o}$. But then $1 \in \mathfrak{a}$ and so $\alpha \in \beta\mathfrak{o}$ implying $\frac{\alpha}{\beta} \in \mathfrak{o}$ which proves the reverse containment and the second identity follows. For the third identity, the forward containment is clear. For the reverse containment, suppose $\frac{\alpha}{\beta} \in \bigcap_{\mathfrak{p}\text{ prime}} \mathfrak{o}_\mathfrak{p}^*$. We have already showed $\frac{\alpha}{\beta} \in \mathfrak{o}$ so it suffices to show $\frac{\alpha}{\beta}$ is a unit. As $\beta \notin \mathfrak{p}$ for any prime $\mathfrak{p}$, it is necessarily not in any maximal integral ideal. Therefore $\beta \in \mathfrak{o}^*$ because the complement of the union of all maximal integral ideals is exactly the set of units. Interchanging the roles of α and β shows that $\alpha \in \mathfrak{o}^*$ as well. Hence $\frac{\alpha}{\beta} \in \mathfrak{o}^*$ which proves the reverse containment and the third equality follows. □

A noetherian integral domain is said to be **local** if it has a unique maximal integral ideal. As we might expect, localizing at a prime will produce a local noetherian integral domain and hence justifies the term localization. We claim that $\mathcal{O}_\mathfrak{p}$ is a local noetherian integral domain with maximal integral ideal $\mathfrak{p}_\mathfrak{p}$. Indeed, $\mathcal{O}_\mathfrak{p}$ is already a noetherian integral domain and it is clear that $\mathfrak{p}_\mathfrak{p}$ is an ideal of $\mathcal{O}_\mathfrak{p}$. By Proposition 5.4.1, the map

$$\mathfrak{q} \to \mathfrak{q}_\mathfrak{p},$$

is a bijection between the primes of $\mathcal{O}$ contained in $\mathfrak{p}$ the primes of $\mathcal{O}_\mathfrak{p}$. As maximal integral ideals are necessarily prime, $\mathfrak{p}_\mathfrak{p}$ is the unique maximal integral ideal of $\mathcal{O}_\mathfrak{p}$. This implies

$$\mathcal{O}_\mathfrak{p}^* = \mathcal{O}_\mathfrak{p} - \mathfrak{p}_\mathfrak{p},$$

because the complement of the union of all maximal integral ideals is exactly the set of units and $\mathfrak{p}_\mathfrak{p}$ is the only one. It follows that $\mathcal{O}_\mathfrak{p}^* + \mathfrak{p}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}^*$. Indeed, the reverse containment is trivial and the forward containment follows since otherwise $\mathcal{O}_\mathfrak{p}^* + \mathfrak{p}_\mathfrak{p} \subseteq \mathfrak{p}_\mathfrak{p}$ which would imply $\mathcal{O}_\mathfrak{p}^* \subseteq \mathfrak{p}_\mathfrak{p}$ and this is impossible because $\mathfrak{p}_\mathfrak{p}$ is maximal (hence proper). In other words, the sum of a unit of $\mathcal{O}_\mathfrak{p}$ and an element of $\mathfrak{p}_\mathfrak{p}$ is a unit of $\mathcal{O}_\mathfrak{p}$. Also, if $\mathfrak{p}$ itself is maximal we can say more:

Proposition 5.4.4. *Let $\mathcal{O}$ be a noetherian integral domain and $\mathfrak{p}$ be a prime. Then there is an embedding*

$$\mathcal{O}/\mathfrak{p} \to \mathcal{O}_\mathfrak{p}/\mathfrak{p}_\mathfrak{p},$$

identifying $\mathcal{O}_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ with the field of fractions of $\mathcal{O}/\mathfrak{p}$. In particular, if $\mathfrak{p}$ is maximal we have

$$\mathcal{O}/\mathfrak{p}^n \cong \mathcal{O}_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}^n,$$

for all $n \geq 1$.

Proof. Let $n \geq 1$ and consider the homomorphism

$$\phi : \mathcal{O}/\mathfrak{p}^n \to \mathcal{O}_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}^n \qquad \alpha \pmod{\mathfrak{p}^n} \mapsto \alpha \pmod{\mathfrak{p}_\mathfrak{p}^n}.$$

By Proposition 5.4.1, $\mathfrak{p} = \mathfrak{p}_\mathfrak{p} \cap \mathcal{O}$ and therefore this map is an embedding when $n = 1$. As $\mathfrak{p}_\mathfrak{p}$ is maximal, $\mathcal{O}_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is a field and thus must be the field of fractions of $\mathcal{O}/\mathfrak{p}$ under this embedding. It remains to prove the last statement so suppose $\mathfrak{p}$ is maximal. We will show ϕ is both injective and surjective which will finish the proof. For injectivity, it suffices to prove $\ker \phi = 0$. So let $\alpha \in \mathcal{O}$ be a representative of a class in $\mathcal{O}/\mathfrak{p}$ such that such that $\alpha \in \mathfrak{p}_\mathfrak{p}^n$. Then $\alpha = \frac{\beta}{\delta}$ with $\beta \in \mathfrak{p}^n$ and $\delta \notin \mathfrak{p}$. Thus β is a representative of the zero class in $\mathcal{O}/\mathfrak{p}^n$ and so α must be too which implies $\ker \phi = 0$ and injectivity follows. For surjectivity, we first claim that the image of every $\delta \in \mathcal{O} - \mathfrak{p}$ is a unit in $\mathcal{O}/\mathfrak{p}^n$ which equivalent to the fact that $\mathfrak{p}^n + \delta\mathcal{O} = \mathcal{O}$ since there is then a $\gamma \in \mathcal{O}$ such that $\delta - \gamma \in \mathfrak{p}^n$. For $n = 1$, maximality of $\mathfrak{p}$ implies $\mathfrak{p} + \delta\mathcal{O} = \mathcal{O}$. We now argue by induction, so suppose the claim holds for $\mathcal{O}/\mathfrak{p}^{n-1}$. Then $\mathfrak{p}^{n-1} + \delta\mathcal{O} = \mathcal{O}$ whence $\mathfrak{p}^n + \delta\mathfrak{p} = \mathfrak{p}$ and therefore $\mathfrak{p}^n + \delta\mathcal{O} = \mathcal{O}$ by the primality of $\mathfrak{p}$. Now let $\frac{\alpha}{\delta} \in \mathcal{O}_\mathfrak{p}$ be a representative of a class in $\mathcal{O}_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}^n$. As $\delta \notin \mathfrak{p}$, what we have just proved shows that the class represented by δ in $\mathcal{O}/\mathfrak{p}^n$ is invertible. Hence $\frac{\alpha}{\delta}$ represents a class in $\mathcal{O}/\mathfrak{p}^n$ which proves surjectivity. □

A **discrete valuation ring** is a principal ideal domain with a unique maximal integral ideal (recall that principal ideal domains are necessarily noetherian integral domains). In other words, a discrete valuation ring is a local principal ideal domain and is a particularly simple example of a local noetherian integral domain. As principal ideal domains are integrally closed by Lemma 5.1.1, discrete valuation rings are necessarily Dedekind domains. In fact, they are Dedekind domains with exactly one prime by

Proposition 5.3.3. If $\mathcal{O}$ is a discrete valuation ring and $\mathfrak{p}$ is its maximal integral ideal (necessarily prime) then $\mathfrak{p}$ is of the form $\mathfrak{p} = \pi\mathcal{O}$ for some prime $\pi \in \mathcal{O}$. We call π a **uniformizer** of $\mathcal{O}$ and it is uniquely defined up to multiplication by units of $\mathcal{O}$. As every element not in $\mathfrak{p}$ is a unit (since $\mathcal{O}$ is local) and $\mathcal{O}$ is a unique factorization domain (because it is a principal ideal domain), it follows that every $\alpha \in \mathcal{O}$ is of the form $\alpha = \varepsilon\pi^n$ for some $\varepsilon \in \mathcal{O}^*$ and $n \geq 0$. In particular, π is the only prime of $\mathcal{O}$ up to multiplication by units and every integral ideal of $\mathcal{O}$ is of the form $\mathfrak{p}^n$ for some $n \geq 0$. As $\mathcal{O}$ is a principal ideal domain, it follows that $\mathfrak{p}$ is also the only prime of $\mathcal{O}$. Thus $\mathfrak{p}$ is both the unique maximal integral ideal and prime $\mathcal{O}$. Moreover, if K is the field of fractions of $\mathcal{O}$ then every nonzero $\kappa \in K$ can be uniquely expressed as

$$\kappa = \varepsilon\pi^n,$$

for some $\varepsilon \in \mathcal{O}^*$ and $n \in \mathbb{Z}$. The **valuation** v associated to $\mathcal{O}$ on K is the function defined by

$$v : K \to \mathbb{Z} \cup \{\infty\} \qquad \kappa \mapsto v(\kappa) = \begin{cases} n & \text{if } \kappa = \varepsilon\pi^n, \\ \infty & \text{if } \kappa = 0. \end{cases}$$

We call $v(\kappa)$ the **valuation** of κ with respect to $\mathcal{O}$. Note that $v(\kappa) = 0$ if and only if κ is a unit in $\mathcal{O}$. If $\kappa \neq 0$ then the $v(\kappa)$ is characterized by the equation

$$\kappa\mathcal{O} = \mathfrak{p}^{v(\kappa)}, \tag{5.5}$$

since $\kappa = \varepsilon\pi^{v(\kappa)}$. Moreover, if $\kappa = \varepsilon\pi^n$ and $\eta = \delta\pi^m$, we have

$$\kappa\eta = \varepsilon\delta\pi^{n+m} \quad \text{and} \quad \kappa + \eta = (\varepsilon\pi^{n-\min(n,m)} + \delta\pi^{m-\min(n,m)})\pi^{\min(n,m)},$$

where $\varepsilon\pi^{n-\min(n,m)} + \delta\pi^{m-\min(n,m)}$ is a unit of $\mathcal{O}$ because one of the exponents of π is zero so that it is a sum of a unit of $\mathcal{O}$ and an element of $\mathfrak{p}$. Therefore v satisfies the properties

$$v(\kappa\eta) = v(\kappa) + v(\eta) \quad \text{and} \quad v(\kappa + \eta) \geq \min(v(\kappa), v(\eta)). \tag{5.6}$$

In view of the first identity in Equation (5.6), v restricted to K^* is a surjective homomorphism. Discrete valuation rings, and hence valuation themselves, arise as as localizations of Dedekind domains at primes. Before we deduce this, we will establish a few facts. The first is that localization respects inversion of fractional ideals:

Proposition 5.4.5. *Let $\mathcal{O}$ be a Dedekind domain with field of fractions K and $D \subseteq \mathcal{O} - \{0\}$ be a multiplicative subset. Then for any fractional ideal $\mathfrak{f}$, we have*

$$\mathfrak{f}^{-1}D^{-1} = (\mathfrak{f}D^{-1})^{-1}.$$

Proof. In light of the fact that every fractional ideal $\mathfrak{f}$ is of the form $\mathfrak{f} = \frac{1}{\delta}\mathfrak{a}$ for some nonzero $\delta \in \mathcal{O}$ and integral ideal $\mathfrak{a}$, we have $\mathfrak{f}^{-1} = \delta\mathfrak{a}^{-1}$ and so it suffices to prove the claim for integral ideals. We will show $\mathfrak{a}^{-1}D^{-1} = (\mathfrak{a}D^{-1})^{-1}$. For the forward inclusion, let $\frac{\alpha}{\delta} \in \mathfrak{a}^{-1}D^{-1}$. Since $\alpha \in \mathfrak{a}^{-1}$, Proposition 5.3.2 implies $\alpha\mathfrak{a} \subseteq \mathcal{O}$ and therefore $\frac{\alpha}{\delta}\mathfrak{a}D^{-1} \subseteq \mathcal{O}D^{-1}$. Invoking Proposition 5.3.2 again shows $\frac{\alpha}{\delta} \in (\mathfrak{a}D^{-1})^{-1}$ which proves the forward inclusion. For the reverse inclusion, suppose $\kappa \in (\mathfrak{a}D^{-1})^{-1}$. By Proposition 5.3.2, $\kappa\mathfrak{a}D^{-1} \subseteq \mathcal{O}D^{-1}$ and multiplying by $\mathfrak{a}^{-1}$ shows $\kappa \in \mathfrak{a}^{-1}D^{-1}$ since $\mathfrak{a}$ is an ideal of $\mathcal{O}$. This proves the reverse inclusion. □

Our second fact is that that the localization of a Dedekind domain is again a Dedekind domain:

Proposition 5.4.6. *Let $\mathcal{O}$ be a Dedekind domain and $D \subseteq \mathcal{O} - \{0\}$ be a multiplicative subset. Then $\mathcal{O}D^{-1}$ is a Dedekind domain.*

Proof. We have seen that $\mathcal{O}D^{-1}$ is a noetherian integral domain since $\mathcal{O}$ is. $\mathcal{O}D^{-1}$ is also integrally closed by Proposition 5.4.2. It remains to show that every prime of $\mathcal{O}D^{-1}$ is maximal. Letting $\mathfrak{p}D^{-1}$ be such prime, we see that it must be maximal because $\mathfrak{p}$ is and the bijections in Proposition 5.4.1 are inclusion-preserving.□

As an immediate consequence of Proposition 5.4.6, the localization $\mathcal{O}_\mathfrak{p}$ at any prime $\mathfrak{p}$ is a Dedekind domain if $\mathcal{O}$ is. We can now show that localizing a Dedekind domain at a prime produces a discrete valuation ring. Actually, we will prove the following stronger statement:

Theorem 5.4.1. *Let $\mathcal{O}$ be a noetherian integral domain. Then $\mathcal{O}$ is a Dedekind domain if and only if $\mathcal{O}_\mathfrak{p}$ is a discrete valuation ring for all primes $\mathfrak{p}$.*

Proof. Recall that $\mathcal{O}_\mathfrak{p}$ is local for any prime $\mathfrak{p}$ of $\mathcal{O}$ and so $\mathfrak{p}_\mathfrak{p}$ is the unique maximal integral ideal of $\mathcal{O}_\mathfrak{p}$. For the forward implication, suppose $\mathcal{O}$ is a Dedekind domain. Then $\mathcal{O}_\mathfrak{p}$ is as well by Proposition 5.4.6. Since primes are maximal in Dedekind domains, $\mathfrak{p}_\mathfrak{p}$ is also the only prime of $\mathcal{O}_\mathfrak{p}$. By uniqueness of prime factorizations of fractional ideals, there exists $\pi \in \mathfrak{p}_\mathfrak{p} - \mathfrak{p}_\mathfrak{p}^2$ and hence $\mathfrak{p}_\mathfrak{p} = \pi\mathcal{O}_\mathfrak{p}$ which further implies $\mathfrak{p}_\mathfrak{p}^n = \pi^n\mathcal{O}_\mathfrak{p}$ for all $n \geq 1$. As these are the only integral ideals of $\mathcal{O}_\mathfrak{p}$, we see that $\mathcal{O}_\mathfrak{p}$ is a principal ideal domain and hence a discrete valuation ring. This proves the forward implication. For the reverse implication, suppose all of the localizations $\mathcal{O}_\mathfrak{p}$ are discrete valuation rings. Then they are principal ideal domains and hence integrally closed by Lemma 5.1.1. As the intersection of integrally closed rings is clearly integrally closed, it follows from Proposition 5.4.3 that $\mathcal{O}$ is integrally closed and therefore an integrally closed integral domain by our assumptions. As $\mathcal{O}$ is noetherian by assumption, it remains to show that every prime $\mathfrak{q}$ of $\mathcal{O}$ is maximal. Since $\mathcal{O}$ is noetherian (which we recall is equivalent to every nonempty collection of ideals having a maximal element), $\mathfrak{q} \subseteq \mathfrak{p}$ for some maximal integral ideal $\mathfrak{p}$ which is necessarily prime. Under the inclusion-preserving bijections in Proposition 5.4.1, we have

$$\mathfrak{q} = \mathfrak{q}_\mathfrak{p} \cap \mathcal{O} = \mathfrak{p}_\mathfrak{p} \cap \mathcal{O} = \mathfrak{p},$$

where the middle equality holds because $\mathfrak{q}_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}$ as $\mathcal{O}_\mathfrak{p}$ has a unique prime which is also the unique maximal integral ideal as we have seen. Hence $\mathfrak{q}$ itself must be maximal. □

Proposition 5.4.3 and Theorem 5.4.1 together form a powerful simplification tool for Dedekind domains $\mathcal{O}$. For if we want to prove a property about a fractional ideal $\mathfrak{f}$ of $\mathcal{O}$, Proposition 5.4.3 implies that it suffices to show this property holds for the corresponding fractional ideals $\mathfrak{f}_\mathfrak{p}$ in the localizations $\mathcal{O}_\mathfrak{p}$ at all primes $\mathfrak{p}$ and is preserved under intersections. Moreover, the localizations $\mathcal{O}_\mathfrak{p}$ are discrete valuation rings by Theorem 5.4.1 in addition to being Dedekind domains by Proposition 5.4.6. In particular, the localizations $\mathcal{O}_\mathfrak{p}$ are also principal ideal domains. This latter fact is often immensely helpful. In any case, let $v_\mathfrak{p}$ denote the valuation of $\mathcal{O}_\mathfrak{p}$. We call $v_\mathfrak{p}$ the **valuation** associated to the prime $\mathfrak{p}$ of $\mathcal{O}$. These valuations are intimately connected to the prime factorization of principal fractional ideals. Indeed, the prime factorization of fractional ideals implies that for any $\kappa \in K^*$, we have

$$\kappa\mathcal{O} = \prod_{\mathfrak{q} \text{ prime}} \mathfrak{q}^{e_\mathfrak{q}},$$

where the product is taken over all primes $\mathfrak{q}$ of $\mathcal{O}$, $e_\mathfrak{q} \in \mathbb{Z}$ for all $\mathfrak{q}$, and all but finitely many of the $e_\mathfrak{q}$ are zero. We claim that $v_\mathfrak{p}(\kappa) = e_\mathfrak{p}$ for all primes $\mathfrak{p}$. To see this, first observe that if $\mathfrak{p}$ and $\mathfrak{q}$ are distinct primes then we have $\mathfrak{q}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$. Indeed, by uniqueness of prime factorizations of fractional ideals choose $\alpha \in \mathfrak{q} - \mathfrak{p}$.

Then $\alpha \in \mathfrak{q}_\mathfrak{p} - \mathfrak{p}_\mathfrak{p}$ and hence α is invertible in $\mathcal{O}_\mathfrak{p}$ because $\mathcal{O}_\mathfrak{p}^* = \mathcal{O}_\mathfrak{p} - \mathfrak{p}_\mathfrak{p}$. Therefore $1 \in \mathfrak{q}_\mathfrak{p}$ and so the integral ideal $\mathfrak{q}_\mathfrak{p}$ must be $\mathcal{O}_\mathfrak{p}$. This fact and the prime factorization of $\kappa\mathcal{O}$ together imply

$$\kappa\mathcal{O}_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}^{e_\mathfrak{p}},$$

and we readily see that $v_\mathfrak{p}(\kappa) = e_\mathfrak{p}$ by Equation (5.5). In particular, $v_\mathfrak{p}(\kappa) = 0$ for all but finitely many primes $\mathfrak{p}$.

Remark 5.4.2. *The valuation $v_\mathfrak{p}$ associated to the prime $\mathfrak{p}$ of $\mathcal{O}$ is sometimes called an* ***exponential valuation****.*

Continue to let $\mathcal{O}$ be a Dedekind domain with field of fractions K and let X be a set of all but finitely many primes of $\mathcal{O}$. Then $\mathcal{O}(X)$ is a Dedekind domain by Proposition 5.4.6 and by Proposition 5.4.1 the primes $\mathfrak{p}_X$ of $\mathcal{O}(X)$ are of the form $\mathfrak{p}_X = \mathfrak{p}(X)$ for $\mathfrak{p} \in X$. Moreover, $\mathcal{O}$ and $\mathcal{O}(X)$ have the same localizations at $\mathfrak{p}$ and $\mathfrak{p}_X$ respectively. That is,

$$\mathcal{O}_\mathfrak{p} = \mathcal{O}(X)_{\mathfrak{p}_X}. \tag{5.7}$$

Indeed, observe that

$$\mathcal{O}(X)_{\mathfrak{p}_X} = \left\{ \frac{\alpha\gamma}{\beta\delta} \in K : \alpha, \beta, \gamma, \delta \in \mathcal{O} \text{ with } \delta \not\equiv 0 \pmod{\mathfrak{p}} \text{ and } \beta, \gamma \not\equiv 0 \pmod{\mathfrak{p}} \text{ for some } \mathfrak{p} \in X \right\},$$

which is exactly $\mathcal{O}_\mathfrak{p}$. From Equation (5.7) and the fact that the primes $\mathfrak{p}_X$ are the only primes of $\mathcal{O}(X)$, the prime factorization of a fractional ideal $\mathfrak{f}(X)$ of $\mathcal{O}(X)$, for a fractional ideal $\mathfrak{f}$ of $\mathcal{O}$, is obtained from that of $\mathfrak{f}$ by removing those primes factors corresponding to primes not in X. Also, from Proposition 5.4.3 and Equation (5.7) we conclude that the original fractional ideal can be recovered by taking the intersections of $\mathfrak{f}(X)_\mathfrak{p}$ for all $\mathfrak{p} \notin X$. Moreover, the ideal class and unit groups of $\mathcal{O}(X)$ are related to those of $\mathcal{O}$ via the following exact sequence:

Proposition 5.4.7. *Let $\mathcal{O}$ be a Dedekind domain with field of fractions K and let X be set of all but finitely many primes of $\mathcal{O}$. Then the sequence*

$$1 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{O}(X)^* \longrightarrow \bigoplus_{\mathfrak{p} \notin X} K^*/\mathcal{O}_\mathfrak{p}^* \xrightarrow{\prod_{\mathfrak{p} \notin X} \mathfrak{p}^{v_\mathfrak{p}}} \mathrm{Cl}(\mathcal{O}) \xrightarrow{\mathcal{O}(X)} \mathrm{Cl}(\mathcal{O}(X)) \longrightarrow 1,$$

where the fourth map takes the representative $(\kappa_\mathfrak{p})_{\mathfrak{p} \notin X}$ of a class in $\bigoplus_{\mathfrak{p} \notin X} K^*/\mathcal{O}_\mathfrak{p}^*$ to the class represented by $\prod_{\mathfrak{p} \notin X} \mathfrak{p}^{v_\mathfrak{p}(\kappa_\mathfrak{p})}$ in $\mathrm{Cl}(\mathcal{O})$ and the fifth map takes the representative $\mathfrak{a}$ of an ideal class in $\mathrm{Cl}(\mathcal{O})$ to the ideal class represented by $\mathfrak{a}(X)$ in $\mathrm{Cl}(\mathcal{O}(X))$, is exact. Moreover, $K^*/\mathcal{O}_\mathfrak{p}^* \cong \mathbb{Z}$ for all primes $\mathfrak{p}$ of $\mathcal{O}$.

Proof. The sequence is exact at $\mathcal{O}^*$ because the second map is injective. For exactness at $\mathcal{O}(X)^*$, it is clear that the image of the second map belongs to the kernel of the third map. So suppose $\alpha \in \mathcal{O}(X)^*$ belongs to the kernel of the third map. Then $\alpha \in \mathcal{O}_\mathfrak{p}^*$ for all $\mathfrak{p} \notin X$ and for all $\mathfrak{p} \in X$ as well by Equation (5.7). It follows from Proposition 5.4.3 that $\alpha \in \mathcal{O}^*$ so it is in the image of the second map which proves exactness at $\mathcal{O}(X)^*$. For exactness at $\bigoplus_{\mathfrak{p} \notin X} K^*/\mathcal{O}_\mathfrak{p}^*$, from Equation (5.7) again we see that the image of the third map is contained in the kernel of the fourth. So suppose $(\kappa_\mathfrak{p})_{\mathfrak{p} \notin X}$ is a representative of a class in $\bigoplus_{\mathfrak{p} \notin X} K^*/\mathcal{O}_\mathfrak{p}^*$ that is contained in the kernel of the fourth map. Then there is a $\kappa \in K^*$ such that

$$\prod_{\mathfrak{p} \notin X} \mathfrak{p}^{v_\mathfrak{p}(\kappa_\mathfrak{p})} = \kappa\mathcal{O} = \prod_{\mathfrak{p} \text{ prime}} \mathfrak{p}^{v_\mathfrak{p}(\kappa)}.$$

By the prime factorization of fractional ideals, $v_\mathfrak{p}(\kappa) = 0$ for all $\mathfrak{p} \in X$ and $v_\mathfrak{p}(\kappa) = v_\mathfrak{p}(\kappa_\mathfrak{p})$ for all $\mathfrak{p} \notin X$. As $v_\mathfrak{p}(\kappa) = 0$ for all $\mathfrak{p} \in X$, we have $\kappa \in \mathcal{o}_\mathfrak{p}^*$ for all $\mathfrak{p} \in X$. Because the primes of $\mathcal{o}(X)$ are $\mathfrak{p}_X$ for $\mathfrak{p} \in X$, Equation (5.7) and Proposition 5.4.3 together imply that $\bigcap_{\mathfrak{p}\in X} \mathcal{o}_\mathfrak{p}^* = \mathcal{o}(X)^*$ and therefore $\kappa \in \mathcal{o}(X)^*$. As $v_\mathfrak{p}(\kappa) = v_\mathfrak{p}(\kappa_\mathfrak{p})$ for all $\mathfrak{p} \notin X$, we have $\kappa \equiv \kappa_\mathfrak{p} \pmod{\mathcal{o}_\mathfrak{p}^*}$ for all $\mathfrak{p} \notin X$ because they have the same power of a uniformizer for $\mathcal{o}_\mathfrak{p}^*$. We have shown that $\kappa \in \mathcal{o}(X)^*$ and $\kappa \equiv \kappa_\mathfrak{p} \pmod{\mathcal{o}_\mathfrak{p}^*}$ for all $\mathfrak{p} \notin X$ which means κ maps to $(\kappa_\mathfrak{p})_{\mathfrak{p}\notin X}$ under the third map. In other words, $(\kappa_\mathfrak{p})_{\mathfrak{p}\notin X}$ is in the image of the third map proving exactness at $\bigoplus_{\mathfrak{p}\notin X} K^*/\mathcal{o}_\mathfrak{p}^*$. For exactness at $\mathrm{Cl}(\mathcal{o})$, we first show that $\mathfrak{p}(X) = \mathcal{o}(X)$ for any prime $\mathfrak{p} \notin X$. Indeed, by maximality we can choose $\alpha \in \mathfrak{p} - \bigcup_{\mathfrak{q}\in X} \mathfrak{q}$. This fores α to be invertible in $\mathcal{o}(X)$ so that $1 \in \mathfrak{p}(X)$ and hence the ideal $\mathfrak{p}(X)$ must be $\mathcal{o}(X)$. It follows that the image of the fourth map is contained in the kernel of the fifth. Now suppose $\mathfrak{a}$ is an integral ideal representing a class in $\mathrm{Cl}(\mathcal{o})$ that is contained in the kernel of the fifth map. Then there is a $\kappa \in K^*$ such that

$$\mathfrak{a}(X) = \kappa\mathcal{o}(X),$$

In view of Equation (5.7) and Proposition 5.4.3 again, taking the intersection with the localizations $\mathcal{o}_\mathfrak{p}$ for all $\mathfrak{p} \notin X$ shows that

$$\mathfrak{a} = \kappa\mathcal{o},$$

and therefore $(v_\mathfrak{p}(\kappa))_{\mathfrak{p}\notin X}$ is a representative of a class in $\bigoplus_{\mathfrak{p}\notin X} K^*/\mathcal{o}_\mathfrak{p}^*$ whose image under the fourth map is $\mathfrak{a}$. In other words, $\mathfrak{a}$ is in the image of the third map proving exactness at $\mathrm{Cl}(\mathcal{o})$. Lastly, to show exactness $\mathrm{Cl}(\mathcal{o}(X))$ recall that every ideal class can be represented by an integral ideal and that the primes $\mathfrak{p}_X$ of $\mathcal{o}(X)$ are of the form $\mathfrak{p}_X = \mathfrak{p}(X)$ for $\mathfrak{p} \in X$. Then the fifth map is surjective since the primes of $\mathcal{o}$ and $\mathcal{o}(X)$ generate their corresponding ideal class groups by the prime factorization of fractional ideals. Therefore exactness holds at $\mathrm{Cl}(\mathcal{o}(X))$ which completes exactness of the sequence. The last statement follows from the first isomorphism theorem since the valuation $v_\mathfrak{p}$ restricted to K^* is a surjective homomorphism and the kernel is precisely the set of units of $\mathcal{o}_\mathfrak{p}$. This completes the proof. □

We now turn to the setting of a number field K. We let S denote a finite set of primes of $\mathcal{O}_K$ and let X denote the set of all primes that do not belong to S. The **ring of S-integers** $\mathcal{O}_K^S$ of K is defined by

$$\mathcal{O}_K^S = \mathcal{O}_K(X).$$

We call any $\alpha \in \mathcal{O}_K^S$ an **algebraic S-integer**. The **S-class group** $\mathrm{Cl}^S(K)$ of K is the ideal class group of $\mathcal{O}_K^S$. The **S-class number** h_K^S of K is the class number of $\mathrm{Cl}^S(K)$. The **S-unit group** of K is the unit group $(\mathcal{O}_K^S)^*$ of $\mathcal{O}_K^S$ and we call any element of $(\mathcal{O}_K^S)^*$ an **S-unit** of K.

5.5 Dedekind Extensions

Having discussed the prime factorization of fractional ideals in Dedekind domains, we now turn to discussing how primes factor when considered in a larger Dedekind domain. Let $\mathcal{o}$ be a Dedekind domain with field of fractions K, L/K be a finite separable extension, and $\mathcal{O}$ be the integral closure of $\mathcal{o}$ in L. We first show that $\mathcal{O}$ is also a Dedekind domain:

Proposition 5.5.1. *Let $\mathcal{o}$ be a Dedekind domain with field of fractions K, L/K be a finite separable extension, and $\mathcal{O}$ be the integral closure of $\mathcal{o}$ in L. Then $\mathcal{O}$ is a Dedekind domain.*

Proof. $\mathcal{O}$ is an integral domain because it is a subring of L and is integrally closed since it is the integral closure of $\mathcal{o}$. We now show $\mathcal{O}$ is noetherian. Let the degree of L/K be n and let $\lambda_1, \ldots, \lambda_n$ be a basis for L/K. By Proposition 5.1.3 we may multiply by a nonzero element of $\mathcal{o}$, if necessary, to ensure that this

basis is contained in $\mathcal{O}$. Then $d_{L/K}(\lambda_1, \ldots, \lambda_n)$ is nonzero by Proposition 5.2.5 and Lemma 5.2.2 implies that

$$d_{L/K}(\lambda_1, \ldots, \lambda_n)\mathcal{O} \subseteq o\lambda_1 + \cdots + o\lambda_n.$$

Thus $\mathcal{O}$ is a finitely generated o-module of rank at most n. In particular, every ideal of $\mathcal{O}$ is also a finitely generated o-module and therefore also a finitely generated $\mathcal{O}$-module. It remains to show that every nonzero prime ideal is maximal. Letting $\mathfrak{P}$ be a nonzero prime ideal, it suffices to show $\mathcal{O}/\mathfrak{P}$ is a field. To this end, consider the homomorphism

$$\phi : o \to \mathcal{O}/\mathfrak{P} \qquad \alpha \mapsto \alpha \pmod{\mathfrak{p}}.$$

Then $\ker \phi = \mathfrak{P} \cap o$ and we claim $\mathfrak{P} \cap o$ is a nonzero prime ideal of o. It is clearly an ideal of o and is prime because $\mathfrak{P}$ is. To see that it is nonzero, let $\alpha \in \mathfrak{P}$ be nonzero. As α is algebraic over o, we have

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0,$$

for some $n \geq 1$ and $a_i \in o$ for $0 \leq i \leq n-1$. Taking n minimal, we have $a_0 \neq 0$. Isolating a_0 shows that $a_0 \in \mathfrak{P}$ and hence $a_0 \in \mathfrak{P} \cap o$. Therefore $\mathfrak{P} \cap o = \mathfrak{p}$ for some prime $\mathfrak{p}$. Hence $\ker \phi = \mathfrak{p}$ and by the first isomorphism theorem, ϕ induces an injection $\phi : \mathbb{F}_\mathfrak{p} \to \mathcal{O}/\mathfrak{P}$. As $\mathcal{O}$ is the integral closure of o in L, it is obtained from o by forming a polynomial ring with algebraic elements in L. The latter injection then shows that $\mathcal{O}/\mathfrak{P}$ is obtained from $\mathbb{F}_\mathfrak{p}$ by adjoining these algebraic elements reduced modulo $\mathfrak{P}$. Their reductions are seen to be algebraic over $\mathbb{F}_\mathfrak{p}$ by reducing their minimal polynomials over K, viewed as elements of $\mathcal{O}[x]$, modulo $\mathfrak{P}$ upon recalling that the coefficients of which are in o by Proposition 5.1.3. Hence $\mathcal{O}/\mathfrak{P}$ is obtained from $\mathbb{F}_\mathfrak{p}$ by adjoining algebraic elements to $\mathbb{F}_\mathfrak{p}$ (since $\mathbb{F}_\mathfrak{p}[\overline{\alpha}] = \mathbb{F}_\mathfrak{p}(\overline{\alpha})$ if $\overline{\alpha}$ is algebraic over $\mathbb{F}_\mathfrak{p}$) and is therefore a field. □

Under the assumptions of Proposition 5.5.1, it follows from Proposition 5.1.3 that L is the field of fractions of $\mathcal{O}$. We say that a ring extension $\mathcal{O}/o$ is a **Dedekind extension** of a finite separable extension L/K if $\mathcal{O}$ and o are Dedekind domains whose field of fractions are L and K respectively and $\mathcal{O}$ is the integral closure of o in L. This immediately implies

$$\mathcal{O} \cap K = o.$$

Moreover, if $\mathcal{O}$ is also a free o-module then $\mathcal{O}/o$ admits an integral basis. In light of Proposition 5.4.6, $\mathcal{O}D^{-1}/oD^{-1}$ is also a Dedekind extension of L/K for any multiplicative subset $D \subseteq o - \{0\}$. We call $\mathcal{O}D^{-1}/oD^{-1}$ the **localization** of $\mathcal{O}/o$ at D. In the case $D = o - \mathfrak{p}$ for a prime $\mathfrak{p}$ of o, we call $\mathcal{O}_\mathfrak{p}/o_\mathfrak{p}$ the **localization** of $\mathcal{O}/o$ at $\mathfrak{p}$. We say that a Dedekind extension $\mathcal{O}/o$ is **local** if o is a discrete valuation ring. As o is a principal ideal domain, Theorem 5.2.1 implies that $\mathcal{O}/o$ admits an integral basis if it is local. In addition, as o has a unique prime we see that $\mathcal{O}$ has finitely many primes since they all must be above the prime of o. By Proposition 5.3.3 this forces $\mathcal{O}$ to be a principal ideal domain as well. In particular, localizing the Dedekind extension $\mathcal{O}/o$ at $\mathfrak{p}$ produces the local Dedekind extension $\mathcal{O}_\mathfrak{p}/o_\mathfrak{p}$. With this phrasing, the proof of Proposition 5.5.1 gives the following corollary:

Corollary 5.5.1. *Let $\mathcal{O}/o$ be a Dedekind extension of a degree n separable extension L/K. Then $\mathcal{O}$ is a finitely generated o-module of rank at most n and every prime $\mathfrak{P}$ of $\mathcal{O}$ satisfies*

$$\mathfrak{P} \cap o = \mathfrak{p},$$

for some prime $\mathfrak{p}$ of o. Moreover, $\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}$ is a finite extension of degree at most n.

Proof. The first statement was deduced in the proof of Proposition 5.5.1 along with the fact that there is an injection $\mathbb{F}_\mathfrak{p} \to \mathbb{F}_\mathfrak{P}$. From this injection it follows that $\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}$ is an extension and since $\mathcal{O}$ is a finitely generated o-module of rank at most n, $\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}$ has degree at most n. □

Continue to let $\mathcal{O}/o$ be a Dedekind extension of a degree n separable extension L/K. Note that if $\mathfrak{f}$ is a fractional ideal of o then $\mathfrak{f}\mathcal{O}$ is a fractional ideal of $\mathcal{O}$ because $\mathfrak{f}$ is a finitely generated o-module and hence a finitely generated $\mathcal{O}$-module. In particular, $\mathfrak{a}\mathcal{O}$ is an integral ideal of $\mathcal{O}$ for any integral ideal $\mathfrak{a}$ of o. Now let $\mathfrak{P}$ and $\mathfrak{p}$ be primes of $\mathcal{O}$ and o respectively. We say that $\mathfrak{P}$ is **above** $\mathfrak{p}$, or equivalently, $\mathfrak{p}$ is **below** $\mathfrak{P}$ if

$$\mathfrak{P} \cap o = \mathfrak{p}.$$

Then Corollary 5.5.1 implies that every prime of $\mathcal{O}$ is above exactly one prime of o. If $\mathfrak{P}$ is above $\mathfrak{p}$ then $\mathfrak{P} \mid \mathfrak{p}\mathcal{O}$. Indeed, since $\mathfrak{p} \subseteq \mathfrak{P}$ we have $\mathfrak{p}\mathcal{O} \subseteq \mathfrak{P}$ which is to say that $\mathfrak{P}$ divides $\mathfrak{p}\mathcal{O}$. This implies that only finitely many primes $\mathfrak{P}$ can lie above a prime $\mathfrak{p}$ and they are exactly the prime factors of $\mathfrak{p}\mathcal{O}$. Moreover, every prime of o is below at least one prime of $\mathcal{O}$. To see this, it suffices to show that the integral ideal $\mathfrak{p}\mathcal{O}$ satisfies

$$\mathfrak{p}\mathcal{O} \neq \mathcal{O}.$$

By the prime factorization of fractional ideals, choosing $\alpha \in \mathfrak{p} - \mathfrak{p}^2$ we can write $\alpha o = \mathfrak{p}\mathfrak{a}$ for some integral ideal $\mathfrak{a}$ relatively prime to $\mathfrak{p}$. Then $\mathfrak{a} + \mathfrak{p} = o$ and so there exist $\beta \in \mathfrak{a}$ and $\gamma \in \mathfrak{p}$ such that $\beta + \gamma = 1$. Then $\beta\mathfrak{p} \subseteq \alpha o$ and $\beta \notin \mathfrak{p}$ because otherwise $1 \in \mathfrak{p}$ which is absurd since $\mathfrak{p}$ is proper. Now if $\mathfrak{p}\mathcal{O} = \mathcal{O}$, it would follow that $\beta\mathcal{O} \subseteq \alpha\mathcal{O}$ which would imply $\beta = \alpha\delta$ for some $\delta \in \mathcal{O}$. Hence $\delta = \frac{\beta}{\alpha} \in K$ because K is the field of fractions of o. As $\mathcal{O} \cap K = o$, we also find that $\delta \in o$. But since $\alpha \in \mathfrak{p}$, it follows that $\beta \in \mathfrak{p}$ which is a contraction. Therefore $\mathfrak{p}\mathcal{O} \neq \mathcal{O}$. In addition, we have

$$\mathfrak{p}\mathcal{O} \cap o = \mathfrak{p}.$$

The reverse inclusion is obvious. For the forward inclusion, as $\mathfrak{p}\mathcal{O} \neq \mathcal{O}$ there exists a prime factor $\mathfrak{P}$ of $\mathfrak{p}\mathcal{O}$ so that $\mathfrak{p}\mathcal{O} \subseteq \mathfrak{P}$. But then $\mathfrak{P}$ is above $\mathfrak{p}$ so that $\mathfrak{P} \cap o = \mathfrak{p}$ and the forward inclusion follows since $\mathfrak{p}\mathcal{O} \subseteq \mathfrak{P}$. We illustrate the relationship between $\mathfrak{P}$ and $\mathfrak{p}$ via the following extension:

$$\begin{array}{c} \mathfrak{P} \subset \mathcal{O} \subseteq L \\ \Big| \\ \mathfrak{p} \subset o \subseteq K. \end{array}$$

We have the residue class fields $\mathbb{F}_\mathfrak{P}$ and $\mathbb{F}_\mathfrak{p}$. The former is a finite extension of the latter of degree at most n by Corollary 5.5.1. Accordingly, we call $\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}$ the **residue class extension** of $\mathcal{O}/o$ for $\mathfrak{P}$. Actually since Corollary 5.5.1 implies $\mathcal{O}$ is at finitely generated o-module of rank at most n, the quotient ring $\mathfrak{B}/\mathfrak{A}$ is a $\mathbb{F}_\mathfrak{p}$-vector space of dimension at most n for any integral ideals $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{p} \subseteq \mathfrak{A} \subseteq \mathfrak{B}$. In any case, we define the **inertia degree** $f_\mathfrak{p}(\mathfrak{P})$ of $\mathfrak{P}$ relative to $\mathfrak{p}$ by

$$f_\mathfrak{p}(\mathfrak{P}) = [\mathbb{F}_\mathfrak{P} : \mathbb{F}_\mathfrak{p}].$$

That is, $f_\mathfrak{p}(\mathfrak{P})$ is the dimension of the residue field $\mathbb{F}_\mathfrak{P}$ as a $\mathbb{F}_\mathfrak{p}$-vector space. The **ramification index** $e_\mathfrak{p}(\mathfrak{P})$ is the power of $\mathfrak{P}$ appearing in the prime factorization of $\mathfrak{p}\mathcal{O}$. If $\mathfrak{p}\mathcal{O}$ has prime factors $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ then the prime factorization of $\mathfrak{p}\mathcal{O}$ is

$$\mathfrak{p}\mathcal{O} = \mathfrak{P}_1^{e_\mathfrak{p}(\mathfrak{P}_1)} \cdots \mathfrak{P}_r^{e_\mathfrak{p}(\mathfrak{P}_r)}.$$

All of this is preserved under localization as the following proposition shows:

Proposition 5.5.2. *Let $\mathcal{O}/o$ be a Dedekind extension of a finite separable extension L/K and let $D \subseteq o - \{0\}$ be a multiplicative subset. Then for the Dedekind extension $\mathcal{O}D^{-1}/oD^{-1}$, $\mathfrak{P}D^{-1}$ is above $\mathfrak{p}D^{-1}$ if and only if $\mathfrak{P}$ is above $\mathfrak{p}$. Moreover, there are isomorphisms*

$$\mathbb{F}_{\mathfrak{P}D^{-1}} \cong \mathbb{F}_{\mathfrak{P}} \quad \textit{and} \quad \mathbb{F}_{\mathfrak{p}D^{-1}} \cong \mathbb{F}_{\mathfrak{p}}.$$

Lastly, we have equalities

$$f_{\mathfrak{p}D^{-1}}(\mathfrak{P}D^{-1}) = f_{\mathfrak{p}}(\mathfrak{P}) \quad \textit{and} \quad e_{\mathfrak{p}D^{-1}}(\mathfrak{P}D^{-1}) = e_{\mathfrak{p}}(\mathfrak{P}).$$

Proof. Note that $\mathfrak{P}$ and $\mathfrak{p}$ are necessarily primes disjoint from D. On the one hand, suppose $\mathfrak{P}$ is above $\mathfrak{p}$. Then $\mathfrak{P} \cap o = \mathfrak{p}$ and applying the first bijection in Proposition 5.4.1 gives

$$\mathfrak{P}D^{-1} \cap oD^{-1} = \mathfrak{p}D^{-1}.$$

Hence $\mathfrak{P}D^{-1}$ is above $\mathfrak{p}D^{-1}$. On the other hand, suppose $\mathfrak{P}D^{-1}$ is above $\mathfrak{p}D^{-1}$. Then $\mathfrak{P}D^{-1} \cap oD^{-1} = \mathfrak{p}D^{-1}$ and applying the second bijection in Proposition 5.4.1 gives

$$\mathfrak{P} \cap o = \mathfrak{p}.$$

(recall $o \subseteq \mathcal{O}$) so that $\mathfrak{P}$ is above $\mathfrak{p}$. This proves the first statement. For the second statement, consider the homomorphisms

$$\Phi : \mathcal{O} \to \mathbb{F}_{\mathfrak{P}D^{-1}} \qquad \alpha \mapsto \alpha \pmod{\mathfrak{P}D^{-1}},$$

and

$$\phi : o \to \mathbb{F}_{\mathfrak{p}D^{-1}} \qquad \alpha \mapsto \alpha \pmod{\mathfrak{p}D^{-1}}.$$

They are surjections since $\mathcal{O}\mathfrak{P}D^{-1} = \mathcal{O}D^{-1}\mathfrak{P}D^{-1}$ and $o\mathfrak{p}D^{-1} = oD^{-1}\mathfrak{p}D^{-1}$. Moreover, $\ker\Phi = \mathfrak{P}$ and $\ker\phi = \mathfrak{p}$. Therefore the isomorphisms follow from the first isomorphism theorem. This proves the second statement. In fact, these two isomorphisms together give

$$f_{\mathfrak{p}D^{-1}}(\mathfrak{P}D^{-1}) = f_{\mathfrak{p}}(\mathfrak{P}),$$

by the definition of the inertia degrees. Finally, we have

$$e_{\mathfrak{p}D^{-1}}(\mathfrak{P}D^{-1}) = e_{\mathfrak{p}}(\mathfrak{P}),$$

because the first statement implies that the prime factors of $\mathfrak{p}D^{-1}\mathcal{O}$ correspond to those of $\mathfrak{p}\mathcal{O}$ and hence their powers must be the same. □

Let us now introduce towers of Dedekind extensions. We say that $\mathcal{O}'/\mathcal{O}/o$ is a **tower** of Dedekind extensions for a finite separable tower of extensions $M/L/K$ if $\mathcal{O}'/\mathcal{O}$ and $\mathcal{O}/o$ are Dedekind extensions for M/L and L/K respectively. Now let $\mathfrak{P}'$, $\mathfrak{P}$, and $\mathfrak{p}$ be primes of $\mathcal{O}'$, $\mathcal{O}$, and o respectively with $\mathfrak{P}'$ above $\mathfrak{P}$ and $\mathfrak{P}$ above $\mathfrak{p}$. Then we have the residue class field extensions $\mathbb{F}_{\mathfrak{P}'}/\mathbb{F}_{\mathfrak{P}}$ and $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$. Moreover, $\mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})} \mid\mid \mathfrak{p}\mathcal{O}$ and $\mathfrak{P}^{e_{\mathfrak{P}}(\mathfrak{P}')} \mid\mid \mathfrak{P}\mathcal{O}'$. Then

$$e_{\mathfrak{p}}(\mathfrak{P}') = e_{\mathfrak{p}}(\mathfrak{P})e_{\mathfrak{P}}(\mathfrak{P}') \quad \text{and} \quad f_{\mathfrak{p}}(\mathfrak{P}') = f_{\mathfrak{p}}(\mathfrak{P})f_{\mathfrak{P}}(\mathfrak{P}'). \tag{5.8}$$

In other words, the ramification indices and inertia degrees are multiplicative with respect to towers of field extensions. We illustrate this relationship via the following tower of extensions:

$$\begin{array}{c} \mathfrak{P}' \subset \mathcal{O}' \subseteq M \\ | \\ \mathfrak{P} \subset \mathcal{O} \subseteq L \\ | \\ \mathfrak{p} \subset \mathcal{o} \subseteq K. \end{array}$$

The inertia degrees and ramification indices satisfy a simple relationship to the degree of L/K. First, we will prove a useful lemma:

Lemma 5.5.1. *Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of a degree n separable extension L/K. Suppose $\mathfrak{P}$ is a prime above $\mathfrak{p}$. Then for any $e \geq 1$, we have*

$$\mathcal{O}/\mathfrak{P}^e \cong \bigoplus_{0 \leq i \leq e-1} \mathbb{F}_{\mathfrak{P}}.$$

In particular, the dimension of $\mathcal{O}/\mathfrak{P}^e$ as a $\mathbb{F}_{\mathfrak{p}}$-vector space is $ef_{\mathfrak{p}}(\mathfrak{P})$.

Proof. Consider the descending chain

$$\mathcal{O}/\mathfrak{P}^e \supseteq \mathfrak{P}/\mathfrak{P}^e \supseteq \cdots \supseteq \mathfrak{P}^{e-1}/\mathfrak{P}^e \supseteq \mathfrak{P}^e/\mathfrak{P}^e,$$

of $\mathbb{F}_{\mathfrak{p}}$-vector spaces. By the third isomorphism theorem, these quotients are of the form $\mathfrak{P}^i/\mathfrak{P}^{i+1}$ for $0 \leq i \leq e-1$ and are all isomorphic to $\mathbb{F}_{\mathfrak{P}}$ by Lemma 5.3.3. Therefore we have a decomposition

$$\mathfrak{P}^i/\mathfrak{P}^e \cong \mathbb{F}_{\mathfrak{P}} \oplus (\mathfrak{P}^{i+1}/\mathfrak{P}^e),$$

for all i. Iteratively applying this isomorphism $e-1$ times gives

$$\mathcal{O}/\mathfrak{P}^e \cong \bigoplus_{0 \leq i \leq e-1} \mathbb{F}_{\mathfrak{P}}.$$

This proves the first statement. Since the dimension of $\mathbb{F}_{\mathfrak{P}}$ as a $\mathbb{F}_{\mathfrak{p}}$-vector space is $f_{\mathfrak{p}}(\mathfrak{P})$ by definition, it follows that $\mathcal{O}/\mathfrak{P}^e$ is a $\mathbb{F}_{\mathfrak{p}}$-vector space of dimension $ef_{\mathfrak{p}}(\mathfrak{P})$. This proves the second statement. □

We now describe the relationship between inertia degrees and ramification indices which is known as the **fundamental equality**:

Theorem (Fundamental equality). *Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of a degree n separable extension L/K. Suppose $\mathfrak{p}$ is a prime of $\mathcal{o}$ and $\mathfrak{p}\mathcal{O}$ has prime factorization*

$$\mathfrak{p}\mathcal{O} = \mathfrak{P}_1^{e_{\mathfrak{p}}(\mathfrak{P}_1)} \cdots \mathfrak{P}_r^{e_{\mathfrak{p}}(\mathfrak{P}_r)}.$$

Then

$$n = \sum_{1 \leq i \leq r} e_{\mathfrak{p}}(\mathfrak{P}_i) f_{\mathfrak{p}}(\mathfrak{P}_i).$$

Proof. Since distinct primes are relatively prime, the Chinese remainder theorem implies that

$$\mathcal{O}/\mathfrak{p}\mathcal{O} \cong \bigoplus_{1\leq i\leq r} \mathcal{O}/\mathfrak{P}_i^{e_\mathfrak{p}(\mathfrak{P}_i)}.$$

As $\mathcal{O}/\mathfrak{p}\mathcal{O}$ and $\mathcal{O}/\mathfrak{P}_i^{e_\mathfrak{p}(\mathfrak{P}_i)}$ are $\mathbb{F}_\mathfrak{p}$-vector spaces for all i, it suffices to show $\mathbb{F}_\mathfrak{P}$ is of dimension n and $\mathcal{O}/\mathfrak{P}_i^{e_\mathfrak{p}(\mathfrak{P}_i)}$ is of dimension $e_\mathfrak{p}(\mathfrak{P}_i)f_\mathfrak{p}(\mathfrak{P}_i)$ for all i. For $\mathcal{O}/\mathfrak{p}\mathcal{O}$, we already know it is a $\mathbb{F}_\mathfrak{p}$-vector space of dimension at most n (since $\mathcal{O}$ is a finitely generated o-module of rank at most n by Corollary 5.5.1 and $\mathfrak{p} \subseteq \mathfrak{p}\mathcal{O}$). Therefore we must show that the dimension is exactly n. Let $\overline{\lambda_1}, \ldots, \overline{\lambda_m}$ be a basis for $\mathcal{O}/\mathfrak{p}\mathcal{O}$ as a $\mathbb{F}_\mathfrak{p}$-vector space and let $\lambda_1, \ldots, \lambda_m$ be any lift of this basis to $\mathcal{O}$. As $m \leq n$, it suffices to show $\lambda_1, \ldots, \lambda_m$ spans L/K and hence $m = n$. Let $M = \lambda_1 o + \cdots + \lambda_m o$ and set $N = \mathcal{O}/M$. Then $\mathcal{O} = M + \mathfrak{p}\mathcal{O}$ since $\lambda_1, \ldots, \lambda_m$ is a lift a basis for $(\mathcal{O}/\mathfrak{p}\mathcal{O})/\mathbb{F}_\mathfrak{p}$, and hence $N = \mathfrak{p}N$. As $\mathcal{O}$ is a finitely generated o-module of rank at most n by Corollary 5.5.1, so is N. So let $\omega_1, \ldots, \omega_r$ be generators. As $N = \mathfrak{p}N$, we have

$$\omega_i = \sum_{1\leq j\leq r} \alpha_{i,j}\omega_j,$$

for some $\alpha_{i,j} \in \mathfrak{p}$ for $1 \leq i, j \leq r$. These r equations are equivalent to the identity

$$((\alpha_{i,j})_{i,j} - I) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_r \end{pmatrix} = \mathbf{0}.$$

Let $d = \det((\alpha_{i,j})_{i,j} - I)$. Then $d \neq 0$ because expanding the determinant shows $d \equiv (-1)^r \pmod{\mathfrak{p}}$ as $\alpha_{i,j} \in \mathfrak{p}$ for all i and j. Multiplying on the left by the adjugate $\operatorname{adj}(\alpha_{i,j})_{i,j} - I)$ of $(\alpha_{i,j})_{i,j} - I$ and recalling that a matrix times its adjugate is its determinant times the identity, we obtain

$$d \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_r \end{pmatrix} = \mathbf{0}.$$

Hence multiplication by d annihilates N which is to say that $d\mathcal{O} \subseteq M$. Equivalently,

$$d\mathcal{O} \subseteq \lambda_1 o + \cdots + \lambda_m o.$$

By Proposition 5.1.3 and that $d \neq 0$, multiplication by K shows $L = \lambda_1 K + \cdots + \lambda_m K$ (as the reverse containment is trivial). Hence $\lambda_1, \ldots, \lambda_m$ spans L/K so that $m = n$ and $\mathcal{O}/\mathfrak{p}\mathcal{O}$ is a $\mathbb{F}_\mathfrak{p}$-vector space of dimension n. For $\mathcal{O}/\mathfrak{P}_i^{e_\mathfrak{p}(\mathfrak{P}_i)}$, the dimensionality claim follows from Lemma 5.5.1. So our dimension computations combine to give

$$n = \sum_{1\leq i\leq r} e_\mathfrak{p}(\mathfrak{P}_i)f_\mathfrak{p}(\mathfrak{P}_i).$$

□

We also classify primes according to extremal cases of the fundamental equality. We say $\mathfrak{p}$ is **inert** in $\mathcal{O}/o$ if $r = 1$ so that there is a single prime $\mathfrak{P}$ above $\mathfrak{p}$ so that $e_\mathfrak{p}(\mathfrak{P}) = 1$ and $f_\mathfrak{p}(\mathfrak{P}) = n$ by the fundamental equality. Then

$$\mathfrak{p}\mathcal{O} = \mathfrak{P},$$

which means $\mathfrak{p}\mathcal{O}$ is prime. We say $\mathfrak{p}$ is **totally split** in $\mathcal{O}/o$ if $r = n$ so that there are primes $\mathfrak{P}_i$ above $\mathfrak{p}$ with $e_\mathfrak{p}(\mathfrak{P}_i) = f_\mathfrak{p}(\mathfrak{P}_i) = 1$ for $1 \leq i \leq n$ by the fundamental equality. Hence

$$\mathfrak{p}\mathcal{O} = \mathfrak{P}_1 \cdots \mathfrak{P}_n.$$

In terms of the inertia degrees, being inert or totally split in $\mathcal{O}/o$ are antithetical properties. In particular, the smaller the inertia degrees are the greater the tendency for $\mathfrak{p}\mathcal{O}$ to factor into distinct primes. Now let us introduce ramification. If $\mathfrak{P}$ is a prime of $\mathcal{O}$ above $\mathfrak{p}$, we say that $\mathfrak{P}$ is **unramified** in $\mathcal{O}/o$ if $e_{\mathfrak{p}}(\mathfrak{P}) = 1$ and the residue class extension $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$ is separable. Otherwise, we say $\mathfrak{P}$ is **ramified** in $\mathcal{O}/o$. Moreover, we say $\mathfrak{P}$ is **totally ramified** in $\mathcal{O}/o$ if in addition to being ramified we have $f_{\mathfrak{p}}(\mathfrak{P}) = 1$. Similarly, we say that a prime $\mathfrak{p}$ of o is **unramified** in $\mathcal{O}/o$ if every prime $\mathfrak{P}$ above it is unramified and is **ramified** in $\mathcal{O}/o$ otherwise. The Dedekind extension $\mathcal{O}/o$ itself is said to be **unramified** if every prime of o is unramified in $\mathcal{O}/o$ and is said to be **ramified** otherwise.

Remark 5.5.1. *We will see that it is an exceptional phenomena for a prime $\mathfrak{p}$ of o to ramify in $\mathcal{O}/o$. Therefore it is typical that $\mathfrak{p}$ is either inert or totally split in $\mathcal{O}/o$.*

Unfortunately, there is no general way to see how $\mathfrak{p}\mathcal{O}$ factors for an arbitrary Dedekind extension $\mathcal{O}/o$ of a finite separable extension L/K. However, we can make some progress in this respect. Let θ be a primitive element for L/K so that $L = K(\theta)$. By Proposition 5.1.3 we can multiply by a nonzero element of o, if necessary, to ensure that $\theta \in \mathcal{O}$ and hence its minimal polynomial $m_\theta(x)$ over K has coefficients in o. We then define the **conductor** $\mathfrak{Q}_{\mathcal{O}/o}$ of $\mathcal{O}/o$ relative to θ by

$$\mathfrak{Q}_{\mathcal{O}/o} = \{\alpha \in \mathcal{O} : \alpha\mathcal{O} \subseteq o[\theta]\}.$$

This is an integral ideal of $\mathcal{O}$ provided it is nonzero. Since $\mathcal{O}$ is a finitely generated o module by Corollary 5.5.1, let $\omega_1, \ldots, \omega_r$ be generators. As L/K is algebraic, θ is algebraic over K and hence $L = K[\theta]$. Then

$$\omega_i = \sum_{1 \le j \le r} \kappa_{i,j}\theta^j,$$

with $\kappa_{i,j} \in K$ for $1 \le i, j \le r$. As K is the field of fractions of o, $\kappa_{i,j} = \frac{\alpha_{i,j}}{\delta_{i,j}}$ with $\alpha_{i,j}, \delta_{i,j} \in o$ for all i and j. Setting $\delta = \prod_{1 \le i,j \le r} \delta_{i,j}$, we have that δ is a nonzero element of o (hence $\mathcal{O}$ as well) with $\delta\omega_i \in o[\theta]$ for all i. As $\omega_1, \ldots, \omega_r$ generate $\mathcal{O}$ as a o-module, it follows that $\delta \in \mathfrak{Q}_{\mathcal{O}/o}$. Then $\mathfrak{Q}_{\mathcal{O}/o}$ is an integral ideal of $\mathcal{O}$ as claimed. The **Dedekind-Kummer theorem** describes the factorization of $\mathfrak{p}\mathcal{O}$ provided it is relatively prime to the conductor $\mathfrak{Q}_{\mathcal{O}/o}$:

Theorem (Dedekind-Kummer theorem). *Let $\mathcal{O}/o$ be a Dedekind extension of a degree n separable extension L/K, θ be a primitive element of L/K contained in $\mathcal{O}$ with minimal polynomial $m_\theta(x)$ over K, and $\mathfrak{p}$ be a prime of o such that $\mathfrak{p}\mathcal{O}$ is relatively prime to $\mathfrak{Q}_{\mathcal{O}/o}$. Suppose*

$$\overline{m_\theta}(x) = \overline{m_1}(x)^{e_1} \cdots \overline{m_r}(x)^{e_r},$$

is the prime factorization of $\overline{m_\theta}(x)$ in $\mathbb{F}_{\mathfrak{p}}[x]$. Let $m_i(x)$ be any lift of $\overline{m_i}(x)$ to $o[x]$ and set

$$\mathfrak{P}_i = \mathfrak{p}\mathcal{O} + m_i(\theta)\mathcal{O},$$

for $1 \le i \le r$. Then $\mathfrak{P}_i$ is a prime of $\mathcal{O}$ for all i,

$$\mathfrak{p}\mathcal{O} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r},$$

is the prime factorization of $\mathfrak{p}\mathcal{O}$, and $f_{\mathfrak{p}}(\mathfrak{P}_i) = \deg(\overline{m_i}(x))$ for all i.

Proof. First consider the homomorphism

$$\phi : o[\theta] \to \mathcal{O}/\mathfrak{p}\mathcal{O} \qquad \alpha \mapsto \alpha \pmod{\mathfrak{p}\mathcal{O}}.$$

We have $\mathfrak{p}\mathcal{O} + \mathfrak{A}_{\mathcal{O}/\mathcal{o}} = \mathcal{O}$ because $\mathfrak{p}\mathcal{O}$ is relatively prime to $\mathfrak{A}_{\mathcal{O}/\mathcal{o}}$. As $\mathfrak{A}_{\mathcal{O}/\mathcal{o}} \subseteq \mathcal{o}[\theta]$, it follows that $\mathfrak{p}\mathcal{O} + \mathcal{o}[\theta] = \mathcal{O}$ which shows ϕ is surjective. Now $\ker\phi = \mathcal{o}[\theta] \cap \mathfrak{p}\mathcal{O}$ and we claim $\mathcal{o}[\theta] \cap \mathfrak{p}\mathcal{O} = \mathfrak{p}\mathcal{o}[\theta]$. The reverse inclusion is clear since $\mathfrak{p}$ is an integral ideal of $\mathcal{o}$. For the forward inclusion, intersecting both sides of $\mathfrak{p}\mathcal{O} + \mathfrak{A}_{\mathcal{O}/\mathcal{o}} = \mathcal{O}$ with $\mathcal{o}$ gives $\mathfrak{p} \cap \mathfrak{A}_{\mathcal{O}/\mathcal{o}} = \mathcal{o}$ because $\mathfrak{p}\mathcal{O} \cap \mathcal{o} = \mathfrak{p}$. Hence

$$\mathcal{o}[\theta] \cap \mathfrak{p}\mathcal{O} = (\mathfrak{p} \cap \mathfrak{A}_{\mathcal{O}/\mathcal{o}})(\mathcal{o}[\theta] \cap \mathfrak{p}\mathcal{O}) = (\mathfrak{p}\mathcal{o}[\theta] \cap \mathfrak{p}\mathcal{O}) + (\mathfrak{A}_{\mathcal{O}/\mathcal{o}}\mathcal{o}[\theta] \cap \mathfrak{A}_{\mathcal{O}/\mathcal{o}}\mathfrak{p}\mathcal{O}) \subseteq \mathfrak{p}\mathcal{o}[\theta],$$

where the inclusion follows by the definition of the conductor $\mathfrak{A}_{\mathcal{O}/\mathcal{o}}$. This proves the reverse inclusion so that $\ker\phi = \mathfrak{p}\mathcal{o}[\theta]$. By first isomorphism theorem, we obtain

$$\mathcal{o}[\theta]/\mathfrak{p}\mathcal{o}[\theta] \cong \mathcal{O}/\mathfrak{p}\mathcal{O}.$$

Since $m_\theta(x)$ is the minimal polynomial for θ over K, we have an isomorphism $\mathcal{o}[x]/m_\theta(x)\mathcal{o}[x] \cong \mathcal{o}[\theta]$ given by evaluation at θ. Then we have the chain of isomorphism

$$\mathcal{o}[\theta]/\mathfrak{p}\mathcal{o}[\theta] \cong (\mathcal{o}[x]/m_\theta(x)\mathcal{o}[x])/(\mathfrak{p}(\mathcal{o}[x]/m_\theta(x)\mathcal{o}[x])) \cong \mathcal{o}[x]/(\mathfrak{p}\mathcal{o}[x] + m_\theta(x)\mathcal{o}[x]) \cong \mathbb{F}_\mathfrak{p}[x]/\overline{m_\theta}(x)\mathbb{F}_\mathfrak{p}[x],$$

where the second and third isomorphisms follow by taking $\mathcal{o}[x]/(\mathfrak{p}\mathcal{o}[x] + m_\theta(x)\mathcal{o}[x])$ and reducing elements of $\mathcal{o}[x]$ modulo $m_\theta(x)\mathcal{o}[x]$ or their coefficients modulo $\mathfrak{p}$ respectively. Therefore the inverse isomorphism is given by sending any representative $\overline{f}(x)$ of a class in $\mathbb{F}_\mathfrak{p}[x]/\overline{m_\theta}(x)\mathbb{F}_\mathfrak{p}[x]$ to a lift $f(x)$ in $\mathcal{o}[x]$ and then to $\overline{f(\theta)}$ by reducing $f(\theta)$ modulo $\mathfrak{p}\mathcal{O}$. Now set $A = \mathbb{F}_\mathfrak{p}[x]/\overline{m_\theta}(x)\mathbb{F}_\mathfrak{p}[x]$. The Chinese remainder theorem gives an isomorphism

$$A \cong \bigoplus_{1\le i\le r} \mathbb{F}_\mathfrak{p}[x]/\overline{m_i}(x)^{e_i}\mathbb{F}_\mathfrak{p}[x].$$

As $\overline{m_i}(x)$ is irreducible, $\overline{m_i}(x)\mathbb{F}_\mathfrak{p}[x]$ is maximal and hence $\mathbb{F}_\mathfrak{p}[x]/\overline{m_i}(x)\mathbb{F}_\mathfrak{p}[x]$ is a field. By the third isomorphism theorem, $\overline{m_i}(x)\mathbb{F}_\mathfrak{p}[x]/\overline{m_i}(x)^{e_i}\mathbb{F}_\mathfrak{p}[x]$ is a maximal ideal of $\mathbb{F}_\mathfrak{p}[x]/\overline{m_i}(x)^{e_i}\mathbb{F}_\mathfrak{p}[x]$. It follows that the maximal ideals of A are precisely the $\overline{m_i}(x)A$ and we have an isomorphism

$$A/\overline{m_i}(x)A \cong \mathbb{F}_\mathfrak{p}[x]/\overline{m_i}(x)\mathbb{F}_\mathfrak{p}[x],$$

for all i. Via the isomorphism $\mathcal{O}/\mathfrak{p}\mathcal{O} \cong A$ described above, the maximal ideals of $\mathcal{O}/\mathfrak{p}\mathcal{O}$ are exactly $\overline{m_i(\theta)}(\mathcal{O}/\mathfrak{p}\mathcal{O})$. We now show that the $\mathfrak{P}_i$ are prime. To see this, consider the surjective homomorphism

$$\pi : \mathcal{O} \to \mathcal{O}/\mathfrak{p}\mathcal{O} \qquad \alpha \mapsto \alpha \pmod{\mathfrak{p}\mathcal{O}}.$$

Then the image of $\mathfrak{P}_i$ under π is $\overline{m_i(\theta)}(\mathcal{O}/\mathfrak{p}\mathcal{O})$. As this ideal is maximal and hence prime, the preimage $\mathfrak{P}_i$ is prime too. Moreover, the $\mathfrak{P}_i$ are all distinct since the $\overline{m_i(\theta)}\mathcal{O}/\mathfrak{p}\mathcal{O}$ are which are all distinct because the $\overline{m_i}(x)A$ are (using the isomorphism $\mathcal{O}/\mathfrak{p}\mathcal{O} \cong A$). In particular, they are also relatively prime. By construction, $\mathfrak{P}_i \subseteq \mathfrak{p}\mathcal{O}$ so that the $\mathfrak{P}_i$ are prime factors of $\mathfrak{p}\mathcal{O}$. These are the only prime factors of $\mathfrak{p}\mathcal{O}$ because the image of any prime under π contained in $\mathfrak{p}\mathcal{O}$ must be a maximal ideal of $\mathcal{O}/\mathfrak{p}\mathcal{O}$ (since primes are maximal and by the fourth isomorphism theorem) and every maximal ideal is one of the $\overline{m_i(\theta)}(\mathcal{O}/\mathfrak{p}\mathcal{O})$. Together, all of this means that $\mathfrak{p}\mathcal{O}$ admits the prime factorization

$$\mathfrak{p}\mathcal{O} = \mathfrak{P}_1^{e_\mathfrak{p}(\mathfrak{P}_1)} \dots \mathfrak{P}_r^{e_\mathfrak{p}(\mathfrak{P}_r)},$$

for some ramification indices $e_\mathfrak{p}(\mathfrak{P}_i)$. We will be done if we can show $e_\mathfrak{p}(\mathfrak{P}_i) = e_i$ and $f_\mathfrak{p}(\mathfrak{P}_i) = \deg(\overline{m_i}(x))$ for all i. To accomplish this, observe that we have an isomorphism

$$\mathbb{F}_{\mathfrak{P}_i} \cong (\mathcal{O}/\mathfrak{p}\mathcal{O})/(\overline{m_i(\theta)}(\mathcal{O}/\mathfrak{p}\mathcal{O})) \cong \mathbb{F}_\mathfrak{p}[x]/\overline{m_i}(x)\mathbb{F}_\mathfrak{p}[x],$$

where the first isomorphism follow by taking $\mathbb{F}_{\mathfrak{P}_i}$ and reducing $\mathcal{O}$ modulo $\mathfrak{p}$ and the second isomorphism follows from $\mathcal{O}/\mathfrak{p}\mathcal{O} \cong A$ and that the image of the maximal ideal $\overline{m_i(\theta)}(\mathcal{O}/\mathfrak{p}\mathcal{O})$ under this isomorphism is $\overline{m_i}(x)A$. Now $\mathbb{F}_\mathfrak{p}[x]/\overline{m_i}(x)\mathbb{F}_\mathfrak{p}[x]$ is a $\mathbb{F}_\mathfrak{p}$-vector space of degree $\deg(\overline{m_i}(x))$. Hence $f_p(\mathfrak{P}_i) = \deg(\overline{m_i}(x))$ for all i as desired. The ideal $\overline{m_i}(x)^{e_i}A$ under the isomorphism $A \cong \mathcal{O}/\mathfrak{p}\mathcal{O}$ is the ideal $\overline{m_i(\theta)}^{e_i}(\mathcal{O}/\mathfrak{p}\mathcal{O})$. As the image of $\mathfrak{P}_i$ under π is $\overline{m_i(\theta)}(\mathcal{O}/\mathfrak{p}\mathcal{O})$, we have that $\mathfrak{P}_i^{e_i}$ is contained in the preimage of $\overline{m_i(\theta)}^{e_i}(\mathcal{O}/\mathfrak{p}\mathcal{O})$ under π. As $\overline{m_\theta(\theta)}(\mathcal{O}/\mathfrak{p}\mathcal{O}) = 0$, it follows that

$$\mathfrak{p}\mathcal{O} = \pi^{-1}(0) \supseteq \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}.$$

Since the $\mathfrak{P}_i$ are prime, we have $e_\mathfrak{p}(\mathfrak{P}_i) \leq e_i$ for all i. But the fundamental equality then gives

$$n = \sum_{1 \leq i \leq r} e_\mathfrak{p}(\mathfrak{P}_i) f_p(\mathfrak{P}_i) \leq \sum_{1 \leq i \leq r} e_i f_p(\mathfrak{P}_i) \leq \sum_{1 \leq i \leq r} e_i \deg(\overline{m_i}(x)) \leq n,$$

where the last equality follows by the prime factorization of $\overline{m_\theta}(x)$ and that $\deg(\overline{m_\theta}(x)) = \deg(m_\theta(x))$ because $m_\theta(x)$ is monic. This shows $e_\mathfrak{p}(\mathfrak{P}_i) = e_i$ for all i which completes the proof. □

Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of a degree n separable extension L/K. The Dedekind-Kummer theorem allows us to compute the prime factorization of $\mathfrak{p}\mathcal{O}$ provided this integral ideal is relatively prime to the conductor $\mathfrak{O}_{\mathcal{O}/\mathcal{o}}$. By the prime factorization of fractional ideals, $\mathfrak{O}_{\mathcal{O}/\mathcal{o}}$ has finitely many prime factors so we only have to avoid finitely many primes of $\mathcal{O}$. In fact, if the conductor is $\mathcal{O}$ then we do not have to avoid any primes at all. This occurs when $\mathcal{O}/\mathcal{o}$ is monogenic. Indeed, Suppose $\mathcal{O} = \mathcal{o}[\alpha]$ for some $\alpha \in \mathcal{O}$. Then $1, \alpha, \ldots, \alpha^{n-1}$ is an integral basis for $\mathcal{O}/\mathcal{o}$ and is necessarily a basis for L/K. But then α is also a primitive element for L/K which implies

$$\mathfrak{O}_{\mathcal{O}/\mathcal{o}} = \mathcal{O}.$$

We now turn to the setting of a number field K. Every prime $\mathfrak{p}$ of K is above some prime p (recall $\mathbb{Z}$ is a principal ideal domain so we are referring to primes by their generator). Then the residue class extension of $\mathcal{O}_K/\mathbb{Z}$ for $\mathfrak{p}$ is $\mathbb{F}_\mathfrak{p}/\mathbb{F}_p$. Also, we write $f_p(\mathfrak{p})$ and $e_p(\mathfrak{p})$ for the inertia degree and ramification index of $\mathfrak{p}$ relative to p respectively. Moreover, all of the residue class extensions are separable since $\mathbb{F}_p$ is a perfect field. This implies $\mathfrak{p}$ is ramified in $\mathcal{O}_K/\mathbb{Z}$ if and only if $e_p(\mathfrak{p}) = 1$. If θ is a primitive element for $K/\mathbb{Q}$ contained in $\mathcal{O}_K$, then the **conductor** $\mathfrak{q}_K$ of K relative to θ is the conductor of $\mathcal{O}_K/\mathbb{Z}$ relative to θ. Even though K admits an integral basis, $\mathcal{O}_K$ is not necessarily monogenic so that $\mathfrak{q}_K$ may not be $\mathcal{O}_K$.

5.6 Galois Extensions

Much more can be said about the ramification of primes in a Dedekind extension $\mathcal{O}/\mathcal{o}$ when the associated extension L/K is Galois. Since L/K is assumed to be finite and separable, this amounts to further assuming L/K is normal. In this case, the elements of the Galois group $\mathrm{Gal}(L/K) = \mathrm{Hom}_K(L, \overline{K})$. Then by Proposition 5.2.1, we have

$$\mathrm{N}_{L/K}(\lambda) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\lambda) \quad \text{and} \quad \mathrm{Tr}_{L/K}(\lambda) = \sum_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\lambda),$$

for all $\lambda \in L$. As $\mathcal{o} \subseteq K$, we see that $\mathrm{Gal}(L/K)$ fixes $\mathcal{o}$ pointwise and hence every fractional ideal of $\mathcal{o}$ as well. Now let $\sigma \in \mathrm{Gal}(L/K)$ and $\alpha \in \mathcal{O}$. Then

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0,$$

for some $n \geq 1$ and $a_i \in \mathfrak{o}$ for $0 \leq i \leq n-1$. Applying σ gives

$$\sigma(\alpha)^n + a_{n-1}\sigma(\alpha)^{n-1} + \cdots + a_0 = 0,$$

because σ is a K-embedding. This shows $\sigma(\alpha) \in \mathcal{O}$ and therefore the Galois group $\mathrm{Gal}(L/K)$ acts on $\mathcal{O}$. In particular, each automorphism σ of $\mathrm{Gal}(L/K)$ restricts to an automorphism of $\mathcal{O}$ and therefore $\sigma(\mathfrak{P})$ is a prime of $\mathcal{O}$ if $\mathfrak{P}$. Moreover, if $\mathfrak{P}$ is above $\mathfrak{p}$ then so is $\sigma(\mathfrak{P})$. Indeed, just observe that

$$\sigma(\mathfrak{P}) \cap \mathfrak{o} = \sigma(\mathfrak{P} \cap \mathfrak{o}) = \sigma(\mathfrak{p}) = \mathfrak{p}.$$

Accordingly, we say that $\sigma(\mathfrak{P})$ is **conjugate** to $\mathfrak{P}$. It turns out that the Galois group acts transitively on the set of primes above a given prime:

Proposition 5.6.1. *Let $\mathcal{O}/\mathfrak{o}$ be a Dedekind extension of a finite Galois extension L/K and let $\mathfrak{p}$ be a prime of $\mathfrak{o}$. Then $\mathrm{Gal}(L/K)$ acts transitively on the set of primes above $\mathfrak{p}$. Moreover, if $\mathfrak{p}\mathcal{O}$ has prime factorization*

$$\mathfrak{p}\mathcal{O} = \mathfrak{P}_1^{e_\mathfrak{p}(\mathfrak{P}_1)} \cdots \mathfrak{P}_r^{e_\mathfrak{p}(\mathfrak{P}_r)},$$

then

$$f_\mathfrak{p}(\mathfrak{P}_1) = \cdots = f_\mathfrak{p}(\mathfrak{P}_r) \quad \textit{and} \quad e_\mathfrak{p}(\mathfrak{P}_1) = \cdots = e_\mathfrak{p}(\mathfrak{P}_r).$$

Proof. Let $\mathfrak{P}_i$ and $\mathfrak{P}_j$ for $1 \leq i < j \leq r$ be two distinct primes above $\mathfrak{p}$. Assume by contraction that $\sigma(\mathfrak{P}_i) \neq \mathfrak{P}_j$ for any $\sigma \in \mathrm{Gal}(L/K)$. Since distinct primes are relatively prime, the Chinese remainder theorem implies the existence of an $\alpha \in \mathcal{O}$ such that

$$\alpha \equiv 1 \pmod{\sigma(\mathfrak{P}_i)} \quad \text{and} \quad \alpha \equiv 0 \pmod{\mathfrak{P}_j},$$

for all $\sigma \in \mathrm{Gal}(L/K)$. Now recall that $\mathrm{N}_{L/K}(\alpha) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\alpha)$ by Proposition 5.2.1 and $\mathrm{N}_{L/K}(\alpha) \in \mathfrak{o}$ by Proposition 5.2.2. Moreover,

$$\mathfrak{P}_i \cap \mathfrak{o} = \mathfrak{p} = \mathfrak{P}_j \cap \mathfrak{o}.$$

On the one hand, $\alpha \notin \sigma(\mathfrak{P}_i)$ and hence $\sigma(\alpha) \notin \mathfrak{P}_i$ for all $\sigma \in \mathrm{Gal}(L/K)$ (recall $\mathrm{Gal}(L/K)$ is a group so σ has an inverse σ^{-1}). But then $\mathrm{N}_{L/K}(\alpha) \notin \mathfrak{P}_i$ for otherwise $\sigma(\alpha) \in \mathfrak{P}_i$ for some $\sigma \in \mathrm{Gal}(L/K)$ by primality of $\mathfrak{P}_i$. It must be the case that $\mathrm{N}_{L/K}(\alpha) \notin \mathfrak{p}$ because $\mathfrak{P}_i$ is above $\mathfrak{p}$. On the other hand, $\alpha \in \mathfrak{P}_j$ and so $\mathrm{N}_{L/K}(\alpha) \in \mathfrak{P}_j$ since $\sigma(\alpha) \in \mathfrak{P}_j$ when σ is the identity and $\mathfrak{P}_j$ is an integral ideal. But then $\mathrm{N}_{L/K}(\alpha) \in \mathfrak{p}$ because $\mathfrak{P}_j$ is above $\mathfrak{p}$. This gives a contradiction and therefore the action is transitive. We now show that the inertia degrees and ramification indices of $\mathfrak{P}_i$ and $\mathfrak{P}_j$ are equal which will complete the proof. By what we have just proved, there exists a $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\mathfrak{P}_i) = \mathfrak{P}_j$. Since σ is an automorphism of $\mathcal{O}$, it induces an isomorphism

$$\mathbb{F}_{\mathfrak{P}_i} \cong \mathbb{F}_{\mathfrak{P}_j}.$$

Therefore the inertia degrees of $\mathfrak{P}_i$ and $\mathfrak{P}_j$ are equal. For the ramification indices, recall that σ is an automorphism of $\mathcal{O}$ fixing $\mathfrak{o}$, and hence $\mathfrak{p}$ too, pointwise. Thus $\sigma(\mathfrak{p}\mathcal{O}) = \mathfrak{p}\mathcal{O}$ which implies that $\mathfrak{P}_i^e \mid\mid \mathfrak{p}\mathcal{O}$ if and only if $\mathfrak{P}_j^e \mid\mid \mathfrak{p}\mathcal{O}$. Therefore the ramification indices of $\mathfrak{P}_i$ and $\mathfrak{P}_j$ are equal as well. This completes the proof □

Another way to phrase Proposition 5.6.1 is that the primes above $\mathfrak{p}$ are all conjugate to each other and their inertia degrees and ramification indices are all equal. We call the common inertia degree f the **inertia degree** of $\mathfrak{p}$ and the common ramification index e the **ramification index** of $\mathfrak{p}$. If there are r primes above $\mathfrak{p}$, the fundamental equality takes the particularly simple form

$$n = ref.$$

Continue to let $\mathfrak{P}$ be a prime above $\mathfrak{p}$. We define the **decomposition group** $D_{\mathfrak{p}}(\mathfrak{P})$ of $\mathfrak{P}$ by

$$D_{\mathfrak{p}}(\mathfrak{P}) = \{\tau \in \mathrm{Gal}(L/K) : \tau(\mathfrak{P}) = \mathfrak{P}\}.$$

Equivalently, $D_{\mathfrak{p}}(\mathfrak{P})$ is the stabilizer subgroup of $\mathfrak{P}$ in $\mathrm{Gal}(L/K)$. The associated **decomposition field** $L^{D_{\mathfrak{p}}(\mathfrak{P})}$ of $\mathfrak{P}$ is defined to be

$$L^{D_{\mathfrak{p}}(\mathfrak{P})} = \{\lambda \in L : \tau(\lambda) = \lambda \text{ for all } \tau \in D_{\mathfrak{p}}(\mathfrak{P})\}.$$

In other words, the decomposition field of $\mathfrak{P}$ is the fixed field of L by $D_{\mathfrak{p}}(\mathfrak{P})$. In particular, the fundamental theorem of Galois theory gives

$$\mathrm{Gal}(L/L^{D_{\mathfrak{p}}(\mathfrak{P})}) = D_{\mathfrak{p}}(\mathfrak{P}).$$

We let $\mathcal{O}^D$ denote the integral closure of o in $L^{D_{\mathfrak{p}}(\mathfrak{P})}$ so that $\mathcal{O}/\mathcal{O}^D/o$ is a tower of Dedekind extensions. Also, we will write $\mathfrak{P}^D$ for the prime of $L^{D_{\mathfrak{p}}(\mathfrak{P})}$ below $\mathfrak{P}$. Then we can illustrate this relationship via the following tower of extensions:

$$\begin{array}{c} \mathfrak{P} \subset \mathcal{O} \subseteq L \\ | \\ \mathfrak{P}^D \subset \mathcal{O}^D \subseteq L^{D_{\mathfrak{p}}(\mathfrak{P})} \\ | \\ \mathfrak{p} \subset o \subseteq K. \end{array}$$

Any of the decomposition groups $D_{\mathfrak{p}}(\mathfrak{P})$ encode how $\mathfrak{p}\mathcal{O}$ splits into distinct prime factors in $\mathcal{O}$. Indeed, by the orbit-stabilizer theorem the number of cosets in $\mathrm{Gal}(L/K)/D_{\mathfrak{p}}(\mathfrak{P})$ is equal to the size of the orbit of $\mathfrak{P}$ under the action of $\mathrm{Gal}(L/K)$. As the action is transitive by Proposition 5.6.1, $\sigma(\mathfrak{P})$ runs over the primes above $\mathfrak{p}$ as σ runs over a complete set of representatives for this coset space. It follows that the number of prime factors of $\mathfrak{p}\mathcal{O}$ is equal to the index $|\mathrm{Gal}(L/K)/D_{\mathfrak{p}}(\mathfrak{P})|$. In particular, $\mathfrak{p}$ is inert in $\mathcal{O}/o$ if and only if $D_{\mathfrak{p}}(\mathfrak{P}) = \mathrm{Gal}(L/K)$ which is equivalent to $L^{D_{\mathfrak{p}}(\mathfrak{P})} = K$. Antithetically, $\mathfrak{p}$ is totally split in $\mathcal{O}/o$ if and only if $D_{\mathfrak{p}}(\mathfrak{P}) = \{\mathrm{id}\}$ which is equivalent to $L^{D_{\mathfrak{p}}(\mathfrak{P})} = L$. More generally, the fundamental equality implies $n = ef|\mathrm{Gal}(L/K)/D_{\mathfrak{p}}(\mathfrak{P})|$ which is to say that

$$|D_{\mathfrak{p}}(\mathfrak{P})| = ef. \tag{5.9}$$

Then $L/L^{D_{\mathfrak{p}}(\mathfrak{P})}$ has degree ef and $L^{D_{\mathfrak{p}}(\mathfrak{P})}/K$ has degree equal to the number of distinct prime factors of $\mathfrak{p}\mathcal{O}$ which is $|\mathrm{Gal}(L/K)/D_{\mathfrak{p}}(\mathfrak{P})|$. If $\mathfrak{p}$ inert in $\mathcal{O}/o$ then $n = ef$ implying $e = 1$ and $n = f$ as we have seen. If $\mathfrak{p}$ is totally split in $\mathcal{O}/o$ then $1 = ef$ so that $e = f = 1$ as we have also seen. Moreover, the decomposition group of a conjugate prime $\sigma(\mathfrak{P})$ to $\mathfrak{P}$ is the conjugate of decomposition group of $\mathfrak{P}$ by σ. In other words,

$$D_{\mathfrak{p}}(\sigma(\mathfrak{P})) = \sigma D_{\mathfrak{p}}(\mathfrak{P})\sigma^{-1}.$$

This is simply because $\tau(\mathfrak{P}) = \mathfrak{P}$ if and only if $\sigma\tau\sigma^{-1}(\sigma(\mathfrak{P})) = \sigma(\mathfrak{P})$. Also, the fundamental theorem of Galois theory implies that $D_{\mathfrak{p}}(\sigma(\mathfrak{P}))$ is normal if and only if $L^{D_{\mathfrak{p}}(\sigma(\mathfrak{P}))}/K$ is Galois in which case

$$\mathrm{Gal}(L^{D_{\mathfrak{p}}(\sigma(\mathfrak{P}))}/K) \cong \mathrm{Gal}(L/K)/D_{\mathfrak{p}}(\sigma(\mathfrak{P})).$$

Regardless, Proposition 5.6.1 shows that the prime factorization of $\mathfrak{p}\mathcal{O}$ takes the form

$$\mathfrak{p}\mathcal{O} = \left(\prod_{\sigma \in \mathrm{Gal}(L/K)/D_{\mathfrak{p}}(\mathfrak{P})} \sigma(\mathfrak{P}) \right)^{e}.$$

The decomposition field is aptly named because it contains all of the information about the different prime factors that $\mathfrak{p}\mathcal{O}$ splits into in $\mathcal{O}$. In particular, the primes $\mathfrak{P}^D$ are inert in $\mathcal{O}/\mathcal{O}^D$ as the following proposition shows:

Proposition 5.6.2. *Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of a degree n Galois extension L/K, let $\mathfrak{p}$ be a prime of $\mathcal{o}$ with inertia degree f and ramification index e, and let $\mathfrak{P}$ be a prime of $\mathcal{O}$ above $\mathfrak{p}$. Then the following hold:*

(i) $\mathfrak{P}^D$ is inert in $\mathcal{O}/\mathcal{O}^D$.

(ii) $e_{\mathfrak{p}}(\mathfrak{P}^D) = f_{\mathfrak{p}}(\mathfrak{P}^D) = 1$, $e_{\mathfrak{P}^D}(\mathfrak{P}) = e$, and $f_{\mathfrak{P}^D}(\mathfrak{P}) = f$.

Proof. We will prove the statements separately:

(i) By the fundamental theorem of Galois theory, $\mathrm{Gal}(L/L^{D_{\mathfrak{p}}(\mathfrak{P})}) = D_{\mathfrak{p}}(\mathfrak{P})$. It follows from Proposition 5.6.1 that the primes of $\mathcal{O}$ above $\mathfrak{P}^D$ are of the form $\sigma(\mathfrak{P})$ for $\sigma \in D_{\mathfrak{p}}(\mathfrak{P})$. But for these σ we have $\sigma(\mathfrak{P}) = \mathfrak{P}$. Therefore $\mathfrak{P}^D$ is inert $\mathcal{O}/\mathcal{O}^D$. This proves (i).

(ii) Recall $\mathrm{Gal}(L/L^{D_{\mathfrak{p}}(\mathfrak{P})}) = D_{\mathfrak{p}}(\mathfrak{P})$ by the fundamental theorem of Galois theory. On the one hand, since $|\mathrm{Gal}(L/L^{D_{\mathfrak{p}}(\mathfrak{P})})|$ is the degree of $L/L^{D_{\mathfrak{p}}(\mathfrak{P})}$ and $\mathfrak{P}^D$ is inert in $\mathcal{O}/\mathcal{O}^D$, the fundamental equality gives $|D_{\mathfrak{p}}(\mathfrak{P})| = e_{\mathfrak{P}^D}(\mathfrak{P}) f_{\mathfrak{P}^D}(\mathfrak{P})$. Then Equation (5.9) gives

$$ef = e_{\mathfrak{P}^D}(\mathfrak{P}) f_{\mathfrak{P}^D}(\mathfrak{P}).$$

On the other hand, as $\mathcal{O}/\mathcal{O}^D/\mathcal{o}$ is a tower of Dedekind extensions, Equation (5.8) implies $e = e_{\mathfrak{P}^D}(\mathfrak{P}) e_{\mathfrak{p}}(\mathfrak{P}^D)$ and $f = f_{\mathfrak{P}^D}(\mathfrak{P}) f_{\mathfrak{p}}(\mathfrak{P}^D)$. Thus

$$ef = e_{\mathfrak{P}^D}(\mathfrak{P}) e_{\mathfrak{p}}(\mathfrak{P}^D) f_{\mathfrak{P}^D}(\mathfrak{P}) f_{\mathfrak{p}}(\mathfrak{P}^D).$$

These two identities together imply $e_{\mathfrak{p}}(\mathfrak{P}^D) = f_{\mathfrak{p}}(\mathfrak{P}^D) = 1$. It follows that $e_{\mathfrak{P}^D}(\mathfrak{P}) = e$ and $f_{\mathfrak{P}^D}(\mathfrak{P}) = f$. This proves (ii). □

We can view Proposition 5.6.2 (i) as a statement that $\mathfrak{p}\mathcal{O}^D$ encodes all of the information about the distinct prime factors $\mathfrak{p}\mathcal{O}$ splits into in $\mathcal{O}$. Moreover, Proposition 5.6.2 (ii) can be interpreted as the fact that $\mathcal{O}^D/\mathcal{o}$ contains no information about the ramification index or inertia degree of $\mathfrak{p}$ in $\mathcal{O}/\mathcal{o}$. All of that information is contained in $\mathcal{O}/\mathcal{O}^D$. We need to do more work to unpack this information. Since any $\tau \in D_{\mathfrak{p}}(\mathfrak{P})$ leaves $\mathcal{O}$ and $\mathfrak{P}$ invariant, it induces an automorphism $\overline{\tau}$ of the residue class field $\mathbb{F}_{\mathfrak{P}}$ defined by

$$\overline{\tau} : \mathbb{F}_{\mathfrak{P}} \to \mathbb{F}_{\mathfrak{P}} \qquad \alpha \mapsto \tau(\alpha) \pmod{\mathfrak{P}}.$$

We then obtain a homomorphism

$$T_{\mathfrak{p}}^{\mathfrak{P}} : D_{\mathfrak{p}}(\mathfrak{P}) \to \mathrm{Aut}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}) \qquad \tau \mapsto \overline{\tau}.$$

The following proposition shows that the residue class extensions $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$ are normal and that $T_{\mathfrak{p}}^{\mathfrak{P}}$ is surjective:

Proposition 5.6.3. *Let $\mathcal{O}/o$ be a Dedekind extension of a finite Galois extension L/K, let $\mathfrak{p}$ be a prime of o, and let $\mathfrak{P}$ be a prime of $\mathcal{O}$ above $\mathfrak{p}$. Then the residue class extension $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$ is normal. Moreover, the homomorphism $T_{\mathfrak{p}}^{\mathfrak{P}}$ is surjective.*

Proof. By Proposition 5.6.2 (ii), $f_{\mathfrak{p}}(\mathfrak{P}^D) = 1$ so that we have an isomorphism $\mathbb{F}_{\mathfrak{P}^D} \cong \mathbb{F}_{\mathfrak{p}}$. Under this isomorphism we may assume $L^{D_{\mathfrak{p}}(\mathfrak{P})} = K$ which is to say $D_{\mathfrak{p}}(\mathfrak{P}) = \mathrm{Gal}(L/K)$. We will now prove that $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$ is normal. So let $\alpha \in \mathcal{O}$ and suppose $m_\alpha(x)$ and $m_{\overline{\alpha}}(x)$ are the minimal polynomials of α and $\overline{\alpha}$ over K and $\mathbb{F}_{\mathfrak{p}}$ respectively. Consider the reduction $\overline{m_\alpha}(x)$ of $m_\alpha(x)$, viewed as an element of $\mathcal{O}[x]$, modulo $\mathfrak{P}$ upon recalling that the coefficients of which are in o by Proposition 5.1.3. As α is a representative of $\overline{\alpha}$, we find that $\overline{\alpha}$ is a root of $\overline{m_\alpha}(x)$ and thus $m_{\overline{\alpha}}(x)$ divides $\overline{m_\alpha}(x)$ in $\mathbb{F}_{\mathfrak{P}}[x]$. Since L/K is normal (as it is Galois), $m_\alpha(x)$ splits into linear factors over L. These linear factors must also belong to $\mathcal{O}[x]$ since their roots are elements of $\mathcal{O}$ which itself is integrally closed. It follows that $\overline{m_\alpha}(x)$ splits into linear factors over $\mathbb{F}_{\mathfrak{P}}$. As $m_{\overline{\alpha}}(x)$ divides $\overline{m_\alpha}(x)$ in $\mathbb{F}_{\mathfrak{P}}[x]$, the same is true for $m_{\overline{\alpha}}(x)$. Thus $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$ is normal. We now prove the surjectivity of $T_{\mathfrak{p}}^{\mathfrak{P}}$. Consider the maximal separable subextension of $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$. Since this extension is finite (because the residue class extension is), it is simple by the primitive element theorem. Letting $\theta \in \mathcal{O}$ be such that $\overline{\theta}$ is a primitive element, the maximal separable subextension is of the form $\mathbb{F}_{\mathfrak{p}}(\overline{\theta})/\mathbb{F}_{\mathfrak{p}}$. Now let $\overline{\tau} \in \mathrm{Aut}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})$. Since $\overline{\tau}$ fixes $\mathbb{F}_{\mathfrak{p}}$ pointwise, $\overline{\tau}(\overline{\theta})$ is a root of $m_{\overline{\theta}}(x)$. As $m_{\overline{\theta}}(x)$ divides $\overline{m_\theta}(x)$ in $\mathbb{F}_{\mathfrak{P}}[x]$, we find that $\overline{\tau}(\overline{\theta})$ is also a root of $\overline{m_\theta}(x)$. Therefore there is a root θ' of $m_\theta(x)$ that is also a representative of $\overline{\tau}(\overline{\theta})$. Because L/K is Galois, we can find $\tau \in \mathrm{Gal}(L/K)$ such that $\tau(\theta) = \theta'$. Moreover, $\tau \in D_{\mathfrak{p}}(\mathfrak{P})$ since $D_{\mathfrak{p}}(\mathfrak{P}) = \mathrm{Gal}(L/K)$. Then $T_{\mathfrak{p}}^{\mathfrak{P}}(\tau) = \overline{\tau}$ because they both take $\overline{\theta}$ to the same element, hence act the same on $\mathbb{F}_{\mathfrak{p}}(\overline{\theta})/\mathbb{F}_{\mathfrak{p}}$, and $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}(\overline{\theta})$ is purely inseparable so that it has trivial automorphism group. This proves $T_{\mathfrak{p}}^{\mathfrak{P}}$ is surjective completing the proof. □

Accordingly, we define the **inertia group** $I_{\mathfrak{p}}(\mathfrak{P})$ of $\mathfrak{P}$ by

$$I_{\mathfrak{p}}(\mathfrak{P}) = \ker T_{\mathfrak{p}}^{\mathfrak{P}}.$$

Then $I_{\mathfrak{p}}(\mathfrak{P})$ is a normal subgroup of $D_{\mathfrak{p}}(\mathfrak{P})$. The associated **inertia field** $L^{I_{\mathfrak{p}}(\mathfrak{P})}$ of $\mathfrak{P}$ is defined to be

$$L^{I_{\mathfrak{p}}(\mathfrak{P})} = \{\lambda \in L : \tau(\lambda) = \lambda \text{ for all } \tau \in I_{\mathfrak{p}}(\mathfrak{P})\}.$$

In other words, the inertia field of $\mathfrak{P}$ is the fixed field of L by $I_{\mathfrak{p}}(\mathfrak{P})$. In particular, the fundamental theorem of Galois theory gives

$$\mathrm{Gal}(L/L^{I_{\mathfrak{p}}(\mathfrak{P})}) = I_{\mathfrak{p}}(\mathfrak{P}).$$

Since $I_{\mathfrak{p}}(\mathfrak{P})$ is a subgroup of $D_{\mathfrak{p}}(\mathfrak{P})$, the fundamental theorem of Galois theory implies that $L^{I_{\mathfrak{p}}(\mathfrak{P})}$ is an intermediate field of $L/L^{D_{\mathfrak{p}}(\mathfrak{P})}$. We let $\mathcal{O}^I$ denote the integral closure of o in $L^{I_{\mathfrak{p}}(\mathfrak{P})}$ so that $\mathcal{O}/\mathcal{O}^I/\mathcal{O}^D$ is a tower of Dedekind extensions. Also, we will write $\mathfrak{P}^I$ for the prime of $L^{I_{\mathfrak{p}}(\mathfrak{P})}$ below $\mathfrak{P}$. Then we can illustrate this relationship via the following tower of extensions:

$$\begin{array}{c} \mathfrak{P} \subset \mathcal{O} \subseteq L \\ | \\ \mathfrak{P}^I \subset \mathcal{O}^I \subseteq L^{I_{\mathfrak{p}}(\mathfrak{P})} \\ | \\ \mathfrak{P}^D \subset \mathcal{O}^D \subseteq L^{D_{\mathfrak{p}}(\mathfrak{P})}. \end{array}$$

The inertia group $I_{\mathfrak{p}}(\mathfrak{P})$ is closely related to the automorphism group of the residue class extension $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$. To see this, first recall that $\mathrm{Gal}(L/L^{D_{\mathfrak{p}}(\mathfrak{P})}) = D_{\mathfrak{p}}(\mathfrak{P})$ by the fundamental theorem of Galois theory. Then $I_{\mathfrak{p}}(\mathfrak{P})$ is a normal subgroup of $\mathrm{Gal}(L/L^{D_{\mathfrak{p}}(\mathfrak{P})})$ and the fundamental theorem of Galois theory again implies $L^{I_{\mathfrak{p}}(\mathfrak{P})}/L^{D_{\mathfrak{p}}(\mathfrak{P})}$ is Galois with

$$\mathrm{Gal}(L^{I_{\mathfrak{p}}(\mathfrak{P})}/L^{D_{\mathfrak{p}}(\mathfrak{P})}) \cong D_{\mathfrak{p}}(\mathfrak{P})/I_{\mathfrak{p}}(\mathfrak{P}).$$

Since $T_{\mathfrak{p}}^{\mathfrak{P}}$ is surjective by Proposition 5.6.3, the first isomorphism theorem induces an isomorphism

$$\mathrm{Gal}(L^{I_{\mathfrak{p}}(\mathfrak{P})}/L^{D_{\mathfrak{p}}(\mathfrak{P})}) \cong \mathrm{Aut}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}).$$

Now let e and f be the ramification index and inertia degree of $\mathfrak{p}$ respectively. The inertia group $I_{\mathfrak{p}}(\mathfrak{P})$ encodes e and f provided the residue class extension $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$ is separable. Indeed, Proposition 5.6.3 implies $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$ is Galois. Then $\mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}) = \mathrm{Aut}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})$ and it follows that $|\mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})| = f$. Then Equation (5.9) and our two isomorphisms together give

$$|\mathrm{Gal}(L^{I_{\mathfrak{p}}(\mathfrak{P})}/L^{D_{\mathfrak{p}}(\mathfrak{P})})| = f \quad \text{and} \quad |I_{\mathfrak{p}}(\mathfrak{P})| = e. \tag{5.10}$$

Then $L^{I_{\mathfrak{p}}(\mathfrak{P})}/L^{D_{\mathfrak{p}}(\mathfrak{P})}$ has degree f and $L/L^{I_{\mathfrak{p}}(\mathfrak{P})}$ has degree e. The decomposition and inertia groups fit into the follow exact sequence

Proposition 5.6.4. *Let $\mathcal{O}/o$ be a Dedekind extension of a finite Galois extension L/K, let $\mathfrak{p}$ be a prime of o, and let $\mathfrak{P}$ be a prime of $\mathcal{O}$ above $\mathfrak{p}$. Then the sequence*

$$1 \longrightarrow I_{\mathfrak{p}}(\mathfrak{P}) \longrightarrow D_{\mathfrak{p}}(\mathfrak{P}) \xrightarrow{T_{\mathfrak{p}}^{\mathfrak{P}}} \mathrm{Aut}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}) \longrightarrow 1,$$

is exact.

Proof. The sequence is exact at $I_{\mathfrak{p}}(\mathfrak{P})$ because the second map is injective. Exactness at $D_{\mathfrak{p}}(\mathfrak{P})$ follows by the definition of $I_{\mathfrak{p}}(\mathfrak{P})$. Lastly, we have exactness at $\mathrm{Aut}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})$ since $T_{\mathfrak{p}}^{\mathfrak{P}}$ is surjective by Proposition 5.6.3. □

We can say more when the residue class extension $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$ is separable as the following proposition shows:

Proposition 5.6.5. *Let $\mathcal{O}/o$ be a Dedekind extension of a finite Galois extension L/K, let $\mathfrak{p}$ be a prime of o with inertia degree f and ramification index e, and let $\mathfrak{P}$ be a prime of $\mathcal{O}$ above $\mathfrak{p}$. Moreover, suppose the residue class extension $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$ is separable. Then the following hold:*

(i) $e_{\mathfrak{P}^I}(\mathfrak{P}) = e$ and $f_{\mathfrak{P}^I}(\mathfrak{P}) = 1$.

(ii) $e_{\mathfrak{P}^D}(\mathfrak{P}^I) = 1$ and $f_{\mathfrak{P}^D}(\mathfrak{P}^I) = f$.

Proof. First observe that $I_{\mathfrak{P}^I}(\mathfrak{P}) = I_{\mathfrak{p}}(\mathfrak{P})$. Indeed, recall that $\mathrm{Gal}(L/L^{I_{\mathfrak{p}}(\mathfrak{P})}) = I_{\mathfrak{p}}(\mathfrak{P})$. Then Proposition 5.6.1 implies that the primes of $\mathcal{O}$ above $\mathfrak{P}^I$ are of the form $\sigma(\mathfrak{P})$ for $\sigma \in I_{\mathfrak{p}}(\mathfrak{P})$. But for these σ we have $\sigma(\mathfrak{P}) = \mathfrak{P}$. This shows that $\mathfrak{P}^I$ is inert in $\mathcal{O}/\mathcal{O}^I$. Therefore $D_{\mathfrak{P}^I}(\mathfrak{P}) = I_{\mathfrak{p}}(\mathfrak{P})$. Since $\ker T_{\mathfrak{p}}^{\mathfrak{P}} \subseteq \ker T_{\mathfrak{P}^I}^{\mathfrak{P}}$ (because any automorphism that is the identity on $\mathbb{F}_{\mathfrak{P}}$ is also the identity on the subfield $\mathbb{F}_{\mathfrak{P}^I}$), we conclude that $I_{\mathfrak{P}^I}(\mathfrak{P}) = I_{\mathfrak{p}}(\mathfrak{P})$. We now prove the statements:

(i) Since $\mathrm{Gal}(L/L^{I_{\mathfrak{p}}(\mathfrak{P})}) = I_{\mathfrak{p}}(\mathfrak{P})$ and $\mathfrak{P}^I$ is inert in $\mathcal{O}/\mathcal{O}^I$, we have

$$e = e_{\mathfrak{P}^I}(\mathfrak{P}) f_{\mathfrak{P}^I}(\mathfrak{P}),$$

by the fundamental equality and Equation (5.10). As $I_{\mathfrak{P}^I}(\mathfrak{P}) = I_{\mathfrak{p}}(\mathfrak{P})$, it follows from Proposition 5.6.3 that $\mathrm{Aut}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{P}^I}) = \{\mathrm{id}\}$. Therefore $e_{\mathfrak{P}^I}(\mathfrak{P}) = e$ and $f_{\mathfrak{P}^I}(\mathfrak{P}) = 1$ proving (i).

(ii) Combining Equation (5.8) and Proposition 5.6.2 (ii) together gives

$$e = e_{\mathfrak{P}^D}(\mathfrak{P}^I)e_{\mathfrak{P}^I}(\mathfrak{P}) \quad \text{and} \quad f = f_{\mathfrak{P}^D}(\mathfrak{P}^I)f_{\mathfrak{P}^I}(\mathfrak{P}).$$

Then (i) implies (ii). □

In the case the residue class extension $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$ is separable, we fit the decomposition and inertia fields into the following tower of extensions:

$$\begin{array}{c} \mathfrak{P} \subset \mathcal{O} \subseteq L \\ {\scriptstyle e}\,\big|\,{\scriptstyle 1} \\ \mathfrak{P}^I \subset \mathcal{O}^I \subseteq L^{I_{\mathfrak{p}}(\mathfrak{P})} \\ {\scriptstyle 1}\,\big|\,{\scriptstyle f} \\ \mathfrak{P}^D \subset \mathcal{O}^D \subseteq L^{D_{\mathfrak{p}}(\mathfrak{P})} \\ {\scriptstyle 1}\,\big|\,{\scriptstyle 1} \\ \mathfrak{p} \subset o \subseteq K. \end{array}$$

The labels on left and right of the extensions represent the corresponding ramification indices and inertia degrees respectively which come from Propositions 5.6.2 and 5.6.5. Then we see that the extension $L^{D_{\mathfrak{p}}(\mathfrak{P})}/K$ contains all of the information about the distinct prime factors $\mathfrak{p}\mathcal{O}$ splits into in $\mathcal{O}$, the extension $L^{I_{\mathfrak{p}}(\mathfrak{P})}/L^{D_{\mathfrak{p}}(\mathfrak{P})}$ contains all of the information about the corresponding inertia degrees, and the extension $L/L^{I_{\mathfrak{p}}(\mathfrak{P})}$ contains all of the information about the corresponding ramification indices. In particular, $\mathfrak{p}$ is unramified in $\mathcal{O}/o$ if and only if $L^{I_{\mathfrak{p}}(\mathfrak{P})} = L$ which is equivalent to $I_{\mathfrak{p}}(\mathfrak{P}) = \{\mathrm{id}\}$.

5.7 The Different and Discriminant

Let $\mathcal{O}/o$ be a Dedekind extension of a degree n separable extension L/K. It is exceptionally rare for a prime $\mathfrak{p}$ of K to ramify. We will construct two integral ideals, one of $\mathcal{O}$ and the other of o which will tell us which primes ramify in L or K. These integral ideals are the different and discriminant respectively. To describe them, we will need the concept of lattices in Dedekind domains. We say that $\mathfrak{L}$ is a **o-lattice** if it is a finitely generated o-submodule of L. Moreover, we say that $\mathfrak{L}$ is **complete** if it spans L/K. That is, $\mathfrak{L}$ contains a basis of L/K.

Remark 5.7.1. *A $\mathbb{Z}$-lattice is an integral lattice and a complete $\mathbb{Z}$-lattice is a complete integral lattice.*

A o-lattice $\mathfrak{L}$ need not be a fractional ideal of o since it does not need to be a $\mathcal{O}$-submodule of L. However, every fractional ideal $\mathfrak{F}$ of $\mathcal{O}$ is a complete o-lattice. Indeed, by Corollary 5.5.1 $\mathcal{O}$ is a finitely generated o-module and so $\mathfrak{F}$ is a finitely generated o-module as well. Moreover, $\mathfrak{F}$ contains a basis of L/K. For if $\lambda_1, \ldots, \lambda_n$ is a basis for L/K we may use Proposition 5.1.3 to multiply by a nonzero element of o, if necessary, to ensure that this basis is contained in $\mathcal{O}$. Choosing any nonzero $\alpha \in \mathfrak{F}$, $\alpha\lambda_1, \ldots, \alpha\lambda_n$ is a basis for L/K inside $\mathfrak{F}$.

Remark 5.7.2. *A o-lattice $\mathcal{L}$ need not be a free o-submodule of L. Indeed, we are only guaranteed that $\mathcal{L}$ is a finitely generated o-submodule of L not that it is a free finitely generated o-submodule of L.*

Recall by Lemma 5.2.1 that there is a nondegenerate symmetric bilinear form on K given by

$$\mathrm{Tr}_{L/K} : L \times L \to K \qquad (\lambda, \eta) \mapsto \mathrm{Tr}_{L/K}(\lambda\eta).$$

We call this bilinear form the **trace form** of L/K. The trace form makes L into a nondegenerate inner product space over K and so every basis $\lambda_1, \ldots, \lambda_n$ admits a dual basis $\lambda_1^\vee, \ldots, \lambda_n^\vee$ with respect to $\mathrm{Tr}_{L/K}$ defined by

$$\mathrm{Tr}_{L/K}(\lambda_i \lambda_j^\vee) = \delta_{i,j},$$

for $1 \leq i, j \leq n$. The trace form will allow us to introduce duals. Indeed, if $\mathfrak{F}$ is a fractional ideal $\mathcal{O}$, the **dual** $\mathfrak{F}^\vee$ of $\mathfrak{F}$ is defined by

$$\mathfrak{F}^\vee = \{\lambda \in L : \mathrm{Tr}_{L/K}(\lambda\mathfrak{F}) \subseteq \mathfrak{o}\}.$$

We say that $\mathfrak{F}$ is **self-dual** if $\mathfrak{F}^\vee = \mathfrak{F}$. The following proposition shows that the dual $\mathfrak{F}^\vee$ is indeed a fractional ideal:

Proposition 5.7.1. *Let $\mathcal{O}/\mathfrak{o}$ be a Dedekind extension of a degree n separable extension L/K and let $\mathfrak{F}$ be a fractional ideal of $\mathcal{O}$. Then $\mathfrak{F}^\vee$ is also a fractional ideal of $\mathcal{O}$ and*

$$\mathfrak{F}^\vee = \mathfrak{F}^{-1}\mathcal{O}^\vee.$$

Proof. We will first show that $\mathfrak{F}^\vee$ is a finitely generated $\mathfrak{o}$-submodule of L. Let $\lambda_1, \ldots, \lambda_n$ be a basis for L/K. Using Proposition 5.1.3 to multiply by a nonzero element of $\mathfrak{o}$, if necessary, we can ensure that this basis is contained in $\mathcal{O}$. Now choose some nonzero $\alpha \in \mathfrak{F} \cap \mathfrak{o}$ (such an element exists since every prime $\mathfrak{P}$ of $\mathcal{O}$ is above a prime $\mathfrak{p}$ of $\mathfrak{o}$ and $\mathfrak{F}$ is of the form $\mathfrak{F} = \frac{1}{\delta}\mathfrak{A}$ for some nonzero $\delta \in \mathcal{O}$ and integral ideal $\mathfrak{A}$ so that $\mathfrak{A} \subseteq \mathfrak{F}$). Now suppose $\lambda \in \mathfrak{F}^\vee$ and write $\lambda = \kappa_1\lambda_1 + \cdots + \kappa_n\lambda_n$ with $\kappa_i \in K$ for $1 \leq i \leq n$. Then linearity of the trace implies

$$\sum_{1 \leq j \leq n} \alpha\kappa_j \,\mathrm{Tr}_{L/K}(\lambda_i\lambda_j) = \mathrm{Tr}_{L/K}(\alpha\lambda_i\lambda),$$

for $1 \leq i \leq n$. These n equations are equivalent to the identity

$$\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n) \begin{pmatrix} \alpha\kappa_1 \\ \vdots \\ \alpha\kappa_n \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}_{L/K}(\alpha\lambda_1\lambda) \\ \vdots \\ \mathrm{Tr}_{L/K}(\alpha\lambda_n\lambda) \end{pmatrix}.$$

Multiplying on the left by the adjugate $\mathrm{adj}(\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n))$ of $\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n)$ and recalling that a matrix times its adjugate is its determinant times the identity, we see that

$$d_{L/K}(\lambda_1, \ldots, \lambda_n) \begin{pmatrix} \alpha\kappa_1 \\ \vdots \\ \alpha\kappa_n \end{pmatrix} = \mathrm{adj}(\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n)) \begin{pmatrix} \mathrm{Tr}_{L/K}(\alpha\lambda_1\lambda) \\ \vdots \\ \mathrm{Tr}_{L/K}(\alpha\lambda_n\lambda) \end{pmatrix}.$$

Since $\lambda_1, \ldots, \lambda_n$ is a basis for L/K that is contained in $\mathcal{O}$, the matrix $\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n)$ has entries in $\mathfrak{o}$ by Proposition 5.2.2 and therefore $\mathrm{adj}(\mathrm{Tr}_{L/K}(\lambda_1, \ldots, \lambda_n))$ does too. As $\alpha\lambda_i \in \mathfrak{F}$ (because $\lambda_j \in \mathcal{O}$ for all j) and $\lambda \in \mathfrak{F}^\vee$, Proposition 5.2.2 again implies $\mathrm{Tr}_{L/K}(\alpha\lambda_i\lambda) \in \mathfrak{o}$ for all i. So the right-hand side has entires in $\mathfrak{o}$ and hence the left-hand side must as well. This means $d_{L/K}(\lambda_1, \ldots, \lambda_n)\alpha\kappa_i \in A$ for all i. Since $\lambda \in \mathfrak{F}^\vee$ was arbitrary, we have

$$\alpha d_{L/K}(\lambda_1, \ldots, \lambda_n)\mathfrak{F}^\vee \subseteq \mathcal{O}.$$

As $\mathcal{O}$ is a finitely generated $\mathfrak{o}$-module by Corollary 5.5.1, it follows that $\mathfrak{F}^\vee$ is a finitely generated $\mathfrak{o}$-submodule of L. Therefore $\mathfrak{F}^\vee$ is also a finitely generated $\mathcal{O}$-submodule of L if it is preserved under

multiplication by $\mathcal{O}$. Let $\alpha \in \mathcal{O}$ and $\beta \in \mathfrak{F}^\vee$. Then we must show $\alpha\beta \in \mathfrak{F}^\vee$. To see this, observe that $\mathrm{Tr}_{L/K}(\alpha\beta\mathfrak{F}) \subseteq \mathrm{Tr}_{L/K}(\beta\mathfrak{F}) \subseteq \mathfrak{o}$ by Proposition 5.2.2 since $\alpha\mathfrak{F} \subseteq \mathfrak{F}$ and $\beta \in \mathfrak{F}^\vee$. Therefore $\alpha\beta \in \mathfrak{F}^\vee$ and hence $\mathfrak{F}^\vee$ is a fractional ideal proving the first statement. To prove the second statement we will show containment in both directions. For the forward containment, first suppose $\alpha \in \mathfrak{F}^\vee$ and $\beta \in \mathfrak{F}$. Then $\mathrm{Tr}_{L/K}(\alpha\beta\mathcal{O}) \subseteq \mathrm{Tr}_{L/K}(\alpha\mathfrak{F}) \subseteq \mathfrak{o}$ by Proposition 5.2.2 since $\beta\mathcal{O} \subseteq \mathfrak{F}$ and $\alpha \in \mathfrak{F}^\vee$. Hence $\alpha\beta \in \mathcal{O}^\vee$ so that $\mathfrak{F}^\vee\mathfrak{F} \subseteq \mathcal{O}^\vee$ and therefore $\mathfrak{F}^\vee \subseteq \mathfrak{F}^{-1}\mathcal{O}^\vee$. This proves the forward containment. For the reverse containment, suppose $\alpha \in \mathfrak{F}^{-1}$ and $\beta \in \mathcal{O}^\vee$. Then $\mathrm{Tr}_{L/K}(\alpha\beta\mathfrak{F}) \subseteq \mathrm{Tr}_{L/K}(\beta\mathcal{O}) \subseteq \mathfrak{o}$ by Proposition 5.2.2 since $\alpha\mathfrak{F} \subseteq \mathcal{O}$ and $\beta \in \mathcal{O}^\vee$. Thus $\alpha\beta \in \mathfrak{F}^\vee$ implying $\mathfrak{F}^{-1}\mathcal{O}^\vee \subseteq \mathfrak{F}^\vee$ and proving the reverse containment. This completes the proof. □

As the dual $\mathfrak{F}^\vee$ is a fractional ideal by Proposition 5.7.1, it is also a complete $\mathfrak{o}$-lattice. As we might hope, localization respects duals:

Proposition 5.7.2. *Let $\mathcal{O}/\mathfrak{o}$ be a Dedekind extension of a degree n separable extension L/K and let $D \subseteq \mathfrak{o} - \{0\}$ be a multiplicative subset. Then for any fractional ideal $\mathfrak{F}$ of $\mathcal{O}$, we have*

$$(\mathfrak{F}D^{-1})^\vee = (\mathfrak{F}^\vee)D^{-1}.$$

Proof. For the forward containment, suppose $\lambda \in (\mathfrak{F}D^{-1})^\vee$. Then $\mathrm{Tr}_{L/K}(\lambda\mathfrak{F}D^{-1}) \subseteq \mathfrak{o}D^{-1}$. As $D \subset K$, linearity of the trace implies $\lambda = \frac{\alpha}{\delta}$ with $\mathrm{Tr}_{L/K}(\alpha\mathfrak{F})\delta \subseteq \mathfrak{o}D^{-1}$ so that $\alpha \in \mathfrak{F}^\vee$ and $\delta \in D$. Hence $\lambda \in (\mathfrak{F}^\vee)D^{-1}$ and the forward containment follows. For the reverse containment, suppose $\frac{\alpha}{\delta} \in (\mathfrak{F}^\vee)D^{-1}$. Then $\alpha \in \mathfrak{F}^\vee$ so that $\mathrm{Tr}_{L/K}(\alpha\mathfrak{F}) \subseteq \mathfrak{o}$ and $\delta \in D$. As $D \subset K$, linearity of the trace implies $\mathrm{Tr}_{L/K}(\lambda\mathfrak{F}) \subseteq \mathfrak{o}D^{-1}$ with $\lambda = \frac{\alpha}{\delta}$. Hence $\frac{\alpha}{\delta} \in (\mathfrak{F}D^{-1})^\vee$ and the reverse containment holds. □

We will now introduce the different and the discriminant. We define the **complement** $\mathfrak{C}_{\mathcal{O}/\mathfrak{o}}$ of $\mathcal{O}/\mathfrak{o}$ by

$$\mathfrak{C}_{\mathcal{O}/\mathfrak{o}} = \mathcal{O}^\vee.$$

This is a fractional ideal by Proposition 5.7.1. The **different** $\mathfrak{D}_{\mathcal{O}/\mathfrak{o}}$ of $\mathcal{O}/\mathfrak{o}$ is defined to be the inverse of the complement $\mathfrak{C}_{\mathcal{O}/\mathfrak{o}}$:

$$\mathfrak{D}_{\mathcal{O}/\mathfrak{o}} = (\mathcal{O}^\vee)^{-1}.$$

As $\mathcal{O} \subseteq \mathcal{O}^\vee$ by Proposition 5.2.2, it follows from Proposition 5.3.2 that $\mathfrak{D}_{\mathcal{O}/\mathfrak{o}}$ is an integral ideal and

$$\mathfrak{D}_{\mathcal{O}/\mathfrak{o}} = \{\lambda \in L : \lambda\mathcal{O}^\vee \subseteq \mathcal{O}\}.$$

Also, Lemma 5.3.5 gives the first isomorphism in the following chain:

$$\mathcal{O}/\mathfrak{D}_{\mathcal{O}/\mathfrak{o}} \cong \mathfrak{D}_{\mathcal{O}/\mathfrak{o}}^{-1}/\mathcal{O} \cong \mathcal{O}^\vee/\mathcal{O}. \tag{5.11}$$

From this chain of isomorphisms we find that

$$\mathfrak{D}_{\mathcal{O}/\mathfrak{o}} \subseteq \mathcal{O} \subseteq \mathcal{O}^\vee.$$

Therefore the different $\mathfrak{D}_{\mathcal{O}/\mathfrak{o}}$ is a measure of how much $\mathcal{O}$ fails to be self-dual for if $\mathfrak{D}_{\mathcal{O}/\mathfrak{o}} = \mathcal{O}^\vee$ then $\mathcal{O}$ must be self-dual. Moreover, by Proposition 5.7.1 we can express the dual fractional ideal $\mathfrak{F}^\vee$ of a fractional ideal $\mathfrak{F}$ of $\mathfrak{o}$ in terms of the different as

$$\mathfrak{F}^\vee = \mathfrak{F}^{-1}\mathfrak{D}_{\mathcal{O}/\mathfrak{o}}^{-1}.$$

Now for the discriminant. We define the **discriminant** $\mathfrak{d}_{\mathcal{O}/\mathfrak{o}}$ of $\mathcal{O}/\mathfrak{o}$ to be the ideal of $\mathfrak{o}$ generated by all discriminants $d_{L/K}(\alpha_1, \ldots, \alpha_n)$ as $\alpha_1, \ldots, \alpha_n$ runs over all bases of L/K contained in $\mathcal{O}$. Note that $\mathfrak{d}_{\mathcal{O}/\mathfrak{o}}$

is necessarily an integral ideal. Recall that if $\mathcal{O}/o$ admits an integral then $\mathcal{O}$ is a free o-module of rank n. In this case, bases of L/K contained in $\mathcal{O}$ are precisely the integral bases of $\mathcal{O}/o$. Then

$$\mathfrak{d}_{\mathcal{O}/o} = d_o(\mathcal{O})o,$$

so that the discriminant $\mathfrak{d}_{\mathcal{O}/o}$ is generated by the discriminant $d_o(\mathcal{O})$. In particular, $\mathfrak{d}_{\mathcal{O}/o}$ is a principal integral ideal if $\mathcal{O}/o$ admits an integral basis. By Theorem 5.2.1 this will hold if o is a principal ideal domain but not necessarily in general. The different and discriminant also respect localization:

Proposition 5.7.3. *Let $\mathcal{O}/o$ be a Dedekind extension of a degree n separable extension L/K and let $D \subseteq o - \{0\}$ be a multiplicative subset. Then*

$$\mathfrak{D}_{\mathcal{O}D^{-1}/oD^{-1}} = \mathfrak{D}_{\mathcal{O}/o}D^{-1} \quad \textit{and} \quad \mathfrak{d}_{\mathcal{O}D^{-1}/oD^{-1}} = \mathfrak{d}_{\mathcal{O}/o}D^{-1}$$

Proof. For the first identity, it is equivalent to show

$$((\mathcal{O}D^{-1})^\vee)^{-1} = (\mathcal{O}^\vee)^{-1}D^{-1},$$

by definition of the different. Applying Propositions 5.4.5 and 5.7.2, we see that the right-had side is equal to the left-hand side as desired. For the second identity, we will show containment in both directions. For the forward inclusion, suppose $\frac{\alpha_1}{\delta_1}, \dots, \frac{\alpha_n}{\delta_n}$ is a basis of L/K contained in $\mathcal{O}D^{-1}$. Setting $\delta = \delta_1 \cdots \delta_n$, we see that $\frac{\alpha_1\delta}{\delta_1}, \dots, \frac{\alpha_n\delta}{\delta_n}$ is a basis of L/K contained in $\mathcal{O}$. As $D \subset K$ and $d_{L/K}\left(\frac{\alpha_1\delta}{\delta_1}, \dots, \frac{\alpha_n\delta}{\delta_n}\right) \in \mathfrak{d}_{\mathcal{O}/o}$, linearity of the trace implies $d_{L/K}\left(\frac{\alpha_1}{\delta_1}, \dots, \frac{\alpha_n}{\delta_n}\right) \in \mathfrak{d}_{\mathcal{O}/o}D^{-1}$. This shows the forward containment. For the reverse containment, suppose $\alpha_1, \dots, \alpha_n$ is a basis of L/K contained in $\mathcal{O}$ and let $\delta \in D^{-1}$. Then $\frac{\alpha_1}{\delta}, \dots, \frac{\alpha_n}{\delta}$ is a basis for L/K contained in $\mathcal{O}D^{-1}$. As $D \subseteq K$ and $d_{L/K}\left(\frac{\alpha_1}{\delta}, \dots, \frac{\alpha_n}{\delta}\right) \in \mathfrak{d}_{\mathcal{O}D^{-1}/oD^{-1}}$, linearity of the trace again implies $d_{L/K}(\alpha_1 \dots, \alpha_n) \in \mathfrak{d}_{\mathcal{O}D^{-1}/oD^{-1}}$ proving the reverse containment. □

We want to show that a prime $\mathfrak{P}$ of $\mathcal{O}$ ramifies in $\mathcal{O}/o$ if and only if it divides the different $\mathfrak{D}_{\mathcal{O}/o}$ provided the residue class extensions are separable. We first require a lemma:

Lemma 5.7.1. *Let $\mathcal{O}/o$ be a Dedekind extension of a degree n separable extension L/K and let $\mathfrak{A}$ be an integral ideal of $\mathcal{O}$. Then $\mathfrak{A} \mid \mathfrak{D}_{\mathcal{O}/o}$ if and only if $\mathrm{Tr}_{L/K}(\mathfrak{A}^{-1}) \subseteq o$.*

Proof. $\mathfrak{A}$ divides $\mathfrak{D}_{\mathcal{O}/o}$ if and only if $\mathfrak{D}_{\mathcal{O}/o} \subseteq \mathfrak{A}$. This equivalent to $(\mathcal{O}^\vee)^{-1} \subseteq \mathfrak{A}$ and inverting shows the further equivalence $\mathfrak{A}^{-1} \subseteq \mathcal{O}^\vee$. This last containment is equivalent to $\mathrm{Tr}_{L/K}(\mathfrak{A}^{-1}) \subseteq o$ which completes the proof. □

With Lemma 5.7.1 in hand, we can now prove our claim:

Theorem 5.7.1. *Let $\mathcal{O}/o$ be a Dedekind extension of a degree n separable extension L/K and assume that all residue class extensions are separable. Then a prime $\mathfrak{P}$ of $\mathcal{O}$ ramifies in $\mathcal{O}/o$ if and only if $\mathfrak{P}$ divides $\mathfrak{D}_{\mathcal{O}/o}$. In particular, if $\mathfrak{P}$ is above $\mathfrak{p}$ then the following hold:*

(i) $\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})} \mid \mathfrak{D}_{\mathcal{O}/o}$ if and only if $e_\mathfrak{p}(\mathfrak{P}) \equiv 0 \pmod{\mathfrak{p}}$.

(ii) $\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})-1} \mid\mid \mathfrak{D}_{\mathcal{O}/o}$ if and only if $e_\mathfrak{p}(\mathfrak{P}) \not\equiv 0 \pmod{\mathfrak{p}}$.

Proof. We first show that (i) and (ii) together imply that $\mathfrak{P}$ ramifies in $\mathcal{O}/o$ if and only if $\mathfrak{P} \mid \mathfrak{D}_{\mathcal{O}/o}$. Since the residue class extensions are separable by assumption, $\mathfrak{P}$ ramifies in $\mathcal{O}/o$ if and only if $e_{\mathfrak{p}}(\mathfrak{P}) \geq 2$. Therefore (i) and (ii) together show that $\mathfrak{P}$ ramifies in $\mathcal{O}/o$ if and only if $\mathfrak{P} \mid \mathfrak{D}_{\mathcal{O}/o}$ (note that $1 \not\equiv 0 \pmod{\mathfrak{p}}$ for any prime $\mathfrak{p}$). Therefore it remains to prove (i) and (ii). In view of Propositions 5.5.2 and 5.7.3, it suffices to assume $\mathcal{O}/o$ is a local Dedekind extension. Therefore o is a discrete valuation ring, $\mathcal{O}$ is a principal ideal domain, and $\mathcal{O}/o$ admits an integral basis $\alpha_1, \ldots, \alpha_n$ making $\mathcal{O}$ a free o-module of rank n. We will first show $\mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})-1} \mid \mathfrak{D}_{\mathcal{O}/o}$ independent of $e_{\mathfrak{p}}(\mathfrak{P})$ modulo $\mathfrak{p}$ (note that this is satisfied by both (i) and (ii)). To this end, write $\mathfrak{p}\mathcal{O} = \mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})-1}\mathfrak{A}$ for some integral ideal $\mathfrak{A}$ of $\mathcal{O}$ so that $\mathfrak{P} \mid\mid \mathfrak{A}$. By Lemma 5.7.1, $\mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})-1} \mid \mathfrak{D}_{\mathcal{O}/o}$ if and only if $\mathrm{Tr}_{L/K}(\mathfrak{P}^{1-e_{\mathfrak{p}}(\mathfrak{P})}) \subseteq o$. Since $\mathfrak{P}^{1-e_{\mathfrak{p}}(\mathfrak{P})} = \mathfrak{p}^{-1}\mathfrak{A}$, linearity of the trace implies that this is equivalent to $\mathrm{Tr}_{L/K}(\mathfrak{A}) \subseteq \mathfrak{p}$. By Proposition 5.2.8, $\mathrm{Tr}_{K/L}(\mathfrak{A}) = \mathrm{Tr}_{\mathcal{O}/o}(\mathfrak{A})$ and so it is further equivalent to show

$$\mathrm{Tr}_{\mathcal{O}/o}(\mathfrak{A}) \subseteq \mathfrak{p},$$

Let $\alpha \in \mathfrak{A}$ and T_α be the multiplication by α map. Then $\mathfrak{p}\mathcal{O}$ is T_α-invariant because it is an ideal of $\mathcal{O}$. This induces a multiplication by $\overline{\alpha}$ map $T_{\overline{\alpha}}$ on $\mathcal{O}/\mathfrak{p}\mathcal{O}$ as a $\mathbb{F}_{\mathfrak{p}}$-vector space. As the classes $\overline{\alpha_1}, \ldots, \overline{\alpha_n}$ are a basis for $\mathcal{O}/\mathfrak{p}\mathcal{O}$ as a $\mathbb{F}_{\mathfrak{p}}$-vector space, it follows that

$$\mathrm{Tr}_{\mathcal{O}/o}(\alpha) \equiv \mathrm{Tr}_{(\mathcal{O}/\mathfrak{p}\mathcal{O})/\mathbb{F}_{\mathfrak{p}}}(\overline{\alpha}) \pmod{\mathfrak{p}}.$$

Therefore it is yet further equivalent to show

$$\mathrm{Tr}_{(\mathcal{O}/\mathfrak{p}\mathcal{O})/\mathbb{F}_{\mathfrak{p}}}(\overline{\alpha}) \equiv 0 \pmod{\mathfrak{p}},$$

for all $\alpha \in \mathfrak{A}$. To prove this, observe that $\overline{\alpha}$ runs over to the subring $\mathfrak{A}/\mathfrak{p}\mathcal{O}$ of $\mathcal{O}/\mathfrak{p}\mathcal{O}$ as α runs over $\mathfrak{A}$. By construction, $\mathfrak{A}$ is divisible by every prime factor of $\mathfrak{p}\mathcal{O}$ and so a power, say k, of $\mathfrak{A}$ is divisible by $\mathfrak{p}\mathcal{O}$. But this means $T_{\overline{\alpha}}^k$ is the zero operator so that it is nilpotent. As nilpotent maps have trace zero, $\mathrm{Tr}_{(\mathcal{O}/\mathfrak{p}\mathcal{O})/\mathbb{F}_{\mathfrak{p}}}(\overline{\alpha}) = 0$. But then $\mathrm{Tr}_{\mathcal{O}/o}(\alpha) \equiv 0 \pmod{p}$ and, as α was arbitrary, the claim is justified. We now prove (i) and (ii):

(i) Begin by writing $\mathfrak{p}\mathcal{O} = \mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})}\mathfrak{A}$ for some integral ideal $\mathfrak{A}$ of $\mathcal{O}$ so that $\mathfrak{A}$ is relatively prime to $\mathfrak{P}$. Arguing as before, we find that $\mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})} \mid \mathfrak{D}_{\mathcal{O}/o}$ if and only if

$$\mathrm{Tr}_{(\mathcal{O}/\mathfrak{p}\mathcal{O})/\mathbb{F}_{\mathfrak{p}}}(\overline{\alpha}) = 0,$$

for all $\alpha \in \mathfrak{A}$. Since $\mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})}$ and $\mathfrak{A}$ are relatively prime, the Chinese remainder theorem implies

$$\mathcal{O}/\mathfrak{p}\mathcal{O} \cong (\mathcal{O}/\mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})}) \oplus (\mathcal{O}/\mathfrak{A}).$$

So there exists $\beta, \gamma \in \mathcal{O}$ such that $\beta \equiv \alpha \pmod{\mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})}}$ and $\gamma \equiv \alpha \pmod{\mathfrak{A}}$. Then Equation (5.2) implies

$$\mathrm{Tr}_{(\mathcal{O}/\mathfrak{p}\mathcal{O})/\mathbb{F}_{\mathfrak{p}}}(\overline{\alpha}) = \mathrm{Tr}_{(\mathcal{O}/\mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})})/\mathbb{F}_{\mathfrak{p}}}(\overline{\beta}) + \mathrm{Tr}_{(\mathcal{O}/\mathfrak{A})/\mathbb{F}_{\mathfrak{p}}}(\overline{\gamma}).$$

As $\alpha \in \mathfrak{A}$, $\overline{\gamma} = 0$ and so $\mathrm{Tr}_{(\mathcal{O}/\mathfrak{A})/\mathbb{F}_{\mathfrak{p}}}(\overline{\gamma}) = 0$. Therefore it is further equivalent to show

$$\mathrm{Tr}_{(\mathcal{O}/\mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})})/\mathbb{F}_{\mathfrak{p}}}(\overline{\beta}) = 0,$$

for all $\beta \in \mathcal{O}$. We will show that this occurs if and only if $e_{\mathfrak{p}}(\mathfrak{P}) \equiv 0 \pmod{\mathfrak{p}}$. So let $\beta \in \mathcal{O}$. By Lemma 5.5.1, we have an isomorphism

$$\mathcal{O}/\mathfrak{P}^{e_{\mathfrak{p}}(\mathfrak{P})} \cong \bigoplus_{0 \leq e \leq e_{\mathfrak{p}}(\mathfrak{P})-1} \mathcal{O}/\mathfrak{P}.$$

Therefore there exists $\beta_e \in \mathcal{O}$ such that $\beta \equiv \beta_e \pmod{\mathfrak{P}}$ for all e. But then $\mathrm{Tr}_{\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}}(\overline{\beta}_e) = \mathrm{Tr}_{\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}}(\overline{\beta})$ for all e, and combining with Equation (5.2) gives

$$\mathrm{Tr}_{(\mathcal{O}/\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})})/\mathbb{F}_\mathfrak{p}}(\overline{\beta}) = e_\mathfrak{p}(\mathfrak{P})\,\mathrm{Tr}_{\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}}(\overline{\beta}),$$

which we recall is an element of $\mathbb{F}_\mathfrak{p}$. As the residue class extensions are assumed to be separable, Lemma 5.2.1 implies that $\mathrm{Tr}_{\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}}(\overline{\beta})$ cannot be zero for all $\beta \in \mathcal{O}$. So it must be the case that $\mathrm{Tr}_{(\mathcal{O}/\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})})/\mathbb{F}_\mathfrak{p}}(\overline{\beta}) = 0$ for all $\beta \in \mathcal{O}$ if and only if $e_\mathfrak{p}(\mathfrak{P}) \equiv 0 \pmod{\mathfrak{p}}$. This proves (i).

(ii) As we have already shown $\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})-1} \mid \mathfrak{D}_{\mathcal{O}/\mathcal{o}}$, (ii) follows from (i). □

Note that Theorem 5.7.1 does not tell us the degree to which $\mathfrak{P}$ divides $\mathfrak{D}_{\mathcal{O}/\mathcal{o}}$ in the case $e_\mathfrak{p}(\mathfrak{P}) \equiv 0 \pmod{p}$. It only tells us that the degree is at least $e_\mathfrak{p}(\mathfrak{P})$. Also, only finitely many primes can ramify in L as a corollary of Theorem 5.7.1:

Corollary 5.7.1. *Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of a finite separable extension L/K and assume that all residue class extensions are separable. Then only finitely many primes ramify in L.*

Proof. There are only finitely many prime factors of $\mathfrak{D}_{\mathcal{O}/\mathcal{o}}$ by the prime factorization of fractional ideals. Therefore only finitely many primes can ramify in L by Theorem 5.7.1. □

Similarly, a prime $\mathfrak{p}$ of $\mathcal{o}$ ramifies in $\mathcal{O}/\mathcal{o}$ if and only if it divides the discriminant $\mathfrak{d}_{\mathcal{O}/\mathcal{o}}$ provided the residue class extensions are separable:

Theorem 5.7.2. *Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of a degree n separable extension L/K and assume that all residue class extensions are separable. Then a prime $\mathfrak{p}$ of $\mathcal{o}$ ramifies in $\mathcal{O}/\mathcal{o}$ if and only if $\mathfrak{p}$ divides $\mathfrak{d}_{\mathcal{O}/\mathcal{o}}$.*

Proof. As the residue class extensions are assumed to be separable, $\mathfrak{p}$ ramifies in $\mathcal{O}/\mathcal{o}$ if and only if $e_\mathfrak{p}(\mathfrak{P}) \geq 2$ for some prime $\mathfrak{P}$ above $\mathfrak{p}$. In view of Propositions 5.5.2 and 5.7.3, it suffices to assume $\mathcal{O}/\mathcal{o}$ is a local Dedekind extension. Therefore $\mathcal{o}$ is a discrete valuation ring, $\mathcal{O}$ is a principal ideal domain, and $\mathcal{O}/\mathcal{o}$ admits an integral basis $\alpha_1, \ldots, \alpha_n$ making $\mathcal{O}$ a free $\mathcal{o}$-module of rank n. Then, as we have see, $\mathfrak{d}_{\mathcal{O}/\mathcal{o}}$ is principal and

$$\mathfrak{d}_{\mathcal{O}/\mathcal{o}} = d_\mathcal{o}(\mathcal{O})\mathcal{o}.$$

Therefore $\mathfrak{p}$ divides $\mathfrak{d}_{\mathcal{O}/\mathcal{o}}$ if and only if $d_\mathcal{o}(\mathcal{O}) \equiv 0 \pmod{\mathfrak{p}}$. Since $\overline{\alpha_1}, \ldots, \overline{\alpha_n}$ is a basis for $\mathcal{O}/\mathfrak{p}\mathcal{O}$ as a $\mathbb{F}_\mathfrak{p}$-vector space, it follows that

$$d_\mathcal{o}(\mathcal{O}) \equiv d_{\mathbb{F}_\mathfrak{p}}(\mathcal{O}/\mathfrak{p}\mathcal{O}) \pmod{\mathfrak{p}}.$$

So $\mathfrak{p}$ divides $\mathfrak{d}_{\mathcal{O}/\mathcal{o}}$ if and only if $d_{\mathbb{F}_\mathfrak{p}}(\mathcal{O}/\mathfrak{p}\mathcal{O}) = 0$. Now suppose $\mathfrak{p}\mathcal{O}$ has prime factorization

$$\mathfrak{p}\mathcal{O} = \mathfrak{P}_1^{e_\mathfrak{p}(\mathfrak{P}_1)} \cdots \mathfrak{P}_r^{e_\mathfrak{p}(\mathfrak{P}_r)}.$$

As the $\mathfrak{P}_1^{e_\mathfrak{p}(\mathfrak{P}_1)}, \ldots, \mathfrak{P}_r^{e_\mathfrak{p}(\mathfrak{P}_r)}$ are pairwise relatively prime, the Chinese remainder theorem gives

$$\mathcal{O}/\mathfrak{p}\mathcal{O} \cong \bigoplus_{1 \leq i \leq r} \mathcal{O}/\mathfrak{P}_i^{e_\mathfrak{p}(\mathfrak{P}_i)}.$$

Then Proposition 5.2.4 further implies

$$d_{\mathbb{F}_\mathfrak{p}}(\mathcal{O}/\mathfrak{p}\mathcal{O}) = \prod_{1 \leq i \leq r} d_{\mathbb{F}_\mathfrak{p}}(\mathcal{O}/\mathfrak{P}_i^{e_\mathfrak{p}(\mathfrak{P}_i)}),$$

Therefore $d_{\mathbb{F}_\mathfrak{p}}(\mathcal{O}/\mathfrak{p}\mathcal{O}) = 0$ if and only if $d_{\mathbb{F}_\mathfrak{p}}(\mathcal{O}/\mathfrak{P}_i^{e_\mathfrak{p}(\mathfrak{P}_i)}) = 0$ for some i. We will prove that this occurs if and only if $e_\mathfrak{p}(\mathfrak{P}_i) \geq 2$ which will complete the proof because this is exactly when $\mathfrak{p}$ ramifies. So let $\mathfrak{P}$ be above $\mathfrak{p}$ and first suppose $e_\mathfrak{p}(\mathfrak{P}) \geq 2$. Then we need to show $d_{\mathbb{F}_\mathfrak{p}}(\mathcal{O}/\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})}) = 0$. By uniqueness of prime factorizations of fractional ideals, there exists $\beta_1 \in \mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})-1} - \mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})}$. Then $\beta_1^2 \in \mathfrak{P}^{2(e_\mathfrak{p}(\mathfrak{P})-1)} \subseteq \mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})}$ because $e_\mathfrak{p}(\mathfrak{P}) \geq 2$. By construction, $\overline{\beta_1}$ in $\mathcal{O}/\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})}$ is nonzero and such that $\overline{\beta_1}^2 = 0$. Since $\mathcal{O}/\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})}$ is a $\mathbb{F}_\mathfrak{p}$-vector space of dimension $f_\mathfrak{p}(\mathfrak{P})$, there exists a basis of the form $\overline{\beta_1}, \ldots, \overline{\beta_{f_\mathfrak{p}(\mathfrak{P})}}$. Now

$$\mathrm{Tr}_{(\mathcal{O}/\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})})/\mathbb{F}_\mathfrak{p}}(\overline{\beta_1\beta_j}) = 0,$$

for $1 \leq j \leq f_\mathfrak{p}(\mathfrak{P})$ since $T_{\overline{\beta_1\beta_j}}$ is nilpotent because $T^2_{\overline{\beta_1\beta_j}}$ is the zero operator as $\overline{\beta_1}^2 = 0$. But then the first row of $\mathrm{Tr}_{(\mathcal{O}/\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})})/\mathbb{F}_\mathfrak{p}}(\overline{\beta_1}, \ldots, \overline{\beta_{f_\mathfrak{p}(\mathfrak{P})}})$ is zero and hence $d_{\mathbb{F}_\mathfrak{p}}(\mathcal{O}/\mathfrak{P}^{e_\mathfrak{p}(\mathfrak{P})}) = 0$. Now suppose $e_\mathfrak{p}(\mathfrak{P}) = 1$. Then it remains to show $d_{\mathbb{F}_\mathfrak{p}}(\mathbb{F}_\mathfrak{P}) \neq 0$. Since the residue class extension $\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p}$ is separable by assumption, Proposition 5.2.5 implies $d_{\mathbb{F}_\mathfrak{p}}(\mathbb{F}_\mathfrak{P}) \neq 0$. This completes the proof. □

As a corollary of Theorem 5.7.2 we see that only finitely many primes can ramify in K:

Corollary 5.7.2. *Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of a degree n separable extension L/K and assume that all residue class extensions are separable. Then only finitely many primes ramify in K.*

Proof. There are only finitely many prime factors of $\mathfrak{d}_{\mathcal{O}/\mathcal{o}}$ by the prime factorization of fractional ideals. Therefore only finitely many primes can ramify in K by Theorem 5.7.2. □

Now consider the case of a number field K of degree n. The **complement** $\mathfrak{C}_K$ of K is the complement of $\mathcal{O}_K/\mathbb{Z}$, the **different** $\mathfrak{D}_K$ of K is the different of $\mathcal{O}_K/\mathbb{Z}$, and the **discriminant** $\mathfrak{d}_K$ of K is the discriminant of $\mathcal{O}_K/\mathbb{Z}$. Since $\mathbb{Z}$ is a principal ideal domain, the $\mathfrak{d}_K$ is related to the discriminant Δ_K by

$$\mathfrak{d}_K = \Delta_K\mathcal{O}_K.$$

As all of the residue class extensions are separable, it follows from Theorems 5.7.1 and 5.7.2 that a prime of K ramifies in $\mathcal{O}_K/\mathbb{Z}$ if and only if it divides $\mathfrak{D}_K$ and a prime of $\mathbb{Q}$ ramifies in $\mathcal{O}_K/\mathbb{Z}$ if and only if it divides $|\Delta_K|$. Moreover, finitely many primes of K and $\mathbb{Q}$ ramify by Corollaries 5.7.1 and 5.7.2.

5.8 The Ideal Norm

Let $\mathcal{O}/\mathcal{o}$ be a Dedekind extension of a degree n extension L/K. If $\mathfrak{P}$ a prime of L is above the prime $\mathfrak{p}$ of K, we define the **norm** $\mathrm{N}_{\mathcal{O}/\mathcal{o}}(\mathfrak{P})$ of $\mathfrak{P}$ by

$$\mathrm{N}_{\mathcal{O}/\mathcal{o}}(\mathfrak{P}) = \mathfrak{p}^{f_\mathfrak{p}(\mathfrak{P})}.$$

Setting $\mathrm{N}_{\mathcal{O}/\mathcal{o}}(\mathcal{O}) = \mathcal{o}$, we extend this multiplicatively to all fractional ideals of $\mathcal{O}$. This induces a homomorphism

$$\mathrm{N}_{\mathcal{O}/\mathcal{o}} : I_\mathcal{O} \to I_\mathcal{o} \qquad \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r} \mapsto \mathfrak{p}_1^{e_1 f_{\mathfrak{p}_1}(\mathfrak{P}_1)} \cdots \mathfrak{p}_r^{e_r f_{\mathfrak{p}_r}(\mathfrak{P}_r)},$$

called the **ideal norm** of $\mathcal{O}/\mathcal{o}$. It follows from the fundamental equality and multiplicatively of the ideal norm that

$$\mathrm{N}_{\mathcal{O}/\mathcal{o}}(\mathfrak{f}\mathcal{O}) = \mathfrak{f}^n,$$

for any fractional ideal $\mathfrak{f}$ of $\mathcal{o}$.

Remark 5.8.1. *The ideal norm can be though of as an almost inverse to the mapping that sends a fractional ideal $\mathfrak{f}$ of $\mathcal{O}$ to the fractional ideal $\mathfrak{f}\mathcal{O}$ of $\mathcal{O}$.*

In the case of a degree n number field K, the **ideal norm** N_K of K is the ideal norm of $\mathcal{O}_K/\mathbb{Z}$. As $\mathbb{Z}$ is a principal ideal domain, every fractional ideal is principal. Therefore, $\mathrm{N}_K(\mathfrak{f})$ is generated by an $r_\mathfrak{f} \in \mathbb{Q}^*$ for every fractional ideal $\mathfrak{f}$ of $\mathcal{O}_K$. We define the **norm** $\mathrm{N}_K(\mathfrak{f})$ of $\mathfrak{f}$ by

$$\mathrm{N}_K(\mathfrak{f}) = |r_\mathfrak{f}|.$$

Since the ideal norm is multiplicative, we obtain a homomorphism

$$\mathrm{N}_K : I_K \to \mathbb{Q}^* \qquad \mathfrak{f} \mapsto |r_\mathfrak{f}|,$$

called the **norm** of K. For an integral ideal $\mathfrak{a}$, there is a simple relationship between the norm $\mathfrak{a}$ and the quotient $\mathcal{O}_K/\mathfrak{a}$:

Proposition 5.8.1. *Let K be a number field of degree n. Then for any integral ideal $\mathfrak{a}$, we have*

$$\mathrm{N}_K(\mathfrak{a}) = |\mathcal{O}_K/\mathfrak{a}|.$$

Moreover, any $\alpha \in \mathcal{O}_K$ satisfies

$$\mathrm{N}_K(\alpha\mathcal{O}_K) = |\mathrm{N}_K(\alpha)|.$$

Proof. Since K admits an integral basis, $\mathcal{O}_K$ is a free abelian group of rank n as is any fractional ideal. In particular, $|\mathcal{O}_K/\mathfrak{a}|$ is finite. As the Chinese remainder theorem implies

$$|\mathcal{O}_K/\mathfrak{a}\mathfrak{b}| = |\mathcal{O}_K/\mathfrak{a}||\mathcal{O}_K/\mathfrak{b}|,$$

whenever $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime, it suffices to prove the claim in the case of a prime power. So let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ above p and $e \geq 1$. On the one hand, $\mathrm{N}_K(\mathfrak{p}^e) = p^{ef_p(\mathfrak{p})}$ by definition of the norm. On the other hand, Lemma 5.5.1 implies $\mathcal{O}_K/\mathfrak{p}^e$ is an $\mathbb{F}_p$-vector space of dimension $ef_p(\mathfrak{p})$ so that $|\mathcal{O}_K/\mathfrak{p}^e| = p^{ef_p(\mathfrak{p})}$. Together, we have

$$\mathrm{N}_K(\mathfrak{p}^e) = |\mathcal{O}_K/\mathfrak{p}^e|,$$

and the first statement follows. For the second statement, we just have to show that $|\mathrm{N}_K(\alpha)| = |\mathcal{O}_K/\alpha\mathcal{O}_K|$. To this end, let $\alpha_1, \ldots, \alpha_n$ be an integral basis for K. Writing

$$\alpha = \sum_{1 \leq i \leq n} a_i\alpha_i,$$

with $a_i \in \mathbb{Z}$, we see that $a_1\alpha_1, \ldots, a_n\alpha_n$ is a basis for $\alpha\mathcal{O}_K$. In particular, the base change matrix from $\alpha_1, \ldots, \alpha_n$ to this basis is a diagonal matrix with the a_i on the diagonal. Then on the one hand, we have $|\mathcal{O}_K/\alpha\mathcal{O}_K| = |a_1 \cdots a_n|$ by Proposition C.1.1 again. On the other hand, the multiplication by α map in terms of the basis $a_1\alpha_1, \ldots, a_n\alpha_n$ has matrix representation

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix},$$

and so $\mathrm{N}_K(\alpha) = a_1 \cdots a_n$. Hence

$$|\mathcal{O}_K/\alpha\mathcal{O}_K| = |\mathrm{N}_K(\alpha)|,$$

as desired. □

As a consequence of Proposition 5.8.1 and Lagrange's theorem, $\mathrm{N}_K(\mathfrak{a}) \in \mathfrak{a}$ for any integral ideal $\mathfrak{a}$ and therefore for every fractional ideal as well (recall every such fractional ideal is of the form $\frac{1}{\delta}\mathfrak{a}$ for some nonzero $\delta \in \mathbb{Z}$ and integral ideal $\mathfrak{a}$). Computing the norm of the discriminant Δ_K is also an easy matter:

Proposition 5.8.2. *Let K be a number field of degree n. Then*

$$\mathrm{N}_K(\mathfrak{D}_K) = |\Delta_K|.$$

Proof. From Equation (5.11) we have an isomorphism

$$\mathcal{O}_K/\mathfrak{D}_K \cong \mathcal{O}_K^\vee/\mathcal{O}_K.$$

Therefore $\mathrm{N}_K(\mathfrak{D}_K) = |\mathcal{O}_K^\vee/\mathcal{O}_K|$ by Proposition 5.8.1. Now let $\alpha_1, \dots, \alpha_n$ be an integral basis for $\mathcal{O}_K$. Then the dual basis $\alpha_1^\vee, \dots, \alpha_n^\vee$ is a basis for $\mathcal{O}_K^\vee$ and we have

$$\alpha_i^\vee = \sum_{1 \le j \le n} \mathrm{Tr}_K(\alpha_i \alpha_j)\alpha_j.$$

But then the base change matrix from $\alpha_1, \dots, \alpha_n$ to $\alpha_1^\vee, \dots, \alpha_n^\vee$ is $\mathrm{Tr}_K(\alpha_1, \dots, \alpha_n)$. The claim follows by Proposition C.1.1 and the definition of Δ_K. □

We can now compute the norm of a dual ideal:

Corollary 5.8.1. *Let K be a number field and $\mathfrak{f}$ be a fractional ideal. Then*

$$\mathrm{N}_K(\mathfrak{f}^\vee) = \frac{\mathrm{N}_K(\mathfrak{f}^{-1})}{|\Delta_K|}.$$

Proof. This follows immediately from $\mathfrak{f}^\vee = \mathfrak{f}^{-1}\mathfrak{D}_K^{-1}$, multiplicativity of the norm, and Proposition 5.8.2. □

Lastly, let $a_K(m)$ denote the number of integral ideals of norm m. Because the ideal norm is multiplicative so is $a_K(m)$. Moreover, we have the following result:

Proposition 5.8.3. *Let K be a number field of degree n. Then $a_K(m) \le \sigma_0(m)^n$.*

Proof. Let $\mathfrak{a}$ be an integral ideal of norm m. First suppose $m = p^k$ for some prime p and $k \ge 0$. As there are at most n primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ above p with inertia degrees $f_p(\mathfrak{p}_1), \dots, f_p(\mathfrak{p}_n)$ respectively, we have

$$\mathrm{N}_K(\mathfrak{a}) = p^{e_1 f_p(\mathfrak{p}_1)} \cdots p^{e_n f_p(\mathfrak{p}_n)},$$

for some integers $0 \le e_i \le k$ for $1 \le i \le n$. Therefore the number of possibilities is equivalent to the number of solutions

$$e_1 f_p(\mathfrak{p}_1) + \cdots + e_n f_p(\mathfrak{p}_n) = k,$$

which is at most $\sigma_0(p^k)^n = (k+1)^n$. This proves the claim in the case m is a prime power. By multiplicativity of $a_K(m)$ and the divisor function, it follows that the number of integral ideals of norm m is at most $\sigma_0(m)^n$ as desired. □

As a consequence of Proposition A.3.1, we have the slightly weaker estimate $a_K(n) \ll_\varepsilon n^\varepsilon$.

Chapter 6

Geometry of Number Fields

We can apply geometric tools to a number field K by leveraging the distinct embeddings of K into $\overline{\mathbb{Q}}$. This theory is developed under the umbrella of Minkowski space and we will use it to prove two crucial results about number fields: finiteness of the class number and Dirichlet's unit theorem.

6.1 Minkowski Space

Let K be number field of degree n and let $\sigma \in \mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$. Then either σ is real or complex and if it is complex it has a paired $\mathbb{Q}$-embedding $\overline{\sigma} \in \mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$ given by the conjugate of σ. Accordingly, let r_1 and $2r_2$ be the number of real and complex $\mathbb{Q}$-embeddings of K into $\overline{\mathbb{Q}}$ respectively. We call the pair (r_1, r_2) the **signature** of K and it satisfies the relation

$$n = r_1 + 2r_2.$$

Setting

$$K_{\mathbb{C}} = \mathbb{C}^n,$$

we see that $K_{\mathbb{C}}$ is a $\mathbb{C}$-algebra and also a complex Hilbert space with respect to $\langle \cdot, \cdot \rangle_{K_{\mathbb{C}}}$ which we take to be the standard complex inner product. We denote the associated Lebesgue measure by $d\lambda_{K_{\mathbb{C}}}$. We define the **canonical embedding** j of K to be the $\mathbb{Q}$-embedding

$$j : K \to K_{\mathbb{C}} \qquad \kappa \mapsto (\sigma(\kappa))_{\sigma},$$

where σ runs over $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$. Consider the conjugation map

$$F : \mathbb{C} \to \mathbb{C} \qquad z \mapsto \overline{z}.$$

This induces an automorphism

$$F : K_{\mathbb{C}} \to K_{\mathbb{C}} \qquad (z_{\sigma})_{\sigma} \mapsto (\overline{z_{\overline{\sigma}}})_{\sigma},$$

that is clearly an involution. The inner product $\langle \cdot, \cdot \rangle_{K_{\mathbb{C}}}$ is also F-equivariant since any $\mathbf{z}, \mathbf{w} \in K_{\mathbb{C}}$ satisfy

$$\langle F(\mathbf{z}), F(\mathbf{z}) \rangle_{K_{\mathbb{C}}} = \langle (\overline{z_{\overline{\sigma}}})_{\sigma}, (\overline{w_{\overline{\sigma}}})_{\sigma} \rangle_{K_{\mathbb{C}}} = \sum_{\sigma} \overline{z_{\overline{\sigma}} w_{\overline{\sigma}}} = \overline{\sum_{\sigma} z_{\sigma} w_{\sigma}} = F(\langle \mathbf{z}, \mathbf{w} \rangle_{K_{\mathbb{C}}}),$$

where in the third equality we have used the fact that the complex $\mathbb{Q}$-embeddings of K into $\overline{\mathbb{Q}}$ come in conjugate pairs. On $K_{\mathbb{C}}$ we also have linear maps

$$\mathrm{Tr}_{K_{\mathbb{C}}}((z_{\sigma})_{\sigma}) = \sum_{\sigma} z_{\sigma} \quad \text{and} \quad \mathrm{N}_{K_{\mathbb{C}}}((z_{\sigma})_{\sigma}) = \prod_{\sigma} z_{\sigma},$$

and clearly induce homomorphisms

$$\mathrm{Tr}_{K_\mathbb{C}} : K_\mathbb{C} \to \mathbb{C} \quad \text{and} \quad \mathrm{N}_{K_\mathbb{C}} : K_\mathbb{C}^* \to \mathbb{C}^*.$$

The composition of j with $\mathrm{Tr}_{K_\mathbb{C}}$ and $\mathrm{N}_{K_\mathbb{C}}$ are Tr_K and N_K respectively since

$$\mathrm{Tr}_{K_\mathbb{C}}(j(\kappa)) = \sum_\sigma \sigma(\kappa) = \mathrm{Tr}_K(\kappa) \quad \text{and} \quad \mathrm{N}_{K_\mathbb{C}}(j(\kappa)) = \prod_\sigma \sigma(\kappa) = \mathrm{N}_K(\kappa),$$

where the last equality in each chain follow by Proposition 5.2.1. We now define the **Minkowski space** $K_\mathbb{R}$ of K by

$$K_\mathbb{R} = \{(z_\sigma)_\sigma \in K_\mathbb{C} : F((z_\sigma)_\sigma) = (z_\sigma)_\sigma\}.$$

In other words, $K_\mathbb{R}$ consists of all of the F-invariant points of $K_\mathbb{C}$. That is, $\mathbf{z} \in K_\mathbb{R}$ if and only if $F(\mathbf{z}) = \mathbf{z}$ or equivalently $z_{\overline{\sigma}} = \overline{z_\sigma}$ for all σ. In particular, $j(K) \subset K_\mathbb{R}$ because $\overline{\sigma}(\kappa) = \overline{\sigma(\kappa)}$ by definition of $\overline{\sigma}$. We denote the restriction of the inner product $\langle \cdot, \cdot \rangle_{K_\mathbb{C}}$ on $K_\mathbb{C}$ to $K_\mathbb{R}$ by $\langle \cdot, \cdot \rangle_{K_\mathbb{R}}$. Note that $K_\mathbb{R}$ is an $\mathbb{R}$-algebra. Moreover, the inner product $\langle \cdot, \cdot \rangle_{K_\mathbb{R}}$ turns $K_\mathbb{R}$ into a real Hilbert space. Indeed, for any $\mathbf{z}, \mathbf{w} \in K_\mathbb{R}$ the conjugate symmetry and F-equivariance of the inner product together give

$$\overline{\langle \mathbf{z}, \mathbf{w} \rangle} = F(\langle \mathbf{w}, \mathbf{z} \rangle) = \langle F(\mathbf{z}), F(\mathbf{w}) \rangle = \langle \mathbf{z}, \mathbf{w} \rangle,$$

so that $\langle \mathbf{z}, \mathbf{w} \rangle \in \mathbb{R}$ is real. Accordingly, we call $\langle \cdot, \cdot \rangle_{K_\mathbb{R}}$ the **Minkowski inner product**. We denote the restriction of the Lebesgue measure $d\lambda_\mathbb{C}$ to $K_\mathbb{R}$ by $d\lambda_{K_\mathbb{R}}$ which is also the Lebesgue measure associated to $\langle \cdot, \cdot \rangle_{K_\mathbb{R}}$. We call $d\lambda_{K_\mathbb{R}}$ the **Minkowski measure**. Lastly, we denote the restrictions of $\mathrm{Tr}_{K_\mathbb{C}}$ and $\mathrm{N}_{K_\mathbb{C}}$ to $K_\mathbb{R}$ by $\mathrm{Tr}_{K_\mathbb{R}}$ and $\mathrm{N}_{K_\mathbb{R}}$ respectively and call these maps the **Minkowski trace** and **Minkowski norm** respectively. We also have homomorphisms

$$\mathrm{Tr}_{K_\mathbb{R}} : K_\mathbb{R} \to \mathbb{R} \quad \text{and} \quad \mathrm{N}_{K_\mathbb{R}} : K_\mathbb{R}^* \to \mathbb{R}^*,$$

$\mathbf{z} \in K_\mathbb{R}$ if and only if $z_{\overline{\sigma}} = \overline{z_\sigma}$ for all σ. As $j(K) \subset K_\mathbb{R}$, the compositions of j with the Minkowski trace and Minkowski norm are the field trace and field norm respectively. We can now give a more explicit description of $K_\mathbb{R}$ and to do this we setup some notation. Let ρ run over the real $\mathbb{Q}$-embeddings of $\mathrm{Hom}_\mathbb{Q}(K, \overline{\mathbb{Q}})$ and let τ run over a complete set of representatives of the pairs of complex $\mathbb{Q}$-embeddings of $\mathrm{Hom}_\mathbb{Q}(K, \overline{\mathbb{Q}})$. For any $\sigma \in \mathrm{Hom}_\mathbb{Q}(K, \overline{\mathbb{Q}})$, let $N_\sigma = 1, 2$ according to if σ is real or complex respectively. So $N_\rho = 1$ and $N_\tau = 2$. As $K_\mathbb{R}$ consists of exactly the F-invariant points of $K_\mathbb{C}$, we have

$$K_\mathbb{R} = \{(z_\sigma)_\sigma \in K_\mathbb{C} : z_\rho \in \mathbb{R} \text{ and } z_{\overline{\tau}} = \overline{z_\tau} \text{ for all } \rho \text{ and } \tau\}.$$

We now describe an explicit isomorphism from the Minkowski space and $\mathbb{R}^n$:

Proposition 6.1.1. *Let K be a number field of degree n and signature (r_1, r_2). Also let σ run over $\mathrm{Hom}_\mathbb{Q}(K, \overline{\mathbb{Q}})$, ρ run over all such real $\mathbb{Q}$-embeddings, and τ run over a complete set of representatives of all such pairs of complex $\mathbb{Q}$-embeddings. Then there is an isomorphism*

$$K_\mathbb{R} \to \mathbb{R}^n \qquad z_\sigma \mapsto x_\sigma = \begin{cases} z_\sigma & \text{if } \sigma = \rho, \\ \mathrm{Re}(z_\sigma) & \text{if } \sigma = \tau, \\ \mathrm{Im}(z_\sigma) & \text{if } \sigma = \overline{\tau}. \end{cases}$$

In particular, $K_\mathbb{R}$ is a n-dimensional real vector space. Moreover, the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ induced by the Minkowski inner product is given by

$$\langle \mathbf{x}, \mathbf{x}' \rangle = \sum_\sigma N_\sigma x_\sigma x'_\sigma,$$

for any $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$.

Proof. This map is an isomorphism since it is invertible and linear in each component. Since there are n elements of $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$, we see that $K_{\mathbb{R}}$ is an n-dimensional real vector space. We will now prove the statement about the inner product. Let $(z_\sigma)_\sigma$ and $(z'_\sigma)_\sigma$ be elements of $K_{\mathbb{R}}$ and let $(x_\sigma)_\sigma$ and $(x'_\sigma)_\sigma$ be the corresponding elements in $\mathbb{R}^n$. If $\sigma = \rho$ then

$$x_\rho = z_\rho \quad \text{and} \quad x'_\rho = z'_\rho,$$

and thus

$$x_\rho x'_\rho = z_\rho \overline{z'_\rho}.$$

If $\sigma = \tau$ then

$$x_\tau = \mathrm{Re}(z_\tau), \quad x_{\overline{\tau}} = \mathrm{Im}(z_\tau), \quad x'_\tau = \mathrm{Re}(z'_\tau), \text{and} \quad x'_{\overline{\tau}} = \mathrm{Im}(z'_\tau),$$

and hence

$$2(x_\tau x'_\tau + x_{\overline{\tau}} x'_{\overline{\tau}}) = 2(\mathrm{Re}(z_\tau)\mathrm{Re}(z'_\tau) + \mathrm{Im}(z_\tau)\mathrm{Im}(z'_\tau)) = 2\mathrm{Re}(z_\tau \overline{z'_\tau}) = z_\tau \overline{z'_\tau} + z_{\overline{\tau}} \overline{z'_{\overline{\tau}}}.$$

This proves the claim about the inner product. □

Define the **Minkowski embedding** σ_K of K by

$$\sigma_K : K \to \mathbb{R}^n \qquad \kappa \mapsto (\rho_1(\kappa), \ldots, \rho_{r_1}(k), \mathrm{Re}(\tau_1(\kappa)), \mathrm{Im}(\tau_1(\kappa)), \ldots, \mathrm{Re}(\tau_{r_2}(\kappa)), \mathrm{Im}(\tau_{r_2}(\kappa))),$$

where $\rho_1, \ldots, \rho_{r_1}$ are the real $\mathbb{Q}$-embeddings of $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$and $\tau_1, \ldots, \tau_{r_2}$ are representatives of pairs of complex $\mathbb{Q}$-embeddings of $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$. The Minkowski embedding σ_K is then a $\mathbb{Q}$-embedding of K into $\mathbb{R}^n$ since it is the composition of the canonical embedding j (whose image under K is in $K_{\mathbb{R}}$ as we have noted) and the isomorphism established by Proposition 6.1.1. It is also independent of the choice of representatives $\tau_1, \ldots, \tau_{r_2}$ since the complex $\mathbb{Q}$-embeddings of $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$ occur in conjugate pairs. As j is a $\mathbb{Q}$-embedding and any fractional ideal $\mathfrak{f}$ is a complete lattice, $j(\mathfrak{f})$ is a complete lattice in $K_{\mathbb{R}}$. We now determine the covolume of $j(\mathfrak{f})$:

Proposition 6.1.2. *Let K be a number field with signature (r_1, r_2). Then $V_{j(\mathfrak{f})}$ is the absolute value of the determinant of any embedding matrix for $\mathfrak{f}$. In particular,*

$$V_{j(\mathfrak{f})} = \mathrm{N}_K(\mathfrak{f})\sqrt{|\Delta_K|},$$

and

$$V_{j(\mathcal{O}_K)} = \sqrt{|\Delta_K|}.$$

Proof. The last statement follows from the first two by taking $\mathfrak{f} = \mathcal{O}_K$ so it suffices to prove the first two statements. Let $\kappa_1, \ldots, \kappa_n$ be a basis for $\mathfrak{f}$ and let $\sigma_1, \ldots, \sigma_n$ be the elements of $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$. Then the associated generator matrix P for $j(\mathfrak{f})$ is given by

$$P = \begin{pmatrix} \sigma_1(\kappa_1) & \cdots & \sigma_1(\kappa_n) \\ \vdots & & \vdots \\ \sigma_n(\kappa_1) & \cdots & \sigma_n(\kappa_n) \end{pmatrix} = M(\kappa_1, \ldots, \kappa_n),$$

which is an embedding matrix for $\mathfrak{f}$. Hence

$$V_{j(\mathfrak{f})} = |\det(M(\kappa_1, \ldots, \kappa_n))|,$$

proving the first statement. We will be done if we can show

$$|\det(M(\kappa_1,\dots,\kappa_n))| = \mathrm{N}_K(\mathfrak{f})|\det(M(\alpha_1,\dots,\alpha_n))|,$$

for any integral basis $\alpha_1,\dots,\alpha_n$ since $|\det(M(\alpha_1,\dots,\alpha_n))| = \sqrt{|\Delta_K|}$ (as we have seen). As $\mathfrak{f}$ is a fractional ideal, there exists a nonzero $\delta \in \mathcal{O}_K$ and an integral ideal $\mathfrak{a}$ such that

$$\mathfrak{f} = \frac{1}{\delta}\mathfrak{a}.$$

Then $\delta\kappa_1,\dots,\delta\kappa_n$ is a basis for $\mathfrak{a}$. By Propositions 5.8.1 and C.1.1, we have

$$|\det(M(\delta\kappa_1,\dots,\delta\kappa_n))| = \mathrm{N}_K(\mathfrak{a})|\det(M(\alpha_1,\dots,\alpha_n))|,$$

since they together show that $\mathrm{N}_K(\mathfrak{a})$ is the absolute value of the determinant of the base change matrix from $\kappa_1,\dots,\kappa_n$ to $\delta\kappa_1,\dots,\delta\kappa_n$. Similarly,

$$|\det(M(\delta\kappa_1,\dots,\delta\kappa_n))| = |\mathrm{N}_K(\delta)||\det(M(\kappa_1,\dots,\kappa_n))|,$$

since Propositions 5.8.1 and C.1.1 together show that $|\mathrm{N}_K(\delta)|$ is the absolute value of the determinant of the base change matrix from $\kappa_1,\dots,\kappa_n$ to $\delta\kappa_1,\dots,\delta\kappa_n$. As $\mathrm{N}_K(\mathfrak{f}) = \frac{\mathrm{N}_K(\mathfrak{a})}{|\mathrm{N}_K(\delta)|}$ by multiplicativity of the norm and Proposition 5.8.1, these two identities for $|\det(M(\delta\kappa_1,\dots,\delta\kappa_n))|$ together imply the claim. □

Now as σ_K is also a $\mathbb{Q}$-embedding, $\sigma_K(\mathfrak{f})$ is a complete lattice in $\mathbb{R}^n$. As a corollary of Proposition 6.1.2, we can determine the covolume of $\sigma_K(\mathfrak{f})$:

Corollary 6.1.1. *Let K be a number field with signature (r_1, r_2). Then*

$$V_{\sigma_K(\mathfrak{f})} = \mathrm{N}_K(\mathfrak{f})\frac{\sqrt{|\Delta_K|}}{2^{r_2}}.$$

In particular,

$$V_{\sigma_K(\mathcal{O}_K)} = \frac{\sqrt{|\Delta_K|}}{2^{r_2}}.$$

Proof. The second statement follows from the first by taking $\mathfrak{f} = \mathcal{O}_K$ so it suffices to prove the first statement. Let $\kappa_1,\dots,\kappa_n$ be a basis for $\mathfrak{f}$, $\rho_1,\dots,\rho_{r_1}$ be the real $\mathbb{Q}$-embeddings of $\mathrm{Hom}_{\mathbb{Q}}(K,\overline{\mathbb{Q}})$ and $\tau_1,\dots,\tau_{r_2}$ be a complete set of representatives of pairs of complex $\mathbb{Q}$-embeddings of $\mathrm{Hom}_{\mathbb{Q}}(K,\overline{\mathbb{Q}})$. Then the associated generator matrix P for $\sigma_K(\mathfrak{f})$ is

$$P = \begin{pmatrix} \rho_1(\kappa_1) & \cdots & \rho_{r_1}(\kappa_1) & \mathrm{Re}(\tau_1(\kappa_1)) & \mathrm{Im}(\tau_1(\kappa_1)) & \cdots & \mathrm{Re}(\tau_{r_2}(\kappa_1)) & \mathrm{Im}(\tau_{r_2}(\kappa_1)) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_1(\kappa_n) & \cdots & \rho_{r_1}(\kappa_n) & \mathrm{Re}(\tau_1(\kappa_n)) & \mathrm{Im}(\tau_1(\kappa_n)) & \cdots & \mathrm{Re}(\tau_{r_2}(\kappa_n)) & \mathrm{Im}(\tau_{r_2}(\kappa_n)) \end{pmatrix}^t.$$

By Proposition 6.1.2 we are done if the absolute value of the determinant of this matrix is 2^{-r_2} times the determinate of an embedding matrix for $\mathfrak{f}$. To show this, first add an i multiple of the imaginary columns to their corresponding real columns and then apply the identity $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$ to the imaginary columns to obtain

$$P' = \begin{pmatrix} \rho_1(\kappa_1) & \cdots & \rho_{r_1}(\kappa_1) & \tau_1(\kappa_1) & \frac{\tau_1(\kappa_1)-\overline{\tau_1}(\kappa_1)}{2i} & \cdots & \tau_{r_2}(\kappa_1) & \frac{\tau_{r_2}(\kappa_1)-\overline{\tau_{r_2}}(\kappa_1)}{2i} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_1(\kappa_n) & \cdots & \rho_{r_1}(\kappa_n) & \tau_1(\kappa_n) & \frac{\tau_1(\kappa_n)-\overline{\tau_1}(\kappa_n)}{2i} & \cdots & \tau_{r_2}(\kappa_n) & \frac{\tau_{r_2}(\kappa_n)-\overline{\tau_{r_2}}(\kappa_n)}{2i} \end{pmatrix}^t.$$

Since P' differs from P by column addition, their determinants are the same. Multiplying the imaginary columns of P' by $-2i$ and then adding the corresponding columns to annihilate the negative terms results in

$$P'' = \begin{pmatrix} \rho_1(\kappa_1) & \cdots & \rho_{r_1}(\kappa_1) & \tau_1(\kappa_1) & \overline{\tau_1}(\kappa_1) & \cdots & \tau_{r_2}(\kappa_1) & \overline{\tau_{r_2}}(\kappa_1) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_1(\kappa_n) & \cdots & \rho_{r_1}(\kappa_n) & \tau_1(\kappa_n) & \overline{\tau_1}(\kappa_n) & \cdots & \tau_{r_2}(\kappa_n) & \overline{\tau_{r_2}}(\kappa_n) \end{pmatrix}^t.$$

As P'' differs from P' by column addition and column scaling of which there were r_2 many of factor $-2i$, the determinant of P'' is $(-2i)^{-r_2}$ that of P'. Altogether,

$$V_{\mathfrak{f}} = |\det(P)| = |\det(P')| = |(-2i)^{-r_2}\det(P'')| = 2^{-r_2}|\det(P'')|.$$

Since the complex $\mathbb{Q}$-embeddings of $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$ occur in conjugate pairs and $\kappa_1, \ldots, \kappa_n$ is a basis for K, we see that $P'' = M(\kappa_1, \ldots, \kappa_n)$ is an embedding matrix for $\mathfrak{f}$. □

6.2 Finiteness of the Class Number

Recall that the class number h_K is a measure of how much the ring of integers $\mathcal{O}_K$ fails to be a principal ideal domain. From Remark 5.3.2, the class number of an arbitrary Dedekind domain need not be finite. However, we will show that the class number h_K for a number field K is always finite and thus $\mathcal{O}_K$ only has finite failure from being a principal ideal domain:

Theorem 6.2.1. *Let K be a number field of degree n and signature (r_1, r_2). Also, let $X \subseteq \mathbb{R}^n$ be a compact convex symmetric set and set $M = \max_{\mathbf{x}\in X}\left(\prod_{1\le i\le n}|x_i|\right)$. Then every ideal class contains an integral ideal $\mathfrak{a}$ satisfying*

$$\mathrm{N}_K(\mathfrak{a}) \le \frac{2^{r_1+r_2}M}{\mathrm{Vol}(X)}\sqrt{|\Delta_K|}.$$

Moreover, the ideal class group $\mathrm{Cl}(K)$ is finite so that the class number h_K is too.

Proof. Let $\mathfrak{f}$ be a fractional ideal, and set

$$\lambda^n = 2^n \frac{V_{\sigma_K(\mathfrak{f}^{-1})}}{\mathrm{Vol}(X)},$$

for any $n \ge 1$. Then by construction,

$$\mathrm{Vol}(\lambda X) = \lambda^n \mathrm{Vol}(X) = 2^n V_{\sigma_K(\mathfrak{f}^{-1})}.$$

By Minkowski's lattice point theorem, there exists a nonzero $\alpha \in \mathfrak{f}^{-1}$ such that $\sigma_K(\alpha) \in \sigma_K(\mathfrak{f}^{-1})$ and $\sigma_K(\alpha) \in \lambda X$. Since $\alpha \in \mathfrak{f}^{-1}$, $\alpha\mathfrak{f} \subseteq \mathcal{O}_K$ so that $\alpha\mathfrak{f}$ is an integral ideal in the same ideal class as $\mathfrak{f}$. Now let σ run over $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$. Since the norm is multiplicative, we have

$$\mathrm{N}_K(\alpha\mathfrak{f}) = |\mathrm{N}_K(\alpha)|\,\mathrm{N}_K(\mathfrak{f}) = \left|\prod_{\sigma}\sigma(\alpha)\right|\mathrm{N}_K(\mathfrak{f}) \le \lambda^n M\,\mathrm{N}_K(\mathfrak{f}),$$

where in the first equality we have applied multiplicativity of the norm and Proposition 5.8.1, in the second we have used Proposition 5.2.1, and the inequality follows since $\sigma_K(\alpha) \in \lambda X$. This inequality, our choice of λ^n, and Corollary 6.1.1 together give

$$\mathrm{N}_K(\alpha\mathfrak{f}) \le \lambda^n M\,\mathrm{N}_K(\mathfrak{f}) = 2^n M\,\mathrm{N}_K(\mathfrak{f})\frac{V_{\sigma_K(\mathfrak{f}^{-1})}}{\mathrm{Vol}(X)} = 2^n M\frac{\sqrt{|\Delta_K|}}{2^{r_2}\mathrm{Vol}(X)} = \frac{2^{r_1+r_2}M}{\mathrm{Vol}(X)}\sqrt{|\Delta_K|},$$

which proves the first statement since the fractional ideal $\mathfrak{f}$ was arbitrary. We now prove that the class group is finite. By what we have just proved, we can find a complete set of representatives of $\mathrm{Cl}(K)$ consisting of integral ideals of bounded norm. Since the norm is multiplicative, the prime factors of these representatives have bounded norm as well. As we have seen, the norm of a prime integral ideal is exactly the prime p below it. Thus the norms of these prime factors are bounded primes p. As there are finitely many prime integral ideals above any prime p (because $p\mathcal{O}_K$ factors into a product of prime integral ideals and these are exactly the prime integral ideals above p), it follows that these representatives have finitely many prime factors. Altogether this means that there are finitely many representatives. Hence $\mathrm{Cl}(K)$ is finite and so the class number h_K is too. □

We would like to obtain an explicit bound in Theorem 6.2.1 by making a choice for the set X. To obtain a bound that is not too large, we need to ensure that the volume of X is large while the constant M is small. The following lemma dictates our choice of X and computes its volume:

Lemma 6.2.1. *Suppose n is a positive integer and write $n = r_1 + 2r_2$ for some nonnegative integers r_1 and r_2. Let $X \subset \mathbb{R}^n$ to be the compact convex symmetric set given by*

$$X = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{1 \le i \le r_1} |x_i| + 2 \sum_{\substack{1 \le j \le r_2 \\ j \equiv 1 \pmod 2}} \sqrt{x_{r_1+j}^2 + x_{r_1+j+1}^2} \le n \right\}.$$

Then

$$\mathrm{Vol}(X) = \frac{n^n}{n!} 2^{r_1} \left(\frac{\pi}{2}\right)^{r_2}.$$

Proof. Making the change of variables $x_{r_1+j} \mapsto u_j \sin(\theta_j)$ and $x_{r_1+j+1} \mapsto u_j \cos(\theta_j)$ for all j gives

$$\mathrm{Vol}(X) = \int_{X'} u_1 \cdots u_{r_2} \, dx_1 \cdots dx_{r_1} \, du_1 \theta_1 \cdots du_{r_2} \theta_{r_2},$$

where

$$X' = \left\{ (x_1, \ldots, x_{r_1}, u_1, \theta_1, \ldots, u_{r_2}, \theta_{r_2}) : \sum_{1 \le i \le r_1} |x_i| + 2 \sum_{1 \le j \le r_2} u_j \le n \right\}.$$

Since the integrand is independent of the θ_j, we have

$$\mathrm{Vol}(X) = (2\pi)^{r_2} \int_{X'} u_1 \cdots u_{r_2} \, dx_1 \cdots dx_{r_1} \, du_1 \cdots du_{r_2}.$$

Making the change of variables $u_j \mapsto \frac{u_j}{2}$ for all j and using the fact that the integrand is symmetric in the x_i for all i gives

$$\mathrm{Vol}(X) = 2^{r_1} \left(\frac{\pi}{2}\right)^{r_2} \int_{X''} u_1 \cdots u_{r_2} \, dx_1 \cdots dx_{r_1} \, du_1 \cdots du_{r_2}, \tag{6.1}$$

where

$$X'' = \left\{ (x_1, \ldots, x_{r_1}, u_1, \ldots, u_{r_2}) : \sum_{1 \le i \le r_1} x_i + \sum_{1 \le j \le r_2} u_j \le n \right\}.$$

To compute the remaining integral, for nonnegative integers ℓ and k and $t \ge 0$, we let

$$X''_{\ell,k}(t) = \left\{ (x_1, \ldots, x_\ell, u_1, \ldots, u_k) : \sum_{1 \le i \le \ell} x_i + \sum_{1 \le j \le k} u_j \le t \right\},$$

and set

$$I_{\ell,k}(t) = \int_{X''_{\ell,k}(t)} u_1 \cdots u_\ell \, dx_1 \cdots dx_n \, du_1 \cdots du_k.$$

Then we have to compute $I_{r_1,r_2}(n)$. To this end, the change of variables $x_i \mapsto tx_i$ and $u_j \mapsto tu_j$ for all i and j gives

$$I_{\ell,k}(t) = t^{\ell+2k} I_{\ell,k}(1). \tag{6.2}$$

Now note that the condition

$$\sum_{1\le i\le \ell} x_i + \sum_{1\le j\le k} u_j \le t,$$

is equivalent to

$$\sum_{1\le i\le \ell-1} x_i + \sum_{1\le j\le k} u_j \le t - x_i.$$

This fact together with the Fubini–Tonelli theorem and Equation (6.2) gives

$$I_{\ell,k}(1) = \int_0^1 I_{\ell-1,k}(1-x_\ell)\, dx_\ell = \int_0^1 (1-x_\ell)^{\ell-1+2k} I_{\ell-1,k}(1)\, dx_\ell = \frac{1}{\ell+2k} I_{\ell-1,k}(1).$$

Repeating this procedure $\ell - 1$ times results in

$$I_{\ell,k}(1) = \frac{1}{(\ell+2k)\cdots(2k+1)} I_{0,k}(1). \tag{6.3}$$

Similarly, the condition

$$\sum_{1\le j\le k} u_j \le t,$$

is equivalent to

$$\sum_{1\le j\le k-1} u_j \le t - u_k.$$

This fact together with the Fubini–Tonelli theorem, Equation (6.2), and Proposition 1.7.4, gives

$$I_{0,k}(1) = \int_0^1 u_k I_{0,k-1}(1-u_k)\, du_k = \int_0^1 u_k (1-u_k)^{2k-2} I_{0,k-1}(1)\, du_k = B(1,2k-1) I_{0,k-1}(1) = \frac{1}{2k} I_{0,k-1}(1).$$

Repeating this procedure $k-1$ times results in

$$I_{0,k}(1) = \frac{1}{k!}, \tag{6.4}$$

since $I_{0,0}(1) = 1$. Combining Equations (6.2) to (6.4) we find that

$$I_{\ell,k}(t) = t^{\ell+2k} \frac{1}{(\ell+2k)!}.$$

In particular, $I_{r_1,r_2}(n) = \frac{n^n}{n!}$ and from Equation (6.1) we obtain

$$\mathrm{Vol}(X) = \frac{n^n}{n!} 2^{r_1} \left(\frac{\pi}{2}\right)^{r_2},$$

as desired. □

Observe that the set X in Lemma 6.2.1 just consists of those points in $\mathbb{R}^n$ whose induced norm corresponding to the induced Minkowski inner product is at most n (recall Proposition 6.1.1). We can now obtain an explicit bound in Theorem 6.2.1 known as the **Minkowski bound**:

Theorem (Minkowski bound). *Let K be a number field of degree n and signature (r_1, r_2). Then every ideal class contains an integral ideal $\mathfrak{a}$ satisfying*

$$\mathrm{N}_K(\mathfrak{a}) \leq \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} \sqrt{|\Delta_K|}.$$

Proof. Let X be given by

$$X = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{1 \leq i \leq r_1} |x_i| + 2 \sum_{\substack{1 \leq j \leq r_2 \\ j \equiv 1 \pmod 2}} \sqrt{x_{r_1+j}^2 + x_{r_1+j+1}^2} \leq n \right\}.$$

Then Theorem 6.2.1 and Lemma 6.2.1 together give

$$\mathrm{N}_K(\mathfrak{a}) \leq M \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} \sqrt{|\Delta_K|},$$

where $M = \max_{\mathbf{x} \in X} \left(\prod_{1 \leq \ell \leq n} |x_\ell| \right)$. But for all $\mathbf{x} \in X$, the arithmetic-geometric mean inequality gives

$$\left(\prod_{1 \leq \ell \leq n} |x_\ell| \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{1 \leq \ell \leq n} |x_\ell| \leq 1,$$

where the second inequality holds by the definition of X. Hence $M \leq 1$ and this completes the proof. □

As a corollary we can obtain a lower bound for the discriminant of a number field and show that every number field K, other than $\mathbb{Q}$, has at least one ramified prime in $\mathcal{O}_K/\mathbb{Z}$:

Corollary 6.2.1. *Let K be a number field of degree n. Then*

$$|\Delta_K| \geq \left(\frac{\pi}{4}\right)^{\frac{n}{2}} \frac{n^n}{n!}.$$

In particular, there is at least one ramified prime in $\mathcal{O}_K/\mathbb{Z}$ provided the degree of K is at least 2.

Proof. Since the norm of every integral ideal is at least 1, $\pi < 4$, and r_2 is at most n, the desired inequality follows immediately from Minkowski's bound. Now suppose $n \geq 2$. In the case $n = 2$, the lower bound is larger than 1 so that $|\Delta_K|$ is at least 2 for every quadratic number field. As $n^n \geq n!$ for all $n \geq 1$ (which easily follows by induction), $\left(\frac{\pi}{4}\right)^{\frac{n}{2}} \frac{n^n}{n!}$ is an increasing function in n. Therefore $|\Delta_K| \geq 2$ for all $n \geq 2$ so that $|\Delta_K|$ has a prime divisor. This means at least one prime of K ramifies in $\mathcal{O}_K/\mathbb{Z}$. □

Generally speaking, the class number h_K is one of the most difficult pieces of arithmetic data about K to compute. For example, it is still unknown if there are infinitely many number fields of class number 1, that is, number fields such that their ring of integers are principal ideal domains.

6.3 Dirichlet's Unit Theorem

Let K be a number field of signature (r_1, r_2). We define the **rank** r_K of K to be

$$r_K = r_1 + r_2 - 1.$$

This will be an important in the following. Another very important piece of arithmetic data about K is the structure of the unit group $\mathcal{O}_K^*$. Note that $\mathcal{O}_K^*$ is closed under conjugation because $\mathrm{N}_K(\alpha) = \mathrm{N}_K(\overline{\alpha})$ as a consequence of Proposition 5.2.1. Moreover, since the norm of a unit is ± 1, we can express the unit group as

$$\mathcal{O}_K^* = \{\varepsilon \in \mathcal{O}_K : \mathrm{N}_K(\varepsilon) = \pm 1\}.$$

Our main goal will be to describe the group structure of $\mathcal{O}_K^*$ completely. Let Ω denote the group of all roots of unity. We will set $\mu(K) = \mathcal{O}_K^* \cap \Omega$ so that $\mu(K)$ is the subgroup of $\mathcal{O}_K^*$ consisting of all of the roots of unity in K. Clearly $\{\pm 1\} \subseteq \mu(K)$. In fact, $\mu(K)$ is finite since any root of unity in K is a root of $x^n - 1$ and thus an n-th root of unity. We set

$$w_K = |\mu(K)|.$$

Our goal will be to show that $\mathcal{O}_K^*$ is a direct product of $\mu(K)$ and a free abelian group of rank r_K. Determining that the rank of the free group is exactly r_K will be the most difficult part of the proof. We will require a map on K^* that transitions between the field trace and the field norm. Let $\rho_1, \ldots, \rho_{r_1}$ be the real $\mathbb{Q}$-embeddings of $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$ and $\tau_1, \ldots, \tau_{r_2}$ be a complete set of representatives of pairs of complex $\mathbb{Q}$-embeddings of $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$. First consider the map

$$\ell : K_{\mathbb{R}}^* \to \mathbb{R}^{r_K+1} \qquad (z_\sigma)_\sigma \mapsto (\log|z_{\rho_1}|, \ldots, \log|z_{\rho_{r_1}}|, 2\log|z_{\tau_1}|, \ldots, 2\log|z_{\tau_{r_2}}|).$$

This map is independent of the choice of representatives $\tau_1, \ldots, \tau_{r_2}$ because $z_{\overline{\tau}} = \overline{z_\tau}$ for $\mathbf{z} \in K_{\mathbb{R}}$. We define the **logarithmic embedding** $\log_K$ of K by

$$\log_K : K^* \to \mathbb{R}^{r_K+1} \qquad \kappa \mapsto (\log|\rho_1(\kappa)|, \ldots, \log|\rho_{r_1}(\kappa)|, 2\log|\tau_1(\kappa)|, \ldots, 2\log|\tau_{r_2}(\kappa)|).$$

Then $\log_K$ is just the restriction of j to K^* composed with ℓ. Since ℓ is a homomorphism and j is a $\mathbb{Q}$-embedding, $\log_K$ is a homomorphism. We distinguish the subsets

$$S = \{\mathbf{x} \in \mathbb{R}_+^{r_K+1} : \mathrm{N}_{\mathbb{R}^{r_K+1}/\mathbb{R}}(\mathbf{x}) = 1\} \quad \text{and} \quad H = \{\mathbf{x} \in \mathbb{R}^{r_K+1} : \mathrm{Tr}_{\mathbb{R}^{r_K+1}/\mathbb{R}}(\mathbf{x}) = 0\},$$

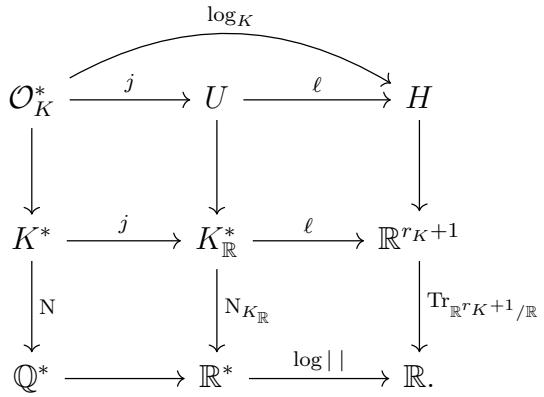
called the **norm-one hypersurface** and **trace-zero hyperplane** of $\mathbb{R}^{r_K+1}$ respectively. Note that H is an r_K-dimensional subspace of $\mathbb{R}^{r_K+1}$. We will also make use of the subset

$$U = \{\mathbf{z} \in K_{\mathbb{R}} : \mathrm{N}_{K_{\mathbb{R}}}(\mathbf{z}) = \pm 1\}.$$

Let λ denote the restriction of $\log_K$ to $\mathcal{O}_K^*$ and set

$$\Lambda = \log_K(\mathcal{O}_K^*),$$

so that Λ is the image of λ. We call Λ the **unit lattice** of K. It is not immediately obvious that Λ is a lattice, but we will show this and more. Observe that ℓ takes U into H since $\mathrm{Tr}_{\mathbb{R}^{r_K+1}/\mathbb{R}}(\ell(\mathbf{z})) = \log|\mathrm{N}_{K_{\mathbb{R}}}(\mathbf{z})| = 1$. In particular, $\Lambda \subset H$ since $j(\mathcal{O}_K^*) \subset U$. All of this data can be collected into the following commutative diagram:



We first show that the logarithmic embedding fits into an exact sequence:

Proposition 6.3.1. *Let K be a number field. Then the sequence*

$$1 \longrightarrow \mu(K) \longrightarrow \mathcal{O}_K^* \xrightarrow{\lambda} \Lambda \longrightarrow 0,$$

is exact.

Proof. Exactness of this sequence means that $\mu(K)$ is the kernel of λ. If $\omega \in \mu(K)$ then $|\omega| = 1$ and it follows that $\lambda(\omega) = 0$. Therefore $\ker\lambda$ contains $\mu(K)$. To see that this is all of $\ker\lambda$, suppose $\varepsilon \in \mathcal{O}_K^*$ is such that $\lambda(\varepsilon) = 0$. But then every component of $j(\varepsilon)$ is has absolute value 1 and therefore belongs to a bounded subset of the Minkowski space $K_{\mathbb{R}}$. Since $\mathcal{O}_K^*$ is a subgroup of $\mathcal{O}_K$ and $j(\mathcal{O}_K)$ is a complete lattice in $K_{\mathbb{R}}$ (as we have already remarked), $j(\mathcal{O}_K^*)$ is a lattice in $K_{\mathbb{R}}$. But then $j(\varepsilon)$ belongs to a discrete set by Proposition 1.5.4. Together, $j(\varepsilon)$ belongs to a discrete and bounded set and hence is necessarily finite. Since j is a $\mathbb{Q}$-embedding, it follows that the subgroup $\ker\lambda$ of $\mathcal{O}_K^*$ contains finitely many elements and hence only roots of unity because $\ker\lambda \subset K \subset \mathbb{C}$. Thus $\ker\lambda = \mu(K)$. □

Our aim now is to show that the unit lattice Λ is a a free abelian group of rank r_K. For this, we will require a lemma:

Lemma 6.3.1. *Let K be a number field. There are finitely many elements in $\mathcal{O}_K$ of a given norm up to multiplication by units.*

Proof. Recall that the norm of an algebraic integer is an integer and that the elements of norm ± 1 are exactly the units of K. Therefore it suffices to prove the claim for norm $n \geq 2$ (the norm is only zero for zero itself). Further recall that $\mathcal{O}_K/n\mathcal{O}_K$ is finite (with $\mathrm{N}_K(n\mathcal{O}_K)$ many elements). Therefore, it suffices to show that in each coset there is at most one element of norm n up to multiplication by units. To show this, suppose α and β are two representatives in the same class are are of norm n. Writing $\alpha = \beta + n\gamma$ for some $n\gamma \in n\mathcal{O}_K$, we have

$$\frac{\alpha}{\beta} = 1 + \frac{n}{\beta}\gamma = 1 + \frac{\mathrm{N}_K(\beta)}{\beta}\gamma,$$

which is an element of $\mathcal{O}_K$ because $\frac{\mathrm{N}_K(\beta)}{\beta}$ is since $\mathrm{N}_K(\beta) \in \beta\mathcal{O}_K$ as any integral ideal contains it norm. Hence $\frac{\alpha}{\beta} \in \mathcal{O}_K$, and interchanging the roles of α and β shows that $\frac{\beta}{\alpha} \in \mathcal{O}_K$ too. But then $\frac{\alpha}{\beta}$ is a unit in $\mathcal{O}_K$ and thus α and β differ up to multiplication by a unit. □

Recall that $\Lambda \subset H$. We will show Λ is a lattice in H, actually a complete lattice, and compute its rank as a free abelian group:

Theorem 6.3.1. *Let K be a number field of degree n and signature (r_1, r_2). Then the unit lattice Λ is a complete lattice in the trace-zero hyperplane H of $\mathbb{R}^{r_K+1}$. In particular, Λ is a free abelian group of rank r_K.*

Proof. Throughout, let σ run over $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$, ρ run over all such real $\mathbb{Q}$-embeddings, and τ run over a complete set of representatives of all such pairs of complex $\mathbb{Q}$-embeddings. We first prove Λ is a lattice in H. Since λ is a homomorphism (because $\log_K$ is), Λ is a subgroup of H. So by Proposition 1.5.4, Λ is a lattice if and only if it is discrete. In particular, we will show that for any $c > 0$, the bounded region

$$X = \{\mathbf{x} \in \mathbb{R}^{r_K+1} : |x_\rho| \leq c \text{ and } |x_\tau| \leq 2c \text{ for all } \rho \text{ and } \tau\},$$

contains only finitely many points of Λ. The preimage of X under ℓ is

$$\ell^{-1}(X) = \{\mathbf{z} \in K_{\mathbb{R}} : e^{-c} \leq |z_\sigma| \leq e^c \text{ for all } \sigma\},$$

and hence it contains finitely many points of $j(\mathcal{O}_K^*)$ because this is a subset of the lattice $j(\mathcal{O}_K)$. It follows that X contains finitely many points of Λ (as the preimage of $\mathbf{x}$ in $\mathbb{R}^n$ under ℓ contains 2^n points of $K_{\mathbb{R}}$) so that Λ is discrete and thus a lattice. We will now show that Λ is a complete lattice in H and since H is an r_K-dimensional real vector space, this will also prove the claim about the rank of Λ. By Proposition 1.5.5 it suffices to show that there is a bounded subset M of the trace-zero hyperplane H whose translates by Λ cover H. Actually, since ℓ is surjective it suffices to construct a bounded subset T of U such that

$$U = \bigcup_{\varepsilon \in \mathcal{O}_K^*} j(\varepsilon)T.$$

Indeed, if such T exists then any $\mathbf{z} \in T$ satisfies $|\,\mathrm{N}_{K_{\mathbb{R}}}(\mathbf{z})| = \prod_\sigma |z_\sigma| = 1$ and hence each z_σ is bounded above and away from zero because T is bounded. Setting $M = \ell(T)$, it follows that M is also bounded (because $\log|z|$ is continuous) and

$$H = \bigcup_{\lambda} (M + \lambda).$$

It now suffices to construct such a subset T. For every σ, fix constants $c_\sigma > 0$ satisfying

$$c_\sigma = c_{\overline{\sigma}} \quad \text{and} \quad \prod_\sigma c_\sigma > \left(\frac{2}{\pi}\right)^{r_2} \sqrt{|D_K|},$$

and set $C = \prod_\sigma c_\sigma$. Now consider the bounded subset

$$Z = \{\mathbf{z} \in K_{\mathbb{R}} : |z_\sigma| < c_\sigma \text{ for all } \sigma\}.$$

For any $\mathbf{w} \in U$, we have

$$\mathbf{w}Z = \{\mathbf{z} \in K_{\mathbb{R}} : |z_\sigma| < |w_\sigma| c_\sigma \text{ for all } \sigma\},$$

and $\prod_\sigma |w_\sigma| c_\sigma = \mathrm{N}_{K_{\mathbb{R}}}(\mathbf{w})C = C$ so that $\mathbf{w}Z$ is also bounded. By Proposition 6.1.1, the volume of $\mathbf{w}Z$ is 2^{r_2} times the volume of

$$X = \{\mathbf{x} \in \mathbb{R}^n : |x_\rho| < c_\rho \text{ and } x_\tau^2 + x_{\overline{\tau}}^2 < c_\tau^2 \text{ for all } \rho \text{ and } \tau\},$$

which is $\prod_\rho (2c_\rho) \prod_\tau (\pi c_\tau^2) = C 2^{r_1} \pi^{r_2}$ because X is the product of r_1 many intervals each of length $2c_\rho$ and r_2 many disks each of of radius c_τ. Thus $\mathrm{Vol}(\mathbf{w}Z) = C 2^{r_K+1} \pi^{r_2}$. By Proposition 6.1.2, $V_{j(\mathcal{O}_K)} = \sqrt{|D_K|}$ and our choice of C gives

$$\mathrm{Vol}(\mathbf{w}Z) > 2^n V_{j(\mathcal{O}_K)}.$$

Since $j(\mathcal{O}_K)$ is a complete lattice in $K_{\mathbb{R}}$, Minkowski's lattice point theorem implies that there exists some nonzero $\alpha \in \mathcal{O}_K$ such that $j(\alpha) \in \mathbf{w}Z$. Now by Lemma 6.3.1, there exist finitely many nonzero elements $\alpha_1, \ldots, \alpha_m$ of $\mathcal{O}_K$ such that every $\alpha \in \mathcal{O}_K$ with $0 < \mathrm{N}_K(\alpha) \leq C$ is equal to α_i for some $1 \leq i \leq m$ by multiplying by a unit. Set

$$T = U \cap \left(\bigcup_{1 \leq i \leq m} j(\alpha_i)^{-1} Z \right).$$

Then T is a bounded subset of U since Z is a bounded subset of $K_{\mathbb{R}}$ (and thus the $j(\alpha_i)^{-1}Z$ are too). We now claim that

$$U = \bigcup_{\varepsilon \in \mathcal{O}_K^*} j(\varepsilon) T.$$

Indeed, since $\mathbf{w}^{-1} \in U$ for any $\mathbf{w} \in U$ we have shown implies that there exists a nonzero $\alpha \in \mathcal{O}_K$ such that $j(\alpha) \in \mathbf{w}^{-1}Z$. Hence $j(\alpha) = \mathbf{w}^{-1}\mathbf{z}$ for some $\mathbf{z} \in Z$. But as

$$|\,\mathrm{N}_K(\alpha)| = |\,\mathrm{N}_{K_{\mathbb{R}}}(j(\alpha))| = |\,\mathrm{N}_{K_{\mathbb{R}}}(\mathbf{w}^{-1}\mathbf{z})| = |\,\mathrm{N}_{K_{\mathbb{R}}}(\mathbf{z})| < C,$$

it follows that there exists an α_i and $\varepsilon \in \mathcal{O}_K^*$ such that $\alpha_i = \alpha\varepsilon$. Writing $\mathbf{w} = j(\alpha)^{-1}\mathbf{z}$ (recall that $K_{\mathbb{R}}$ is commutative), we have

$$\mathbf{w} = j(\alpha)^{-1}\mathbf{z} = j(\alpha_i\varepsilon^{-1})^{-1}\mathbf{z} = j(\varepsilon)j(\alpha_i)^{-1}\mathbf{z},$$

where the last equality holds because j is a $\mathbb{Q}$-embedding. As $\mathbf{w}, j(\varepsilon) \in U$, we see that $j(\alpha_i)^{-1}\mathbf{z} \in U$ and thus $j(\alpha_i)^{-1}\mathbf{z} \in T$. But then $\mathbf{w} \in j(\varepsilon)T \subset U$ as desired. □

By Theorem 6.3.1, there exist elements $\varepsilon_1, \ldots, \varepsilon_{r_K}$ of $\mathcal{O}_K^*$ such that $\lambda(\varepsilon_1), \ldots, \lambda(\varepsilon_{r_K})$ is a basis for the unit lattice Λ by taking the preimage under j of any basis for Λ. We say that $\varepsilon_1, \ldots, \varepsilon_{r_K}$ are a **system of fundamental units** of K and we call any such element a **fundamental unit** for K. The structure theorem for $\mathcal{O}_K^*$ known as **Dirichlet's unit theorem** says that $\mathcal{O}_K^*$ is a product of a root of unity in K and powers of fundamental units:

Theorem (Dirichlet's unit theorem). *Let K be a number field of signature (r_1, r_2). Then*

$$\mathcal{O}_K^* \cong \mu(K) \times \mathbb{Z}^{r_K}.$$

In particular, if $\varepsilon_1, \ldots, \varepsilon_{r_K}$ is a system of fundamental units for K then any unit ε of K is of the form

$$\varepsilon = \omega \varepsilon_1^{\nu_1} \cdots \varepsilon_{r_K}^{\nu_{r_K}},$$

for some $\omega \in \mu(K)$ and $\nu_i \in \mathbb{Z}$ for $1 \leq i \leq r_K$.

Proof. By Proposition 6.3.1 we have an exact sequence

$$1 \longrightarrow \mu(K) \longrightarrow \mathcal{O}_K^* \xrightarrow{\lambda} \Lambda \longrightarrow 0,$$

and by Theorem 6.3.1 we know that Λ is a free abelian group of rank r_K. Let $\varepsilon_1, \ldots, \varepsilon_{r_K}$ be a system of fundamental units for K and let E be the subgroup of $\mathcal{O}_K^*$ generated by them. Then λ induces an isomorphism between E and Λ so that $\mu(K) \cap E = \{1\}$ because the sequence is exact. As $\mu(K)$ and E are subgroups, $\mu(K) \cap E = \{1\}$ implies $\mathcal{O}_K^* \cong \mu(K) \times E$. Since $E \cong \mathbb{Z}^{r_K}$ (because $\Lambda \cong \mathbb{Z}^{r_K}$), we have $\mathcal{O}_K^* \cong \mu(K) \times \mathbb{Z}^{r_K}$. Moreover, from the isomorphism $E \cong \mathbb{Z}^{r_K}$ we see that any element of E is of the form $\varepsilon_1^{\nu_1} \cdots \varepsilon_{r_K}^{\nu_{r_K}}$ with $\nu_i \in \mathbb{Z}$ for $1 \leq i \leq r_K$. This completes the proof. □

Note that Dirichlet's unit theorem also implies that Λ is a complete lattice in H (by the definition of λ). In other words, we may reference this result instead of Theorem 6.3.1. We will now discuss the covolume V_Λ of Λ. Let $\varepsilon_1, \ldots, \varepsilon_{r_K}$ be a system of fundamental units for K. Then $\lambda(\varepsilon_1), \ldots, \lambda(\varepsilon_{r_K})$ is a basis for Λ. Setting

$$\lambda_0 = \frac{1}{\sqrt{r_K+1}}\mathbf{1},$$

we see that λ_0 is a unit vector in $\mathbb{R}^{r_K+1}$ since $||\mathbf{1}|| = r_K+1$ and is orthogonal to H because $\mathrm{Tr}_{\mathbb{R}^{r_K+1}/\mathbb{R}}(\lambda_0) = \sqrt{r_K+1}$ is nonzero. As λ_0 is orthogonal to $\Lambda \subset H$, we see that $\lambda_0, \lambda(\varepsilon_1), \ldots, \lambda(\varepsilon_{r_K})$ is a basis for the complete lattice $\Lambda' = \mathbb{Z}\lambda_0 + \Lambda$ in $\mathbb{R}^{r_K+1}$. Since λ_0 is a unit vector, the volume of the fundamental domain for Λ' in $\mathbb{R}^{r_K+1}$ is equal to the volume of the fundamental domain for Λ in $\mathbb{H}$. By Proposition 1.5.3, the corresponding covolumes are equal which is to say $V_\Lambda = V_{\Lambda'}$. So it suffices to compute $V_{\Lambda'}$. The generator matrix P for Λ' associated to the basis $\lambda_0, \lambda(\varepsilon_1), \ldots, \lambda(\varepsilon_{r_K})$ is given by

$$P = \begin{pmatrix} \frac{1}{\sqrt{r_K+1}} & \lambda(\varepsilon_1)_1 & \cdots & \lambda(\varepsilon_{r_K})_1 \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{r_K+1}} & \lambda(\varepsilon_1)_{r_K+1} & \cdots & \lambda(\varepsilon_{r_K})_{r_K+1} \end{pmatrix}.$$

Adding all of the rows to a fixed row results in

$$P' = \begin{pmatrix} \frac{1}{\sqrt{r_K+1}} & \lambda(\varepsilon_1)_1 & \cdots & \lambda(\varepsilon_{r_K})_1 \\ \vdots & \vdots & & \vdots \\ \sqrt{r_K+1} & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{r_K+1}} & \lambda(\varepsilon_1)_{r_K+1} & \cdots & \lambda(\varepsilon_{r_K})_{r_K+1} \end{pmatrix},$$

because $\lambda(\varepsilon) \in H$ for all $\varepsilon \in \mathcal{O}_K^*$. As P' differs from P be row addition, their determinants are the same. Cofactor expanding along the row with all zeros except the first entry, and that this row is arbitrary, shows

$$V_\Lambda = \sqrt{r_K+1}R_K, \tag{6.5}$$

where R_K is the absolute value of the determinant of any rank r_K minor of

$$\begin{pmatrix} \lambda(\varepsilon_1)_1 & \cdots & \lambda(\varepsilon_{r_K})_1 \\ \vdots & & \vdots \\ \lambda(\varepsilon_1)_{r_K+1} & \cdots & \lambda(\varepsilon_{r_K})_{r_K+1} \end{pmatrix}.$$

We call R_K the **regulator** of K. Since V_Λ is independent of the choice of basis, the regulator R_K is independent of any choice of a system of fundamental units for K. Moreover, since H is a real inner product space we see that the regulator is roughly a measure of the density of the fundamental units in K (recall Proposition 1.5.3). The smaller the regulator the more dense the fundamental units are.

Chapter 7

Quadratic and Cyclotomic Number Fields

We provide a more detailed discussion of number fields with particularly simple structure. Namely, we discuss quadratic and cyclotomic number fields because they are monogenic.

7.1 Quadratic Number Fields

We will classify and discuss the structure of quadratic number fields. Note that since any degree 2 extension is normal, quadratic number fields are automatically Galois. We first show that quadratic number fields are exactly those where we adjoin the square-root of a square-free integer other than 0 or 1:

Proposition 7.1.1. *Every quadratic number field K is of the form $K = \mathbb{Q}(\sqrt{d})$ for some square-free integer d other than 0 or 1.*

Proof. Suppose K is a quadratic number field. In particular, $K/\mathbb{Q}$ is separable so by the primitive element theorem there exists $\theta \in K$ such that $K = \mathbb{Q}(\theta)$. The minimal polynomial $m_\theta(x)$ of θ over $\mathbb{Q}$ is of the form

$$m_\theta(x) = x^2 + ax + b,$$

for $a, b \in \mathbb{Q}$. Then the quadratic formula gives

$$\theta = -\frac{a}{2} \pm \frac{\sqrt{q}}{2},$$

where $q = a^2 - 4b \in \mathbb{Q}$. Clearly $q \neq 0$ and $q \neq 1$ for otherwise $\theta \in \mathbb{Q}$. It follows that $K = \mathbb{Q}(\sqrt{q})$. Write $q = \frac{n}{m}$ for relatively prime $n, m \in \mathbb{Z}$ and set $d = m^2 q = nm \in \mathbb{Z}$. Then d is square-free, $d \neq 0$, and $d \neq 1$. Moreover, $\sqrt{d} = m\sqrt{q}$ so that $K = \mathbb{Q}(\sqrt{d})$. □

From Proposition 7.1.1, we see that the d for a quadratic number field $\mathbb{Q}(\sqrt{d})$ satisfies $d \equiv 1, 2, 3 \pmod 4$ (otherwise d is not square-free). Moreover, any element of a quadratic number field is of the form $a + b\sqrt{d}$ with $a, b \in \mathbb{Q}$ and for some square-free d other than 0 or 1. We say that a quadratic number field $\mathbb{Q}(\sqrt{d})$ is **real** if $d > 0$ and **imaginary** if $d < 0$. Now $\mathbb{Q}(\sqrt{d})$ is real or imaginary according to if $\sqrt{d}$ is real or purely imaginary so that the two elements σ_1 and σ_2 of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{d}), \overline{\mathbb{Q}})$ are

$$\sigma_1(a + b\sqrt{d}) = a + b\sqrt{d} \quad \text{and} \quad \sigma_2(a + b\sqrt{d}) = a - b\sqrt{d},$$

because the roots of the minimal polynomial for $\sqrt{d}$ over $\mathbb{Q}$ are $\pm\sqrt{d}$. In particular, the signature is $(2, 0)$ or $(0, 1)$ according to if $\mathbb{Q}(\sqrt{d})$ is real or imaginary. We write $\mathrm{Tr}_d = \mathrm{Tr}_{\mathbb{Q}(\sqrt{d})}$ and $\mathrm{N}_d = \mathrm{N}_{\mathbb{Q}(\sqrt{d})}$ for the

field trace and field norm of $\mathbb{Q}(\sqrt{d})$ respectively. Then Proposition 5.2.1 shows that the trace and norm of $\kappa = a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d})$ are given by

$$\mathrm{Tr}_d(\kappa) = 2a \quad \text{and} \quad \mathrm{N}_d(\kappa) = a^2 - b^2 d.$$

We will now begin describing the ring of integers, discriminant, and the factorization of primes in $\mathbb{Q}(\sqrt{d})$. For simplicity, we will write $\mathcal{O}_d = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ and $\Delta_d = \Delta_{\mathbb{Q}(\sqrt{d})}$. The ring of integers has a particularly simple description since quadratic number fields are monogenic as the following proposition shows:

Proposition 7.1.2. *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field. Then $\mathbb{Q}(\sqrt{d})$ is monogenic where*

$$\mathcal{O}_d = \begin{cases} \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{if } d \equiv 1 \pmod 4, \\ \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

Proof. Let $\alpha = a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d})$ be an algebraic integer. If $b = 0$ then $\alpha \in \mathbb{Q}$ and since the only elements of $\mathbb{Q}$ that are algebraic integers are the integers themselves we must have that α is an integer. Now suppose $b \neq 0$. Then the minimal polynomial of α over $\mathbb{Q}$ is

$$m_\alpha(x) = x^2 + 2ax + (a^2 - b^2 d) = (x - (a + b\sqrt{d}))(x - (a - b\sqrt{d})).$$

As α is an algebraic integer, $2a \in \mathbb{Z}$ and $a^2 - b^2 d \in \mathbb{Z}$ (note that these are the trace and norm of α respectively). In particular, $(2a)^2 + (2b)^2 d \in \mathbb{Z}$ and hence $(2b)^2 \in \mathbb{Z}$ is as well. But as $b \in \mathbb{Q}$, it must be the case that $2b \in \mathbb{Z}$. If $2a = n + 1$ is odd then n is even. We compute

$$a^2 - b^2 d = \left(\frac{n+1}{2}\right)^2 - b^2 d = \frac{n^2 + 2n + 1 + 4b^2 d}{4},$$

and since the right-hand side must be an integer $b \notin \mathbb{Z}$. For if $b \in \mathbb{Z}$, the numerator of the right-hand side is equivalent to 1 modulo 4 because n is even. As $2b \in \mathbb{Z}$ it follows that $2b$ must be odd so set $2b = m + 1$ with m even. Again, we compute

$$a^2 - b^2 d = \left(\frac{n+1}{2}\right)^2 - \left(\frac{m+1}{2}\right)^2 d = \frac{n^2 + 2n + 1 - d(m^2 + 2m + 1)}{4},$$

and since the right-hand side must be an integer the numerator must be divisible by 4. As n and m are even, this is equivalent to $d \equiv 1 \pmod 4$. So we have shown $2a$ or $2b$ is odd if and only if $d \equiv 1 \pmod 4$. Thus if $d \equiv 1 \pmod 4$, we have $a = \frac{a'}{2}$ and $b = \frac{b'}{2}$ for some $a', b' \in \mathbb{Z}$ and hence $\alpha \in \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$. Otherwise, $d \equiv 2, 3 \pmod 4$ (because d is square-free) so that $2a$ and $2b$ are both even, $a, b \in \mathbb{Z}$, and therefore $\alpha \in \mathbb{Z}[\sqrt{d}]$. We have now shown that $\mathcal{O}_d \subseteq \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$ and $\mathcal{O}_d \subseteq \mathbb{Z}[\sqrt{d}]$ according to if $d \equiv 1 \pmod 4$ or $d \equiv 2, 3 \pmod 4$ respectively. For the reverse containment, it suffices to show that $\frac{1+\sqrt{d}}{2}$ and $\sqrt{d}$ are algebraic integers according to if $d \equiv 1 \pmod 4$ or $d \equiv 2, 3 \pmod 4$ respectively since $\mathcal{O}_K$ is a ring. Indeed they are since their minimal polynomials over $\mathbb{Q}$ are

$$m_{\frac{1+\sqrt{d}}{2}}(x) = x^2 - x + \frac{1-d}{4} \quad \text{and} \quad m_{\sqrt{d}}(x) = x^2 - d,$$

where $\frac{1-d}{4} \in \mathbb{Z}$ because $d \equiv 1 \pmod 4$. □

It follows from Proposition 7.1.2 that

$$1, \frac{1+\sqrt{d}}{2} \quad \text{and} \quad 1, \sqrt{d},$$

are integral bases for $\mathcal{O}_d$ according to if $d \equiv 1 \pmod 4$ or $d \equiv 2, 3 \pmod 4$ respectively. Let us now show that the discriminants quadratic number fields are exactly the fundamental discriminants other than 1:

Proposition 7.1.3. *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field. Then*

$$\Delta_d = \begin{cases} d & \text{if } d \equiv 1 \pmod 4, \\ 4d & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

In particular, the discriminants quadratic number fields are exactly the fundamental discriminants other than 1*.*

Proof. Let σ_1 and σ_2 be the two elements of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{d}), \overline{\mathbb{Q}})$ where σ_1 is the identity and σ_2 is given by sending $\sqrt{d}$ to its conjugate. If $d \equiv 1 \pmod 4$, an integral basis for $\mathcal{O}_d$ is $1, \frac{1+\sqrt{d}}{2}$. In this case, the embedding matrix is

$$M\left(1, \frac{1+\sqrt{d}}{2}\right) = \begin{pmatrix} 1 & \frac{1+\sqrt{d}}{2} \\ 1 & \frac{1-\sqrt{d}}{2} \end{pmatrix},$$

and thus $\Delta_d = d$. If $d \equiv 2, 3 \pmod 4$, an integral basis for $\mathcal{O}_d$ is $1, \sqrt{d}$. In this case, the embedding matrix is

$$M(1, \sqrt{d}) = \begin{pmatrix} 1 & \sqrt{d} \\ 1 & -\sqrt{d} \end{pmatrix},$$

and hence $\Delta_d = 4d$. This proves the first statement and the second statement is clear since d is square-free and not 0 or 1. □

From now on, we will write fundamental discriminants other than 1 as Δ_d instead of D to clarify the connection to quadratic number fields. We will now discuss the factorization of $p\mathcal{O}_d$ in the quadratic number field $\mathbb{Q}(\sqrt{d})$ for a prime p. Since $\mathbb{Q}(\sqrt{d})$ is a degree 2 extension, the fundamental equality (or Proposition 5.6.1 using that $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ is Galois) implies that p is either inert, totally ramified, or totally split. In other words, there are three possible cases for how $p\mathcal{O}_d$ factors:

$$p\mathcal{O}_d = \mathfrak{p}, \quad p\mathcal{O}_d = \mathfrak{p}^2, \quad \text{and} \quad p\mathcal{O}_d = \mathfrak{p}\mathfrak{q},$$

for some primes $\mathfrak{p}$ and $\mathfrak{q}$, according to if p is inert, totally ramified, or totally split. Since $\mathbb{Q}(\sqrt{d})$ is monogenic by Proposition 7.1.2, the conductor is $\mathcal{O}_K$ and so we can describe the factorization using the Dedekind-Kummer theorem for every prime p. In particular, we will connect the prime factorization to the quadratic character χ_{Δ_d} associated to the fundamental discriminant Δ_d:

Proposition 7.1.4. *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field and let χ_{Δ_d} be the quadratic character given by the fundamental discriminant Δ_d. Then for any prime p, we have*

$$\chi_{\Delta_d}(p) = \begin{cases} 1 & \text{if } p \text{ is totally split}, \\ -1 & \text{if } p \text{ is inert}, \\ 0 & \text{if } p \text{ is totally ramified}. \end{cases}$$

Proof. Recall that p is ramified if and only if it divides $|\Delta_d|$ and note that this is exactly when $\chi_{\Delta_d}(p) = 0$. Therefore it suffices to prove the cases when p is totally split or inert. First suppose $d \equiv 1 \pmod 4$ so that $\mathcal{O}_d = \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$ and $\Delta_d = d$ by Propositions 7.1.2 and 7.1.3. The minimal polynomial $m_{\frac{1+\sqrt{d}}{2}}(x)$ for $\frac{1+\sqrt{d}}{2}$ over $\mathbb{Q}$ is

$$m_{\frac{1+\sqrt{d}}{2}}(x) = x^2 - x + \frac{1-d}{4},$$

where $\frac{1-d}{4} \in \mathbb{Z}$ because $d \equiv 1 \pmod 4$. The reduction of $m_{\frac{1+\sqrt{d}}{2}}(x)$ modulo p is either irreducible, factors into two distinct linear factors, or is a square, and Dedekind-Kummer theorem implies that this is equivalent to p being inert, totally split, or totally ramified accordingly because the prime factorization of fractional ideals is unique. First suppose $p \neq 2$. Then from the quadratic formula, $m_{\frac{1+\sqrt{d}}{2}}(x)$ reduces modulo p as

$$m_{\frac{1+\sqrt{d}}{2}}(x) \equiv \left(x - \frac{1+\sqrt{d}}{2}\right)\left(x - \frac{1-\sqrt{d}}{2}\right) \pmod p,$$

if and only if the roots $\frac{1\pm\sqrt{d}}{2}$ are elements of $\mathbb{F}_p$ and is otherwise irreducible. As $p \neq 2$, these factors are distinct. Moreover, $\frac{1\pm\sqrt{d}}{2}$ is an element of $\mathbb{F}_p$ if and only if d is a square modulo p and hence p is totally split or inert according to if $\chi_d(p) = \pm 1$. Now suppose $p = 2$. Since $m_{\frac{1+\sqrt{d}}{2}}(x)$ has a nonzero linear term with an odd coefficient, it reduces modulo 2 as

$$m_{\frac{1+\sqrt{d}}{2}}(x) \equiv x(x-1) \pmod 2,$$

if and only if $\frac{1-d}{4} \equiv 0 \pmod 2$ and is otherwise irreducible. Clearly these factors are distinct. Now observe $\frac{1-d}{4} \equiv 0 \pmod 2$ is equivalent to $d \equiv 1 \pmod 8$ provided $d > 0$ and $d \equiv 7 \pmod 8$ provided $d < 0$ and thus p is totally split or inert according to if $\chi_d(2) = \pm 1$. This completes the argument in the case $d \equiv 1 \pmod 4$. Now suppose $d \equiv 2, 3 \pmod 4$ so that $\mathcal{O}_d = \mathbb{Z}[\sqrt{d}]$ and $\Delta_d = 4d$ by Propositions 7.1.2 and 7.1.3. The minimal polynomial $m_{\sqrt{d}}(x)$ for $\sqrt{d}$ over $\mathbb{Q}$ is

$$m_{\sqrt{d}}(x) = x^2 - d.$$

As $\Delta_d = 4d$, we see that 2 is ramified and therefore we may assume $p \neq 2$. Similarly, the reduction of $m_{\sqrt{d}}(x)$ modulo p is either irreducible, factors into two distinct linear factors, or is a square, and Dedekind-Kummer theorem implies that this is equivalent to p being inert, totally split, or totally ramified accordingly because the prime factorization of fractional ideals is unique. As $p \neq 2$, the quadratic formula implies that $m_{\sqrt{d}}(x)$ reduces modulo p as

$$m_{\sqrt{d}}(x) \equiv (x - \sqrt{d})(x + \sqrt{d}) \pmod p,$$

if and only if the roots $\pm\sqrt{d}$ are elements of $\mathbb{F}_p$. As $p \neq 2$, these factors are distinct. Moreover, $\sqrt{d}$ is an element of $\mathbb{F}_p$ if and only if d and hence $4d$ are squares modulo p so that p is totally split or inert according to if $\chi_{4d}(p) = \pm 1$. This completes the verification in the case $d \equiv 2, 3 \pmod 4$. □

From Proposition 7.1.4, the factorization of primes in $\mathbb{Q}(\sqrt{d})$ is controlled by the quadratic character χ_{Δ_d} associated to the fundamental discriminant Δ_d. In other words, the factorization of p depends completely upon if Δ_d is a square modulo p. While splitting of primes can be explicitly described for quadratic number fields, the class number is a significantly more difficult problem. We will write $h_d = h_{\mathbb{Q}(\sqrt{d})}$. The **class number problem** was originally introduced by Gauss and aims to classify all quadratic number fields of a given class number:

Open Problem (Class number problem). *For a fixed $n \geq 1$, classify all quadratic number fields $\mathbb{Q}(\sqrt{d})$ of class number n.*

Some progress has been made toward the class number problem. In 1801, Gauss found nine imaginary quadratic numbers fields of class number 1 (see [Gau01]). They are listed according to d as follows:

$$d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}.$$

Gauss also conjectured that these are the only imaginary quadratic numbers fields of class number 1. An argument was presented by Heegner in 1952 (see [Hee52]) which was correct up to some minor flaws. Baker and Stark both independently gave independent proofs in the mid 1960's (see [Bak67, Sta67]) resulting in the following theorem which solves the class number problem for imaginary quadratic number fields in the case $n = 1$:

Theorem 7.1.1. *If $\mathbb{Q}(\sqrt{d})$ is an imaginary quadratic number field of class number 1 then*

$$d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}.$$

Equivalently, an imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$ has class number 1 if and only if

$$\Delta_d \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\}.$$

As for real quadratic fields, we know much less. In the same 1801 paper of Gauss (see [Gau01]), he conjectured that there should be infinitely many real quadratic fields and that the class number should remain unbounded:

Conjecture 7.1.1. *There are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{d})$ that have class number 1. Moreover,*

$$\lim_{d \to \infty} h_d = \infty.$$

While the class number problem remains quite out of reach, the structure of the unit group is much easier to classify. Write $\mu(d) = \mu(\mathbb{Q}(\sqrt{d}))$ and $w_d = w_{\mathbb{Q}(\sqrt{d})}$. In fact, by Dirichlet's unit theorem we only need to understand the roots of unity μ_d of $\mathbb{Q}(\sqrt{d})$. In all but two cases, $\mu(d) = \langle -1 \rangle$:

Proposition 7.1.5. *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field. Then*

$$\mu(d) = \begin{cases} \langle i \rangle & \text{if } d = -1, \\ \langle \omega_6 \rangle & \text{if } d = -3, \\ \langle -1 \rangle & \text{otherwise,} \end{cases} \quad \text{and} \quad w_d = \begin{cases} 4 & \text{if } d = -1, \\ 6 & \text{if } d = -3, \\ 2 & \text{otherwise,} \end{cases}$$

where ω_6 is primitive 6-th root of unity. In particular,

$$\mathcal{O}_d^* = \begin{cases} \mu(d) \times \langle \varepsilon_d \rangle & \text{if } d > 0, \\ \mu(d) & \text{if } d < 0, \end{cases}$$

where ε_d is a fundamental unit.

Proof. First suppose $d > 0$. Then $\mathbb{Q}(\sqrt{d}) \subset \mathbb{R}$ and thus $\mu(d) = \langle -1 \rangle$ since these are the only roots of unity in $\mathbb{R}$ and clearly they are in $\mathbb{Q}(\sqrt{d})$. Now suppose $d < 0$. Then $\mathbb{Q}(\sqrt{d})$ is imaginary and its signature is $(0, 1)$. Recall that $\alpha \in \mathcal{O}_d$ is a unit if and only if the norm of α is ± 1. Actually, since $d < 0$ the definition of the norm shows that the norm is always nonnegative. Hence α is a unit if and only if its norm is 1. First suppose $d \equiv 2, 3 \pmod 4$. Then Proposition 7.1.2 implies

$$\alpha = a + b\sqrt{d},$$

for some $a, b \in \mathbb{Z}$, and α is a unit if and only if

$$\mathrm{N}_d(\alpha) = a^2 - b^2 d = a^2 + b^2|d| = 1.$$

Since $|d| \equiv d \equiv 2, 3 \pmod 4$, this happens if and only if $b = 0$ unless $d = -1$. In the former case, $d < 0$ and $\alpha = a$ is a unit if and only if $a^2 = 1$ which is to say that $\alpha = \pm 1$. In the latter case, $d = -1$ and $\alpha = a + bi$ with $b \neq 0$ (for otherwise we are in the former case) is a unit if and only if $a^2 + b^2 = 1$ which means $a = \pm 1$ and $b = 0$ or $a = 0$ and $b = \pm 1$ so that α runs over the 4-th roots of unity. Altogether, we have shown that $\mu(d) = \langle -1 \rangle$ provided $d \equiv 2, 3 \pmod 4$ unless $d = -1$ in which case $\mu(-1) = \langle i \rangle$. Now suppose $d \equiv 1 \pmod 4$. Then Proposition 7.1.2 implies

$$\alpha = a + b\frac{1 + \sqrt{d}}{2} = \frac{2a + b}{2} + \frac{b}{2}\sqrt{d},$$

for some $a, b \in \mathbb{Z}$, and α is a unit if and only if

$$\mathrm{N}_d(\alpha) = \frac{4a^2 + 4ab + b^2}{4} - \frac{b^2}{4}d = a^2 + ab + (1 + |d|)\frac{b^2}{4} = 1.$$

Since $|d| \equiv d \equiv 1 \pmod 4$, this happens if and only if $b = 0$ or $d = -3$ (if $b \neq 0$ then $a^2 + ab + (1 + |d|)\frac{b^2}{4} > 1$ for such d unless $d = -3$). In the former case, $\alpha = a$ is a unit if and only if $a^2 = 1$ which means $\alpha = \pm 1$. In the latter case, $\alpha = a + b\frac{1+\sqrt{-3}}{2}$ is a unit if and only if $a^2 + ab + b^2 = 1$ which happens if $a = \pm 1$ and $b = 0$, $a = 0$ and $b = \pm 1$, $a = 1$ and $b = -1$, or $a = -1$ and $b = 1$ so that α runs over the 6-th roots of unity. This shows $\mu(d) = \langle -1 \rangle$ provided $d \equiv 1 \pmod 4$ unless $d = -3$ in which case $\mu(-3) = \langle \omega_6 \rangle$. This proves the claim about $\mu(d)$ in all cases and the statement about w_d follows immediately. To prove the last statement, the signature of $\mathbb{Q}(\sqrt{d})$ is $(2, 0)$ or $(0, 1)$ according to if $d > 0$ or $d < 0$. Applying Dirichlet's unit theorem completes the proof. □

Lastly, we discuss the regulator. Set $R_d = R_{\mathbb{Q}(\sqrt{d})}$. Then we have the following proposition:

Proposition 7.1.6. *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field. Then*

$$R_d = \begin{cases} \log|\varepsilon_d| & \text{if } d > 0, \\ 1 & \text{if } d < 0, \end{cases}$$

where ε_d is a fundamental unit.

Proof. If $d > 0$ then Proposition 7.1.5 implies that a system of fundamental units for $\mathbb{Q}(\sqrt{d})$ is given by a single fundamental unit ε_d. Since the signature of $\mathbb{Q}(\sqrt{d})$ is $(2, 0)$, we have $\lambda(\varepsilon_d) = (\log|\varepsilon_d|, \log|\varepsilon_d|)$ and therefore $R_K = \log|\varepsilon_d|$. If $d < 0$ then Proposition 7.1.5 implies that there are no fundamental units and thus $R_K = 1$. □

7.2 Cyclotomic Number Fields

Let ω be a primitive n-th root of unity. We call $\mathbb{Q}(\omega)$ the n-th **cyclotomic field**. Note that $\mathbb{Q}(\omega)$ is independent of the choice of primitive root ω since $\mathbb{Q}(\omega)$ contains all n-th roots of unity. As ω is a root of $x^n - 1$, we see that $\mathbb{Q}(\omega)/\mathbb{Q}$ is a finite extension of degree at most n. In particular, $\mathbb{Q}(\omega)$ is a number field. More generally, we say that a number field K is **cyclotomic** if K is the n-th cyclotomic field for some $n \geq 1$. That is, $K = \mathbb{Q}(\omega)$ for some primitive n-th root of unity ω. In any case, our aim is to study the structure of cyclotomic number fields $\mathbb{Q}(\omega)$. Our first step is to compute the degree of $\mathbb{Q}(\omega)$ which is the degree of the minimal polynomial of ω over $\mathbb{Q}$. Accordingly, we define the n-th **cyclotomic polynomial** $\Phi_n(x)$ by

$$\Phi_n(x) = \prod_{k \in (\mathbb{Z}/n\mathbb{Z})^*} (x - \omega^k).$$

That is, $\Phi_n(x)$ is the polynomial whose roots are the primitive n-th roots of unity. It is clearly monic, of degree $\varphi(n)$, and divides $x^n - 1$. As every n-th root of unity is a primitive d-th root of unity for some $d \mid n$, we also find that

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \tag{7.1}$$

Clearly $\Phi_1(x) = x - 1$ and $\Phi_2(x) = x + 1$. When $n = p$ for a prime p, Equation (7.1) implies

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + 1.$$

More generally, writing $n = p^k$ and inducting on k using Equation (7.1) gives

$$\Phi_{p^k}(x) = \frac{x^{p^k} - 1}{x^{p^{k-1}} - 1} = \frac{x^{p^k} - 1}{x^{p^{k-1}} - 1} = x^{p^{k-1}(p-1)} + x^{p^{k-1}(p-2)} + \cdots + 1. \tag{7.2}$$

Observe from Equation (7.2) that $\Phi_{p^k}(x)$ has coefficients in $\mathbb{Z}$. This is true for a general cyclotomic polynomial $\Phi_n(x)$ in addition to irreducibility over $\mathbb{Z}$ as the following proposition shows:

Proposition 7.2.1. *$\Phi_n(x)$ has coefficients in and is irreducible over $\mathbb{Z}$.*

Proof. We first show $\Phi_n(x)$ has coefficients in $\mathbb{Z}$ and we will argue by induction. The claim is true for $n = 1$ since $\Phi_1(x) = x - 1$. So assume by induction that it is true for all $1 \leq d < n$. In view of Equation (7.1), we have

$$x^n - 1 = \Phi_n(x) \prod_{\substack{d|n \\ d<n}} \Phi_d(x),$$

and $\prod_{\substack{d|n \\ d<n}} \Phi_d(x)$ has coefficients in $\mathbb{Z}$. Therefore $\prod_{\substack{d|n \\ d<n}} \Phi_d(x)$ divides $x^n - 1$ in $\mathbb{Q}[x]$ and hence in $\mathbb{Z}[x]$ as well by Gauss's lemma. Thus $\Phi_n(x)$ has coefficients in $\mathbb{Z}$ as desired. We now show $\Phi_n(x)$ is irreducible over $\mathbb{Z}$. So suppose

$$\Phi_n(x) = f(x)g(x),$$

for monic polynomials $f(x), g(x) \in \mathbb{Z}[x]$ (recall $\Phi_n(x)$ is monic) with $f(x)$ irreducible. Then it suffices to show $f(x) = \Phi_n(x)$. Now let ω be a root of $f(x)$. Then ω is also a root of $\Phi_n(x)$ and necessarily a primitive n-th roots of unity. Since $f(x)$ is monic and irreducible it is necessarily the minimal polynomial of ω over $\mathbb{Q}$. Now let p be any prime not dividing n. Then ω^p is also a primitive n-th root of unity and hence a root of either $f(x)$ or $g(x)$. Suppose ω^p is a root of $g(x)$. Then ω is a root of $g(x^p)$, and since $f(x)$ is

the minimal polynomial of ω over $\mathbb{Q}$, $f(x)$ divides $g(x^p)$ in $\mathbb{Q}[x]$. By Gauss's lemma, it follows that $f(x)$ divides $g(x^p)$ in $\mathbb{Z}[x]$ too. Therefore

$$g(x^p) = f(x)h(x),$$

for a monic polynomial $h(x) \in \mathbb{Z}[x]$. Reducing this factorization modulo p, we obtain

$$\overline{g}(x^p) \equiv \overline{g}(x)^p \equiv \overline{f}(x)\overline{h}(x) \pmod{p},$$

where the first congruence holds since $\overline{g}(x^p) = \overline{g}(x)^p$ in $\mathbb{F}_p[x]$ (recall Fermat's little theorem and that the characteristic of $\mathbb{F}_p$ is p). As $p \geq 2$, this equivalence shows that $\overline{f}(x)$ and $\overline{h}(x)$ must have a common factor. In other words, $\overline{g}(x^p)$ has a multiple root in $\mathbb{F}_p$ and therefore $\overline{g}(x)$ does as well. Reducing the factorization for $\Phi_n(x)$ modulo p gives

$$\overline{\Phi_n}(x) \equiv \overline{f}(x)\overline{g}(x) \pmod{p}.$$

Then $\overline{\Phi_n}(x)$ has a multiple root in $\mathbb{F}_p$ since $\overline{g}(x)$ does. As $\overline{\Phi_n}(x)$ divides $x^n - 1$ (because $\Phi_n(x)$ does and $x^n - 1$ is itself reduced modulo p), it follows that $x^n - 1$ has a multiple root in $\mathbb{F}_p$. This is impossible since $x^n - 1$ has n distinct roots in $\overline{\mathbb{F}_p}$ as p does not divide n (recall that the derivative of $x^n - 1$ is nx^{n-1} which is relatively prime to p). It follows that ω^p cannot be a root of $g(x)$ and is therefore a root of $f(x)$. Now let $k \in (\mathbb{Z}/n\mathbb{Z})^*$ and write $k = p_1 p_2 \cdots p_k$ as a product of primes not dividing n. Then $\omega^k = \omega^{p_1 p_2 \cdots p_k}$ is a root of $f(x)$ and hence every primitive n-th root of unity is a root of $f(x)$. Thus $f(x) = \Phi_n(x)$ which proves $\Phi_n(x)$ is irreducible over $\mathbb{Z}$. □

Since $\Phi_n(x)$ is monic, Proposition 7.2.1 implies that $\Phi_n(x)$ is the minimal polynomial of ω over $\mathbb{Q}$ and hence of every primitive n-th root of unity over $\mathbb{Q}$. It follows that the degree of $\mathbb{Q}(\omega)$ is $\varphi(n)$ because this is the degree of $\Phi_n(x)$. This implies $\mathbb{Q}(\omega)$ is the splitting field of $\Phi_n(x)$ over $\mathbb{Q}$ because if one primitive n-th root of unity belongs to a field then they all do (as they are powers of each other). In particular, $\mathbb{Q}(\omega)/\mathbb{Q}$ is normal and hence Galois. Moreover, every primitive n-root of unity is an algebraic integer since $\Phi_n(x)$ also has coefficients in $\mathbb{Z}$ by Proposition 7.2.1. We now turn to the question of the ring of integers of $\mathbb{Q}(\omega)$. For convenience write $\mathcal{O}_\omega = \mathcal{O}_{\mathbb{Q}(\omega)}$ and set

$$\mathfrak{p}_\omega = (1 - \omega)\mathcal{O}_\omega.$$

We will first prove a useful lemma which shows that $\mathfrak{p}_\omega$ is a prime of $\mathbb{Q}(\omega)$ and more in the case n is a prime power:

Lemma 7.2.1. *Let $\mathbb{Q}(\omega)$ be the cyclotomic number field generated by a primitive p^e-th root of unity ω with for some prime p and $e \geq 1$. Then*

$$p\mathcal{O}_\omega = \mathfrak{p}_\omega^{\varphi(p^e)}.$$

In particular, $\mathfrak{p}_\omega$ is a prime above p with $f_p(\mathfrak{p}_\omega) = 1$. Moreover, $1, \omega, \ldots, \omega^{\varphi(p^e)-1}$ is a basis for $\mathbb{Q}(\omega)/\mathbb{Q}$ with

$$d_{\mathbb{Q}(\omega)/\mathbb{Q}}(1, \omega, \ldots, \omega^{\varphi(p^e)-1}) = \pm p^{\varphi(p^e)e - p^{e-1}}.$$

Proof. In view of the definition of $\Phi_{p^e}(x)$ and Equation (7.2), we have

$$x^{p^{e-1}(p-1)} + x^{p^{e-1}(p-2)} + \cdots + 1 = \prod_{k \in (\mathbb{Z}/p^e\mathbb{Z})^*} (x - \omega^k).$$

Setting $x = 1$ gives

$$p = \prod_{k \in (\mathbb{Z}/p^e\mathbb{Z})^*} (1 - \omega^k).$$

In the case $e = 1$, ω is a primitive p-th root of unity. Then $\mathrm{N}_{\mathbb{Q}(\omega)/\mathbb{Q}}(1-\omega) = p$ by Proposition 5.2.1 since $\mathbb{Q}(\omega)/\mathbb{Q}$ is Galois. In any case, the factors $1-\omega^k$ are clearly algebraic integers because ω is (as a consequence of Proposition 7.2.1). Then

$$\varepsilon_k = \frac{1-\omega^k}{1-\omega} = \omega^{k-1} + \omega^{k-2} + \cdots + 1,$$

is also an algebraic integer and satisfies $1-\omega^k = \varepsilon_k(1-\omega)$. Moreover,

$$\varepsilon_k^{-1} = \frac{1-\omega}{1-\omega^k} = \frac{1-\omega^{k\overline{k}}}{1-\omega^k} = \omega^{k(\overline{k}-1)} + \omega^{k(\overline{k}-2)} + \cdots + 1,$$

is also an algebraic integer. This means ε_k is a unit in $\mathcal{O}_\omega$. So upon setting $\varepsilon = \prod_{k\in(\mathbb{Z}/p^e\mathbb{Z})^*} \varepsilon_k$, we conclude that

$$p = \varepsilon(1-\omega)^{\varphi(p^e)},$$

and therefore

$$p\mathcal{O}_\omega = \mathfrak{p}_\omega^{\varphi(p^e)}.$$

Since the degree of $\mathbb{Q}(\omega)$ is $\varphi(p^e)$, the fundamental equality implies that $\mathfrak{p}_\omega$ is prime (otherwise any prime factor has ramification index at least $\varphi(p^e)$) and that $f_p(\mathfrak{p}_\omega) = 1$. This proves the first two statements. For the last two statements, $1, \omega, \ldots, \omega^{\varphi(p^e)-1}$ is a basis for $\mathbb{Q}(\omega)/\mathbb{Q}$ since ω is a primitive element for $\mathbb{Q}(\omega)/\mathbb{Q}$. Now let $\omega_1, \ldots, \omega_{\varphi(p^e)}$ be the conjugates of ω with $\omega_1 = \omega$. Then

$$\Phi_{p^e}(x) = \prod_{1\le i\le\varphi(p^e)} (x-\omega_i).$$

Now Equation (5.4) and Proposition 5.2.1 (since $\mathbb{Q}(\omega)/\mathbb{Q}$ is Galois) give the first and last equalities in the following chain respectively:

$$d(1,\lambda,\ldots,\lambda^{\varphi(p^e)}) = \pm \prod_{\substack{1\le i,j\le\varphi(p^e)\\ i\ne j}} (\omega_i-\omega_j)^2 = \pm \prod_{1\le i\le\varphi(p^e)} \Phi'_{p^e}(\omega_i) = \pm\,\mathrm{N}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\Phi'_{p^e}(\omega)).$$

It remains to show $\mathrm{N}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\Phi'_{p^e}(\omega)) = \pm p^{p^{(e-1)(ep-e-1)}}$. To this end, Equation (7.2) implies

$$(x^{p^{e-1}}-1)\Phi_{p^e}(x) = x^{p^e}-1,$$

and differentiating gives

$$\left(p^{e-1}-1\right)x^{p^{e-1}-1}\Phi_{p^e}(x) + \left(x^{p^{e-1}}-1\right)\Phi'_{p^e}(x) = p^e x^{p^e-1}.$$

Now set $x = \omega$ and let $\xi = \omega^{p^{e-1}}$ to obtain

$$(\xi-1)\,\Phi'_{p^e}(\omega) = p^e\omega^{-1},$$

where ξ is a primitive p-th root of unity. As $\mathrm{N}_{\mathbb{Q}(\xi)/\mathbb{Q}}(1-\xi) = p$ from our previous work, we compute

$$\begin{aligned}
\mathrm{N}_{\mathbb{Q}(\omega)/\mathbb{Q}}(1-\xi) &= \prod_{k\in(\mathbb{Z}/p^e\mathbb{Z})^*}(1-\xi^k) \\
&= \omega^{p+2p+\cdots+(p^{e-1}-1)p}\left(\prod_{k\in(\mathbb{Z}/p\mathbb{Z})^*}(1-\xi^k)\right)^{p^{e-1}} \\
&= \omega^{\frac{p^n(p^{n-1}-1)}{2}}\left(\prod_{k\in(\mathbb{Z}/p\mathbb{Z})^*}(1-\xi^k)\right)^{p^{e-1}} \\
&= \left(\prod_{k\in(\mathbb{Z}/p\mathbb{Z})^*}(1-\xi^k)\right)^{p^{e-1}} \\
&= \mathrm{N}_{\mathbb{Q}(\xi)/\mathbb{Q}}(1-\xi)^{p^{e-1}} \\
&= p^{p^{e-1}},
\end{aligned}$$

where the first and second to last equalities follow by Proposition 5.2.1 since $\mathbb{Q}(\omega)/\mathbb{Q}$ and $\mathbb{Q}(\xi)/\mathbb{Q}$ are Galois. Thus $\mathrm{N}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\xi-1) = \pm p^{p^{e-1}}$. Our previous identity is equivalent to

$$\Phi'_{p^e}(\omega) = \frac{p^e\omega^{-1}}{(\xi-1)},$$

and multiplicativity of the norm together with Proposition 5.2.3 give

$$\mathrm{N}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\Phi'_{p^e}(\omega)) = \frac{p^{\varphi(p^e)e}\,\mathrm{N}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\omega^{-1})}{\mathrm{N}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\xi-1)} = \pm p^{\varphi(p^e)e-p^{e-1}}.$$

This completes the proof. □

Note that Lemma 7.2.1 says $\mathfrak{p}_\omega$ is totally ramified. With Lemma 7.2.1 we can prove $\mathcal{O}_\omega$ is monogenic in full generality:

Proposition 7.2.2. *Let $\mathbb{Q}(\omega)$ be the cyclotomic number field generated by a primitive n-th root of unity ω. Then $\mathbb{Q}(\omega)$ is monogenic where*

$$\mathcal{O}_\omega = \mathbb{Z}[\omega].$$

Proof. The claim is trivial when $n = 1$ so assume $n \geq 2$. We will now prove the claim when $n = p^e$ for a prime p and $e \geq 1$. By Lemma 7.2.1, $1, \omega, \ldots, \omega^{\varphi(p^e)-1}$ is a basis for $\mathbb{Q}(\omega)/\mathbb{Q}$ and

$$d_{\mathbb{Q}(\omega)/\mathbb{Q}}(1, \omega, \ldots, \omega^{\varphi(p^e)-1}) = \pm p^{\varphi(p^e)e-p^{e-1}}.$$

Then Lemma 5.2.2 implies

$$p^{\varphi(p^e)e-p^{e-1}}\mathcal{O}_\omega \subseteq \mathbb{Z}[\omega] \subseteq \mathcal{O}_\omega.$$

Moreover, $\mathbb{F}_{\mathfrak{p}_\omega} \cong \mathbb{F}_p$ since $\mathfrak{p}_\omega$ is a prime above p with $f_p(\mathfrak{p}_\omega) = 1$ by Lemma 7.2.1. Therefore $\mathcal{O}_\omega = \mathbb{Z} + \mathfrak{p}_\omega$ which implies

$$\mathcal{O}_\omega = \mathbb{Z}[\omega] + \mathfrak{p}_\omega.$$

Multiplying by $1-\omega$ gives $\mathfrak{p}_\omega = (1-\omega)\mathbb{Z}[\omega] + \mathfrak{p}_\omega^2$. Combining with the previous identity results in

$$\mathcal{O}_\omega = \mathbb{Z}[\omega] + \mathfrak{p}_\omega^2,$$

because $(1-\omega)\mathbb{Z}[\omega] \subseteq \mathbb{Z}[\omega]$. Iterating this procedure gives

$$\mathcal{O}_\omega = \mathbb{Z}[\omega] + \mathfrak{p}_\omega^t,$$

for any $t \geq 1$. Taking $t = \varphi(p^e)(\varphi(p^e)e - p^{e-1})$ shows that

$$\mathcal{O}_\omega = \mathbb{Z}[\omega] + p^{\varphi(p^e)e-p^{e-1}}\mathcal{O}_\omega = \mathbb{Z}[\omega],$$

because $p\mathcal{O}_\omega = \mathfrak{p}_\omega^{\varphi(p^e)}$ by Lemma 7.2.1 and $p^{\varphi(p^e)e-p^{e-1}}\mathcal{O}_\omega \subseteq \mathbb{Z}[\omega]$. This proves the claim in the case n is a prime power. For the general case, let $n = p_1^{e_1}\cdots p_r^{e_r}$ be the prime factorization of n. Then $\omega_i = \omega^{\frac{n}{p_i^{e_i}}}$ is a primitive $p_i^{e_i}$-th root of unity for $1 \leq i \leq r$ and $\omega = \omega_1 \cdots \omega_r$. This factorization of ω implies

$$\mathbb{Q}(\omega) = \mathbb{Q}(\omega_1)\cdots\mathbb{Q}(\omega_r),$$

and since $p_1^{e_1}, \ldots, p_r^{e_r}$ are pairwise relatively prime, we have

$$\mathbb{Q}(\omega_1)\cdots\mathbb{Q}(\omega_{i-1}) \cap \mathbb{Q}(\omega_i) = \mathbb{Q},$$

for all i (since the degree of $(\mathbb{Q}(\omega_1)\cdots\mathbb{Q}(\omega_{i-1}) \cap \mathbb{Q}(\omega_i))/\mathbb{Q}$ must divide both $p_1^{e_1}\cdots p_{i-1}^{e_{i-1}}$ and $p_i^{e_i}$ and thus is 1). This also implies $\mathbb{Q}(\omega_1)\cdots\mathbb{Q}(\omega_{i-1})$ and $\mathbb{Q}(\omega_i)$ are linearly disjoint over $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ for all i by Proposition C.2.2 because $\mathbb{Q}(\omega_1)\cdots\mathbb{Q}(\omega_{i-1})/\mathbb{Q}$ and $\mathbb{Q}(\omega_i)/\mathbb{Q}$ are both Galois. As the discriminants of $d_{\mathbb{Q}(\omega_1)/\mathbb{Q}}(1, \omega_1, \ldots, \omega_1^{\varphi(p_1^{e_1})-1}), \ldots, d_{\mathbb{Q}(\omega_r)/\mathbb{Q}}(1, \omega_r, \ldots, \omega_r^{\varphi(p_r^{e_r})-1})$ are pairwise relatively prime, successive applications of Proposition 5.2.7 shows that $1, \omega, \ldots, \omega^{\varphi(n)-1}$ is an integral basis for $\mathbb{Q}(\omega)$. This means

$$\mathcal{O}_\omega = \mathbb{Z}[\omega],$$

as desired. □

We can now leverage the Dedekind-Kummer theorem to prove how $p\mathcal{O}_\omega$ decomposes in $\mathcal{O}_\omega$ for any prime p:

Proposition 7.2.3. *Let $\mathbb{Q}(\omega)$ be the cyclotomic number field generated by a primitive n-th root of unity ω. For every prime p, let $e_p \geq 0$ and be such that $p^{e_p} \mid\mid n$ and $f_p \geq 1$ be the smallest positive integer such that*

$$p^{f_p} \equiv 1 \pmod{\frac{n}{p^{e_p}}}.$$

Then if $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the prime factors of $p\mathcal{O}_\omega$ we have

$$p\mathcal{O}_\omega = (\mathfrak{p}_1 \cdots \mathfrak{p}_r)^{\varphi(p^{e_p})},$$

and

$$f_p(\mathfrak{p}_1) = \cdots = f_p(\mathfrak{p}_r) = f_p.$$

Proof. Since $\mathbb{Q}(\omega)$ is monogenic by Proposition 7.2.2, the conductor of $\mathbb{Q}(\omega)$ relative to ω is $\mathcal{O}_\omega$. Therefore we may apply the Dedekind-Kummer theorem to every prime p and as $\Phi_n(x)$ is the minimal polynomial for ω over $\mathbb{Q}$, we simply have to show the prime factorization

$$\overline{\Phi_n}(x) = (\overline{m_1}(x) \cdots \overline{m_r}(x))^{\varphi(p^{e_p})},$$

in $\mathbb{F}_p[x]$ for distinct irreducibles $\overline{m_1}(x), \ldots, \overline{m_r}(x)$ of degree f_p. To this end, let $n = p^e m$ and let ξ and η be primitive p^e-th and m-th roots of unity respectively. Then $\Phi_n(x)$ can be expressed as

$$\Phi_n(x) = \prod_{\substack{k \in (\mathbb{Z}/p^e\mathbb{Z})^* \\ \ell \in (\mathbb{Z}/m\mathbb{Z})^*}} (x - \xi^k \eta^\ell).$$

Recall that

$$(x-1)^{p^{e_p}} \equiv x^{p^{e_p}} - 1 \pmod{p}.$$

Taking $x = \xi$ gives $(\xi - 1)^{p^{e_p}} \equiv 0 \pmod{p}$ and thus $\xi \equiv 1 \pmod{p}$. Then from our expression of $\Phi_n(x)$, we find that

$$\Phi_n(x) \equiv \prod_{1 \leq \ell \leq (\mathbb{Z}/m\mathbb{Z})^*} (1 - \eta^\ell)^{\varphi(p^e)} \equiv \Phi_m(x)^{\varphi(p^{e_p})} \pmod{p}.$$

This is to say

$$\overline{\Phi_n}(x) = \overline{\Phi_m}(x)^{\varphi(p^{e_p})}.$$

Therefore it suffices to show

$$\overline{\Phi_m}(x) = \overline{m_1}(x) \cdots \overline{m_r}(x),$$

and that $\overline{m_1}(x), \ldots, \overline{m_r}(x)$ are all of degree f_p. Now let $\mathfrak{p}$ be a prime above p. Then $\mathbb{F}_\mathfrak{p}$ is an $\mathbb{F}_p$-vector space and therefore has characteristic p. It follows that $x^m - 1$ has m distinct roots in in $\overline{\mathbb{F}_p}$ since p does not divide m (recall that the derivative of $x^m - 1$ is mx^{m-1} which is relatively prime to p). Therefore $x^m - 1$ does not have a multiple root in $\mathbb{F}_\mathfrak{p}$. As $\mathcal{O}_\omega$ contains all the m-th roots of unity by Proposition 7.2.2 and that m divides n, it follows that $\mathbb{F}_\mathfrak{p}$ does as well so the surjective homomorphism

$$\pi : \mathcal{O}_\omega \to \mathcal{O}_\omega/\mathfrak{p} \qquad \alpha \mapsto \alpha \pmod{\mathfrak{p}},$$

maps the primitive m-th roots of unity onto themselves bijectively. This implies that the roots of $\overline{\Phi_m}(x)$ are exactly the primitive m-th roots of unity. Now the smallest extension of $\mathbb{F}_p$ containing all the primitive m-th roots of unity is $\mathbb{F}_{p^{f_p}}$ because its multiplicative group $\mathbb{F}_{p^{f_p}}^*$ is cyclic (as the multiplicative group of any finite field is cyclic), of order $p^{f_p} - 1$ dividing m by assumption, and with f_p minimal. As $\overline{\Phi_m}(x)$ divides $x^n - 1$ in $\mathbb{F}_p[x]$ (because $\Phi_m(x)$ divides $x^m - 1$ in $\mathbb{Z}[x]$), $\overline{\Phi_m}(x)$ has no multiple roots in $\mathbb{F}_p$. Therefore it factors as

$$\overline{\Phi_m}(x) = \overline{m_1}(x) \cdots \overline{m_r}(x),$$

in $\mathbb{F}_p[x]$ for distinct irreducibles $\overline{m_1}(x), \ldots, \overline{m_r}(x)$. These are also necessarily monic because $\Phi_m(x)$, and hence $\overline{\Phi_m}(x)$, is. Moreover, as the roots of $\overline{\Phi_m}(x)$ are the primitive m-th roots of unity, each factor $\overline{m_i}(x)$ is the minimal polynomial of a primitive m-th root of unity in $\mathbb{F}_{p^{f_p}}$ for $1 \leq i \leq r$. The degree of this minimal polynomial is necessarily the degree of $\mathbb{F}_{p^{f_p}}/\mathbb{F}_p$ which is f_p. Therefore $\overline{m_1}(x), \ldots, \overline{m_r}(x)$ are all of degree f_p completing the proof. □

Chapter 8

L-functions of Number Fields

We introduce the L-functions attached to number fields known as Dedekind zeta functions. From the analytic properties of Dedekind zeta functions, we prove the infamous analytic class number formula and the Dirichlet class number formula. We then use the Dirichlet class number formula to deduce estimates for the class number of quadratic number fields.

8.1 Dedekind Zeta Functions

The Definition and Euler Product

We can associate and L-function to every number field. Let K be a number field of degree d and signature (r_1, r_2). The **Dedekind zeta function** $\zeta_K(s)$ is defined by the following Dirichlet series

$$\zeta_K(s) = \sum_{n \geq 1} \frac{a_K(n)}{n^s}.$$

We say $\zeta_K(s)$ is **quadratic**, **cubic**, etc. if K is. Similarly, we say $\zeta_K(s)$ is **cyclotomic** if K is.

Remark 8.1.1. *If we sum over integral ideals of K instead, we can write*

$$\zeta_K(s) = \sum_{\mathfrak{a} \text{ integral}} \frac{1}{\mathrm{N}_K(\mathfrak{a})^s}.$$

We will see that $\zeta_K(s)$ is a Selberg class L-function (note that $\zeta_K(s) = \zeta(s)$ when $d = 1$). We have already seen that the coefficients $a_K(n)$ are multiplicative and satisfy $a_K(n) \ll_\varepsilon n^\varepsilon$. By Proposition 2.2.1, $\zeta_K(s)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits the following infinite product expression:

$$\zeta_K(s) = \prod_p \left(\sum_{n \geq 0} \frac{a_K(p^n)}{p^{hs}} \right).$$

Recall that each prime $\mathfrak{p}$ lies above a prime p and $\mathrm{N}_K(\mathfrak{p})$ is a power of p. As the norm is multiplicative, the prime factorization of fractional ideals implies that our infinite product can be expressed as

$$\zeta_K(s) = \prod_p \prod_{\mathfrak{p} \text{ above } p} (1 - \mathrm{N}_K(\mathfrak{p}) p^{-s})^{-1} = \prod_p \prod_{1 \leq i \leq r_p} (1 - p^{-f_p(\mathfrak{p}_i)s})^{-1},$$

where r_p is the number of primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_{r_p}$ above p. As finitely many primes ramify (because only those dividing $|\Delta_K|$ do), the fundamental equality implies that all but finitely many primes p satisfy

$$d = \sum_{1 \leq i \leq r_p} f_p(\mathfrak{p}_i).$$

In particular, at least one prime does. Therefore, letting $\omega_{p,i}$ be a primitive $f_p(\mathfrak{p}_i)$-th root of unity, we have the following degree d Euler product:

$$\zeta_K(s) = \prod_p \prod_{\substack{1 \leq i \leq r_p \\ 1 \leq k \leq f_p(\mathfrak{p}_i)}} (1 - \omega_{p,i}^k p^{-s})^{-1}.$$

The local factor at p is

$$\zeta_{K,p}(s) = \prod_{\substack{1 \leq i \leq r_p \\ 1 \leq k \leq f_p(\mathfrak{p}_i)}} (1 - \omega_{p,i}^k p^{-s})^{-1},$$

with local roots $\omega_{p,i}^k$ and $\sum_{1 \leq i \leq r_p} (e_p(\mathfrak{p}_i) - 1) f_p(\mathfrak{p}_i)$ many local roots 0.

The Integral Representation: Part I

We will find an integral representation for $\zeta_K(s)$. To do this, let σ run over $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$ and let τ run over a complete set of representatives for the real and pairs of complex $\mathbb{Q}$-embeddings. Also recall that $N_\tau = 1, 2$ depending upon if τ is real or complex. It will be useful to let

$$\mathbf{N} = (N_\tau)_\tau, \quad \mathbf{N}^\vee = (3 - N_\tau)_\tau, \quad \boldsymbol{\alpha} = (|\tau(\alpha)|)_\tau, \quad \text{and} \quad \boldsymbol{j(\alpha)} = (|\sigma(\alpha)|)_\sigma,$$

for all $\alpha \in K$. With this notation,

$$\boldsymbol{\alpha}^{\mathbf{N}} = |\,\mathrm{N}_K(\alpha)|.$$

Now consider the function

$$\omega_K(\mathbf{z}) = \sum_{n \geq 1} a_K(n) e^{\pi i n^{\frac{2}{d}} \langle \mathbf{z}, \mathbf{N} \rangle},$$

defined for $\mathbf{z} \in \mathbb{H}^{r_K+1}$. It is locally absolutely uniformly convergent in this region by the Weierstrass M-test. Taking $\varepsilon < 1$, the bound $a_K(n) \ll_\varepsilon n^\varepsilon$ gives

$$\omega_K(\mathbf{z}) = O\left(\sum_{n \geq 1} n^{r_2+1} e^{-\pi n^{\frac{2}{d}} ||\mathbf{y}||_\infty} \right) = O(e^{-\pi ||\mathbf{y}||_\infty}),$$

where the second equality holds because each term is of smaller order than the next so that the series is bounded by a constant times the first term. Hence $\omega_K(\mathbf{z})$ exhibits exponential decay. Now consider the following Mellin transform:

$$\int_{\mathbb{R}_+^{r_K+1}} \omega_K(i\mathbf{y}) \mathbf{y}^{\frac{s\mathbf{N}}{2}} \frac{d\mathbf{y}}{\mathbf{y}}.$$

By the exponential decay of $\omega_K(\mathbf{z})$, this integral is locally absolutely uniformly convergent for $\sigma > 1$ and hence defines an analytic function there. Then we compute

$$\begin{aligned}
\int_{\mathbb{R}_+^{r_K+1}} \omega_K(i\mathbf{y})\mathbf{y}^{\frac{s\mathbf{N}}{2}}\,\frac{d\mathbf{y}}{\mathbf{y}} &= \int_{\mathbb{R}_+^{r_K+1}} \sum_{n\geq 1} a_K(n)e^{-\pi n^{\frac{2}{d}}\langle \mathbf{y},\mathbf{N}\rangle}\mathbf{y}^{\frac{s\mathbf{N}}{2}}\,\frac{d\mathbf{y}}{\mathbf{y}} \\
&= \sum_{n\geq 1}\int_{\mathbb{R}_+^{r_K+1}} a_K(n)e^{-\pi n^{\frac{2}{d}}\langle \mathbf{y},\mathbf{N}\rangle}\mathbf{y}^{\frac{s\mathbf{N}}{2}}\,\frac{d\mathbf{y}}{\mathbf{y}} && \text{FTT} \\
&= \sum_{n\geq 1}\frac{a_K(n)}{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}n^s}\int_{\mathbb{R}_+^{r_K+1}} e^{-\langle \mathbf{y},\mathbf{1}\rangle}\mathbf{y}^{\frac{s\mathbf{N}}{2}}\,\frac{d\mathbf{y}}{\mathbf{y}} && \mathbf{y}\mapsto\frac{\mathbf{y}}{\mathbf{N}\pi n^{\frac{2}{d}}} \\
&= \frac{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}}\sum_{n\geq 1}\frac{a_K(n)}{n^s} \\
&= \frac{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}}\zeta_K(s).
\end{aligned}$$

Therefore we have an integral representation

$$\zeta_K(s) = \frac{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}}{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}\int_{\mathbb{R}_+^{r_K+1}} \omega_K(i\mathbf{y})\mathbf{y}^{\frac{s\mathbf{N}}{2}}\,\frac{d\mathbf{y}}{\mathbf{y}}.$$

We will manipulate the remaining integral to write it as a different Mellin transform. This will take a fair amount of work. As $\omega_K(\mathbf{z})$ converges absolutely we may sum over integral ideals of K and obtain

$$\int_{\mathbb{R}_+^{r_K+1}} \sum_{\mathfrak{a}\text{ integral}} e^{\pi i\,\mathrm{N}_K(\mathfrak{a})^{\frac{2}{d}}\langle i\mathbf{y},\mathbf{N}\rangle}\mathbf{y}^{\frac{s\mathbf{N}}{2}}\,\frac{d\mathbf{y}}{\mathbf{y}}.$$

Now if $\mathfrak{a}$ is an integral ideal then $\mathcal{O}_K^*$ acts on $\mathfrak{a}-\{0\}$ by multiplication. Therefore we have an orbit space $(\mathfrak{a}-\{0\})/\mathcal{O}_K^*$. The following lemma establishes a bijection between this orbit space and integral ideals in an ideal class:

Lemma 8.1.1. *Let K be a number field. Suppose $\mathfrak{a}$ is an integral ideal and $\mathfrak{K}$ is the ideal class of $\mathfrak{a}^{-1}$. Then there is a bijection*

$$(\mathfrak{a}-\{0\})/\mathcal{O}_K^* \to \{\mathfrak{b}\in\mathfrak{K} : \mathfrak{b}\text{ is an integral ideal}\} \qquad \alpha\mapsto\alpha\mathfrak{a}^{-1}.$$

Proof. We first show that this map is well-defined. If $\alpha\in\mathfrak{a}$ is nonzero then $\alpha\mathfrak{a}^{-1} = \alpha\mathcal{O}_K\mathfrak{a}^{-1}\subseteq\mathcal{O}_K$ by Proposition 5.3.2 so that $\alpha\mathfrak{a}^{-1}$ is an integral ideal belonging to $\mathfrak{K}$ (because it differs from $\mathfrak{a}^{-1}$ by a principal ideal). Therefore the map is well-defined. To show injectivity, suppose $\alpha\mathfrak{a}^{-1} = \beta\mathfrak{a}^{-1}$ for some $\alpha,\beta\in(\mathfrak{a}-\{0\})/\mathcal{O}_K^*$. Multiplying by $\mathfrak{a}$, we obtain $\alpha\mathcal{O}_K = \beta\mathcal{O}_K$ and hence $\alpha\beta^{-1}\in\mathcal{O}_K^*$ from which injectivity follows. For surjectivity, the definition of $\mathfrak{K}$ implies that any integral ideal $\mathfrak{b}\in\mathfrak{K}$ satisfies $\mathfrak{b} = \alpha\mathfrak{a}^{-1}$ for some nonzero $\alpha\in\mathcal{O}_K$ but then $\alpha\in\mathfrak{a}\mathfrak{b}\subseteq\mathfrak{a}$. □

By Minkowski theorem, any ideal class is can be represented by an integral ideal $\mathfrak{a}$. So let $\mathfrak{a}_1,\ldots,\mathfrak{a}_{h_K}$ be a complete set of representatives of $\mathrm{Cl}(K)$ consisting only of integral ideals. As the ideal class group is a group, we see that $\mathfrak{a}_1^{-1},\ldots,\mathfrak{a}_{h_K}^{-1}$ is also a complete set of representatives. By Lemma 8.1.1, this shows that every integral ideal is uniquely of the form $\alpha\mathfrak{a}_i^{-1}$ for some $1\leq i\leq h_K$ and $\alpha\in\mathfrak{a}_i^*/\mathcal{O}_K^*$ and thus has norm $\mathrm{N}_K(\alpha\mathfrak{a}_i^{-1})$. Then we may further rewrite our integral as

$$\sum_{\substack{\mathfrak{a}\in\mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}}\sum_{\alpha\in(\mathfrak{a}-\{0\})/\mathcal{O}_K^*}\int_{\mathbb{R}_+^{r_K+1}} e^{\pi i\,\mathrm{N}_K(\alpha\mathfrak{a}^{-1})^{\frac{2}{d}}\langle i\mathbf{y},\mathbf{N}\rangle}\mathbf{y}^{\frac{s\mathbf{N}}{2}}\,\frac{d\mathbf{y}}{\mathbf{y}},$$

where we have interchange the two sums and integral by the Fubini–Tonelli theorem. Now perform the change of variables $\mathbf{y} \mapsto \boldsymbol{\alpha}^2\mathbf{y}$ to obtain

$$\sum_{\substack{\mathfrak{a}\in \mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}} \sum_{\alpha\in(\mathfrak{a}-\{0\})/\mathcal{O}_K^*} \int_{\mathbb{R}_+^{r_K+1}} e^{\pi i\, \mathrm{N}_K(\alpha\mathfrak{a}^{-1})^{\frac{2}{d}}\langle \boldsymbol{\alpha}^2 i\mathbf{y},\mathbf{N}\rangle} (\boldsymbol{\alpha}^2\mathbf{y})^{\frac{s\mathbf{N}}{2}}\, \frac{d\mathbf{y}}{\mathbf{y}}.$$

We will now alter the region of integration. Observe that any $\mathbf{y}\in\mathbb{R}_+^{r_K+1}$ can be written in the form

$$\mathbf{y} = y^{\frac{1}{d}}\mathbf{x},$$

where

$$y = \mathrm{N}_{\mathbb{R}^{r_K+1}/\mathbb{R}}(\mathbf{y})^{\frac{d}{r_K+1}} \quad \text{and} \quad \mathbf{x} = \frac{\mathbf{y}}{\mathrm{N}_{\mathbb{R}^{r_K+1}/\mathbb{R}}(\mathbf{y})^{\frac{1}{r_K+1}}}.$$

Therefore $\mathbb{R}_+^{r_K+1}$ decomposes as

$$\mathbb{R}_+^{r_K+1} = (0,\infty)\times S,$$

where S is the norm-one hypersurface of $\mathbb{R}^{r_K+1}$. As S is a multiplicative subgroup of $\mathbb{R}_+^{r_K+1}$, the measure $\frac{d\mathbf{y}}{\mathbf{y}}$ on $\mathbb{R}_+^{r_K+1}$ factors as

$$\frac{d\mathbf{y}}{\mathbf{y}} = \frac{d\mathbf{x}\, dy}{y},$$

where $d\mathbf{x}$ (which can also be written as $\frac{d\mathbf{x}}{\mathbf{x}}$ because $\mathbf{x}\in S$) is the restriction of the measure $\frac{d\mathbf{y}}{\mathbf{y}}$ to S. Therefore our expression takes the form

$$\sum_{\substack{\mathfrak{a}\in \mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}} \sum_{\alpha\in(\mathfrak{a}-\{0\})/\mathcal{O}_K^*} \int_0^\infty \int_S e^{\pi i\, \mathrm{N}_K(\alpha\mathfrak{a}^{-1})^{\frac{2}{d}}\left\langle \boldsymbol{\alpha}^2 i y^{\frac{1}{d}}\mathbf{x},\mathbf{N}\right\rangle} \boldsymbol{\alpha}^{s\mathbf{N}} y^{\frac{s}{2}}\, \frac{d\mathbf{x}\, dy}{y}.$$

Now consider the inverse isomorphisms

$$\log : \mathbb{R}_+^{r_K+1} \to \mathbb{R}^{r_K+1} \qquad \text{and} \quad e : \mathbb{R}^{r_K+1} \to \mathbb{R}_+^{r_K+1}.$$

In particular, they restricts to inverse bijections between the norm-one hypersurface S and the trace-zero hyperplane H ($\log(S) = H$ and $e^H = S$). By Dirichlet's unit theorem, $\mathbf{N}^\vee\Lambda$ is a complete lattice in H and therefore has fundamental domain $\mathbf{N}^\vee\mathcal{M}$ where $\mathcal{M}$ is a fundamental domain for Λ. Then

$$H = \bigcup_{\lambda\in\Lambda} (\mathbf{N}^\vee\mathcal{M} + \mathbf{N}^\vee\lambda).$$

Setting $\mathcal{F} = e^{\mathbf{N}^\vee\mathcal{M}}$, we see that $\mathcal{F}$ is a fundamental domain for the properly discontinuous action of Λ on S by multiplication by $e^{\mathbf{N}^\vee\lambda}$ for all $\lambda\in\Lambda$. In particular,

$$S = \bigcup_{\lambda\in\Lambda} e^{\mathbf{N}^\vee\lambda}\mathcal{F}.$$

By Proposition 6.3.1, we may express this decomposition as

$$S = \bigcup_{\varepsilon\in\mathcal{O}_K^*/\mu(K)} e^{\mathbf{N}^\vee\lambda(\varepsilon)}\mathcal{F}.$$

Then folding yields

$$\frac{1}{w_K}\sum_{\substack{\mathfrak{a}\in\mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}}\sum_{\alpha\in(\mathfrak{a}-\{0\})/\mathcal{O}_K^*}\sum_{\varepsilon\in\mathcal{O}_K^*}\int_0^\infty\int_{\mathcal{F}} e^{\pi i\,\mathrm{N}_K(\alpha\mathfrak{a}^{-1})^{\frac{2}{d}}\left\langle \boldsymbol{\alpha}^2 e^{\mathbf{N}^\vee\lambda(\varepsilon)} iy^{\frac{1}{d}}\mathbf{x},\mathbf{N}\right\rangle}\boldsymbol{\alpha}^{s\mathbf{N}}y^{\frac{s}{2}}\,\frac{d\mathbf{x}\,dy}{y},$$

because $e^{\mathbf{N}^\vee\lambda(\varepsilon)}=\mathbf{1}$ if $\varepsilon\in\mu(K)$ and we have interchanged the sum and integrals by the Fubini–Tonelli theorem. As $\boldsymbol{\varepsilon}^{\mathbf{N}}=|\,\mathrm{N}_K(\varepsilon)|=1$ and $\mathrm{N}_K(\varepsilon)=\pm1$ for all $\varepsilon\in\mathcal{O}_K^*$, we can use multiplicativity of the norm and Proposition 5.8.1 to write our expression as

$$\frac{1}{w_K}\sum_{\substack{\mathfrak{a}\in\mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}}\sum_{\alpha\in(\mathfrak{a}-\{0\})/\mathcal{O}_K^*}\sum_{\varepsilon\in\mathcal{O}_K^*}\int_0^\infty\int_{\mathcal{F}} e^{\pi i\,\mathrm{N}_K(\alpha\varepsilon\mathfrak{a}^{-1})^{\frac{2}{d}}\left\langle \boldsymbol{\alpha}^2 e^{\mathbf{N}\lambda(\varepsilon)} iy^{\frac{1}{d}}\mathbf{x},\mathbf{N}\right\rangle}(\boldsymbol{\alpha}\boldsymbol{\varepsilon})^{s\mathbf{N}}y^{\frac{s}{2}}\,\frac{d\mathbf{x}\,dy}{y}.$$

By the definitions of λ and $\mathbf{N}^\vee$, we have $\boldsymbol{\alpha}^2e^{\mathbf{N}\lambda(\varepsilon)}=(\boldsymbol{\alpha}\boldsymbol{\varepsilon})^2$ and therefore we may combine the innermost two sums to obtain

$$\frac{1}{w_K}\sum_{\substack{\mathfrak{a}\in\mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}}\sum_{\alpha\in\mathfrak{a}-\{0\}}\int_0^\infty\int_{\mathcal{F}} e^{\pi i\,\mathrm{N}_K(\alpha\mathfrak{a}^{-1})^{\frac{2}{d}}\left\langle \boldsymbol{\alpha}^2 iy^{\frac{1}{d}}\mathbf{x},\mathbf{N}\right\rangle}\boldsymbol{\alpha}^{s\mathbf{N}}y^{\frac{s}{2}}\,\frac{d\mathbf{x}\,dy}{y}.$$

Performing the change of variables $y\mapsto\frac{y}{\boldsymbol{\alpha}^{2\mathbf{N}}}$ gives

$$\frac{1}{w_K}\sum_{\substack{\mathfrak{a}\in\mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}}\sum_{\alpha\in\mathfrak{a}-\{0\}}\int_0^\infty\int_{\mathcal{F}} e^{\pi i\,\mathrm{N}_K(\mathfrak{a}^{-1})^{\frac{2}{d}}\left\langle \boldsymbol{\alpha}^2 iy^{\frac{1}{d}}\mathbf{x},\mathbf{N}\right\rangle}y^{\frac{s}{2}}\,\frac{d\mathbf{x}\,dy}{y},$$

Interchanging the the two integrals with the two sums by the Fubini–Tonelli theorem and noting that $\langle\mathbf{N},\boldsymbol{\alpha}^2\rangle=\langle\boldsymbol{j}(\boldsymbol{\alpha}),\boldsymbol{j}(\boldsymbol{\alpha})\rangle$, our expression becomes

$$\int_0^\infty\int_{\mathcal{F}}\frac{1}{w_K}\sum_{\substack{\mathfrak{a}\in\mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}}\sum_{j(\alpha)\in j(\mathfrak{a})-\{\mathbf{0}\}} e^{\pi i\,\mathrm{N}_K(\mathfrak{a}^{-1})^{\frac{2}{d}}\left\langle iy^{\frac{1}{d}}\mathbf{x}\boldsymbol{j}(\boldsymbol{\alpha}),\boldsymbol{j}(\boldsymbol{\alpha})\right\rangle}y^{\frac{s}{2}}\,\frac{d\mathbf{x}\,dy}{y}.$$

We compactly express this as

$$\int_0^\infty\omega_{\mathcal{F}}\left(iy^{\frac{1}{d}}\right)y^{\frac{s}{2}}\,\frac{dy}{y},$$

where we have defined $\omega_{\mathcal{F}}(z)$ by

$$\omega_{\mathcal{F}}(z)=\int_{\mathcal{F}}\frac{1}{w_K}\sum_{\substack{\mathfrak{a}\in\mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}}\sum_{j(\alpha)\in j(\mathfrak{a})-\{\mathbf{0}\}} e^{\pi i\,\mathrm{N}_K(\mathfrak{a}^{-1})^{\frac{2}{d}}\langle z\mathbf{x}\boldsymbol{j}(\boldsymbol{\alpha}),\boldsymbol{j}(\boldsymbol{\alpha})\rangle}\,d\mathbf{x},$$

for $z\in\mathbb{H}$. It is locally absolutely uniformly convergent in this region by the Weierstrass M-test and that $\mathcal{F}$ is compact (because $\mathbf{N}^\vee\mathcal{M}$ is and $\log(\mathbf{x})$ is continuous). At last, this gives a final integral representation

$$\zeta_K(s)=\frac{\pi^{\frac{r_1s}{2}}(2\pi)^{r_2s}}{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}\int_0^\infty\omega_{\mathcal{F}}\left(iy^{\frac{1}{d}}\right)y^{\frac{s}{2}}\,\frac{dy}{y},\tag{8.1}$$

where the integral is a Mellin transform. We cannot proceed until we obtain a functional equation for $\omega_{\mathcal{F}}(z)$. So we will make a detour and come back to the integral representation after.

The Hecke Theta Function

The **Hecke theta function** $\vartheta_{\mathfrak{f}}(z\mathbf{x})$ attached to a fractional ideal $\mathfrak{f}$ of K, is defined by

$$\vartheta_{\mathfrak{f}}(z\mathbf{x}) = \sum_{j(\alpha)\in j(\mathfrak{f})} e^{2\pi i \,\mathrm{N}_K(\mathfrak{f}^{-1})^{\frac{2}{d}}\langle z\mathbf{x}\boldsymbol{j(\alpha)},\boldsymbol{j(\alpha)}\rangle},$$

for $z \in \mathbb{H}$ and $\mathbf{x} \in S$. It is locally absolutely uniformly convergent in this region by the Weierstrass M-test. Moreover,

$$\vartheta_{\mathfrak{f}}(z\mathbf{x}) - 1 = O\left(\sum_{j(\alpha)\in j(\mathfrak{f})-\{\mathbf{0}\}} e^{-2\pi y||\mathbf{x}||_\infty \,\mathrm{N}_K(\mathfrak{f}^{-1})^{\frac{2}{d}}\langle \boldsymbol{j(\alpha)},\boldsymbol{j(\alpha)}\rangle}\right) = O(e^{-2\pi y||\mathbf{x}||_\infty}),$$

where the second equality follows since $\langle \boldsymbol{j(\alpha)}, \boldsymbol{j(\alpha)}\rangle \geq d|\,\mathrm{N}_K(\alpha)|^{\frac{2}{d}}$ by the arithmetic-geometric inequality provided α is nonzero, and that $|\,\mathrm{N}_K(\alpha)|\,\mathrm{N}_K(\mathfrak{f})$ is a positive integer for all nonzero $\alpha \in \mathfrak{f}$ (as $\alpha\mathfrak{f}^{-1}$ is an integral ideal by Proposition 5.3.2), which combined show that each term is of smaller order than the single term where $\mathrm{N}_K(\alpha\mathfrak{f}^{-1}) = 1$ (when $\alpha\mathfrak{f}^{-1} = \mathcal{O}_K$) corresponding to $\alpha = \mathrm{N}_K(\mathfrak{f})$ so that the sum is bounded by a constant times this term. The relationship to $\omega_{\mathcal{F}}(z)$ is given by

$$\omega_{\mathcal{F}}(z) = \frac{1}{w_K}\int_{\mathcal{F}} \sum_{\substack{\mathfrak{a}\in \mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}} \vartheta_{\mathfrak{a}}\left(\frac{z\mathbf{x}}{2}\right) d\mathbf{x} - \frac{h_K}{w_K}\int_{\mathcal{F}} d\mathbf{x}.$$

The essential fact we will need is a functional equation for the Hecke theta function:

Theorem 8.1.1. *Let K be a number field of degree d and let $\mathfrak{f}$ be a fractional ideal. For any $z \in \mathbb{H}$ and $\mathbf{x} \in S$,*

$$\vartheta_{\mathfrak{f}}(z\mathbf{x}) = \frac{1}{(-2|\Delta_K|^{\frac{1}{d}}iz)^{\frac{d}{2}}}\vartheta_{\mathfrak{f}^\vee}\left(-\frac{\mathbf{x}}{4|\Delta_K|^{\frac{2}{d}}z}\right).$$

Proof. We will apply the Poisson summation formula to

$$\vartheta_{\mathfrak{f}}(z\mathbf{x}) = \sum_{j(\alpha)\in j(\mathfrak{f})} e^{2\pi i \,\mathrm{N}_K(\mathfrak{f}^{-1})^{\frac{2}{d}}\langle z\mathbf{x}\boldsymbol{j(\alpha)},\boldsymbol{j(\alpha)}\rangle}.$$

To do this, we compute the Fourier transform of the summand and by the identity theorem it suffices to verify this for $z = iy$ with $y > 0$ and $\mathbf{x} = \mathbf{1}$. So set

$$f(\mathbf{x}) = e^{-2\pi y \,\mathrm{N}_K(\mathfrak{f}^{-1})^{\frac{2}{d}}\langle \mathbf{x},\mathbf{x}\rangle}.$$

Then $f(\mathbf{x})$ is of Schwarz class. By Corollary 1.6.2 and multiplicativity of the norm, we have

$$(\mathcal{F}f)(\mathbf{t}) = \mathrm{N}_K(\mathfrak{f})\frac{e^{-\frac{\pi\,\mathrm{N}_K(\mathfrak{f})^{\frac{2}{d}}\langle \mathbf{t},\mathbf{t}\rangle}{2y}}}{(2y)^{\frac{d}{2}}}.$$

By the Poisson summation formula and the identity theorem, we compute

$$\begin{aligned}
\vartheta_{\mathfrak{f}}(z\mathbf{x}) &= \sum_{j(\alpha)\in j(\mathfrak{f})} e^{2\pi i\, \mathrm{N}_K(\mathfrak{f}^{-1})^{\frac{2}{d}}\langle z\mathbf{x}\boldsymbol{j}(\boldsymbol{\alpha}),\boldsymbol{j}(\boldsymbol{\alpha})\rangle} \\
&= \frac{1}{V_{j(\mathfrak{f})}} \sum_{\alpha^\vee\in\mathfrak{f}^\vee} \mathrm{N}_K(\mathfrak{f}) \frac{e^{-\frac{\pi\,\mathrm{N}_K(\mathfrak{f})^{\frac{2}{d}}\langle z\mathbf{x}\boldsymbol{j}(\boldsymbol{\alpha}^\vee),\boldsymbol{j}(\boldsymbol{\alpha}^\vee)\rangle}{-2iz}}}{(-2iz)^{\frac{d}{2}}} \\
&= \frac{1}{\mathrm{N}_K(\mathfrak{f})\sqrt{|\Delta_K|}} \sum_{\alpha^\vee\in\mathfrak{f}^\vee} \mathrm{N}_K(\mathfrak{f}) \frac{e^{-\frac{\pi\,\mathrm{N}_K(\mathfrak{f})^{\frac{2}{d}}\langle z\mathbf{x}\boldsymbol{j}(\boldsymbol{\alpha}^\vee),\boldsymbol{j}(\boldsymbol{\alpha}^\vee)\rangle}{-2iz}}}{(-2iz)^{\frac{d}{2}}} && \text{Proposition 6.1.2} \\
&= \frac{1}{(-2|\Delta_K|^{\frac{1}{d}}iz)^{\frac{d}{2}}} \sum_{\alpha^\vee\in\mathfrak{f}^\vee} e^{-\frac{\pi\,\mathrm{N}_K(\mathfrak{f})^{\frac{2}{d}}\langle z\mathbf{x}\boldsymbol{j}(\boldsymbol{\alpha}^\vee),\boldsymbol{j}(\boldsymbol{\alpha}^\vee)\rangle}{-2iz}} \\
&= \frac{1}{(-2|\Delta_K|^{\frac{1}{d}}iz)^{\frac{d}{2}}} \sum_{\alpha^\vee\in\mathfrak{f}^\vee} e^{-\frac{\pi\left(\mathrm{N}\left((\mathfrak{f}^\vee)^{-1}\right)|\Delta|_K^{-1}\right)^{\frac{2}{d}}\langle z\mathbf{x}\boldsymbol{j}(\boldsymbol{\alpha}^\vee),\boldsymbol{j}(\boldsymbol{\alpha}^\vee)\rangle}{-2iz}} && \text{Corollary 5.8.1} \\
&= \frac{1}{(-2|\Delta_K|^{\frac{1}{d}}iz)^{\frac{d}{2}}} \sum_{\alpha^\vee\in\mathfrak{f}^\vee} e^{2\pi i\,\mathrm{N}\left((\mathfrak{f}^\vee)^{-1}\right)^{\frac{2}{d}}\left\langle\left(-\frac{\mathbf{x}}{4|\Delta_K|^{\frac{2}{d}}z}\right)\boldsymbol{j}(\boldsymbol{\alpha}),\boldsymbol{j}(\boldsymbol{\alpha})\right\rangle} \\
&= \frac{1}{(-2|\Delta_K|^{\frac{1}{d}}iz)^{\frac{d}{2}}} \vartheta_{\mathfrak{f}^\vee}\left(-\frac{\mathbf{x}}{4|\Delta_K|^{\frac{2}{d}}z}\right).
\end{aligned}$$

□

We will use Theorem 8.1.1 to analytically continue $\zeta_K(s)$.

The Integral Representation: Part II

Returning to $\zeta_K(s)$, split the integral in Equation (8.1) into two pieces by writing

$$\int_0^\infty \omega_{\mathcal{F}}\left(iy^{\frac{1}{d}}\right) y^{\frac{s}{2}}\,\frac{dy}{y} = \int_0^{\frac{1}{|\Delta_K|}} \omega_{\mathcal{F}}\left(iy^{\frac{1}{d}}\right) y^{\frac{s}{2}}\,\frac{dy}{y} + \int_{\frac{1}{|\Delta_K|}}^\infty \omega_{\mathcal{F}}\left(iy^{\frac{1}{d}}\right) y^{\frac{s}{2}}\,\frac{dy}{y}. \tag{8.2}$$

We will rewrite the first piece in the same form and symmetrize the result as much as possible. Perform the change of variables $y \mapsto \frac{1}{|\Delta_K|^2 y}$ to the first piece to obtain

$$\int_{\frac{1}{|\Delta_K|}}^\infty \omega_{\mathcal{F}}\left(\frac{i}{(|\Delta_K|^2 y)^{\frac{1}{d}}}\right)(|\Delta_K|^2 y)^{-\frac{s}{2}}\,\frac{dy}{y}.$$

Now we compute

$$\begin{aligned}
\omega_{\mathcal{F}}\left(\frac{i}{(|\Delta_K|^2 y)^{\frac{1}{d}}}\right) &= \omega_{\mathcal{F}}\left(-\frac{1}{|\Delta_K|^{\frac{2}{d}} i y^{\frac{1}{d}}}\right) \\
&= \frac{1}{w_K}\int_{\mathcal{F}} \sum_{\substack{\mathfrak{a}\in \mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}} \vartheta_{\mathfrak{a}}\left(-\frac{\mathbf{x}}{2|\Delta_K|^{\frac{2}{d}} i y^{\frac{1}{d}}}\right) d\mathbf{x} - \frac{h_K}{w_K}\int_{\mathcal{F}} d\mathbf{x} \\
&= \frac{1}{w_K}\int_{\mathcal{F}} \sum_{\substack{\mathfrak{a}\in \mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}} \sqrt{|\Delta_K| y}\,\vartheta_{\mathfrak{a}}\left(\frac{i y^{\frac{1}{d}}\mathbf{x}}{2}\right) d\mathbf{x} - \frac{h_K}{w_K}\int_{\mathcal{F}} d\mathbf{x} \qquad \text{Theorem 8.1.1} \\
&= \frac{\sqrt{|\Delta_K| y}}{w_K}\int_{\mathcal{F}} \sum_{\substack{\mathfrak{a}\in \mathrm{Cl}(K)\\ \mathfrak{a}\text{ integral}}} \vartheta_{\mathfrak{a}}\left(\frac{i y^{\frac{1}{d}}\mathbf{x}}{2}\right) d\mathbf{x} - \frac{h_K}{w_K}\int_{\mathcal{F}} d\mathbf{x} \\
&= \sqrt{|\Delta_K| y}\,\omega_{\mathcal{F}}(i y^{\frac{1}{d}}) + \frac{\sqrt{|\Delta|_K y}\, h_K}{w_K}\int_{\mathcal{F}} d\mathbf{x} - \frac{h_K}{w_K}\int_{\mathcal{F}} d\mathbf{x}.
\end{aligned}$$

This chain implies that our first piece can be expressed as

$$\int_{\frac{1}{|\Delta_K|}}^{\infty}\left(\sqrt{|\Delta_K| y}\,\omega_{\mathcal{F}}(i y^{\frac{1}{d}}) + \frac{\sqrt{|\Delta|_K y}\, h_K}{w_K}\int_{\mathcal{F}} d\mathbf{x} - \frac{h_K}{w_K}\int_{\mathcal{F}} d\mathbf{x}\right)(|\Delta_K|^2 y)^{-\frac{s}{2}}\,\frac{dy}{y},$$

which is further equivalent to

$$|\Delta_K|^{\frac{1}{2}-s}\int_{\frac{1}{|\Delta_K|}}^{\infty} \omega_{\mathcal{F}}(i y^{\frac{1}{d}}) y^{\frac{1-s}{2}}\,\frac{dy}{y} - \frac{2h_K}{\omega_K\sqrt{|\Delta|_K}\, s(1-s)}\int_{\mathcal{F}} d\mathbf{x},$$

since, upon interchanging the integrals in the last two pieces by the Fubini–Tonelli theorem, the inner integrals are $\frac{2}{\sqrt{|\Delta|_K}(1-s)} - \frac{2}{\sqrt{|\Delta|_K}s} = -\frac{2}{\sqrt{|\Delta|_K}s(1-s)}$. Substituting this expression back into Equation (8.2) and combining with Equation (8.1) gives the integral representation

$$\begin{aligned}
\zeta_K(s) = \frac{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}}{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}\Bigg[&- \frac{2h_K}{\omega_K\sqrt{|\Delta|_K}\, s(1-s)}\int_{\mathcal{F}} d\mathbf{x} + |\Delta_K|^{\frac{1}{2}-s}\int_{\frac{1}{|\Delta_K|}}^{\infty} \omega_{\mathcal{F}}(i y^{\frac{1}{d}}) y^{\frac{1-s}{2}}\,\frac{dy}{y} \\
&+ \int_{\frac{1}{|\Delta_K|}}^{\infty} \omega_{\mathcal{F}}\left(i y^{\frac{1}{d}}\right) y^{\frac{s}{2}}\,\frac{dy}{y}\Bigg].
\end{aligned}$$

It remains to compute the first integral on the right-hand side. The following lemma gives the result:

Lemma 8.1.2. *Let K be a number field of signature (r_1, r_2) and set $\mathcal{F} = e^{\mathbf{N}^\vee\mathcal{M}}$ where $\mathcal{M}$ is a fundamental domain for Λ in the trace-zero hyperplane H. Then*

$$\int_{\mathcal{F}} d\mathbf{x} = 2^{r_1-1} R_K.$$

Proof. Let d be the degree of K. Begin by observing that

$$\int_{\mathcal{F}} d\mathbf{x}\int_1^{\varepsilon}\int_{\mathcal{F}} \frac{d\mathbf{x}\,dy}{y} = \int_{[1,e]\times\mathcal{F}} \frac{d\mathbf{y}}{\mathbf{y}},$$

in view of the decomposition $\mathbb{R}_+^{r_K+1} = (0,\infty) \times S$ with $\mathbf{y} = y^{\frac{1}{d}}\mathbf{x}$ and $\frac{d\mathbf{y}}{\mathbf{y}} = \frac{d\mathbf{x}\,dy}{y}$. Under the change of variables $\mathbf{y} \to e^{2\mathbf{y}}$, we obtain

$$2^{r_K+1} \int_{\left[0,\frac{1}{2}\right]\times \frac{\mathcal{M}}{\mathbf{N}}} d\mathbf{y},$$

because the absolute value of the determinant of the Jacobian matrix is $2^{r_K+1}e^{\langle \mathbf{y},\mathbf{1}\rangle}$. The remaining integral is just a $2^{-(r_2+1)}$ multiple of the volume of $\mathcal{M}$. Letting $\varepsilon_1, \ldots, \varepsilon_{r_K}$ be a system of fundamental units for K, Dirichlet's unit theorem implies that $\lambda(\varepsilon_1), \ldots, \lambda(\varepsilon_{r_K})$ is a basis for Λ. Then by Proposition 1.5.3, the volume of $\mathcal{M}$ is just the absolute value of the determinant of the generator matrix P for $\lambda(\varepsilon_1), \ldots, \lambda(\varepsilon_{r_K})$. But this is R_K by definition of the regulator. It follows that our last expression evaluates to $2^{r_1-1}R_K$ as claimed. □

From Lemma 8.1.2, we obtain the integral representation

$$\begin{aligned}\zeta_K(s) = \frac{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}}{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}\Bigg[&- \frac{2^{r_1}h_K R_K}{\omega_K\sqrt{|\Delta|_K}s(1-s)} + |\Delta_K|^{\frac{1}{2}-s}\int_{\frac{1}{|\Delta_K|}}^{\infty} \omega_{\mathcal{F}}(iy^{\frac{1}{d}})y^{\frac{1-s}{2}}\,\frac{dy}{y} \\ &+ \int_{\frac{1}{|\Delta_K|}}^{\infty} \omega_{\mathcal{F}}\left(iy^{\frac{1}{d}}\right) y^{\frac{s}{2}}\,\frac{dy}{y}\Bigg].\end{aligned} \tag{8.3}$$

This integral representation will give analytic continuation. Indeed, everything outside the brackets is entire. The integrands exhibit exponential decay and therefore are locally absolutely uniformly convergent on $\mathbb{C}$. The fractional term is holomorphic except for simple poles at $s = 0$ and $s = 1$. The meromorphic continuation to $\mathbb{C}$ follows with possible simple poles at $s = 0$ and $s = 1$. Actually, there is no pole at $s = 0$. To see this, observe that $\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}$ has at least a simple pole at $s = 0$ (since r_1 or r_2 is positive) and so its reciprocal has at least a simple zero. This cancels the corresponding simple pole of $-\frac{2^{r_1}h_K R_K}{\omega_K s(1-s)}$ so that $\zeta_K(s)$ has a removable singularity and thus is holomorphic at $s = 0$. At $s = 1$, $\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}$ is nonzero and so $\zeta_K(s)$ has a simple pole. Therefore $\zeta_K(s)$ has meromorphic continuation to all of $\mathbb{C}$ with a simple pole at $s = 1$.

The Functional Equation

An immediate consequence of applying the symmetry $s \mapsto 1-s$ to Equation (8.3) is the following functional equation:

$$|\Delta_K|^{\frac{s}{2}}\frac{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}}\zeta_K(s) = |\Delta_K|^{\frac{1-s}{2}}\frac{\Gamma\left(\frac{(1-s)}{2}\right)^{r_1}\Gamma(1-s)^{r_2}}{\pi^{\frac{r_1(1-s)}{2}}(2\pi)^{r_2(1-s)}}\zeta_K(1-s).$$

Using the Legendre duplication formula, we see that

$$\begin{aligned}\frac{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}} &= \frac{1}{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}2^{r_2(1-s)}\pi^{\frac{r_2}{2}}}\Gamma\left(\frac{s}{2}\right)^{r_1+r_2}\Gamma\left(\frac{s+1}{2}\right)^{r_2} \\ &= \frac{1}{2^{r_2}\pi^{\frac{ds}{2}+\frac{r_2}{2}}}\Gamma\left(\frac{s}{2}\right)^{r_1+r_2}\Gamma\left(\frac{s+1}{2}\right)^{r_2} \\ &= \frac{1}{2^{r_2}\pi^{\frac{r_2}{2}}}\pi^{-\frac{ds}{2}}\Gamma\left(\frac{s}{2}\right)^{r_1+r_2}\Gamma\left(\frac{s+1}{2}\right)^{r_2}.\end{aligned}$$

The constant factor in front is independent of s and so can be canceled in the functional equation. Therefore we identify the gamma factor to be

$$\gamma(s,\zeta_K) = \pi^{-\frac{ds}{2}}\Gamma\left(\frac{s}{2}\right)^{r_1+r_2}\Gamma\left(\frac{s+1}{2}\right)^{r_2},$$

with $\kappa_i = 0$ for $1 \leq i \leq r_1 + r_2$ and $\kappa_i = 1$ for $r_1 + r_2 + 1 \leq i \leq d$ the local roots at infinity. Clearly these satisfy the required bounds. The conductor is $q(\zeta_K) = |\Delta_K|$ and so p is ramified if and only if it is ramified in $\mathcal{O}_K/\mathbb{Z}$. Note that if p is unramified then $\sum_{1\leq i\leq r_p}(e_p(\mathfrak{p}_i)-1)f_p(\mathfrak{p}_i) = 0$ so that all local roots are of the form $\omega_{p,i}^k \neq 0$. The completed Dedekind zeta function is

$$\Lambda(s,\zeta_K) = |\Delta_K|^{\frac{s}{2}}\pi^{-\frac{ds}{2}}\Gamma\left(\frac{s}{2}\right)^{r_1+r_2}\Gamma\left(\frac{s+1}{2}\right)^{r_2}\zeta_K(s),$$

with functional equation

$$\Lambda(s,\zeta_K) = \Lambda(1-s,\zeta_K).$$

This is the function equation of $\zeta_K(s)$ and from it we see that the root number is $\varepsilon(\zeta_K) = 1$ and that $\zeta_K(s)$ is self-dual. To see that the order is 1, multiply by $(s-1)$ to clear the polar divisor. As the integrals in Equation (8.3) are locally absolutely uniformly convergent, computing the order amounts to estimating the gamma factor. And because the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s,\zeta_K)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

Therefore the reciprocal of the gamma factor is also of order 1 and it follows that

$$(s-1)\zeta_K(s) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

This shows $(s-1)\zeta_K(s)$ is of order 1 and hence $\zeta_K(s)$ is too after removing the polar divisor. It now remains to compute the residue of $\zeta_K(s)$ at $s=1$. This result is known as the **analytic class number formula**:

Theorem (Analytic class number formula). *Let K be a number field of signature (r_1, r_2). Then*

$$\operatorname*{Res}_{s=1}\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K\sqrt{|\Delta_K|}}.$$

Proof. The only term in Equation (8.3) contributing to the pole is $-\frac{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}}{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}\frac{2^{r_1}h_K R_K}{\omega_K\sqrt{|\Delta|_K}s(1-s)}$. Observe that

$$\lim_{s\to 1}\frac{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}}{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}} = (2\pi)^{r_2},$$

because $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(1) = 1$. Therefore

$$\begin{aligned}
\operatorname*{Res}_{s=1}\zeta_K(s) &= \operatorname*{Res}_{s=1}\left(-\frac{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}}{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}\frac{2^{r_1}h_K R_K}{\omega_K\sqrt{|\Delta|_K}s(1-s)}\right)\\
&= \lim_{s\to 1}\left(\frac{\pi^{\frac{r_1 s}{2}}(2\pi)^{r_2 s}}{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}\frac{2^{r_1}h_K R_K}{\omega_K\sqrt{|\Delta|_K}s}\right)\\
&= \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K\sqrt{|\Delta_K|}}.
\end{aligned}$$

□

The usefulness of the analytic class number formula is that is allows for the estimation of the class number in terms of approximating the residue of the Dedekind zeta function at $s = 1$. We collect our work in the following theorem:

Theorem 8.1.2. *Let K be a number field of degree d, signature (r_1, r_2), and for each prime p let $\omega_{p,i}$ be a primitive $f_p(\mathfrak{p}_i)$-th root of unity for each prime $\mathfrak{p}_i$ above p. Then $\zeta_K(s)$ is a Selberg class L-function with degree d Euler product given by*

$$\zeta_K(s) = \prod_p \prod_{\substack{1 \leq i \leq r_p \\ 1 \leq k \leq f_p(\mathfrak{p}_i)}} (1 - \omega_{p,i}^k p^{-s})^{-1}.$$

Moreover, it admits meromorphic continuation to $\mathbb{C}$, possesses the functional equation

$$|\Delta_K|^{\frac{s}{2}} \pi^{-\frac{ds}{2}} \Gamma\left(\frac{s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{s+1}{2}\right)^{r_2} \zeta_K(s) = \Lambda(s, \zeta_K) = \Lambda(1-s, \zeta_K),$$

and has a simple pole at $s = 1$ of residue $\frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{|\Delta_K|}}$.

8.2 Quadratic Dedekind Zeta Functions

Given our discussion of quadratic number fields, it is a simple matter to determine quadratic Dedekind zeta function explicitly:

Theorem 8.2.1. *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field and let χ_{Δ_d} be the quadratic character associated to the fundamental discriminant Δ_d. Then*

$$\zeta_{\mathbb{Q}(\sqrt{d})}(s) = \zeta(s) L(s, \chi_{\Delta_d}).$$

Proof. By the identity theorem it suffices to prove this for $\sigma > 1$. Using Proposition 7.1.4, the Euler product of $\zeta_{\mathbb{Q}(\sqrt{d})}(s)$ is given by

$$\begin{aligned}
\zeta_{\mathbb{Q}(\sqrt{d})}(s) &= \prod_{\substack{p \\ \chi_{\Delta_d}(p)=1}} (1-p^{-s})^{-1}(1-p^{-s})^{-1} \prod_{\substack{p \\ \chi_{\Delta_d}(p)=-1}} (1-p^{-s})^{-1}(1+p^{-s})^{-1} \prod_{\substack{p \\ \chi_{\Delta_d}(p)=0}} (1-p^{-s})^{-1} \\
&= \zeta(s) \prod_{\substack{p \\ \chi_{\Delta_d}(p)=1}} (1-p^{-s})^{-1} \prod_{\substack{p \\ \chi_{\Delta_d}(p)=-1}} (1+p^{-s})^{-1} \prod_{\substack{p \\ \chi_{\Delta_d}(p)=0}} 1 \\
&= \zeta(s) L(s, \chi_{\Delta_d}).
\end{aligned}$$

This completes the proof. □

Almost immediately, we can prove an interesting expression for $a_{\mathbb{Q}(\sqrt{d})}(n)$:

Proposition 8.2.1. *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field and let χ_{Δ_d} be the quadratic character associated to the fundamental discriminant Δ_d. Then*

$$a_{\mathbb{Q}(\sqrt{d})}(n) = \sum_{d|n} \chi_{\Delta_d}(d).$$

Proof. This follows from Proposition 2.1.1 and Remark 8.1.1 since the coefficients of the L-series of $\zeta(s)L(s,\chi_{\Delta_d})$ are $\sum_{d|n}\chi_{\Delta_d}(d)$. □

Since the residue of $\zeta(s)$ at its pole is 1, the analytic class number formula gives a relationship between the value of Dirichlet L-functions at $s=1$ and the class number of quadratic number fields. This is known as the **Dirichlet class number formula**:

Corollary (Dirichlet class number formula). *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field and let χ_{Δ_d} be the quadratic character associated to the fundamental discriminant Δ_d. Then*

$$L(1,\chi_{\Delta_d}) = \frac{2h_d\log|\varepsilon_d|}{\sqrt{|\Delta_d|}} \quad \text{or} \quad L(1,\chi_{\Delta_d}) = \frac{2\pi h_d}{w_d\sqrt{|\Delta_d|}},$$

where ε_d is a fundamental unit, according to if $d>0$ or $d<0$.

Proof. Let (r_1,r_2) be the signature of $\mathbb{Q}(\sqrt{d})$. Combining Theorem 8.2.1 with the analytic class number formula and that the residue of $\zeta(s)$ at its pole is 1 gives

$$L(1,\chi_{\Delta_d}) = \frac{2^{r_1}(2\pi)^{r_2}h_dR_d}{w_d\sqrt{|\Delta_d|}}.$$

The result follows by Proposition 7.1.5, Proposition 7.1.6, and that (r_1,r_2) is $(2,0)$ or $(0,1)$ according to if $d>0$ or $d<0$. □

As $h_d \geq 1$ and $w_d \leq 6$ (by Proposition 7.1.5), the Dirichlet class number formula gives the estimate

$$L(1,\chi_{\Delta_d}) \gg \frac{\log|\varepsilon_d|}{\sqrt{\Delta_d}} \quad \text{or} \quad L(1,\chi_{\Delta_d}) \gg \frac{1}{\sqrt{\Delta_d}}, \tag{8.4}$$

where ε_d is a fundamental unit, according to if $d>0$ or $d<0$. By Theorem 1.3.2, this is really a result about Dirichlet L-functions attached to primitive quadratic characters. Note that while Equation (8.4) is significantly weaker than Siegel's theorem, the implicit constant is not ineffective.

8.3 The Class Number of Quadratic Number Fields

The Dirichlet class number formula gives a relationship between the class number h_d of the quadratic number field $\mathbb{Q}(\sqrt{d})$ and the value of the Dirichlet L-function $L(s,\chi_{\Delta_d})$ at $s=1$ where χ_{Δ_d} is the quadratic character given by the Kronecker symbol. As χ_{Δ_d} is a primitive quadratic character of conductor $|\Delta_d|>1$ (recall Theorem 1.3.2), we know from Theorem 4.3.1 that $L(1,\chi_{\Delta_d})$ is finite and nonzero. It is interesting to know whether or not this value is computable in general so that we may obtain another formula for the class number. We will actually obtain a formula for $L(1,\chi)$ where χ is any primitive character χ of conductor $q>1$. The computation is fairly straightforward and only requires some basic properties of Ramanujan and Gauss sums that we have already developed. The idea is to rewrite the character values $\chi(n)$ so that we can collapse the infinite series into a Taylor series. Our result is the following:

Theorem 8.3.1. *Let χ be a primitive Dirichlet character with conductor $q>1$. Then*

$$L(1,\chi) = -\frac{\tau(\chi)}{q}\sum_{a \pmod q}\overline{\chi}(a)\log\sin\left(\frac{\pi a}{q}\right) \quad \text{or} \quad L(1,\chi) = \frac{\pi i\tau(\chi)}{q^2}\sum_{a \pmod q}\overline{\chi}(a)a,$$

according to whether χ is even or odd.

Proof. Recall that the Dirichlet L-series $L(s,\chi)$ converges for $\sigma > 0$ and thus at $s = 1$. First compute

$$\begin{aligned}
\chi(n) &= \frac{1}{\tau(\overline{\chi})} \sum_{a \pmod q} \overline{\chi}(a) e^{\frac{2\pi i a n}{q}} && \text{Corollary 1.4.1} \\
&= \frac{\chi(-1)}{\overline{\tau(\chi)}} \sum_{a \pmod q} \overline{\chi}(a) e^{\frac{2\pi i a n}{q}} && \text{Proposition 1.4.2 (i) and } \chi(-1)^2 = 1 \\
&= \frac{\chi(-1)\tau(\chi)}{\tau(\chi)\overline{\tau(\chi)}} \sum_{a \pmod q} \overline{\chi}(a) e^{\frac{2\pi i a n}{q}} \\
&= \frac{\chi(-1)\tau(\chi)}{q} \sum_{a \pmod q} \overline{\chi}(a) e^{\frac{2\pi i a n}{q}} && \text{Theorem 1.4.1.}
\end{aligned}$$

Substituting the above result into the definition of $L(1,\chi)$, we find that

$$\begin{aligned}
L(1,\chi) &= \sum_{n \geq 1} \frac{1}{n} \left(\frac{\chi(-1)\tau(\chi)}{q} \sum_{a \pmod q} \overline{\chi}(a) e^{\frac{2\pi i a n}{q}} \right) \\
&= \frac{\chi(-1)\tau(\chi)}{q} \sum_{a \pmod q} \overline{\chi}(a) \sum_{n \geq 1} \frac{e^{\frac{2\pi i a n}{q}}}{n} \\
&= \frac{\chi(-1)\tau(\chi)}{q} \sum_{a \pmod q} \overline{\chi}(a) \log\left(\left(1 - e^{\frac{2\pi i a}{q}}\right)^{-1} \right),
\end{aligned} \tag{8.5}$$

where in the last line we have used the Taylor series of the logarithm. We will now simplify the last expression in Equation (8.5). Since $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$, we have

$$1 - e^{\frac{2\pi i a}{q}} = -2ie^{\frac{\pi i a}{q}} \left(\frac{e^{\frac{\pi i a}{q}} - e^{-\frac{\pi i a}{q}}}{2i} \right) = -2ie^{\frac{\pi i a}{q}} \sin\left(\frac{\pi a}{q}\right).$$

Therefore the last expression in Equation (8.5) becomes

$$\frac{\chi(-1)\tau(\chi)}{q} \sum_{a \pmod q} \overline{\chi}(a) \log\left(\left(-2ie^{\frac{\pi i a}{q}} \sin\left(\frac{\pi a}{q}\right) \right)^{-1} \right).$$

As a is taken modulo q, we have $0 < \frac{\pi a}{q} < \pi$ so that $\sin\left(\frac{\pi a}{q}\right)$ is never negative. Therefore we can split up the logarithm term and obtain

$$-\frac{\chi(-1)\tau(\chi)}{q} \left(\log(-2i) \sum_{a \pmod q} \overline{\chi}(a) + \frac{\pi i}{q} \sum_{a \pmod q} \overline{\chi}(a) a + \sum_{a \pmod q} \overline{\chi}(a) \log\sin\left(\frac{\pi a}{q}\right) \right).$$

By the Dirichlet orthogonality relations (namely Corollary 1.3.1 (i)), the first sum above vanishes. Therefore

$$L(1,\chi) = -\frac{\chi(-1)\tau(\chi)}{q} \left(\frac{\pi i}{q} \sum_{a \pmod q} \chi(a) a + \sum_{a \pmod q} \chi(a) \log\sin\left(\frac{\pi a}{q}\right) \right). \tag{8.6}$$

Equation (8.6) simplifies in that one of the two sums vanish depending on if χ is even or odd. For the first sum in Equation (8.6), the change of variables $a \to -a$ shows that

$$\frac{\pi i}{q} \sum_{a \pmod q} \chi(a)a = -\frac{\chi(-1)\pi i}{q} \sum_{a \pmod q} \chi(a)a.$$

Hence this sum vanishes if χ is even which proves the even case. For the second sum in Equation (8.6), the change of variables $a \to q - a$ shows that

$$\sum_{a \pmod q} \chi(a) \log \sin\left(\frac{\pi a}{q}\right) = \chi(-1) \sum_{a \pmod q} \chi(a) \log \sin\left(\frac{\pi a}{q}\right).$$

Therefore this sum vanishes if χ is odd proving in the odd case and completing the proof. □

Remark 8.3.1. *Theorem 8.3.1 encodes some interesting identities. For example, if χ is the non-principal Dirichlet character modulo 4 then χ is uniquely defined by $\chi(1) = 1$ and $\chi(3) = \chi(-1) = -1$. In particular, χ is odd and its conductor is 4. Now*

$$\tau(\chi) = \sum_{a \pmod 4} \chi(a)e^{\frac{2\pi i a}{4}} = e^{\frac{2\pi i}{4}} - e^{\frac{6\pi i}{4}} = i - (-i) = 2i,$$

and so by Theorem 8.3.1, we get

$$L(1, \chi) = \frac{\pi i \tau(\chi)}{16}(1 - 3) = \frac{\pi}{4}.$$

Expanding $L(1, \chi)$ gives

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4},$$

*which is the famous **Madhava–Leibniz formula** for π.*

From the definition of χ_{Δ_d}, we see that χ_{Δ_d} is even or odd according to if $d > 0$ or $d < 0$ (recall Proposition 7.1.3). This gives an explicit formula for the class number of quadratic number fields:

Corollary 8.3.1. *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field. Then,*

$$h_d = -\frac{\tau(\chi)}{2\log|\varepsilon_d|\sqrt{|\Delta_d|}} \sum_{a \pmod{|\Delta_d|}} \overline{\chi}(a) \log \sin\left(\frac{\pi a}{|\Delta_d|}\right) \quad \text{or} \quad h_d = \frac{w_d i \tau(\chi)}{2|\Delta_d|^{\frac{3}{2}}} \sum_{a \pmod{|\Delta_d|}} \overline{\chi}(a)a,$$

where ε_d is a fundamental unit, according to if $d > 0$ or $d < 0$.

Proof. This follows from Dirichlet's unit theorem, Theorem 8.3.1, and that χ_{Δ_d} is even or odd according to if $d > 0$ or $d < 0$. □

Having given an explicit formula for the class number of quadratic number fields, we now turn to more useful estimates. By what we have seen, it suffices to obtain bounds for Dirichlet L-functions at $s = 1$ and the tighter these bounds are the tighter our estimates for the class number will be. Upper bounds are not too difficult to obtain:

Proposition 8.3.1. *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field. Then*

$$h_d \log|\varepsilon_d| \ll \sqrt{d}\log(d) \quad \text{or} \quad h_d \ll \sqrt{d}\log(d),$$

where ε_d is a fundamental unit, according to if $d > 0$ or $d < 0$.

Proof. This follows from the Dirichlet class number formula and Lemma 4.4.1 since $\Delta_d \asymp d$ (recall Proposition 7.1.3). □

Effective lower bounds are difficult to obtain. However, if we allow the implicit constant to be ineffective we may use Siegel's theorem:

Proposition 8.3.2. *Let $\mathbb{Q}(\sqrt{d})$ be a quadratic number field. Then there exists a positive constant $c(\varepsilon)$ such that*

$$h_d \log |\varepsilon_d| \geq c(\varepsilon) d^{\frac{1}{2}-\varepsilon} \quad \text{or} \quad h_d \geq c(\varepsilon) d^{\frac{1}{2}-\varepsilon},$$

where ε_d is a fundamental unit, according to if $d > 0$ or $d < 0$.

Proof. This follows from the Dirichlet class number formula and Siegel's theorem since $\Delta_d \asymp d$ (recall Proposition 7.1.3). □

As a corollary we find that there are only finitely many imaginary quadratic number fields of a fixed class number:

Corollary 8.3.2. *Let $n \geq 1$. Then there are finitely many imaginary quadratic number fields $\mathbb{Q}(\sqrt{d})$ of class number n.*

Proof. By Proposition 8.3.2, we see that $h_d \to \infty$ as $d \to -\infty$. The claim follows at once. □

It is in Corollary 8.3.2 that we can see where the ineffectiveness of Siegel's theorem becomes apparent. Indeed, although there are finitely many imaginary quadratic number fields of class number $n \geq 1$, we cannot use the bound

$$h_d \geq c(\varepsilon) d^{\frac{1}{2}-\varepsilon},$$

from Proposition 8.3.2 to reduce this to a finite computation because the constant $c(\varepsilon)$ is ineffective. In other words, we cannot obtain lower bound on d as $d \to -\infty$ that tells us when all class numbers are larger than n even though we know such a bound exists.

Part IV

Holomorphic, Automorphic, and Maass Forms

Chapter 9

Congruence Subgroups and Modular Curves

Every holomorphic or Maass form is a special type of function depending on certain subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. These are the congruence subgroups. The associated modular curve is the quotient of the upper half-space $\mathbb{H}$ by an action of this subgroup. We introduce these topics first as they are the foundation for discussing holomorphic and Maass forms in complete generality.

9.1 Congruence Subgroups

The **modular group** is $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. That is, the modular group is the set of matrices with integer entries and of determinant 1 determined up to sign. The reason we are only interested in these matrices up to sign is because the modular group has a natural action on the upper half-space $\mathbb{H}$ and this action will be invariant under a change in sign. The first result usually proved about the modular group is that it is generated by two matrices:

Proposition 9.1.1.

$$\mathrm{PSL}_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Proof. Set S and T to be the first and second generators respectively. Clearly they belong to $\mathrm{PSL}_2(\mathbb{Z})$. Also, S and T^n for $n \in \mathbb{Z}$ acts on $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$ by

$$S\gamma = S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \quad \text{and} \quad T^n\gamma = T^n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}.$$

In particular, S interchanges the upper-left and lower-left entries of γ up to sign and T^n adds an n multiple of the lower-left entry to the upper-left entry. We have to show $\gamma \in \langle S, T\rangle$ and we will accomplish this by showing that the inverse is in $\langle S, T\rangle$. If $|c| = 0$ then γ is the identity since $\det(\gamma) = 1$ so suppose $|c| \neq 0$. By Euclidean division we can write $a = qc + r$ for some $q \in \mathbb{Z}$ and $|r| < |c|$. Then

$$T^{-q}\gamma = \begin{pmatrix} a-qc & b-qd \\ c & d \end{pmatrix} = \begin{pmatrix} r & b-qd \\ c & d \end{pmatrix}.$$

Multiplying by S yields

$$ST^{-q}\gamma = S\begin{pmatrix} r & b-qd \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ r & b-qd \end{pmatrix},$$

and this matrix has the upper-left entry at least as large as the lower-left entry in norm. Actually the upper-left entry is strictly larger since $|c| > |r|$ by Euclidean division. Therefore if we repeatedly apply this

procedure, it must terminate with the lower-left entry vanishing. But then we have reached the identity matrix. Therefore we have show γ has an inverse in $\langle S, T\rangle$. □

The group $\mathrm{GL}_2^+(\mathbb{Q})$ also decomposes nicely in terms of $\mathrm{PSL}_2(\mathbb{Z})$:

Lemma 9.1.1. *Every $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ can be written as $\alpha = \gamma\eta$ for some $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ and $\eta \in \mathrm{GL}_2^+(\mathbb{Q})$ with $\eta = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$.*

Proof. Let $r \geq 1$ be such that $r\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Z})$. If $c = 0$ then $a \neq 0$ (otherwise α is not invertible) and so $(a, c) \geq 1$. Set $a' = \frac{a}{(a,c)}$ and $c' = \frac{c}{(a,c)}$ so that $a', c' \in \mathbb{Z}$ with $(a', c') = 1$. Then there exists $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ with $\gamma^{-1} = \left(\begin{smallmatrix} * & * \\ -c' & a' \end{smallmatrix}\right)$. Moreover, $\gamma^{-1}r\alpha = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Z})$. The claim is complete upon setting $\eta = \gamma^{-1}\alpha$.□

We will also be interested in special subgroups of the modular group defined by congruence conditions on their entries. For $N \geq 1$, set

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Then $\Gamma(N)$ is the kernel of the homomorphism $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$ given by reducing the coefficients modulo N so it is a normal subgroup with finite index. We call $\Gamma(N)$ the **principal congruence subgroup** of level N. For $\Gamma \leq \mathrm{PSL}_2(\mathbb{Z})$, we say Γ is a **congruence subgroup** if $\Gamma(N) \leq \Gamma$ for some N and the minimal such N is called the **level** of Γ. Note that if $M \mid N$ then $\Gamma(N) \leq \Gamma(M)$. Thus if Γ is a congruence subgroup of level N then $\Gamma(kN) \leq \Gamma$ for all $k \geq 1$. This implies that congruence subgroups are closed under intersection. Also, it turns out that the aforementioned homomorphism is surjective:

Proposition 9.1.2. *The homomorphism $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$ given by reducing the coefficients modulo N is surjective.*

Proof. Suppose $\left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Then $\overline{a}\overline{d} - \overline{b}\overline{c} \equiv 1 \pmod{N}$ so by Bézout's identity (generalized to three integers) $(\overline{c}, \overline{d}, N) = 1$. We claim that there exists s and t such that $c = \overline{c} + sN$, $d = \overline{d} + tN$ with $(c, d) = 1$. Set $g = (\overline{c}, \overline{d})$. Then $(g, N) = 1$ because $(\overline{c}, \overline{d}, N) = 1$. If $\overline{c} = 0$ then set $s = 0$ so $c = 0$ and choose t such that $t \equiv 1 \pmod{p}$ for any prime $p \mid g$ and $t \equiv 0 \pmod{p}$ for any prime $p \nmid g$ and $p \mid \overline{d}$. Such a t exists by the Chinese remainder theorem. Now if $p \mid (c, d)$ then either $p \mid g$ or $p \nmid g$. If $p \mid g$ then $p \mid d - \overline{d} = tN$ which is absurd since $t \equiv 1 \pmod{p}$ and $(t, N) = 1$. If $p \nmid g$ then $p \nmid d - \overline{d} = tN$ but this is also absurd since $t \equiv 0 \pmod{p}$. Therefore $(c, d) = 1$ as claimed. If $\overline{c} = 0$ then $\overline{d} \neq 0$, and we can proceed similarly. Since $(c, d) = 1$ there exists a and b such that $ad - bc = 1$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$ and maps onto $\left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right)$. This proves surjectivity. □

By Proposition 9.1.2, $[\mathrm{PSL}_2(Z) : \Gamma(N)] = |\mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})|$. Since $\Gamma(N) \leq \Gamma$ and $\Gamma(N)$ has finite index in $\mathrm{PSL}_2(\mathbb{Z})$ so does Γ. The subgroups

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\},$$

are particularly important and are congruence subgroups of level N. The latter subgroup is called the **Hecke congruence subgroup** of level N. Note that $\Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N)$. If Γ is a general congruence

subgroup, it is useful to find a generating set for Γ in order to reduce results about Γ to that of the generators. This is usually achieved by performing some sort of Euclidean division argument on the entries of a matrix $\gamma \in \Gamma$ using the supposed generating set to construct the inverse for γ. We will also require a useful lemma which says that congruence subgroups are preserved under conjugation by elements of $\mathrm{GL}_2^+(\mathbb{Q})$ provided we restrict to those elements in $\mathrm{PSL}_2(\mathbb{Z})$:

Lemma 9.1.2. *Let Γ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then $\alpha^{-1}\Gamma\alpha \cap \mathrm{PSL}_2(\mathbb{Z})$ is a congruence subgroup.*

Proof. Recall that if Γ is of level M then $\Gamma(kM) \leq \Gamma$ for every $k \geq 1$. Thus there is an integer $\widetilde{N}$ such that $\Gamma(\widetilde{N}) \leq \Gamma$, $\widetilde{N}\alpha \in \mathrm{GL}_2^+(\mathbb{Z})$, and $\widetilde{N}\alpha \in \mathrm{GL}_2^+(\mathbb{Z})$. Now let $N = \widetilde{N}^3$ and notice that any $\gamma \in \Gamma(N)$ is of the form

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix},$$

for $k_1, \ldots, k_4 \in \mathbb{Z}$. Therefore $\Gamma(N) \subseteq I + N\mathrm{Mat}_2(\mathbb{Z})$. Thus

$$\alpha\Gamma(N)\alpha^{-1} \leq \alpha(I + N\mathrm{Mat}_2(\mathbb{Z}))\alpha^{-1} = I + \widetilde{N}\mathrm{Mat}_2(\mathbb{Z}).$$

As every matrix in $\alpha\Gamma(N)\alpha^{-1}$ has determinant 1 and $\Gamma(\widetilde{N}) \subseteq I + \widetilde{N}\mathrm{Mat}_2(\mathbb{Z})$, it follows that $\alpha\Gamma(N)\alpha^{-1} \leq \Gamma(\widetilde{N})$. As $\Gamma(\widetilde{N}) \leq \Gamma$, we conclude

$$\Gamma(N) \leq \alpha^{-1}\Gamma\alpha,$$

and intersecting with $\mathrm{PSL}_2(\mathbb{Z})$ completes the proof. □

Note that by Lemma 9.1.2, if $\alpha^{-1}\Gamma\alpha \subset \mathrm{PSL}_2(\mathbb{Z})$ then $\alpha^{-1}\Gamma\alpha$ is a congruence subgroup if Γ is. Moreover, since congruence subgroups are closed under intersection, Lemma 9.1.2 further implies that $\alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$ is a congruence subgroup for and two congruence subgroups Γ_1 and Γ_2 any any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$.

9.2 Modular Curves

Recall that $\mathrm{GL}_2^+(\mathbb{Q})$ naturally acts on the Riemann sphere $\hat{\mathbb{C}}$ by Möbius transformations. Explicitly, any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ acts on $z \in \hat{\mathbb{C}}$ by

$$\gamma z = \frac{az + b}{cz + d},$$

where $\gamma\infty = \frac{a}{c}$ and $\gamma\left(-\frac{d}{c}\right) = \infty$. Moreover, recall that this action is a group action, is invariant under scalar multiplication, and acts as automorphisms of $\hat{\mathbb{C}}$. Now observe

$$\mathrm{Im}(\gamma z) = \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{az+b}{cz+d}\frac{c\overline{z}+d}{c\overline{z}+d}\right) = \mathrm{Im}\left(\frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2}\right) = \det(\gamma)\frac{\mathrm{Im}(z)}{|cz+d|^2},$$

where the last equality follows because $\mathrm{Im}(\overline{z}) = -\mathrm{Im}(z)$ and $\det(\gamma) = ad - bc$. Since $\deg(\gamma) > 0$ and $|cz + d|^2 > 0$, γ preserves the sign of the imaginary part of z. So γ preserves the upper half-space $\mathbb{H}$, the lower half-space $\overline{\mathbb{H}}$, and the extended real line $\hat{\mathbb{R}}$ respectively. Moreover, γ restricts to an automorphism on these subspaces since Möbius transformations are automorphisms. In particular, $\mathrm{PSL}_2(\mathbb{Z})$ naturally acts on $\hat{\mathbb{C}}$ by Möbius transformations and preserves the upper half-space. Certain actions of subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ also play important roles. A **Fuchsian group** is any discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ that acts properly discontinuously on $\mathbb{H}$ (see Appendix D.1). It turns out that the modular group is a Fuchsian group (see [DS05] for a proof):

Proposition 9.2.1. *The modular group is a Fuchsian group.*

Note that Proposition 9.2.1 immediately implies that any subgroup of the modular group is also Fuchsian. In particular, all congruence subgroups are Fuchsian. A **modular curve** is a quotient $\Gamma\backslash\mathbb{H}$ of the upper half-space $\mathbb{H}$ by the action of a congruence subgroup Γ. Since Γ is Fuchsian and acts on $\mathbb{H}$ by automorphisms Proposition D.1.1 implies that $\Gamma\backslash\mathbb{H}$ is also connected Hausdorff (recall that $\mathbb{H}$ is connected Hausdorff). In particular, $\Gamma\backslash\mathbb{H}$ admits the fundamental domain

$$\mathcal{F} = \left\{ z \in \mathbb{H} : |\mathrm{Re}(z)| \leq \frac{1}{2} \text{ and } |z| \geq 1 \right\},$$

as the following proposition shows:

Proposition 9.2.2. *$\mathcal{F}$ is a fundamental domain for* $\mathrm{PSL}_2(\mathbb{Z})$.

Proof. Set $\mathrm{PSL}_2(\mathbb{Z}) = \langle S, T \rangle$ where S and T are as in Proposition 9.1.1. We first show any point in $\mathbb{H}$ is $\mathrm{PSL}_2(\mathbb{Z})$-equivalent to a point in $\mathcal{F}$. Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$, we have

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2} = \frac{y}{(cx+d)^2 + (cy)^2}.$$

Since $\det(\gamma) = 1$ we cannot have $c = d = 0$. Then as $y \neq 0$, $|cz+d|^2$ is bounded away from zero and moreover there are finitely many pairs (c, d) such that $|cz+d|^2$ is less than any given upper bound. Therefore there exists $\gamma_0 \in \mathrm{PSL}_2(\mathbb{Z})$ that minimizes $|cz+d|^2$ and hence maximizes $\mathrm{Im}(\gamma_0 z)$. In particular,

$$\mathrm{Im}(S\gamma_0 z) = \frac{\mathrm{Im}(\gamma_0 z)}{|\gamma_0 z|^2} \leq \mathrm{Im}(\gamma_0 z).$$

The inequality above implies $|\gamma_0 z| \geq 1$. Since $\mathrm{Im}(T^n \gamma_0 z) = \mathrm{Im}(\gamma_0 z)$ for all $n \in \mathbb{Z}$, repeating the argument above with $T^n\gamma_0$ in place of γ_0, we see that $|T^n\gamma_0 z| \geq 1$. But T shifts the real part by 1 so we can choose n such that $|\mathrm{Re}(T^n\gamma_0 z)| \leq \frac{1}{2}$. Therefore $T^n\gamma_0 \in \mathrm{PSL}_2(\mathbb{Z})$ sends z into $\mathcal{F}$ as desired. We will now show that if two points in $\mathcal{F}$ are $\mathrm{PSL}_2(\mathbb{Z})$-equivalent via a non-identity element then they lie on the boundary of $\mathcal{F}$. Since $\mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by automorphisms, by Proposition 9.1.1 it suffices to show that S and T map $\mathcal{F}$ outside of $\mathcal{F}$ except for possibly the boundary. This is clear for T since it maps the left boundary line $\left\{ z \in \mathbb{H} : \mathrm{Re}(z) = -\frac{1}{2} \text{ and } |z| \geq 1 \right\}$ to the right boundary line $\left\{ z \in \mathbb{H} : \mathrm{Re}(z) = \frac{1}{2} \text{ and } |z| \geq 1 \right\}$ and every other point of $\mathcal{F}$ is mapped to the right of this line. For S, note that it maps the semicircle $\{z \in \mathbb{H} : |z| = 1\}$ to itself (although not identically) and maps ∞ to zero. Since Möbius transformations send circles to circles and lines to lines it follows that every other point of $\mathcal{F}$ is taken to a point enclosed by the semicircle $\{z \in \mathbb{H} : |z| = 1\}$. Lastly, the interior of $\mathcal{F}$ is a domain since it is open and path-connected. This finishes the proof. □

The region $\mathcal{F}$, shaded in Figure 9.1, is called the **standard fundamental domain**. Figure 9.1 also displays how this fundamental domain changes under the actions of the generators of $\mathrm{PSL}_2(\mathbb{Z})$ as in Proposition 9.1.1. A fundamental domain for any other modular curve can be built from the standard fundamental domain as the following proposition shows (see [Kil15] for a proof):

Proposition 9.2.3. *Let Γ be any congruence subgroup. Then*

$$\mathcal{F}_\Gamma = \bigcup_{\gamma \in \Gamma\backslash\mathrm{PSL}_2(\mathbb{Z})} \gamma\mathcal{F},$$

is a fundamental domain for $\Gamma\backslash\mathbb{H}$.

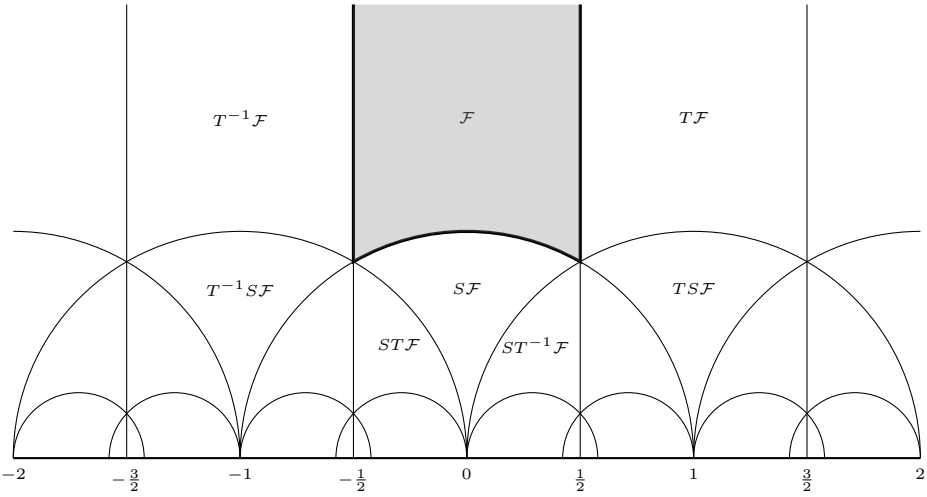


Figure 9.1: The standard fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$.

We might notice that $\mathcal{F}$ in Figure 9.1 is unbounded as it doesn't contain the point ∞. However, if we consider $\mathcal{F} \cup \{\infty\}$ then it would appear that this space is compact. The point ∞ is an example of a cusp and we now make this idea precise. Since any $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ preserves $\hat{\mathbb{R}}$ and γ has integer entries, γ also preserves $\mathbb{Q} \cup \{\infty\}$. A **cusp** of $\Gamma\backslash\mathbb{H}$ is an element of of $\Gamma\backslash(\mathbb{Q} \cup \{\infty\})$. As Γ has finite index in the modular group, there can only be finitely many cusps and the number of cusps is at most the index of Γ. In particular, the Γ-orbit of ∞ is a cusp of $\Gamma\backslash\mathbb{H}$. We denote cusps by gothic characters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ or by representatives of their equivalence classes. For example, we let ∞ denote the cusp $\Gamma\infty$.

Remark 9.2.1. *It turns out that the cusps can be represented as the points needed to make a fundamental domain $\mathcal{F}_\Gamma$ compact as a subset of $\hat{\mathbb{C}}$. To see this, suppose $\mathfrak{a}$ is a limit point of $\mathcal{F}_\Gamma$ that does not belong to $\mathcal{F}_\Gamma$. Then $\mathfrak{a} \in \hat{\mathbb{R}}$. In the case of the standard fundamental domain $\mathcal{F}$, $\mathfrak{a} = \infty$ which is a cusp. Otherwise, $\mathcal{F}_\Gamma$ is a union of images of $\mathcal{F}$ by Proposition 9.2.3 and since $\mathrm{PSL}_2(\mathbb{Z})\infty = \mathbb{Q} \cup \{\infty\}$, we find that $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$.*

Let $\Gamma_\mathfrak{a} \leq \Gamma$ denote the stabilizer subgroup of the cusp $\mathfrak{a}$. For the ∞ cusp, we can describe Γ_∞ explicitly. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ stabilizes ∞ then necessarily $c = 0$ and since $\det(\gamma) = 1$ we must have $a = d = 1$. Therefore $\gamma = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some $b \in \mathbb{Z}$ and γ acts on $\mathbb{H}$ by translation by b. Of course, not every translation is guaranteed to belong to Γ. Letting t be the smallest positive integer such that $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, we have $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \rangle$. In particular, Γ_∞ is an infinite cyclic group. We say that Γ is **reduced at infinity** if $t = 1$ so that $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$. In particular, $\Gamma_1(N)$ and $\Gamma_0(N)$ are reduced at infinity.

Remark 9.2.2. *If Γ is of level N then N is the smallest positive integer such that $\Gamma(N) \leq \Gamma$ so that $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$ is the minimal translation guaranteed to belong to Γ. However, there may be smaller translations so in general $t \leq N$.*

Moreover, for any cusp $\mathfrak{a}$ we have that $\Gamma_\mathfrak{a}$ is also an infinite cyclic group and we denote its generator by $\gamma_\mathfrak{a}$. To see this, if $\mathfrak{a} = \frac{a}{c}$ with $(a, c) = 1$ is a cusp of $\Gamma\backslash\mathbb{H}$ not equivalent to ∞ then there exists an $\sigma_\mathfrak{a} \in \mathrm{PSL}_2(\mathbb{Z})$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$. Indeed, there exists integers d and b such that $ad - bc = 1$ by Bézout's identity and then $\sigma_\mathfrak{a} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is such a matrix. It follows that $\Gamma_\mathfrak{a} = \sigma_\mathfrak{a}\Gamma_\infty\sigma_\mathfrak{a}^{-1}$ and since Γ_∞ is infinite cyclic so is $\Gamma_\mathfrak{a}$. We call any matrix $\sigma_\mathfrak{a} \in \mathrm{PSL}_2(\mathbb{Z})$ satisfying

$$\sigma_\mathfrak{a}\infty = \mathfrak{a} \quad \text{and} \quad \sigma_\mathfrak{a}^{-1}\gamma_\mathfrak{a}\sigma_\mathfrak{a} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

a **scaling matrix** for the cusp $\mathfrak{a}$. Note that $\sigma_{\mathfrak{a}}$ is determined up to composition on the right by an element of Γ_∞. Scaling matrices are useful because they allow us to transfer information at the cusp $\mathfrak{a}$ to the cusp at ∞. Let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$ with scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ respectively. When investigating holomorphic forms, it will be useful to have a double coset decomposition for sets of the form $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$. This is referred to as the **Bruhat decomposition** for Γ:

Theorem (Bruhat decomposition). *Let Γ be any congruence subgroup and let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$ with scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ respectively. Then we have the disjoint decomposition*

$$\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a},\mathfrak{b}}\Omega_\infty \cup \left(\bigcup_{\substack{c\geq 1 \\ d \pmod c}} \Omega_{d/c} \right),$$

where

$$\Omega_\infty = \Gamma_\infty\omega_\infty = \omega_\infty\Gamma_\infty = \Gamma_\infty\omega_\infty\Gamma_\infty \quad \text{and} \quad \Omega_{d/c} = \Gamma_\infty\omega_{d/c}\Gamma_\infty,$$

for some $\omega_\infty = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ and $\omega_{d/c} = \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ with $c \geq 1$ if such a matrix exists otherwise $\Omega_{d/c}$ is empty. Moreover, the entries a and d of $\omega_{d/c}$ are determined modulo c.

Proof. We first show that Ω_∞ is nonempty if and only if $\mathfrak{a} = \mathfrak{b}$. Indeed, if $\omega \in \Omega_\infty$ then $\omega = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}$ for some $\gamma \in \Gamma$. Then

$$\gamma\mathfrak{b} = \sigma_{\mathfrak{a}}\omega\sigma_{\mathfrak{b}}^{-1}\mathfrak{b} = \sigma_{\mathfrak{a}}\omega\infty = \sigma_{\mathfrak{a}}\infty = \mathfrak{a}.$$

This shows that $\mathfrak{a} = \mathfrak{b}$. Conversely, suppose $\mathfrak{a} = \mathfrak{b}$. Then $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ contains $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$ so that Ω_∞ is nonempty. So Ω_∞ is nonempty if and only if $\mathfrak{a} = \mathfrak{b}$. In this case, for any two elements $\omega = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}$ and $\omega' = \sigma_{\mathfrak{a}}^{-1}\gamma'\sigma_{\mathfrak{a}}$ of Ω_∞, we have

$$\gamma'\gamma^{-1}\mathfrak{a} = \sigma_{\mathfrak{a}}\omega'\omega^{-1}\sigma_{\mathfrak{a}}^{-1}\mathfrak{a} = \sigma_{\mathfrak{a}}\omega'\omega^{-1}\infty = \sigma_{\mathfrak{a}}\mathfrak{a}.$$

Hence $\gamma'\gamma^{-1} \in \Gamma_{\mathfrak{a}}$ which implies $\omega'\omega^{-1} = \sigma_{\mathfrak{a}}^{-1}\gamma'\gamma^{-1}\sigma_{\mathfrak{a}} \in \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$. Therefore

$$\Omega_\infty = \Gamma_\infty\omega = \omega\Gamma_\infty = \Gamma_\infty\omega\Gamma_\infty,$$

where the latter two equalities hold because ω is a translation and translations commute. Every other element of $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ belongs to one of the double cosets $\Omega_{d/c}$ with $c \geq 1$ (since we are working in $\mathrm{PSL}_2(\mathbb{Z})$). The relation

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & * \\ c & d \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+cn & * \\ c & d+cm \end{pmatrix},$$

shows that $\Omega_{d/c}$ is determined uniquely by c and $d \pmod c$. Moreover, this relation shows that a and d are determined modulo c. This completes the proof of the theorem. □

Notice that the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ implies

$$\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a},\mathfrak{b}}\omega_\infty \cup \left(\bigcup_{\substack{c\geq 1 \\ d \pmod c}} \omega_{d/c}\Gamma_\infty \right),$$

where it is understood that the coset $\omega_{d/c}\Gamma_\infty$ is empty if the double coset $\Omega_{d/c}$ is too. This shows that every element of $\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ corresponds to a unique $(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}$ with $c \geq 1$, $d \in \mathbb{Z}$, and $(c,d) = 1$, and

additionally the pair $(0,1)$ if and only if $\mathfrak{a} = \mathfrak{b}$ (this pair corresponds to ω_∞). Of course, this correspondence need not be surjective since many of the double cosets $\Omega_{d/c}$ may be empty. To track such c and d for which $\Omega_{d/c}$ is nonempty, let $\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ and $\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ be the sets given by

$$\mathcal{C}_{\mathfrak{a},\mathfrak{b}} = \left\{ c \geq 1 : \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\} \quad \text{and} \quad \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c) = \left\{ d \pmod{c} : \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\}.$$

Then $\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ and $\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ are precisely the sets of c and d take modulo c such that $\Omega_{d/c}$ is nonempty.

Remark 9.2.3. *The Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ implies*

$$\Gamma_\infty \backslash \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a},\mathfrak{b}}\omega_\infty \cup \left(\bigcup_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \omega_{d/c}\Gamma_\infty \right),$$

where none of the cosets $\omega_{d/c}\Gamma_\infty$ are empty. In particular, $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ if and only if (c,d) is a pair with $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $d \in \mathbb{Z}$, and $d \pmod{c} \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$, or additionally $(0,1)$ if $\mathfrak{a} = \mathfrak{b}$.

We will now introduce Kloosterman and Salié sums associated to cusps. We being with the Kloosterman sums. Let $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ be scaling matrices for the cusps $\mathfrak{a}$ and $\mathfrak{b}$ respectively. Then for any $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ and $n, m \in \mathbb{Z}$, the **generalized Kloosterman sum** $K_{\mathfrak{a},\mathfrak{b}}(n,m,c)$ relative to $\mathfrak{a}$ and $\mathfrak{b}$ is defined by

$$K_{\mathfrak{a},\mathfrak{b}}(n,m,c) = \sum_{d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)} e^{\frac{2\pi i(an+\overline{a}m)}{c}},$$

where a has been determined by $ad - bc = 1$. This sum is well-defined by the Bruhat decomposition because a is determined modulo c. In general, $K_{\mathfrak{a},\mathfrak{b}}(n,m,c)$ is not independent of the scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$. However, if $n = m = 0$ we trivially see by the Bruhat decomposition for Γ that

$$K_{\mathfrak{a},\mathfrak{b}}(0,0,c) = \left| \left\{ d \pmod{c} : \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\} \right|,$$

which is independent of the scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$. Moreover, for $\Gamma = \Gamma_1(1)$ we have $\mathfrak{a} = \mathfrak{b} = \infty$ and the Bruhat decomposition for $\Gamma_1(1)$ implies

$$K_{\infty,\infty}(n,m,c) = K(n,m,c),$$

is the usual Kloosterman sum. Therefore if $\mathfrak{a} = \mathfrak{b} = \infty$, we will suppress these dependencies accordingly. The Salié sums are defined in a similar manner. Let $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ be scaling matrices for the cusps $\mathfrak{a}$ and $\mathfrak{b}$ respectively. Then for any $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, and $n, m \in \mathbb{Z}$, and Dirichlet character χ with conductor $q \mid c$, the **generalized Salié sum** $S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c)$ relative to $\mathfrak{a}$ and $\mathfrak{b}$ is defined by

$$S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c) = \sum_{d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)} \chi(a) e^{\frac{2\pi i(an+\overline{a}m)}{c}},$$

where a has been determined by $ad - bc = 1$. This sum is well-defined by the Bruhat decomposition because a is determined modulo c. Like the generalized Kloosterman sum, $S_{\mathfrak{a},\mathfrak{b}}(n,m,c)$ need not independent of the scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$. Moreover, for $\Gamma = \Gamma_1(1)$ we have $\mathfrak{a} = \mathfrak{b} = \infty$ and the Bruhat decomposition for $\Gamma_1(1)$ implies

$$S_{\chi,\infty,\infty}(n,m,c) = S_\chi(n,m,c),$$

is the usual Salié sum. Therefore if $\mathfrak{a} = \mathfrak{b} = \infty$, we will suppress these dependencies accordingly.

9.3 The Hyperbolic Measure

We will also need to integrate over $\Gamma\backslash\mathbb{H}$. In order to do this, we require a measure on $\mathbb{H}$. Our choice of measure will be the **hyperbolic measure** $d\mu$ given by

$$d\mu = d\mu(z) = \frac{dx\,dy}{y^2}.$$

The most important property about the hyperbolic measure is that it is $\mathrm{GL}_2^+(\mathbb{R})$-invariant (see [DS05] for a proof):

Proposition 9.3.1. *The hyperbolic measure $d\mu$ is $\mathrm{GL}_2^+(\mathbb{R})$-invariant.*

As this fact will be used so frequently any time we integrate, we will not mention it explicitly. A particularly important fact is that if Γ is a congruence subgroup then $d\mu$ is Γ-invariant. One of the reasons this is useful is because we can apply the unfolding/folding method to many integrals. The most common instance is when we are integrating the sum $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma z)$ of some holomorphic function $f(z)$ over a fundamental domain $\mathcal{F}_\Gamma$ for $\Gamma\backslash\mathbb{H}$. Indeed, $\mathbb{H} = \bigcup_{\gamma\in\Gamma}\gamma\mathcal{F}_\Gamma$ and so $\Gamma_\infty\backslash\mathbb{H} = \bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma\mathcal{F}_\Gamma$. Since $\mathcal{F}_\Gamma$ is a fundamental domain, the conditions of the unfolding/folding method are satisfied and it follows that

$$\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma z)\,d\mu = \int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\,d\mu,$$

provided either side is absolutely convergent. It is also worth highlighting another fact. Any $\alpha\in\mathrm{GL}_2^+(\mathbb{Q})$ acts as an automorphism of $\mathbb{H}$ which implies that it induces a bijection between $\alpha^{-1}\Gamma\alpha\backslash\mathbb{H}$ and $\Gamma\backslash\mathbb{H}$ and hence between the fundamental domains $\mathcal{F}_{\alpha^{-1}\Gamma\alpha}$ and $\mathcal{F}_\Gamma$. Thus the change of variables $z\mapsto\alpha z$ transforms the fundamental domain $\mathcal{F}_\Gamma$ into $\mathcal{F}_{\alpha^{-1}\Gamma\alpha}$. Therefore

$$\int_{\mathcal{F}_\Gamma} f(z)\,d\mu = \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} f(\alpha z)\,d\mu,$$

provided either side is bounded. Now let us discuss the volume of $\Gamma\backslash\mathbb{H}$. We define the **volume** V_Γ of $\Gamma\backslash\mathbb{H}$ by

$$V_\Gamma = \int_{\mathcal{F}_\Gamma} d\mu.$$

In other words, V_Γ is the volume of the fundamental domain $\mathcal{F}_\Gamma$ with respect to the hyperbolic measure. Also, if $\mathcal{F}_\Gamma = \mathcal{F}$ we write $V_\Gamma = V$. Since the integrand is Γ-invariant, V_Γ is independent of the choice of fundamental domain. Using Proposition 9.2.2, we have

$$V = \int_{\mathcal{F}} d\mu = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\sqrt{1-x^2}}^{\infty}\frac{dy\,dx}{y^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x)\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\pi}{3}.$$

Therefore V is finite. There is also a simple relation between V_Γ and the index of Γ in $\mathrm{PSL}_2(\mathbb{Z})$:

$$V_\Gamma = [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma]V, \tag{9.1}$$

which follows immediately from Proposition 9.2.3. Moreover, V_Γ is finite for every congruence subgroup Γ by Equation (9.1) and that congruence subgroups have finite index in the modular group. A particularly nice application of this fact is that any integral of the form

$$\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\,d\mu,$$

is absolutely convergent provided $f(z)$ is bounded. That is, bounded functions are absolutely convergent over $\mathcal{F}_\Gamma$ with respect to $d\mu$. Moreover, we have a useful lemma:

Lemma 9.3.1. *Let Γ be a congruence subgroup and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. If $\alpha^{-1}\Gamma\alpha \subseteq \mathrm{PSL}_2(\mathbb{Z})$ then $V_{\alpha^{-1}\Gamma\alpha} = V_\Gamma$ and $[\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma\alpha] = [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma]$.*

Proof. The first statement follows from the chain

$$V_{\alpha^{-1}\Gamma\alpha} = \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} d\mu = \int_{\mathcal{F}_\Gamma} d\mu = V_\Gamma,$$

where the middle equality is justified by making the change of variables $z \mapsto \alpha^{-1}z$. The second statement is now immediate from Equation (9.1). □

Chapter 10

The Theory of Holomorphic Forms

Holomorphic forms are special classes of functions on the upper half-space $\mathbb{H}$ of the complex plane. They are holomorphic, have a transformation law with respect to a congruence subgroup, and satisfy a growth condition. We will introduce these forms in a general context. Throughout we assume that all of our congruence subgroups are reduced at infinity.

10.1 Holomorphic Forms

Define $j(\gamma, z)$ by

$$j(\gamma, z) = (cz + d),$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ and $z \in \mathbb{H}$. There is a very useful property that $j(\gamma, z)$ satisfies. To state it, let $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$. Then

$$\gamma\gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a'a + b'c & a'b + b'b \\ c'a + d'c & c'b + d'd \end{pmatrix},$$

and we have

$$\begin{aligned} j(\gamma', \gamma z) j(\gamma, z) &= \left(c' \frac{az + b}{cz + d} + d' \right) (cz + d) \\ &= (c'(az + b) + d'(cz + d)) \\ &= (c'a + d'c)z + c'b + d'd) \\ &= j(\gamma'\gamma, z). \end{aligned}$$

In short, we have relation

$$j(\gamma'\gamma, z) = j(\gamma', \gamma z) j(\gamma, z),$$

which is called the **cocycle condition** for $j(\gamma, z)$. For any integer $k \geq 1$ and any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ we define the **slash operator** $|_{j,k}\gamma : C(\mathbb{H}) \to C(\mathbb{H})$ to be the linear operator given by

$$(f|_{j,k}\gamma)(z) = \deg(\gamma)^{k-1} j(\gamma, z)^{-k} f(\gamma z).$$

If j is clear from content we will suppress this dependence accordingly. Note that if $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, the slash operator takes the simpler form

$$(f|_{j,k}\gamma)(z) = j(\gamma, z)^{-k} f(\gamma z).$$

The cocycle condition implies that the slash operator is multiplicative. Indeed, if $\gamma, \gamma' \in \mathrm{GL}_2^+(\mathbb{Q})$ then

$$\begin{aligned}((f|_{j,k}\gamma')|_{j,k}\gamma)(z) &= \deg(\gamma)^{k-1} j(\gamma, z)^{-k} (f|_{j,k}\gamma')(\gamma z) \\ &= \deg(\gamma'\gamma)^{k-1} j(\gamma', \gamma z)^{-k} j(\gamma, z)^{-k} f(\gamma'\gamma z) \\ &= \deg(\gamma'\gamma)^{k-1} j(\gamma'\gamma, z)^{-k} f(\gamma'\gamma z) && \text{cocycle condition} \\ &= (f|_{j,k}\gamma'\gamma)(z).\end{aligned}$$

If an operator commutes with the slash operators $|_{j,k}\gamma$ for every $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, we say that it is **invariant**. We will now introduce holomorphic forms. Let Γ be a congruence subgroup of level N and let χ be a Dirichlet character of conductor $q \mid N$, and set $\chi(\gamma) = \chi(d)$ for all for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is **holomorphic form** (or **modular form**) on $\Gamma\backslash\mathbb{H}$ of **weight** k, **level** N, and **character** χ if the following properties are satisfied:

(i) f is holomorphic on $\mathbb{H}$.

(ii) $(f|_{j,k}\gamma)(z) = \chi(\gamma) f(z)$ for all $\gamma \in \Gamma$.

(iii) $(f|_{j,k}\alpha)(z) = O(1)$ for all $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$ (or equivalently $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$).

We say f is a (holomorphic) **cusp form** if the additional property is satisfied:

(iv) For all cusps $\mathfrak{a}$ and any $y > 0$, we have

$$\int_0^1 (f|_k\sigma_{\mathfrak{a}})(x + iy)\, dx = 0.$$

Property (ii) is called the **modularity condition** and we say f is **modular**. In particular, f is a function on $\mathcal{F}_\Gamma$. The modularity condition can equivalently be expressed as

$$f(\gamma z) = \chi(\gamma) j(\gamma, z)^k f(z).$$

Property (iii) is called the **growth condition** for holomorphic forms and we say f is **holomorphic at the cusps**. Clearly we only need to verify the growth condition on a set of scaling matrices for the cusps. To see the equivalence in the growth condition, Lemma 9.1.1 implies that we may write $\alpha = \gamma\eta$ for some $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ and $\eta \in \mathrm{GL}_2^+(\mathbb{Q})$ with $\eta = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. This decomposition together with the cocycle condition gives

$$j(\alpha, z) = j(\gamma, \eta z),$$

and it follows that $(f|_{j,k}\alpha)(z) = O(1)$ for all $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ which proves the forward implication. The reverse implication is trivial since $\mathrm{PSL}_2(\mathbb{Z}) \subset \mathrm{GL}_2^+(\mathbb{Q})$. Holomorphic forms also admit Fourier series. Indeed, modularity implies

$$f(z+1) = f\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z\right) = f(z),$$

so that f is 1-periodic. Let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the $\mathfrak{a}$ cusp. As Lemma 9.1.2 implies $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$ is a congruence subgroup, it follows by the cocycle condition that $f|_k\sigma_{\mathfrak{a}}$ is a holomorphic form on $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}\backslash\mathbb{H}$ of the same weight and character as f. In particular, $f|_k\sigma_{\mathfrak{a}}$ is 1-periodic. Note that this means we only need to verify the growth condition as $y \to \infty$. As $f|_k\sigma_{\mathfrak{a}}$ is 1-periodic it admits a Fourier series given by

$$(f|_k\sigma_{\mathfrak{a}})(z) = \sum_{n \geq 0} a_{\mathfrak{a}}(n, y) e^{2\pi i n x},$$

where the sum is only over $n \geq 0$ because holomorphy at the cusps implies that $f|_k\sigma_{\mathfrak{a}}$ is bounded. As f is smooth (since it is holomorphic), it converges uniformly to its Fourier series everywhere. We can simplify the Fourier coefficients $a_{\mathfrak{a}}(n, y)$. To see this, since $f|_k\sigma_{\mathfrak{a}}$ is holomorphic it satisfies the first order Cauchy-Riemann equations so that

$$\frac{1}{2}\left(\frac{\partial f|_k\sigma_{\mathfrak{a}}}{\partial x} + i\frac{\partial f|_k\sigma_{\mathfrak{a}}}{\partial y}\right) = 0.$$

Substituting in the Fourier series and equating coefficients we obtain the ODE

$$2\pi n a_{\mathfrak{a}}(n, y) + a_{\mathfrak{a},y}(n, y) = 0,$$

Solving this ODE by separation of variables, we see that there exists an $a_{\mathfrak{a}}(n)$ such that

$$a_{\mathfrak{a}}(n, y) = a_{\mathfrak{a}}(n)e^{-2\pi ny}.$$

The coefficients $a_{\mathfrak{a}}(n)$ are the only part of the Fourier series depending on the implicit congruence subgroup Γ. Using these coefficients instead, f admits a **Fourier series** at the $\mathfrak{a}$ cusp given by

$$(f|_k\sigma_{\mathfrak{a}})(z) = \sum_{n\geq 0} a_{\mathfrak{a}}(n)e^{2\pi inz},$$

with **Fourier coefficients** $a_{\mathfrak{a}}(n)$. If $\mathfrak{a} = \infty$, we will drop this dependence and in this case $f|_k\sigma_{\mathfrak{a}} = f$. Moreover, property (iv) implies that f is a cusp form if and only if $a_{\mathfrak{a}}(n) = 0$ for every cusp $\mathfrak{a}$. We can also easily derive a bound for the size of the Fourier coefficients of cusp forms. To see this, note that $\left|(f|_k\sigma_{\mathfrak{a}})(z)\mathrm{Im}(z)^{\frac{k}{2}}\right|$ is $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$-invariant by the modularity of $f|_k\sigma_{\mathfrak{a}}$, the cocycle condition, the identity $\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}z)^{\frac{k}{2}} = \frac{\mathrm{Im}(z)^{\frac{k}{2}}}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}},z)|^k}$, and that $|\chi(\gamma)| = 1$. Moreover, this function is bounded on $\mathcal{F}_{\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}}$ because f is a cusp form. Then $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$-invariance implies $\left|(f|_k\sigma_{\mathfrak{a}})(z)\mathrm{Im}(z)^{\frac{k}{2}}\right|$ is bounded on $\mathbb{H}$. From the definition of Fourier series, it follows that

$$a_{\mathfrak{a}}(n)\mathrm{Im}(z)^{\frac{k}{2}} = \int_0^1 (f|_k\sigma_{\mathfrak{a}})(z)\mathrm{Im}(z)^{\frac{k}{2}}e^{-2\pi inz}\,dx \ll \int_0^1 e^{2\pi ny}\,dx \ll e^{2\pi ny}.$$

Upon setting $y = \frac{1}{n}$, the last expression is constant and we obtain

$$a_{\mathfrak{a}}(n) \ll n^{\frac{k}{2}}.$$

This bound is known as the **Hecke bound** for holomorphic forms. It follows from the Hecke bound that

$$(f|_k\sigma_{\mathfrak{a}})(z) = O\left(\sum_{n\geq 1} n^{\frac{k}{2}}e^{-2\pi ny}\right) = O(e^{-2\pi y}),$$

where the second equality holds because each term is of smaller order than the next so that the series is bounded by a constant times its first term. This implies that $(f|_k\sigma_{\mathfrak{a}})(z)$ exhibits exponential decay. Accordingly, we say that f exhibits **exponential decay at the cusps**. Observe that $(f|_k\sigma_{\mathfrak{a}})$ is then bounded on $\mathbb{H}$ and, in particular, f is bounded on $\mathbb{H}$.

10.2 Poincaré and Eisenstein Series

Let Γ be a congruence subgroup of level N. We will introduce two important classes of holomorphic forms on $\Gamma\backslash\mathbb{H}$ namely the Poincaré and Eisenstein series. Let $m \geq 0$, $k \geq 4$, χ be a Dirichlet character with conductor $q \mid N$, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. We define the m-th (holomorphic) **Poincaré series** $P_{m,k,\chi,\mathfrak{a}}(z)$ of weight k with character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp by

$$P_{m,k,\chi,\mathfrak{a}}(z) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma) j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k} e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z}.$$

We call m the **index** of $P_{m,k,\chi,\mathfrak{a}}(z)$. If χ is the trivial character or $\mathfrak{a} = \infty$, we will drop these dependencies accordingly.

Remark 10.2.1. *The reason why we restrict to $k \geq 4$ is because for $k = 0, 2$ the Poincaré series need not converge (see Proposition B.8.1).*

We first verify that $P_{m,k,\chi,\mathfrak{a}}(z)$ is well-defined. It suffices to show that the summands are independent of the representatives γ and $\sigma_{\mathfrak{a}}$. To see that $\overline{\chi}(\gamma)$ is independent of γ, recall that $\Gamma_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}$ and let $\gamma' = \sigma_{\mathfrak{a}}\eta_\infty\sigma_{\mathfrak{a}}^{-1}\gamma$ with $\eta_\infty = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_\infty$. Then

$$\overline{\chi}(\gamma') = \overline{\chi}(\sigma_{\mathfrak{a}}\eta_\infty\sigma_{\mathfrak{a}}^{-1}\gamma) = \overline{\chi}(\sigma_{\mathfrak{a}})\chi(\eta_\infty)\overline{\chi}(\sigma_{\mathfrak{a}})^{-1}\overline{\chi}(\gamma) = \overline{\chi}(\gamma),$$

verifying that $\overline{\chi}(\gamma)$ is independent of the representative γ. As the set of representatives for the scaling matrix $\sigma_{\mathfrak{a}}$ is $\sigma_{\mathfrak{a}}\Gamma_\infty$ and the set of representatives for γ is $\Gamma_{\mathfrak{a}}\gamma$, we see that the set of representatives for $\sigma_{\mathfrak{a}}^{-1}\gamma$ is $\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\gamma$. But as $\Gamma_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}$, this set of representatives is $\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}\gamma$ and therefore it remains to verify independence from multiplication on the left by an element of Γ_∞ namely η_∞. The cocycle relation gives

$$j(\eta_\infty\sigma_{\mathfrak{a}}^{-1}\gamma, z) = j(\eta_\infty, \sigma_{\mathfrak{a}}^{-1}\gamma z) j(\sigma_{\mathfrak{a}}^{-1}\gamma, z) = j(\sigma_{\mathfrak{a}}^{-1}\gamma, z),$$

where the last equality follows because $j(\eta_\infty, \sigma_{\mathfrak{a}}^{-1}\gamma z) = 1$. This verifies that $j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)$ is independent of the representatives γ and $\sigma_{\mathfrak{a}}$. Moreover, we have

$$e^{2\pi i m \eta_\infty \sigma_{\mathfrak{a}}^{-1}\gamma z} = e^{2\pi i m(\sigma_{\mathfrak{a}}^{-1}\gamma z + n)} = e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma} e^{2\pi i m n z} = e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma},$$

which verifies that $e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z}$ is independent of the representatives γ and $\sigma_{\mathfrak{a}}$. Therefore $P_{m,k,\chi,\mathfrak{a}}(z)$ is well-defined. To see that $P_{m,k,\chi,\mathfrak{a}}(z)$ is holomorphic on $\mathbb{H}$, first note that $|e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z}| = e^{-2\pi m \mathrm{Im}(\mathfrak{a}^{-1}\gamma z)} < 1$. Then the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma$ gives

$$P_{m,k,\chi,\mathfrak{a}}(z) \ll \sum_{(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \frac{1}{|cz + d|^k}.$$

As $k \geq 4$, this latter series is locally absolutely uniformly convergent for $z \in \mathbb{H}$ by Proposition B.8.1. Hence $P_{m,k,\chi,\mathfrak{a}}(z)$ does too and so it is holomorphic on $\mathbb{H}$. We now verify modularity for $P_{m,k,\chi,\mathfrak{a}}(z)$. This

is just a computation:

$$\begin{aligned}
P_{m,k,\chi,\mathfrak{a}}(\gamma z) &= \sum_{\gamma' \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\chi}(\gamma') j(\sigma_{\mathfrak{a}}^{-1}\gamma', \gamma z)^{-k} j(\gamma, z)^{-k} e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \sum_{\gamma' \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\chi}(\gamma') \left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma, z)}{j(\gamma, z)} \right)^{-k} j(\gamma, z)^{-k} e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= j(\gamma, z)^k \sum_{\gamma' \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\chi}(\gamma') j(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma, z)^{-k} e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \chi(\gamma) j(\gamma, z)^k \sum_{\gamma' \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\chi}(\gamma'\gamma) j(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma, z)^{-k} e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \chi(\gamma) j(\gamma, z)^k \sum_{\gamma' \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\chi}(\gamma') j(\sigma_{\mathfrak{a}}^{-1}\gamma', z)^{-k} e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma' z} \\
&= \chi(\gamma) P_{m,k,\chi,\mathfrak{a}}(z),
\end{aligned}$$

where in the second line we have used the cocycle condition and in the second to last line we have made the change of variables $\gamma' \mapsto \gamma'\gamma^{-1}$. To verify the growth condition, we will need a technical lemma:

Lemma 10.2.1. *Let $a, b > 0$ be reals and consider the half-strip*

$$S_{a,b} = \{z \in \mathbb{H} : |x| \leq a \text{ and } y \geq b\}.$$

Then there is a $\delta \in (0,1)$ such that

$$|nz + m| \geq \delta |ni + m|,$$

for all $n, m \in \mathbb{Z}$ and all $z \in S_{a,b}$.

Proof. If $n = 0$ then any δ is sufficient. If $n \neq 0$ then the desired inequality is equivalent to

$$\left| \frac{z + \frac{m}{n}}{i + \frac{n}{m}} \right| \geq \delta.$$

So consider the function

$$f(z, r) = \left| \frac{z + r}{i + r} \right|,$$

for $z \in S_{a,b}$ and $r \in \mathbb{R}$. It suffices to show $f(z, r) \geq \delta$. As $z \in \mathbb{H}$, $z - r \neq 0$ so that $f(z, r)$ is continuous and positive on $S_{a,b} \times \mathbb{R}$. Now let $Y > b$ and consider the region

$$S_{a,b}^Y = \{z \in \mathbb{H} : |x| \leq a \text{ and } b \leq y \leq Y\}.$$

We claim that there exists a Y such that if $y > Y$ and $|x| > Y$ then $f(z, r)^2 > \frac{1}{4}$. Indeed, we compute

$$f(z, r)^2 = \frac{(z + r)(\overline{z} + r)}{(i + r)(-i + r)} = \frac{|z|^2 + 2xr + r^2}{1 + r^2} \geq \frac{y + r^2}{1 + r^2},$$

where in the inequality we have used the bound $|z|^2 \geq y$ and that x is bounded. Now $\frac{r^2}{1+r^2} \to 1$ as $r \to \pm\infty$ so there exists a Y such that $|r| > Y$ implies $\frac{r^2}{1+r^2} \geq \frac{1}{4}$. Then

$$\frac{y + r^2}{1 + r^2} \geq \frac{y}{1 + r^2} + \frac{r^2}{1 + r^2} \geq \frac{y}{1 + r^2} + \frac{1}{4} > \frac{1}{4}.$$

It follows that $f(z, r) > \frac{1}{2}$ outside of $S_{a,b}^Y \times [-Y, Y]$. But this latter region is compact and so $f(z, r)$ obtains a minimum δ' on it. Setting $\delta = \min\left(\frac{1}{2}, \delta'\right)$ completes the proof. □

We can now verify the growth condition for $P_{m,k,\chi,\mathfrak{a}}(z)$. Let $\sigma_{\mathfrak{b}}$ be a scaling matrix for the cusp $\mathfrak{b}$. Then the bound $|e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z}| = e^{-2\pi m\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)} < 1$, cocycle condition, and Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ together give

$$j(\sigma_{\mathfrak{b}},z)^{-k}P_{m,k,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}}z) \ll \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}}\frac{1}{|cz+d|^k}.$$

Now decompose this last sum as

$$\sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}}\frac{1}{|cz+d|^k} = \sum_{d\neq 0}\frac{1}{d^k} + \sum_{c\neq 0}\sum_{d\in\mathbb{Z}}\frac{1}{|cz+d|^k} = 2\sum_{d\geq 1}\frac{1}{d^k} + 2\sum_{c\geq 1}\sum_{d\in\mathbb{Z}}\frac{1}{|cz+d|^k}.$$

Since the first sum is absolutely uniformly bounded, it suffices to show that the double sum is too. To see this, let $y \geq 1$ and δ be as in Lemma 10.2.1. Then for any integer $N \geq 1$ we can write

$$\begin{aligned}\sum_{c\geq 1}\sum_{d\in\mathbb{Z}}\frac{1}{|cz+d|^k} &= \sum_{c+|d|\leq N}\frac{1}{|cz+d|^k} + \sum_{c+|d|>N}\frac{1}{|cz+d|^k}\\ &\leq \sum_{c+|d|\leq N}\frac{1}{|cz+d|^k} + \sum_{c+|d|>N}\frac{1}{(\delta|ci+d|)^k}\\ &\leq \sum_{c+|d|\leq N}\frac{1}{|cz+d|^k} + \frac{1}{\delta^k}\sum_{c+|d|>N}\frac{1}{|ci+d|^k}.\end{aligned}$$

As $\sum_{c\geq 1}\sum_{d\in\mathbb{Z}}\frac{1}{|ci+d|^k}$ converges by Proposition B.8.1, the second sum tends to zero as $N\to\infty$. As for the first sum, it is finite and each term is bounded. Thus the double sum is absolutely uniformly bounded. This verifies the growth condition. We collect this work as a theorem:

Theorem 10.2.1. *Let $m \geq 0$, $k \geq 4$, χ be a Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. The Poincaré series*

$$P_{m,k,\chi,\mathfrak{a}}(z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-k}e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z},$$

is a weight k holomorphic form with character χ on $\Gamma\backslash\mathbb{H}$.

For $m = 0$, we write $E_{k,\chi,\mathfrak{a}}(z) = P_{0,k,\chi,\mathfrak{a}}(z)$ and call $E_{k,\chi,\mathfrak{a}}(z)$ the (holomorphic) **Eisenstein series** of weight k and character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp. It is defined by

$$E_{k,\chi,\mathfrak{a}}(z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\chi}(\gamma)j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-k}.$$

If χ is the trivial character or $\mathfrak{a} = \infty$, we will drop these dependencies accordingly. In particular, we have already verified the following theorem:

Theorem 10.2.2. *Let $k \geq 4$, χ be Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. The Eisenstein series*

$$E_{k,\chi,\mathfrak{a}}(z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\chi}(\gamma)j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-k},$$

is a weight k holomorphic form with character χ on $\Gamma\backslash\mathbb{H}$.

We will now compute the Fourier series of the Poincaré series with positive index:

Proposition 10.2.1. *Let $m \geq 1$, $k \geq 4$, χ be Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$. The Fourier series of $P_{m,k,\chi,\mathfrak{a}}(z)$ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{b}$ cusp is given by*

$$(P_{m,k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t\geq 1}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t} + \left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1}\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{2\pi i^{-k}}{c}J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right)S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c)\right)e^{2\pi itz}.$$

Proof. From the cocycle condition, the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$, and Remark 9.2.3, we have

$$(P_{m,k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \delta_{\mathfrak{a},\mathfrak{b}}e^{2\pi imz} + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}},d\in\mathbb{Z}\\ d\pmod c\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\overline{\chi}(d)\frac{e^{2\pi im\left(\frac{a}{c}-\frac{1}{c^2z+cd}\right)}}{(cz+d)^k},$$

where a has been determined modulo c by $ad-bc=1$ and we have used the fact that

$$\frac{a}{c} - \frac{1}{c^2z+cd} = \frac{az+b}{cz+d}.$$

Summing over all pairs (c,d) with $c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $d\in\mathbb{Z}$, and $d\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is the same as summing over all triples (c,ℓ,r) with $c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $\ell\in\mathbb{Z}$, and r taken modulo c with $r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$. Indeed, this is seen by writing $d=c\ell+r$. Moreover, since $ad-bc=1$ we have $a(c\ell+r)-bc=1$ which further implies that $ar\equiv 1 \pmod c$. So we may take a to be the inverse for r modulo c. Then

$$\begin{aligned}
\sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}},d\in\mathbb{Z}\\ d\pmod c\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\overline{\chi}(d)\frac{e^{2\pi im\left(\frac{a}{c}-\frac{1}{c^2z+cd}\right)}}{(cz+d)^k} &= \sum_{(c,\ell,r)}\overline{\chi}(c\ell+r)\frac{e^{2\pi im\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k}\\
&= \sum_{(c,\ell,r)}\overline{\chi}(r)\frac{e^{2\pi im\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k}\\
&= \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\sum_{\ell\in\mathbb{Z}}\overline{\chi}(r)\frac{e^{2\pi im\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k}\\
&= \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\overline{\chi}(r)\sum_{\ell\in\mathbb{Z}}\frac{e^{2\pi im\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k},
\end{aligned}$$

where on the right-hand side it is understood that we are summing over all triples (c,ℓ,r) with the prescribed properties and the second line holds since χ has conductor diving the level and $d\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is determined modulo c. Now let

$$I_{c,r}(z) = \sum_{\ell\in\mathbb{Z}}\frac{e^{2\pi im\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k}.$$

We will apply the Poisson summation formula to $I_{c,r}(z)$. By the identity theorem it suffices to apply the Poisson summation formula for $z=iy$ with $y>0$. Accordingly, let $f(x)$ be given by

$$f(x) = \frac{e^{2\pi im\left(\frac{a}{c}-\frac{1}{c^2x+cr+ic^2y}\right)}}{(cx+r+icy)^k}.$$

Then $f(x)$ is absolutely integrable on $\mathbb{R}$ because it exhibits polynomial decay of order $k > 1$. We compute its Fourier transform

$$(\mathcal{F}f)(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx = \int_{-\infty}^{\infty} \frac{e^{2\pi im\left(\frac{a}{c} - \frac{1}{c^2x+cr+ic^2y}\right)}}{(cx+r+icy)^k} e^{-2\pi itx}\,dx.$$

Complexify the integral to get

$$\int_{\mathrm{Im}(z)=0} \frac{e^{2\pi im\left(\frac{a}{c} - \frac{1}{c^2z+cr+ic^2y}\right)}}{(cz+r+icy)^k} e^{-2\pi itz}\,dz.$$

Now make the change of variables $z \mapsto z - \frac{r}{c} - icy$ to obtain

$$e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}-2\pi ty} \int_{\mathrm{Im}(z)=y} \frac{e^{-\frac{2\pi im}{c^2z}}}{(cz)^k} e^{-2\pi itz}\,dz.$$

The integrand is meromorphic with a pole at $z = 0$. Moreover, we have

$$\frac{1}{(cz)^k} \ll \frac{1}{|cz|^k}, \quad e^{-\frac{2\pi im}{c^2z}} \ll e^{-\frac{2\pi m\mathrm{Im}(z)}{|cz|^2}}, \quad \text{and} \quad e^{-2\pi itz} \ll e^{2\pi ty}.$$

The first expression has polynomial decay, the second expression is bounded, and the third expression exhibits exponential decay if and only if $t < 0$ and when $t = 0$ it is bounded. So when $t \leq 0$ we may take the limit as $\mathrm{Im}(z) \to \infty$ by shifting the line of integration and conclude that the integral vanishes. It remains to compute the integral for $t \geq 1$. To do this, make the change of variables $z \mapsto -\frac{z}{2\pi it}$ to the last integral to rewrite it as

$$\begin{aligned}
-\frac{1}{2\pi it} \int_{(2\pi ty)} \frac{e^{-\frac{4\pi^2 mt}{c^2z}}}{\left(-\frac{cz}{2\pi it}\right)^k} e^z\,dz &= -\frac{1}{2\pi it} \int_{(2\pi ty)} \left(-\frac{2\pi it}{cz}\right)^k e^{z-\frac{4\pi^2mt}{c^2z}}\,dz \\
&= \frac{(-2\pi it)^{k-1}}{c^k} \int_{(2\pi ty)} z^{-k} e^{z-\frac{4\pi^2mt}{c^2z}}\,dz \\
&= \frac{(-2\pi it)^{k-1}}{c^k} \int_{-\infty}^{(0^+)} z^{-k} e^{z-\frac{4\pi^2mt}{c^2z}}\,dz \\
&= \frac{2\pi i^{-k}}{c} \left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right),
\end{aligned}$$

where in the second to last line we have homotoped the line of integration to a Hankel contour about the negative real axis and in the last line we have used the Schläflin integral representation for the J-Bessel function (see Appendix B.6). So in total we obtain

$$(\mathcal{F}f)(t) = \begin{cases} \left(\frac{2\pi i^{-k}}{c}\left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right) e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right) e^{-2\pi ty} & \text{if } t \geq 1, \\ 0 & \text{if } t \leq 0. \end{cases}$$

By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z) = \sum_{t\geq 1} \left(\frac{2\pi i^{-k}}{c}\left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right) e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right) e^{2\pi itz}.$$

Plugging this back into the Poincaré series gives a form of the Fourier series:

$$\begin{aligned}(P_{m,k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) &= \delta_{\mathfrak{a},\mathfrak{b}}e^{2\pi imz} + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r)\sum_{t\geq 1}\left(\frac{2\pi i^{-k}}{c}\left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1}J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right)e^{2\pi imz}\\ &= \sum_{t\geq 1}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t} + \left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1}\sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}}\overline{\chi}(r)\frac{2\pi i^{-k}}{c}J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right)e^{2\pi itz}\\ &= \sum_{t\geq 1}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t} + \left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1}\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{2\pi i^{-k}}{c}J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right)\sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\overline{\chi}(r)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right)e^{2\pi itz}.\end{aligned}$$

We will simplify the innermost sum. Since a is the inverse for r modulo c, the innermost sum above becomes

$$\sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\overline{\chi}(r)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}} = \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\overline{\chi}(\overline{a})e^{2\pi im\frac{a}{c}+2\pi it\frac{\overline{a}}{c}} = \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}\chi(a)e^{\frac{2\pi i(am+\overline{a}t)}{c}} = S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c).$$

So at last, we obtain our desired Fourier series:

$$(P_{m,k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t\geq 1}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t} + \left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1}\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{2\pi i^{-k}}{c}J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right)S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c)\right)e^{2\pi itz}.$$

□

An immediate consequence of Proposition 10.2.1 is that the Poincaré series $P_{m,k,\chi,\mathfrak{a}}(z)$ with positive index are cusp forms. In a similar manner, we can obtain the Fourier series of the Eisenstein series too:

Proposition 10.2.2. *Let $k \geq 4$, χ be Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$. The Fourier series of $E_{k,\chi,\mathfrak{a}}(z)$ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{b}$ cusp is given by*

$$(E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t\geq 0}\left(\delta_{\mathfrak{a},\mathfrak{b}} + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{(-2\pi it)^k}{(k-1)!c^k}S_{\chi,\mathfrak{a},\mathfrak{b}}(0,t,c)\right)e^{2\pi itz}.$$

Proof. From the cocycle condition, the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$, and Remark 9.2.3, we have

$$(E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \delta_{\mathfrak{a},\mathfrak{b}} + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}},d\in\mathbb{Z}\\ d\pmod c\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\overline{\chi}(d)\frac{1}{(cz+d)^k},$$

where a has been determined modulo c by $ad-bc=1$ and we have used the fact that

$$\frac{a}{c} - \frac{1}{c^2z+cd} = \frac{az+b}{cz+d}.$$

Summing over all pairs (c,d) with $c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $d\in\mathbb{Z}$, and $d\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is the same as summing over all triples (c,ℓ,r) with $c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $\ell\in\mathbb{Z}$, and r taken modulo c with $r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$. Indeed, this is seen by writing

$d = c\ell + r$. Moreover, since $ad - bc = 1$ we have $a(c\ell + r) - bc = 1$ which further implies that $ar \equiv 1 \pmod{c}$. So we may take a to be the inverse for r modulo c. Then

$$\begin{aligned}\sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}, d \in \mathbb{Z} \\ d \pmod{c} \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(d) \frac{1}{(cz+d)^k} &= \sum_{(c,\ell,r)} \overline{\chi}(c\ell + r) \frac{1}{(cz + c\ell + r)^k} \\ &= \sum_{(c,\ell,r)} \overline{\chi}(r) \frac{1}{(cz + c\ell + r)^k} \\ &= \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \sum_{\ell \in \mathbb{Z}} \overline{\chi}(r) \frac{1}{(cz + c\ell + r)^k} \\ &= \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(r) \sum_{\ell \in \mathbb{Z}} \frac{1}{(cz + c\ell + r)^k},\end{aligned}$$

where on the right-hand side it is understood that we are summing over all triples (c, ℓ, r) with the prescribed properties and the second line holds since χ has conductor diving the level and $d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is determined modulo c. Now let

$$I_{c,r}(z) = \sum_{\ell \in \mathbb{Z}} \frac{1}{(cz + c\ell + r)^k}.$$

We apply the Poisson summation formula to $I_{c,r}(z)$. By the identity theorem it suffices to apply the Poisson summation formula for $z = iy$ with $y > 0$. So let $f(x)$ be given by

$$f(x) = \frac{1}{(cx + r + icy)^k}.$$

Then $f(x)$ is absolutely integrable on $\mathbb{R}$ because it exhibits polynomial decay of order $k > 1$. We compute its Fourier transform

$$(\mathcal{F}f)(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x}\, dx = \int_{-\infty}^{\infty} \frac{e^{-2\pi i t x}}{(cx + r + icy)^k}\, dx.$$

Complexify the integral to get

$$\int_{\mathrm{Im}(z)=0} \frac{e^{-2\pi i t z}}{(cz + r + icy)^k}\, dz.$$

Now make the change of variables $z \mapsto z - \frac{r}{c} - icy$ to obtain

$$e^{2\pi i t \frac{r}{c} - 2\pi t y} \int_{\mathrm{Im}(z)=y} \frac{e^{-2\pi i t z}}{(cz)^k}\, dz.$$

The integrand is meromorphic with a pole at $z = 0$. Moreover, we have

$$\frac{1}{(cz)^k} \ll \frac{1}{|cz|^k} \quad \text{and} \quad e^{-2\pi i t z} \ll e^{2\pi t y}.$$

The first expression has polynomial decay while the expression exhibits exponential decay if and only if $t < 0$ and when $t = 0$ it is bounded. So when $t \leq 0$ we may take the limit as $\mathrm{Im}(z) \to \infty$ by shifting the

line of integration and conclude that the integral vanishes. It remains to compute the integral for $t \geq 1$. To do this, make the change of variables $z \mapsto -\frac{z}{2\pi it}$ to the last integral to rewrite it as

$$-\frac{1}{2\pi it}\int_{(2\pi ty)} \frac{e^z}{\left(-\frac{cz}{2\pi it}\right)^k}\,dz = -\frac{1}{2\pi it}\int_{(2\pi ty)} \left(-\frac{2\pi it}{cz}\right)^k e^z\,dz = \frac{(-2\pi it)^{k-1}}{c^k}\int_{(2\pi ty)} \frac{e^z}{z^k}\,dz.$$

The integrand of the last integral has a pole of order k at $z = 0$. To find the residue, the Laurent series of $\frac{e^z}{z^k}$ is

$$\frac{e^z}{z^k} = \sum_{n\geq 0} \frac{z^{n-k}}{n!},$$

and thus the residue of the integrand is $\frac{1}{(k-1)!}$. In shifting the line of integration to $(-x)$, for some $x > 0$, we pass by this pole and obtain

$$\frac{(-2\pi it)^k}{(k-1)!c^k} + \int_{(-x)} \frac{e^z}{z^k}\,dz.$$

Moreover, we have

$$\frac{1}{z^k} \ll \frac{1}{|z|^k} \quad \text{and} \quad e^z \ll e^x.$$

The first expression has polynomial decay while the second expression exhibits exponential decay provided $x < 0$. Therefore we make take the limit as $x \to \infty$ by shifting the line of integration again and conclude that the latter integral vanishes. Altogether, we have show that

$$(\mathcal{F}f)(t) = \begin{cases} \left(\frac{(-2\pi it)^k}{(k-1)!c^k} e^{2\pi it\frac{r}{c}}\right) e^{-2\pi ty} & \text{if } t \geq 1, \\ 0 & \text{if } t \leq 0. \end{cases}$$

By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z) = \sum_{t\geq 1} \left(\frac{(-2\pi it)^k}{(k-1)!c^k} e^{2\pi it\frac{r}{c}}\right) e^{2\pi itz}.$$

Substituting this back into the Eisenstein series gives a form of the Fourier series:

$$\begin{aligned}(E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) &= \delta_{\mathfrak{a},\mathfrak{b}} + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r) \sum_{t\geq 1} \left(\frac{(-2\pi it)^k}{(k-1)!c^k} e^{2\pi it\frac{r}{c}}\right) e^{2\pi imz} \\ &= \sum_{t\geq 0} \left(\delta_{\mathfrak{a},\mathfrak{b}} + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r) \frac{(-2\pi it)^k}{(k-1)!c^k} e^{2\pi it\frac{r}{c}}\right) e^{2\pi itz} \\ &= \sum_{t\geq 0} \left(\delta_{\mathfrak{a},\mathfrak{b}} + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{(-2\pi it)^k}{(k-1)!c^k} \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(r) e^{2\pi it\frac{r}{c}}\right) e^{2\pi itz}.\end{aligned}$$

We will simplify the innermost sum. Since a is the inverse for r modulo c, the innermost sum above becomes

$$\sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(r) e^{2\pi it\frac{r}{c}} = \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(\overline{a}) e^{2\pi it\frac{\overline{a}}{c}} = \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \chi(a) e^{\frac{2\pi i\overline{a}t}{c}} = S_{\chi,\mathfrak{a},\mathfrak{b}}(0,t,c).$$

So at last, we obtain our desired Fourier series:

$$(E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t\geq 0}\left(\delta_{\mathfrak{a},\mathfrak{b}} + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{(-2\pi it)^k}{(k-1)!c^k} S_{\chi,\mathfrak{a},\mathfrak{b}}(0,t,c)\right) e^{2\pi itz}.$$

□

An interesting observation from Proposition 10.2.2 is that $E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}}$ is necessarily a cusp form unless $\mathfrak{a} = \mathfrak{b}$.

10.3 Inner Product Spaces of Holomorphic Forms

Let $\mathcal{H}_k(\Gamma,\chi)$ denote the complex vector space of all weight k holomorphic forms with character χ on $\Gamma\backslash\mathbb{H}$ and $\mathcal{S}_k(\Gamma,\chi)$ denote the associated subspace of cusp forms. Moreover, if χ is the trivial character, we will suppress the dependence upon χ. Note that if Γ_1 and Γ_2 are two congruence subgroups such that $\Gamma_1 \leq \Gamma_2$ then we have the inclusion

$$\mathcal{H}_k(\Gamma_2,\chi) \subseteq \mathcal{H}_k(\Gamma_1,\chi),$$

and this respects the subspace of cusp forms. So in general, the smaller the congruence subgroup the more holomorphic forms there are. Our goal is to turn $\mathcal{S}_k(\Gamma,\chi)$ into a complex Hilbert space to which we can apply a linear theory. To this end, for $f,g\in\mathcal{S}_k(\Gamma,\chi)$ define their **Petersson inner product** by

$$\langle f,g\rangle_\Gamma = \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu.$$

If the congruence subgroup is clear from context we will suppress the dependence upon Γ. Since f and g have exponential decay at the cusps, the integrand is bounded and thus is locally absolutely uniformly convergent because we are integrating over a region of finite volume. The integrand is also Γ-invariant so that the integral is independent of the choice of fundamental domain. These two facts together imply that the Petersson inner product is well-defined. We will continue to use this notation even if f and g do not belong to $\mathcal{S}_k(\Gamma,\chi)$ provided the integral is locally absolutely uniformly convergent. A basic property of the Petersson inner product is that it is invariant with respect to the slash operator:

Proposition 10.3.1. *For any $f,g\in\mathcal{S}_k(\Gamma,\chi)$ and $\alpha\in\mathrm{PSL}_2(\mathbb{Z})$, we have*

$$\langle f|_k\alpha, g|_k\alpha\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f,g\rangle_\Gamma.$$

Proof. This is just a computation:

$$\begin{aligned}
\langle f|_k\alpha, g|_k\alpha\rangle_{\alpha^{-1}\Gamma\alpha} &= \frac{1}{V_{\alpha^{-1}\Gamma\alpha}}\int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} (f|_k\alpha)(z)\overline{(g|_k\alpha)(z)}\mathrm{Im}(z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} (f|_k\alpha)(z)\overline{(g|_k\alpha)(z)}\mathrm{Im}(z)^k\,d\mu && \text{Lemma 9.3.1} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} |j(\alpha,z)|^{-2k} f(\alpha z)\overline{g(\alpha z)}\mathrm{Im}(z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |j(\alpha,z)|^{-2k} f(z)\overline{g(z)}\mathrm{Im}(\alpha z)^k\,d\mu && z\mapsto\alpha^{-1}z \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu \\
&= \langle f,g\rangle_\Gamma.
\end{aligned}$$

□

To show that $\mathcal{S}_k(\Gamma,\chi)$ is a complex Hilbert space, we will need a dimensionality result (see [DS05] for a proof in the case χ is trivial):

Theorem 10.3.1. *$\mathcal{H}_k(\Gamma,\chi)$ is finite dimensional.*

We can now show that the Petersson inner product turns $\mathcal{S}_k(\Gamma,\chi)$ into a complex Hilbert space:

Proposition 10.3.2. *$\mathcal{S}_k(\Gamma,\chi)$ is a complex Hilbert space with respect to the Petersson inner product.*

Proof. Let $f,g\in\mathcal{S}_k(\Gamma,\chi)$. Linearity of the integral immediately implies that the Petersson inner product is linear on $\mathcal{S}_k(\Gamma,\chi)$. It is also positive definite since

$$\langle f,f\rangle=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}f(z)\overline{f(z)}\mathrm{Im}(z)^k\,d\mu=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}|f(z)|^2\mathrm{Im}(z)^k\,d\mu\geq 0,$$

with equality if and only if f is identically zero. To see that it is conjugate symmetric, observe

$$\begin{aligned}\overline{\langle g,f\rangle}&=\overline{\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}g(z)\overline{f(z)}\mathrm{Im}(z)^k\,d\mu}\\&=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\overline{g(z)}f(z)\mathrm{Im}(z)^k\,\overline{d\mu}\\&=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\overline{g(z)}f(z)\mathrm{Im}(z)^k\,d\mu\qquad\qquad d\mu=\frac{dx\,dy}{y^2}\\&=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}f(z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu\\&=\langle f,g\rangle.\end{aligned}$$

So the Petersson inner product turns $\mathcal{S}_k(\Gamma,\chi)$ into a complex inner product space. Since $\mathcal{S}_k(\Gamma,\chi)$ is finite dimensional by Theorem 10.3.1, it follows immediately that $\mathcal{S}_k(\Gamma,\chi)$ is a complex Hilbert space. □

Now suppose $f\in\mathcal{S}_k(\Gamma,\chi)$ has Fourier coefficients $a_{\mathfrak{a}}(n)$ at the $\mathfrak{a}$ cusp. Define linear functionals $\phi_{m,k,\chi,\mathfrak{a}}:\mathcal{S}_k(\Gamma,\chi)\to\mathbb{C}$ by

$$\phi_{m,k,\chi,\mathfrak{a}}(f)=a_{\mathfrak{a}}(m).$$

Since $\mathcal{S}_k(\Gamma,\chi)$ is a finite dimensional complex Hilbert space, the Riesz representation theorem implies that there exists unique $v_{m,k,\chi,\mathfrak{a}}\in\mathcal{S}_k(\Gamma,\chi)$ such that

$$\langle f,v_{m,k,\chi,\mathfrak{a}}\rangle=\phi_{m,k,\chi,\mathfrak{a}}(f)=a_{\mathfrak{a}}(m).$$

We would like to know what these cusp forms are. It turns out that $v_{m,k,\chi,\mathfrak{a}}(z)$ will be the Poincaré series $P_{m,k,\chi,\mathfrak{a}}(z)$ of positive index up to a constant. To deduce this, we will need the very useful identity

$$\int_0^1 e^{2\pi i(n-m)x}\,dx=\delta_{n-m,0},\tag{10.1}$$

where $n,m\in\mathbb{Z}$. We will prove the following theorem:

Theorem 10.3.2. *Let $f\in\mathcal{S}_k(\Gamma,\chi)$ have Fourier coefficients $a_{\mathfrak{a}}(n)$ at the $\mathfrak{a}$ cusp of $\Gamma\backslash\mathbb{H}$. Then*

$$\langle f,P_{m,k,\chi,\mathfrak{a}}\rangle=\frac{\Gamma(k-1)}{V_\Gamma(4\pi m)^{k-1}}a_{\mathfrak{a}}(m),$$

for all $m\geq 1$.

Proof. We compute the inner product as follows:

$$\begin{aligned}
\langle f, P_{m,k,\chi,\mathfrak{a}}\rangle &= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{P_{m,k,\chi,\mathfrak{a}}(z)}\mathrm{Im}(z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\chi(\gamma)\overline{j(\sigma_\mathfrak{a}^{-1}\gamma,z)^{-k}}f(z)e^{-2\pi i m\overline{\sigma_\mathfrak{a}^{-1}\gamma z}}\mathrm{Im}(z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\chi(\gamma)j(\sigma_\mathfrak{a}^{-1}\gamma,z)^k f(z)e^{-2\pi i m\overline{\sigma_\mathfrak{a}^{-1}\gamma z}}\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\left(\frac{j(\sigma_\mathfrak{a}^{-1}\gamma,z)}{j(\gamma,z)}\right)^k f(\gamma z)e^{-2\pi i m\overline{\sigma_\mathfrak{a}^{-1}\gamma z}}\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)^k\,d\mu && \text{modularity} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}j(\sigma_\mathfrak{a},\sigma_\mathfrak{a}^{-1}\gamma z)^{-k}f(\gamma z)e^{-2\pi i m\overline{\sigma_\mathfrak{a}^{-1}\gamma z}}\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)^k\,d\mu && \text{cocycle condition} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}j(\sigma_\mathfrak{a},\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a}z)^{-k}f(\gamma\sigma_\mathfrak{a}z)e^{-2\pi i m\overline{\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a}z}}\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a}z)^k\,d\mu && z\mapsto\sigma_\mathfrak{a}z \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\infty\backslash\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}j(\sigma_\mathfrak{a},\gamma z)^{-k}f(\sigma_\mathfrak{a}\gamma z)e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(\gamma z)^k\,d\mu && \gamma\mapsto\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{a}^{-1} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\infty\backslash\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}(f|_k\sigma_\mathfrak{a})(\gamma z)e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(\gamma z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\Gamma_\infty\backslash\mathbb{H}}(f|_k\sigma_\mathfrak{a})(z)e^{-2\pi i m\bar{z}}\mathrm{Im}(z)^k\,d\mu && \text{unfolding.}
\end{aligned}$$

Substituting in the Fourier series of f at the $\mathfrak{a}$ cusp, we can finish the computation:

$$\begin{aligned}
\frac{1}{V_\Gamma}\int_{\Gamma_\infty\backslash\mathbb{H}}(f|_k\sigma_\mathfrak{a})(z)e^{-2\pi i m\bar{z}}\mathrm{Im}(z)^k\,d\mu &= \frac{1}{V_\Gamma}\int_0^\infty\int_0^1\sum_{n\geq 1}a_\mathfrak{a}(n)e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^k\,\frac{dx\,dy}{y^2} \\
&= \frac{1}{V_\Gamma}\int_0^\infty\sum_{n\geq 1}\int_0^1 a_\mathfrak{a}(n)e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^k\,\frac{dx\,dy}{y^2} && \text{FTT} \\
&= \frac{1}{V_\Gamma}\int_0^\infty a_\mathfrak{a}(m)e^{-4\pi m y}y^k\,\frac{dy}{y^2},
\end{aligned}$$

where the last line follows because Equation (10.1) implies that the inner integral cuts off all the terms except the diagonal $n=m$. Then

$$\begin{aligned}
\frac{1}{V_\Gamma}\int_0^\infty a_\mathfrak{a}(m)e^{-4\pi m y}y^k\,\frac{dy}{y^2} &= \frac{a_\mathfrak{a}(m)}{V_\Gamma}\int_0^\infty e^{-4\pi m y}y^{k-1}\,\frac{dy}{y} \\
&= \frac{a_\mathfrak{a}(m)}{V_\Gamma}\int_0^\infty e^{-4\pi m y}y^{k-1}\,\frac{dy}{y} && y\mapsto\frac{y}{4\pi m} \\
&= \frac{a_\mathfrak{a}(m)}{V_\Gamma(4\pi m)^{k-1}}\int_0^\infty e^{-y}y^{k-1}\,\frac{dy}{y} \\
&= \frac{\Gamma(k-1)}{V_\Gamma(4\pi m)^{k-1}}a_\mathfrak{a}(m).
\end{aligned}$$

This completes the proof. □

It follows immediately from Theorem 10.3.2 that

$$v_{m,k,\chi,\mathfrak{a}}(z) = \frac{V_\Gamma (4\pi m)^{k-1}}{\Gamma(k-1)} P_{m,k,\chi,\mathfrak{a}}(z).$$

Thus the Poincaré series $P_{m,k,\chi,\mathfrak{a}}(z)$ of positive index extract the Fourier coefficients of f at different cusps up to a constant. With Theorem 10.3.2 in hand we can prove the following result:

Theorem 10.3.3. *The Poincaré series of positive index span $\mathcal{S}_k(\Gamma, \chi)$.*

Proof. Let $f \in \mathcal{S}_k(\Gamma, \chi)$ have Fourier coefficients $a_\mathfrak{a}(n)$ at the $\mathfrak{a}$ cusp. Since $\Gamma(k-1) \neq 0$, Theorem 10.3.2 implies $\langle f, P_{m,k,\chi,\mathfrak{a}} \rangle = 0$ if and only if $a_\mathfrak{a}(m) = 0$. So f is orthogonal to all the Poincaré series $P_{m,k,\chi,\mathfrak{a}}$ of positive index if and only if every Fourier coefficient $a_\mathfrak{a}(m)$ is zero. But this happens if and only if f is identically zero. □

10.4 Double Coset Operators

We are ready to introduce a class of general operators, depending upon double cosets, on a congruence subgroup Γ of level N. We will use these operators to define the diamond and Hecke operators. For $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, consider the double coset

$$\Gamma_1 \alpha \Gamma_2 = \{\gamma_1 \alpha \gamma_2 : \gamma_1 \in \Gamma_1 \text{ and } \gamma_2 \in \Gamma_2\}.$$

Then Γ_1 acts on the set $\Gamma_1 \alpha \Gamma_2$ by left multiplication so that it decomposes into a disjoin union of orbit spaces. Thus

$$\Gamma_1 \alpha \Gamma_2 = \bigcup_{\beta \in \Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2} \Gamma_1 \beta,$$

where the sum is over the orbit representatives β. However, in order for these operators to be well-defined it is necessary that the orbit decomposition above is a finite union. This is indeed the case and we will require a lemma which gives a way to describe the orbit representatives for $\Gamma_1 \alpha \Gamma_2$ in terms of coset representatives:

Lemma 10.4.1. *Let Γ_1 and Γ_2 be congruence subgroups and let let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Set $\Gamma_3 = \alpha^{-1}\Gamma_1 \alpha \cap \Gamma_2$. Then left multiplication map*

$$\Gamma_2 \to \Gamma_1 \alpha \Gamma_2 \qquad \gamma_2 \mapsto \alpha \gamma_2,$$

induces a bijection from the coset space $\Gamma_3 \backslash \Gamma_2$ to the orbit space $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$.

Proof. We will show that the induced map is both surjective and injective. For surjectivity, the orbit representatives β of $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ are of the form $\beta = \gamma_1 \alpha \gamma_2$ for some $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. Since Γ_1 is acting on $\Gamma_1 \alpha \Gamma_2$ by left multiplication, β can be written as $\beta = \alpha \gamma_2'$ for some $\gamma_2' \in \Gamma_2$. This shows that the induced map is a surjection. To prove injectivity, let $\gamma_2, \gamma_2' \in \Gamma_2$ be such that the orbit space representatives $\alpha\gamma_2$ and $\alpha\gamma_2'$ are equivalent. That is,

$$\Gamma_1 \alpha \gamma_2 = \Gamma_1 \alpha \gamma_2'.$$

This implies $\alpha \gamma_2 (\gamma_2')^{-1} \in \Gamma_1 \alpha$ and so $\gamma_2 (\gamma_2')^{-1} \in \alpha^{-1} \Gamma_1 \alpha$. But we also have $\gamma_2 (\gamma_2')^{-1} \in \Gamma_2$ and these two facts together imply $\gamma_2 (\gamma_2')^{-1} \in \Gamma_3$. Hence

$$\Gamma_3 \gamma_2 = \Gamma_3 \gamma_2',$$

which shows that the induced map is also an injection. □

With this lemma in hand, we can prove that the orbit decomposition of $\Gamma_1\alpha\Gamma_2$ is finite:

Proposition 10.4.1. *Let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then the orbit decomposition*

$$\Gamma_1\alpha\Gamma_2 = \bigcup_j \Gamma_1\beta_j,$$

with respect to the action of Γ_1 by left multiplication, is a finite union.

Proof. Let $\Gamma_3 = \alpha\Gamma_1\alpha^{-1} \cap \Gamma_2$. Then Γ_3 acts on Γ_2 by left multiplication. By Lemma 10.4.1, the number of orbits of $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ is the same as the number of cosets of $\Gamma_3\backslash\Gamma_2$ which is $[\Gamma_2 : \Gamma_3]$. By Lemma 9.1.2, $\alpha^{-1}\Gamma_1\alpha \cap \mathrm{PSL}_2(\mathbb{Z})$ is a congruence subgroup and hence $[\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma_1\alpha \cap \mathrm{PSL}_2(\mathbb{Z})]$ is finite. As $\Gamma_2 = \mathrm{PSL}_2(\mathbb{Z})\cap\Gamma_2$ and $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha\cap\mathrm{PSL}_2(\mathbb{Z})\cap\Gamma_2$, it follows that $[\Gamma_2 : \Gamma_3] \leq [\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma_1\alpha\cap\mathrm{PSL}_2(\mathbb{Z})]$ completing the proof. □

In light of Proposition 10.4.1, we will denote the orbit representatives by β_j to make it clear that there are finitely many. We can now introduce our double coset operators. Let Γ_1 and Γ_2 be two congruence subgroups and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. We define the **double coset operator** $[\Gamma_1\alpha\Gamma_2]_k$ on $\mathcal{S}_k(\Gamma_1)$ to be the linear operator given by

$$(f[\Gamma_1\alpha\Gamma_2]_k)(z) = \sum_j (f|_k\beta_j)(z) = \sum_j \det(\beta_j)^{k-1} j(\beta_j, z)^{-k} f(\beta_j z),$$

By Proposition 10.4.1 this sum is finite. It remains to check that $f[\Gamma_1\alpha\Gamma_2]_k$ is well-defined. Indeed, if β_j and β_j' belong to the same orbit then $\beta_j'\beta_j^{-1} \in \Gamma_1$. But then as $f \in \mathcal{S}_k(\Gamma_1)$, is it invariant under the $|_k\beta_j'\beta_j^{-1}$ operator so that

$$(f|_k\beta_j)(z) = ((f|_k\beta_j'\beta_j^{-1})|_k\beta_j)(z) = (f|_k\beta_j')(z),$$

and therefore the $[\Gamma_1\alpha\Gamma_2]_k$ operator is well-defined. Actually, the map $[\Gamma_1\alpha\Gamma_2]_k$ preserves holomorphic forms:

Proposition 10.4.2. *For any two congruence subgroups Γ_1 and Γ_2, $[\Gamma_1\alpha\Gamma_2]_k$ maps $\mathcal{S}_k(\Gamma_1)$ into $\mathcal{S}_k(\Gamma_2)$.*

Proof. Holomorphy is immediate since the sum in $f[\Gamma_1\alpha\Gamma_2]_k$ is finite by Proposition 10.4.1. For modularity, let $\gamma \in \Gamma_2$. Then

$$\begin{aligned}
(f[\Gamma_1\alpha\Gamma_2]_k)(\gamma z) &= \sum_j \det(\beta_j)^{k-1} j(\beta_j, \gamma z)^{-k} f(\beta_j\gamma z) \\
&= \sum_j \det(\beta_j\gamma)^{k-1} j(\beta_j, \gamma z)^{-k} f(\beta_j\gamma z) && \det(\gamma) = 1 \\
&= \sum_j \det(\beta_j\gamma)^{k-1} \left(\frac{j(\gamma, z)}{j(\beta_j\gamma, z)}\right)^k f(\beta_j\gamma z) && \text{cocycle condition} \\
&= j(\gamma, z)^k \sum_j \det(\beta_j\gamma)^{k-1} j(\beta_j\gamma, z)^{-k} f(\beta_j\gamma z) \\
&= j(\gamma, z)^k \sum_j \det(\beta_j)^{k-1} j(\beta_j, z)^{-k} f(\beta_j z) && \beta_j \mapsto \beta_j\gamma^{-1} \\
&= j(\gamma, z)^k \sum_j (f|_k\beta_j)(z) \\
&= j(\gamma, z)^k (f[\Gamma_1\alpha\Gamma_2]_k)(z).
\end{aligned}$$

This proves the modularity of $f[\Gamma_1\alpha\Gamma_2]_k$. For the growth condition, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$ of $\Gamma_2\backslash\mathbb{H}$. For any orbit representative β_j, $\beta_j\sigma_{\mathfrak{a}}$ takes ∞ to an element of $\mathbb{Q}\cup\{\infty\}$ since $\beta_j \in \mathrm{GL}_2^+(\mathbb{Q})$. In other words, $\beta_j\sigma_{\mathfrak{a}}\infty = \mathfrak{b}$ for some cusp $\mathfrak{b}$ of $\Gamma_1\backslash\mathbb{H}$. Then by the cocycle condition, we have

$$j(\sigma_{\mathfrak{a}}, z)^{-k}(f[\Gamma_1\alpha\Gamma_2]_k)(\sigma_{\mathfrak{a}}z) = \sum_j \det(\beta_j)^{k-1} j(\beta_j\sigma_{\mathfrak{a}}, z)^{-k} f(\beta_j\sigma_{\mathfrak{a}}z),$$

and the growth condition follows from that of f. In particular, $f[\Gamma_1\alpha\Gamma_2]_k$ is a cusp form since f is. □

The double coset operators are the most basic types of operators on holomorphic forms. They are the building blocks needed to define the more important diamond and Hecke operators.

10.5 Diamond and Hecke Operators

The diamond and Hecke operators are special linear operators that are used to construct a linear theory of holomorphic forms. They will also help us understand the Fourier coefficients. Throughout this discussion, we will obtain corresponding results for holomorphic forms with nontrivial characters. We will discuss the diamond operator first. To define them, we need to consider both the congruence subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$. Recall that $\Gamma_1(N) \leq \Gamma_0(N)$ and consider the map

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^* \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod N,$$

(d is invertible modulo N since $c \equiv 0 \pmod N$ and $ad - bc = 1$). This is a surjective homomorphism and its kernel is exactly $\Gamma_1(N)$ so that $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ and $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$. Letting $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $f \in \mathcal{S}_k(\Gamma_1)$, consider $\left(f\left[\Gamma_1(N)\alpha\Gamma_1(N)\right]_k\right)(z)$. This is only dependent upon the lower-right entry d of α taken modulo N. To see this, since $\Gamma_1(N)$ is normal in $\Gamma_0(N)$, $\Gamma_1(N)\alpha = \alpha\Gamma_1(N)$ so that $\Gamma_1(N)\alpha\Gamma_1(N) = \alpha\Gamma_1(N)$ and hence there is only one representative for the orbit decomposition. Therefore

$$\left(f\left[\Gamma_1(N)\alpha\Gamma_1(N)\right]_k\right)(z) = (f|_k\alpha)(z).$$

This induces an action of $\Gamma_0(N)$ on $\mathcal{S}_k(\Gamma_1)$ and since $\Gamma_1(N)$ acts trivially, this is really an action of $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$. In other words, we have an induced action that depends only upon the lower-right entry d of α taken modulo N. So for any d modulo N, we define the **diamond operator** $\langle d\rangle : \mathcal{S}_k(\Gamma_1(N)) \to \mathcal{S}_k(\Gamma_1(N))$ to be the linear operator given by

$$(\langle d\rangle f)(z) = (f|_k\alpha)(z),$$

for any $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Our discussion above has already shown that the diamond operators $\langle d\rangle$ are well-defined. Moreover, the diamond operators are also invertible with $\langle\overline{d}\rangle$ serving as an inverse and α^{-1} as a representative for the definition. Also, since the operator $|_k\alpha$ is multiplicative and

$$\begin{pmatrix} * & * \\ 0 & d \end{pmatrix}\begin{pmatrix} * & * \\ 0 & e \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & de \end{pmatrix} \pmod N,$$

the diamond operators are multiplicative. One reason the diamond operators are useful is that they decompose $\mathcal{S}_k(\Gamma_1(N))$ into eigenspaces. For any Dirichlet character χ modulo N, we let

$$\mathcal{S}_k(N,\chi) = \{f \in \mathcal{S}_k(\Gamma_1(N)) : \langle d\rangle f = \chi(d)f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^*\},$$

be the χ-eigenspace. Also let $\mathcal{S}_k(N,\chi)$ be the corresponding subspace of cusp forms. Then $\mathcal{S}_k(\Gamma_1(N))$ admits a decomposition into these eigenspaces:

Proposition 10.5.1. *We have a direct sum decomposition*

$$\mathcal{S}_k(\Gamma_1(N)) = \bigoplus_{\chi \pmod N} \mathcal{S}_k(N, \chi).$$

Proof. The diamond operators give a representation of $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$ over $\mathcal{S}_k(\Gamma_1(N))$. Explicitly,

$$\Phi : (\mathbb{Z}/N\mathbb{Z})^* \times \mathcal{S}_k(\Gamma_1(N)) \to \mathcal{S}_k(\Gamma_1(N)) \qquad (d, f) \mapsto \langle d \rangle f.$$

But any representation of a finite abelian group over $\mathbb{C}$ is completely reducible with respect to the characters of the group and every irreducible subrepresentation is 1-dimensional (see Theorem C.4.1). Since the characters of $(\mathbb{Z}/N\mathbb{Z})^*$ are given by Dirichlet characters, the decomposition as a direct sum follows. □

Proposition 10.5.1 shows that the diamond operators sieve holomorphic forms on $\Gamma_1(N)\backslash\mathbb{H}$ with trivial character in terms of holomorphic forms on $\Gamma_0(N)\backslash\mathbb{H}$ with nontrivial characters. In particular, $\mathcal{S}_k(N, \chi) = \mathcal{S}_k(\Gamma_0(N), \chi)$. So by Proposition 10.5.1, we have

$$\mathcal{S}_k(\Gamma_1(N)) = \bigoplus_{\chi \pmod N} \mathcal{S}_k(\Gamma_0(N), \chi).$$

This fact clarifies why it is necessary to consider holomorphic forms with nontrivial characters. Now it is time to define the Hecke operators. For a prime p, we define the p-th **Hecke operator** $T_p : \mathcal{S}_k(\Gamma_1(N)) \to \mathcal{S}_k(\Gamma_1(N))$ to be the linear operator given by

$$(T_p f)(z) = \left(\left[\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) \right]_k f \right)(z).$$

We will start discussing properties of the diamond and Hecke operators, but first we prove an important lemma classifying the double coset $\Gamma_1(N) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma_1(N)$:

Lemma 10.5.1. *Let p be a prime. Then*

$$\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) = \left\{ \gamma \in \mathrm{Mat}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & p \end{pmatrix} \pmod N \text{ and } \det(\gamma) = p \right\}.$$

Proof. For the forward containment, it is clear that any element of $\Gamma_1(N) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma_1(N)$ has determinant p and that

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \pmod N,$$

so the forward containment holds. For the reverse containment, let L be the set of 2×1 column vectors with entries in $\mathbb{Z}$, in particular $L \cong \mathbb{Z}^2$, and set

$$L_0 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in L : y \equiv 0 \pmod N \right\}.$$

Then $\mathrm{Mat}_2(\mathbb{Z})$ acts on L_0 on the left by matrix multiplication. Now suppose $\gamma \in \mathrm{Mat}_2(\mathbb{Z})$ is such that $\gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & p \end{smallmatrix}\right) \pmod N$ and $\det(\gamma) = p$. Clearly L_0 has index N in L. As the action of γ on L_0 multiplies the lower entry by p, γL_0 has index p in L_0. These two facts together imply

$$[L : \gamma L_0] = [L : L_0][L_0; \gamma L_0] = Np.$$

As γL_0 is a finitely generated abelian group (L is a rank 2 free module over $\mathbb{Z}$ and γL_0 is a subgroup), the structure theorem for finitely generated abelian groups implies that there exists a basis u, v of L with $\det(u, v) = 1$, so $L = u\mathbb{Z} \oplus v\mathbb{Z}$, and positive integers m and n with $m \mid n$, $mn = Np$, and such that $\gamma L_0 = mu\mathbb{Z} \oplus nv\mathbb{Z}$. Moreover, as $m \mid n$ any element of γL_0 is zero modulo m. Now $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in L_0$ and by our choice of γ, we have $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \gamma L_0$ as well. Thus $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \pmod{m}$. This forces $m = 1$ and $n = Np$ so that $\gamma L_0 = u\mathbb{Z} \oplus Npv\mathbb{Z}$. As there are unique subgroups of index N and p between $\mathbb{Z}$ and $Np\mathbb{Z}$, namely $N\mathbb{Z}$ and $p\mathbb{Z}$ respectively, we have

$$L_0 = u\mathbb{Z} \oplus Nv\mathbb{Z}, \quad \gamma L = u\mathbb{Z} \oplus pv\mathbb{Z}, \quad \text{and} \quad \gamma L_0 = u\mathbb{Z} \oplus Npv\mathbb{Z}.$$

Now let $\gamma_1 = (u, v)$. Since $u \in L_0$, we have that $\gamma_1 \in \Gamma_0(N)$. Now set $\gamma_2 = \left(\gamma_1 \left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\right)^{-1} \gamma$. Then $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ with $\det(\gamma_2) = 1$. It follows that

$$\gamma = \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \gamma_2.$$

Let e_1 and e_2 be the standard basis vectors of L. Then $\gamma e_1 \in \gamma L_0$ implies that $\gamma e_1 = au + pcv$ with $a, c \in \mathbb{Z}$ and $N \mid c$. Similarly, $\gamma e_2 \in \gamma L$ implies that $\gamma e_2 = bu + pdv$ with $b, d \in \mathbb{Z}$. Letting $u = \left(\begin{smallmatrix}u_1\\u_2\end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix}v_1\\v_2\end{smallmatrix}\right)$, these facts together imply that $\gamma_2 = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ and the above identity can be written in the form

$$\gamma = \begin{pmatrix} au_1 + pcv_1 & bu_1 + pdv_1 \\ au_2 + pcv_2 & bu_2 + pdv_2 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

As $\det(\gamma_2) = 1$ and $N \mid c$, we conclude that $\gamma_2 \in \Gamma_0(N)$. Hence $\gamma \in \Gamma_0(N) \left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right) \Gamma_0(N)$. Moreover, as $\gamma_1, \gamma_2 \in \Gamma_0(N)$ it follows that they belong to $\Gamma_1(N)$ if u_1 or a is equivalent to 1 modulo N respectively. But as $\gamma \equiv \left(\begin{smallmatrix}1&*\\0&p\end{smallmatrix}\right) \pmod{N}$ and $N \mid c$, we have $au_1 \equiv 1 \pmod{N}$. Thus if γ_1 or γ_2 belongs to $\Gamma_1(N)$ then they both do. So to complete the reverse containment it suffices to show

$$\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N) = \Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N).$$

The reverse containment in this double coset equality is clear since $\Gamma_1(N) \subseteq \Gamma_0(N)$. For the forward containment, let $\gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \Gamma_0(N)$. Then we need to show that there exists a matrix $\delta \in \Gamma_1(N)$ such that

$$\gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N) = \delta \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N).$$

Equivalently, there exists a matrix $\delta' \in \Gamma_1(N)$ (that is the inverse of δ) such that

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \delta' \gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in \Gamma_0(N).$$

If $p \mid N$, taking $\delta' = \left(\begin{smallmatrix}cd+1&-1\\-cd&1\end{smallmatrix}\right) \in \Gamma_1(N)$ gives

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \delta' \gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} cd+1 & -1 \\ -cd & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} * & * \\ p^{-1}c(1-ad) & * \end{pmatrix},$$

which belongs to $\Gamma_0(N)$ because $p \mid N$, $N \mid c$, and $N \mid 1 - ad$ (as $ad - bc = 1$). If $p \nmid N$ then the Chinese remainder theorem implies that there exists a $d' \in \mathbb{Z}$ with $d' \equiv 1 \pmod{N}$ and $d' \equiv -a \pmod{p}$. Necessarily $(c, d') = 1$ because $N \mid c$. Thus there exists δ' of the form $\delta' = \left(\begin{smallmatrix}a'&b'\\c&d'\end{smallmatrix}\right) \in \Gamma_1(N)$, and we have

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \delta' \gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} a' & b' \\ c & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} * & * \\ p^{-1}c(a+d') & * \end{pmatrix},$$

which again belongs to $\Gamma_0(N)$ because $p \mid d' + a$ and $N \mid c$. This prove the forward containment, and hence the original reverse containment, thus completing the proof. □

With Lemma 10.5.1, it is not too hard to see that the diamond and Hecke operators commute:

Proposition 10.5.2. *For every $d \in (\mathbb{Z}/N\mathbb{Z})^*$ and prime p, the diamond operators $\langle d \rangle$ and Hecke operators T_p on $\mathcal{S}_k(\Gamma_1(N))$ commute:*

$$\langle d \rangle T_p = T_p \langle d \rangle$$

Proof. Let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we have

$$\gamma\alpha\gamma^{-1} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & (p-1)ab \\ 0 & p \end{pmatrix} \pmod{N},$$

because $c \equiv 0 \pmod{N}$, $ad - bc = 1$, and $ad \equiv 1 \pmod{N}$. By Lemma 10.5.1, $\gamma\alpha\gamma^{-1} \in \Gamma_1(N)\alpha\Gamma_1(N)$ and so we can use this representative in place of α. On the one hand,

$$\Gamma_1(N)\alpha\Gamma_1(N) = \bigcup_j \Gamma_1(N)\beta_j.$$

On the other hand, using $\gamma\alpha\gamma^{-1}$ in place of α and the normality of $\Gamma_1(N)$ in $\Gamma_0(N)$, we have

$$\Gamma_1(N)\alpha\Gamma_1(N) = \Gamma_1(N)\gamma\alpha\gamma^{-1}\Gamma_1(N) = \gamma\Gamma_1(N)\alpha\Gamma_1(N)\gamma^{-1} = \gamma\bigcup_j \Gamma_1(N)\beta_j\gamma^{-1} = \bigcup_j \Gamma_1(N)\gamma\beta_j\gamma^{-1}.$$

Upon comparing these two decompositions of $\Gamma_1(N)\alpha\Gamma_1(N)$ gives

$$\bigcup_j \Gamma_1(N)\beta_j = \bigcup_j \Gamma_1(N)\gamma\beta_j\gamma^{-1}.$$

Now let $f \in \mathcal{S}_k(\Gamma_1(N))$. Then this equivalence of unions implies

$$\langle d \rangle T_p f = \sum_j f|_k \beta_j \gamma = \sum_j f|_k \gamma\beta_j = T_p\langle d \rangle f.$$

□

Using Lemma 10.5.1 we can obtain an explicit description of the Hecke operator T_p:

Proposition 10.5.3. *Let $f \in \mathcal{S}_k(\Gamma_1(N))$. Then the Hecke operator T_p acts on f as follows:*

$$(T_p f)(z) = \begin{cases} \displaystyle\sum_{j \pmod p} \left(f\Big|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right)(z) + \left(f\Big|_k \begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right)(z) & \text{if } p \nmid N, \\ \displaystyle\sum_{j \pmod p} \left(f\Big|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right)(z) & \text{if } p \mid N, \end{cases}$$

where m and n are chosen such that $\det\left(\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)\right) = 1$.

Proof. Set $\Gamma_3 = \alpha^1\Gamma_1(N)\alpha \cap \Gamma_1(N)$ where $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. Define

$$\beta_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \quad \text{and} \quad \beta_\infty = \begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} pm & n \\ Np & p \end{pmatrix},$$

for j taken modulo p and where m and n are chosen such that $\det\left(\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)\right) = 1$. It suffices to show $\beta_1, \ldots, \beta_{p-1}$ and $\beta_1, \ldots, \beta_{p-1}, \beta_\infty$ are complete sets of orbit representatives for $\Gamma_1(N)\backslash\Gamma_1(N)\alpha\Gamma_1(N)$ depending on if $p \mid N$ or not. To accomplish this, we will find a complete set of coset representatives for $\Gamma_3\backslash\Gamma_1(N)$ and then use Lemma 10.4.1. First we require an explicit description of Γ_3. Let

$$\Gamma^0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod{p} \right\},$$

and define

$$\Gamma_1^0(N,p) = \Gamma_1(N) \cap \Gamma^0(p).$$

We claim $\Gamma_3 = \Gamma_1^0(N,p)$. For the forward containment, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ and observe that

$$\alpha^{-1}\gamma\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} a & pd \\ p^{-1}c & d \end{pmatrix}.$$

If $\alpha^{-1}\gamma\alpha \in \Gamma_3$ then $\alpha^{-1}\gamma\alpha \in \Gamma_1(N)$ and thus $p \mid c$ so that $\alpha^{-1}\gamma\alpha \in \mathrm{PSL}_2(\mathbb{Z})$. Moreover, the previous computation implies $\alpha^{-1}\gamma\alpha \in \Gamma_1^0(N,p)$ and so $\Gamma_3 \subseteq \Gamma_1^0(N,p)$. For the reverse containment, suppose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^1(N,p)$. Then $b = pk$ for some $k \in \mathbb{Z}$. Now observe

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} a & k \\ pc & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \alpha^{-1}\gamma\alpha,$$

where $\gamma = \left(\begin{smallmatrix} a & k \\ pc & d \end{smallmatrix}\right)$. As $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ we conclude $\gamma \in \Gamma_1(N)$ as well. Now let

$$\alpha_j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_\infty = \begin{pmatrix} pm & n \\ N & 1 \end{pmatrix},$$

for j taken modulo p and where m and n are as before. Clearly $\alpha_j \in \Gamma_1(N)$ for all j. As $pm - Nn = 1$, we have $pm \equiv 1 \pmod{N}$ so that $\alpha_\infty \in \Gamma_1(N)$ as well. We claim that $\alpha_1, \ldots, \alpha_{p-1}$ and $\alpha_1, \ldots, \alpha_{p-1}, \alpha_\infty$ are sets of coset representatives for $\Gamma_3\backslash\Gamma_1(N)$ depending on if $p \mid N$ or not. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ and consider

$$\gamma\alpha_j^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b - aj \\ c & d - cj \end{pmatrix}.$$

As $\gamma\alpha_j^{-1} \in \Gamma_1(N)$ because both γ and γ_i are, $\gamma\alpha_j^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$ for some i if and only if

$$b \equiv aj \pmod{p}.$$

First suppose $p \nmid a$. Then a is invertible modulo p so we may take $j = \overline{a}b \pmod{p}$. Now suppose $p \mid a$. If there is some i satisfying $b \equiv ai \pmod{p}$ then we also have $p \mid b$. But as $ad - bc = 1$, this is impossible and so no such i exists. As $a \equiv 1 \pmod{N}$, $p \mid a$ if and only if $p \nmid N$. In this case consider instead

$$\gamma\alpha_\infty^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -n \\ -N & pm \end{pmatrix} = \begin{pmatrix} a - Nb & pmb - na \\ c - Nd & pmd - nc \end{pmatrix}.$$

Since $p \mid a$, we have $pmb - na \equiv 0 \mod p$ so that $\gamma\alpha_\infty^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$. Altogether, we have shown that $\alpha_1, \ldots, \alpha_{p-1}$ and $\alpha_1, \ldots, \alpha_{p-1}, \alpha_\infty$ are sets of coset representatives for $\Gamma_3\backslash\Gamma_1(N)$ depending on if $p \mid N$ or not. To show they are complete sets, we need to show that no two representatives belong to the same coset. To this end, suppose j and j' are distinct, taken modulo p, and consider

$$\alpha_j\alpha_{j'}^{-1} = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & j' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & j - j' \\ 0 & 1 \end{pmatrix}.$$

Then $\alpha_j\alpha_{j'}^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$ if and only if $j - j' \equiv 0 \pmod{p}$. This is impossible since j and j' are distinct. Hence α_j and $\alpha_{j'}$ represent distinct cosets. Now consider

$$\alpha_j\alpha_\infty^{-1} = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -n \\ -N & pm \end{pmatrix} = \begin{pmatrix} 1-Nj & pmj-n \\ -N & pm \end{pmatrix}.$$

Now $\alpha_j\alpha_\infty^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$ if and only if $pmj - n \equiv 0 \pmod{p}$. This is impossible since $pm - Nn = 1$ implies $p \nmid n$. Therefore α_j and α_∞ represent distinct cosets. It follows that $\alpha_1, \ldots, \alpha_{p-1}$ and $\alpha_1, \ldots, \alpha_{p-1}, \alpha_\infty$ are complete sets of coset representatives for $\Gamma_3\backslash\Gamma_1(N)$ depending on if $p \mid N$ or not. As

$$\alpha\alpha_j = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} = \beta_j \quad \text{and} \quad \alpha\alpha_\infty = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} pm & n \\ N & 1 \end{pmatrix} = \begin{pmatrix} pm & n \\ Np & p \end{pmatrix} = \beta_\infty,$$

Lemma 10.4.1 finishes the proof. □

This explicit definition of T_p can be used to compute how the Hecke operators act on the Fourier coefficients of a holomorphic form:

Proposition 10.5.4. *Let $f \in \mathcal{S}_k(\Gamma_1(N))$ have Fourier coefficients $a_n(f)$. Then for all primes p, we have*

$$(T_p f)(z) = \sum_{n\geq 1}\left(a_{np}(f) + \chi_{N,0}(p)p^{k-1}a_{\frac{n}{p}}(\langle p\rangle f)\right)e^{2\pi inz},$$

is the Fourier series of $T_p f$ where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid n$. Moreover, if $f \in \mathcal{S}_k(N,\chi)$ then $T_p f \in \mathcal{S}_k(N,\chi)$ and

$$(T_p f)(z) = \sum_{n\geq 1}\left(a_{np}(f) + \chi(p)p^{k-1}a_{\frac{n}{p}}(f)\right)e^{2\pi inz},$$

where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid n$.

Proof. In view of Proposition 10.5.1 and the linearity of the Hecke operators, it suffices to assume $f \in \mathcal{S}_k(N,\chi)$. By Proposition 10.5.2, $T_p f \in \mathcal{S}_k(N,\chi)$. It remains to verify the second formula. Observe that

$$\left(f\Big|_k\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right)(z) = \frac{1}{p}\sum_{n\geq 1}a_n(f)e^{\frac{2\pi in(z+j)}{p}}.$$

Summing over all j modulo p gives

$$\sum_{j \pmod p}\left(f\Big|_k\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right)(z) = \sum_{n\geq 1}a_n(f)e^{\frac{2\pi inz}{p}}\frac{1}{p}\sum_{j \pmod p}e^{\frac{2\pi nij}{p}}.$$

If $p \nmid N$ then the inner sum vanishes because it is the sum over all p-th roots of unity. If $p \mid N$ then the sum is also p. Therefore

$$\sum_{j \pmod p}\left(f\Big|_k\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right)(z) = \sum_{n\geq 1}a_{np}(f)e^{2\pi inz}.$$

If $p \mid N$ then Proposition 10.5.3 implies

$$(T_p f)(z) = \sum_{n\geq 1}a_{np}(f)e^{2\pi inz}, \tag{10.2}$$

which is the claimed Fourier series of $T_p f$. If $p \nmid N$ then we have the additional term

$$\begin{aligned}\left(f\Big|_k \begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right)(z) &= \left(\langle p\rangle f\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right)(z) \\ &= p^{k-1}(\langle p\rangle f)(pz) \\ &= \sum_{n\geq 1} p^{k-1} a_n(\langle p\rangle f) e^{2\pi i n p z} \\ &= \sum_{n\geq 1} \chi(p) p^{k-1} a_n(f) e^{2\pi i n p z},\end{aligned}$$

where the first equality holds because $\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right) \in \Gamma_0(N)$ and the last equality holds because $\langle p\rangle f = \chi(p) f$. In this case, Proposition 10.5.3 gives

$$(T_p f)(z) = \sum_{n\geq 1} a_{np}(f) + \chi(p) p^{k-1} a_{\frac{n}{p}}(f) e^{2\pi i n z}.$$

Since $\chi(p) = 0$ if $p \mid N$, these two cases can be expressed together as

$$(T_p f)(z) = \sum_{n\geq 1} \left(a_{np}(f) + \chi(p) p^{k-1} a_{\frac{n}{p}}(f)\right) e^{2\pi i n z}.$$

□

We now mention the crucial result about Hecke operators which is that they form a simultaneously commuting family with the diamond operators:

Proposition 10.5.5. *Let p and q be primes and $d, e \in (\mathbb{Z}/N\mathbb{Z})^*$. The Hecke operators T_p and T_q and diamond operators $\langle d\rangle$ and $\langle e\rangle$ on $\mathcal{S}_k(\Gamma_1(N))$ form a simultaneously commuting family:*

$$T_p T_q = T_q T_p, \quad \langle d\rangle T_p = T_p \langle d\rangle, \quad \text{and} \quad \langle d\rangle\langle e\rangle = \langle e\rangle\langle d\rangle.$$

Proof. Showing the diamond and Hecke operators commute was Proposition 10.5.2. To show commutativity of the diamond operators, let $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\eta = \left(\begin{smallmatrix} * & * \\ * & e \end{smallmatrix}\right) \in \Gamma_0(N)$. Then

$$\gamma\eta \equiv \begin{pmatrix} * & * \\ 0 & de \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & ed \end{pmatrix} \equiv \eta\gamma \pmod{N}.$$

Therefore $\gamma\eta|_k = \eta\gamma|_k$ as operators and so for any $f \in \mathcal{S}_k(\Gamma_1(N))$, we have

$$\langle d\rangle\langle e\rangle f = f|_k \gamma\eta_k = f|_k \eta\gamma_k = \langle e\rangle\langle d\rangle f.$$

We now show that the Hecke operators commute. In view of Proposition 10.5.1 and linearity of the Hecke operators, it suffices to prove this for $f \in \mathcal{S}_k(N, \chi)$. Applying Proposition 10.5.4 twice, for any $n \geq 1$ we compute

$$\begin{aligned}a_n(T_p T_q f) &= a_{np}(T_q f) + \chi(p) p^{k-1} a_{\frac{n}{p}}(T_q f) \\ &= a_{npq}(f) + \chi(q) q^{k-1} a_{\frac{np}{q}}(f) + \chi(p) p^{k-1}\left(a_{\frac{nq}{p}}(f) + \chi(q) q^{k-1} a_{\frac{n}{pq}}(f)\right) \\ &= a_{npq}(f) + \chi(q) q^{k-1} a_{\frac{np}{q}}(f) + \chi(p) p^{k-1} a_{\frac{nq}{p}}(f) + \chi(pq)(pq)^{k-1} a_{\frac{n}{pq}}(f).\end{aligned}$$

The last expression is symmetric in p and q so that $a_n(T_p T_q f) = a_n(T_q T_p f)$ for all $n \geq 1$. Since all of the Fourier coefficients are equal, we get

$$T_p T_q f = T_q T_p f.$$

□

We can use Proposition 10.5.5 to construct diamond operators $\langle m\rangle$ and Hecke operators T_m for all $m \geq 1$. The **diamond operator** $\langle m\rangle : \mathcal{S}_k(\Gamma_1(N)) \to \mathcal{S}_k(\Gamma_1(N))$ is defined to be the linear operator given by

$$\langle m\rangle = \begin{cases} \langle m\rangle \text{ with } m \text{ taken modulo } N & \text{if } (m,N)=1, \\ 0 & \text{if } (m,N)>1. \end{cases}$$

Now for the Hecke operators. If $m = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ is the prime decomposition of m then we define the m-th **Hecke operator** $T_m : \mathcal{S}_k(\Gamma_1(N)) \to \mathcal{S}_k(\Gamma_1(N))$ to be the linear operator given by

$$T_m = \prod_{1\leq i\leq k} T_{p_i^{r_i}},$$

where T_{p^r} is defined inductively by

$$T_{p^r} = \begin{cases} T_pT_{p^{r-1}} - p^{k-1}\langle p\rangle T_{p^{r-2}} & \text{if } p \nmid N, \\ T_p^r & \text{if } p \mid N, \end{cases}$$

for all $r \geq 2$. Note that when $m = 1$, the product is empty and so T_1 is the identity operator. By Proposition 10.5.5, the Hecke operators T_m are multiplicative but not completely multiplicative in m. Moreover, they commute with the diamond operators $\langle m\rangle$. Using these definitions, Propositions 10.5.4 and 10.5.5, a more general formula for how the Hecke operators T_m act on Fourier coefficients can be derived:

Proposition 10.5.6. *Let $f \in \mathcal{S}_k(\Gamma_1(N))$ have Fourier coefficients $a_n(f)$. Then for $m \geq 1$ with $(m,N) = 1$, we have*

$$(T_mf)(z) = \sum_{n\geq 1}\left(\sum_{d\mid(n,m)} d^{k-1}a_{\frac{nm}{d^2}}(\langle d\rangle f)\right)e^{2\pi inz},$$

is the Fourier series of T_mf. Moreover, if $f \in \mathcal{S}_k(N,\chi)$ then

$$(T_mf)(z) = \sum_{n\geq 1}\left(\sum_{d\mid(n,m)} \chi(d)d^{k-1}a_{\frac{nm}{d^2}}(f)\right)e^{2\pi inz}.$$

Proof. In view of Proposition 10.5.1 and linearity of the Hecke operators, we may assume $f \in \mathcal{S}_k(N,\chi)$. Therefore we only need to verify the second formula. When $m = 1$ the result is obvious and when $m = p$, we have

$$\sum_{d\mid(n,p)} \chi(d)d^{k-1}a_{\frac{np}{d^2}}(f) = a_{np}(f) + \chi(p)p^{k-1}a_{\frac{n}{p}}(f),$$

which is the result obtained from Proposition 10.5.4. By induction assume that the desired formula holds for $m = 1, p, \ldots, p^{r-1}$. Using the definition of T_{p^r} and Proposition 10.5.4, for any $n \geq 1$ we compute

$$\begin{aligned} a_n(T_{p^r}f) &= a_n(T_pT_{p^{r-1}}f) - \chi(p)p^{k-1}a_n(T_{p^{r-2}}f) \\ &= a_{np}(T_{p^{r-1}}f) + \chi(p)p^{k-1}a_{\frac{n}{p}}(T_{p^{r-1}}f) - \chi(p)p^{k-1}a_n(T_{p^{r-2}}f). \end{aligned}$$

By our induction hypothesis, this last expression is

$$\sum_{d\mid(np,p^{r-1})} \chi(d)d^{k-1}a_{\frac{np^r}{d^2}}(f) + \chi(p)p^{k-1}\sum_{d\mid\left(\frac{n}{p},p^{r-1}\right)} \chi(d)d^{k-1}a_{\frac{np^{r-2}}{d^2}}(f) - \chi(p)p^{k-1}\sum_{d\mid(n,p^{r-2})} \chi(d)d^{k-1}a_{\frac{np^{r-2}}{d^2}}(f).$$

Write the first sum as

$$\sum_{d|(np,p^{r-1})} \chi(d)d^{k-1}a_{\frac{np^r}{d^2}}(f) = a_{np^r}(f) + \sum_{d|(n,p^{r-2})} \chi(d)d^{k-1}a_{\frac{np^{r-2}}{d^2}}(f),$$

and observe that the sum on the right-hand side cancels the entire third term above. Therefore our expression reduces to

$$\begin{aligned} a_{np^r}(f) + \chi(p)p^{k-1} \sum_{d|\left(\frac{n}{p},p^{r-1}\right)} \chi(d)d^{k-1}a_{\frac{np^{r-2}}{d^2}}(f) &= a_{np^r}(f) + \sum_{d|\left(\frac{n}{p},p^{r-1}\right)} \chi(dp)(dp)^{k-1}a_{\frac{np^{r-2}}{d^2}}(f) \\ &= a_{np^r}(f) + \sum_{\substack{d|(n,p^r) \\ d\neq 1}} \chi(d)d^{k-1}a_{\frac{np^r}{d^2}}(f) \\ &= \sum_{d|(n,p^r)} \chi(d)d^{k-1}a_{\frac{np^r}{d^2}}(f), \end{aligned}$$

where in the second line we have performed the change of variables $dp \mapsto d$ in the sum. This proves the claim when $m = p^r$ for all $r \geq 0$. By multiplicativity of the Hecke operators, it suffices to prove the claim when $m = p^r q^s$ for another prime q and some $s \geq 0$. We compute

$$\begin{aligned} a_n(T_{p^r q^s} f) &= a_n(T_{p^r} T_{q^s} f) \\ &= \sum_{d_1|(n,p^r)} \chi(d_1)d_1^{k-1}a_{\frac{np^r}{d_1^2}}(T_{q^s} f) \\ &= \sum_{d_1|(n,p^r)} \chi(d_1)d_1^{k-1} \sum_{d_2|\left(\frac{np^r}{d_1^2},q^s\right)} \chi(d_2)d_2^{k-1}a_{\frac{np^r q^s}{(d_1 d_2)^2}}(f) \\ &= \sum_{d_1|(n,p^r)} \sum_{d_2|\left(\frac{np^r}{d_1^2},q^s\right)} \chi(d_1 d_2)(d_1 d_2)^{k-1}a_{\frac{np^r q^s}{(d_1 d_2)^2}}(f). \end{aligned}$$

Summing over pairs (d_1, d_2) of divisors of (n, p^r) and $\left(\frac{np^r}{d^2}, q^s\right)$ respectively is the same as summing over divisors d of $(n, p^r q^s)$. Indeed, because p and q are relative prime, any such d is of the form $d = d_1 d_2$ where $d_1 \mid (n, p^r)$ and $d_2 \mid \left(\frac{np^r}{d_2^2}, q^s\right)$. Therefore the double sum becomes

$$\sum_{d|(n,p^r q^s)} \chi(d)d^{k-1}a_{\frac{np^r q^s}{d^2}}(f).$$

This completes the proof. □

The diamond and Hecke operators turn out to be normal on the subspace of cusp forms. To prove this fact, we will require a lemma:

Lemma 10.5.2. *Let Γ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then there exist $\beta_1, \ldots, \beta_n \in \mathrm{GL}_2^+(\mathbb{Q})$, where $n = [\Gamma : \alpha^{-1}\Gamma\alpha \cap \Gamma] = [\Gamma : \alpha\Gamma\alpha^{-1} \cap \Gamma]$, and such that*

$$\Gamma\alpha\Gamma = \bigcup_j \Gamma\beta_j = \bigcup_j \beta_j\Gamma.$$

Proof. Apply Lemma 9.3.1 with the congruence subgroup $\alpha\Gamma\alpha^{-1}\cap\Gamma$ in place of Γ to get

$$[\mathrm{PSL}_2(\mathbb{Z}):\alpha^{-1}\Gamma\alpha\cap\Gamma]=[\mathrm{PSL}_2(\mathbb{Z}):\alpha\Gamma\alpha^{-1}\cap\Gamma].$$

Dividing both sides by $[\mathrm{PSL}_2(\mathbb{Z}):\Gamma]$ gives

$$[\Gamma:\alpha^{-1}\Gamma\alpha\cap\Gamma]=[\Gamma:\alpha\Gamma\alpha^{-1}\cap\Gamma].$$

Therefore we can find coset representatives $\gamma_1,\dots,\gamma_n\in\Gamma$ and $\widetilde{\gamma}_1,\dots,\widetilde{\gamma}_n\in\Gamma$ such that

$$\Gamma=\bigcup_j(\alpha^{-1}\Gamma\alpha\cap\Gamma)\gamma_j=\bigcup_j(\alpha^{-1}\Gamma\alpha\cap\Gamma)\widetilde{\gamma}_j^{-1}.$$

Invoking Lemma 10.4.1 twice, we can express each of these coset decompositions as an orbit decomposition:

$$\bigcup_j(\alpha^{-1}\Gamma\alpha\cap\Gamma)\gamma_j=\bigcup_j\Gamma\alpha\gamma_j\quad\text{and}\quad\bigcup_j(\alpha^{-1}\Gamma\alpha\cap\Gamma)\widetilde{\gamma}_j^{-1}=\bigcup_j\Gamma\alpha^{-1}\widetilde{\gamma}_j^{-1}.$$

It follows that

$$\Gamma=\bigcup_j\Gamma\alpha\gamma_j=\bigcup_j\widetilde{\gamma}_j\alpha\Gamma.$$

For each j the orbit spaces $\Gamma\alpha\gamma_j$ and $\widetilde{\gamma}_j\alpha\Gamma$ have nonempty intersection. For if they did we would have $\Gamma\alpha\gamma_j\subseteq\bigcup_{i\neq j}\widetilde{\gamma}_i\alpha\Gamma$ and thus $\Gamma\alpha\Gamma\subseteq\bigcup_{i\neq j}\widetilde{\gamma}_i\alpha\Gamma$. This contradicts the previous decomposition of Γ. Therefore we can find representatives $\beta_j\in\Gamma\alpha\gamma_j\cap\widetilde{\gamma}_j\alpha\Gamma$ for every j. Then β_j

$$\Gamma=\bigcup_j\Gamma\beta_j=\bigcup_j\beta_j\Gamma.$$

□

We can use Lemma 10.5.2 to compute adjoints:

Proposition 10.5.7. *Let Γ be a congruence subgroup and let $\alpha\in\mathrm{GL}_2^+(\mathbb{Q})$. Set $\alpha'=\det(\alpha)\alpha^{-1}$. Then the following are true:*

(i) If $\alpha^{-1}\Gamma\alpha\subseteq\mathrm{PSL}_2(\mathbb{Z})$ then for all $f\in\mathcal{S}_k(\Gamma)$ and $g\in\mathcal{S}_k(\alpha^{-1}\Gamma\alpha)$, we have

$$\langle f|_k\alpha,g\rangle_{\alpha^{-1}\Gamma\alpha}=\langle f,g|_k\alpha'\rangle_\Gamma.$$

(ii) For all $f,g\in\mathcal{S}_k(\Gamma)$, we have

$$\langle f[\Gamma\alpha\Gamma]_k,g\rangle=\langle f,g[\Gamma\alpha'\Gamma]_k\rangle.$$

In particular, if $\alpha^{-1}\Gamma\alpha=\Gamma$ then $|_k\alpha^*=|_k\alpha'$ and $[\Gamma\alpha\Gamma]_k^*=[\Gamma\alpha'\Gamma]_k$ as operators.

Proof. To prove (i) we first compute

$$\begin{aligned}
\langle f|_k\alpha,g\rangle_{\alpha^{-1}\Gamma\alpha}&=\frac{1}{V_{\alpha^{-1}\Gamma\alpha}}\int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}}(f|_k\alpha)(z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu\\
&=\frac{1}{V_{\alpha^{-1}\Gamma\alpha}}\int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}}\det(\alpha)^{k-1}j(\alpha,z)^{-k}f(\alpha z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu\\
&=\frac{1}{V_\Gamma}\int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}}\det(\alpha)^{k-1}j(\alpha,z)^{-k}f(\alpha z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu && \text{Lemma 9.3.1}\\
&=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\det(\alpha)^{k-1}j(\alpha,\alpha^{-1}z)^{-k}f(z)\overline{g(\alpha^{-1}z)}\mathrm{Im}(\alpha^{-1}z)^k\,d\mu && z\mapsto\alpha^{-1}z.
\end{aligned}$$

As α' acts as α^{-1} on $\mathbb{H}$, this latter integral is equivalent to

$$\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \det(\alpha)^{k-1} j(\alpha,\alpha' z)^{-k} f(z)\overline{g(\alpha' z)}\mathrm{Im}(\alpha' z)^k \, d\mu.$$

Moreover, applying the cocycle condition and the identities $\mathrm{Im}(\alpha' z) = \det(\alpha')\frac{\mathrm{Im}(z)}{|j(\alpha',z)|^2}$, $j(\alpha\alpha', z) = \det(\alpha)$, and $\det(\alpha') = \det(\alpha)$ together, we can further rewrite the integral as

$$\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \det(\alpha')^{k-1}\overline{j(\alpha', z)^{-k}} f(z)\overline{g(\alpha' z)}\mathrm{Im}(z)^k \, d\mu.$$

Reversing the first computation in the start of the proof but applied to this integral shows that that

$$\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \det(\alpha')^{k-1}\overline{j(\alpha', z)^{-k}} f(z)\overline{g(\alpha' z)}\mathrm{Im}(z)^k \, d\mu = \langle f, g|_k\alpha'\rangle_\Gamma,$$

which completes the proof of (i). To prove (ii), Lemma 10.5.2 implies the coset decomposition $\Gamma\alpha\Gamma = \bigcup_j \Gamma\beta_j$ so that we can use the β_j as representatives for the $[\Gamma\alpha\Gamma]_k$ operator. As the β_j also satisfy $\Gamma\alpha\Gamma = \bigcup_j \beta_j\Gamma$, upon inverting β_j and noting that $\beta_j \in \Gamma\alpha$, we obtain $\Gamma\alpha^{-1}\Gamma = \bigcup_j \Gamma\beta_j^{-1}$. Since scalar multiplication commutes with matrices and the matrices in Γ have determinant 1, we conclude that $\Gamma\alpha'\Gamma = \bigcup_j \Gamma\beta_j'$ where $\beta_j' = \det(\beta_j)\beta_j^{-1}$ (also $\det(\beta_j) = \det(\alpha)$). So we can use the β_j' as representatives in the $[\Gamma\alpha'\Gamma]_k$ operator. The statement now follows from (i). The last statement is obvious. □

We can now prove that the diamond and Hecke operators are normal:

Proposition 10.5.8. *On $\mathcal{S}_k(\Gamma_1(N))$, the diamond operators $\langle m\rangle$ and Hecke operators T_m are normal for all $m \geq 1$ with $(m, N) = 1$. Moreover, their adjoints are given by*

$$\langle m\rangle^* = \langle \overline{m}\rangle \quad \textit{and} \quad T_p^* = \langle \overline{p}\rangle T_p.$$

Proof. By multiplicativity of the diamond and Hecke operators, it suffices to prove the two formulas when $m = p$ for a prime $p \nmid N$. We will prove the adjoint formula for diamond operators first. Let $\alpha = \left(\begin{smallmatrix}\overline{p} & * \\ * & p\end{smallmatrix}\right) \in \Gamma_0(N)$ and $\alpha' = \det(\alpha)\alpha^{-1} = \left(\begin{smallmatrix}p & * \\ * & \overline{p}\end{smallmatrix}\right) \in \Gamma_0(N)$. Then Proposition 10.5.7 gives

$$\langle p\rangle^* = |_k\alpha' = \langle \overline{p}\rangle.$$

This proves the adjoint formula for the diamond operators and normality follows from multiplicativity. For the Hecke operators, let $\beta_j = \left(\begin{smallmatrix}1 & j \\ 0 & p\end{smallmatrix}\right)$ and $\beta_\infty = \left(\begin{smallmatrix}pm & n \\ Np & p\end{smallmatrix}\right)$ for j taken modulo p and where m and n are chosen such that $\det\left(\left(\begin{smallmatrix}m & n \\ N & p\end{smallmatrix}\right)\right) = 1$. By Proposition 10.5.3, $\beta_1, \ldots, \beta_{p-1}, \beta_\infty$ is a complete set of orbit representatives for T_p. Now set $\alpha = \left(\begin{smallmatrix}1 & 0 \\ 0 & p\end{smallmatrix}\right)$ and $\alpha' = \det(\alpha)\alpha^{-1} = \left(\begin{smallmatrix}p & 0 \\ 0 & 1\end{smallmatrix}\right)$. Observe that

$$\begin{pmatrix}1 & n \\ N & pm\end{pmatrix}^{-1} \alpha \begin{pmatrix}p & n \\ N & m\end{pmatrix} = \begin{pmatrix}mp & -n \\ -N & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & p\end{pmatrix}\begin{pmatrix}p & n \\ N & m\end{pmatrix} = \alpha'.$$

As $\left(\begin{smallmatrix}1 & n \\ N & pm\end{smallmatrix}\right) \in \Gamma_1(N)$ (note that $pm \equiv 1 \pmod N$ since $pm - Nn = 1$) and $\left(\begin{smallmatrix}p & n \\ N & m\end{smallmatrix}\right) \in \Gamma_0(N)$, the fact that $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ yields

$$\Gamma_1(N)\alpha'\Gamma_1(N) = \Gamma_1(N)\alpha\Gamma_1(N)\begin{pmatrix}p & n \\ N & m\end{pmatrix}.$$

As $m \equiv \overline{p} \pmod N$, the above identity and Proposition 10.5.7 together give

$$T_p^* = [\Gamma_1(N)\alpha'\Gamma_1(N)]_k = \langle \overline{p}\rangle T_p.$$

This proves the adjoint formula for the Hecke operators and normality follows from multiplicativity. □

Note that on $\mathcal{S}_k(\Gamma_1(1))$, all of the diamond operators are the identity and therefore $T_p^* = T_p$ for all primes p. That is, the Hecke operators are self-adjoint (as are the diamond operators since they are the identity). Now suppose f is a non-constant cusp form. Let the eigenvalue of T_m for f be $\lambda_f(m)$. We say that the $\lambda_f(m)$ are the **Hecke eigenvalues** of f. If f is a simultaneous eigenfunction for all diamond operators $\langle m \rangle$ and Hecke operators T_m with $(m, N) = 1$, we call f an **eigenform**. If the condition $(m, N) = 1$ can be dropped, so that f is a simultaneous eigenfunction for all diamond and Hecke operators, we say f is a **Hecke eigenform**. In particular, on $\Gamma_1(1)\backslash\mathbb{H}$ all eigenforms are Hecke eigenforms. Now let f have Fourier coefficients $a_n(f)$. If f is a Hecke eigenform then Proposition 10.5.6 immediately implies that the first Fourier coefficient of $T_m f$ is $a_m(f)$ and so

$$a_m(f) = \lambda_f(m) a_1(f),$$

for all $m \geq 1$. Therefore we cannot have $a_1(f) = 0$ for this would mean f is constant. So we can normalize f by dividing by $a_1(f)$ which guarantees that this Fourier coefficient is 1. It follows that

$$a_m(f) = \lambda_f(m),$$

for all $m \geq 1$. This normalization is called the **Hecke normalization** of f. The **Petersson normalization** of f is where we normalize so that $\langle f, f \rangle = 1$. In particular, any orthonormal basis of $\mathcal{S}_k(\Gamma_1(N))$ is Petersson normalized. From the spectral theorem we derive an important corollary:

Theorem 10.5.1. *$\mathcal{S}_k(\Gamma_1(N))$ admits an orthonormal basis of eigenforms.*

Proof. By Theorem 10.3.1, $\mathcal{S}_k(\Gamma_1(N))$ is finite dimensional. The claim then follows from the spectral theorem along with Propositions 10.5.5 and 10.5.8. □

The Hecke eigenvalues of Hecke eigenforms satisfy certain relations known as the **Hecke relations** for holomorphic forms:

Proposition (Hecke relations, holomorphic). *Let $f \in \mathcal{S}_k(N, \chi)$ be a Hecke eigenform with Hecke eigenvalues $\lambda_f(m)$. Then the Hecke eigenvalues are multiplicative and satisfy*

$$\lambda_f(n)\lambda_f(m) = \sum_{d|(n,m)} \chi(d) d^{k-1} \lambda_f\left(\frac{nm}{d^2}\right) \quad \text{and} \quad \lambda_f(nm) = \sum_{d|(n,m)} \mu(d)\chi(d) d^{k-1} \lambda_f\left(\frac{n}{d}\right)\lambda_f\left(\frac{m}{d}\right),$$

for all $n, m \geq 1$ with $(nm, N) = 1$. Moreover,

$$\lambda_f(p^r) = \lambda_f(p)^r,$$

for all $p \mid N$ and $r \geq 2$.

Proof. If necessary, Hecke normalize f. The multiplicativity of the Hecke eigenvalues now follows from the multiplicity of the Hecke operators. The first identity follows from computing the n-th Fourier coefficient of $T_m f$ in two different ways. On the one hand, use that f is a Hecke eigenform to get $\lambda_f(n)\lambda_f(m)$. On the other hand, use Proposition 10.5.6 to obtain $\sum_{d|(n,m)} \chi(d) d^{k-1} \lambda_f\left(\frac{nm}{d^2}\right)$. For the second identity, computing the p-th Fourier coefficient of T_p in two different ways just as before, we have

$$\chi(p) p^{k-1} = \lambda_f(p)^2 - \lambda_f(p^2),$$

provided $(p, N) = 1$. The second identity now follows from the first because $\lambda_f(n)$ is a specially multiplicative function in n (see Theorem A.2.1). The third identity follows from inspecting the first Fourier coefficient of $T_{p^r} f$ and noting that $T_{p^r} = T_p^r$ provided $p \mid N$. □

As an immediate consequence of the Hecke relations, the Hecke operators satisfy analogous relations:

Corollary 10.5.1. *The Hecke operators are multiplicative and satisfy*

$$T_n T_m = \sum_{d|(n,m)} \chi(d) d^{k-1} T_{\frac{nm}{d^2}} \quad \text{and} \quad T_{nm} = \sum_{d|(n,m)} \mu(d)\chi(d) d^{k-1} T_{\frac{n}{d}} T_{\frac{m}{d}},$$

for all $n, m \geq 1$ with $(nm, N) = 1$.

Proof. This is immediate from Theorem 10.5.1 and the Hecke relations. □

The identities in Corollary 10.5.1 can also be established directly. Moreover, the first identity is symmetric in n and m so it can be used to show that the Hecke operators commute.

10.6 Atkin-Lehner Theory

So far, our entire theory of holomorphic forms has started with a fixed congruence subgroup of some level. Atkin-Lehner theory, or the theory of oldforms and newforms, allows us to discuss holomorphic forms in the context of moving between levels. In this setting, we will only deal with congruence subgroups of the form $\Gamma_1(N)$ and $\Gamma_0(N)$. The easiest way lift a holomorphic form from a smaller level to a larger level is to observe that if $M \mid N$ then $\Gamma_1(N) \leq \Gamma_1(M)$ so there is a natural inclusion $\mathcal{S}_k(\Gamma_1(M)) \subseteq \mathcal{S}_k(\Gamma_1(N))$. There is a less trivial way of lifting from $\mathcal{S}_k(\Gamma_1(M))$ to $\mathcal{S}_k(\Gamma_1(N))$. For any $d \mid \frac{N}{M}$, let $\alpha_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $f \in \mathcal{S}_k(\Gamma_1(M))$, we consider

$$(f|_k \alpha_d)(z) = \det(\alpha_d)^{k-1} j(\alpha_d, z)^{-k} f(\alpha_d z) = d^{k-1} f(dz).$$

It turns out that $|_k \alpha_d$ maps $\mathcal{S}_k(\Gamma_1(M))$ into $\mathcal{S}_k(\Gamma_1(N))$ and more:

Proposition 10.6.1. *Let M and N be positive integers such that $M \mid N$. For any $d \mid \frac{N}{M}$, $|_k \alpha_d$ maps $\mathcal{S}_k(\Gamma_1(M))$ into $\mathcal{S}_k(\Gamma_1(N))$. In particular, $|_k \alpha_d$ takes $\mathcal{S}_k(M, \chi)$ into $\mathcal{S}_k(N, \chi)$.*

Proof. By Proposition 10.5.1, it suffices to verify only the latter statement so we may assume $f \in \mathcal{S}_k(M, \chi)$. It is clear that holomorphy is satisfied for $f|_k \alpha_d$. To verify modularity, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d' \end{smallmatrix}\right) \in \Gamma_1(N)$. Then

$$\alpha_d \gamma \alpha_d^{-1} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d' \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & bd \\ d^{-1}c & d' \end{pmatrix} = \gamma',$$

where $\gamma' = \left(\begin{smallmatrix} a & bd \\ d^{-1}c & d' \end{smallmatrix}\right)$. Since $c \equiv 0 \pmod{N}$ and $d \mid \frac{N}{M}$, we deduce that $d^{-1}c \equiv 0 \pmod{M}$. So $\gamma' \in \Gamma_1(M)$ and therefore $\alpha_d \Gamma_1(N) \alpha_d^{-1} \subseteq \Gamma_1(M)$, or equivalently, $\Gamma_1(N) \subseteq \alpha_d^{-1} \Gamma_1(M) \alpha_d$. Writing $\gamma = \alpha_d^{-1} \gamma' \alpha_d$, we see that $\chi(\gamma') = \chi(\gamma)$ and $j(\gamma', \alpha_d z) = j(\gamma, z)$ and so

$$\begin{aligned}
(f|_k \alpha_d)(\gamma z) &= d^{k-1} f(d\gamma z) \\
&= d^{k-1} f(d\alpha_d^{-1} \gamma' \alpha_d z) \\
&= d^{k-1} f\left(\gamma' \alpha_d z\right) \\
&= \chi(\gamma') j(\gamma', \alpha_d z) d^{k-1} f(\alpha_d z) && \text{modularity} \\
&= \chi(\gamma) j(\gamma, z) d^{k-1} f(\alpha_d z) \\
&= \chi(\gamma) j(\gamma, z) d^{k-1} f(dz) \\
&= \chi(\gamma) j(\gamma, z) (f|_k \alpha_d)(z).
\end{aligned}$$

This verifies the modularity of $f|_k\alpha_d$. For the growth condition, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$ of $\Gamma_1(M)\backslash\mathbb{H}$. Then $\alpha_d\sigma_{\mathfrak{a}}$ takes ∞ to an element of $\mathbb{Q}\cup\{\infty\}$ since $\alpha_d \in \mathrm{GL}_2^+(\mathbb{Q})$. In other words, $\alpha_d\sigma_{\mathfrak{a}}\infty = \mathfrak{b}$ for some cusp $\mathfrak{b}$ of $\Gamma_1(N)\backslash\mathbb{H}$. Then the cocycle condition implies

$$j(\sigma_{\mathfrak{a}}, z)^{-k}(f|_k\alpha_d)(\sigma_{\mathfrak{a}}z) = \det(\alpha_d)^{k-1}j(\alpha_d\sigma_{\mathfrak{a}}, z)^{-k}f(\alpha_d\sigma_{\mathfrak{a}}z),$$

and the growth condition follows from that of f. In particular, $f|_k\alpha_d$ is a cusp form since f is. □

We can now define oldforms and newforms. For each divisor d of N, set

$$i_d : \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{d}\right)\right) \oplus \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{d}\right)\right) \to \mathcal{S}_k(\Gamma_1(N)) \qquad (f,g) \mapsto f + g|_k\alpha_d.$$

This map is well-defined by Proposition 10.6.1. The subspace of **oldforms** of level N is

$$\mathcal{S}_k^{\mathrm{old}}(\Gamma_1(N)) = \bigoplus_{p|N} \mathrm{Im}(i_p),$$

and the subspace of **newforms** of level N is

$$\mathcal{S}_k^{\mathrm{new}}(\Gamma_1(N)) = \mathcal{S}_k^{\mathrm{old}}(\Gamma_1(N))^{\perp}.$$

An element of these subspaces is called an **oldform** or **newform** respectively. Note that there are no oldforms of level 1. We will need a useful operator for the study of oldforms and newforms. Let

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

and note that $\det(W_N) = N$. We define the **Atkin-Lehner operator** ω_N to be the linear operator on $\mathcal{S}_k(\Gamma_1(N))$ given by

$$(\omega_N f)(z) = N^{1-\frac{k}{2}}(f|_k W_N)(z) = N^{\frac{k}{2}}j(W_N, z)^{-k}f(W_N z) = (\sqrt{N}z)^{-k}f\left(-\frac{1}{Nz}\right).$$

As W_N is invertible, so is the Atkin-Lehner operator ω_N. It is not difficult to see how ω_N acts on $\mathcal{S}_k(\Gamma_1(N))$:

Proposition 10.6.2. *ω_N maps $\mathcal{S}_k(\Gamma_1(N))$ into itself. In particular, ω_N takes $\mathcal{S}_k(N,\chi)$ into $\mathcal{S}_k(N,\overline{\chi})$ and preserves the subspaces of oldforms and newforms. Moreover, ω_N is self-adjoint and*

$$\omega_N^2 f = (-1)^k f.$$

Proof. In light of Proposition 10.5.1, the first statement is a consequence of the latter ones. Therefore we may assume $f \in \mathcal{S}_k(N,\chi)$. Holomorphy is obvious. For modularity, note that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we have

$$W_N\gamma W_N^{-1} = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & N^{-1} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -N^{-1}c \\ -Nb & a \end{pmatrix} = \gamma',$$

where $\gamma' = \left(\begin{smallmatrix} d & -N^{-1}c \\ -Nb & a \end{smallmatrix}\right) \in \Gamma_0(N)$. Thus $W_N\gamma = \gamma' W_N$ and it follows that

$$\begin{aligned}
(\omega_N f)(\gamma z) &= (\sqrt{N}\gamma z)^{-k} f(W_N \gamma z) \\
&= (\sqrt{N}\gamma z)^{-k} f(\gamma' W_N z) \\
&= \chi(\gamma')\left(\sqrt{N}\frac{az+b}{cz+d}\right)^{-k}\left(\frac{b}{z}+a\right)^{k} f\left(-\frac{1}{Nz}\right) && \text{modularity} \\
&= \chi(\gamma')\left(\sqrt{N}\frac{az+b}{cz+d}\right)^{-k}\left(\frac{z}{az+b}\right)^{-k} f\left(-\frac{1}{Nz}\right) \\
&= \chi(\gamma') j(\gamma, z)^k (\sqrt{N}z)^{-k} f\left(-\frac{1}{Nz}\right) \\
&= \overline{\chi}(\gamma) j(\gamma, z)^k (\sqrt{N}z)^{-k} f\left(-\frac{1}{Nz}\right) && ad \equiv 1 \pmod{N} \\
&= \overline{\chi}(\gamma) j(\gamma, z)^k (\omega_N f)(z).
\end{aligned}$$

This verifies modularity of $\omega_N f$. As for the growth condition, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$. Then $W_N\sigma_{\mathfrak{a}}$ takes ∞ to an element of $\mathbb{Q} \cup \{\infty\}$ since $W_N \in \mathrm{GL}_2^+(\mathbb{Q})$. In other words, $W_N\sigma_{\mathfrak{a}}\infty = \mathfrak{b}$ for some cusp $\mathfrak{b}$. Then the cocycle condition implies

$$j(\sigma_{\mathfrak{a}}, z)^{-k}(\omega_N f)(\sigma_{\mathfrak{a}} z) = N^{\frac{k}{2}} j(W_N\sigma_{\mathfrak{a}}, z)^{-k} f(W_N \sigma_{\mathfrak{a}} z),$$

and the growth condition follows from that of f. In particular, $\omega_N f$ is a cusp form because f is. It follows that $\omega_N f \in \mathcal{S}_k(N, \overline{\chi})$. We now show that ω_N is self-adjoint. Indeed, since $\omega_N' = \det(W_N)\omega_N^{-1} = \omega_N$ (recall we are working in $\mathrm{PSL}_2(\mathbb{Z})$) Proposition 10.5.7 implies

$$\omega_N^* = N^{1-\frac{k}{2}}|_k W_N' = \omega_N.$$

Thus ω_N is self-adjoint. We now show that ω_N preserves the subspaces of oldforms and newforms. To show that ω_N preserves $\mathcal{S}_k^{\mathrm{old}}(\Gamma_1(N))$, let $h = f + g|_k\alpha_p$ be in the image of i_p, with $p \mid N$, so that $i_p(f, g) = h$. Then it suffices to show

$$\omega_N i_p(f, g) = i_p(\omega_{\frac{N}{p}} p^{\frac{3k}{2}-1} g, \omega_{\frac{N}{p}} p^{1-\frac{k}{2}} f),$$

which will follow from the formulas

$$\omega_N f = (\omega_{\frac{N}{p}} p^{1-\frac{k}{2}} f)|_k \alpha_p \quad \text{and} \quad \omega_N(g|_k\alpha_p) = \omega_{\frac{N}{p}} p^{\frac{3k}{2}-1} g.$$

Both formulas follow immediately from the identities

$$W_N = W_{\frac{N}{p}}\alpha_p \quad \text{and} \quad \alpha_p W_N = p W_{\frac{N}{p}}.$$

Thus ω_N preserves the subspace of oldforms. To see that ω_N preserves the subspace of newforms as well, let f and g be a newform and an oldform respectively. The fact that ω_N is self-adjoint and preserves the subspace of oldforms gives

$$\langle \omega_N f, g\rangle = \langle f, \omega_N g\rangle = 0.$$

Hence $\omega_N f$ must also be a newform and so ω_N preserves the subspace of newforms. It remains to prove the formula. For this, observe $W_N^2 z = z$ and $j(W_N^2, z) = (-N)^k$ so that

$$(\omega_N^2 f)(z) = N^{2-k}(f|_k W_N^2)(z) = N^k j(W_N^2, z) f(W_N^2 z) = (-1)^k f(z). \qquad \square$$

Proposition 10.6.2 shows that ω_N is an involution if k is even and is at most of order 4. We now need to understand how the Atkin-Lehner operator interacts with the diamond and Hecke operators:

Proposition 10.6.3. *On $\mathcal{S}_k(\Gamma_1(N))$, the diamond operators $\langle m\rangle$ and Hecke operators T_m satisfy the following adjoint formulas for all $m \geq 1$:*

$$\langle m\rangle^* = \omega_N \langle m\rangle \omega_N^{-1} \quad \textit{and} \quad T_m^* = \omega_N T_m \omega_N^{-1}.$$

Proof. By multiplicativity of the diamond and Hecke operators, it suffices to prove the two adjoint formulas when $m = p$ for a prime p. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$. Then

$$W_N \gamma W_N^{-1} = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & N^{-1} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -N^{-1}c \\ -Nb & a \end{pmatrix} = \gamma',$$

where $\gamma' = \left(\begin{smallmatrix} d & -N^{-1}c \\ -Nb & a \end{smallmatrix}\right) \in \Gamma_1(N)$. In other words, W_N normalizes $\Gamma_1(N)$ so that $W_N \Gamma_1(N) W_N^{-1} = \Gamma_1(N)$. As $\Gamma_0(N) \leq \Gamma_1(N)$, the same holds for $\Gamma_0(N)$ as well. For the diamond operators, the formula is obvious when $p \mid N$ since $\langle p\rangle$ is the zero operator. So suppose $p \nmid N$ and let $\alpha = \left(\begin{smallmatrix} \overline{p} & * \\ * & p \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\alpha' = \det(\alpha)\alpha^{-1} = \left(\begin{smallmatrix} p & * \\ * & \overline{p} \end{smallmatrix}\right) \in \Gamma_0(N)$. As $W_N \alpha W_N^{-1} = \alpha'$, Proposition 10.5.7 gives

$$\langle p\rangle^* = |_k \alpha' = \omega_N \langle p\rangle \omega_N^{-1}.$$

This is the desired adjoint formula for the diamond operators when $p \mid N$. Thus the adjoint formula holds for all the diamond operators. For the Hecke operators, let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ and set $\alpha' = \det(\alpha)\alpha^{-1} = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $W_N \alpha W_N^{-1} = \alpha'$, and as W_N normalizes $\Gamma_1(N)$, Proposition 10.5.7 gives

$$T_p^* = [\Gamma_1(N)\alpha'\Gamma_1(N)]_k = \omega_N T_p \omega_N^{-1}.$$

This is the desired adjoint formula for the Hecke operators. □

It turns out that the spaces of oldforms and newforms are invariant under the diamond and Hecke operators:

Proposition 10.6.4. *On $\mathcal{S}_k(\Gamma_1(N))$, the diamond operators $\langle m\rangle$ and Hecke operators T_m preserve the subspaces of oldforms and newforms for all $m \geq 1$.*

Proof. By multiplicativity of the diamond and Hecke operators, it suffices to prove this when m is prime. Moreover, if $N = 1$ the result is trivial because there are no oldforms. Therefore we may assume $N > 1$ and hence consider a prime p with $p \mid N$. We will first show that the diamond and Hecke operators preserve $\mathcal{S}_k^{\text{old}}(\Gamma_1(N))$. Let $h = f + g|_k \alpha_p$ be in the image of i_p so that $i_p(f, g) = h$ and let q be a prime. For the diamond operators, it suffices to show

$$\langle q\rangle i_p(f, g) = i_p(\langle q\rangle f, \langle q\rangle g),$$

which will follow from the formulas

$$\langle q\rangle f = \langle q\rangle f \quad \text{and} \quad \langle q\rangle(g|_k \alpha_p) = (\langle q\rangle g)|_k \alpha_p.$$

In both of these formulas, the diamond operators on the left-hand sides and right-hand sides are on levels N and $\frac{N}{p}$ respectively. Both formulas are trivial if $q \mid N$ because then $\langle q\rangle$ is the zero operator. So suppose $q \nmid N$. If $\alpha = \left(\begin{smallmatrix} * & * \\ * & q \end{smallmatrix}\right) \in \Gamma_0(N)$ then $\alpha \in \Gamma_0\left(\frac{N}{p}\right)$ and the first formula follows. The second formula follows from the identity

$$\begin{pmatrix} * & * \\ * & q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & * \\ * & q \end{pmatrix}.$$

Therefore the diamond operators preserve $\mathcal{S}_k^{\text{old}}(\Gamma_1(N))$. The case for the Hecke operators is slightly more involved. Now consider the Hecke operator T_q and suppose $q \neq p$. For this Hecke operator, it suffices to show

$$T_q i_p(f,g) = i_p(T_q f, T_q g),$$

which will follow from the formulas

$$T_q f = T_q f \quad \text{and} \quad T_q(g|_k \alpha_p) = (T_q g)|_k \alpha_p.$$

In both of these formulas, the Hecke operators on the left-hand sides and right-hand sides are on levels N and $\frac{N}{p}$ respectively. The first formula follows immediately from Proposition 10.5.4 since this shows that the action of the Hecke operators on the Fourier coefficients is identical. For the second formula, by Proposition 10.5.1 we may assume that $g \in \mathcal{S}\left(\frac{N}{p}, \chi\right)$. Then $g|_k \alpha_p \in \mathcal{S}_k(N,\chi)$ (recall Proposition 10.6.1) and the second formula follows by comparing Fourier coefficients using Proposition 10.5.4. Thus the Hecke operators T_q, save for $q = p$, preserve $\mathcal{S}_k^{\text{old}}(\Gamma_1(N))$. Lastly, we consider the Hecke operator T_p. For this Hecke operator, it suffices to show

$$T_p i_p(f,g) = i_p(T_p f + p^{k-1} g, -\langle p \rangle f),$$

which will follow from the formulas

$$T_p f = T_p f - (\langle p \rangle f)|_k \alpha_p \quad \text{and} \quad T_p(g|_k \alpha_p) = p^{k-1} g.$$

In both of these formulas, the diamond and Hecke operators on the left-hand sides and right-hand sides are on levels N and $\frac{N}{p}$ respectively. Since $p \mid N$, Proposition 10.5.3 gives

$$T_p f = \sum_{j \pmod p} f\Big|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} = T_p f - (\langle p \rangle f)|_k \alpha_p,$$

where the second equality follows regardless of whether $p \mid \frac{N}{p}$ or not because either $\langle p \rangle$ is the zero operator on level $\frac{N}{p}$ (in the case $p \mid \frac{N}{p}$) or it annihilates the extra term in the Hecke operator T_p on level $\frac{N}{p}$ (in the case $p \nmid \frac{N}{p}$). Therefore the first formula holds. Similarly, since $p \mid N$, Proposition 10.5.3 gives

$$T_p(g|_k \alpha_p)(z) = \sum_{j \pmod p} g\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} = \sum_{j \pmod p} g\Big|_k \begin{pmatrix} p & pj \\ 0 & p \end{pmatrix} = p^{k-1} g,$$

where the last equality follows because g is 1-periodic. This shows that the second formula holds. Thus the Hecke operator T_p preserves $\mathcal{S}_k^{\text{old}}(\Gamma_1(N))$. We have now shown that all of the diamond and Hecke operators preserve $\mathcal{S}_k^{\text{old}}(\Gamma_1(N))$. To show that they also preserve $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$, let f and g be a newform and an oldform respectively. Then for any $m \geq 1$, Proposition 10.6.3 gives

$$\langle \langle m \rangle f, g \rangle = \langle f, \omega_N \langle m \rangle \omega_N^{-1} g \rangle = 0 \quad \text{and} \quad \langle T_m f, g \rangle = \langle f, \omega_N T_m \omega_N^{-1} g \rangle = 0,$$

where the second equality in either identity holds because the diamond and Hecke operators preserve $\mathcal{S}_k^{\text{old}}(\Gamma_1(N))$ and so does the Atkin-Lehner operator ω_N (and hence its inverse as well) by Proposition 10.6.2. It follows that the diamond and Hecke operators preserve $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ and this completes the proof. □

As a corollary, we deduce that these subspaces admit orthogonal bases of eigenforms:

Corollary 10.6.1. *$\mathcal{S}_k^{\text{old}}(\Gamma_1(N))$ and $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ admit orthonormal bases of eigenforms.*

Proof. This follows immediately from Theorem 10.5.1 and Proposition 10.6.4 □

Something quite amazing happens for the subspace in $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$; the condition $(m, N) = 1$ for eigenforms in a base can be removed. Therefore the eigenforms are actually eigenfunctions for all of the diamond and Hecke operators. We require a preliminary result whose proof is quite involved but it is not beyond the scope of this text (see [DS05] for a proof):

Lemma 10.6.1. *If $f \in \mathcal{S}_k(\Gamma_1(N))$ has Fourier coefficients $a_n(f)$ and is such that $a_n(f) = 0$ for all $n \geq 1$ whenever $(n, N) = 1$ then*

$$f = \sum_{p|N} p^{k-1} f_p|_k \alpha_p,$$

for some $f_p \in \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{p}\right)\right)$.

The important observation to make about Lemma 10.6.1 is that if $f \in \mathcal{S}_k(\Gamma_1(N))$ is such that its n-th Fourier coefficients vanish when n is relatively prime to the level then f must be an oldform. With this lemma we can prove the main theorem about $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$. The introduction of some language will be useful for the statement and its proof. We say that f is a **primitive Hecke eigenform** if it is a nonzero Hecke normalized Hecke eigenform in $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$. We can now prove the main result about newforms which is that Hecke eigenforms exist:

Theorem 10.6.1. *Let $f \in \mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ be an eigenform. Then the following hold:*

(i) f is a Hecke eigenform.

(ii) If g is any cusp form with the same Hecke eigenvalues at all primes then $g = cf$ for some nonzero $c \in \mathbb{C}$.

Moreover, the primitive Hecke eigenforms in $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ form an orthogonal basis and each such eigenform lies in an eigenspace $\mathcal{S}_k(N, \chi)$.

Proof. First suppose $f \in \mathcal{S}_k(\Gamma_1(N))$ is an eigenform with Fourier coefficients $a_n(f)$. For $m \geq 1$ with $(m, N) = 1$, there exists $\lambda_f(m), \mu_f(m) \in \mathbb{C}$ such that $T_m f = \lambda_f(m) f$ and $\langle m \rangle f = \mu_f(m) f$. Actually, $\langle m \rangle f = \mu_f(m) f$ holds for all $m \geq 1$ because $\langle m \rangle$ is the zero operator if $(m, N) > 1$ and in this case we can take $\mu_f(m) = 0$. If we set $\chi(n) = \mu_f(m)$ then χ is a Dirichlet character modulo N. This follows because multiplicativity of $\langle m \rangle$ implies the same for χ and χ is N-periodic since $\langle m \rangle$ is N-periodic ($\langle m \rangle$ is defined by m taken modulo N). But then $\langle m \rangle f = \chi(m) f$ so that $f \in \mathcal{S}_k(N, \chi)$. As f is an eigenform, we also have $a_m(f) = \lambda_f(m) a_1(f)$ provided $(m, N) = 1$. So if $a_1(f) = 0$, Lemma 10.6.1 implies $f \in \mathcal{S}_k^{\text{old}}(\Gamma_1(N))$. With this fact in hand, we can prove the statements.

(i) The claim is trivial if f is zero, so assume otherwise. If $f \in \mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ then $f \notin \mathcal{S}_k^{\text{old}}(\Gamma_1(N))$ and so by what we have shown $a_1(f) \neq 0$. Therefore we may Hecke normalize f so that $a_1(f) = 1$ and $a_m(f) = \lambda_f(m)$. Now set $g_m = T_m f - \lambda_f(m) f$ for any $m \geq 1$. By Proposition 10.6.4, $g_m \in \mathcal{S}_k^{\text{new}}(\Gamma_1(N))$. Moreover, g_m is an eigenform and its first Fourier coefficient is zero. But then $g_m \in \mathcal{S}_k^{\text{old}}(\Gamma_1(N))$ too and so $g_m = 0$ because $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ and $\mathcal{S}_k^{\text{old}}(\Gamma_1(N))$ are orthogonal subspaces. This means $T_m f = \lambda_f(m) f$ for any $m \geq 1$. Therefore f is a primitive Hecke eigenform and so is a Hecke eigenform before Hecke normalization.

(ii) Suppose g has the same Hecke eigenvalues at all primes. By multiplicativity of the Hecke operators, g is a Hecke eigenform. After Hecke normalization, f and g have the same Fourier coefficients and so are identical. It follows that before Hecke normalization $f = cg$ for some nonzero $c \in \mathbb{C}$.

Note that our initial remarks together with (i) show that each primitive Hecke eigenform f belongs to some eigenspace $\mathcal{S}_k(N, \chi)$. By Corollary 10.6.1, $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ admits an orthogonal basis of eigenforms which by (i) are Hecke eigenforms. As $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ is finite dimensional (because $\mathcal{S}_k(\Gamma_1(N))$ is), it follows that all of the primitive Hecke eigenforms form an orthogonal basis for $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ if we can show that they are linearly independent. So suppose to the contrary that we have a nontrivial linear relation

$$\sum_{1 \leq i \leq r} c_i f_i = 0,$$

for some primitive Hecke eigenforms f_i, nonzero constants c_i, and with r minimal. Note that $r \geq 2$ for else we do not have a nontrivial linear relation. Letting $m \geq 1$ applying the operator $T_m - \lambda_{f_1}(m)$ to our nontrivial linear relation gives

$$\sum_{2 \leq i \leq r} c_i(\lambda_{f_i}(m) - \lambda_{f_1}(m)) f_i = 0,$$

which has one less term. Since r was chosen to be minimal, this implies $\lambda_{f_i}(m) - \lambda_{f_1}(m)$ for all i. But m was arbitrary, so $f_i = f_1$ for all i by (ii). Hence $r = 1$ which is a contradiction. □

Statement (i) in Theorem 10.6.1 implies that primitive Hecke eigenforms satisfy the Hecke relations for all $n, m \geq 1$. Statement (ii) is known as **multiplicity one** for holomorphic forms and can be interpreted as saying that a basis of newforms for $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ contains one element per set of eigenvalues for the Hecke operators. We will now discuss conjugate cusp forms. For any $f \in \mathcal{S}_k(N, \chi)$, we define the **conjugate** $\overline{f}$ of f by

$$\overline{f}(z) = \overline{f(-\overline{z})}.$$

Note that if f has Fourier coefficients $a_n(f)$ then $\overline{f}$ has Fourier coefficients $\overline{a_n(f)}$. It turns out that $\overline{f}$ is indeed a holomorphic cusp form and behaves well with respect to the Hecke operators:

Proposition 10.6.5. *If $f \in \mathcal{S}_k(N, \chi)$ then $\overline{f} \in \mathcal{S}_k(N, \overline{\chi})$. Moreover,*

$$T_m \overline{f} = \overline{T_m f},$$

for all $m \geq 1$ with $(m, N) = 1$. In particular, if f is an eigenform with Hecke eigenvalues $\lambda_f(m)$ then $\overline{f}$ is too but with Hecke eigenvalues $\overline{\lambda_f(m)}$.

Proof. Holomorphy is clear so we need next need to verify the modularity. For this, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and note that $\gamma = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then we compute

$$\begin{aligned}
\overline{f}(\gamma z) &= \overline{f(-\overline{\gamma z})} \\
&= \overline{f(\gamma'(-\overline{z}))} \\
&= \overline{\chi(\gamma) j(\gamma', -\overline{z})^k f(-\overline{z})} \\
&= \overline{\chi(\gamma) j(\gamma, z)^k f(-\overline{z})} \\
&= \overline{\chi}(\gamma) j(\gamma, z)^k \overline{f(-\overline{z})} \\
&= \overline{\chi}(\gamma) j(\gamma, z)^k \overline{f}(z),
\end{aligned}$$

which proves modularity. The growth condition follows immediately from that of f and thus $\overline{f}$ is a cusp form since f is. This proves the first statement. The second statement is immediate from Proposition 10.5.6. For the last statement, suppose $f \in \mathcal{S}_k(N,\chi)$ is an eigenform with Hecke eigenvalues $\lambda_f(m)$. Then by what we have already shown,

$$(T_m\overline{f})(z) = (\overline{T_m f})(z) = \overline{T_m f(-\overline{z})} = \overline{\lambda_f(m) f(-\overline{z})} = \overline{\lambda_f(m) f}(z),$$

which completes the proof. □

In conjunction with Theorem 10.6.1, Proposition 10.6.5 implies that the primitive Hecke eigenforms in $\mathcal{S}_k^{\text{new}}(\Gamma_1(N))$ are conjugate invariant and if $f \in \mathcal{S}_k(N,\chi)$ is such an eigenform then $\overline{f} \in \mathcal{S}_k(N,\overline{\chi})$ is as well. The crucial fact we need is how $\omega_N f$ is related to $\overline{f}$ when f is a primitive Hecke eigenform:

Proposition 10.6.6. *If $f \in \mathcal{S}_k(N,\chi)$ is a primitive Hecke eigenform then*

$$\omega_N f = \omega_N(f)\overline{f},$$

where $\overline{f} \in \mathcal{S}_k(N,\overline{\chi})$ is a primitive Hecke eigenform and $\omega_N(f) \in \mathbb{C}$ is nonzero with $|\omega_N(f)| = 1$.

Proof. Let f have Hecke eigenvalues $\lambda_f(m)$. On the one hand, Theorem 10.6.1 and Proposition 10.6.5 together imply that $\overline{f}$ is a primitive Hecke eigenform with Hecke eigenvalues $\overline{\lambda_f(m)}$. One the other hand, Proposition 10.6.3 implies $\omega_N T_m = T_m^* \omega_N$ for all $m \geq 1$. Then

$$\langle T_m \omega_N f, \omega_N f\rangle = \langle \omega_N f, T_m^* \omega_N f\rangle = \langle \omega_N f, \omega_N T_m f\rangle = \langle \omega_N f, \lambda_m(f)\omega_N f\rangle = \overline{\lambda_f(m)}\langle \omega_N f, \omega_N f\rangle,$$

and it follows that $T_m \omega_N f = \overline{\lambda_f(m)}\omega_N f$. In other words, $\omega_N f$ is a Hecke eigenform with Hecke eigenvalues $\overline{\lambda_f(m)}$. Then multiplicity one gives

$$\omega_N f = \omega_N(f)\overline{f},$$

for some nonzero $\omega_N(f) \in \mathbb{C}$. Actually, by Proposition 10.6.2 we see that ω_N is at most of order 4 so necessarily $|\omega_N(f)| = 1$. □

10.7 The Ramanujan-Petersson Conjecture

We will now discuss a famous conjecture about the size of the Hecke eigenvalues of primitive Hecke eigenforms. Historically the conjecture was born from conjectures made about the **modular discriminant** Δ given by

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2),$$

which is a weight 12 primitive Hecke eigenform on $\Gamma_1(1)\backslash\mathbb{H}$ (see [DS05] for a proof). Therefore it is natural to begin our discussion here. It can be shown that the Fourier series of the modular discriminant is

$$\Delta(z) = \sum_{n\geq 1} \tau(n) e^{2\pi i n z},$$

where the $\tau(n)$ are integers with $\tau(1) = 1$ and $\tau(2) = -24$ (see [DS05] for a proof). The function $\tau : \mathbb{Z}_+ \to \mathbb{Z}$ is called **Ramanujan's τ function**. Ramanujan himself studied this function in his 1916 paper (see [Ram16]), and computed $\tau(n)$ for $1 \leq n \leq 30$. From these computations he conjectured that τ should satisfy the following three properties:

(i) If $(n, m) = 1$ then $\tau(nm) = \tau(n)\tau(m)$.

(ii) $\tau(p^n) = \tau(p^{n-1})\tau(p) - p^{11}\tau(p^{n-2})$ for all prime p.

(iii) $|\tau(p)| \leq 2p^{\frac{11}{2}}$ for all prime p.

Note that (i) means τ is multiplicative. Moreover, (i) and (ii) are strikingly similar to the properties satisfied by the Hecke operators. In fact, (i) and (ii) are follow from the fact that Δ is a Hecke eigenform. Property (iii) turned out to be drastically more difficult to prove and is known as the classical **Ramanujan-Petersson conjecture**. To state the Ramanujan-Petersson conjecture for holomorphic forms, suppose $f \in \mathcal{S}_k(N, \chi)$ is a primitive Hecke eigenform with Hecke eigenvalues $\lambda_f(m)$. For each prime p, consider the polynomial

$$1 - \lambda_f(p)p^{-\frac{k-1}{2}}p^{-s} + \chi(p)p^{-2s}.$$

We call this the p-th **Hecke polynomial** of f. We call the roots $\alpha_1(p)$ and $\alpha_2(p)$ the p-th **Hecke roots** of f. From this quadratic, we have

$$\alpha_1(p) + \alpha_2(p) = \lambda_f(p)p^{-\frac{k-1}{2}} \quad \text{and} \quad \alpha_1(p)\alpha_2(p) = \chi(p).$$

Then the more general **Ramanujan-Petersson conjecture** for holomorphic forms is following statement:

Theorem (Ramanujan-Petersson conjecture, holomorphic). *Let $f \in \mathcal{S}_k(N, \chi)$ be a primitive Hecke eigenform with Hecke eigenvalues $\lambda_f(m)$ and let $\alpha_1(p)$ and $\alpha_2(p)$ be the p-th Hecke roots of f. Then for all primes p, we have*

$$|\lambda_f(p)| \leq 2p^{\frac{k-1}{2}}.$$

Moreover, if $p \nmid N$ then

$$|\alpha_1(p)| = |\alpha_2(p)| = 1.$$

In the 1970's Deligne proved the Ramanujan-Petersson conjecture (see [Del71] and [Del74] for the full proof). The argument is significantly beyond the scope of this text, and in actuality follows from Deligne's work on the Weil conjectures (except in the case $k = 1$ which requires a modified argument). This requires understanding classical algebraic topology and ℓ-acid cohomology in addition to number theory. As such, the proof of the Ramanujan-Petersson conjecture has been one of the biggest advances in number theory in recent decades. From the Hecke relations and the Ramanujan-Petersson conjecture, we have the more general bound $\lambda_f(m) \ll \sigma_0(m)m^{\frac{k-1}{2}}$. Then Proposition A.3.1 gives the slightly weaker estimate $\lambda_f(m) \ll_\varepsilon m^{\frac{k-1}{2}+\varepsilon}$.

10.8 Twists of Holomorphic Forms

We can also twist holomorphic forms by Dirichlet characters. Let $f \in \mathcal{S}_k(N, \chi)$ have Fourier series

$$f(z) = \sum_{n\geq 1} a_n(f)e^{2\pi inz},$$

and let ψ be a Dirichlet character modulo M. We define the **twisted holomorphic form** $f \otimes \psi$ of f twisted by ψ by the Fourier series

$$(f \otimes \psi)(z) = \sum_{n\geq 1} a_n(f)\psi(n)e^{2\pi inz}.$$

In order for $f \otimes \psi$ to be well-defined, we need to prove that it is a holomorphic form. The following proposition proves this and more when ψ is primitive:

Proposition 10.8.1. *Suppose $f \in \mathcal{S}_k(N, \chi)$ and ψ is a primitive Dirichlet character of conductor q. Then $f \otimes \psi \in \mathcal{S}_k(Nq^2, \chi\psi^2)$.*

Proof. By Corollary 1.4.1, we can write

$$\begin{aligned}(f \otimes \psi)(z) &= \sum_{n \geq 1} a_n(f)\psi(n)e^{2\pi inz} \\ &= \sum_{n \geq 1} a_n(f) \left(\frac{1}{\tau(\overline{\psi})} \sum_{r \pmod q} \overline{\psi}(r) e^{\frac{2\pi irn}{q}} \right) e^{2\pi inz} \\ &= \frac{1}{\tau(\overline{\psi})} \sum_{r \pmod q} \overline{\psi}(r) \sum_{n \geq 1} a_n(f) e^{2\pi in\left(z + \frac{r}{q}\right)} \\ &= \frac{1}{\tau(\overline{\psi})} \sum_{r \pmod q} \overline{\psi}(r) f\left(z + \frac{r}{q}\right).\end{aligned}$$

From this last expression, holomorphy is immediate since the sum is finite. For modularity, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(Nq^2)$ and set $\gamma_r = \left(\begin{smallmatrix} 1 & \frac{r}{q} \\ 0 & 1 \end{smallmatrix}\right)$ for every r modulo q. Then for r and r' modulo q, we compute

$$\gamma_r \gamma \gamma_{r'}^{-1} = \begin{pmatrix} 1 & \frac{r}{q} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -\frac{r'}{q} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + \frac{cr}{q} & b - \frac{ar' - dr}{q} - \frac{crr'}{q^2} \\ c & d - \frac{cr'}{q} \end{pmatrix}.$$

Since $c \equiv 0 \pmod{Nq^2}$, if we choose r' (for each r) such that $ar' \equiv dr \pmod q$ then $\gamma_r \gamma \gamma_r^{-1} \in \Gamma_0(N)$. Such a choice exists and is unique by Bézout's identity because a and d are relatively prime to q as $ad \equiv 1 \pmod{Nq^2}$. Making this choice and setting $\eta_r = \gamma_r \gamma \gamma_{r'}^{-1}$, we compute

$$f\left(\gamma z + \frac{r}{q}\right) = f(\gamma_r \gamma z) = f(\eta_r \gamma_{r'} z) = \chi(\eta_r) j(\eta_r, \gamma_{r'} z)^k f(\gamma_{r'} z) = \chi(\eta_r) j(\eta_r, \gamma_{r'} z)^k f\left(z + \frac{r'}{q}\right).$$

Moreover,

$$\chi(\eta_r) j(\eta_r, \gamma_{r'} z) = \chi\left(d - \frac{cr'}{q}\right)\left(c\gamma_{r'} z + d - \frac{cr'}{q}\right) = \chi(d)(cz + d) = \chi(\gamma) j(\gamma, z).$$

Together these two computations imply

$$f\left(\gamma z + \frac{r}{q}\right) = \chi(\gamma) j(\gamma, z)^k f\left(z + \frac{r'}{q}\right).$$

Now, as $ar' \equiv dr \pmod q$ and $ad \equiv 1 \pmod q$, we have

$$\overline{\psi}(r) = \overline{\psi}(a\bar{d}r') = \psi^2(d)\overline{\psi}(r') = \psi^2(\gamma)\overline{\psi}(r').$$

Putting everything together,

$$\begin{aligned}(f \otimes \psi)(\gamma z) &= \frac{1}{\tau(\overline{\psi})} \sum_{r \pmod q} \overline{\psi}(r) f\left(\gamma z + \frac{r}{q}\right) \\ &= \chi(\gamma) j(\gamma, z)^k \frac{1}{\tau(\overline{\psi})} \sum_{r \pmod q} \overline{\psi}(r) f\left(z + \frac{r'}{q}\right) \\ &= \chi\psi^2(\gamma) j(\gamma, z)^k \frac{1}{\tau(\overline{\psi})} \sum_{r \pmod q} \overline{\psi}(r') f\left(z + \frac{r'}{q}\right) \\ &= \chi\psi^2(\gamma) j(\gamma, z)^k (f \otimes \psi)(z).\end{aligned}$$

from which the modularity of $f \otimes \psi$ follows. For the growth condition, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$ of $\Gamma_0(Nq^2)\backslash\mathbb{H}$. As $\gamma_r \in \mathrm{GL}_2^+(\mathbb{Q})$, $\gamma_r\sigma_{\mathfrak{a}}$ takes ∞ to an element of $\mathbb{Q} \cup \{\infty\}$. Thus $\gamma_r\sigma_{\mathfrak{a}}\infty = \mathfrak{b}$ for some cusp $\mathfrak{b}$ of $\Gamma_0(N)\backslash\mathbb{H}$. Then as $j(\gamma_r, \sigma_{\mathfrak{a}}z) = 1$, our previous work and the cocycle condition together imply

$$j(\sigma_{\mathfrak{a}}, z)^{-k}(f \otimes \psi)(\sigma_{\mathfrak{a}}z) = \frac{1}{\tau(\overline{\psi})} \sum_{r \pmod q} \overline{\psi}(r) j(\gamma_r\sigma_{\mathfrak{a}}, z)^{-k} f(\gamma_r\sigma_{\mathfrak{a}}z),$$

and the growth condition follows from that of f. Thus $f \otimes \psi$ is a cusp form since f is. □

The generalization of Proposition 10.8.1 to all characters is slightly more involved. To this end, define operators U_p and V_p on $\mathcal{S}_k(\Gamma_1(N))$ to be the linear operators given by

$$(U_pf)(z) = \sum_{n\geq 1} a_{np}(f)e^{2\pi inz},$$

and

$$(V_pf)(z) = \sum_{n\geq 1} a_n(f)e^{2\pi inpz},$$

if f has Fourier series

$$f(z) = \sum_{n\geq 1} a_n(f)e^{2\pi inz}.$$

We will show that both U_p and V_p map $\mathcal{S}_k(\Gamma_1(N))$ into $\mathcal{S}_k(\Gamma_1(Np))$ and more:

Lemma 10.8.1. *For any prime p, U_p and V_p map $\mathcal{S}_k(\Gamma_1(N))$ into $\mathcal{S}_k(\Gamma_1(Np))$. In particular, U_p and V_p map $\mathcal{S}_k(N, \chi)$ into $\mathcal{S}_k(Np, \chi\chi_{p,0})$.*

Proof. In light of Proposition 10.5.1, the first statement follows from the second. As $N \mid Np$, $\Gamma_1(Np) \leq \Gamma_1(N)$ so that $f \in \mathcal{S}_k(\Gamma_1(Np))$ if $f \in \mathcal{S}_k(\Gamma_1(N))$. Now suppose $f \in \mathcal{S}_k(N, \chi)$. Similarly, $N \mid Np$ implies $\Gamma_0(Np) \leq \Gamma_0(N)$ so that $f \in \mathcal{S}_k(Np, \chi\chi_{p,0})$ for the modulus Np character $\chi\chi_{p,0}$. Therefore we may assume $f \in \mathcal{S}_k(Np, \chi\chi_{p,0})$. Now consider U_p. As $p \mid Np$, Equation (10.2) implies $U_p = T_p$ is the p-th Hecke operator on $\mathcal{S}_k(\Gamma_1(Np))$ and the claim follows from the definition of the Hecke operators and Proposition 10.5.4. Now consider V_p. We have

$$(V_pf)(z) = f(pz),$$

and the claim follows by regarding $f \in \mathcal{S}_k(Np, \chi\chi_{p,0})$ and that $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ lies in the center of $\mathrm{PSL}_2(\mathbb{Z})$. □

We can now generalize Proposition 10.8.1 to all characters:

Proposition 10.8.2. *Suppose $f \in \mathcal{S}_k(N, \chi)$ and ψ is a Dirichlet character modulo M. Then $f \otimes \psi \in \mathcal{S}_k(NM^2, \chi\psi^2)$.*

Proof. Let $\widetilde{\psi}$ be the primitive character of conductor q inducing ψ. Then $\psi = \widetilde{\psi}\psi_{\frac{M}{q},0}$. As $\psi_{\frac{M}{q},0} = \prod_{p\mid\frac{M}{q}} \psi_{p,0}$, it suffices to prove the claim when ψ is primitive and when $\psi = \psi_{p,0}$. The primitive case follows from Proposition 10.8.1. So suppose $\psi = \psi_{p,0}$. Then

$$f \otimes \psi_{p,0} = f - V_pU_pf.$$

Now by Lemma 10.8.1, $V_pU_pf \in \mathcal{S}_k(Np^2, \chi\psi_{p,0}^2)$ (where we have written $\psi_{p,0}^2$ in place of $\chi_{p,0}$). Since we also have $f \in \mathcal{S}_k(Np^2, \chi\psi_{p,0}^2)$ (because $N \mid Np^2$ so that $\Gamma_1(Np^2) \leq \Gamma_1(N)$ and $\Gamma_0(Np^2) \leq \Gamma_0(N)$ and again writing $\psi_{p,0}^2$ in place of $\chi_{p,0}$), it follows that $f \otimes \psi_{p,0} \in \mathcal{S}_k(Np^2, \chi\psi_{p,0}^2)$. This proves the claim in the case $\psi = \psi_{p,0}$ and thus completes the proof. □

In particular, Proposition 10.8.2 shows that $f \otimes \psi$ is well-defined for any Dirichlet character ψ.

Chapter 11

The Theory of Automorphic and Maass Forms

Maass forms are the non-holomorphic analog to holomorphic forms. They are analytic, eigenfunctions for a differential operator, invariant with respect to a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, and satisfy a growth condition. The closely related automorphic forms satisfy fewer conditions and are necessary for the discussion of Maass forms in full generality. We introduce both Maass forms, automorphic forms, and their general theory. Throughout we assume that all of our congruence subgroups are reduced at infinity.

11.1 Automorphic and Maass Forms

Define $\varepsilon(\gamma, z)$ by

$$\varepsilon(\gamma, z) = \left(\frac{cz+d}{|cz+d|}\right),$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ and $z \in \mathbb{H}$. Note that $|\varepsilon(\gamma, z)| = 1$. Moreover, we have the relation

$$\varepsilon(\gamma, z) = \left(\frac{j(\gamma, z)}{|j(\gamma, z)|}\right).$$

As a consequence, the cocycle condition for $j(\gamma, z)$ implies

$$\varepsilon(\gamma'\gamma, z) = \varepsilon(\gamma', \gamma z)\varepsilon(\gamma, z),$$

and this is called the **cocycle condition** for $\varepsilon(\gamma, z)$. For any $k \in \mathbb{Z}$ and any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ we define the **slash operator** $|_{\varepsilon,k} : C(\mathbb{H}) \to C(\mathbb{H})$ to be the linear operator given by

$$(f|_{\varepsilon,k}\gamma)(z) = \det(\gamma)^{-1}\varepsilon(\gamma, z)^{-k} f(\gamma z).$$

If ε is clear from content we will suppress this dependence accordingly. Note that if $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, the slash operator takes the simpler form

$$(f|_{\varepsilon,k}\gamma)(z) = \varepsilon(\gamma, z)^{-k} f(\gamma z).$$

The cocycle condition implies that the slash operator is multiplicative. Indeed, if $\gamma, \gamma' \in \mathrm{GL}_2^+(\mathbb{Q})$ then

$$\begin{aligned}
((f|_{\varepsilon,k}\gamma')|_{\varepsilon,k}\gamma)(z) &= \det(\gamma)^{-1}\varepsilon(\gamma, z)^{-k}(f|_{\varepsilon,k}\gamma')(\gamma z) \\
&= \det(\gamma'\gamma)^{-1}\varepsilon(\gamma', \gamma z)^{-k}\varepsilon(\gamma, z)^{-k} f(\gamma'\gamma z) \\
&= \det(\gamma'\gamma)^{-1}\varepsilon(\gamma'\gamma, z)^{-k} f(\gamma'\gamma z) && \text{cocycle condition} \\
&= (f|_{\varepsilon,k}\gamma'\gamma)(z).
\end{aligned}$$

Any operator that commutes with the slash operators $|_{\varepsilon,k}\gamma$ for every $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ is said to be **invariant**. We define differential operators $R_k : C^\infty(\mathbb{H}) \to C^\infty(\mathbb{H})$ and $L_k : C^\infty(\mathbb{H}) \to C^\infty(\mathbb{H})$ to be the linear operators given by

$$R_k = \frac{k}{2} + y\left(i\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \quad \text{and} \quad L_k = \frac{k}{2} + y\left(i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right).$$

We call these operators the **Maass differential operators**. In particular, R_k is the **Maass raising operator** and L_k is the **Mass lowering operator**. The **Laplace operator** $\Delta_k : C^\infty(\mathbb{H}) \to C^\infty(\mathbb{H})$ is the linear operator given by

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}.$$

When $k = 0$, we will suppress this dependence. Note that Δ is the usual Laplace operator on $\mathbb{H}$. Expanding the products $R_{k-2}L_k$ and $L_{k+2}R_k$ and invoking the Cauchy-Riemann equation

$$\frac{\partial}{\partial y} = i\frac{\partial}{\partial x},$$

we arrive at the identities

$$\Delta_k = -R_{k-2}L_k + \frac{k}{2}\left(1 - \frac{k}{2}\right) \quad \text{and} \quad \Delta_k = -L_{k+2}R_k - \frac{k}{2}\left(1 - \frac{k}{2}\right).$$

The Mass differential operators and the Laplace operator satisfy important relations (see [Bum97] for a proof):

Proposition 11.1.1. *The Laplace operator Δ_k is invariant. That is,*

$$\Delta_k(f|_{\varepsilon,k}\gamma) = \Delta_k(f)|_{\varepsilon,k}\gamma,$$

for all $f \in C^\infty(\mathbb{H})$ and $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. Moreover, the Maass differential operators R_k and L_k satisfy

$$(R_k f)|_{\varepsilon,k+2}\gamma = R_k(f|_{\varepsilon,k}\gamma) \quad \textit{and} \quad (R_k f)|_{\varepsilon,k-2}\gamma = L_k(f|_{\varepsilon,k}\gamma).$$

We will now introduce automorphic functions, automorphic forms, and Maass forms. Let Γ be a congruence subgroup of level N that is reduced at infinity and let χ be a Dirichlet character of conductor $q \mid N$. Set $\chi(\gamma) = \chi(d)$ for all for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. First up are the automorphic functions. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is an **automorphic function** of **weight** k, **level** N, and **character** χ if following property is satisfied:

(i) $(f|_{\varepsilon,k}\gamma)(z) = \chi(\gamma)f(z)$ for all $\gamma \in \Gamma$.

We call property (i) is called the **automorphy condition** and we say that f is **automorphic**. The automorphy condition can equivalently be expressed as

$$f(\gamma z) = \chi(\gamma)\varepsilon(\gamma, z)^k f(z).$$

Note that automorphic functions admit Fourier series. Indeed, automorphy implies

$$f(z+1) = f\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z\right) = f(z),$$

so that f is 1-periodic. Let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the $\mathfrak{a}$ cusp. As Lemma 9.1.2 implies $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$ is a congruence subgroup, it follows by the cocycle condition that $f|_k\sigma_{\mathfrak{a}}$ is an automorphic function on $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}\backslash\mathbb{H}$ of the same weight and character as f. In particular, $f|_k\sigma_{\mathfrak{a}}$ is 1-periodic. Thus f admits a **Fourier series** at the $\mathfrak{a}$ cusp given by

$$(f|_k\sigma_{\mathfrak{a}})(z) = \sum_{n\in\mathbb{Z}} a_{\mathfrak{a}}(n,y)e^{2\pi inx},$$

with **Fourier coefficients** $a_{\mathfrak{a}}(n,y)$. Observe that the sum is over all $n \in \mathbb{Z}$ since f may be unbounded as $z \to \infty$. If $\mathfrak{a} = \infty$, we will drop this dependence and in this case $(f|_k\sigma_{\mathfrak{a}}) = f$. Unlike holomorphic forms, this Fourier series need not converge pointwise anywhere. Next up are the automorphic forms. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is an **automorphic form** of **weight** k, **eigenvalue** λ, **level** N, and **character** χ if following properties are satisfied:

(i) f is smooth on $\mathbb{H}$.

(ii) $(f|_{\varepsilon,k}\gamma)(z) = \chi(\gamma)f(z)$ for all $\gamma \in \Gamma$.

(iii) f is an eigenfunction for Δ_k with eigenvalue λ.

In property (iii), we will often let $s \in \mathbb{C}$ be such that $\lambda = s(1-s)$ and write $\lambda = \lambda(s)$ so that the eigenvalue can be determined by s. It turns turns out that Property (i) is implied by (iii). This is because Δ_k is an elliptic operator and any eigenfunction of an elliptic operator is automatically analytic and hence smooth (see [Eva22] for a proof in the weight zero case and [DFI02] for notes on the general case). Let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the $\mathfrak{a}$ cusp. As automorphic forms are automorphic functions, it follows by Proposition 11.1.1 that $f|_k\sigma_{\mathfrak{a}}$ is an automorphic form on $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}\backslash\mathbb{H}$ of the same weight, eigenvalue, and character as f. Moreover, $f|_k\sigma_{\mathfrak{a}}$ is also 1-periodic and so f admits a **Fourier series** at the $\mathfrak{a}$ cusp given by

$$(f|_k\sigma_{\mathfrak{a}})(z) = \sum_{n\in\mathbb{Z}} a_{\mathfrak{a}}(n,y,s)e^{2\pi inx},$$

with **Fourier coefficients** $a_{\mathfrak{a}}(n,y,s)$. If $\mathfrak{a} = \infty$ or s is fixed, we will drop these dependencies accordingly and in this case $f|_k\sigma_{\mathfrak{a}} = f$. As f (and hence $f|_k\sigma_{\mathfrak{a}}$ too) is smooth, it converges uniformly to its Fourier series everywhere. The Fourier coefficients $a_{\mathfrak{a}}(n,y,s)$ are mostly determined by Δ_k. To see this, since $f|_k\sigma_{\mathfrak{a}}$ is smooth we may differentiate the Fourier series of $f|_k\sigma_{\mathfrak{a}}$ termwise. The fact that $f|_k\sigma_{\mathfrak{a}}$ is an eigenfunction for Δ_k with eigenvalue $\lambda(s)$ gives the ODE

$$(4\pi^2n^2y^2 - 2\pi nky)a_{\mathfrak{a}}(n,y,s) - y^2a_{\mathfrak{a},yy}(n,y,s) = \lambda(s)a_{\mathfrak{a}}(n,y,s).$$

If $n \neq 0$, this is a Whittaker equation. To see this, first put the ODE in homogeneous form

$$y^2a_{\mathfrak{a},yy}(n,y,s) - (4\pi^2n^2y^2 - 2\pi nky - \lambda(s))a_{\mathfrak{a}}(n,y,s) = 0.$$

Now make the change of variables $y \mapsto \frac{y}{4\pi|n|}$ to get

$$y^2a_{\mathfrak{a},yy}(n,4\pi|n|y,s) - \left(\frac{y^2}{4} - \operatorname{sgn}(n)\frac{k}{2}y - \lambda(s)\right)a_{\mathfrak{a}}(n,4\pi|n|y,s) = 0,$$

where $\operatorname{sgn}(n) = \pm 1$ if n is positive or negative respectively. Diving by y^2 results in

$$a_{\mathfrak{a},yy}(n,4\pi|n|y,s) + \left(\frac{1}{4} - \frac{\operatorname{sgn}(n)\frac{k}{2}}{y} - \frac{\lambda(s)}{y^2}\right)a_{\mathfrak{a}}(n,4\pi|n|y,s) = 0.$$

As $\lambda(s) = s(1-s) = \frac{1}{4} - (s-\frac{1}{2})^2$, the above equation becomes

$$a_{\mathfrak{a},yy}(n, 4\pi|n|y, s) + \left(\frac{1}{4} - \frac{\operatorname{sgn}(n)\frac{k}{2}}{y} - \frac{\frac{1}{4} - \left(s-\frac{1}{2}\right)^2}{y^2}\right) a_{\mathfrak{a}}(n, 4\pi|n|y, s) = 0.$$

This is the Whittaker equation (see Appendix B.7). Since f has moderate growth at the cusps, the general solution is the Whittaker function $W_{\operatorname{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y)$. Therefore

$$a_{\mathfrak{a}}(n, y, s) = a_{\mathfrak{a}}(n, s) W_{\operatorname{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y),$$

for some coefficients $a_{\mathfrak{a}}(n, s)$. If $n = 0$ then the differential equation is a second order linear ODE which is

$$-y^2 a_{\mathfrak{a},yy}(0, y, s) = \lambda(s) a_{\mathfrak{a}}(0, y, s).$$

This is a Cauchy-Euler equation, and since s and $1-s$ are the two roots of $z^2 - z + \lambda$, the general solution is

$$a_{\mathfrak{a}}(0, y, s) = a_{\mathfrak{a}}^+(s) y^s + a_{\mathfrak{a}}^-(s) y^{1-s},$$

The coefficients $a_{\mathfrak{a}}(n, s)$ and $a_{\mathfrak{a}}^{\pm}(s)$ are the only part of the Fourier series that actually depend on the implicit congruence subgroup Γ. Using these coefficients, f admits **Fourier-Whittaker series** at the $\mathfrak{a}$ cusp given by

$$(f|_k \sigma_{\mathfrak{a}})(z) = a_{\mathfrak{a}}^+(s) y^s + a_{\mathfrak{a}}^-(s) y^{1-s} + \sum_{n \neq 0} a_{\mathfrak{a}}(n, s) W_{\operatorname{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y) e^{2\pi i n x},$$

with **Fourier-Whittaker coefficients** $a_{\mathfrak{a}}^{\pm}(s)$ and $a_{\mathfrak{a}}(n, s)$. If $\mathfrak{a} = \infty$ or s is fixed, we will drop these dependencies accordingly and in this case $f|_k \sigma_{\mathfrak{a}} = f$. As f (and hence $f|_k \sigma_{\mathfrak{a}}$ too) is smooth, it converges uniformly to its Fourier-Whittaker series everywhere. Last up are the Maass forms. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is a **Maass form** on $\Gamma\backslash\mathbb{H}$ of **weight** k, **eigenvalue** λ, **level** N, and **character** χ if the following properties are satisfied:

(i) f is smooth on $\mathbb{H}$.

(ii) $(f|_{\varepsilon,k}\gamma)(z) = \chi(\gamma) f(z)$ for all $\gamma \in \Gamma$.

(iii) f is an eigenfunction for Δ_k with eigenvalue λ.

(iv) $(f|_{\varepsilon,k}\alpha)(z) = O(y^n)$ for some $n \geq 1$ and all $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$ (or equivalently $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$).

We say f is a (Maass) **cusp form** if the additional property is satisfied:

(v) For all cusps $\mathfrak{a}$ and any $y > 0$, we have

$$\int_0^1 (f|_k \sigma_{\mathfrak{a}})(x + iy)\, dx = 0.$$

Property (iv) is called the **growth condition** for Maass forms and we say f has **moderate growth at the cusps**. Clearly we only need to verify the growth condition on a set of scaling matrices for the cusps. Moreover, the equivalence in the growth condition follows exactly in the same way as for holomorphic forms. Indeed, Lemma 9.1.1 implies that we may write $\alpha = \gamma\eta$ for some $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ and $\eta \in \mathrm{GL}_2^+(\mathbb{Q})$ with $\eta = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. This decomposition along with the cocycle condition gives

$$\varepsilon(\alpha, z) = \varepsilon(\gamma, \eta z),$$

and it follows that $(f|_{\varepsilon,k}\alpha)(z) = o(e^{2\pi y})$ for all $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ which proves the forward implication. The reverse implication is trivial since $\mathrm{PSL}_2(\mathbb{Z}) \subset \mathrm{GL}_2^+(\mathbb{Q})$.

Remark 11.1.1. *Holomorphic forms embed into Maass forms. If $f(z)$ is a weight k holomorphic form on $\Gamma\backslash\mathbb{H}$ then $F(z) = \mathrm{Im}(4\pi z)^{\frac{k}{2}} f(z)$ is a weight k Maass form on $\Gamma\backslash\mathbb{H}$. This is because $\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|j(\gamma,z)|^2}$ and $F(z)$ clearly has polynomial growth in $\mathrm{Im}(z)$. Moreover, as $f(z)$ is holomorphic, it satisfies the Cauchy-Riemann equations so that*

$$L_k(F(z)) = \left(\frac{k}{2} + y\left(i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\right)(F(z)) = \frac{k}{2}F(z) - \frac{k}{2}F(z) = 0.$$

Therefore

$$\Delta_k(F(z)) = \left(-R_{k-2}L_k + \frac{k}{2}\left(1 - \frac{k}{2}\right)\right)(F(z)) = \frac{k}{2}\left(1 - \frac{k}{2}\right)F(z).$$

This means $F(z)$ is an eigenfunction for Δ_k with eigenvalue $\lambda\left(\frac{k}{2}\right)$.

One might expect that the Maass raising and lowering operators R_k and L_k act by changing the weight of a Maass form by ± 2 respectively. This is indeed the case:

Proposition 11.1.2. *If f is a weight k Maass form on $\Gamma\backslash\mathbb{H}$ then $R_k f$ and $L_k f$ are Maass forms on $\Gamma\backslash\mathbb{H}$ of weight $k+2$ and $k-2$ respectively and of the same eigenvalue, level, and character as f.*

Proof. This is immediate from the definition of Maass forms and Proposition 11.1.1. □

As a consequence of Proposition 11.1.2, it suffices to study Maass forms of weights $k = 0, 1$ although imposing this additional restriction is usually unnecessary. Let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the $\mathfrak{a}$ cusp. As Maass forms are automorphic forms, it follows by Proposition 11.1.1 that $f|_k\sigma_{\mathfrak{a}}$ is an automorphic form on $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}\backslash\mathbb{H}$ of the same weight, eigenvalue, and character as f. Moreover, $f|_k\sigma_{\mathfrak{a}}$ is also 1-periodic. Note that this means we only need to verify the growth condition as $y \to \infty$. As $f|_k\sigma_{\mathfrak{a}}$ is 1-periodic, f admits a **Fourier-Whittaker series** at the $\mathfrak{a}$ cusp given by

$$(f|_k\sigma_{\mathfrak{a}})(z) = a_{\mathfrak{a}}^+(s)y^s + a_{\mathfrak{a}}^-(s)y^{1-s} + \sum_{n\neq 0} a_{\mathfrak{a}}(n,s)W_{\mathrm{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)e^{2\pi inx},$$

with **Fourier-Whittaker coefficients** $a_{\mathfrak{a}}^{\pm}(s)$ and $a_{\mathfrak{a}}(n,s)$. As f (and hence $f|_k\sigma_{\mathfrak{a}}$ too) is smooth, it converges uniformly to its Fourier-Whittaker series everywhere. Moreover, property (v) implies that f is a cusp form if and only if $a_{\mathfrak{a}}^{\pm}(s) = 0$ for every cusp $\mathfrak{a}$. It is useful to specify the Whittaker function in the case of weight zero Maass forms. When $k = 0$, Theorem B.7.1 implies that the Fourier-Whittaker series of f at the $\mathfrak{a}$ cusp takes the form

$$(f|_k\sigma_{\mathfrak{a}})(z) = a_{\mathfrak{a}}^+(s)y^s + a_{\mathfrak{a}}^-(s)y^{1-s} + \sum_{n\neq 0} a_{\mathfrak{a}}(n,s)\sqrt{4|n|y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}.$$

In any case, we can also easily derive a bound for the size of the Fourier-Whittaker coefficients of cusp forms. Fix some $Y > 0$ and consider

$$\int_{\Gamma_\infty\backslash\mathbb{H}_Y} |(f|_k\sigma_{\mathfrak{a}})(z)|^2\, d\mu,$$

where $\mathbb{H}_Y$ is the half-plane defined by $Y \leq \mathrm{Im}(z) \leq 2Y$. This integral converges since $\Gamma_\infty\backslash\mathbb{H}_Y$ is compact. Substituting in the Fourier-Whittaker series of f at the $\mathfrak{a}$ cusp, this integral can be expressed as

$$\int_Y^{2Y}\int_0^1 \sum_{n,m\neq 0} a_{\mathfrak{a}}(n,s)\overline{a_{\mathfrak{a}}(m,s)}W_{\mathrm{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)\overline{W_{\mathrm{sgn}(m)\frac{k}{2},s-\frac{1}{2}}(4\pi|m|y)}e^{2\pi i(n-m)x}\,\frac{dy}{y^2}.$$

Appealing to the Fubini–Tonelli theorem, we can interchange the sum and the two integrals. Upon making this interchange, the identity Equation (10.1) implies that the inner integral cuts off all of the terms except the diagonal $n = m$, resulting in

$$\sum_{n\neq 0}\int_Y^{2Y} |a_{\mathfrak{a}}(n,s)|^2 |W_{\operatorname{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)|^2\,\frac{dy}{y^2}.$$

In particular, we see that this is a sum of nonnegative terms. Retaining only a single term in the sum, we have

$$|a_{\mathfrak{a}}(n,s)|^2\int_Y^{2Y} |W_{\operatorname{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)|^2\,\frac{dy}{y^2} \ll \int_{\Gamma_\infty\backslash\mathbb{H}_Y} |(f|_k\sigma_{\mathfrak{a}})(z)|^2\,d\mu.$$

Moreover, $|(f|_k\sigma_{\mathfrak{a}})(z)|^2$ is bounded on $\Gamma_\infty\backslash\mathbb{H}_Y$ because this space is compact, so that

$$\int_{\Gamma_\infty\backslash\mathbb{H}_Y} |(f|_k\sigma_{\mathfrak{a}})(z)|^2\,d\mu \ll \int_Y^{2Y}\int_0^1 \frac{dx\,dy}{y^2} \ll \frac{1}{Y}.$$

Putting these two estimates together gives

$$|a_{\mathfrak{a}}(n,s)|^2\int_Y^{2Y} |W_{\operatorname{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)|^2\,\frac{dy}{y^2} \ll \frac{1}{Y}.$$

Taking $Y = \frac{1}{|n|}$ and making the change of variables $y \mapsto \frac{y}{|n|}$, we obtain

$$a_{\mathfrak{a}}(n,s) \ll 1.$$

This bound is known as the **Hecke bound** for Maass forms. Using Lemma B.7.1, we have the estimate $W_{\operatorname{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y) = O((|n|y)^{\frac{k}{2}}e^{-2\pi|n|y})$. This estimate together with the Hecke bound gives

$$(f|_k\sigma_{\mathfrak{a}})(z) = O\left(y^{\frac{k}{2}}\sum_{n\neq 0}|n|^{\frac{k+1}{2}}e^{-2\pi|n|y}\right) = O\left(y^{\frac{k}{2}}\sum_{n\geq 1}n^{\frac{k+1}{2}}e^{-2\pi ny}\right) = O(y^{\frac{k}{2}}e^{-2\pi y}),$$

where the last equality holds because each term is of smaller order than the next so that the series is bounded by a constant times its first term. It follows that $(f|_k\sigma_{\mathfrak{a}})(z)$ exhibits exponential decay. Accordingly, we say that f exhibits **exponential decay at the cusps**. Observe that $f|_k\sigma_{\mathfrak{a}}$ is then bounded on $\mathbb{H}$ and, in particular, f is bounded on $\mathbb{H}$.

11.2 Poincaré and Eisenstein Series

Let Γ be a congruence subgroup of level N. We will introduce two important classes of automorphic functions on $\Gamma\backslash\mathbb{H}$ namely the Poincaré and Eisenstein series. The Eisenstein series will be Maass forms while the Poincaré series will only be automorphic functions. Both of these classes are defined on a larger space $\mathbb{H}\times\{s\in\mathbb{C}:\sigma>1\}$ and hence are functions of two variables. Let $m\geq 0$, $k\geq 0$, χ be a Dirichlet character with conductor $q\mid N$, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. Then the m-th (automorphic) **Poincaré series** $P_{m,k,\chi,\mathfrak{a}}(z,s)$ of weight k and character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp is defined by

$$P_{m,k,\chi,\mathfrak{a}}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-k}\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}.$$

We call m the **index** of $P_{m,k,\chi,\mathfrak{a}}(z,s)$. If $k=0$, χ is the trivial character, or $\mathfrak{a}=\infty$, we will drop these dependencies accordingly. We first show that $P_{m,k,\chi,\mathfrak{a}}(z,s)$ is well-defined. It suffices to show that the summands are independent of the representatives γ and $\sigma_{\mathfrak{a}}$. This has already been accomplished when we introduced the holomorphic Poincaré series for $\overline{\chi}(\gamma)$ and $e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}$. Now just as with the holomorphic Poincaré series, the set of representatives of $\sigma_{\mathfrak{a}}^{-1}\gamma$ is $\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}\gamma$ and it remains to verify independence from multiplication on the left by an element of Γ_∞ namely η_∞. The cocycle relation implies

$$\varepsilon(\eta_\infty\sigma_{\mathfrak{a}}^{-1}\gamma,z)=\varepsilon(\eta_\infty,\sigma_{\mathfrak{a}}^{-1}\gamma z)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma,z)=\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma,z),$$

where the last equality follows because $\varepsilon(\eta_\infty,\sigma_{\mathfrak{a}}^{-1}\gamma z)=1$ as $j(\eta_\infty,\sigma_{\mathfrak{a}}^{-1}\gamma z)=1$. Thus $\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma,z)$ is independent of the representatives γ and $\sigma_{\mathfrak{a}}$. Lastly, we have

$$\mathrm{Im}(\eta_\infty\sigma_{\mathfrak{a}}^{-1}\gamma z)=\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z),$$

because η_∞ does not affect the imaginary part as it acts by translation. Therefore $\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)$ is independent of the representatives γ and $\sigma_{\mathfrak{a}}$ as well. We conclude that $P_{m,k,\chi,\mathfrak{a}}(z,s)$ is well-defined. We claim $P_{m,k,\chi,\mathfrak{a}}(z,s)$ is also locally absolutely uniformly convergent for $z\in\mathbb{H}$ and $\sigma>1$. To see this, first recall that $|e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}|=e^{-2\pi m\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)}<1$. Then the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma$ yields

$$P_{m,k,\chi,\mathfrak{a}}(z,s)\ll\sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}}\frac{\mathrm{Im}(z)^\sigma}{|cz+d|^{2\sigma}},$$

and this latter series is locally absolutely uniformly convergent for $z\in\mathbb{H}$ and $\sigma>1$ by Proposition B.8.1. Hence the same holds for $P_{m,k,\chi,\mathfrak{a}}(z,s)$. Verifying automorphy amounts to a computation:

$$\begin{aligned}
P_{m,k,\chi,\mathfrak{a}}(\gamma z,s)&=\sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\chi}(\gamma')\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma',\gamma z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z}\\
&=\sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\chi}(\gamma')\left(\frac{\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma,z)}{\varepsilon(\gamma,z)}\right)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z}\\
&=\varepsilon(\gamma,z)^k\sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\chi}(\gamma')\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma,z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z}\\
&=\chi(\gamma)\varepsilon(\gamma,z)^k\sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\chi}(\gamma')\overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma,z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z}\\
&=\chi(\gamma)\varepsilon(\gamma,z)^k\sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\chi}(\gamma'\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma,z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z}\\
&=\chi(\gamma)\varepsilon(\gamma,z)^k\sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\chi}(\gamma')\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma',z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma' z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z}\\
&=\chi(\gamma)\varepsilon(\gamma,z)^k P_{m,k,\chi,\mathfrak{a}}(z,s),
\end{aligned}$$

where in the second line we have used the cocycle condition and in the second to last line we have made the change of variables $\gamma'\mapsto\gamma'\gamma^{-1}$. As for the growth condition, let $\sigma_{\mathfrak{b}}$ be a scaling matrix for the cusp $\mathfrak{b}$. Then the bound $|e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z}|=e^{-2\pi m\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)}<1$, cocycle condition, and the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ together give

$$\varepsilon(\sigma_{\mathfrak{b}},z)^{-k}P_{m,k,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s)\ll\mathrm{Im}(z)^\sigma\sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}}\frac{1}{|cz+d|^{2\sigma}}.$$

Now decompose this sum as

$$\sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{|cz+d|^{2\sigma}} = \sum_{d\neq 0} \frac{1}{d^{2\sigma}} + \sum_{c\neq 0}\sum_{d\in\mathbb{Z}} \frac{1}{|cz+d|^{2\sigma}} = 2\sum_{d\geq 1} \frac{1}{d^{2\sigma}} + 2\sum_{c\geq 1}\sum_{d\in\mathbb{Z}} \frac{1}{|cz+d|^{2\sigma}}.$$

Notice that the first sum is absolutely uniformly bounded provided $\sigma > 1$. Moreover, the exact same argument as for holomorphic Eisenstein series shows that the second sum is too. So for all $\mathrm{Im}(z) \geq 1$ and $\sigma > 1$, we have

$$\varepsilon(\sigma_{\mathfrak{b}}, z)^{-k} P_{m,k,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) \ll \mathrm{Im}(z)^{\sigma} = o(e^{2\pi \mathrm{Im}(z)}),$$

provided $\mathrm{Im}(z) \geq 1$ and $\sigma > 1$. This verifies the growth condition. We collect this work as a theorem:

Theorem 11.2.1. *Let $m \geq 0$, $k \geq 0$, χ be a Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. For $\sigma > 1$, the Poincaré series*

$$P_{m,k,\chi,\mathfrak{a}}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z},$$

is a smooth automorphic function on $\Gamma\backslash\mathbb{H}$.

For $m = 0$, we write $E_{k,\chi,\mathfrak{a}}(z,s) = P_{0,k,\chi,\mathfrak{a}}(z,s)$ and call $E_{k,\chi}(z)$ the (Maass) **Eisenstein series** of weight k and character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp. It is defined by

$$E_{k,\chi,\mathfrak{a}}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s.$$

If $k = 0$, χ is the trivial character, or $\mathfrak{a} = \infty$, we will drop these dependencies accordingly. It turns out that $E_{k,\chi,\mathfrak{a}}(z,s)$ is actually a Maass form. The only thing left to verify is that $E_{k,\chi,\mathfrak{a}}(z,s)$ is an eigenfunction for Δ_k. To see this, first observe that

$$\Delta_k(y^s) = \left(-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}\right)(y^s) = \lambda(s)y^s.$$

Therefore $\mathrm{Im}(z)^s$ is an eigenfunction for Δ_k with eigenvalue $\lambda(s)$. Since Δ_k is invariant, we have

$$\Delta_k((\mathrm{Im}(\cdot)^s|_{\varepsilon,k}\gamma)(z)) = ((\Delta_k \mathrm{Im}(\cdot)^s)|_{\varepsilon,k}\gamma)(z) = \lambda(s)(\mathrm{Im}(\cdot)^s|_{\varepsilon,k}\gamma)(z),$$

and so $(\mathrm{Im}(\cdot)^s|_{\varepsilon,k}\gamma)(z) = \varepsilon(\gamma,z)^{-k}\mathrm{Im}(\gamma z)^s$ is also an eigenfunction for Δ_k with eigenvalue $\lambda(s)$ for all $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$. It follows that

$$\Delta_k(E_{k,\chi,\mathfrak{a}}(z,s)) = \lambda(s)E_{k,\chi,\mathfrak{a}}(z,s),$$

which shows $E_{k,\chi,\mathfrak{a}}(z,s)$ is also an eigenfunction for Δ_k with eigenvalue $\lambda(s)$. We collect this work as a theorem:

Theorem 11.2.2. *Let $k \geq 0$, χ be a Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. For $\sigma > 1$, the Eisenstein series*

$$E_{k,\chi,\mathfrak{a}}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s,$$

is a weight k Maass form with eigenvalue $\lambda(s)$ and character χ on $\Gamma\backslash\mathbb{H}$.

11.3 Inner Product Spaces of Automorphic Functions

Let $\mathcal{A}_k(\Gamma,\chi)$ denote the complex vector space of all weight k automorphic functions with character χ on $\Gamma\backslash\mathbb{H}$ and let $\mathcal{A}_{k,\lambda}(\Gamma,\chi)$, $\mathcal{M}_{k,\lambda}(\Gamma,\chi)$, and $\mathcal{C}_{k,\lambda}(\Gamma,\chi)$ denote the associated subspaces of automorphic functions, Maass forms, and cusp forms of eigenvalue λ respectively. If $k=0$ or χ is the trivial character, we will suppress these dependencies. Note that if Γ_1 and Γ_2 are two congruence subgroups such that $\Gamma_1 \leq \Gamma_2$ then we have the inclusion

$$\mathcal{A}_k(\Gamma_2,\chi) \subseteq \mathcal{A}_k(\Gamma_1,\chi),$$

and this respects the subspaces of automorphic forms, Maass forms, and cusp forms. So in general, the smaller the congruence subgroup the more automorphic functions there are. Our goal is to construct a complex Hilbert space containing $\mathcal{C}_{k,\lambda}(\Gamma,\chi)$ for which we can apply a linear theory. The natural space to consider is the L^2-space for automorphic functions. We define the L^2-norm $||\cdot||_\Gamma$ for $f \in \mathcal{A}_k(\Gamma,\chi)$ by

$$||f||_\Gamma = \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |f(z)|^2\, d\mu\right)^{\frac{1}{2}}.$$

If the congruence subgroup is clear from context we will suppress the dependence upon Γ. As f is automorphic, the norm is independent of the choice of fundamental domain and hence well-defined. Let $\mathcal{L}_k(\Gamma,\chi)$ be the subspace of $\mathcal{A}_k(\Gamma,\chi)$ consisting of those functions with bounded L^2-norm and let $\mathcal{L}_{k,\lambda}(\Gamma,\chi)$ denote the associated subspace of automorphic forms. Moreover, if χ is the trivial character or if $k=0$, we will suppress these dependencies accordingly. Since this is an L^2-space, $\mathcal{L}_k(\Gamma,\chi)$ is an induced complex inner product space (because the parallelogram law is satisfied). In particular, for any $f,g \in \mathcal{L}_k(\Gamma,\chi)$ we define their **Petersson inner product** to be

$$\langle f,g\rangle_\Gamma = \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}\, d\mu.$$

If the congruence subgroup is clear from context we will suppress the dependence upon Γ. The integral is locally absolutely uniformly convergent by the Cauchy–Schwarz inequality and that $f,g \in \mathcal{L}_k(\Gamma,\chi)$. As f and g are automorphic, the integral is independent of the choice of fundamental domain. These two facts imply that the Petersson inner product is well-defined. We will continue to use this notion even if f and g do not belong to $\mathcal{L}_k(\Gamma,\chi)$ provided the integral is locally absolutely uniformly convergent. Just as was the case for holomorphic forms, the Petersson inner product is invariant with respect to the slash operator:

Proposition 11.3.1. *For any $f,g \in \mathcal{L}_k(\Gamma,\chi)$ and $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$, we have*

$$\langle f|_k\alpha, g|_k\alpha\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f,g\rangle_\Gamma.$$

Proof. The argument used in the proof of Proposition 10.3.1 holds verbatim. □

More importantly, the Petersson inner product turns $\mathcal{L}_k(\Gamma,\chi)$ into a complex Hilbert space:

Proposition 11.3.2. *$\mathcal{L}_k(\Gamma,\chi)$ is a complex Hilbert space with respect to the Petersson inner product.*

Proof. Let $f,g \in \mathcal{L}_k(\Gamma,\chi)$. Linearity of the integral immediately implies that the Petersson inner product is linear on $\mathcal{L}_k(\Gamma,\chi)$. It is also positive definite since

$$\langle f,f\rangle = \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{f(z)}\, d\mu = \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |f(z)|^2\, d\mu \geq 0,$$

with equality if and only if f is identically zero. To see that it is conjugate symmetric, observe

$$\begin{aligned}
\overline{\langle g, f\rangle} &= \overline{\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} g(z)\overline{f(z)}\,d\mu} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \overline{g(z)}f(z)\,\overline{d\mu} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \overline{g(z)}f(z)\,d\mu \qquad\qquad d\mu = \frac{dx\,dy}{y^2} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}\,d\mu \\
&= \langle f, g\rangle.
\end{aligned}$$

So the Petersson inner product turns $\mathcal{L}_k(\Gamma,\chi)$ into a complex inner product space. We now show that $\mathcal{L}_k(\Gamma,\chi)$ is complete. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{L}_k(\Gamma,\chi)$. Then $||f_n - f_m|| \to 0$ as $n, m \to \infty$. But

$$||f_n - f_m|| = \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |f_n(z) - f_m(z)|^2\,d\mu\right)^{\frac{1}{2}},$$

and this integral tends to zero if and only if $|f_n(z) - f_m(z)| \to 0$ as $n, m \to \infty$. Therefore $\lim_{n\to\infty} f_n(z)$ exists and we define the limiting function f by $f(z) = \lim_{n\to\infty} f_n(z)$. We claim that f is automorphic. Indeed, as the f_n are automorphic, we have

$$f(\gamma z) = \lim_{n\to\infty} f_n(\gamma z) = \lim_{n\to\infty} \chi(\gamma)\varepsilon(\gamma,z)^k f_n(z) = \chi(\gamma)\varepsilon(\gamma,z)^k \lim_{n\to\infty} f_n(z) = \chi(\gamma)\varepsilon(\gamma,z)^k f(z),$$

for any $\gamma \in \Gamma$. Also, $||f|| < \infty$. To see this, since $(f_n)_{n\geq 1}$ is Cauchy we know $(||f_n||)_{n\geq 1}$ converges. In particular, $\lim_{n\to\infty} ||f_n|| < \infty$. But

$$\lim_{n\to\infty} ||f_n|| = \lim_{n\to\infty} \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |f_n(z)|^2\,d\mu\right)^{\frac{1}{2}} = \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \left|\lim_{n\to\infty} f_n(z)\right|^2\,d\mu\right)^{\frac{1}{2}} = \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |f(z)|^2\,d\mu\right)^{\frac{1}{2}} = ||f||,$$

where the second equality holds by the dominated convergence theorem. Hence $||f|| < \infty$ as desired and so $f \in \mathcal{L}_k(\Gamma,\chi)$. We now show that $f_n \to f$ in the L^2-norm. Indeed,

$$||f(z) - f_n(z)|| = \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |f(z) - f_n(z)|^2\,d\mu\right)^{\frac{1}{2}},$$

and it follows that $||f(z) - f_n(z)|| \to 0$ as $n \to \infty$ so that the Cauchy sequence $(f_n)_{n\geq 1}$ converges. □

We will need two more subspaces. Let $\mathcal{B}_k(\Gamma,\chi)$ be the subspace of $\mathcal{A}_k(\Gamma,\chi)$ such that f is smooth and bounded and let $\mathcal{D}_k(\Gamma,\chi)$ be the subspace of $\mathcal{A}_k(\Gamma,\chi)$ such that f and $\Delta_k f$ are smooth and bounded. If χ is the trivial character or if $k = 0$, we will suppress the dependencies accordingly. Since boundedness on $\mathbb{H}$ implies square-integrability over $\mathcal{F}_\Gamma$, we have the following chain of inclusions:

$$\mathcal{D}_k(\Gamma,\chi) \subseteq \mathcal{B}_k(\Gamma,\chi) \subseteq \mathcal{L}_k(\Gamma,\chi) \subseteq \mathcal{A}_k(\Gamma,\chi).$$

Moreover, $\mathcal{D}_k(\Gamma,\chi)$ is almost all of $\mathcal{L}_k(\Gamma,\chi)$ as the following proposition shows:

Proposition 11.3.3. *$\mathcal{D}_k(\Gamma,\chi)$ is dense in $\mathcal{L}_k(\Gamma,\chi)$.*

Proof. Note that $\mathcal{D}_k(\Gamma,\chi)$ is an algebra of functions that vanish at infinity. We will show that $\mathcal{D}_k(\Gamma,\chi)$ is nowhere vanishing, separates points, and self-adjoint. For nowhere vanishing fix a $z\in\mathbb{H}$. Let φ_z be a bump function defined on some sufficiently small neighborhood U_z of z. Then

$$\Phi(v)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\overline{\chi}(\gamma)\varepsilon(\gamma,v)^{-k}\varphi_z(\gamma v),$$

belongs to $\mathcal{D}_k(\Gamma,\chi)$ and is nonzero at z (the automorphy follows exactly as in the case of Eisenstein series). We now show $\mathcal{D}_k(\Gamma,\chi)$ separates points. To see this, consider two distinct points $z,w\in\mathbb{H}$. Let $U_{z,w}$ be a small neighborhood of z not containing w. Then $\Phi_z\mid_{U_{z,w}}$ belongs to $\mathcal{D}_k(\Gamma,\chi)$ with $\Phi_z\mid_{U_{z,w}}(z)\neq 0$ and $\varphi_z\mid_{U_{z,w}}(w)=0$. To see that $\mathcal{D}_k(\Gamma,\chi)$ is self-adjoint, recall that conjugation is smooth and commutes with partial derivatives so that if f belongs to $\mathcal{D}_k(\Gamma,\chi)$ then so does $\overline{f}$. Therefore the Stone–Weierstrass theorem for complex functions defined on locally compact Hausdorff spaces (as $\mathbb{H}$ is a locally compact Hausdorff space) implies that $\mathcal{D}_k(\Gamma,\chi)$ is dense in $C_0(\mathbb{H})$ with the supremum norm. Note that $\mathcal{L}_k(\Gamma,\chi)\subseteq C_0(\mathbb{H})$. Now we show $\mathcal{D}_k(\Gamma,\chi)$ is dense in $\mathcal{L}_k(\Gamma,\chi)$. Let $f\in\mathcal{L}_k(\Gamma,\chi)$. By what we have just shown, there exists a sequence $(f_n)_{n\geq 1}$ in $\mathcal{D}_k(\Gamma,\chi)$ converging to f in the supremum norm. But

$$||f-f_n||=\left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}|f(z)-f_n(z)|^2\,d\mu\right)^{\frac{1}{2}}\leq\left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sup_{z\in\mathcal{F}_\Gamma}|f(z)-f_n(z)|^2\,d\mu\right)^{\frac{1}{2}},$$

and the last expression tends to zero as $n\to\infty$ because $f_n\to f$ in the supremum norm. □

As $\mathcal{D}_k(\Gamma,\chi)\subseteq\mathcal{B}_k(\Gamma,\chi)$, Proposition 11.3.3 implies that $\mathcal{B}_k(\Gamma,\chi)$ is dense in $\mathcal{L}_k(\Gamma,\chi)$ too. It can be shown that the Laplace operator Δ_k is bounded from below and symmetric on $\mathcal{D}_k(\Gamma,\chi)$ and hence admits a self-adjoint extension to $\mathcal{L}_k(\Gamma,\chi)$ (see [Iwa02] for a proof in the weight zero case and [DFI02] for notes on the general case):

Proposition 11.3.4. *On $\mathcal{L}_k(\Gamma,\chi)$, the Laplace operator Δ_k is bounded from below by $\lambda\left(\frac{|k|}{2}\right)$ and self-adjoint.*

In particular, Δ is positive. If we suppose $f\in\mathcal{L}_k(\Gamma,\chi)$ is an eigenfunction for Δ_k with eigenvalue λ then Proposition 11.3.4 implies λ is real and $\lambda\geq\lambda\left(\frac{|k|}{2}\right)$. Since $\lambda=s(1-s)$ and λ is real, s and $1-s$ are either conjugates or real. In the former case, $s=1-\overline{s}$ and we find that

$$\sigma=1-\sigma\quad\text{and}\quad t=t.$$

Therefore $s=\frac{1}{2}+it$. In the later case, s is real. It follows that in either case, we may write $\lambda=\frac{1}{4}+r^2$ and $s=\frac{1}{2}+\nu$ for unique r and ν with r real or purely imaginary and ν purely imaginary or real corresponding to the two cases respectively. In particular, we also have $\lambda=\frac{1}{4}-\nu^2$ and $\nu=ir$. We refer to r and ν as the **spectral parameter** and **type** of f respectively. We collect the ways of expressing λ below:

$$\lambda=s(1-s)=\frac{1}{4}+r^2=\frac{1}{4}-\nu^2.$$

Therefore to specific λ it suffices to specify either s, the spectral parameter r, or the type ν. We will often replace λ with one of these parameters in $\mathcal{A}_{k,\lambda}(\Gamma,\chi)$, $\mathcal{M}_{k,\lambda}(\Gamma,\chi)$, $\mathcal{C}_{k,\lambda}(\Gamma,\chi)$, and $\mathcal{L}_{k,\lambda}(\Gamma,\chi)$.

Remark 11.3.1. *In the case of embedding weight k holomorphic forms into Maass forms, we have*

$$\lambda=\frac{k}{2}\left(1-\frac{k}{2}\right)=\frac{1}{4}+\left(i\frac{1-k}{2}\right)^2+\frac{1}{4}-\left(\frac{1-k}{2}\right)^2,$$

so that $r=i\frac{1-k}{2}$ and $\nu=\frac{k-1}{2}$.

We now introduce variations of the Poincaré and Eisenstein series. Let $m \geq 0$, $k \geq 0$, χ be a Dirichlet character with conductor $q \mid N$, $\sigma_{\mathfrak{a}}$ be a scaling matrix for the $\mathfrak{a}$ cusp, and $\psi(y)$ be a smooth function such that $\psi(y) \ll_\varepsilon y^{1+\varepsilon}$ as $y \to 0$ Then the m-th (automorphic) **Poincaré series** $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ of weight k and character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp and with respect to $\psi(y)$ is defined by

$$P_{m,k,\chi,\mathfrak{a}}(z,\psi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}.$$

If $k = 0$, χ is the trivial character, or $\mathfrak{a} = \infty$, we will drop these dependencies accordingly. Moreover, if $\psi(y)$ is a bump function, we say that $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is **incomplete**. We claim that $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is well-defined. This is easy to see as we have already showed $\overline{\chi}(\gamma)$, $\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}$, $\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)$, and $e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}$, are all independent of representatives for γ and $\sigma_{\mathfrak{a}}$ when discussing the automorphic Poincaré series. So $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is well-defined. We claim $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is also locally absolutely uniformly convergent for $z \in \mathbb{H}$. To see this, we require a technical lemma:

Lemma 11.3.1. *For any compact subset K of $\mathbb{H}$, there are finitely many pairs $(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}$, with $c \neq 0$, for which*

$$\frac{\mathrm{Im}(z)}{|cz+d|^2} > 1,$$

for all $z \in K$.

Proof. Let $\beta = \sup_{z\in K} |z|$. As $|cz+d| \geq |cz| > 0$ and $\mathrm{Im}(z) < |z|$, we have

$$\frac{\mathrm{Im}(z)}{|cz+d|^2} \leq \frac{1}{|c^2 z|} \leq \frac{1}{|c|^2\beta}.$$

So if $\frac{\mathrm{Im}(z)}{|cz+d|^2} > 1$ then $\frac{1}{|c|^2\beta} > 1$ which is to say $|c| < \frac{1}{\sqrt{\beta}}$ and therefore $|c|$ is bounded. On the other hand, $|cz+d| \geq |d| \geq 0$. Excluding the finitely many terms $(c,0)$, we may assume $|d| > 0$. In this case, similarly

$$\frac{\mathrm{Im}(z)}{|cz+d|^2} \leq \left|\frac{z}{d^2}\right| \leq \frac{\beta}{|d|^2}.$$

So if $\frac{\mathrm{Im}(z)}{|cz+d|^2} > 1$ then $\frac{\beta}{|d|^2} > 1$ which is to say $|d| < \sqrt{\beta}$. So $|d|$ is also bounded. Since both $|c|$ and $|d|$ are bounded, the claim follows. □

Now we are ready to show that $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is locally absolutely uniformly convergent for $z \in \mathbb{H}$. Let K be a compact subset of $\mathbb{H}$. Then it suffices to show $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is absolutely uniformly convergent on K. The bound $|e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}| = e^{-2\pi m\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)} < 1$ and the Bruhat decomposition applied to $\sigma_{\mathfrak{a}}^{-1}\Gamma$ together give

$$P_{m,k,\chi,\mathfrak{a}}(z,\psi) \ll \psi(\mathrm{Im}(z)) + \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right).$$

It now further suffices to show that the latter series above is absolutely uniformly convergent on K. By Lemma 11.3.1, there are all but finitely many terms in the sum with $\psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right) \ll_\varepsilon \left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)^{1+\varepsilon}$. But the finitely many other terms are all uniformly bounded on K because $\psi(y)$ is continuous (as it is smooth). Therefore

$$\sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right) \ll \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)^{1+\varepsilon} \ll \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)^{1+\varepsilon},$$

and this last series is locally absolutely uniformly convergent for $z \in \mathbb{H}$ by Proposition B.8.1. It follows that $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is too. Actually, we can do better if $\psi(y)$ is a bump function since finitely many terms will be nonzero. Indeed, $\sigma_{\mathfrak{a}}^{-1}\Gamma$ is a Fuchsian group because it is a subset of the modular group. So from $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\backslash\Gamma = \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma$ we see that $\{\sigma_{\mathfrak{a}}^{-1}\gamma z : \gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma\}$ is discrete. Since $\mathrm{Im}(z)$ is an open map it takes discrete sets to discrete sets so that $\{\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z) : \gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma\}$ is also discrete. Now $\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))$ is nonzero if and only if $\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z) \in \mathrm{Supp}(\psi)$ and $\{\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z) : \gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma\} \cap \mathrm{Supp}(\psi)$ is finite as it is a discrete subset of a compact set (since $\psi(y)$ has compact support). Hence finitely many of the terms are nonzero. Moreover, the compact support of $\psi(y)$ then implies that $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is also compactly supported (since the function $\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))$ is continuous and $\mathbb{C}$ is Hausdorff) and hence bounded on $\mathbb{H}$. As a consequence, $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is L^2-integrable. We collect this work as a theorem:

Theorem 11.3.1. *Let $m \geq 0$, $k \geq 0$, χ be a Dirichlet character with conductor dividing the level, $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$, and $\psi(y)$ be a smooth function such that $\psi(y) \ll_\varepsilon y^{1+\varepsilon}$ as $y \to 0$. The Poincaré series*

$$P_{m,k,\chi,\mathfrak{a}}(z,\psi) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z},$$

is a smooth automorphic function on $\Gamma\backslash\mathbb{H}$. If $\psi(y)$ is a bump function, $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is L^2-integrable.

For $m = 0$, we write $E_{k,\chi,\mathfrak{a}}(z,\psi) = P_{0,k,\chi,\mathfrak{a}}(z,\psi)$ and call $E_{k,\chi,\mathfrak{a}}(z,\psi)$ the (automorphic) **Eisenstein series** of weight k and character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp and with respect to $\psi(y)$. It is defined by

$$E_{k,\chi,\mathfrak{a}}(z,\psi) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)).$$

If $k = 0$, χ is the trivial character, or $\mathfrak{a} = \infty$, we will drop these dependencies accordingly. Moreover, if $\psi(y)$ is a bump function, we say that $E_{k,\chi,\mathfrak{a}}(z,\psi)$ is **incomplete**. We have already verified the following theorem:

Theorem 11.3.2. *Let $k \geq 0$, χ be a Dirichlet character with conductor dividing the level, $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$, and $\psi(y)$ be a smooth function such that $\psi(y) \ll_\varepsilon y^{1+\varepsilon}$ as $y \to 0$. The Eisenstein series*

$$E_{k,\chi,\mathfrak{a}}(z,\psi) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)),$$

is a smooth automorphic function on $\Gamma\backslash\mathbb{H}$. If $\psi(y)$ is a bump function, $E_{k,\chi,\mathfrak{a}}(z,\psi)$ is also L^2-integrable.

Unfortunately, the Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,\psi)$ fail to be Maass forms because they are not eigenfunctions for the Laplace operator. This is because compactly supported functions cannot be analytic (which as we have already mentioned is implied for any eigenfunction of the Laplace operator). However, the incomplete Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,\psi)$ are L^2-integrable where as the Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ are not. This is the advantage in working with incomplete Eisenstein series. We will compute their inner product against an arbitrary element of $\mathcal{B}_k(\Gamma,\chi)$. Let $f \in \mathcal{B}_k(\Gamma,\chi)$ and consider $E_{k,\chi,\mathfrak{a}}(\cdot,\psi)$. We compute their inner product

as follows:

$$\begin{aligned}
\langle f, E_{k,\chi,\mathfrak{a}}(\cdot,\psi)\rangle &= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{E_{k,\chi,\mathfrak{a}}(z,\psi)}\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\chi(\gamma)\overline{\varepsilon(\sigma_\mathfrak{a}^{-1}\gamma,z)^{-k}}f(z)\overline{\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))}\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\chi(\gamma)\varepsilon(\sigma_\mathfrak{a}^{-1}\gamma,z)^{k}f(z)\overline{\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))}\,d\mu && \frac{\overline{\varepsilon(\sigma_\mathfrak{a}^{-1}\gamma,z)}}{\varepsilon(\sigma_\mathfrak{a}^{-1}\gamma,z)}=1 \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\left(\frac{\varepsilon(\sigma_\mathfrak{a}^{-1}\gamma,z)}{\varepsilon(\gamma,z)}\right)^k f(\gamma z)\overline{\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))}\,d\mu && \text{automorphy} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\varepsilon(\sigma_\mathfrak{a},\sigma_\mathfrak{a}^{-1}\gamma z)^{-k}f(\gamma z)\overline{\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))}\,d\mu && \text{cocycle condition} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\varepsilon(\sigma_\mathfrak{a},\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a} z)^{-k}f(\gamma\sigma_\mathfrak{a} z)\overline{\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a} z))}\,d\mu && z\mapsto\sigma_\mathfrak{a} z \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\infty\backslash\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}\varepsilon(\sigma_\mathfrak{a},\gamma z)^{-k}f(\sigma_\mathfrak{a}\gamma z)\overline{\psi(\mathrm{Im}(\gamma z))}\,d\mu && \gamma\mapsto\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{a}^{-1} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\infty\backslash\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}(f|_k\sigma_\mathfrak{a})(\gamma z)\overline{\psi(\mathrm{Im}(\gamma z))}\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\Gamma_\infty\backslash\mathbb{H}}(f|_k\sigma_\mathfrak{a})(z)\overline{\psi(\mathrm{Im}(z))}\,d\mu && \text{unfolding.}
\end{aligned}$$

Substituting in the Fourier series of f at the $\mathfrak{a}$ cusp, we obtain

$$\frac{1}{V_\Gamma}\int_0^\infty\int_0^1\left(\sum_{n\in\mathbb{Z}}a_\mathfrak{a}(n,y)e^{2\pi inx}\right)\overline{\psi(y)}\,\frac{dx\,dy}{y^2}.$$

By the Fubini–Tonelli theorem, we can interchange the sum and the two integrals. Upon making this interchange, the identity Equation (10.1) implies that the inner integral cuts off all of the terms in the sum except the diagonal $n=0$, resulting in

$$\frac{1}{V_\Gamma}\int_0^\infty a_\mathfrak{a}(0,y)\overline{\psi(y)}\,\frac{dy}{y^2}.$$

This latter integral is precisely the constant term in the Fourier series of f at the $\mathfrak{a}$ cusp. It follows that f is orthogonal to $\mathcal{E}_k(\Gamma,\chi)$ if and only if $a_\mathfrak{a}(0,y)=0$ for all cusps $\mathfrak{a}$. To state this property in another way, let $\mathcal{E}_k(\Gamma,\chi)$ and $\mathcal{C}_k(\Gamma,\chi)$ denote the subspaces of $\mathcal{B}_k(\Gamma,\chi)$ generated by such forms respectively. Moreover, let $\mathcal{C}_{k,\nu}(\Gamma,\chi)$ and $\mathcal{A}_{k,\nu}(\Gamma,\chi)$ denote the corresponding subspaces of $\mathcal{C}_k(\Gamma,\chi)$ and $\mathcal{A}_k(\Gamma,\chi)$ whose type is ν. If $k=0$ or χ is the trivial character, we will suppress these dependencies. Then we have shown that

$$\mathcal{B}_k(\Gamma,\chi)=\mathcal{E}_k(\Gamma,\chi)\oplus\mathcal{C}_k(\Gamma,\chi).$$

Moreover, as $\mathcal{B}_k(\Gamma,\chi)$ is dense in $\mathcal{L}_k(\Gamma,\chi)$, we have

$$\mathcal{L}_k(\Gamma,\chi)=\overline{\mathcal{E}_k(\Gamma,\chi)}\oplus\overline{\mathcal{C}_k(\Gamma,\chi)},$$

where the closure is with respect to the topology induced by the L^2-norm. The essential fact is that $\mathcal{C}_k(\Gamma,\chi)$ will turn out to be the subspace of weight k cusp forms with character χ on $\Gamma\backslash\mathbb{H}$ and the corresponding subspaces $\mathcal{C}_{k,\nu}(\Gamma,\chi)$ are finite dimensional. Thus all cusp forms are L^2-integrable and we can apply a linear theory to $\mathcal{C}_{k,\nu}(\Gamma,\chi)$.

11.4 Spectral Theory of the Laplace Operator

We are now ready to discuss the spectral theory of the Laplace operator Δ_k. What we want to do is to decompose $\mathcal{L}_k(\Gamma,\chi)$ into subspaces invariant under Δ_k such that on each subspace Δ_k has either pure point spectrum, absolutely continuous spectrum, or residual spectrum. Although the proof is beyond the scope of this text, the spectral resolution of the Laplace operator on $\mathcal{C}_k(\Gamma,\chi)$ is as follows (see [Iwa02] for a proof in the weight zero case and [DFI02] for notes on the general case):

Theorem 11.4.1. *The Laplace operator Δ_k has pure point spectrum on $\mathcal{C}_k(\Gamma,\chi)$. The corresponding subspaces $\mathcal{C}_{k,\nu}(\Gamma,\chi)$ are finite dimensional and mutually orthogonal. Letting $u_1,u_2,\dots$ be an orthonormal basis of cusp forms for $\mathcal{C}_k(\Gamma,\chi)$, every $f\in\mathcal{C}_k(\Gamma,\chi)$ admits a series of the form*

$$f(z)=\sum_{j\geq 1}\langle f,u_j\rangle u_j(z),$$

which is locally absolutely uniformly convergent if $f\in\mathcal{D}_k(\Gamma,\chi)$ and convergent in the L^2-norm otherwise.

We will now discuss the spectrum of the Laplace operator on $\mathcal{E}_k(\Gamma,\chi)$. Essential is the meromorphic continuation of the Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ (see [Iwa02] for a proof in the weight zero case and [DFI02] for notes on the general case):

Theorem 11.4.2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$. The Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ admits meromorphic continuation to $\mathbb{C}$, via a Fourier-Whittaker series at the $\mathfrak{b}$ cusp given by*

$$E_{k,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s)=\delta_{\mathfrak{a},\mathfrak{b}}y^s+\tau_{\mathfrak{a},\mathfrak{b}}(s)y^{1-s}+\sum_{n\neq 0}\tau_{\mathfrak{a},\mathfrak{b}}(n,s)W_{\operatorname{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)e^{2\pi inx},$$

where $\tau_{\mathfrak{a},\mathfrak{b}}(s)$ and $\tau_{\mathfrak{a},\mathfrak{b}}(n,s)$ are meromorphic functions.

The Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ also satisfy a functional equation. To state it we need some notation. Fix an ordering of the cusps $\mathfrak{a}$ of $\Gamma\backslash\mathbb{H}$ and define

$$\mathcal{E}(z,s)=(E_{k,\chi,\mathfrak{a}}(z,s))_{\mathfrak{a}}^t\quad\text{and}\quad\Phi(s)=(\tau_{\mathfrak{a},\mathfrak{b}}(s))_{\mathfrak{a},\mathfrak{b}}.$$

In other words, $\mathcal{E}(z,s)$ is the column vector of the Eisenstein series and $\Phi(s)$ is the square matrix of meromorphic functions $\tau_{\mathfrak{a},\mathfrak{b}}(s)$ described in Theorem 11.4.2. Then we have the following (see [Iwa02] for a proof in the weight zero case and [DFI02] for notes on the general case):

Theorem 11.4.3. *The Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ of weight k and character χ on $\Gamma\backslash\mathbb{H}$ satisfy the functional equation*

$$\mathcal{E}(z,s)=\Phi(s)\mathcal{E}(z,1-s).$$

The matrix $\Phi(s)$ is symmetric and satisfies the functional equation

$$\Phi(s)\Phi(1-s)=I.$$

Moreover, it is unitary on the line $\sigma=\frac{1}{2}$ and Hermitian if s is real.

As $\Phi(s)$ is symmetric by Theorem 11.4.3, if $\mathfrak{a} = \infty$ or $\mathfrak{b} = \infty$, we will suppress these dependencies for $\tau_{\mathfrak{a},\mathfrak{b}}$. Understanding the poles of $\tau_{\mathfrak{a},\mathfrak{b}}$ are also important for understanding the poles of the Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ (see [Iwa02] for a proof in the weight zero case and [DFI02] for notes on the general case):

Theorem 11.4.4. *The functions $\tau_{\mathfrak{a},\mathfrak{b}}(s)$ are meromorphic for $\sigma \geq \frac{1}{2}$ with a finite number of simple poles in the interval $(\frac{1}{2}, 1]$. A pole of $\tau_{\mathfrak{a},\mathfrak{b}}(s)$ is also a pole of $\tau_{\mathfrak{a},\mathfrak{a}}(s)$. Moreover, the poles of $E_{k,\chi,\mathfrak{a}}(z,s)$ are among the poles of $\tau_{\mathfrak{a},\mathfrak{a}}(s)$, $E_{k,\chi,\mathfrak{a}}(z,s)$ has no poles on the line $\sigma = \frac{1}{2}$, and the residues of $E_{k,\chi,\mathfrak{a}}(z,s)$ are Maass forms in $\mathcal{E}_k(\Gamma,\chi)$.*

To begin decomposing $\mathcal{E}_k(\Gamma,\chi)$, consider the subspace $C_0^\infty(\mathbb{R}_{>0})$ of $\mathcal{L}^2(\mathbb{R}_{>0})$ with the normalized standard complex inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^\infty f(r)\overline{g(r)}\,dr,$$

for any $f, g \in C_0^\infty(\mathbb{R}_{>0})$. For each cusps $\mathfrak{a}$ of $\Gamma\backslash\mathbb{H}$ we associate the **Eisenstein transform** $E_{k,\chi,\mathfrak{a}} : C_0^\infty(\mathbb{R}_{>0}) \to \mathcal{A}_k(\Gamma,\chi)$ defined by

$$(E_{k,\chi,\mathfrak{a}}f)(z) = \frac{1}{4\pi}\int_0^\infty f(r)E_{k,\chi,\mathfrak{a}}\left(z, \frac{1}{2} + ir\right)\,dr.$$

Clearly $E_{k,\chi,\mathfrak{a}}f$ is automorphic because $E_{k,\chi,\mathfrak{a}}(z,s)$ is. It is not too hard to show the following (see [Iwa02] for a proof in the weight zero case and [DFI02] for notes on the general case):

Proposition 11.4.1. *If $f \in C_0^\infty(\mathbb{R}_{>0})$ then $E_{\mathfrak{a}}f$ is L^2-integrable over $\mathcal{F}_\Gamma$. That is, $E_{k,\chi,\mathfrak{a}}$ maps $C_0^\infty(\mathbb{R}_{>0})$ into $\mathcal{L}_k(\Gamma,\chi)$. Moreover,*

$$\langle E_{k,\chi,\mathfrak{a}}f, E_{k,\chi,\mathfrak{b}}g\rangle = \delta_{\mathfrak{a},\mathfrak{b}}\langle f, g\rangle,$$

for any $f, g \in C_0^\infty(\mathbb{R}_{>0})$ and any two cusps $\mathfrak{a}$ and $\mathfrak{b}$.

We let $\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ denote the image of the Eisenstein transform $E_{k,\chi,\mathfrak{a}}$. We call $\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ the **Eisenstein space** of $E_{k,\chi,\mathfrak{a}}(z,s)$. An immediate consequence of Proposition 11.4.1 is that the Eisenstein spaces for distinct cusps are orthogonal. Moreover, since $E_{k,\chi,\mathfrak{a}}\left(z, \frac{1}{2} + ir\right)$ is an eigenfunction for the Laplace operator with eigenvalue $\lambda = \frac{1}{4} + r^2$, and f and $E_{k,\chi,\mathfrak{a}}\left(z, \frac{1}{2} + ir\right)$ are smooth, the Leibniz integral rule implies

$$\Delta E_{k,\chi,\mathfrak{a}} = E_{k,\chi,\mathfrak{a}}M,$$

where $M : C_0^\infty(\mathbb{R}_{>0}) \to C_0^\infty(\mathbb{R}_{>0})$ is the multiplication operator given by

$$(Mf)(r) = \left(\frac{1}{4} + r^2\right)f(r),$$

for all $f \in C_0^\infty(\mathbb{R}_{>0})$. Therefore if $E_{k,\chi,\mathfrak{a}}f$ belongs to $\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ then so does $E_{k,\chi,\mathfrak{a}}(Mf)$. But as $f, Mf \in C_0^\infty(\mathbb{R}_{>0})$, this means $\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ is invariant under the Laplace operator. While the Eisenstein spaces are invariant, they do not make up all of $\mathcal{E}_k(\Gamma,\chi)$. By Theorem 11.4.4, the residues of the Eisenstein series belong to $\mathcal{E}_k(\Gamma,\chi)$. Let $\mathcal{R}_k(\Gamma,\chi)$ denote the subspace generated by the residues of these Eisenstein series. We call an element of $\mathcal{R}_k(\Gamma,\chi)$ a (residual) **Maass form** (by Theorem 11.4.4 they are Maass forms). Also let $\mathcal{R}_{k,s_j}(\Gamma,\chi)$ denote the subspace generated by those residues taken at $s = s_j$. For both of these subspaces, if χ is the trivial character of if $k = 0$, we will suppress the dependencies accordingly. Since there are finitely many cusps of $\Gamma\backslash\mathbb{H}$, each $\mathcal{R}_{k,s_j}(\Gamma,\chi)$ is finite dimensional. As the number of residues in $(\frac{1}{2}, 1]$ is finite by Theorem 11.4.4, it follows that $\mathcal{R}_k(\Gamma,\chi)$ is finite dimensional too. So $\mathcal{R}_k(\Gamma,\chi)$ decomposes as

$$\mathcal{R}_k(\Gamma,\chi) = \bigoplus_{\frac{1}{2} < s_j \leq 1} \mathcal{R}_{k,s_j}(\Gamma,\chi).$$

This decomposition is orthogonal because the Maass forms belonging to distinct subspaces $\mathcal{R}_{k,s_j}(\Gamma,\chi)$ have distinct eigenvalues and eigenfunctions of self-adjoint operators are orthogonal (recall that Δ_k is self-adjoint by Proposition 11.3.4). Also, each subspace $\mathcal{R}_{k,s_j}(\Gamma,\chi)$ is clearly invariant under the Laplace operator because its elements are Maass forms. The residual forms are particularly simple in the weight zero case (see [Iwa02] for a proof):

Proposition 11.4.2. *There is only one residual form in $\mathcal{R}(\Gamma,\chi)$. It is obtained from the residue at $s=1$ and it is a constant function.*

We are now ready for the spectral resolution. Although the proof is beyond the scope of this text, the spectral resolution of the Laplace operator on $\mathcal{E}_k(\Gamma,\chi)$ is as follows (see [Iwa02] for a proof in the weight zero case and [DFI02] for notes on the general case):

Theorem 11.4.5. *$\mathcal{E}_k(\Gamma,\chi)$ admits the orthogonal decomposition*

$$\mathcal{E}_k(\Gamma,\chi)=\mathcal{R}_k(\Gamma,\chi)\oplus\left(\bigoplus_{\mathfrak{a}}\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)\right),$$

where the direct sum is over the cusps of $\Gamma\backslash\mathbb{H}$. The Laplace operator Δ_k has discrete spectrum on $\mathcal{R}_k(\Gamma,\chi)$ in the interval $[0,\frac{1}{4})$ and has pure continuous spectrum on each Eisenstein space $\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ covering the interval $\left[\frac{1}{4},\infty\right)$ uniformly with multiplicity one. Letting $u_1,u_2,\dots$ be an orthonormal basis residual Maass forms for $\mathcal{R}_k(\Gamma,\chi)$, every $f\in\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ admits a decomposition of the form

$$f(z)=\sum_{j\geq1}\langle f,u_j\rangle u_j(z)+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\left\langle f,E_{k,\chi,\mathfrak{a}}\left(\cdot,\frac{1}{2}+\nu\right)\right\rangle E_{k,\chi,\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)\,dr.$$

The series and integrals are locally absolutely uniformly convergent if $f\in\mathcal{D}_k(\Gamma,\chi)$ and convergent in the L^2-norm otherwise.

Combining Theorems 11.4.1 and 11.4.5 gives the full spectral resolution of $\mathcal{L}_k(\Gamma,\chi)$:

Theorem 11.4.6. *$\mathcal{B}_k(\Gamma,\chi)$ admits the orthogonal decomposition*

$$\mathcal{B}_k(\Gamma,\chi)=\mathcal{C}_k(\Gamma,\chi)\oplus\mathcal{R}_k(\Gamma,\chi)\oplus\left(\bigoplus_{\mathfrak{a}}\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)\right),$$

where the sum is over all cusps of $\Gamma\backslash\mathbb{H}$. The Laplace operator has pure point spectrum on $\mathcal{C}_k(\Gamma,\chi)$, discrete spectrum on $\mathcal{R}_k(\Gamma,\chi)$, and absolutely continuous spectrum on $\mathcal{E}_k(\Gamma,\chi)$. Letting $u_1,u_2,\dots$ be an orthonormal basis of Maass forms for $\mathcal{C}_k(\Gamma,\chi)\oplus\mathcal{R}_k(\Gamma,\chi)$, any $f\in\mathcal{L}_k(\Gamma,\chi)$ has a series of the form

$$f(z)=\sum_{j\geq1}\langle f,u_j\rangle u_j(z)+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\left\langle f,E_{k,\chi,\mathfrak{a}}\left(\cdot,\frac{1}{2}+\nu\right)\right\rangle E_{k,\chi,\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)\,dr,$$

which is locally absolutely uniformly convergent if $f\in\mathcal{D}_k(\Gamma,\chi)$ and convergent in the L^2-norm otherwise. Moreover,

$$\mathcal{L}_k(\Gamma,\chi)=\overline{\mathcal{C}_k(\Gamma,\chi)}\oplus\overline{\mathcal{R}_k(\Gamma,\chi)}\oplus\left(\bigoplus_{\mathfrak{a}}\overline{\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)}\right),$$

where the closure is with respect to the topology induced by the L^2-norm.

Proof. Combine Theorems 11.4.1 and 11.4.5 and use the fact that $\mathcal{B}_k(\Gamma,\chi)=\mathcal{E}_k(\Gamma,\chi)\oplus\mathcal{C}_k(\Gamma,\chi)$ for the first statement. The last statement holds because $\mathcal{B}_k(\Gamma,\chi)$ is dense in $\mathcal{L}_k(\Gamma,\chi)$. □

11.5 Double Coset Operators

We can extend the theory of double coset operators to Maass form just as we did for holomorphic forms. For any two congruence subgroups Γ_1 and Γ_2 (not necessarily of the same level) and any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we define the **double coset operator** $[\Gamma_1 \alpha \Gamma_2]_k$ to be the linear operator on $\mathcal{C}_{k,\nu}(\Gamma_1)$ given by

$$(f[\Gamma_1 \alpha \Gamma_2]_k)(z) = \sum_j (f|_k \beta_j)(z) = \sum_j \det(\beta_j)^{-1} \varepsilon(\beta_j, z)^{-k} f(\beta_j z).$$

As was the case for holomorphic forms, Proposition 10.4.1 implies that this sum is finite. It remains to check that $f[\Gamma_1 \alpha \Gamma_2]_k$ is well-defined. Indeed, if β_j and β_j' belong to the same orbit then $\beta_j' \beta_j^{-1} \in \Gamma_1$. But then as $f \in \mathcal{C}_{k,\nu}(\Gamma_1)$, is it invariant under the $|_k \beta_j' \beta_j^{-1}$ operator so that

$$(f|_k \beta_j)(z) = ((f|_k \beta_j' \beta_j^{-1})|_k \beta_j)(z) = (f|_k \beta_j')(z),$$

and therefore the $[\Gamma_1 \alpha \Gamma_2]_k$ operator is well-defined. There is also an analogous statement about the double coset operators for Maass forms:

Proposition 11.5.1. *For any two congruence subgroups Γ_1 and Γ_2, $[\Gamma_1 \alpha \Gamma_2]_k$ maps $\mathcal{C}_{k,\nu}(\Gamma_1)$ into $\mathcal{C}_{k,\nu}(\Gamma_2)$.*

Proof. Arguing as in the proof of Proposition 10.4.2 with smoothness replacing holomorphy, automorphy replacing modularity, and the analogous growth condition for Maass forms, the only piece left to verify is that $f[\Gamma_1 \alpha \Gamma_2]_k$ is an eigenfunction for Δ with eigenvalue λ if f is. This is easy since the invariance of Δ implies

$$\Delta(f[\Gamma_1 \alpha \Gamma_2]_k)(z) = \sum_j \Delta(f|_k \beta_j)(z) = \lambda \sum_j \det(\beta_j)^{-1} \varepsilon(\beta_j, z)^{-k} f(\beta_j z) = \lambda (f[\Gamma_1 \alpha \Gamma_2]_k)(z).$$

Thus $f[\Gamma_1 \alpha \Gamma_2]_k$ is an eigenfunction for Δ with eigenvalue λ. This completes the proof. □

11.6 Diamond and Hecke Operators

Extending the theory of diamond operators and Hecke operators is also fairly straightforward. To see this, we have already shown that $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ so that

$$\left(f\left[\Gamma_1(N) \alpha \Gamma_1(N)\right]_k\right)(z) = (f|_k \alpha)(z),$$

for any $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Therefore, for any d taken modulo N, we define the **diamond operator** $\langle d \rangle : \mathcal{C}_{k,\nu}(\Gamma_1(N)) \to \mathcal{C}_{k,\nu}(\Gamma_1(N))$ to be the linear operative given by

$$(\langle d \rangle f)(z) = (f|_k \alpha)(z),$$

for any $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$. As for holomorphic forms, the diamond operators are multiplicative and invertible. They also decompose $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ into eigenspaces. For any Dirichlet character modulo N, let

$$\mathcal{C}_{k,\nu}(N, \chi) = \{f \in \mathcal{C}_{k,\nu}(\Gamma_1(N)) : \langle d \rangle f = \chi(d) f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^*\},$$

be the χ-eigenspace. Also let $\mathcal{C}_{k,\nu}(N, \chi)$ be the corresponding subspace of cusp forms. Then $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ admits a decomposition into these eigenspaces:

Proposition 11.6.1. *We have a direct sum decomposition*

$$\mathcal{C}_{k,\nu}(\Gamma_1(N)) = \bigoplus_{\chi \pmod N} \mathcal{C}_{k,\nu}(N,\chi).$$

Proof. The argument used in the proof of Proposition 10.5.1 holds verbatim. □

Just as for holomorphic forms, Proposition 11.6.1 shows that the diamond operators sieve Maass forms on $\Gamma_1(N)\backslash\mathbb{H}$ with trivial character in terms of Maass forms on $\Gamma_0(N)\backslash\mathbb{H}$ with nontrivial characters. Precisely, $\mathcal{C}_{k,\nu}(N,\chi) = \mathcal{C}_{k,\nu}(\Gamma_0(N),\chi)$ and $\mathcal{C}_{k,\nu}(N,\chi) = \mathcal{C}_{k,\nu}(\Gamma_0(N),\chi)$. So by Proposition 11.6.1, we have

$$\mathcal{C}_{k,\nu}(\Gamma_1(N)) = \bigoplus_{\chi \pmod N} \mathcal{C}_{k,\nu}(\Gamma_0(N),\chi).$$

As for holomorphic forms, this decomposition helps clarify why we consider Maass forms with nontrivial characters. We define the Hecke operators in the same way as for holomorphic forms. For a prime p, we define the p-th **Hecke operator** $T_p : \mathcal{C}_{k,\nu}(\Gamma_1(N)) \to \mathcal{C}_{k,\nu}(\Gamma_1(N))$ to be the linear operator given by

$$(T_p f)(z) = \left(f \left[\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) \right]_k \right)(z).$$

The diamond and Hecke operators commute:

Proposition 11.6.2. *For every $d \in (\mathbb{Z}/N\mathbb{Z})^*$ and prime p, the diamond operators $\langle d\rangle$ and Hecke operators T_p on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ commute:*

$$\langle d\rangle T_p = T_p \langle d\rangle$$

Proof. The argument used in the proof of Proposition 10.5.2 holds verbatim. □

Exactly as for holomorphic forms, Lemma 10.5.1 will give an explicit description of the Hecke operator T_p:

Proposition 11.6.3. *Let $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$. Then the Hecke operator T_p acts on f as follows:*

$$(T_p f)(z) = \begin{cases} \displaystyle\sum_{j \pmod p} \left(f\Big|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right)(z) + \left(f\Big|_k \begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right)(z) & \text{if } p \nmid N, \\ \displaystyle\sum_{j \pmod p} \left(f\Big|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right)(z) & \text{if } p \mid N, \end{cases}$$

where m and n are chosen such that $\det \left(\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix} \right) \right) = 1$.

Proof. The argument used in the proof of Proposition 10.5.3 holds verbatim. □

We use Proposition 11.6.3 to understand how the Hecke operators act on the Fourier-Whittaker coefficients of Maass forms:

Proposition 11.6.4. *Let $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$ have Fourier-Whittaker coefficients $a_n(f)$. Then for all primes p,*

$$(T_p f)(z) = \sum_{n \neq 0} \left(a_{np}(f) + \chi_{N,0}(p) p^{-1} a_{\frac{n}{p}}(\langle p\rangle f) \right) W_{\operatorname{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y) e^{2\pi i n x},$$

is the Fourier-Whittaker series of $T_p f$ where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid |n|$. Moreover, if $f \in \mathcal{C}_{k,\nu}(N,\chi)$ then $T_p f \in \mathcal{C}_{k,\nu}(N,\chi)$ and

$$(T_p f)(z) = \sum_{n\neq 0} \left(a_{np}(f) + \chi(p) p^{-1} a_{\frac{n}{p}}(f) \right) W_{\operatorname{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y) e^{2\pi i n x},$$

where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid |n|$.

Proof. Argue as in the proof of Proposition 10.5.4. □

As for holomorphic forms, the Hecke operators form a simultaneously commuting family with the diamond operators:

Proposition 11.6.5. *Let p and q be primes and $d, e \in (\mathbb{Z}/N\mathbb{Z})^*$. Then the Hecke operators T_p and T_q and diamond operators $\langle d\rangle$ and $\langle e\rangle$ on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ form a simultaneously commuting family:*

$$T_p T_q = T_q T_p, \quad \langle d\rangle T_p = T_p \langle d\rangle, \quad \text{and} \quad \langle d\rangle\langle e\rangle = \langle e\rangle\langle d\rangle.$$

Proof. Argue as in the proof of Proposition 10.5.5. □

We use Proposition 11.6.5 to construct diamond operators $\langle m\rangle$ and Hecke operators T_m for all $m \geq 1$ exactly as for holomorphic forms. Explicitly, the **diamond operator** $\langle m\rangle : \mathcal{C}_{k,\nu}(\Gamma_1(N)) \to \mathcal{C}_{k,\nu}(\Gamma_1(N))$ is defined to be the linear operator given by

$$\langle m\rangle = \begin{cases} \langle m\rangle \text{ with } m \text{ taken modulo } N & \text{if } (m,N) = 1, \\ 0 & \text{if } (m,N) > 1. \end{cases}$$

For the Hecke operators, if $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the prime decomposition of m then the m-th **Hecke operator** $T_m : \mathcal{C}_{k,\nu}(\Gamma,\chi) \to \mathcal{C}_{k,\nu}(\Gamma,\chi)$ is the linear operator given by

$$T_m = \prod_{1\leq i\leq k} T_{p_i^{r_i}},$$

where T_{p^r} is defined inductively by

$$T_{p^r} = \begin{cases} T_p T_{p^{r-1}} - p^{-1}\langle p\rangle T_{p^{r-2}} & \text{if } p \nmid N, \\ T_p^r & \text{if } p \mid N, \end{cases}$$

for all $r \geq 2$. Note that when $m = 1$, the product is empty and so T_1 is the identity operator. By Proposition 11.6.5, the Hecke operators T_m are multiplicative but not completely multiplicative in m and they commute with the diamond operators $\langle m\rangle$. Moreover, a more general formula for how the Hecke operators T_m act on the Fourier-Whittaker coefficients can be derived:

Proposition 11.6.6. *Let $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$ have Fourier-Whittaker coefficients $a_n(f)$. Then for $m \geq 1$ with $(m,N) = 1$,*

$$(T_m f)(z) = \sum_{n\neq 0} \left(\sum_{d|(n,m)} d^{-1} a_{\frac{nm}{d^2}}(\langle d\rangle f) \right) W_{\operatorname{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y) e^{2\pi i n x},$$

is the Fourier-Whittaker series of $T_m f$. Moreover, if $f \in \mathcal{C}_{k,\nu}(N,\chi)$ then

$$(T_m f)(z) = \sum_{n\neq 0} \left(\sum_{d|(n,m)} \chi(d) d^{-1} a_{\frac{nm}{d^2}}(f) \right) W_{\operatorname{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y) e^{2\pi i n x}.$$

Proof. Argue as in the proof of Proposition 10.5.6. □

The diamond and Hecke operators turn out to be normal on the subspace of cusp forms. Just as with holomorphic forms, we can use Lemma 10.5.2 to compute adjoints:

Proposition 11.6.7. *Let Γ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Set $\alpha' = \det(\alpha)\alpha^{-1}$. Then the following are true:*

(i) If $\alpha^{-1}\Gamma\alpha \subseteq \mathrm{PSL}_2(\mathbb{Z})$ then for all $f \in \mathcal{C}_{k,\nu}(\Gamma, \chi)$ and $g \in \mathcal{C}_{k,\nu}(\alpha^{-1}\Gamma\alpha)$, we have

$$\langle f|_k\alpha, g\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g|_k\alpha'\rangle_\Gamma.$$

(ii) For all $f, g \in \mathcal{C}_{k,\nu}(\Gamma, \chi)$, we have

$$\langle f[\Gamma\alpha\Gamma]_k, g\rangle = \langle f, g[\Gamma\alpha'\Gamma]_k\rangle.$$

In particular, if $\alpha^{-1}\Gamma\alpha = \Gamma$ then $|_k\alpha^* = |_k\alpha'$ and $[\Gamma\alpha\Gamma]_k^* = [\Gamma\alpha'\Gamma]_k$ as operators.

Proof. Argue as in the proof of Proposition 10.5.7. □

We can now prove that the diamond and Hecke operators are normal:

Proposition 11.6.8. *On $\mathcal{C}_{k,\nu}(\Gamma_1(N))$, the diamond operators $\langle m\rangle$ and Hecke operators T_m are normal for all $m \geq 1$ with $(m, N) = 1$. Moreover, their adjoints are given by*

$$\langle m\rangle^* = \langle \overline{m}\rangle \quad \text{and} \quad T_p^* = \langle \overline{p}\rangle T_p.$$

Proof. The argument used in the proof of Proposition 10.5.8 holds verbatim. □

Just as for holomorphic forms, all of the diamond operators on $\mathcal{C}_{k,\nu}(\Gamma_1(1))$ are the identity and therefore $T_p^* = T_p$ for all primes p. So the Hecke operators are self-adjoint (as are the diamond operators since they are the identity). We need one last operator since cusp forms have Fourier-Whittaker coefficients for all $n \neq 0$. Let $X : \mathcal{C}_{k,\nu}(\Gamma_1(N)) \to \mathcal{C}_{k,\nu}(\Gamma_1(N))$ be the linear operator defined by

$$(Xf)(z) = f(-\overline{z}).$$

As $-\overline{z} = -x + iy$, X acts as reflection with respect to x. Then define the parity **Hecke operator** T_{-1} to be the linear operator on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ given by

$$T_{-1} = X \prod_{\substack{-k<\ell<k \\ \ell\equiv k \pmod 2}} L_k.$$

We will also set

$$\delta(\nu, k) = \frac{\Gamma\left(\nu + \frac{1-k}{2}\right)}{\Gamma\left(\nu + \frac{1+k}{2}\right)}.$$

Notice that $\delta(\nu, 0) = 1$. The parity Hecke operator acts as an involution on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ and more as the following proposition shows (see [DFI02] for a proof):

Proposition 11.6.9. *T_{-1} is an involution on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$. In particular, T_{-1} is an involution on $\mathcal{C}_{k,\nu}(N,\chi)$ as well. If $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$ has Fourier-Whittaker coefficients $a_n(f)$ then*

$$(T_{-1}f)(z) = \sum_{n\geq 1} a_n(f)\delta(\nu,k)^{-1}W_{-\frac{k}{2},\nu}(4\pi|n|y)e^{-2\pi inx} + a_{-n}(f)\delta(\nu,k)W_{\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx},$$

is the Fourier-Whittaker series of $T_{-1}f$. Moreover, T_{-1} commutes with the diamond operators $\langle m\rangle$ and Hecke operators T_m for all $m \geq 1$, is normal, and its adjoint is given by

$$T_{-1}^* = -T_{-1}.$$

Let $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$. As T_{-1} is an involution, the only possible eigenvalues are ± 1. Accordingly, we say that $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$ is **even** if $T_{-1}f = f$ and is **odd** if $T_{-1}f = -f$. Then by Proposition 11.6.9, we have

$$a_{-n}(f) = \pm a_n(f)\delta(\nu,k)^{-1} = \pm a_n(f)\frac{\Gamma\left(\nu+\frac{1+k}{2}\right)}{\Gamma\left(\nu+\frac{1-k}{2}\right)},$$

for all $n \geq 1$ and with $\pm$ according to if f is even or odd. Thus the Fourier-Whittaker series of f takes the form

$$f(z) = \sum_{n\geq 1} a_n(f)\left(\delta(\nu,k)^{-1}W_{-\frac{k}{2},\nu}(4\pi|n|y)e^{-2\pi inx} \pm W_{\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx}\right),$$

with $\pm$ according to if f is even or odd. If the weight is zero, the Fourier-Whittaker series drastically simplifies via, Theorem B.7.1, the identity $\delta(\nu,0) = 1$, and the exponential identities for sine and cosine, so that we obtain

$$f(z) = a^+y^{\frac{1}{2}+\nu} + a^-y^{\frac{1}{2}-\nu} + \sum_{n\geq 1} a_n(f)\sqrt{4|n|y}K_\nu(2\pi|n|y)\,\mathrm{SC}(2\pi nx),$$

where $\mathrm{SC}(x) = \cos(x)$ if f is even and $\mathrm{SC}(x) = i\sin(x)$ if f is odd. The benefit of working with even and odd forms is that it suffices to determine non-constant Fourier-Whittaker coefficients for $n \geq 1$ instead of $n \neq 0$. Now suppose f is a non-constant cusp form. Let the eigenvalue of T_m for f be $\lambda_f(m)$. We say that the $\lambda_f(m)$ are the **Hecke eigenvalues** of f. Just as for holomorphic forms, if f is a Maass form on $\Gamma_1(N)\backslash\mathbb{H}$ that is a simultaneous eigenfunction for all diamond operators $\langle m\rangle$ and Hecke operators T_m with $(m,N) = 1$, we call f an **eigenform**. If the condition $(m,N) = 1$ can be dropped, so that f is a simultaneous eigenfunction for all diamond and Hecke operators, we say f is a **Hecke-Maass eigenform**. In particular, on $\Gamma_1(1)\backslash\mathbb{H}$ all eigenforms are Hecke-Maass eigenforms. Now let f have Fourier-Whittaker coefficients $a_n(f)$. As for holomorphic forms, if f a Hecke-Maass eigenform Proposition 11.6.6 immediately implies that the first Fourier-Whittaker coefficient of T_mf is $a_m(f)$ and so

$$a_m(f) = \lambda_f(m)a_1(f),$$

for all $m \geq 1$. Therefore we cannot have $a_1(f) = 0$ for this would mean f is constant. So we can normalize f by dividing by $a_1(f)$ which guarantees that this Fourier-Whittaker coefficient is 1. It follows that

$$a_m(f) = \lambda_f(m),$$

for all $m \geq 1$. This normalization is called the **Hecke normalization** of f. The **Petersson normalization** of f is where we normalize so that $\langle f,f\rangle = 1$. In particular, any orthonormal basis of $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ is Petersson normalized. From the spectral theorem we have an analogous corollary as for holomorphic forms:

Theorem 11.6.1. *$\mathcal{C}_{k,\nu}(\Gamma_1(N))$ admits an orthonormal basis of eigenforms.*

Proof. By Theorem 11.4.1, $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ is finite dimensional. The claim then follows from the spectral theorem along with Propositions 11.6.5 and 11.6.8. □

Also, just as in the holomorphic setting, we have **Hecke relations** for Maass forms:

Proposition (Hecke relations, Maass). *Let $f \in \mathcal{C}_{k,\nu}(N,\chi)$ be a Hecke-Maass eigenform with Hecke eigenvalues $\lambda_f(m)$. Then the Hecke eigenvalues are multiplicative and satisfy*

$$\lambda_f(n)\lambda_f(m) = \sum_{d|(n,m)} \chi(d)d^{-1}\lambda_f\left(\frac{nm}{d^2}\right) \quad \text{and} \quad \lambda_f(nm) = \sum_{d|(n,m)} \mu(d)\chi(d)d^{-1}\lambda_f\left(\frac{n}{d}\right)\lambda_f\left(\frac{m}{d}\right),$$

for all $n, m \geq 1$ with $(nm, N) = 1$. Moreover,

$$\lambda_f(p^r) = \lambda_f(p)^r,$$

for all $p \mid N$ and $r \geq 2$.

Proof. The argument used in the proof of the Hecke relations for holomorphic forms holds verbatim. □

As an immediate consequence of the Hecke relations, the Hecke operators satisfy analogous relations:

Corollary 11.6.1. *The Hecke operators are multiplicative and satisfy*

$$T_nT_m = \sum_{d|(n,m)} \chi(d)d^{-1}T_{\frac{nm}{d^2}} \quad \text{and} \quad T_{nm} = \sum_{d|(n,m)} \mu(d)\chi(d)d^{-1}T_{\frac{n}{d}}T_{\frac{m}{d}},$$

for all $n, m \geq 1$ with $(nm, N) = 1$.

Proof. The argument used in the proof of Corollary 10.5.1 holds verbatim. □

Just as for holomorphic forms, the identities in Corollary 11.6.1 can also be established directly and the first identity can be used to show that the Hecke operators commute.

11.7 Atkin-Lehner Theory

There is also an Atkin-Lehner theory for Maass form. As with holomorphic forms, we will only deal with congruence subgroups of the form $\Gamma_1(N)$ or $\Gamma_0(N)$ and cusp forms on the corresponding modular curves. The trivial way to lift Maass forms from a smaller level to a larger level is via the natural inclusion $\mathcal{C}_{k,\nu}(\Gamma_1(M)) \subseteq \mathcal{C}_{k,\nu}(\Gamma_1(N))$ provided $M \mid N$ which follows from $\Gamma_1(N) \leq \Gamma_1(M)$. Alternatively, for any $d \mid \frac{N}{M}$, let $\alpha_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $f \in \mathcal{C}_{k,\nu}(\Gamma_1(M))$ then

$$(f|_k\alpha_d)(z) = d^{-1}\varepsilon(\alpha_d, z)^{-k}f(\alpha_d z) = d^{-1}f(dz).$$

Similar to holomorphic forms, $|_k\alpha_d$ maps $\mathcal{C}_{k,\nu}(\Gamma_1(M))$ into $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ and more:

Proposition 11.7.1. *Let M and N be positive integers such that $M \mid N$. For any $d \mid \frac{N}{M}$, $|_k\alpha_d$ maps $\mathcal{C}_{k,\nu}(\Gamma_1(M))$ into $\mathcal{C}_{k,\nu}(\Gamma_1(N))$. In particular, $|_k\alpha_d$ takes $\mathcal{C}_{k,\nu}(M,\chi)$ into $\mathcal{C}_{k,\nu}(N,\chi)$.*

Proof. Arguing as in the proof of Proposition 10.6.1 with smoothness replacing holomorphy, automorphy replacing modularity, and the analogous growth condition for Maass forms, the only piece left to verify is that $f[\alpha_d]$ is an eigenfunction for Δ with eigenvalue λ if f is. This is easy since the invariance of Δ implies

$$\Delta(f|_k\alpha_d)(z) = \lambda d^{-1} f(dz) = \lambda(f|_k\alpha_d)(z).$$

Therefore $f|_k\alpha_d$ is an eigenfunction for Δ with eigenvalue λ. This completes the proof. □

We can now define oldforms and newforms. For each divisor d of N, set

$$i_d : \mathcal{C}_{k,\nu}\left(\Gamma_1\left(\frac{N}{d}\right)\right) \oplus \mathcal{C}_{k,\nu}\left(\Gamma_1\left(\frac{N}{d}\right)\right) \to \mathcal{C}_{k,\nu}(\Gamma_1(N)) \qquad (f,g) \mapsto f + g|_k\alpha_d.$$

This map is well-defined by Proposition 11.7.1. The subspace of **oldforms** of level N is

$$\mathcal{C}^{\text{old}}_{k,\nu}(\Gamma_1(N)) = \bigoplus_{p|N} \text{Im}(i_p),$$

and the subspace of **newforms** of level N is

$$\mathcal{C}^{\text{new}}_{k,\nu}(\Gamma_1(N)) = \mathcal{C}^{\text{old}}_{k,\nu}(\Gamma_1(N))^{\perp}.$$

An element of these subspaces is called an **oldform** or **newform** respectively. Note that there are no oldforms of level 1. Just as with holomorphic forms, we need a useful operator. Recall the matrix

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

with $\det(W_N) = N$. We define the **Atkin-Lehner operator** ω_N to be the linear operator on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ given by

$$(\omega_N f)(z) = N(f|_k W_N)(z) = \varepsilon(W_N, z)^{-k} f(W_N z) = \left(\frac{z}{|z|}\right)^{-k} f\left(-\frac{1}{Nz}\right).$$

It is not too difficult to see how ω_N acts on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$:

Proposition 11.7.2. *ω_N maps $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ into itself. In particular, ω_N takes $\mathcal{C}_{k,\nu}(N,\chi)$ into $\mathcal{C}_{k,\nu}(N,\overline{\chi})$. Moreover, ω_N is self-adjoint and*

$$\omega_N^2 f = (-1)^k f.$$

Proof. Arguing as in the proof of Proposition 10.6.2 with smoothness replacing holomorphy, automorphy replacing modularity, and the analogous growth condition for Maass forms, the only piece left to verify is that $\omega_N f$ is an eigenfunction for Δ with eigenvalue λ if f is. This is easy since the invariance of Δ implies

$$\Delta(\omega_N f)(z) = \lambda \left(\frac{z}{|z|}\right)^{-k} f(W_N z) = \lambda(\omega_N f)(z).$$

Thus $\omega_N f$ is an eigenfunction for Δ with eigenvalue λ. This completes the proof. □

Proposition 11.7.2 shows that ω_N is an involution if k is even and is at most of order 4. As with holomorphic forms, we need to understand how the Atkin-Lehner operator interacts with the diamond and Hecke operators:

Proposition 11.7.3. *On $\mathcal{C}_{k,\nu}(\Gamma_1(N))$, the diamond operators $\langle m\rangle$ and Hecke operators T_m satisfy the following adjoint formulas for all $m \geq 1$:*

$$\langle m\rangle^* = \omega_N \langle m\rangle \omega_N^{-1} \quad \text{and} \quad T_m^* = \omega_N T_m \omega_N^{-1}.$$

Proof. The argument used in the proof of Proposition 10.6.3 holds verbatim. □

It turns out that the spaces of oldforms and newforms are invariant under the diamond and Hecke operators (argue as in the proof of Proposition 10.6.4):

Proposition 11.7.4. *On $\mathcal{C}_{k,\nu}(\Gamma_1(N))$, the diamond operators $\langle m\rangle$ and Hecke operators T_m preserve the subspaces of oldforms and newforms for all $m \geq 1$.*

Proof. The argument used in the proof of Proposition 10.6.4 holds verbatim. □

As a corollary, these subspaces admit orthogonal bases of eigenforms:

Corollary 11.7.1. *$\mathcal{C}_{k,\nu}^{\text{old}}(\Gamma_1(N))$ and $\mathcal{C}_{k,\nu}^{\text{new}}(\Gamma_1(N))$ admit orthonormal bases of eigenforms.*

Proof. This follows immediately from Theorem 11.6.1 and Proposition 11.7.4 □

We can remove the condition $(m, N) = 1$ for eigenforms in a basis of $\mathcal{C}_{k,\nu}^{\text{new}}(\Gamma_1(N))$ so that the eigenforms are eigenfunctions for all of the diamond and Hecke operators. As for holomorphic forms, we need a preliminary result (argue as in the proof of Lemma 10.6.1 as given in [DS05]):

Lemma 11.7.1. *If $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$ has Fourier-Whittaker coefficients $a_n(f)$ and is such that $a_n(f) = 0$ for all $n \geq 1$ whenever $(n, N) = 1$ then*

$$f = \sum_{p|N} p^{-1} f_p|_k \alpha_p,$$

for some $f_p \in \mathcal{C}_{k,\nu}\left(\Gamma_1\left(\frac{N}{p}\right)\right)$.

As was the case for holomorphic forms, we observe from Lemma 11.7.1 that if $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$ is such that its positive n-th Fourier-Whittaker coefficients vanish when n is relatively prime to the level then f must be an oldform. The main theorem about $\mathcal{C}_{k,\nu}^{\text{new}}(\Gamma_1(N))$ can now be proved. We say that f is a **primitive Hecke-Maass eigenform** if it is a nonzero Hecke normalized Hecke-Maass eigenform in $\mathcal{C}_{k,\nu}^{\text{new}}(\Gamma_1(N))$. We can now prove the main result about newforms which is that Hecke-Maass eigenforms exist:

Theorem 11.7.1. *Let $f \in \mathcal{C}_{k,\nu}^{\text{new}}(\Gamma_1(N))$ be an eigenform. Then the following hold:*

(i) f is a Hecke-Maass eigenform.

(ii) If g is any cusp form with the same Hecke eigenvalues at all primes then $g = cf$ for some nonzero $c \in \mathbb{C}$.

Moreover, the primitive Hecke-Maass eigenforms in $\mathcal{C}_{k,\nu}^{\text{new}}(\Gamma_1(N))$ form an orthogonal basis and each such eigenform lies in an eigenspace $\mathcal{C}_{k,\nu}(N, \chi)$.

Proof. The argument used in the proof of Theorem 10.6.1 holds verbatim. □

Statement (i) in Theorem 11.7.1 implies that primitive Hecke-Maass eigenforms satisfy the Hecke relations for all $n, m \geq 1$. Statement (ii) is referred to as **multiplicity one** for Maass forms. As is the case for holomorphic forms, $\mathcal{C}_{k,\nu}^{\text{new}}(\Gamma_1(N))$ contains one element per set of eigenvalues for the Hecke operators. As a consequence of multiplicity one, all primitive Hecke-Maass eigenforms are either even or odd:

Proposition 11.7.5. *If $f \in \mathcal{C}_{k,\nu}(N, \chi)$ is a primitive Hecke-Maass eigenform then f is either even or odd.*

Proof. By Proposition 11.6.9, the parity Hecke operator T_{-1} commutes with all the Hecke operators. Therefore, if $f \in \mathcal{C}_{k,\nu}(N, \chi)$ is a primitive Hecke-Maass eigenform with Hecke eigenvalues $\lambda_f(m)$ then $T_{-1}f$ is a Hecke eigenform with the same Hecke eigenvalues. Then multiplicity one gives

$$T_{-1}f = cf,$$

for some nonzero $c \in \mathbb{C}$. But as T_{-1} is an involution by Proposition 11.6.9, we must have $c = \pm 1$. □

We now discuss conjugate cusp forms. For any cusp form $f \in \mathcal{C}_{k,\nu}(N, \chi)$, we define the **conjugate** $\overline{f}$ of f by

$$\overline{f}(z) = \overline{f(-\overline{z})}.$$

Note that if f has Fourier-Whittaker coefficients $a_n(f)$ then $\overline{f}$ has Fourier-Whittaker coefficients $\overline{a_n(f)}$ by the conjugate symmetry of the Whittaker function (see Appendix B.7). It turns out that $\overline{f}$ is indeed a cusp form:

Proposition 11.7.6. *If $f \in \mathcal{C}_{k,\nu}(N, \chi)$ then $\overline{f} \in \mathcal{C}_{k,\nu}(N, \overline{\chi})$. Moreover,*

$$T_m\overline{f} = \overline{T_m f},$$

for all $m \geq 1$ with $(m, N) = 1$. In particular, if f is an eigenform with Hecke eigenvalues $\lambda_f(m)$ then $\overline{f}$ is too but with Hecke eigenvalues $\overline{\lambda_f(m)}$.

Proof. Argue as in the proof of Proposition 10.6.5. □

Just as with holomorphic forms, Theorem 11.7.1 and Proposition 11.7.6 together imply that the primitive Hecke-Maass eigenforms in $\mathcal{C}_{k,\nu}^{\text{new}}(\Gamma_1(N))$ are conjugate invariant and if $f \in \mathcal{C}_{k,\nu}(N, \chi)$ is such an eigenform then $\overline{f} \in \mathcal{C}_{k,\nu}(N, \overline{\chi})$ is as well. The crucial fact we need is how $\omega_N f$ is related to $\overline{f}$ when f is a primitive Hecke-Maass eigenform:

Proposition 11.7.7. *If $f \in \mathcal{C}_{k,\nu}(N, \chi)$ is a primitive Hecke-Maass eigenform then*

$$\omega_N f = \omega_N(f)\overline{f},$$

where $\overline{f} \in \mathcal{C}_{k,\nu}(N, \overline{\chi})$ is a primitive Hecke-Maass eigenform and $\omega_N(f) \in \mathbb{C}$ is nonzero with $|\omega_N(f)| = 1$.

Proof. The argument used in the proof of Proposition 10.6.6 holds verbatim. □

11.8 The Ramanujan-Petersson Conjecture

As for the size of the Fourier-Whittaker coefficients of Maass form, much is currently unknown. But there is an analogous conjecture to the one for holomorphic forms. To state it, suppose $f \in \mathcal{C}_{k,\nu}(N,\chi)$ is a primitive Hecke-Maass eigenform with Hecke eigenvalues $\lambda_f(m)$. For each prime p, consider the polynomial

$$1 - \lambda_f(p)p^{-s} + \chi(p)p^{-2s}.$$

We call this the p-th **Hecke polynomial** of f. We call the roots $\alpha_1(p)$ and $\alpha_2(p)$ the p-th **Hecke roots** of f. Then

$$\alpha_1(p) + \alpha_2(p) = \lambda_f(p) \quad \text{and} \quad \alpha_1(p)\alpha_2(p) = \chi(p).$$

The **Ramanujan-Petersson conjecture** for Maass forms is following statement:

Conjecture (Ramanujan-Petersson conjecture, Maass). *Let $f \in \mathcal{C}_{k,\nu}(N,\chi)$ be a primitive Hecke-Maass eigenform with Hecke eigenvalues $\lambda_f(m)$ and let $\alpha_1(p)$ and $\alpha_2(p)$ be the p-th Hecke roots of f. Then for all primes p, we have*

$$|\lambda_f(p)| \leq 2p^{-\frac{1}{2}}.$$

Moreover, if $p \nmid N$ then

$$|\alpha_1(p)| = |\alpha_2(p)| = 1.$$

The Ramanujan-Petersson conjecture has not yet been proved, but there has been partial progress toward the conjecture. The current best bound is $|\lambda_f(p)| \leq 2p^{\frac{7}{64}-\frac{1}{2}}$ due to Kim and Sarnak (see [KRS03] for the proof). Under the Ramanujan-Petersson conjecture, the Hecke relations give the more general bound $\lambda_f(m) \ll \sigma_0(m)m^{-\frac{1}{2}}$. Then Proposition A.3.1 implies the slightly weaker estimate $\lambda_f(m) \ll_\varepsilon m^{\varepsilon-\frac{1}{2}}$. It turns out that the Ramanujan-Petersson conjecture is tightly connected to another conjecture of Selberg about the smallest possible eigenvalue of Maass form on $\Gamma\backslash\mathbb{H}$. Note that the possible eigenvalues are discrete by Theorem 11.4.1 and so there exists a smallest eigenvalue. To state it, recall that if f is a Maass form with eigenvalue λ and spectral parameter r on $\Gamma\backslash\mathbb{H}$ then $\lambda = \frac{1}{4} + r^2$ with either $r \in \mathbb{R}$ or $ir \in [0, \frac{1}{2})$. The **Selberg conjecture** claims that the second case never occurs:

Conjecture (Selberg conjecture). *If λ is the smallest eigenvalue for Maass forms on $\Gamma\backslash\mathbb{H}$ then*

$$\lambda \geq \frac{1}{4}.$$

Selberg was able to achieve a remarkable lower bound using the analytic continuation of a certain Dirichlet series and the Weil bound for Kloosterman sums (see [Iwa02] for a proof):

Theorem 11.8.1. *If λ is the smallest eigenvalue for Maass forms on $\Gamma\backslash\mathbb{H}$ then*

$$\lambda \geq \frac{3}{16}.$$

In the language of automorphic representations, these two conjectures are a consequence of a much larger conjecture (see [BB13] for details).

11.9 Twists of Maass Forms

We can also twist Maass forms by Dirichlet characters. Let $f \in \mathcal{C}_{k,\nu}(N,\chi)$ have Fourier-Whittaker series

$$f(z) = a^+(f)y^{\frac{1}{2}+\nu} + a^-(f)y^{\frac{1}{2}-\nu} + \sum_{n\neq 0} a_n(f) W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx}.$$

and let ψ be a primitive Dirichlet character modulo M. We define the **twisted Maass form** $f \otimes \psi$ of f twisted by ψ by the Fourier-Whittaker series

$$(f\otimes\psi)(z) = a^+(f)\psi(0)y^{\frac{1}{2}+\nu} + a^-(f)\psi(0)y^{\frac{1}{2}-\nu} + \sum_{n\neq 0} a_n(f)\psi(n) W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx}.$$

In order for $f\otimes\psi$ to be well-defined, we need to prove that it is a Maass form. The following proposition proves this and more when ψ is primitive:

Proposition 11.9.1. *Suppose $f \in \mathcal{C}_{k,\nu}(N,\chi)$ and ψ is a primitive Dirichlet character of conductor q. Then $f\otimes\psi \in \mathcal{C}_{k,\nu}(Nq^2, \chi\psi^2)$.*

Proof. Arguing as in the proof of Proposition 10.8.2 with smoothness replacing holomorphy, automorphy replacing modularity, and the analogous growth condition for Maass forms, the only piece left to verify is that $f\otimes\psi$ is an eigenfunction for Δ with eigenvalue λ if f is in the case ψ is primitive. This is easy since the invariance of Δ implies

$$\Delta(f\otimes\chi)(z) = \frac{1}{\tau(\overline{\psi})}\sum_{r \pmod q} \overline{\psi}(r)\Delta(f)\left(z+\frac{r}{q}\right) = \frac{\lambda}{\tau(\overline{\psi})}\sum_{r \pmod q}\overline{\psi}(r) f\left(z+\frac{r}{q}\right) = \lambda(f\otimes\chi)(z).$$

□

The generalization of Proposition 11.9.1 to all characters is slightly more involved. Define operators U_p and V_p on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ to be the linear operators given by

$$(U_p f)(z) = \sum_{n\neq 0} a_{np}(f) W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx},$$

and

$$(V_p f)(z) = \sum_{n\neq 0} a_n(f) W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|py)e^{2\pi inpx},$$

if f has Fourier-Whittaker series

$$f(z) = \sum_{n\neq 0} a_n(f) W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx}.$$

We will show that both U_p and V_p map $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ into $\mathcal{C}_{k,\nu}(\Gamma_1(Np))$ and more:

Lemma 11.9.1. *For any prime p, U_p and V_p map $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ into $\mathcal{C}_{k,\nu}(\Gamma_1(Np))$. In particular, U_p and V_p map $\mathcal{C}_{k,\nu}(N,\chi)$ into $\mathcal{C}_{k,\nu}(Np, \chi\chi_{p,0})$.*

Proof. The argument used in the proof of Lemma 10.8.1 holds verbatim. □

We can now generalize Proposition 11.9.1 to all characters:

Proposition 11.9.2. *Suppose $f \in \mathcal{C}_{k,\nu}(N, \chi)$ and ψ is a Dirichlet character modulo M. Then $f \otimes \psi \in \mathcal{C}_{k,\nu}(NM^2, \chi\psi^2)$.*

Proof. Arguing as in the proof of Proposition 10.8.2, it remains to show that $f \otimes \psi_{p,0}$ is an eigenfunction with eigenvalue λ if f is. As $U_p = T_p$ is the p-th Hecke operator on $\mathcal{C}_{k,\nu}(\Gamma_1(Np))$, U_p commutes with Δ. It is also clear that V_p commutes with Δ. These facts together with

$$f \otimes \psi_{p,0} = f - V_p U_p f,$$

show that $f \otimes \psi_{p,0}$ is an eigenfunction with eigenvalue λ if f is. □

In particular, Proposition 11.9.2 shows that $f \otimes \psi$ is well-defined for any Dirichlet character ψ.

Chapter 12

Trace Formulas

There are various types of formulas that relate the Fourier coefficients of holomorphic, automorphic, and Maass forms. One of the most important such formulas is the Petersson trace formula.

12.1 The Petersson Trace Formula

From Theorem 10.6.1, $\mathcal{S}_k(N,\chi)$ admits an orthonormal basis of Hecke eigenforms. In particular, $\mathcal{S}_k(N,\chi)$ admits a merely orthogonal basis. Denote this basis by $u_1,\dots,u_r$ where r is the dimension of $\mathcal{S}_k(N,\chi)$. Each of these forms admits a Fourier series at the $\mathfrak{a}$ cusp given by

$$(u_j|\sigma_{\mathfrak{a}})(z)=\sum_{n\geq 1}a_{j,\mathfrak{a}}(n)e^{2\pi inz}.$$

The Petersson trace formula is an equation relating the Fourier coefficients $a_{j,\mathfrak{a}}(n)$ and $a_{j,\mathfrak{b}}(n)$ for $1\leq j\leq r$ at two cusps $\mathfrak{a}$ and $\mathfrak{b}$ of $\Gamma_0(N)\backslash\mathbb{H}$ to a sum of J-Bessel functions and Salié sums. To prove the Petersson trace formula we compute the inner product of two Poincaré series $P_{n,k,\chi,\mathfrak{a}}(z)$ and $P_{m,k,\chi,\mathfrak{b}}(z)$ in two different ways. One way is geometric in nature while the other is spectral. Since Theorem 10.3.2 says that $\langle P_{n,k,\chi,\mathfrak{a}},P_{m,k,\chi,\mathfrak{b}}\rangle$ extracts the m-th Fourier coefficient of $P_{n,k,\chi,\mathfrak{a}}$ up to a constant, the Petersson trace formula amounts to computing the m-th Fourier coefficient of $P_{n,k,\chi,\mathfrak{a}}$ in two different ways. We will begin with the geometric method first. This is easy as we have already computed the Fourier series of the Poincaré series. Applying Theorem 10.3.2 to the Fourier series in Proposition 10.2.1 gives

$$\langle P_{n,k,\chi,\mathfrak{a}},P_{m,k,\chi,\mathfrak{b}}\rangle=\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m}+\left(\frac{\sqrt{m}}{\sqrt{n}}\right)^{k-1}\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{2\pi i^{-k}}{c}J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right)S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c)\right).$$

This is the first half of the Petersson trace formula. To obtain the second half, we use the fact that $u_1,\dots,u_r$ is an orthogonal basis for $\mathcal{S}_k(N,\chi)$ and Theorem 10.3.2 to write

$$P_{n,k,\chi,\mathfrak{a}}(z)=\sum_{1\leq j\leq r}\frac{\langle P_{n,k,\chi,\mathfrak{a}},u_j\rangle}{\langle u_j,u_j\rangle}u_j(z)=\sum_{1\leq j\leq r}\frac{\overline{\langle u_j,P_{n,k,\chi,\mathfrak{a}}\rangle}}{\langle u_j,u_j\rangle}u_j(z)=\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi n)^{k-1}}\sum_{1\leq j\leq r}\frac{\overline{a_{j,\mathfrak{a}}(n)}}{\langle u_j,u_j\rangle}u_j(z).$$

This last expression is the spectral decomposition of $P_{n,k,\chi,\mathfrak{a}}(z)$ in terms of the basis $u_1,\dots,u_r$. So if we apply Theorem 10.3.2 to this last expression, we obtain

$$\langle P_{n,k,\chi,\mathfrak{a}},P_{m,k,\chi,\mathfrak{b}}\rangle=\left(\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi\sqrt{nm})^{k-1}}\right)^2\sum_{1\leq j\leq r}\frac{\overline{a_{j,\mathfrak{a}}(n)}a_{j,\mathfrak{b}}(m)}{\langle u_j,u_j\rangle},$$

which is the second half of the Petersson trace formula. Equating the first and second halves and canceling the common $\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}}$ factor gives

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi n)^{k-1}} \sum_{1\le j\le r} \frac{\overline{a_{j,\mathfrak{a}}(n)}a_{j,\mathfrak{b}}(m)}{\langle u_j, u_j\rangle} = \delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m} + \left(\frac{\sqrt{m}}{\sqrt{n}}\right)^{k-1} \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c).$$

Since $\left(\frac{\sqrt{m}}{\sqrt{n}}\right)^{k-1} = 1$ when $n = m$, we can factor this term out of the entire right-hand side and cancel it resulting in the **Petersson trace formula** relative to the $\mathfrak{a}$ and $\mathfrak{b}$ cusps:

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi\sqrt{nm})^{k-1}} \sum_{1\le j\le r} \frac{\overline{a_{j,\mathfrak{a}}(n)}a_{j,\mathfrak{b}}(m)}{\langle u_j, u_j\rangle} = \delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m} + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c).$$

We refer to the left-hand side side as the **spectral side** and the right-hand side as the **geometric side**. We collect our work as a theorem:

Theorem (Petersson trace formula). *Let $u_1,\dots,u_r$ be an orthogonal basis of Hecke eigenforms for $\mathcal{S}_k(N,\chi)$ with Fourier coefficients $a_{j,\mathfrak{a}}(n)$ at the $\mathfrak{a}$ cusp. Then for any positive integers $n, m \ge 1$ and any two cusps $\mathfrak{a}$ and $\mathfrak{b}$, we have*

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi\sqrt{nm})^{k-1}} \sum_{1\le j\le r} \frac{\overline{a_{j,\mathfrak{a}}(n)}a_{j,\mathfrak{b}}(m)}{\langle u_j, u_j\rangle} = \delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m} + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c).$$

Note that if we take $u_1,\dots,u_r$ to be an orthonormal basis, equivalently the basis elements are Hecke normalized, then $\langle u_j, u_j\rangle = 1$. Regardless, a particularly important case of the Petersson trace formula is when $\mathfrak{a} = \mathfrak{b} = \infty$. For then $\mathcal{C}_{\infty,\infty} = \{c \ge 1 : c \equiv 0 \pmod N\}$, the Salié sum reduces to the usual one, and the Petersson trace formula takes the form

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi\sqrt{nm})^{k-1}} \sum_{1\le j\le r} \frac{\overline{a_j(n)}a_j(m)}{\langle u_j, u_j\rangle} = \delta_{n,m} + \sum_{\substack{c\ge 1\\ c\equiv 0 \pmod N}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_{\chi}(n,m,c).$$

Chapter 13

L-functions of Holomorphic, Automorphic, and Maass Forms

We discuss the L-functions associated to holomorphic, automorphic, and Maass forms. In particular, we develop the theory of Hecke and Hecke-Maass L-functions. We also discuss the Rankin-Selberg method associated to these L-functions. Then we use Rankin-Selberg convolution L-functions of Hecke and Hecke-Maass eigenforms to prove two useful results: strong multiplicity one and the Ramanujan-Petersson conjecture on average. This first result is a strengthening of multiplicity one for holomorphic and Maass forms. The second result is weaker than the Ramanujan-Petersson conjecture for holomorphic or Maass forms but is often a sufficient replacement.

13.1 Hecke L-functions

The Definition and Euler Product

We will investigate the L-functions of holomorphic cusp forms. Let $f \in \mathcal{S}_k(N, \chi)$ and denote its Fourier series by

$$f(z) = \sum_{n \geq 1} a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z},$$

with $a_f(1) = 1$. Thus if f is a Hecke eigenform, the $a_f(n)$ are the Hecke eigenvalues of f normalized so that they are constant on average. The **Hecke L-series** (respectively **Hecke L-function** if it is an L-function) $L(s, f)$ of f is defined by the following Dirichlet series:

$$L(s, f) = \sum_{n \geq 1} \frac{a_f(n)}{n^s}.$$

We will see that $L(s, f)$ is a Selberg class L-function if f is a primitive Hecke eigenform. From now on, we make this assumption about f. The Hecke relations and the Ramanujan-Petersson conjecture for holomorphic forms together imply that the coefficients $a_f(n)$ are multiplicative and satisfy $a_f(n) \ll_\varepsilon n^\varepsilon$. By Proposition 2.2.1, $L(s, f)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits the following infinite product expression:

$$L(s, f) = \prod_p \left(\sum_{n \geq 0} \frac{a_f(p^n)}{p^{ns}} \right).$$

To determine the Euler product, the Hecke relations imply that the coefficients $a_f(n)$ satisfy

$$a_f(p^n) = \begin{cases} a_f(p^{n-1})a_f(p) - \chi(p)a_f(p^{n-2}) & \text{if } p \nmid N, \\ (a_f(p))^n & \text{if } p \mid N, \end{cases} \tag{13.1}$$

for all primes p and $n \geq 2$. We now simplify the factor inside the product using this Equation (13.1). On the one hand, if $p \nmid N$:

$$\begin{aligned} \sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} &= 1 + \frac{a_f(p)}{p^s} + \sum_{n\geq 2} \frac{a_f(p^n)}{p^{ns}} \\ &= 1 + \frac{a_f(p)}{p^s} + \sum_{n\geq 2} \frac{a_f(p^{n-1})a_f(p) - \chi(p)a_f(p^{n-2})}{p^{ns}} \\ &= 1 + \frac{a_f(p)}{p^s} + \frac{a_f(p)}{p^s}\sum_{n\geq 1} \frac{a_f(p^n)}{p^{ns}} - \frac{\chi(p)}{p^{2s}}\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} \\ &= 1 + \left(\frac{a_f(p)}{p^s} - \frac{\chi(p)}{p^{2s}}\right)\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}}. \end{aligned}$$

By isolating the sum we find

$$\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} = \left(1 - \frac{a_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1}.$$

On the other hand, if $p \mid N$ we have

$$\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} = \sum_{n\geq 0} \frac{(a_f(p))^n}{p^{ns}} = \left(1 - a_f(p)p^{-s}\right)^{-1}.$$

Therefore

$$L(s,f) = \prod_{p\nmid N}(1 - a_f(p)p^{-s} + \chi(p)p^{-2s})^{-1} \prod_{p\mid N}(1 - a_f(p)p^{-s})^{-1}.$$

Letting $\alpha_1(p)$ and $\alpha_2(p)$ be the p-th Hecke roots of f, we can express $L(s,f)$ as the following degree 2 Euler product:

$$L(s,f) = \prod_{p}(1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1}.$$

The local factor at p is

$$L_p(s,f) = (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1},$$

with local roots $\alpha_1(p)$ and $\alpha_2(p)$. Upon applying partial fraction decomposition to the local factor, we find

$$\frac{1}{1 - \alpha_1(p)p^{-s}}\frac{1}{1 - \alpha_2(p)p^{-s}} = \frac{\frac{\alpha_1(p)}{\alpha_1(p)-\alpha_2(p)}}{1 - \alpha_1(p)p^{-s}} + \frac{\frac{-\alpha_2(p)}{\alpha_1(p)-\alpha_2(p)}}{1 - \alpha_2(p)p^{-s}}.$$

Expanding both sides as series into Dirichlet series, and comparing coefficients using Proposition 2.1.1, gives

$$a_f(p^n) = \frac{\alpha_1(p)^{n+1} - \alpha_2(p)^{n+1}}{\alpha_1(p) - \alpha_2(p)}. \tag{13.2}$$

The Integral Representation

We now want to find an integral representation for $L(s,f)$. Consider the following Mellin transform:

$$\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}.$$

As f has exponential decay at the cusps, this integral is locally absolutely uniformly convergent for $\sigma>1$ and hence defines an analytic function there. Then we compute

$$\begin{aligned}
\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y} &= \int_0^\infty \sum_{n\geq 1} a_f(n)n^{\frac{k-1}{2}}e^{-2\pi ny}y^{s+\frac{k-1}{2}}\,\frac{dy}{y}\\
&= \sum_{n\geq 1} a_f(n)n^{\frac{k-1}{2}}\int_0^\infty e^{-2\pi ny}y^{s+\frac{k-1}{2}}\,\frac{dy}{y} && \text{FTT}\\
&= \sum_{n\geq 1}\frac{a_f(n)}{(2\pi)^{s+\frac{k-1}{2}}n^s}\int_0^\infty e^{-y}y^{s+\frac{k-1}{2}}\,\frac{dy}{y} && y\mapsto\frac{y}{2\pi n}\\
&= \frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}}\sum_{n\geq 1}\frac{a_f(n)}{n^s}\\
&= \frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}}L(s,f).
\end{aligned}$$

Rewriting, we have an integral representation

$$L(s,f)=\frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma\left(s+\frac{k-1}{2}\right)}\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}. \tag{13.3}$$

Now split the integral on the right-hand side into two pieces by writing

$$\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}=\int_0^{\frac{1}{\sqrt{N}}} f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}+\int_{\frac{1}{\sqrt{N}}}^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}. \tag{13.4}$$

Now we will rewrite the first piece in the same form and symmetrize the result as much as possible. Begin by performing the change of variables $y\mapsto\frac{1}{Ny}$ to the first piece to obtain

$$\int_{\frac{1}{\sqrt{N}}}^\infty f\left(\frac{i}{Ny}\right)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y}.$$

Rewriting in terms of the Atkin-Lehner operator and recalling that $\omega_N f=\omega_N(f)\overline{f}$ by Proposition 10.6.6, we have

$$\begin{aligned}
\int_{\frac{1}{\sqrt{N}}}^\infty f\left(\frac{i}{Ny}\right)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y} &= \int_{\frac{1}{\sqrt{N}}}^\infty f\left(-\frac{1}{iNy}\right)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y}\\
&= \int_{\frac{1}{\sqrt{N}}}^\infty \left(\sqrt{N}iy\right)^k(\omega_N f)(iy)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y}\\
&= \int_{\frac{1}{\sqrt{N}}}^\infty \left(\sqrt{N}iy\right)^k\omega_N(f)\overline{f}(iy)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y}\\
&= \omega_N(f)i^kN^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^\infty \overline{f}(iy)y^{(1-s)-\frac{k-1}{2}}\,\frac{dy}{y}.
\end{aligned}$$

Substituting this result back into Equation (13.4) and combining with Equation (13.3) yields the integral representation

$$L(s,f) = \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma\left(s+\frac{k-1}{2}\right)}\left[\omega_N(f)i^k N^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^{\infty}\overline{f}(iy)y^{(1-s)+\frac{k-1}{2}}\,\frac{dy}{y} + \int_{\frac{1}{\sqrt{N}}}^{\infty} f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}\right]. \tag{13.5}$$

This integral representation will give analytic continuation. To see this, we know everything outside the brackets is entire. The integrands exhibit exponential decay and therefore the integrals are locally absolutely uniformly convergent on $\mathbb{C}$. Hence we have analytic continuation to all of $\mathbb{C}$. In particular, we have shown that $L(s,f)$ has no poles.

The Functional Equation

An immediate consequence of applying the symmetry $s \mapsto 1-s$ to Equation (13.5) is the following functional equation:

$$N^{\frac{s}{2}}\frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}}L(s,f) = \omega_N(f)i^k N^{\frac{1-s}{2}}\frac{\Gamma\left((1-s)+\frac{k-1}{2}\right)}{(2\pi)^{(1-s)+\frac{k-1}{2}}}L(1-s,\overline{f}).$$

Using the Legendre duplication formula, we find that

$$\begin{aligned}\frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}} &= \frac{1}{(2\pi)^{s+\frac{k-1}{2}}2^{1-\left(s+\frac{k-1}{2}\right)}\sqrt{\pi}}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)\\ &= \frac{1}{2\pi^{s+\frac{1}{2}}}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)\\ &= \frac{1}{\sqrt{4\pi}}\pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right).\end{aligned}$$

The constant factor in front is independent of s and so can be canceled in the functional equation. Therefore we identify the gamma factor as

$$\gamma(s,f) = \pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right),$$

with $\kappa_1 = k-1$ and $\kappa_2 = k+1$ the local roots at infinity. The conductor is $q(f) = N$, so the primes dividing the level ramify, and by the Ramanujan-Petersson conjecture for holomorphic forms, $\alpha_1(p) \neq 0$ and $\alpha_2(p) \neq 0$ for all primes $p \nmid N$. The completed L-function is

$$\Lambda(s,f) = N^{\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)L(s,f),$$

with functional equation

$$\Lambda(s,f) = \omega_N(f)i^k\Lambda(1-s,\overline{f}).$$

This is the functional equation of $L(s,f)$. From it, the root number is $\varepsilon(f) = \omega_N(f)i^k$ and we see that $L(s,f)$ has dual $L(s,\overline{f})$. We will now show that $L(s,f)$ is of order 1. Since $L(s,f)$ has no poles, we do not need to clear any polar divisors. As the integrals in Equation (13.5) are locally absolutely uniformly

convergent, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s,f)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So the reciprocal of the gamma factor is also of order 1. Then

$$L(s,f) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So $L(s,f)$ is of order 1. We summarize all of our work into the following theorem:

Theorem 13.1.1. *Let $f \in \mathcal{S}_k(N,\chi)$ be a primitive Hecke eigenform and for every prime p let $\alpha_1(p)$ and $\alpha_2(p)$ be the p-th Hecke roots of f. Then $L(s,f)$ is a Selberg class L-function with degree 2 Euler product given by*

$$L(s,f) = \prod_p (1-\alpha_1(p)p^{-s})^{-1}(1-\alpha_2(p)p^{-s})^{-1}.$$

Moreover, it admits analytic continuation to $\mathbb{C}$ and possesses the functional equation

$$N^{\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)L(s,f) = \Lambda(s,f) = \omega_N(f)i^k\Lambda(1-s,\overline{f}).$$

Beyond Primitivity

We can still obtain analytic continuation of the L-series $L(s,f)$ if f is not a primitive Hecke eigenform. Indeed, since the primitive Hecke eigenforms form a basis for the space of newforms, we can prove the following:

Theorem 13.1.2. *For any $f \in \mathcal{S}_k(\Gamma_1(N))$, $L(s,f)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits analytic continuation to $\mathbb{C}$.*

Proof. If f is a newform, this follows from Theorems 10.6.1 and 13.1.1. We will prove the case for oldforms by induction. The base case is clear since if $N = 1$ there are no oldforms. So assume by induction that the claim holds for all proper divisors of N. As f is an oldform, there is a proper divisor $d \mid N$ with $d > 1$ such that

$$f(z) = g(z) + d^{k-1}h(dz) = g(z) + \prod_{p^r||d}(V_p^r h)(z),$$

for some $g, h \in \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{d}\right)\right)$. Note that $V_p h \in \mathcal{S}_k\left(\Gamma_1\left(\frac{Np}{d}\right)\right)$ by Lemma 10.8.1. Our induction hypothesis then implies that $L(s,g)$ and $L(s,V_p^r h)$, for all $p^r \,||\, d$, admit analytic continuation to $\mathbb{C}$. Thus so does $L(s,f)$ which completes the proof. □

13.2 Hecke-Maass L-functions

The Definition and Euler Product

We will investigate the L-functions of weight zero Maass cusp forms. Let $f \in \mathcal{C}_\nu(N,\chi)$ and denote its Fourier-Whittaker series by

$$f(z) = \sum_{n\geq 1} a_f(n)\left(\sqrt{y}K_\nu(2\pi ny)e^{2\pi inx} + \frac{a_f(-n)}{a_f(n)}\sqrt{y}K_\nu(2\pi ny)e^{-2\pi inx}\right),$$

with $a_f(1) = 1$. Thus if f is a Hecke eigenform then as f is even or odd by Proposition 11.7.5, the Fourier-Whittaker series takes the form

$$f(z) = \sum_{n\geq 1} a_f(n)\sqrt{y}K_\nu(2\pi ny)\,\mathrm{SC}(2\pi nx),$$

and the $a_f(n)$ are the Hecke eigenvalues of f normalized so that they are constant on average. The **Hecke-Maass L-series** (respectively **Hecke-Maass L-function** if it is an L-function) $L(s,f)$ of f is defined by the following Dirichlet series:

$$L(s,f) = \sum_{n\geq 1} \frac{a_f(n)}{n^s}.$$

We will see that $L(s,f)$ is a Selberg class L-function if f is a primitive Hecke-Maass eigenform. From now on, we make this assumption about f. The Ramanujan-Petersson conjecture for Maass forms is not know so $L(s,f)$ has not been proven to be a Selberg class L-function. As it is conjectured to be, throughout we will assume that the Ramanujan-Petersson conjecture for Maass forms holds. The Hecke relations and the Ramanujan-Petersson conjecture for Maass forms together imply that the coefficients $a_f(n)$ are multiplicative and satisfy $a_f(n) \ll_\varepsilon n^\varepsilon$. By Proposition 2.2.1, $L(s,f)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits the following infinite product expression:

$$L(s,f) = \prod_p \left(\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}}\right).$$

To determine the Euler product, the Hecke relations imply that the coefficients $a_f(n)$ satisfy

$$a_f(p^n) = \begin{cases} a_f(p^{n-1})a_f(p) - \chi(p)a_f(p^{n-2}) & \text{if } p \nmid N, \\ (a_f(p))^n & \text{if } p \mid N, \end{cases} \tag{13.6}$$

for all primes p and $n \geq 2$. We now simplify the factor inside the product using this Equation (13.6). On the one hand, if $p \nmid N$:

$$\begin{aligned} \sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} &= 1 + \frac{a_f(p)}{p^s} + \sum_{n\geq 2} \frac{a_f(p^n)}{p^{ns}} \\ &= 1 + \frac{a_f(p)}{p^s} + \sum_{n\geq 2} \frac{a_f(p^{n-1})a_f(p) - \chi(p)a_f(p^{n-2})}{p^{ns}} \\ &= 1 + \frac{a_f(p)}{p^s} + \frac{a_f(p)}{p^s}\sum_{n\geq 1} \frac{a_f(p^n)}{p^{ns}} - \frac{\chi(p)}{p^{2s}}\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} \\ &= 1 + \left(\frac{a_f(p)}{p^s} - \frac{\chi(p)}{p^{2s}}\right)\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}}. \end{aligned}$$

By isolating the sum we find

$$\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} = \left(1 - \frac{a_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1}.$$

On the other hand, if $p \mid N$ we have

$$\sum_{n\geq 0} \frac{a_f(p^n)}{p^{ns}} = \sum_{n\geq 0} \frac{(a_f(p))^n}{p^{ns}} = \left(1 - a_f(p)p^{-s}\right)^{-1}.$$

Therefore

$$L(s,f) = \prod_{p\nmid N}(1 - a_f(p)p^{-s} + \chi(p)p^{-2s})^{-1} \prod_{p|N}(1 - a_f(p)p^{-s})^{-1}.$$

Letting $\alpha_1(p)$ and $\alpha_2(p)$ be the p-th Hecke roots of f, we can express $L(s,f)$ as the following degree 2 Euler product:

$$L(s,f) = \prod_{p}(1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1}.$$

The local factor at p is

$$L_p(s,f) = (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1},$$

with local roots $\alpha_1(p)$ and $\alpha_2(p)$. Upon applying partial fraction decomposition to the local factor, we find

$$\frac{1}{1-\alpha_1(p)p^{-s}}\frac{1}{1-\alpha_2(p)p^{-s}} = \frac{\frac{\alpha_1(p)}{\alpha_1(p)-\alpha_2(p)}}{1-\alpha_1(p)p^{-s}} + \frac{\frac{-\alpha_2(p)}{\alpha_1(p)-\alpha_2(p)}}{1-\alpha_2(p)p^{-s}}.$$

Expanding both sides as series into Dirichlet series, and comparing coefficients using Proposition 2.1.1, gives

$$a_f(p^n) = \frac{\alpha_1(p)^{n+1} - \alpha_2(p)^{n+1}}{\alpha_1(p) - \alpha_2(p)}. \tag{13.7}$$

The Integral Representation

We want to find an integral representation for $L(s,f)$. Recall that f is an eigenfunction for the parity Hecke operator T_{-1} with eigenvalue ± 1. Equivalently, f is even if the eigenvalue is 1 and odd if the eigenvalue is -1. The integral representation will depend upon this parity. To handle both cases simultaneously, let $\delta_f = 0, 1$ according to whether f is even or odd. In other words,

$$\delta_f = \frac{1 - a_f(-1)}{2}.$$

Now consider the following Mellin transform:

$$\int_0^\infty \left(\frac{\partial}{\partial x}^{\delta_f} f\right)(iy) y^{s-\frac{1}{2}+\delta_f}\,\frac{dy}{y}.$$

As f has exponential decay at the cusps, this integral is locally absolutely uniformly convergent for $\sigma > 1$ and hence defines an analytic function there. The derivative operator is present because $\mathrm{SC}(x) = i\sin(x)$ if f is odd. In any case, the smoothness of f implies that we may differentiate its Fourier-Whittaker series termwise to obtain

$$\left(\frac{\partial}{\partial x}^{\delta_f} f\right)(z) = \sum_{n\geq 1} a_f(n)(2\pi i n)^{\delta_f}\sqrt{y}K_\nu(2\pi n y)\cos(2\pi n x).$$

Therefore regardless if f is even or odd, the Fourier-Whittaker series of $\left(\frac{\partial}{\partial x}^{\delta_f} f\right)(z)$ has $\mathrm{SC}(x) = \cos(x)$ and the integral does not vanish identically. Then we compute

$$\begin{aligned}
\int_0^\infty \left(\frac{\partial}{\partial x}^{\delta_f} f\right)(iy) y^{s-\frac{1}{2}+\delta_f} \frac{dy}{y} &= \int_0^\infty \sum_{n\geq 1} a_f(n)(2\pi i n)^{\delta_f} K_\nu(2\pi n y) y^{s+\delta_f} \frac{dy}{y} \\
&= \sum_{n\geq 1} a_f(n)(2\pi i n)^{\delta_f} \int_0^\infty K_\nu(2\pi n y) y^{s+\delta_f} \frac{dy}{y} && \text{FTT} \\
&= \sum_{n\geq 1} \frac{a_f(n)}{(2\pi)^s n^s} i^{\delta_f} \int_0^\infty K_\nu(y) y^{s+\delta_f} \frac{dy}{y} && y \mapsto \frac{y}{2\pi n} \\
&= \frac{\Gamma\left(\frac{s+\delta_f+\nu}{2}\right)\Gamma\left(\frac{s+\delta_f-\nu}{2}\right)}{2^{2-\delta_f}\pi^s(-i)^{\delta_f}} \sum_{n\geq 1} \frac{a_f(n)}{n^s} && \text{Appendix E.1} \\
&= \frac{\Gamma\left(\frac{s+\delta_f+\nu}{2}\right)\Gamma\left(\frac{s+\delta_f-\nu}{2}\right)}{2^{2-\delta_f}\pi^s(-i)^{\delta_f}} L(s,f).
\end{aligned}$$

Rewriting, we have an integral representation

$$L(s,f) = \frac{2^{2-\delta_f}\pi^s(-i)^{\delta_f}}{\Gamma\left(\frac{s+\delta_f+\nu}{2}\right)\Gamma\left(\frac{s+\delta_f-\nu}{2}\right)} \int_0^\infty \left(\frac{\partial}{\partial x}^{\delta_f} f\right)(iy) y^{s-\frac{1}{2}+\delta_f} \frac{dy}{y}. \tag{13.8}$$

Now split the integral on the right-hand side into two pieces by writing

$$\int_0^\infty \left(\frac{\partial}{\partial x}^{\delta_f} f\right)(iy) y^{s-\frac{1}{2}+\delta_f} \frac{dy}{y} = \int_0^{\frac{1}{\sqrt{N}}} \left(\frac{\partial}{\partial x}^{\delta_f} f\right)(iy) y^{s-\frac{1}{2}+\delta_f} \frac{dy}{y} + \int_{\frac{1}{\sqrt{N}}}^\infty \left(\frac{\partial}{\partial x}^{\delta_f} f\right)(iy) y^{s-\frac{1}{2}+\delta_f} \frac{dy}{y}. \tag{13.9}$$

Now we will rewrite the first piece in the same form and symmetrize the result as much as possible. Performing the change of variables $y \mapsto \frac{1}{Ny}$ to the first piece to obtain

$$\int_{\frac{1}{\sqrt{N}}}^\infty \left(\frac{\partial}{\partial x}^{\delta_f} f\right)\left(\frac{i}{Ny}\right)(Ny)^{-s+\frac{1}{2}-\delta_f} \frac{dy}{y}.$$

We will rewrite this in terms of the Atkin-Lehner operator. But first we require an identity that relates $\frac{\partial}{\partial x}^{\delta_f}$ with the Atkin-Lehner operator ω_N. By the identity theorem it suffices verify this for $z \in \mathbb{H}$ with $|z|$ fixed. Observe that $-\frac{1}{Nz} = \frac{-x}{N|z|^2} + \frac{iy}{N|z|^2}$. Now differentiate termwise to see that

$$\begin{aligned}
\left(\frac{\partial}{\partial x}^{\delta_f} \omega_N f\right)(z) &= \left(\frac{\partial}{\partial x}^{\delta_f}\right) f\left(-\frac{1}{Nz}\right) \\
&= \left(\frac{\partial}{\partial x}^{\delta_f}\right) \sum_{n\geq 1} a_f(n) \sqrt{\frac{y}{N|z|^2}} K_\nu(2\pi n y)\, \mathrm{SC}\left(-2\pi n \frac{x}{N|z|^2}\right) \\
&= (-N|z|^2)^{-\delta_f} \sum_{n\geq 1} a_f(n)(2\pi i n)^{\delta_f} \sqrt{\frac{y}{N|z|^2}} K_\nu\left(2\pi n \frac{y}{N|z|^2}\right) \cos\left(-2\pi n \frac{x}{N|z|^2}\right) \\
&= (-N|z|^2)^{-\delta_f} \left(\frac{\partial}{\partial x}^{\delta_f} f\right)\left(-\frac{1}{Nz}\right).
\end{aligned}$$

By the identity theorem, we have

$$\left(\frac{\partial}{\partial x}^{\delta_f} f\right)\left(-\frac{1}{Nz}\right) = (-N|z|^2)^{\delta_f}\left(\frac{\partial}{\partial x}^{\delta_f} \omega_N f\right)(z),$$

Using this identity, rewriting in terms of the Atkin-Lehner operator, and recalling that $\omega_N f = \omega_N(f)\overline{f}$ by Proposition 11.7.7, we have

$$\begin{aligned}
\int_{\frac{1}{\sqrt{N}}}^{\infty}\left(\frac{\partial}{\partial x}^{\delta_f} f\right)\left(\frac{i}{Ny}\right)(Ny)^{-s+\frac{1}{2}-\delta_f}\frac{dy}{y} &= \int_{\frac{1}{\sqrt{N}}}^{\infty}\left(\frac{\partial}{\partial x}^{\delta_f} f\right)\left(-\frac{1}{iNy}\right)(Ny)^{-s+\frac{1}{2}-\delta_f}\frac{dy}{y} \\
&= \int_{\frac{1}{\sqrt{N}}}^{\infty}(-Ny^2)^{\delta_f}\left(\left(\frac{\partial}{\partial x}^{\delta_f}\right)\omega_N f\right)(iy)(Ny)^{-s+\frac{1}{2}-\delta_f}\frac{dy}{y} \\
&= \int_{\frac{1}{\sqrt{N}}}^{\infty}(-Ny^2)^{\delta_f}\omega_N(f)\left(\left(\frac{\partial}{\partial x}^{\delta_f}\right)\overline{f}\right)(iy)(Ny)^{-s+\frac{1}{2}-\delta_f}\frac{dy}{y} \\
&= w_N(f)(-1)^{\delta_f}N^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^{\infty}\left(\left(\frac{\partial}{\partial x}^{\delta_f}\right)\overline{f}\right)(iy)y^{(1-s)-\frac{1}{2}+\delta_f}\frac{dy}{y}.
\end{aligned}$$

Substituting this result back into Equation (13.9) and combining with Equation (13.8) gives the integral representation

$$\begin{aligned}
L(s,f) = \frac{2^{2-\delta_f}\pi^s(-i)^{\delta_f}}{\Gamma\left(\frac{s+\delta_f+\nu}{2}\right)\Gamma\left(\frac{s+\delta_f-\nu}{2}\right)}\Bigg[&w_N(f)(-1)^{\delta_f}N^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^{\infty}\left(\left(\frac{\partial}{\partial x}^{\delta_f}\right)\overline{f}\right)(iy)y^{(1-s)-\frac{1}{2}+\delta_f}\frac{dy}{y} \\
&+\int_{\frac{1}{\sqrt{N}}}^{\infty}\left(\frac{\partial}{\partial x}^{\delta_f} f\right)(iy)y^{s-\frac{1}{2}+\delta_f}\frac{dy}{y}\Bigg].
\end{aligned} \tag{13.10}$$

This integral representation will give analytic continuation. To see this, note that everything outside the brackets is entire. The integrands exhibit exponential decay and therefore the integrals are locally absolutely uniformly convergent on $\mathbb{C}$. Hence we have analytic continuation to all of $\mathbb{C}$. In particular, $L(s,f)$ has no poles.

The Functional Equation

An immediate consequence of applying the symmetry $s \mapsto 1-s$ to Equation (13.10) is the following functional equation:

$$N^{\frac{s}{2}}\frac{\Gamma\left(\frac{s+\delta_f+\nu}{2}\right)\Gamma\left(\frac{s+\delta_f-\nu}{2}\right)}{2^{2-\delta_f}\pi^s(-i)^{\delta_f}}L(s,f) = \omega_N(f)(-1)^{\delta_f}N^{\frac{1-s}{2}}\frac{\Gamma\left(\frac{(1-s)+\delta_f+\nu}{2}\right)\Gamma\left(\frac{(1-s)+\delta_f-\nu}{2}\right)}{2^{2-\delta_f}\pi^{1-s}(-i)^{\delta_f}}L(1-s,\overline{f}).$$

The constant factor in the denominator is independent of s and so can be canceled in the functional equation. Therefore we identify the gamma factor as

$$\gamma(s,f) = \pi^{-s}\Gamma\left(\frac{s+\delta_f+\nu}{2}\right)\Gamma\left(\frac{s+\delta_f-\nu}{2}\right),$$

with $\kappa_1 = \delta_f + \nu$ and $\kappa_2 = \delta_f - \nu$ the local roots at infinity (these are conjugates because ν is either purely imaginary of real). The conductor is $q(f) = N$, so the primes dividing the level ramify, and by

the Ramanujan-Petersson conjecture for Maass forms, $\alpha_1(p) \neq 0$ and $\alpha_2(p) \neq 0$ for all primes $p \nmid N$. The completed L-function is

$$\Lambda(s,f) = N^{\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s+\delta_f+\nu}{2}\right)\Gamma\left(\frac{s+\delta_f-\nu}{2}\right)L(s,f),$$

with functional equation

$$\Lambda(s,f) = \omega_N(f)(-1)^{\delta_f}\Lambda(1-s,\overline{f}).$$

This is the functional equation of $L(s,f)$. From it, the root number is $\varepsilon(f) = \omega_N(f)(-1)^{\delta_f}$ and we see that $L(s,f)$ has dual $L(s,\overline{f})$. We will now show that $L(s,f)$ is of order 1. Since $L(s,f)$ has no poles, we do not need to clear any polar divisors. As the integrals in Equation (13.10) are locally absolutely uniformly convergent, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s,f)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So the reciprocal of the gamma factor is also of order 1. Then

$$L(s,f) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So $L(s,f)$ is of order 1. We summarize all of our work into the following theorem:

Theorem 13.2.1. *For any primitive Hecke-Maass eigenform Let $f \in \mathcal{C}_\nu(N,\chi)$ be a primitive Hecke-Maass eigenform and for every prime p let $\alpha_1(p)$ and $\alpha_2(p)$ be the p-th Hecke roots of f. Then $L(s,f)$ is a Selberg class L-function, provided the Ramanujan-Petersson conjecture for Maass forms holds, with degree 2 Euler product given by*

$$L(s,f) = \prod_p (1-\alpha_1(p)p^{-s})^{-1}(1-\alpha_2(p)p^{-s})^{-1}.$$

Moreover, it admits analytic continuation to $\mathbb{C}$ and possesses the functional equation

$$N^{\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s+\delta_f+\nu}{2}\right)\Gamma\left(\frac{s+\delta_f-\nu}{2}\right)L(s,f) = \Lambda(s,f) = \omega_N(f)(-1)^{\delta_f}\Lambda(1-s,\overline{f}).$$

Beyond Primitivity

We can still obtain analytic continuation of the L-series $L(s,f)$ if f is not a primitive Hecke-Maass eigenform. Similarly to the Hecke L-function case, this holds because the primitive Hecke-Maass eigenforms form a basis for the space of newforms:

Theorem 13.2.2. *For any $f \in \mathcal{C}_\nu(\Gamma_1(N))$, $L(s,f)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits analytic continuation to $\mathbb{C}$ provided the Ramanujan-Petersson conjecture for Maass forms holds.*

Proof. Argue as in the proof of Theorem 13.1.2. □

13.3 The Rankin-Selberg Method

The Definition and Euler Product

The Rankin-Selberg method is a process by which we can construct new L-functions from old ones. Instead of giving the general definition outright, we first provide a full discussion of the method only in the simplest case. Many technical difficulties arise in the fully general setting. Let $f, g \in \mathcal{S}_k(1)$ be primitive Hecke eigenforms with Fourier series

$$f(z) = \sum_{n\geq 1} a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z} \quad \text{and} \quad g(z) = \sum_{n\geq 1} a_g(n) n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

The L-series $L(s, f \times g)$ of f and g is defined by

$$L(s, f\times g) = \sum_{n\geq 1} \frac{a_{f\times g}(n)}{n^s} = \sum_{n\geq 1} \frac{a_f(n)\overline{a_g(n)}}{n^s} = \sum_{n\geq 1} \frac{a_f(n)\overline{a_g(n)}}{n^s}.$$

The **Rankin-Selberg convolution** $L(s, f \otimes g)$ of f and g is defined by

$$L(s, f\otimes g) = \sum_{n\geq 1} \frac{a_{f\otimes g}(n)}{n^s} = \zeta(2s) L(s, f\times g),$$

where $a_{f\otimes g}(n) = \sum_{n=m\ell^2} a_f(m)\overline{a_g(m)}$. Our primary aim will be to show that $L(s, f \otimes g)$ is the Rankin-Selberg convolution of $L(s, f)$ and $L(s, g)$ and is a Selberg class L-function. Since $a_f(n)$ and $a_g(n)$ are both multiplicative so is $a_{f\otimes g}(n)$. Moreover, $a_{f\otimes g}(n) \ll_\varepsilon n^\varepsilon$ because $a_f(n) \ll_\varepsilon n^\varepsilon$ and $a_g(n) \ll_\varepsilon n^\varepsilon$. It follows from Proposition 2.2.1 that $L(s, f \otimes g)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits the following infinite product expression:

$$L(s, f\otimes g) = \zeta(2s) L(s, f\times g) = \prod_p (1-p^{-2s})^{-1} \prod_p \left(\sum_{n\geq 0} \frac{a_f(p^n)\overline{a_g(p^n)}}{p^{ns}} \right).$$

Now let $\alpha_1(p)$ and $\alpha_2(p)$ be the p-th Hecke roots of f while $\beta_1(p)$ and $\beta_2(p)$ are the p-th Hecke roots of g. We will simplify the factor inside the latter product using Equation (13.2):

$$\begin{aligned}
\sum_{n\geq 0} \frac{a_f(p^n)\overline{a_g(p^n)}}{p^{ns}} &= \sum_{n\geq 0} \left(\frac{\alpha_1(p)^{n+1} - \alpha_2(p)^{n+1}}{\alpha_1(p) - \alpha_2(p)} \right) \left(\frac{(\overline{\beta_1(p)})^{n+1} - (\overline{\beta_2(p)})^{n+1}}{\overline{\beta_1(p)} - \overline{\beta_2(p)}} \right) p^{-ns} \\
&= (\alpha_1(p) - \alpha_2(p))^{-1} \left(\overline{\beta_1(p)} - \overline{\beta_2(p)} \right)^{-1} \\
&\cdot \left[\sum_{n\geq 1} \frac{\alpha_1(p)^n (\overline{\beta_1(p)})^n}{p^{(n-1)s}} + \frac{\alpha_2(p)^n (\overline{\beta_2(p)})^n}{p^{(n-1)s}} - \frac{\alpha_1(p)^n (\overline{\beta_2(p)})^n}{p^{(n-1)s}} - \frac{\alpha_2(p)^n (\overline{\beta_1(p)})^n}{p^{(n-1)s}} \right] \\
&= (\alpha_1(p) - \alpha_2(p))^{-1} \left(\overline{\beta_1(p)} - \overline{\beta_2(p)} \right)^{-1} \left[\alpha_1(p)\overline{\beta_1(p)} \left(1 - \alpha_1(p)\overline{\beta_1(p)} p^{-s} \right)^{-1} \right. \\
&+ \alpha_2(p)\overline{\beta_2(p)} \left(1 - \alpha_2(p)\overline{\beta_2(p)} p^{-s} \right)^{-1} - \alpha_1(p)\overline{\beta_2(p)} \left(1 - \alpha_1(p)\overline{\beta_2(p)} p^{-s} \right)^{-1} \\
&\left. - \alpha_2(p)\overline{\beta_1(p)} \left(1 - \alpha_2(p)\overline{\beta_1(p)} p^{-s} \right)^{-1} \right]
\end{aligned}$$

$$
\begin{aligned}
&= (\alpha_1(p)-\alpha_2(p))^{-1}\left(\overline{\beta_1(p)}-\overline{\beta_2(p)}\right)^{-1}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)^{-1}\\
&\cdot\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)^{-1}\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)^{-1}\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)^{-1}\\
&\cdot\Bigg[\alpha_1(p)\overline{\beta_1(p)}\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)\\
&+\alpha_2(p)\overline{\beta_2(p)}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)\\
&-\alpha_1(p)\overline{\beta_2(p)}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)\\
&-\alpha_2(p)\overline{\beta_1(p)}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)\Bigg].
\end{aligned}
$$

The term in the brackets simplifies to

$$\left(1-\alpha_1(p)\alpha_2(p)\overline{\beta_1(p)\beta_2(p)}p^{-2s}\right)(\alpha_1(p)-\alpha_2(p))\left(\overline{\beta_1(p)}-\overline{\beta_2(p)}\right),$$

because all of the other terms are annihilated by symmetry in $\alpha_1(p)$, $\alpha_2(p)$, $\overline{\beta_1(p)}$, and $\overline{\beta_2(p)}$. The Ramanujan-Petersson conjecture for holomorphic forms implies $\alpha_1(p)\alpha_2(p)\overline{\beta_1(p)\beta_2(p)}=1$. Therefore the corresponding factor above is $(1-p^{-2s})$. This factor cancels the local factor at p in the Euler product of $\zeta(2s)$, so that

$$\sum_{n\geq 0}\frac{a_f(p^n)\overline{a_g(p^n)}}{p^{ns}}=\prod_{1\leq j,\ell\leq 2}\left(1-\alpha_j(p)\overline{\beta_\ell(p)}p^{-s}\right)^{-1}.$$

Hence we have the following degree 4 Euler product:

$$L(s,f\otimes g)=\prod_p\prod_{1\leq j,\ell\leq 2}\left(1-\alpha_j(p)\overline{\beta_\ell(p)}p^{-s}\right)^{-1}.$$

The local factor at p is

$$L_p(s,f\otimes g)=\prod_{1\leq j,\ell\leq 2}\left(1-\alpha_j(p)\overline{\beta_\ell(p)}p^{-s}\right)^{-1},$$

with local roots $\alpha_j(p)\overline{\beta_\ell(p)}$.

The Integral Representation: Part I

We now look for an integral representation for $L(s,f\otimes g)$. Consider the following integral:

$$\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\,d\mu.$$

This will turn out to be a Mellin transform as we will soon see. Since f and g have exponential decay, this integral is locally absolutely uniformly convergent for $\sigma>1$ and hence defines an analytic function there.

We have

$$\begin{aligned}
\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\,d\mu &= \int_0^\infty \int_0^1 f(x+iy)\overline{g(x+iy)}y^{s+k}\,\frac{dx\,dy}{y^2} \\
&= \int_0^\infty \int_0^1 \sum_{n,m\geq 1} a_f(n)\overline{a_g(m)}(nm)^{\frac{k-1}{2}} e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^{s+k}\,\frac{dx\,dy}{y^2} \\
&= \int_0^\infty \sum_{n,m\geq 1} \int_0^1 a_f(n)\overline{a_g(m)}(nm)^{\frac{k-1}{2}} e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^{s+k}\,\frac{dx\,dy}{y^2} \quad \text{FTT} \\
&= \int_0^\infty \sum_{n\geq 1} a_f(n)\overline{a_g(n)}n^{k-1}e^{-4\pi ny}y^{s+k}\,\frac{dy}{y^2},
\end{aligned}$$

where the last line follows by Equation (10.1). Observe that this last integral is a Mellin transform. The rest is a computation:

$$\begin{aligned}
\int_0^\infty \sum_{n\geq 1} a_f(n)\overline{a_g(n)}n^{k-1}e^{-4\pi ny}y^{s+k}\,\frac{dy}{y^2} &= \sum_{n\geq 1} a_f(n)\overline{a_g(n)}n^{k-1}\int_0^\infty e^{-4\pi ny}y^{s+k}\,\frac{dy}{y^2} && \text{FTT} \\
&= \sum_{n\geq 1} \frac{a_f(n)\overline{a_g(n)}}{(4\pi)^{s+k-1}n^s}\int_0^\infty e^{-y}y^{s+k-1}\,\frac{dy}{y} && y\mapsto \frac{y}{4\pi n} \\
&= \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}}\sum_{n\geq 1}\frac{a_f(n)\overline{a_g(n)}}{n^s} \\
&= \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}}L(s,f\times g).
\end{aligned}$$

Rewriting, we have an integral representation

$$L(s,f\times g) = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)}\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\,d\mu.$$

We rewrite the integral as follows:

$$\begin{aligned}
\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\,d\mu &= \int_{\mathcal{F}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma z)\overline{g(\gamma z)}\mathrm{Im}(\gamma z)^{s+k}\,d\mu && \text{folding} \\
&= \int_{\mathcal{F}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^k\overline{j(\gamma,z)^k}f(z)\overline{g(z)}\mathrm{Im}(\gamma z)^{s+k}\,d\mu && \text{modularity} \\
&= \int_{\mathcal{F}} f(z)\overline{g(z)}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}|j(\gamma,z)|^{2k}\mathrm{Im}(\gamma z)^{s+k}\,d\mu \\
&= \int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\gamma z)^s\,d\mu \\
&= \int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k E(z,s)\,d\mu.
\end{aligned}$$

Note that $E(z,s)$ is the weight zero Eisenstein series on $\Gamma_1(1)\backslash\mathbb{H}$ at the ∞ cusp. Altogether, this gives the integral representation

$$L(s,f\times g) = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)}\int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k E(z,s)\,d\mu. \tag{13.11}$$

We cannot investigate the integral any further until we understand the Fourier-Whittaker series of $E(z,s)$ and obtain a functional equation. Therefore we will take a necessary detour and return to the integral after.

Explicit Fourier-Whittaker Series of Eisenstein Series

We will compute the Fourier-Whittaker series of $E(z,s)$. To do this we will need the following technical lemma:

Lemma 13.3.1. *For $\sigma > 1$ and $b \in \mathbb{Z}$,*

$$\sum_{m\geq 1} \frac{r(b,m)}{m^{2s}} = \begin{cases} \frac{\zeta(2s-1)}{\zeta(2s)} & \text{if } b = 0, \\ \frac{\sigma_{1-2s}(b)}{\zeta(2s)} & \text{if } b \neq 0. \end{cases}$$

Proof. If $\sigma > 1$ then the desired evaluation of the sum is locally absolutely uniformly convergent because the Riemann zeta function is in that region. Hence the sum will be too provided we prove the identity. Suppose $b = 0$. Then $r(0,m) = \varphi(m)$. Since $\varphi(m)$ is multiplicative we have

$$\sum_{m\geq 1} \frac{\varphi(m)}{m^{2s}} = \prod_p \left(\sum_{k\geq 0} \frac{\varphi(p^k)}{p^{k(2s)}} \right). \tag{13.12}$$

Recalling that $\varphi(p^k) = p^k - p^{k-1}$ for $k \geq 1$, make the following computation:

$$\begin{aligned} \sum_{k\geq 0} \frac{\varphi(p^k)}{p^{k(2s)}} &= 1 + \sum_{k\geq 1} \frac{p^k - p^{k-1}}{p^{k(2s)}} \\ &= \sum_{k\geq 0} \frac{1}{p^{k(2s-1)}} - \frac{1}{p} \sum_{k\geq 1} \frac{1}{p^{k(2s-1)}} \\ &= \sum_{k\geq 0} \frac{1}{p^{k(2s-1)}} - p^{-2s} \sum_{k\geq 0} \frac{1}{p^{k(2s-1)}} \\ &= (1 - p^{-2s}) \sum_{k\geq 0} \frac{1}{p^{k(2s-1)}} \\ &= \frac{1 - p^{-2s}}{1 - p^{-(2s-1)}}. \end{aligned} \tag{13.13}$$

Combining Equations (13.12) and (13.13) gives

$$\sum_{m\geq 1} \frac{\varphi(m)}{m^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)}.$$

Now suppose $b \neq 0$, Proposition 1.4.1 gives the first equality in the following chain:

$$\begin{aligned}\sum_{m\geq 1}\frac{r(b,m)}{m^{2s}} &= \sum_{m\geq 1} m^{-2s}\sum_{\ell|(b,m)}\ell\mu\left(\frac{m}{\ell}\right)\\ &= \sum_{\ell|b}\ell\sum_{m\geq 1}\frac{\mu(m)}{(m\ell)^{2s}}\\ &= \left(\sum_{\ell|b}\ell^{1-2s}\right)\left(\sum_{m\geq 1}\frac{\mu(m)}{m^{2s}}\right)\\ &= \sigma_{1-2s}(b)\sum_{m\geq 1}\frac{\mu(m)}{m^{2s}}\\ &= \frac{\sigma_{1-2s}(b)}{\zeta(2s)} \qquad \text{Proposition A.2.2.}\end{aligned}$$ □

We can now compute the Fourier-Whittaker series of $E(z,s)$:

Proposition 13.3.1. *The Fourier-Whittaker series of $E(z,s)$ is given by*

$$E(z,s) = y^s + y^{1-s}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} + \sum_{t\geq 1}\left(\frac{2\pi^s|t|^{s-\frac{1}{2}}\sigma_{1-2s}(t)}{\Gamma(s)\zeta(2s)}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|t|y)\right)e^{2\pi itx}.$$

Proof. Fix s with $\sigma > 1$. By the Bruhat decomposition for $\Gamma_1(1)$ and Remark 9.2.3, we have

$$E(z,s) = \mathrm{Im}(z)^s + \sum_{\substack{c\geq 1, d\in\mathbb{Z}\\(c,d)=1}}\frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}}.$$

Summing over all pairs $(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}$ with $c\geq 1$, $d\in\mathbb{Z}$, and $(c,d)=1$ is the same as summing over all triples (c,ℓ,r) with $c\geq 1$, $\ell\in\mathbb{Z}$, r taken modulo c, and $(r,c)=1$. This is seen by writing $d = c\ell + r$. Therefore

$$\sum_{\substack{c\geq 1, d\in\mathbb{Z}\\(c,d)=1}}\frac{\mathrm{Im}(z)^s}{|cx+icy+d|^{2s}} = \sum_{(c,\ell,r)}\frac{\mathrm{Im}(z)^s}{|cz+c\ell+r|^{2s}} = \sum_{\substack{c\geq 1\\ r \pmod c}}{}'\sum_{\ell\in\mathbb{Z}}\frac{\mathrm{Im}(z)^s}{|cz+c\ell+r|^{2s}}.$$

where on the right-hand side it is understood that we are summing over all triples (c,ℓ,r) with the prescribed properties. Now let

$$I_{c,r}(z,s) = \sum_{\ell\in\mathbb{Z}}\frac{\mathrm{Im}(z)^s}{|cz+c\ell+r|^{2s}}.$$

We apply the Poisson summation formula to $I_{c,r}(z,s)$. By the identity theorem it suffices to apply the Poisson summation formula for $z = iy$ with $y > 0$. So let $f(x)$ be given by

$$f(x) = \frac{y^s}{|cx+r+icy|^{2s}}.$$

Then $f(x)$ is absolutely integrable on $\mathbb{R}$ because it exhibits polynomial decay of order $\sigma > 1$. We compute the Fourier transform

$$\begin{aligned}
(\mathcal{F}f)(t) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx \\
&= \int_{-\infty}^{\infty} \frac{y^s}{|cx+r+icy|^{2s}} e^{-2\pi itx}\,dx \\
&= \int_{-\infty}^{\infty} \frac{y^s}{((cx+r)^2+(cy)^2)^s} e^{-2\pi itx}\,dx \\
&= e^{2\pi it\frac{r}{c}} \int_{-\infty}^{\infty} \frac{y^s}{((cx)^2+(cy)^2)^s} e^{-2\pi itx}\,dx && x \mapsto x - \frac{r}{c} \\
&= \frac{e^{2\pi it\frac{r}{c}}}{c^{2s}} \int_{-\infty}^{\infty} \frac{y^s}{(x^2+y^2)^s} e^{-2\pi itx}\,dx \\
&= \frac{e^{2\pi it\frac{r}{c}}}{c^{2s}} \int_{-\infty}^{\infty} \frac{y^{s+1}}{((xy)^2+y^2)^s} e^{-2\pi itxy}\,dx && x \mapsto xy \\
&= \frac{e^{2\pi it\frac{r}{c}}}{c^{2s}} \int_{-\infty}^{\infty} \frac{y^{1-s}}{(x^2+1)^s} e^{-2\pi itxy}\,dx.
\end{aligned}$$

Appealing to Appendix E.1 to compute this latter integral, we see that

$$(\mathcal{F}f)(t) = \begin{cases} \frac{y^{1-s}}{c^{2s}} \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} & \text{if } t = 0, \\ \frac{e^{2\pi it\frac{r}{c}}}{c^{2s}} \frac{2\pi^s|t|^{s-\frac{1}{2}}}{\Gamma(s)} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|t|y) & \text{if } t \neq 0. \end{cases}$$

By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z,s) = \frac{y^{1-s}}{c^{2s}} \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + \sum_{t\neq 0} \left(\frac{e^{2\pi it\frac{r}{c}}}{c^{2s}} \frac{2\pi^s|t|^{s-\frac{1}{2}}}{\Gamma(s)} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|t|y) \right) e^{2\pi itx}.$$

Substituting this back into the Eisenstein series gives a form of the Fourier-Whittaker series:

$$\begin{aligned}
E(z,s) &= y^s + \sum_{\substack{c\geq 1 \\ r \pmod c}}{}' \left(\frac{y^{1-s}}{c^{2s}} \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + \sum_{t\geq 1} \left(\frac{e^{2\pi it\frac{r}{c}}}{c^{2s}} \frac{2\pi^s|t|^{s-\frac{1}{2}}}{\Gamma(s)} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|t|y) \right) e^{2\pi itx} \right) \\
&= y^s + y^{1-s} \sum_{\substack{c\geq 1 \\ r \pmod c}}{}' \frac{1}{c^{2s}} \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + \sum_{t\geq 1} \left(\sum_{\substack{c\geq 1 \\ r \pmod c}}{}' \frac{e^{2\pi it\frac{r}{c}}}{c^{2s}} \frac{2\pi^s|t|^{s-\frac{1}{2}}}{\Gamma(s)} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|t|y) \right) e^{2\pi itx} \\
&= y^s + y^{1-s} \sum_{c\geq 1} \frac{r(0,c)}{c^{2s}} \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + \sum_{t\geq 1} \left(\sum_{c\geq 1} \frac{r(t,c)}{c^{2s}} \frac{2\pi^s|t|^{s-\frac{1}{2}}}{\Gamma(s)} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|t|y) \right) e^{2\pi itx}.
\end{aligned}$$

By applying Lemma 13.3.1 to compute the Dirichlet series of Ramanujan sums, we obtain the desired Fourier-Whittaker series:

$$E(z,s) = y^s + y^{1-s} \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} + \sum_{t\geq 1} \left(\frac{2\pi^s|t|^{s-\frac{1}{2}}\sigma_{1-2s}(t)}{\Gamma(s)\zeta(2s)} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|t|y) \right) e^{2\pi itx}.$$

□

Having computed the Fourier-Whittaker series, we would like to obtain a functional equation for $E(z,s)$ under the symmetry $s \mapsto 1-s$. To this end, we define $E^*(z,s)$ by

$$E^*(z,s) = \Lambda(2s,\zeta)E(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s).$$

From Proposition 13.3.1, the Fourier coefficients $a^*(n,y,s)$ of $E^*(z,s)$ in the Fourier series

$$E^*(z,s) = a^*(0,y,s) + \sum_{n\neq 0} a^*(n,y,s)e^{2\pi inx},$$

are given by

$$a^*(n,y,s) = \begin{cases} y^s\pi^{-s}\Gamma(s)\zeta(2s) + y^{1-s}\pi^{-\left(s-\frac{1}{2}\right)}\Gamma(s-\frac{1}{2})\zeta(2s-1) & \text{if } n=0, \\ 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y) & \text{if } n\neq 0. \end{cases}$$

We can now derive a functional equation for $E^*(z,s)$. Using the definition and functional equation for $\Lambda(2s-1,\zeta)$, we can rewrite the second term in the constant coefficient to get

$$a^*(n,y,s) = \begin{cases} y^s\Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta) & \text{if } n=0, \\ 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y) & \text{if } n\neq 0. \end{cases} \tag{13.14}$$

Now observe that the constant coefficient is invariant under $s \mapsto 1-s$. Each $n \neq 0$ coefficient is also invariant under $s \mapsto 1-s$. To see this we will use two facts. First, from Appendix B.6, $K_s(y)$ is invariant under $s \mapsto -s$ and so $K_{s-\frac{1}{2}}(2\pi|n|y)$ is invariant as $s \mapsto 1-s$. Second, for $n \neq 0$ we have

$$|n|^{s-\frac{1}{2}}\sigma_{1-2s}(n) = |n|^{\frac{1}{2}-s}|n|^{2s-1}\sigma_{1-2s}(n) = |n|^{\frac{1}{2}-s}|n|^{2s-1}\sum_{d|n} d^{1-2s} = |n|^{\frac{1}{2}-s}\sum_{d|n}\left(\frac{|n|}{d}\right)^{2s-1} = |n|^{\frac{1}{2}-s}\sigma_{2s-1}(n),$$

where the second to last equality follows by writing $|n|^{2s-1} = \left(\frac{|n|}{d}\right)^{2s-1} d^{2s-1}$ for each $d \mid n$. These two facts together give the invariance of the $n \neq 0$ coefficients under $s \mapsto 1-s$. Altogether, we have shown the following functional equation for $E^*(z,s)$:

$$E^*(z,s) = E^*(z,1-s).$$

We can now obtain meromorphic continuation of $E^*(z,s)$ in s to all of $\mathbb{C}$ for any $z \in \mathbb{H}$. We first write $E^*(z,s)$ as a Fourier-Whittaker series using Equation (13.14):

$$E^*(z,s) = y^s\Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta) + \sum_{n\neq 0} 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}.$$

Since $\Lambda(2s,\zeta)$ is meromorphic on $\mathbb{C}$, the constant term of $E^*(z,s)$ is as well. To finish the meromorphic continuation of $E^*(z,s)$ it now suffices to show

$$\sum_{n\neq 0} 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx},$$

is meromorphic on $\mathbb{C}$. We will actually prove it is locally absolutely uniformly convergent. So let K be a compact subset of $\mathbb{C}$. Then we have to show $E^*(z,s)$ is absolutely convergent on K for any $z \in \mathbb{H}$. To

achieve this we need two bounds, one for $\sigma_{1-2s}(n)$ and one for $K_{s-\frac{1}{2}}(2\pi|n|y)$. For the first bound, we use the estimate $\sigma_0(n) \ll_\varepsilon n^\varepsilon$ (recall Proposition A.3.1) to derive that

$$\sigma_{1-2s}(n) = \sum_{d|n} d^{1-2s} \ll \sigma_0(n) n^{1-2s} \ll_\varepsilon n^{1-2s+\varepsilon}.$$

For the second bound, Lemma B.6.2 implies

$$K_{s-\frac{1}{2}}(2\pi|n|y) \ll e^{-2\pi|n|y}.$$

Using these two bounds, we have

$$\sum_{n\neq 0} 2|n|^{s-\frac{1}{2}} \sigma_{1-2s}(n) \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x} \ll_\varepsilon \sum_{n\geq 1} n^{\frac{1}{2}-s+\varepsilon} \sqrt{y} e^{-2\pi n y}. \tag{13.15}$$

This latter series is absolutely uniformly convergent on K by the Weierstrass M-test. Therefore $E^*(z,s)$ is absolutely convergent on K for any $z \in \mathbb{H}$ and the meromorphic continuation to $\mathbb{C}$ follows. It remains to investigate the poles and residues. We will accomplish this from direct inspection of the Fourier-Whittaker coefficients:

Proposition 13.3.2. *$E^*(z,s)$ has simple poles at $s=0$ and $s=1$, and*

$$\operatorname*{Res}_{s=0} E^*(z,s) = -\frac{1}{2} \quad \textit{and} \quad \operatorname*{Res}_{s=1} E^*(z,s) = \frac{1}{2}.$$

Proof. Since the constant term in the Fourier-Whittaker series of $E^*(z,s)$ is the only non-holomorphic term, poles of $E^*(z,s)$ can only come from that term. So we are reduced to understanding the poles of

$$y^s \Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta). \tag{13.16}$$

Notice $\Lambda(2s,\zeta)$ has simple poles at $s=0$, $s=\frac{1}{2}$ (one from the Riemann zeta function and one from the gamma factor) and no others. It follows that $E^*(z,s)$ has a simple pole at $s=0$ coming from the y^s term in Equation (13.16), and by the functional equation there is also a pole at $s=1$ coming from the y^{1-s} term. At $s=\frac{1}{2}$, both terms in Equation (13.16) have simple poles and we will show that the singularity there is removable. Since the residues of $\Lambda(s,\zeta)$ at its poles are both 1, we have $\operatorname{Res}_{s=\frac{1}{2}} \Lambda(2s) = \frac{1}{2}$ and $\operatorname{Res}_{s=\frac{1}{2}} \Lambda(2(1-s)) = -\frac{1}{2}$. Then

$$\operatorname*{Res}_{s=\frac{1}{2}} E^*(z,s) = \operatorname*{Res}_{s=\frac{1}{2}} y^s \Lambda(2s,\zeta) + \operatorname*{Res}_{s=\frac{1}{2}} y^{1-s}\Lambda(2(1-s),\zeta) = \frac{1}{2}y^{\frac{1}{2}} - \frac{1}{2}y^{\frac{1}{2}} = 0.$$

Hence the singularity at $s=\frac{1}{2}$ is removable. As for the residues at $s=0$ and $s=1$, the functional equation implies that they are negatives of each other. So it suffices to compute the residue at $s=0$. Recall $\zeta(0) = -\frac{1}{2}$ and $\operatorname{Res}_{s=0}\Gamma(s) = 1$. Then together we find

$$\operatorname*{Res}_{s=0} E^*(z,s) = \operatorname*{Res}_{s=0} y^s \Lambda(2s,\zeta) = -\frac{1}{2}.$$

□

This completes our study of $E(z,s)$.

The Integral Representation: Part II

We can now continue with the Rankin-Selberg convolution $L(s, f \otimes g)$. Writing Equation (13.11) in terms of $E^*(z,s)$ and $L(s, f \otimes g)$ results in the integral representation

$$L(s, f \otimes g) = \frac{(4\pi)^{s+k-1}\pi^s}{\Gamma(s+k-1)\Gamma(s)} \int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k E^*(z,s)\, d\mu. \tag{13.17}$$

This integral representation will give analytic continuation. To see this, note that the gamma functions are analytic for $\sigma < 0$. By the functional equation for $E^*(z,s)$, the integral is invariant as $s \mapsto 1-s$. These two facts together give analytic continuation to $\mathbb{C}$ outside of the critical strip. The continuation inside of the critical strip will be meromorphic because of the poles of $E^*(z,s)$. To see this, substituting the Fourier-Whittaker series for $E^*(z,s)$ into Equation (13.17) gives

$$\begin{aligned} L(s, f \otimes g) = \frac{(4\pi)^{s+k-1}\pi^s}{\Gamma(s+k-1)\Gamma(s)} \Bigg[&\int_{\mathcal{F}} f(x+iy)\overline{g(x+iy)} y^k (y^s \Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta)) \frac{dx\,dy}{y^2} \\ &+ \int_{\mathcal{F}} f(x+iy)\overline{g(x+iy)} y^k \sum_{n\neq 0} 2|n|^{s-\frac{1}{2}} \sigma_{1-2s}(n)\sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x} \frac{dx\,dy}{y^2} \Bigg], \end{aligned}$$

and we are reduced to showing that both integrals are locally absolutely uniformly convergent in the critical strip and distance ε away from the poles of $E^*(z,s)$. Indeed, the first integral is locally absolutely uniformly convergent in this region since the exponential decay of f and g imply that the integrand is bounded and we are integrating over a region of finite volume. As for the second integral, since s is in the critical strip $0 \leq \sigma \leq 1$ and so Equation (13.15) implies that it is

$$O_\varepsilon \left(\int_{\mathcal{F}} f(x+iy)\overline{g(x+iy)} y^k \sum_{n\geq 1} n^{\frac{1}{2}+\varepsilon}\sqrt{y} e^{-2\pi n y} \frac{dx\,dy}{y^2} \right).$$

But then

$$\sum_{n\geq 1} n^{\frac{1}{2}+\varepsilon} e^{-2\pi n y} = O(e^{-2\pi y}),$$

because each term is of smaller order than the next so that the series is bounded by a constant times its first term. It follows that the first sum has exponential decay. Together with the exponential decay of f and g, the integrand is bounded and thus is locally absolutely uniformly convergent because we are integrating over a region of finite volume. The meromorphic continuation to the critical strip and hence to all of $\mathbb{C}$ follows. In particular, $L(s, f \otimes g)$ has at most simple poles at $s = 0$ and $s = 1$. Actually, there is no pole at $s = 0$. Indeed, $\Gamma(s)$ has a simple pole at $s = 0$ and therefore its reciprocal has a simple zero. This cancels the simple pole at $s = 0$ coming from $E^*(z,s)$ and therefore $L(s, f \otimes g)$ is has a removable singularity at $s = 0$. So there is at worst a simple pole at $s = 1$.

The Functional Equation

An immediate consequence of applying the symmetry $s \mapsto 1-s$ to Equation (13.17) is the following functional equation:

$$\frac{\Gamma(s+k-1)\Gamma(s)}{(4\pi)^{s+k-1}\pi^s} L(s, f \otimes g) = \frac{\Gamma((1-s)+k-1)\Gamma(1-s)}{(4\pi)^{(1-s)+k-1}\pi^{1-s}} L(1-s, f \otimes g).$$

Applying the Legendre duplication formula, we see that

$$\begin{aligned}\frac{\Gamma(s+k-1)\Gamma(s)}{(4\pi)^{s+k-1}\pi^s} &= \frac{2^{2s+k-3}}{(4\pi)^{s+k-1}\pi^{s+1}}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\\ &= \frac{1}{2^{k+1}\pi^k}\pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right).\end{aligned}$$

The constant factor in front is independent of s and can therefore be canceled in the functional equation. We identify the gamma factor as:

$$\gamma(s, f\otimes g) = \pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right),$$

with $\mu_{1,1} = k-1$, $\mu_{2,2} = k$, $\mu_{1,2} = 0$, and $\mu_{2,1} = 1$ the local roots at infinity. The completed L-function is

$$\Lambda(s, f\otimes g) = \pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L(s, f\otimes g),$$

so the conductor is $q(f\otimes g) = 1$ and no primes ramify. Then

$$\Lambda(s, f\otimes g) = \Lambda(1-s, f\otimes g),$$

is the functional equation of $L(s, f\otimes g)$. In particular, the root number $\varepsilon(f\otimes g) = 1$, and $L(s, f\otimes g)$ is self-dual. We can now show that $L(s, f\otimes g)$ is of order 1. Since the possible pole at $s = 1$ is simple, multiplying by $(s-1)$ clears the possible polar divisor. As the integral in Equation (13.17) is locally absolutely uniformly convergent, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s, f\otimes g)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So the reciprocal of the gamma factor is also of order 1. Then we find that

$$(s-1)L(s, f\otimes g) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

Thus $(s-1)L(s, f\otimes g)$ is of order 1, and so $L(s, f\otimes g)$ is as well after removing the polar divisor. We now compute the residue of $L(s, f\otimes g)$ at $s = 1$. As $V = \frac{\pi}{3}$, Proposition 13.3.2 implies

$$\operatorname*{Res}_{s=1} L(s, f\otimes g) = \frac{4^k\pi^{k+1}}{\Gamma(k)}\int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k(\operatorname*{Res}_{s=1} E^*(z,s))\,d\mu = \frac{4^k\pi^{k+1}V}{2\Gamma(k)}\langle f, g\rangle.$$

By Theorem 10.6.1, $\langle f, g\rangle \neq 0$ if and only if $f = g$. Therefore the pole at $s = 1$ is a removable singularity unless $f = g$. We summarize all of our work into the following theorem:

Theorem 13.3.1. *Let $f, g \in \mathcal{S}_k(1)$ be primitive Hecke eigenforms and for every prime p let $\alpha_1(p)$ and $\alpha_2(p)$ be the p-th Hecke roots of f while $\beta_1(p)$ and $\beta_2(p)$ are the p-th Hecke roots of g. Then $L(s, f\otimes g)$ is the Rankin-Selberg convolution of $L(s, f)$ and $L(s, g)$ and is a Selberg class L-function with degree 4 Euler product given by*

$$L(s, f\otimes g) = \prod_p \prod_{1\le j,\ell\le 2}\left(1-\alpha_j(p)\overline{\beta_\ell(p)}p^{-s}\right)^{-1}.$$

Moreover, it admits meromorphic continuation to $\mathbb{C}$, possesses the functional equation

$$\pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L(s, f\otimes g) = \Lambda(s, f\otimes g) = \Lambda(1-s, f\otimes g),$$

and has a simple pole at $s = 1$ of residue $\frac{4^k\pi^{k+1}V}{2\Gamma(k)}\langle f, g\rangle$ provided $f = g$.

Beyond Level 1

The Rankin-Selberg method is much more complicated for arbitrary primitive Hecke or Hecke-Maass eigenforms, but the argument is essentially the same. Let f and g both be primitive Hecke or Heck-Maass eigenforms with Fourier or Fourier-Whittaker coefficients $a_f(n)$ and $a_g(n)$ respectively. We suppose f has weight k/type ν, level N, and character χ, and g has weight ℓ/type η, level M, and character ψ. The L-series $L(s, f \times g)$ of f and g is defined by

$$L(s, f \times g) = \sum_{n \geq 1} \frac{a_{f\times g}(n)}{n^s} = \sum_{n \geq 1} \frac{a_f(n)\overline{a_g(n)}}{n^s}.$$

The **Rankin-Selberg convolution** $L(s, f \otimes g)$ of f and g is defined by

$$L(s, f \otimes g) = \sum_{n \geq 1} \frac{a_{f\otimes g}(n)}{n^s} = L(2s, \chi\overline{\psi})L(s, f \times g),$$

where $a_{f\otimes g}(n) = \sum_{n=m\ell^2} \chi\overline{\psi}(\ell^2)a_f(m)\overline{a_g(m)}$. The following generalization was obtained by Jacquet in the 1970's as a consequence of a more general framework (see [JL70, Jac72]):

Theorem 13.3.2. *Let f and b both be primitive Hecke or Heck-Maass eigenforms. Also suppose the following:*

(i) f has weight k/type ν, level N, and character χ.

(ii) g has weight ℓ/type η, level M, and character ψ.

(iii) The Ramanujan-Petersson conjecture for Maass forms holds if f or g are Heck-Maass eigenforms.

Then $L(s, f \otimes g)$ is the Rankin-Selberg convolution of $L(s, f)$ and $L(s, g)$ and is a Selberg class L-function.

We make a few remarks about how Theorem 13.3.2 could be proved analogously to the case of two holomorphic cusp forms of level 1. Local absolute uniform convergence for $\sigma > 1$ are proved in the exactly the same way as we have described. The argument for the Euler product is also similar. However, if either $N > 1$ or $M > 1$, the computation becomes more difficult since the local factors at p for $p \mid NM$ change. Moreover, the situation is increasingly complicated if $(N, M) > 1$. The integral representation has a similar argument, but if the weights/types are distinct the resulting Eisenstein series becomes more complicated. In particular, the Eisenstein series is on $\Gamma_0(NM)\backslash\mathbb{H}$ and if $NM > 1$ then there is more than just the cusp at ∞. Therefore the functional equation of the Eisenstein series at the ∞ cusp will reflect into a linear combination of Eisenstein series at the other cusps. This requires the Fourier-Whittaker series of all of these Eisenstein series. Moreover, this procedure can be generalized to remove the primitive Hecke and/or primitive Hecke-Maass eigenform conditions by taking linear combinations, but we won't attempt discussing this further.

13.4 Strong Multiplicity One

Recall that multiplicity one determines and eigenform, up to a constant, by its Hecke eigenvalues at primes. Using Rankin-Selberg convolution L-functions, we can prove **strong multiplicity one** for holomorphic or Maass forms which says that eigenforms are determined by their Hecke eigenvalues at all but finitely many primes:

Theorem (Strong multiplicity one, holomorphic and Maass). *Let f and g both be Hecke or Hecke-Maass eigenforms. If they have the same Hecke eigenvalues for all but finitely many primes p then $f = g$.*

Proof. By Theorems 10.6.1 and 11.7.1 we may assume that f and g are primitive Hecke eigenforms. Denote the Hecke eigenvalues by $\lambda_f(n)$ and $\lambda_g(n)$ respectively. Let S be the set the primes for which $\lambda_f(p) \neq \lambda_g(p)$ including the primes that ramify for $L(s, f)$ and $L(s, g)$. Then S is finite by assumption. As the local factors of $L(s, f \otimes g)$ are holomorphic and nonzero at $s = 1$, the order of the pole of $L(s, f \otimes g)$ is the same as the order of the pole of

$$L(s, f \otimes g) \prod_{p \in S} L_p(s, f \otimes g)^{-1} = \prod_{p \notin S} L_p(s, f \otimes g).$$

But as $\lambda_f(p) = \lambda_g(p)$ for all $p \notin S$, we have

$$\prod_{p \notin S} L_p(s, f \otimes g) = \prod_{p \notin S} L_p(s, f \otimes f),$$

and so

$$L(s, f \otimes g) \prod_{p \in S} L_p(s, f \otimes g)^{-1} = L(s, f \otimes f) \prod_{p \in S} L_p(s, f \otimes f)^{-1}.$$

Since $L(s, f \otimes f)$ has a simple pole at $s = 1$, it follows that $L(s, f \otimes g)$ does too. But then $f = g$. □

13.5 The Ramanujan-Petersson Conjecture on Average

Rankin-Selberg convolution L-functions can also be used to obtain the Ramanujan-Petersson conjecture on average:

Proposition 13.5.1. *Let f be a primitive Hecke or Hecke-Maass eigenform. Then for any $X > 0$, we have*

$$\sum_{n \leq X} |a_f(n)| \ll_\varepsilon X^{1+\varepsilon},$$

Proof. By the Cauchy-Schwarz inequality,

$$\left(\sum_{n \leq X} |a_f(n)| \right)^2 \leq X \sum_{n \leq X} |a_f(n)|^2, \tag{13.18}$$

The Rankin-Selberg square $L(s, f \otimes f)$ is locally absolutely uniformly convergent for $\sigma > \frac{3}{2}$. Therefore it still admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$. By Landau's theorem, the abscissa of absolute convergence of $L(s, f \otimes f)$, and hence $L(s, f \times f)$ too, is 1. By Proposition 2.1.4, we have

$$\sum_{n \leq X} |a_f(n)|^2 \ll_\varepsilon X^{1+\varepsilon}.$$

Substituting this bound into Equation (13.18), we obtain

$$\left(\sum_{n \leq X} |a_f(n)| \right)^2 \ll_\varepsilon X^{2+\varepsilon},$$

and taking the square root yields

$$\sum_{n \leq X} |a_f(n)| \ll_\varepsilon X^{1+\varepsilon}.$$

□

The bound in Proposition 13.5.1 should be compared with the implication $a_f(n) \ll_\varepsilon n^\varepsilon$ that follows from the corresponding Ramanujan-Petersson conjecture. While Proposition 13.5.1 is not useful in the holomorphic form case, it is in the Maass form case. Indeed, recall that if f is a primitive Hecke-Maass eigenform we needed to assume the Ramanujan-Petersson conjecture for Maass forms to ensure $a_f(n) \ll_\varepsilon n^\varepsilon$ so that $L(s, f)$ was locally absolutely uniformly convergent for $\sigma > 1$. However, Propositions 2.1.5 and 13.5.1 now together imply $L(s, f)$ is locally absolutely uniformly convergent for $\sigma > 1$ without this assumption. Often Proposition 13.5.1 is all that is needed for additional applications instead of outright assuming the Ramanujan-Petersson conjecture for Maass forms.

Part V

Subconvexity and Moments

Chapter 14

Subconvexity Results

We discuss a classical subconvexity result for Dirichlet L-functions due to Burgess which relies upon subtle estimates for sums of Dirichlet characters. Necessary to this discussion is the infamous Pólya-Vinogradov inequality which we also prove.

14.1 The Burgess Bound for Dirichlet L-functions

Let χ be a Dirichlet character modulo m. We call the sum

$$S_\chi(M,N) = \sum_{M+1\leq n\leq M+N} \chi(n),$$

for $M \geq 0$ and $N \geq 1$, the **character sum** of χ. If $M = 0$, we write $S_\chi(N) = S_\chi(0,N)$ for simplicity. In the case χ is primitive of conductor $q > 1$, an infamous subconvexity estimate for the Dirichlet L-function $L(s,\chi)$ in the q-aspect was achieved by Burgess in the 1960's (see [Bur63]). The key idea of his argument lies in improved bounds for character sums in certain ranges of M and N relative to powers of q which we will describe. On the one hand, since $|\chi(n)| \leq 1$ for all $n \geq 1$, we have the trivial bound

$$S_\chi(M,N) \ll N. \tag{14.1}$$

On the other hand, by the Dirichlet orthogonality relations (namely Corollary 1.3.1 i) and that χ is non-principal, we have

$$S_\chi(M,N) \ll m. \tag{14.2}$$

Note that Equation (14.2) is only an improvement upon Equation (14.1) when $m \ll N$. It turns out that a much shaper bound than Equation (14.2) can be obtained with very little work. We will first require a small lemma:

Lemma 14.1.1. *Let $f(x)$ be a integrable convex function on $\left(a-\frac{b}{2}, a+\frac{b}{2}\right)$ for some $a \in \mathbb{R}$ and $b > 0$. Then*

$$f(a) \leq \frac{1}{b}\int_{a-\frac{b}{2}}^{a+\frac{b}{2}} f(x)\,dx.$$

Proof. By the definition of convexity,

$$f(a) = f\left(\frac{1}{2}x + \frac{1}{2}(2a-x)\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(2a-x).$$

Integrating the left-hand side over $\left(a-\frac{b}{2}, a+\frac{b}{2}\right)$ yields

$$\int_{a-\frac{b}{2}}^{a+\frac{b}{2}} f(a)\,dx = bf(a),$$

while integrating the right-hand side over $\left[a-\frac{b}{2}, a+\frac{b}{2}\right]$ gives

$$\int_{a-\frac{b}{2}}^{a+\frac{b}{2}} \left(\frac{1}{2}f(x)+\frac{1}{2}f(2a-x)\right)\,dx = \frac{1}{2}\int_{a-\frac{b}{2}}^{a+\frac{b}{2}} f(x)\,dx + \int_{a-\frac{b}{2}}^{a+\frac{b}{2}} \frac{1}{2}f(2a-x)\,dx = \int_{a-\frac{b}{2}}^{a+\frac{b}{2}} f(x)\,dx,$$

upon making the change of variables $x \mapsto 2a-x$ in the second integral. Hence

$$bf(a) \leq \int_{a-\frac{b}{2}}^{a+\frac{b}{2}} f(x)\,dx,$$

which is equivalent to the claim. □

We can now improve upon Equation (14.2). This result is known as the **Pólya-Vinogradov inequality** as it was proved independently by Pólya and Vinogradov in 1918 (see [Pól18] and [Vin18]).

Theorem (Pólya-Vinogradov inequality). *Let χ be a non-principal Dirichlet character modulo m. Then for any $M \geq 0$ and $N \geq 1$,*

$$S_\chi(M,N) \ll \sqrt{m}\log(m).$$

Proof. First suppose χ is primitive of conductor $q > 1$. Using Corollary 1.4.1, we have

$$S_\chi(M,N) = \frac{1}{\tau(\overline{\chi})}\sum_{a \pmod q} \overline{\chi}(a) \sum_{M+1\leq n\leq M+N} e^{\frac{2\pi i a n}{q}}.$$

The inner sum is geometric and evaluates to

$$\sum_{M+1\leq n\leq M+N} e^{\frac{2\pi i a n}{q}} = e^{\frac{2\pi i a(M+1)}{q}}\left(\frac{1-e^{\frac{2\pi i a N}{q}}}{1-e^{\frac{2\pi i a}{q}}}\right) = e^{\frac{2\pi i a\left(M+\frac{N-1}{2}\right)}{q}}\left(\frac{e^{\frac{\pi i a N}{q}}-e^{-\frac{\pi i a N}{q}}}{e^{\frac{\pi i a}{q}}-e^{-\frac{\pi i a}{q}}}\right) = e^{\frac{2\pi i a\left(M+\frac{N-1}{2}\right)}{q}}\frac{\sin\left(\frac{\pi N a}{q}\right)}{\sin\left(\frac{\pi a}{q}\right)},$$

where in the last equality we have made use of the formula $\sin(x) = \frac{e^{ix}+e^{-ix}}{2i}$. Then Theorem 1.4.1, gives

$$S_\chi(M,N) \ll \frac{1}{\sqrt{q}}\sum_{1\leq a\leq q-1}\frac{1}{\sin\left(\frac{\pi a}{q}\right)},$$

where we recall that $\overline{\chi}(0) = 0$. Now the function $\frac{1}{\sin(\pi x)}$ is convex and integrable on $(0,1)$ (its second derivative is $\pi^2\frac{1+\cos^2(\pi x)}{\sin^3(\pi x)} > 0$ on this interval), so from Lemma 14.1.1 we get

$$\frac{1}{\sqrt{q}}\sum_{1\leq a\leq q-1}\frac{1}{\sin\left(\frac{\pi a}{q}\right)} \leq \frac{1}{\sqrt{q}}\sum_{1\leq a\leq q-1} q\int_{\frac{a}{q}-\frac{1}{2q}}^{\frac{a}{q}+\frac{1}{2q}}\frac{1}{\sin(\pi x)}\,dx = \sqrt{q}\int_{\frac{1}{2q}}^{1-\frac{1}{2q}}\frac{1}{\sin(\pi x)}\,dx = 2\sqrt{q}\int_{\frac{1}{2q}}^{\frac{1}{2}}\frac{1}{\sin(\pi x)}\,dx,$$

where the last equality holds because $\sin(\pi x) = \sin(\pi - \pi x) = \sin(\pi(1-x))$ so that the integral of $\frac{1}{\sin(\pi x)}$ is symmetric over $\left(\frac{1}{2q}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1 - \frac{1}{2q}\right)$. Now $\sin(\pi x) \geq 2x$ on the interval $[0, \frac{1}{2}]$ (because $\sin(\pi x) = 2x$ at the boundary and both functions are increasing on the interior), so that

$$2\sqrt{q} \int_{\frac{1}{2q}}^{\frac{1}{2}} \frac{1}{\sin(\pi x)}\, dx \leq q \int_{\frac{1}{2q}}^{\frac{1}{2}} \frac{1}{x}\, dx \ll \sqrt{q}\log(q).$$

Therefore

$$S_\chi(M, N) \ll \sqrt{q}\log(q),$$

as desired. This proves the bound in the case χ is primitive. Now suppose χ is induced by the primitive character $\widetilde{\chi}$ of conductor $q > 1$. Then $q \mid m$, and so we may write $m = kq$ for some $k \geq 1$. Rewrite the sum in terms of $\widetilde{\chi}$ as follows:

$$\begin{aligned}
S_\chi(M, N) &= \sum_{\substack{M+1 \leq n \leq M+N \\ (n,k)=1}} \widetilde{\chi}(n) \\
&= \sum_{M+1 \leq n \leq M+N} \widetilde{\chi}(n) \sum_{d \mid (n,k)} \mu(d) && \text{Proposition A.2.1} \\
&= \sum_{d \mid k} \mu(d) \sum_{\substack{M+1 \leq n \leq M+N \\ d \mid n}} \widetilde{\chi}(n) \\
&= \sum_{d \mid k} \mu(d) \sum_{\frac{M+1}{d} \leq n \leq \frac{M+N}{d}} \widetilde{\chi}(dn) && n \to dn \\
&= \sum_{d \mid k} \mu(d)\widetilde{\chi}(d) \sum_{\left\lfloor \frac{M+1}{d} \right\rfloor \leq n \leq \left\lfloor \frac{M+N}{d} \right\rfloor} \widetilde{\chi}(n).
\end{aligned}$$

The inner sum is $O(\sqrt{q}\log(q))$ by the primitive case and so

$$S_\chi(M, N) \ll \sqrt{q}\log(q) \sum_{d \mid k} |\mu(d)| \ll 2^{\omega(k)}\sqrt{q}\log(q).$$

Since $2^{\omega(k)} \ll \sigma_0(k) \ll_\varepsilon k^\varepsilon$ (see Proposition A.3.1) it follows that $2^{\omega(k)} \ll \sqrt{k}$ upon taking $\varepsilon = \frac{1}{2}$. This bound together with $\log(q) \leq \log(m)$ gives

$$S_\chi(M, N) \ll \sqrt{m}\log(m),$$

as claimed. □

The Pólya-Vinogradov inequality greatly improves upon Equation (14.2) and is particularly useful when N is much larger than m. A slight improvement was made in 1977 by Montgomery and Vaughn under the Riemann hypothesis for Dirichlet L-functions (see [MV77]):

Theorem 14.1.1. *Let χ be a non-principal Dirichlet character modulo m. Then for any $M \geq 0$ and $N \geq 1$,*

$$S_\chi(M, N) \ll \sqrt{m}\log\log(m),$$

provided the Riemann hypothesis for Dirichlet L-functions holds.

While Theorem 14.1.1 is not much of an improvement from the Pólya-Vinogradov inequality, it is sharp due to a result of Paley in 1932 (see [Pal32]). This means that, quite remarkably, the Pólya-Vinogradov inequality is almost optimal. It is also useful to have an estimate that is sharper when N is small compared to m. In 1963, Burgess made progress in this direction by proving the following result (see [Bur63] for a proof which is a generalization of his 1962 papers [Bur62a] and [Bur62b]):

Theorem 14.1.2. *Let χ be a non-principal Dirichlet character modulo $m > 1$ and $M \geq 0$, $N \geq 1$, and $r \geq 1$ be integers. Then*

$$S_\chi(M, N) \ll_\varepsilon N^{1-\frac{1}{r}} m^{\frac{r+1}{4r^2}+\varepsilon},$$

provided m is cube-free or $r = 2$.

Theorem 14.1.2 can be thought of as a blend between Equation (14.1) and the Pólya-Vinogradov inequality. Despite the proof being reasonably short, we do not provide the argument as it requires some machinery beyond the scope of this text. More importantly, Burgess used Theorem 14.1.2 in conjunction with the Pólya-Vinogradov inequality to prove a subconvexity estimate for Dirichlet L-functions in the conductor-aspect:

Theorem 14.1.3. *Let χ be a primitive Dirichlet character of conductor $q > 1$. Then for $0 < \sigma < 1$, we have*

$$L(s, \chi) \ll_{t,\varepsilon} \begin{cases} q^{\frac{4-5\sigma}{8}+\varepsilon} & \text{if } 0 < \sigma \leq \frac{1}{2}, \\ q^{\frac{3-3\sigma}{8}+\varepsilon} & \text{if } \frac{1}{2} \leq \sigma < 1. \end{cases}$$

In particular,

$$L\left(\frac{1}{2} + it, \chi\right) \ll_\varepsilon q^{\frac{3}{16}+\varepsilon}.$$

Proof. Let $0 < \sigma < 1$. Summation by parts (see Corollary B.3.1) gives

$$\sum_{n \leq N} \frac{\chi(n)}{n^s} = S_\chi(N)N^{-s} + \sum_{n \leq N-1} S_\chi(n)(n^{-s} - (n+1)^{-s}),$$

for any $N \geq 1$. By applying the mean value theorem to $f(x) = \frac{1}{x^s}$ on the interval $[n, n+1]$ we have $(n^{-s} - (n+1)^{-s}) \ll_t \frac{1}{n^{\sigma+1}}$, and thus

$$\sum_{n \leq N} \frac{\chi(n)}{n^s} \ll_t |S_\chi(N)|N^{-\sigma} + \sum_{n \leq N-1} \frac{|S_\chi(n)|}{n^{\sigma+1}}.$$

As $|S_\chi(N)||N^{-\sigma}| \to 0$ as $N \to \infty$ (recall Equation (14.2)), taking the limit as $N \to \infty$ results in

$$L(s, \chi) \ll_t \sum_{n \geq 1} \frac{|S_\chi(n)|}{n^{\sigma+1}},$$

where the estimate follows Letting $M > 2$, $N \geq 1$, and decompose the right-hand side, we obtain

$$L(s, \chi) \ll_t \sum_{1 \leq M-1} \frac{|S_\chi(n)|}{n^{\sigma+1}} + \sum_{M \leq n \leq N} \frac{|S_\chi(n)|}{n^{\sigma+1}} + \sum_{n \geq N+1} \frac{|S_\chi(n)|}{n^{\sigma+1}}.$$

Applying Equation (14.1) to the first character sum, Theorem 14.1.2 (in the case $r = 2$) to the second character sum, and the Pólya-Vinogradov inequality to the third character sum, gives

$$L(s, \chi) \ll_{t,\varepsilon} \sum_{1 \leq M-1} \frac{1}{n^\sigma} + q^{\frac{3}{16}+\varepsilon} \sum_{M \leq n \leq N} \frac{1}{n^{\sigma+\frac{1}{2}}} + q^{\frac{1}{2}+\varepsilon} \sum_{n \geq N+1} \frac{1}{n^{\sigma+1}},$$

where we have used that $\log(q) \ll_\varepsilon q^\varepsilon$. Upper bounding the first and third sums with an integral and integrating, we get

$$L(s,\chi) \ll_{t,\varepsilon} M^{1-\sigma} + q^{\frac{3}{16}+\varepsilon} \sum_{M \le n \le N} \frac{1}{n^{\sigma+\frac{1}{2}}} + q^{\frac{1}{2}+\varepsilon} N^{-\sigma},$$

since $0 < \sigma < 1$. Performing the same argument with the remaining sum, we see that

$$\sum_{M \le n \le N} \frac{1}{n^{\sigma+\frac{1}{2}}} \ll \begin{cases} N^{\frac{1}{2}-\sigma} & \text{if } 0 < \sigma < \frac{1}{2}, \\ \log(N) & \text{if } \sigma = \frac{1}{2}, \\ M^{\frac{1}{2}-\sigma} & \text{if } \frac{1}{2} < \sigma < 1. \end{cases}$$

Upon setting $M = \lfloor q^{\frac{3}{8}} \rfloor$ and $N = \lceil q^{\frac{5}{8}} \rceil$, we obtain

$$L(s,\chi) \ll_{t,\varepsilon} \begin{cases} q^{\frac{4-5\sigma}{8}+\varepsilon} & \text{if } 0 < \sigma \le \frac{1}{2}, \\ q^{\frac{3-3\sigma}{8}+\varepsilon} & \text{if } \frac{1}{2} \le \sigma < \frac{1}{2}, \end{cases}$$

because the third term dominates for $0 < \sigma < \frac{1}{2}$, the first term dominates for $\frac{1}{2} < \sigma < 1$, and at $s = \frac{1}{2}$ the first and third terms are balanced and dominate the second term. This proves the first statement. The second follows by taking $\sigma = \frac{1}{2}$. □

In particular, Theorem 14.1.3 shows that the Burgess bound in the conductor-aspect holds for Dirichlet L-functions. This is how the Burgess bound got its name as it was the first subconvexity estimate for Dirichlet L-functions in the conductor-aspect.

Chapter 15

The Katz-Sarnak Philosophy

The Katz-Sarnak philosophy is an idea that certain statistics about families of L-functions should match statistics for random matrices coming from some particular compact matrix group. One starts with some class of zeros to look at, say zeros of an individual L-function high up the critical strip or zeros of for some collection of L-functions low down on the critical strip. Actually, one works with the corresponding unfolded nontrivial zeros since they are evenly spaced on average. Then some class of test functions are introduced to carry out the statistical calculations in order to reveal the similarity with some class of matrices. In the following, we give a loose introduction to the Katz-Sarnak philosophy.

15.1 Montgomery's Pair Correlation Conjecture

The beginning of the connection between random matrix theory and analytic number theory was at Princeton in the 1970s via discussions between Montgomery and Dyson. They found similarities between statistical information about the nontrivial distribution of the zeros of the Riemann zeta function and calculations in random matrix theory about unitary matrices. To do this, they considered the unfolded nontrivial zeros $\rho_{\text{unf}} = \beta + i\omega$ of $\zeta(s)$ with positive ordinate, that is $\omega > 0$, and indexed them according to the size of ordinate. So let $\Omega = (\omega_n)_{n\geq 1}$ denote the increasing sequence of positive ordinates of the unfolded nontrivial zeros of $\zeta(s)$. Montgomery and Dyson considered the **two-point correlation function** $F(\alpha, \beta; \zeta, W)$ for $\zeta(s)$, defined by

$$F(\alpha, \beta; \zeta, W) = \frac{1}{W}|\{(\omega_n, \omega_m) \in \Omega^2 : \omega_n, \omega_m \leq W \text{ and } \omega_n - \omega_m \in [\alpha, \beta]\}|,$$

for any real α and β with $\alpha < \beta$ and $W > 0$. What this function measures is the probability of how close pairs of zeros tend to be with respect to some fixed distance and up to some fixed height. In other words, the correlation between distances of zeros. They wanted to understand if the limiting distribution

$$F(\alpha, \beta; \zeta) = \lim_{W\to\infty} F(\alpha, \beta; \zeta, W),$$

exists and what can be said about it. The following conjecture made by Montgomery, known as Montgomery's pair correlation conjecture, answers this:

Conjecture (Montgomery's pair correlation conjecture). *For any α and β with $\alpha < \beta$, $F(\alpha, \beta; \zeta)$ exists provided the Riemann hypothesis for the Riemann zeta function holds. Moreover,*

$$F(\alpha, \beta; \zeta) = \int_\alpha^\beta \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 + \delta(x)\right) dx,$$

where $\delta(x)$ is the Dirac delta function.

Montgomery's pair correlation conjecture still remains out of reach, but there is very good numerical evidence supporting it from some unpublished work of Odlyzko (see [Odl92]). Dyson recognized that Montgomery's pair correlation conjecture models a similar situation in random matrix theory that he had investigated earlier. Consider an $N \times N$ unitary matrix $A \in U(N)$ with eigenphases θ_n for $1 \leq n \leq N$ denoted in increasing order. Clearly the average density of the eigenphases of A in $[0, 2\pi)$ is $\frac{N}{2\pi}$. For any eigenphase θ, let ϕ be the **unfolded eigenphase** corresponding to θ be defined by

$$\phi = \frac{N}{2\pi}\theta.$$

It follows that the average density of the unfolded eigenphases of A in $[0, N)$ is 1. Let $\Phi = (\phi_n)_{1 \leq n \leq N}$ denote the increasing sequence of unfolded eigenphases of A. We consider the **two-point correlation function** $F(\alpha, \beta; A, U(N))$ for A, defined by

$$F(\alpha, \beta; A, U(N)) = \frac{1}{N}|\{(\phi_n, \phi_m) \in \Phi^2 : \phi_n - \phi_m \in [\alpha, \beta]\}|,$$

for any real α and β with $\alpha < \beta$. Since $U(N)$ has a Haar measure dA, we can compute the global distribution of $F(\alpha, \beta; A, U(N))$ over $U(N)$, namely $F(\alpha, \beta; U(N))$, defined by

$$F(\alpha, \beta; U(N)) = \int_{U(N)} F(\alpha, \beta; A, U(N))\, dA.$$

Analogously, we want to understand if the limiting distribution

$$F(\alpha, \beta; U) = \lim_{N \to \infty} F(\alpha, \beta; U(N)),$$

exists and what can be said about it. Dyson showed the following (see [Dys62] for a proof):

Proposition 15.1.1. *For any real α and β with $\alpha < \beta$, $F(\alpha, \beta; U)$ exists. Moreover,*

$$F(\alpha, \beta; U) = \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 + \delta(x)\right) dx,$$

where $\delta(x)$ is the Dirac delta function.

The right-hand side of Proposition 15.1.1 is exactly the same formula given in Montgomery's pair correlation conjecture. In other words, if Montgomery's pair correlation conjecture is true then the two-point correlation of the unfolded nontrivial zeros of the Riemann zeta function in the limit as we move up the critical line exactly match the two-point correlation of the unfolded eigenphases of unitary matrices in the limit as the size of the matrices increase. In short, statistical information about the Riemann zeta function agrees with statistical information about the eigenvalues of unitary matrices. This is the origin of the Katz-Sarnak philosophy.

15.2 Symmetry Types and Families

Katz and Sarnak generalized the work of Montgomery and Dyson by establishing a connection between families of L-functions and other compact matrix groups. For the ease of categorization, Katz and Sarnak associated a **symmetry type** to each compact matrix group that they studied. The underlying compact matrix group associated to each symmetric type is called a **matrix ensemble**. The symmetry types and associated matrix ensembles are described in the following table:

Symmetry Type	Matrix Ensemble
Unitary (U)	$\mathrm{U}(N)$
Orthogonal (O^+)	$\mathrm{SO}(2N)$
Orthogonal (O^-)	$\mathrm{SO}(2N+1)$
Symplectic (Sp)	$\mathrm{USp}(2N)$

Let $G(N)$ be a matrix ensemble, where G denotes the symmetry type, and let dA denote the Haar measure. For any $A \in G(N)$, let $\Phi = (\phi_n)_{1 \leq n \leq N}$ denote the increasing sequence of unfolded eigenphases of A. Katz and Sarnak considered two local spacing distributions between the unfolded eigenphases of A. The first was the **two-point correlation function** $F(\alpha, \beta; A, G(N))$ for A, defined by

$$F(\alpha, \beta; A, G(N)) = \frac{1}{N} |\{(\phi_n, \phi_m) \in \Phi^2 : \phi_n - \phi_m \in [\alpha, \beta]\}|,$$

for any real α and β with $\alpha < \beta$. They computed the global distribution of $F(\alpha, \beta; A, G(N))$ over $G(N)$, namely $F(\alpha, \beta; G(N))$, defined by

$$F(\alpha, \beta; G(N)) = \int_{G(N)} F(\alpha, \beta; A, G(N))\, dA,$$

and sought to understand if the limiting distribution

$$F(\alpha, \beta; G) = \lim_{N \to \infty} F(\alpha, \beta; G(N)),$$

exists and what can be said about it. Note that in the case $G(N) = U(N)$, this is exactly the two-point correlation function considered by Dyson. Katz and Sarnak succeeded in generalizing Dyson's work (see [KS23] for a proof):

Proposition 15.2.1. *For symmetry type G and any real α and β with $\alpha < \beta$, $F(\alpha, \beta; G)$ exists. Moreover,*

$$F(\alpha, \beta; G) = \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 + \delta(x)\right) dx,$$

where $\delta(x)$ is the Dirac delta function.

The second local spacing distribution was the k-th **consecutive spacing function** for A, defined by

$$\mu_k(\alpha, \beta; A, G(N)) = \frac{1}{N} |\{1 \leq j \leq N : (\phi_{j+k}, \phi_j) \in \Phi^2 \text{ and } \phi_{j+k} - \phi_j \in [\alpha, \beta]\}|,$$

for any real α and β with $\alpha < \beta$ and $k \geq 1$. Again, they proceeded to compute the global distribution over $G(N)$, namely $\mu_k(\alpha, \beta; G(N))$, defined by

$$\mu_k(\alpha, \beta; G(N)) = \int_{G(N)} \mu_k(\alpha, \beta; A, G(N))\, dA,$$

and asked if the limiting distribution

$$\mu_k(\alpha, \beta; G) = \lim_{N \to \infty} \mu_k(\alpha, \beta; G(N)),$$

exists and what can be said about it. They were able the show the following (see [KS23] for a proof):

Proposition 15.2.2. *For any symmetry type G and any real α and β with $\alpha < \beta$ and $k \geq 1$, $\mu_k(\alpha, \beta; G)$ exists. Moreover, it is independent of the particular symmetry type G.*

In particular, Propositions 15.2.1 and 15.2.2 show that the limiting distributions $F(\alpha, \beta; G)$ and $\mu_k(\alpha, \beta; G)$ are both independent of the symmetry type G. However, this symmetry independence is not true for all limiting distributions. Katz and Sarnak also considered a local and global distributions associated to single eigenphases. The local distribution they considered, associated to a single eigenphase, was the **one-level density function** $\Delta(\alpha, \beta; A, G(N))$ for A, defined by

$$\Delta(\alpha, \beta; A, G(N)) = |\{\phi \in \Phi : \phi \in [\alpha, \beta]\}|.$$

They computed the global distribution, namely $\Delta(\alpha, \beta; G(N))$, defined by

$$\Delta(\alpha, \beta; G(N)) = \int_{G(N)} \Delta(\alpha, \beta; A, G(N))\, dA,$$

and asked if the limiting distribution

$$\Delta(\alpha, \beta; G) = \lim_{N \to \infty} \Delta(\alpha, \beta; G(N)),$$

exists and what can be said about it. Precisely, they proved the following (see [KS23] for a proof):

Proposition 15.2.3. *For any symmetry type G and any real α and β with $\alpha < \beta$ and $k \geq 1$, $\Delta(\alpha, \beta; G)$ exists. Moreover, it depends upon the particular symmetry type G.*

The global distribution they considered, associated to a single eigenphase, was the **k-th eigenphase function** $\nu_k(\alpha, \beta, G(N))$ for $G(N)$, defined by

$$\nu_k(\alpha, \beta; G(N)) = dA\left(\{A \in G(N) : \phi_k \in [\alpha, \beta]\}\right),$$

for any real α and β with $\alpha < \beta$ and $k \geq 1$. Again, they asked if the limiting distribution

$$\nu_k(\alpha, \beta; G) = \lim_{N \to \infty} \nu_k(\alpha, \beta; G(N)),$$

exists and what can be said about it. They were able to show the following (see [KS23] for a proof):

Proposition 15.2.4. *For any symmetry type G and any real α and β with $\alpha < \beta$ and $k \geq 1$, $\nu_k(\alpha, \beta; G)$ exists. Moreover, it depends upon the particular symmetry type G.*

In short, Katz and Sarnak studied four limiting distributions $F(\alpha, \beta; G)$, $\mu_k(\alpha, \beta; G)$, $\Delta(\alpha, \beta; G)$, and $\nu_k(\alpha, \beta; G)$. The former two are distributions about collections of eigenphases of unitary matrices and are independent of the symmetry type G while the latter two are distributions about single eigenphases of unitary matrices and depend upon the symmetry type G. Analogous distributions can be defined for families of L-functions. We say that a collection of L-functions $\mathcal{F} = (L(s_\alpha, f_\alpha))_{\alpha \in F}$, for some infinite indexing set $F \subset \mathbb{R}_{\geq 0}$, is a **family** if it is an ordered set with respect to the analytic conductor and if $\mathfrak{q}(s_\alpha, f_\alpha) \to \infty$ as $\alpha \to \infty$. We say that the family $\mathcal{F}$ is **continuous** if $f_\alpha = f_\beta$ for all $\alpha, \beta \in F$ and $s_\alpha = \sigma + it_\alpha$ where t_α is a continuous function of α. Necessarily, $t_\alpha \to \infty$ as $\alpha \to \infty$ and F is a half-open ray. We say that a family $\mathcal{F}$ is **discrete** if $f_\alpha \neq f_\beta$ for all distinct $\alpha, \beta \in F$ and F is discrete. Necessarily, $q(f_\alpha) \to \infty$ as $\alpha \to \infty$ and, reindexing if necessary, $F \subseteq \mathbb{Z}_+$. Katz and Sarnak arrived at a heuristical conjecture known as the **Katz-Sarnak philosophy** in terms of families of L-functions:

Conjecture (Katz-Sarnak Philosophy).

(i) The statistics about collections of eigenphases of a random matrix belonging to a matrix ensemble of symmetry type G, in the limit as the size of the matrix tends to infinity, should model statistics about the nontrivial zeros of a continuous family of L-functions as the heights of the nontrivial zeros tends to infinity.

(ii) The statistics about a single eigenphase of a random matrix belonging to a matrix ensemble of symmetry type G, in the limit as the size of the matrix tends to infinity, should model statistics about a discrete family of L-functions as the size of the conductor tends to infinity.

Determining the symmetry type of a family is generally a difficult task. Below are some well-studied families and their symmetry types (see [CFK+05] for a determination of the symmetry type):

Symmetry Type	Family
Unitary	$\{L(\sigma + it, f) : L(s, f)$ is of Selberg class$\}$ ordered by t
	$\{L(s, \chi) : \chi$ is a primitive character modulo $q\}$ ordered by q
Orthogonal	$\{L(s, f) : f \in \mathcal{S}_k(N, \chi)$ is a primitive Hecke eigenform$\}$ ordered by k
	$\{L(s, f) : f \in \mathcal{S}_k(N, \chi)$ is a primitive Hecke eigenform$\}$ ordered by N
Symplectic	$\{L(s, \chi_{\Delta_d}) : \Delta_d$ is a fundamental discriminant$\}$ ordered by $\lvert\Delta_d\rvert$

15.3 Characteristic Polynomials of Unitary Matrices

The Katz-Sarnak philosophy conjectures that the statistics about nontrivial zeros of L-functions are modeled by the statistics of eigenphases of random unitary matrices. However, there is a striking surface level connection between L-functions and the characteristic polynomials of unitary matrices which we now describe. For $A \in U(N)$, let

$$L(s, A) = \det(I - sA) = \prod_{1 \le n \le N} (1 - se^{i\theta_n}),$$

be the characteristic polynomial of A. It will turn out that $L(s, A)$ has strikingly similar properties to an L-function. The product expression for $L(s, A)$ is clearly analogous to the Euler product expression for an L-function. Upon expanding the product, we obtain

$$L(s, A) = \sum_{0 \le n \le N} a_n s^n,$$

for some coefficients a_n. This expression is analogous to the L-series of an L-function. Of course, as $L(s, A)$ is a polynomial it is analytic on $\mathbb{C}$. Moreover, $L(s, A)$ possesses a functional equation as $s \mapsto \frac{1}{s}$. To see this, first observe that multiplicativity of the determinant gives

$$L(s, A) = (-1)^N \det(A) s^N \det(I - s^{-1}A^{-1}).$$

As A is unitary, $L(s, A^{-1}) = L(s, A^*) = L(s, \overline{A})$. So the above equation can be expressed as

$$L(s, A) = (-1)^N \det(A) s^N L\left(\frac{1}{s}, \overline{A}\right).$$

This is the analog of the functional equation for an L-function under the symmetry $s \mapsto 1-s$. We identify the analogs of the gamma factor and conductor as 1 and N respectively. Letting $\Lambda(s, A)$ be defined by

$$\Lambda(s, A) = s^{-\frac{N}{2}} L(s, A),$$

the functional equation can be expressed as

$$\Lambda(s, A) = (-1)^N \det(A) \Lambda\left(\frac{1}{s}, \overline{A}\right).$$

From it, the analog of root number is seen to be $(-1)^N \det(A)$ and $L(s, A)$ has dual $L(s, \overline{A})$. As the symmetry $s \mapsto \frac{1}{s}$ leaves the unit circle invariant, the unit circle is the analog of the critical line. The fixed point of the symmetry $s \mapsto \frac{1}{s}$ is $s = 1$ which is the analog of the central point. Moreover, as the zeros of $L(s, A)$ are precisely the eigenvalues of A which lie on the unit circle, because A is unitary, the analog of the Riemann hypothesis is true for $L(s, A)$. We also have an analog of the approximate functional equation. By substituting the polynomial representation of $L(s, A)$ into the functional equation, we obtain

$$\sum_{0 \le n \le N} a_n s^n = (-1)^N \det(A) s^N \sum_{0 \le n \le N} \overline{a_n} s^{-n} = (-1)^N \det(A) \sum_{0 \le n \le N} \overline{a_n} s^{N-n}.$$

Comparing coefficients shows that

$$a_n = (-1)^N \det(A) \overline{a_{N-n}},$$

for $0 \le n \le N$. Then

$$L(s, A) = a_{\frac{N}{2}} s^{\frac{N}{2}} + \sum_{0 \le n \le \frac{N}{2}-1} a_n s^n + (-1)^N \det(A) s^N \sum_{0 \le n \le \frac{N}{2}-1} \overline{a_n} s^{-n},$$

or

$$L(s, A) = \sum_{0 \le n \le \frac{N-1}{2}} a_n s^n + (-1)^N \det(A) s^N \sum_{0 \le n \le \frac{N-1}{2}} \overline{a_n} s^{-n},$$

according to if N is even or odd. These equations together are the analog of the approximate functional equation. This similarity between L-functions and the characteristic polynomials of unitary matrices was heavily exploited by Conrey, Farmer, Keating, Rubinstein, and Snaith to make phenomenal conjectures about the moments of L-functions.

Chapter 16

The Theory of Moments

The study of moments of L-functions is essentially the method of studying a single L-function, or a family of L-functions, on average. While this approach leads to weaker results for a single L-function, it is often more malleable and sheds light on the general behavior of these objects. In the following we introduce moments of a single L-function, moments of families, and discuss some of their longstanding conjectures.

16.1 Continuous and Discrete Moments

One of the longstanding goal in the study of L-functions is to prove the Riemann hypothesis for the Riemann zeta function. More generally, we wish to prove the Riemann hypothesis for different types of L-functions. Another longstanding goal would be to prove the Lindelöf hypothesis for the Riemann zeta function, or more generally, any Selberg class L-function. While both the Riemann and Lindelöf hypotheses remain out of reach for any Selberg class L-function $L(s, f)$, the latter is more tractable because the convexity bound gives the estimate

$$L\left(\frac{1}{2}+it, f\right) \ll_{\varepsilon} \mathfrak{q}\left(\frac{1}{2}+it, f\right)^{\delta+\varepsilon},$$

with $\delta = \frac{1}{4}$ whereas the Lindelöf hypothesis claims that we can take $\delta = 0$. As we have already mentioned, breaking the convexity bound $\delta = \frac{1}{4}$ is a very daunting task requiring extremely modern techniques and has only been successful in a few cases. This approach of breaking convexity to study the Lindelöf hypothesis is quite direct. There is another approach to study the Lindelöf hypothesis that is more indirect by considering averages of the L-function in question. This is the study of moments of L-functions. For any L-function $L(s, f)$ and $k \geq 1$, we define the k-th **moment** of $L(s, f)$, namely $M_k(T, f)$, to be

$$M_k(T, f) = \int_0^T \left|L\left(\frac{1}{2}+it, f\right)\right|^k dt,$$

for any $T > 0$. More generally, we can define moments for any continuous or discrete family $\mathcal{F}$ of L-functions. For any $k \geq 1$, we define the k-th **moment** of $\mathcal{F}$, namely $M_k(T, \mathcal{F}; \sigma)$ or $M_k(Q, \mathcal{F}; s)$, to be

$$M_k(T, \mathcal{F}; \sigma) = \int_0^T |L(\sigma+it, f)|^k dt \quad \text{or} \quad M_k(Q, \mathcal{F}; s) = \sum_{\substack{f \in \mathcal{F} \\ \mathfrak{q}(f) \leq Q}} |L(s, f)|^k,$$

according to if $\mathcal{F}$ is continuous or discrete. We call these moments **continuous** and **discrete** respectively. Note that continuous moments only depend upon σ while discrete moments may depend upon s. Moreover, we suppress the dependence upon σ or s respectively if $\sigma = \frac{1}{2}$ or $s = \frac{1}{2}$. So for a continuous family,

$M_k(T, \mathcal{F}) = M_k(T, f)$. Generally speaking, we are usually interested in proving asymptotics for moments in terms of the parameter of the analytic conductor that is approaching infinity for the particular family. For continuous families this parameter is t and for discrete families it is $q(f)$. In the case of $M_k(T, f)$, it is useful to think of this moment as essentially a k-power average version of the Lindelöf hypothesis for $L(s, f)$. There is actually a close connection to the Lindelöf hypothesis since sufficient estimates for $M_{2k}(T, f)$ in T for all $k \geq 1$ is equivalent to the Lindelöf hypothesis as the following proposition shows:

Proposition 16.1.1. *Let $L(s, f)$ be an L-function. The truth of the Lindelöf hypothesis for $L(s, f)$ is equivalent to the estimate*

$$M_{2k}(T, f) \ll_\varepsilon T^{1+\varepsilon},$$

for all $k \geq 1$.

Proof. First suppose the Lindelöf hypothesis for $L(s, f)$ holds. As $\mathfrak{q}\left(\frac{1}{2} + it, f\right) \ll t^{d_f}$, the Lindelöf hypothesis for $L(s, f)$ implies

$$M_{2k}(T, f) \ll \int_0^T t^\varepsilon \, dt \ll T^{1+\varepsilon},$$

as desired. Now suppose that $M_{2k}(T, f) \ll_\varepsilon T^{1+\varepsilon}$ for all $k \geq 1$. If the Lindelöf hypothesis for $L(s, f)$ is false then there is some $0 < \lambda < 1$ and positive unbounded real sequence $(t_n)_{n\geq 1}$ such that

$$\left|L\left(\frac{1}{2} + it_n, f\right)\right| > C\mathfrak{q}\left(\frac{1}{2} + it_n, f\right)^\lambda,$$

for any $C > 0$. In particular,

$$\left|L\left(\frac{1}{2} + it, f\right)\right| > Ct_n^{d_f\lambda}.$$

Now the convexity bound for $L'(s, f)$ implies the weaker estimate $L'\left(\frac{1}{2} + it\right) \leq C't^{d_f}$ for some $C' > 0$ and t bounded away from zero. Then

$$\left|L\left(\frac{1}{2} + it, f\right) - L\left(\frac{1}{2} + it_n, f\right)\right| = \left|\int_{t_n}^t L'\left(\frac{1}{2} + ir, f\right) \, dr\right| < C't_n^{d_f}|t - t_n| < \frac{C}{2}t_n^{d_f\lambda},$$

provided $|t - t_n| \leq t_n^{d_f(\lambda-1)}$ which is valid for sufficiently large n (and thus t_n and t are bounded away from zero). For such t and n, the previous two bounds together imply

$$\left|L\left(\frac{1}{2} + it, f\right)\right| > \frac{C}{2}t_n^{d_f\lambda}.$$

Now take $T = \frac{4}{3}t_n$ so that the interval $(t_n - t_n^{d_f(\lambda-1)}, t_n + t_n^{d_f(\lambda-1)})$ is contained in $\left(\frac{T}{2}, T\right)$ provided we take n larger if necessary. It follows that

$$\int_{\frac{T}{2}}^T \left|L\left(\frac{1}{2} + it, f\right)\right|^{2k} \, dt > \int_{t_n - t_n^{d_f(\lambda-1)}}^{t_n + t_n^{d_f(\lambda-1)}} \left|L\left(\frac{1}{2} + it, f\right)\right|^{2k} \, dt > 2\left(\frac{C}{2}\right)^{2k} t_n^{(2k+1)d_f\lambda - d_f},$$

and hence

$$M_{2k}(T, f) > 2\left(\frac{C}{2}\right)^{2k}\left(\frac{3}{4}\right)^{(2k+1)d_f\lambda - d_f} T^{(2k+1)d_f\lambda - d_f}.$$

This is a contradiction for sufficiently large k which completes the proof. □

The usefulness of Proposition 16.1.1 is that the difficulty of proving the Lindelöf hypothesis for $L(s, f)$ has been transferred to proving sufficient asymptotics for the moments $M_{2k}(T, f)$ each of which should, heuristically speaking, be an easier problem to resolve on its own.

16.2 The Moment Conjectures

Hardy and Littlewood were the first to introduce moments and they did so in the context of the Riemann zeta function. In 1918, they proved an asymptotic for the second moment (see [HL16] for a proof):

$$M_2(T,\zeta) \sim T\log(T).$$

In 1926, Ingham obtained an asymptotic for the fourth moment (see [Ing28] for a proof):

$$M_4(T,\zeta) \sim \frac{1}{2\pi^2} T\log^4(T).$$

Observe that both of these asymptotics are stronger than those in Proposition 16.1.1. In other words, this is evidence in support of the truth of the Lindelöf hypotheses for $\zeta(s)$. Unfortunately, this is where progress significantly halts. No analogous formula have been obtained for moments of any L-function when $k > 2$. In fact, the problem is so intractable that, until recently, there were not even conjectures about what the asymptotics should be. In 2000, Keating and Snaith used the Katz-Sarnak philosophy to put forth a precise conjecture for the asymptotics of the moments of the Riemann zeta function (see [KS00] for details):

Conjecture 16.2.1. *For all $k \geq 1$,*

$$M_{2k}(T,\zeta) \sim \frac{g_k}{(k^2)!} a_k T\log^{k^2}(T),$$

with

$$g_k = (k^2)! \prod_{0\leq j\leq k-1} \frac{j!}{(j+k)!},$$

and

$$a_k = \prod_p \left((1-p^{-1})^{(k-1)^2} \sum_{0\leq j\leq k-1} \binom{k-1}{j}^2 p^{-j} \right).$$

Conjecture 16.2.1 agrees with the results of Hardy and Littlewood for $k = 1$ (note that $g_1 = 1$ and $a_1 = 1$) and Ingham for $k = 2$ (note that $g_2 = 2$ and $a_2 = \zeta(2)^{-1} = \frac{6}{\pi^2}$ because it is the sum of the reciprocals of squares). Monumental progress was made in 2005 when Conrey, Farmer, Keating, Rubinstein, and Snaith used random matrix theory to put forth a procedure for deducing conjectured asymptotic formulas, not just moments of the Riemann zeta function, for many families of L-functions (see [CFK$^+$05]). Moreover, they proved that analogous statistics hold for the associated symmetry types of the families which is in agreement with the Katz-Sarnak philosophy. We will describe some of their conjectures. For a unitary example, they predicted a refined asymptotic for the Riemann zeta function:

Conjecture 16.2.2. *For all $k \geq 1$ and $T \geq 1$,*

$$M_{2k}(T,\zeta) = TP_k(\log(T)) + O_\varepsilon\left(T^{\frac{1}{2}+\varepsilon}\right),$$

where P_k is a polynomial of degree k^2 with leading coefficient $\frac{g_k}{(k^2)!}a_k$ and we have

$$g_k = (k^2)! \prod_{0\leq j\leq k-1} \frac{j!}{(j+k)!},$$

and

$$a_k = \prod_p \left((1-p^{-1})^{(k-1)^2} \sum_{0\leq j\leq k-1} \binom{k-1}{j}^2 p^{-j} \right).$$

Conjecture 16.2.2 has been proven in the cases $k = 1, 2$ (see [CFK+05] for comments). Moreover, this result is clearly stronger than Conjecture 16.2.1 and it gives an exact error of order $O(T^{\frac{1}{2}+\varepsilon})$. This error is roughly on the order of the square-root of the main term. For an orthogonal example, they predicted an asymptotic for a family of Hecke L-functions:

Conjecture 16.2.3. *Let $\mathcal{H}$ be the family of bases of primitive Hecke eigenforms for the space of newforms of weight 2, square-free level q, and ordered by q. For all $k \geq 1$, and $Q \geq 1$,*

$$M_k(Q, \mathcal{H}) = \frac{1}{3} Q R_k(\log(Q)) + O_\varepsilon\left(Q^{\frac{1}{2}+\varepsilon}\right),$$

where R_k is a polynomial of degree $\frac{1}{2}k(k-1)$ with leading coefficient $\frac{g_k}{\left(\frac{1}{2}k(k-1)\right)!}a_k$ and we have

$$g_k = 2^{k-1}\left(\frac{1}{2}k(k-1)\right)! \prod_{1 \leq j \leq k-1} \frac{j!}{(2j)!},$$

and

$$a_k = \prod_{p \nmid Q} (1-p^{-1})^{\frac{1}{2}k(k-1)} \frac{2}{\pi} \int_0^\pi \sin^2\left(\theta \left(\frac{e^{i\theta}(1-e^{i\theta}p^{-\frac{1}{2}})^{-1} - e^{-i\theta}(1-e^{i\theta}p^{-\frac{1}{2}})^{-1}}{e^{i\theta}-e^{-i\theta}}\right)^k\right) d\theta.$$

The main term in Conjecture 16.2.3 has been proved in the cases $k = 1, 2, 3, 4$ when q is prime (see [CFK+05] for comments). For an symplectic example, they predicted an asymptotic for a family of Dirichlet L-functions:

Conjecture 16.2.4. *Let $\mathcal{D}$ be the family of Dirichlet L-functions attached to the quadratic characters χ_{Δ_d} associated to the fundamental discriminants Δ_d and ordered by $|\Delta_d|$. For all $k \geq 1$ and $D \geq 1$,*

$$M_k(D, \mathcal{D}) = \frac{6}{\pi^2} D Q_k(\log(D)) + O_\varepsilon\left(D^{\frac{1}{2}+\varepsilon}\right),$$

where Q_k is a polynomial of degree $\frac{1}{2}k(k+1)$ with leading coefficient $\frac{g_k}{\left(\frac{1}{2}k(k+1)\right)!}a_k$ and we have

$$g_k = \left(\frac{1}{2}k(k+1)\right)! \prod_{1 \leq j \leq k-1} \frac{j!}{(2j)!},$$

and

$$a_k = \prod_p \frac{(1-p^{-1})^{\frac{1}{2}k(k+1)}}{(1+p^{-1})}\left(\frac{(1-p^{-\frac{1}{2}})^{-k} + (1+p^{-\frac{1}{2}})^{-k}}{2} + p^{-1}\right).$$

The main term in Conjecture 16.2.4 has been proved in the cases $k = 1, 2, 3$ (see [CFK+05] for comments). We will now describe **moment recipe** in [CFK+05] used to predict the $2k$-th moment of a family $\mathcal{F}$ of primitive L-functions:

(i) Let $f \in \mathcal{F}$ and set

$$L(s; \alpha_1, \ldots, \alpha_{2k}, f) = L(s+\alpha_1, f) \cdots L(s+\alpha_k, f) L(1-s-\alpha_{k+1}, f) \cdots L(1-s-\alpha_{2k}, f),$$

with $\alpha_\ell \in \mathbb{C}$ for $1 \leq \ell \leq 2k$.

(ii) Replace each L-function with its two sums from the approximate function equation, ignoring the function $V_s(y)$ and possible residue term, and multiply out all of the sums to obtain 2^{2k} many terms. That is, make the substitutions

$$L(s+\alpha_\ell, f) \mapsto \sum_{n\geq 1} \frac{a_f(n)}{n^{s+\alpha_\ell}} + \varepsilon(s+\alpha_\ell, f) \sum_{n\geq 1} \frac{\overline{a_f(n)}}{n^{1-s-\alpha_\ell}},$$

for $1 \leq \ell \leq k$ and

$$L(1-s-\alpha_\ell, f) \mapsto \sum_{n\geq 1} \frac{a_f(n)}{n^{1-s-\alpha_\ell}} + \varepsilon(1-s-\alpha_\ell, f) \sum_{n\geq 1} \frac{\overline{a_f(n)}}{n^{s+\alpha_\ell}},$$

for $k+1 \leq \ell \leq 2k$ respectively. Then expand all of the terms into the form

$$\prod_{i\in\overline{I}} \varepsilon(s+\alpha_i, f) \prod_{j\in\overline{J}} \varepsilon(1-s-\alpha_j, f) \sum_{n_1,\ldots,n_{2k}\geq 1} \frac{\prod_{i\in I} a_f(n_i) \prod_{i\in\overline{I}} \overline{a_f(n_i)} \prod_{j\in J} a_f(n_j) \prod_{j\in\overline{J}} \overline{a_f(n_j)}}{\prod_{i\in I} n_i^{s+\alpha_i} \prod_{i\in\overline{I}} n_i^{1-s-\alpha_i} \prod_{j\in J} n_j^{1-s-\alpha_j} \prod_{j\in\overline{J}} n_j^{s+\alpha_j}},$$

where $I \cup \overline{I} = \{1, \ldots, k\}$ and $J \cup \overline{J} = \{k+1, \ldots, 2k\}$ are partitions.

(iii) Retain only the $\binom{2k}{k} = \sum_{0\leq m\leq k} \binom{k}{m}^2$ terms for which $|\overline{I}| = |\overline{J}|$ (the size m of these subsets satisfies $0 \leq m \leq k$ and there are $\binom{k}{m}$ many choices for each subset of size m). Then apply Stirling's formula to simplify the remaining ratios of gamma factors by replacing them with their asymptotics.

(iv) In each of the $\binom{2k}{k}$ terms, retain only the diagonal term from the sum. That is, keep the summands for which $n_1 = n_2 = \cdots = n_{2k}$. In other words, only retain

$$\prod_{i\in\overline{I}} \varepsilon(s+\alpha_i, f) \prod_{j\in\overline{J}} \varepsilon(1-s-\alpha_j, f) \sum_{n\geq 1} \frac{\prod_{i\in I} a_f(n) \prod_{i\in\overline{I}} \overline{a_f(n)} \prod_{j\in J} a_f(n) \prod_{j\in\overline{J}} \overline{a_f(n)}}{\prod_{i\in I} n^{s+\alpha_i} \prod_{i\in\overline{I}} n^{1-s-\alpha_i} \prod_{j\in J} n^{1-s-\alpha_j} \prod_{j\in\overline{J}} n^{s+\alpha_j}}.$$

Let $M(s; \alpha_1, \ldots, \alpha_{2k}, f)$ denote the resulting function.

(v) Then for any $T \geq 1$ or $Q \geq 1$, the moment recipe predicts that

$$M_{2k}\left(T, \mathcal{F}, \frac{1}{2}\right) = \lim_{\alpha_1,\ldots,\alpha_{2k}\to 0} \int_0^T M\left(\frac{1}{2}+it; \alpha_1, \ldots, \alpha_{2k}, f\right) \left(1 + O\left(T^{\frac{1}{2}+\varepsilon}\right)\right) dt,$$

or

$$M_{2k}\left(Q, \mathcal{F}, \frac{1}{2}\right) = \lim_{\alpha_1,\ldots,\alpha_{2k}\to 0} \sum_{\substack{f\in\mathcal{F} \\ q(f)\leq Q}} M\left(\frac{1}{2}; \alpha_1, \ldots, \alpha_{2k}, f\right) \left(1 + O\left(T^{\frac{1}{2}+\varepsilon}\right)\right),$$

according to if $\mathcal{F}$ is continuous or discrete.

A couple of comments about the moment recipe are necessary. First, while this procedure produces heuristics for the $2k$-moment, we expect the resulting identities in step (v) to hold for the k-moment as well because if k is odd we have $2\left(\frac{k-1}{2}\right) < k < 2\left(\frac{k+1}{2}\right)$. The reason for only retaining the $\binom{2k}{k}$ terms for which $|\overline{I}| = |\overline{J}|$ in step (iii) is because the terms for which $|\overline{I}| \neq |\overline{J}|$ are expected to be near zero. Indeed, Stirling's formula implies

$$\frac{\Gamma(1-s)}{\Gamma(s)} \sim -s^{1-2s}e^{2s-1} \quad \text{and} \quad \frac{\Gamma(s)}{\Gamma(1-s)} \sim -s^{2s-1}e^{1-2s},$$

provided $|\arg(s)| < \pi - \varepsilon$ and $|s| > \delta$ for some $\varepsilon, \delta > 0$. It follows from the definition of $\varepsilon(s, f)$ that

$$\varepsilon(s,f) \sim -q(f)^{2s-1}\left(\frac{s}{\pi e}\right)^{d_f\left(\frac{1}{2}-s\right)} \quad \text{and} \quad \varepsilon(1-s,f) \sim -q(f)^{1-2s}\left(\frac{s}{\pi e}\right)^{d_f\left(s-\frac{1}{2}\right)}.$$

Because of the presence of the $e^{d_f s}$ and $e^{-d_f s}$ factors in these asymptotics and that

$$\int_0^{2\pi} e^{it}\,dt = 0,$$

we expect the terms for which $|\overline{I}| \neq |\overline{J}|$ to be near zero when integrating from 0 to T. In the case $|\overline{I}| = |\overline{J}|$, these exponential factors cancel, in the limit as $\alpha_i \to 0$ for all i, and therefore we do not expect these terms to be near zero. The two equalities in part (v) are known as the **moment conjectures**:

Conjecture 16.2.5 (Moment conjectures). *Let $\mathcal{F}$ be a family of primitive L-functions with $f \in \mathcal{F}$. Then*

$$M_{2k}\left(T, \mathcal{F}, \frac{1}{2}\right) = \lim_{\alpha_1,\ldots,\alpha_{2k}\to 0} \int_0^T M\left(\frac{1}{2}+it; \alpha_1, \ldots, \alpha_{2k}, f\right)\left(1 + O\left(T^{\frac{1}{2}+\varepsilon}\right)\right)\,dt,$$

or

$$M_{2k}\left(Q, \mathcal{F}, \frac{1}{2}\right) = \lim_{\alpha_1,\ldots,\alpha_{2k}\to 0} \sum_{\substack{f\in\mathcal{F}\\ q(f)\leq Q}} M\left(\frac{1}{2}; \alpha_1, \ldots, \alpha_{2k}, f\right)\left(1 + O\left(T^{\frac{1}{2}+\varepsilon}\right)\right),$$

according to if $\mathcal{F}$ is continuous or discrete.

Part VI

Appendices

Appendix A

Number Theory

A.1 Arithmetic Functions

An arithmetic function f is a function $f : \mathbb{N} \to \mathbb{C}$. That is, it takes the positive integers into the complex numbers. We say that f is **additive** if $f(nm) = f(n) + f(m)$ for all positive integers n and m such that $(n, m) = 1$. If this condition simply holds for all n and m then we say f is **completely additive**. Similarly, we say that f is **multiplicative** if $f(nm) = f(n)f(m)$ for all positive integers n and m such that $(n, m) = 1$. If this condition simply holds for all n and m then we say f is **completely multiplicative**. Many important arithmetic functions are either additive, completely additive, multiplicative, or completely multiplicative. Note that if a f is additive or multiplicative then f is uniquely determined by its values on prime powers and if f is completely additive or completely multiplicative then it is uniquely determined by its values on primes. Moreover, if f is additive or completely additive then $f(1) = 0$ and if f is multiplicative or completely multiplicative then $f(1) = 1$. Below is a list defining the most important arithmetic functions (some of these functions are restrictions of common functions but we define them here as arithmetic functions because their domain being $\mathbb{N}$ is important):

(i) The **constant function**: The function $\mathbf{1}(n)$ restricted to all $n \geq 1$. This function is neither additive or multiplicative.

(ii) The **unit function**: The function $e(n)$ defined by

$$e(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

This function is completely multiplicative.

(iii) The **identity function**: The function $\mathrm{id}(n)$ restricted to all $n \geq 1$. This function is completely multiplicative.

(iv) The **logarithm**: The function $\log(n)$ restricted to all $n \geq 1$. This function is completely additive.

(v) The **Möbius function**: The function $\mu(n)$ defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free with an even number of prime factors,} \\ -1 & \text{if } n \text{ is square-free with an odd number of prime factors,} \\ 0 & \text{if } n \text{ is not square-free,} \end{cases}$$

for all $n \geq 1$. This function is multiplicative.

(vi) The **characteristic function of square-free integers**: The square of the Möbius function $\mu^2(n)$ for all $n \geq 1$. This function is multiplicative.

(vii) **Liouville's function**: The function $\lambda(n)$ defined by

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is composed of } k \text{ not necessarily distinct prime factors,} \end{cases}$$

for all $n \geq 1$. This function is completely multiplicative.

(viii) **Euler's totient function**: The function $\varphi(n)$ defined by

$$\varphi(n) = \sum_{m \pmod n}{}' 1,$$

for all $n \geq 1$. This function is multiplicative.

(ix) The **divisor function**: The function $\sigma_0(n)$ defined by

$$\sigma_0(n) = \sum_{d|n} 1,$$

for all $n \geq 1$. This function is multiplicative.

(x) The **sum of divisors function**: The function $\sigma_1(n)$ defined by

$$\sigma_1(n) = \sum_{d|n} d,$$

for all $n \geq 1$. This function is multiplicative.

(xi) The **generalized sum of divisors function**: The function $\sigma_s(n)$ defined by

$$\sigma_s(n) = \sum_{d|n} d^s,$$

for all $n \geq 1$ and any $s \in \mathbb{C}$. This function is multiplicative.

(xii) The **number of distinct prime factors function**: The function $\omega(n)$ defined by

$$\omega(n) = \sum_{p|n} 1,$$

for all $n \geq 1$. This function is additive.

(xiii) The **total number of prime divisors function**: The function $\Omega(n)$ defined by

$$\Omega(n) = \sum_{p^m|n} 1,$$

for all $n \geq 1$ and where $m \geq 1$. This function is completely additive.

(xiv) The **von Mangoldt function**: The function $\Lambda(n)$ defined by

$$\Lambda(n) = \begin{cases} 0 & \text{if } n \text{ is not a prime power,} \\ \log(p) & \text{if } n = p^m \text{ for some prime } p \text{ and integer } m \geq 1, \end{cases}$$

for all $n \geq 1$. This function is neither additive or multiplicative.

If f and g are two arithmetic functions then we can define a new arithmetic function $f * g$ called the **Dirichlet convolution** of f and g defined by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right),$$

for all $n \geq 1$. This is especially useful when f and g are multiplicative:

Proposition A.1.1. *If f and g are multiplicative arithmetic functions then so is their Dirichlet convolution $f * g$.*

A.2 The Möbius Function

Recall that the Möbius function is the arithmetic function μ defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free with an even number of prime factors,} \\ -1 & \text{if } n \text{ is square-free with an odd number of prime factors,} \\ 0 & \text{if } n \text{ is not square-free,} \end{cases}$$

and it is multiplicative. It also satisfies an important summation property:

Proposition A.2.1.

$$\sum_{d|n} \mu(d) = \delta_{n,1}.$$

From this property, the important **Möbius inversion formula** can be derived:

Theorem (Möbius inversion formula). *Suppose f and g are arithmetic functions. Then*

$$g(n) = \sum_{d|n} f(d),$$

for all $n \geq 1$, if and only if

$$f(n) = \sum_{d|n} g(d) \mu\left(\frac{n}{d}\right)$$

for all $n \geq 1$.

In terms of Dirichlet convolution, the Möbius inversion formula is equivalent to stating that $g = f * \mathbf{1}$ if and only if $f = g * \mu$. Using Möbius inversion, the following useful formula can also be derived:

Proposition A.2.2. *For $\sigma > 1$,*

$$\sum_{n \geq 1} \frac{\mu(n)}{n^s} = \zeta(s)^{-1} = \prod_p (1 - p^{-s}).$$

There is also an important similar statement to the Möbius inversion formula that we will need:

Theorem A.2.1. *Let f be an arithmetic function and let B be the completely multiplicative function defined on primes p by*

$$B(p) = f(p)^2 - f(p^2).$$

Then

$$f(n)f(m) = \sum_{d|(n,m)} B(d) f\left(\frac{nm}{d^2}\right),$$

for all $n, m \geq 1$, if and only if

$$f(nm) = \sum_{d|(n,m)} \mu(d)B(d) f\left(\frac{n}{d}\right) f\left(\frac{m}{d}\right),$$

for all $n, m \geq 1$.

Any arithmetic function f satisfying the conditions of Theorem A.2.1 is said to be **specially multiplicative**.

A.3 The Divisor, Sum of Divisors, and Generalized Sum of Divisors Functions

We generalize $\sigma_0(n)$, $\sigma_1(n)$, and $\sigma_s(n)$ to all nonzero $n \in \mathbb{Z}$ by setting

$$\sigma_0(n) = \sigma_0(|n|), \quad \sigma_1(n) = \sigma_1(|n|), \quad \text{and} \quad \sigma_s(n) = \sigma_s(|n|),$$

for all $s \in \mathbb{C}$. It is very useful to know that $\sigma_0(n)$ grows slowly:

Proposition A.3.1.

$$\sigma_0(n) \ll_\varepsilon n^\varepsilon.$$

This is all we really need to know for the sum of divisors function. As for the generalized sum of divisors function, it has the remarkable property that it can be written as a product. To state it, recall that $\mathrm{ord}_p(n)$ is the positive integer satisfying $p^{\mathrm{ord}_p(n)} \mid\mid n$. Then we have the following statement:

Proposition A.3.2. *For $s \neq 0$,*

$$\sigma_s(n) = \prod_{p|n} \frac{p^{(\mathrm{ord}_p(n)+1)s} - 1}{p^s - 1}.$$

A.4 Quadratic Symbols

Let p be an odd prime. We are often interested in when the equation $x^2 = m \pmod p$ is solvable for some $m \in \mathbb{Z}$. The **Legendre symbol** $\left(\frac{m}{p}\right)$ keeps track of this:

$$\left(\frac{m}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod p \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod p \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod p. \end{cases}$$

Euler's criterion gives an alternative expression for Legendre symbol when m is coprime to p:

Proposition (Euler's criterion). *Let p be an odd prime and suppose $m \in \mathbb{Z}$ with $(m, p) = 1$. Then*

$$\left(\frac{m}{p}\right) = m^{\frac{p-1}{2}} \pmod{p}.$$

From the definition and Euler's criterion is it not difficult to show that the Legendre symbol satisfies the following properties:

Proposition A.4.1. *Let p be an odd prime and let $a, b \in \mathbb{Z}$. Then the following hold:*

(i) If $a \equiv b \pmod{p}$ then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

(ii) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

From Proposition A.4.1, to compute the Legendre symbol in general it suffices to know how to compute $\left(\frac{-1}{p}\right)$, $\left(\frac{2}{p}\right)$, and $\left(\frac{q}{p}\right)$ where q is another odd prime. The **supplemental laws of quadratic reciprocity** are formulas for the first two symbols:

Proposition (Supplemental laws of quadratic reciprocity). *Let p be an odd prime. Then the following formulas hold:*

(i)

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

(ii)

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if } p \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } p \equiv 3, 5 \pmod{8}. \end{cases}$$

The **law of quadratic reciprocity** handles the last symbol by relating $\left(\frac{q}{p}\right)$ to $\left(\frac{p}{q}\right)$:

Theorem (Law of quadratic reciprocity). *Let p and q be odd primes. Then*

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{q}{p}\right).$$

We can generalize the Jacobi symbol further by making it multiplicative in the denominator. Let n be a positive odd integer with prime factorization $n = p_1^{r_1} \cdots p_k^{r_k}$ and let $m \in \mathbb{Z}$. The **Jacobi symbol** $\left(\frac{m}{n}\right)$ is defined by

$$\left(\frac{m}{n}\right) = \prod_{1 \le i \le k} \left(\frac{m}{p_i}\right)^{r_i}.$$

When $n = p$ is prime, the Jacobi symbol reduces to the Legendre symbol and the Jacobi symbol is precisely the unique multiplicative extension of the Legendre symbol to all positive odd integers. Accordingly, the Jacobi symbol has the following properties:

Proposition A.4.2. *Let m and n be positive odd integers and let $a, b \in \mathbb{Z}$. Then the following hold:*

(i) *If* $a \equiv b \pmod{p}$ *then* $\left(\frac{a}{n}\right) = \left(\frac{b}{n}\right)$.

(ii) $\left(\frac{a}{mn}\right) = \left(\frac{a}{m}\right)\left(\frac{a}{n}\right)$.

(iii) $\left(\frac{ab}{n}\right) = \left(\frac{a}{n}\right)\left(\frac{b}{n}\right)$.

There is also an associated reciprocity law:

Proposition A.4.3. *Let* m *and* n *be positive odd integers. Then*

$$\left(\frac{m}{n}\right) = (-1)^{\frac{m-1}{2}\frac{n-1}{2}} \left(\frac{n}{m}\right).$$

We can further generalize the Jacobi symbol so that it is valid for all integers. Let $m, n \in \mathbb{Z}$ where $n = up_1^{r_1} \cdots p_k^{r_k}$ is the prime factorization of n with $u = \pm 1$. The **Kronecker symbol** $\left(\frac{m}{n}\right)$ is defined by

$$\left(\frac{m}{n}\right) = \left(\frac{m}{u}\right) \prod_{1 \le i \le k} \left(\frac{m}{p_i}\right)^{r_i},$$

where we set

$$\left(\frac{m}{1}\right) = 1, \quad \left(\frac{m}{-1}\right) = \begin{cases} 1 & \text{if } m \ge 0, \\ -1 & \text{if } m < 0, \end{cases} \quad \left(\frac{m}{0}\right) = \begin{cases} 1 & \text{if } m = \pm 1, \\ 0 & \text{if } m \ne \pm 1, \end{cases}$$

and

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}. \end{cases}$$

When n is a positive odd integer, the Kronecker symbol reduces to the Jacobi symbol. The Kronecker symbol also satisfies a reciprocity law:

Proposition A.4.4. *Let* m *and* n *be integers. Then*

$$\left(\frac{m}{n}\right) = (-1)^{\frac{m^{(2)}-1}{2}\frac{n^{(2)}-1}{2}} \left(\frac{n}{|m|}\right),$$

where $m^{(2)}$ *and* $n^{(2)}$ *are the parts of* m *and* n *relatively prime to* 2 *respectively.*

Appendix B

Analysis

B.1 Local Absolute Uniform Convergence

Throughout, all functions are understood to complex-valued and regions are understood to be those of $\mathbb{R}^n$ or $\mathbb{C}^n$. Often, we are interested in some series

$$\sum_{n\geq 1} f_n(\mathbf{x}),$$

where of analytic functions $f_n(\mathbf{x})$ on some region D. We say that the series above is **locally absolutely uniformly convergent** if

$$\sum_{n\geq 1} |f_n(\mathbf{x})|,$$

converges uniformly on compact subsets of D. A common method to determine if series is absolutely uniformly convergent is to use the Weierstrass M-test:

Theorem (Weierstrass M-test). *Suppose $(f_n(\mathbf{x}))_{n\geq 1}$ is a sequence of functions on a region D and there is a nonnegative sequence $(M_n)_{n\geq 1}$ such that $|f_n(\mathbf{x})| \leq M_n$ for all $n \geq 1$ and*

$$\sum_{n\geq 1} M_n,$$

converges. Then

$$\sum_{n\geq 1} f_n(\mathbf{x}),$$

is absolutely uniformly convergent on D.

Applying the Weierstrass M-test to arbitrary compact subsets on a region shows that the resulting series is locally absolutely uniformly convergent. This mode of convergence is very useful because it is enough to guarantee the series is analytic on D:

Theorem B.1.1. *Suppose $(f_n(\mathbf{x}))_{n\geq 1}$ is a sequence of analytic functions on a region D. Then if*

$$\sum_{n\geq 1} f_n(\mathbf{x}),$$

is locally absolutely uniformly convergent, it is analytic on D.

We can also apply this idea in the case of integrals. Suppose we have an integral

$$\int_D f(\mathbf{x}, \mathbf{y})\, d\mathbf{y},$$

where $f(\mathbf{x}, \mathbf{y})$ is a complex-valued function on some region $D \times S$ and is analytic in $\mathbf{x}$ for every $\mathbf{y}$. The integral is a function of $\mathbf{x}$, and we say that the integral is **locally absolutely uniformly convergent** if

$$\int_S |f(\mathbf{x}, \mathbf{y})|\, d\mathbf{y},$$

is uniformly bounded on compact subsets of D. Similar to the series case, this mode of convergence is very useful because it guarantees the integral is analytic on D:

Theorem B.1.2. *Suppose $f(\mathbf{x}, \mathbf{y})$ is a complex-valued function on some region $D \times S$ and is analytic in $\mathbf{x}$ for every $\mathbf{y}$. Then if*

$$\int_S |f(\mathbf{x}, \mathbf{y})|\, d\mathbf{y},$$

is locally absolutely uniformly convergent, it is analytic on D.

B.2 Interchange of Integrals, Sums & Derivatives

Often, we would like to interchange a limit and a sum or a limit and an integral. This process is not always allowed, but in many instances it is. The **dominated convergence theorem** (DCT) covers the most well-known sufficient condition:

Theorem (Dominated convergence theorem). *The following hold:*

1. *Let $(f_n(\mathbf{x}))_{n\geq 1}$ be a sequence of functions on some region D and such that*

$$\sum_{n\geq 1} f_n(\mathbf{x}),$$

   is absolutely convergent on D. Then for any $\mathbf{x}_0 \in D$, we have

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \sum_{n\geq 1} f_n(\mathbf{x}) = \sum_{n\geq 1} \lim_{\mathbf{x}\to\mathbf{x}_0} f_n(\mathbf{x}).$$

2. *Let $(f_n(\mathbf{x}))_{n\geq 1}$ be a sequence of continuous and integrable functions on some region D. Suppose that that the sequence converges pointwise to a function $f(\mathbf{x})$, and that there is some integrable function $g(\mathbf{x})$ on D such that*

$$|f_n(\mathbf{x})| \leq g(\mathbf{x}),$$

   for all $n \geq 1$ and all $\mathbf{x} \in D$. Then $f(\mathbf{x})$ is integrable on D and

$$\lim_{n\to\infty} \int_D f_n(\mathbf{x})\, d\mathbf{x} = \int_D f(\mathbf{x})\, d\mathbf{x}.$$

We would also like to interchange two sums, two integrals, or a sum and an integral assuming that one expression converges absolutely. This is sufficient as given by the **Fubini–Tonelli theorem** (FTT):

Theorem (Fubini–Tonelli theorem). *The following hold:*

(i) If $(f_{n,m}(\mathbf{x}))_{n,m\geq 1}$ is a sequence of continuous functions then

$$\sum_{(n,m)\in\mathbb{Z}} f_{n,m}(\mathbf{x}) = \sum_{n\geq 1}\sum_{m\geq 1} f_{n,m}(\mathbf{x}) = \sum_{m\geq 1}\sum_{n\geq 1} f_{n,m}(\mathbf{x}),$$

provided any of the expressions are absolutely convergent.

(ii) If $(f_n(\mathbf{x},\mathbf{y}))_n$ is a sequence of continuous and integrable functions on some region $D\times S$ then

$$\int_{D\times S} f(\mathbf{x},\mathbf{y})\,d(\mathbf{x}\times\mathbf{y}) = \int_S\int_D f(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int_D\int_S f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\,d\mathbf{x},$$

provided any of the expressions are absolutely convergent.

(iii) If $(f_n(\mathbf{x}))_{n\geq 1}$ is a sequence of continuous and integrable functions on some region D then

$$\sum_{n\geq 1}\int_D f_n(\mathbf{x})\,d\mathbf{x} = \int_D \sum_{n\geq 1} f_n(\mathbf{x})\,d\mathbf{x},$$

provided either side is absolutely convergent.

Other times we would also like to interchange a derivative and an integral. The **Leibniz integral rule** tells us when this is allowed:

Theorem (Leibniz integral rule). *Suppose $f(\mathbf{x},t)$ is a function on some region $D\times[a(\mathbf{x}),b(\mathbf{x})]$, for some real-valued functions $a(\mathbf{x})$ and $b(\mathbf{x})$, and such that both $f(\mathbf{x},t)$ and its partial derivative $\frac{\partial}{\partial x_i}f(\mathbf{x},t)$ are continuous in $\mathbf{x}$ and t. Also suppose that $a(\mathbf{x})$ and $b(\mathbf{x})$ are continuous with continuous partial derivatives $\frac{\partial}{\partial x_i}a(\mathbf{x})$ and $\frac{\partial}{\partial x_i}b(\mathbf{x})$ for $\mathbf{x}\in D$. Then*

$$\frac{\partial}{\partial x_i}\left(\int_{a(\mathbf{x})}^{b(\mathbf{x})} f(\mathbf{x},t)\,dt\right) = f(\mathbf{x},b(\mathbf{x}))\frac{\partial}{\partial x_i}b(\mathbf{x}) - f(\mathbf{x},a(\mathbf{x}))\frac{\partial}{\partial x_i}a(\mathbf{x}) + \int_{a(\mathbf{x})}^{b(\mathbf{x})}\frac{\partial}{\partial x_i}f(\mathbf{x},t)\,dt,$$

on D.

The Leibniz integral rule is sometimes applied in the case when $a(\mathbf{x}) = a$ and $b(\mathbf{x}) = b$ are constant. In this case, we get the following corollary:

Corollary B.2.1. *Suppose $f(\mathbf{x},t)$ is a function on some region $D\times[a,b]$, for some $a<b$, and such that both $f(\mathbf{x},t)$ and its partial derivative $\frac{\partial}{\partial x_i}f(\mathbf{x},t)$ are continuous in $\mathbf{x}$ and t. Then*

$$\frac{\partial}{\partial x_i}\left(\int_a^b f(\mathbf{x},t)\,dt\right) = \int_a^b \frac{\partial}{\partial x_i}f(\mathbf{x},t)\,dt,$$

on D.

B.3 Summation Formulas

If $(a_n)_{n\geq 1}$ is a sequence of complex numbers, for every $X \geq 0$ set

$$A(X) = \sum_{n \leq X} a_n.$$

Note that $A(X) = 0$ unless $X \geq 1$. The most well-known summation formula is **summation by parts**:

Theorem (Summation by parts). *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of complex numbers. Then for any positive integers N and M with $1 \leq M \leq N$, we have*

$$\sum_{M \leq n \leq N} a_n(b_{n+1} - b_n) = (a_{N+1}b_{N+1} - a_M b_M) - \sum_{M \leq n \leq N} b_{n+1}(a_{n+1} - a_n).$$

Upon iteratively applying summation by parts in the case $M = 1$ gives the following useful corollary:

Corollary B.3.1. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of complex numbers. Then for any positive integer $N \geq 1$, we have*

$$\sum_{n \leq N} a_n b_n = b_N A(N) - \sum_{n \leq N-1} A(n)(b_{n+1} - b_n).$$

There is another useful summation formula known as **Abel's summation formula** that lets one estimate discrete sums by integrals:

Theorem (Abel's summation formula). *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. For every X and Y with $0 \leq X < Y$ and continuously differentiable function $\phi : [X, Y] \to \mathbb{C}$, we have*

$$\sum_{X \leq n \leq Y} a_n \phi(n) = A(Y)\phi(Y) - A(X)\phi(X) - \int_X^Y A(u)\phi'(u)\, du.$$

There are also some useful corollaries. On the one hand, if we let $X = 0$, so that $A(X) = 0$, we get the following:

Corollary B.3.2. *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. For every $Y > 0$ and continuously differentiable function $\phi : [0, Y] \to \mathbb{C}$, we have*

$$\sum_{n \leq Y} a_n \phi(n) = A(Y)\phi(Y) - \int_0^Y A(u)\phi'(u)\, du.$$

On the other hand, if we take the limit as $Y \to \infty$ we obtain:

Corollary B.3.3. *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. For every $X \geq 0$ and continuously differentiable function $\phi : [X, \infty) \to \mathbb{C}$, we have*

$$\sum_{n \geq X} a_n \phi(n) = \lim_{Y \to \infty} A(Y)\phi(Y) - A(X)\phi(X) - \int_X^\infty A(u)\phi'(u)\, du.$$

Lastly, combining these cases by taking $X = 0$ and the limit as $Y \to \infty$, we arrive at the following result:

Corollary B.3.4. *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. For every continuously differentiable function $\phi : [0, \infty) \to \mathbb{C}$, we have*

$$\sum_{n \geq 1} a_n \phi(n) = \lim_{Y \to \infty} A(Y)\phi(Y) - \int_1^\infty A(u)\phi'(u)\, du.$$

B.4 Factorizations, Order & Rank

The **elementary factors**, also referred to as **primary factors**, are the entire functions $E_n(z)$ defined by

$$E_n(z) = \begin{cases} 1 - z & \text{if } n = 0, \\ (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^n}{n}} & \text{if } n \neq 0. \end{cases}$$

If $f(z)$ is an entire function then it admits a factorization in terms of its zeros and the elementary factors. This is called the **Weierstrass factorization** of $f(z)$:

Theorem (Weierstrass factorization). *Let $f(z)$ be an entire function with $\{a_n\}_{n\geq 1}$ the nonzero zeros of $f(z)$ counted with multiplicity. Also suppose that $f(z)$ has a zero of order m at $z = 0$ where it is understood that if $m = 0$ we mean $f(0) \neq 0$ and if $m < 0$ we mean $f(z)$ has a pole of order $|m|$ at $z = 0$. Then there exists an entire function $g(z)$ and sequence of nonnegative integers $(p_n)_{n\geq 1}$ such that*

$$f(z) = z^m e^{g(z)} \prod_{n\geq 1} E_{p_n}\left(\frac{z}{a_n}\right).$$

The Weierstrass factorization of $f(z)$ can be strengthened if $f(z)$ does not grow too fast. We say $f(z)$ is of **finite order** if there exists a $\rho_0 > 0$ such that

$$f(z) \ll e^{|z|^{\rho_0}},$$

for all $z \in \mathbb{C}$. The **order** ρ of $f(z)$ is the infimum of the ρ_0. Let $q = \lfloor \rho \rfloor$. If there is no such ρ_0, $f(z)$ is said to be of **infinite order** and we set $\rho = q = \infty$. Let $\{a_n\}_{n\geq 1}$ be the nonzero zeros of $f(z)$ that are not zero and ordered such that $a_n \to \infty$ as $n \to \infty$ if there are infinitely many zeros. Then we define the **rank** of $f(z)$ to be the smallest positive integer p such that the series

$$\sum_{n\geq 1} \frac{1}{|a_n|^{p+1}},$$

converges. If there is no such integer we set $p = \infty$ and if there are finitely many zeros we set $p = 0$. We set $g = \max(p, q)$ and call g the **genus** of $f(z)$. We can now state the **Hadamard factorization** of $f(z)$:

Theorem (Hadamard factorization). *Let $f(z)$ be an entire function of finite order ρ. If p is the rank and g is the genus then $g \leq \rho$. Moreover, let $\{a_n\}_{n\geq 1}$ be the nonzero zeros of $f(z)$ counted with multiplicity and suppose that $f(z)$ has a zero of order m at $z = 0$ where it is understood that if $m = 0$ we mean $f(0) \neq 0$ and if $m < 0$ we mean $f(z)$ has a pole of order $|m|$ at $z = 0$. Then there exists a polynomial $Q(z)$ of degree at most q such that*

$$f(z) = z^m e^{Q(z)} \prod_{n\geq 1} E_p\left(\frac{z}{a_n}\right).$$

Moreover, the sum

$$\sum_{n\geq 1} \frac{1}{|a_n|^{\rho+\varepsilon}},$$

converges.

B.5 The Phragmén-Lindelöf Convexity Principle

The **Phragmén-Lindelöf convexity principle** is a generic name for extending the maximum modulus principle to unbounded regions. The **Phragmén-Lindelöf convexity principle** for vertical strips is the case when the unbounded region is the vertical strip $a < \sigma < b$:

Theorem (Phragmén-Lindelöf convexity principle, vertical strip). *Suppose $f(s)$ is a holomorphic function on an open neighborhood of the vertical strip $a < \sigma < b$ such that $f(s) \ll e^{|s|^A}$ for some $A \geq 0$. Then the following hold:*

(i) If $|f(s)| \leq M$ for $\sigma = a, b$, that is on the boundary edges of the strip then $|f(s)| \leq M$ for all s in the strip.

(ii) Assume that there is a continuous function $g(t)$ such that

$$f(a+it) \ll g(t)^{\alpha} \quad \text{and} \quad f(b+it) \ll g(t)^{\beta},$$

for all $t \in \mathbb{R}$. Then

$$f(s) \ll g(t)^{\alpha\ell(\sigma)+\beta(1-\ell(\sigma))},$$

where ℓ is the linear function such that $\ell(a) = 1$ and $\ell(b) = 0$.

We will also need a variant. The **Phragmén-Lindelöf convexity principle** for vertical half-strips is the case when the unbounded region is the vertical half-strip $a < \sigma < b$ with $t > c$:

Theorem (Phragmén-Lindelöf convexity principle, vertical half-strip). *Suppose $f(s)$ is a holomorphic function on an open neighborhood of the vertical strip $a < \sigma < b$ with $t > c$ such that $f(s) \ll e^{|s|^A}$ for some $A \geq 0$. Then the following hold:*

(i) If $|f(s)| \leq M$ for $\sigma = a, b$ with $t \geq c$ and $t = c$ with $a \leq \sigma \leq b$, that is on the boundary edges of the half-strip then $|f(s)| \leq M$ for all s in the strip.

(ii) Assume that there is a continuous function $g(t)$ such that

$$f(a+it) \ll g(t)^{\alpha} \quad \text{and} \quad f(b+it) \ll g(t)^{\beta},$$

for all $t \geq c$. Then

$$f(s) \ll g(t)^{\alpha\ell(\sigma)+\beta(1-\ell(\sigma))},$$

where ℓ is the linear function such that $\ell(a) = 1$ and $\ell(b) = 0$.

B.6 Bessel Functions

For any $\nu \in \mathbb{C}$, the **Bessel equation** is the ODE

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0.$$

There are two linearly independent solutions to this equation. One solution is the **Bessel function of the first kind** $J_\nu(z)$ defined by

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}.$$

For integers n, $J_n(z)$ is entire and we have

$$J_n(z) = (-1)^n J_{-n}(z).$$

Otherwise, $J_\nu(z)$ has a pole at $z = 0$ and $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent solutions to the Bessel equation. The other solution is the **Bessel function of the second kind** $Y_\nu(z)$ defined by

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)},$$

for non-integers ν, and for integers n is

$$Y_n(z) = \lim_{\nu \to n} Y_\nu(z).$$

For any integer n, we also have

$$Y_n(z) = (-1)^n Y_{-n}(z).$$

For the J-Bessel function there is also an important integral representation called the **Schläflin integral representation**:

Proposition (Schläflin integral representation for the J-Bessel function). *For any $\nu \in \mathbb{C}$,*

$$J_\nu(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \int_{-\infty}^{(0^+)} t^{-(\nu+1)} e^{t - \frac{z^2}{4t}}\, dt,$$

provided $|\arg(z)| < \frac{\pi}{2}$ and where the contour is the Hankel contour about the negative real axis.

The **modified Bessel equation** is the ODE

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} - (z^2 + \nu^2)w = 0.$$

Like the Bessel equation, there are two linearly independent solutions. One solution is the **modified Bessel function of the first kind** $I_\nu(x)$ given by

$$I_\nu(z) = i^{-\nu} J_\nu(iz) = \sum_{n \geq 0} \frac{1}{n!\Gamma(n + \nu + 1)} \left(\frac{z}{2}\right)^{2n+\nu}.$$

For $n \in \mathbb{Z}$, this solution is symmetric in n which we express as

$$I_n(z) = I_{-n}(z).$$

We also have a useful integral representation in a half-plane:

Proposition B.6.1. *For any $\nu \in \mathbb{C}$,*

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos(t)}\cos(\nu t)\, dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh(t) - \nu t}\, dt,$$

provided $|\arg(z)| < \frac{\pi}{2}$.

From this integral representation we can show the following asymptotic:

Lemma B.6.1. *For any $\nu \in \mathbb{C}$,*

$$I_\nu(z) \sim_\varepsilon \sqrt{\frac{1}{2\pi z}} e^z,$$

provided $|\arg(z)| \leq \frac{\pi}{2} - \varepsilon$.

The other solution is the **modified Bessel function of the second kind** $K_\nu(z)$ defined by

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)},$$

for non-integers ν, and for integers n is

$$K_n(z) = \lim_{\nu \to n} K_\nu(z).$$

This is one of the more important types of Bessel functions as they appear in the Fourier coefficients of certain Eisenstein series. This function is symmetric in ν (even when ν is an integer) which we express as

$$K_\nu(z) = K_{-\nu}(z).$$

We also have a very useful integral representation in a half-plane:

Proposition B.6.2. *For any $\nu \in \mathbb{C}$,*

$$K_\nu(z) = \int_0^\infty e^{-z\cosh(t)} \cosh(\nu t)\, dt,$$

provided $|\arg(z)| < \frac{\pi}{2}$.

From this integral representation it does not take much to show the following asymptotic:

Lemma B.6.2. *For any $\nu \in \mathbb{C}$,*

$$K_\nu(z) \sim_\varepsilon \sqrt{\frac{\pi}{2z}} e^{-z},$$

provided $|\arg(z)| \leq \frac{\pi}{2} - \varepsilon$.

B.7 Whittaker Functions

For $\kappa, \mu \in \mathbb{C}$, the **Whittaker equation** is the ODE

$$\frac{dw}{dz} + \left(\frac{1}{4} - \frac{\kappa}{z} - \frac{\frac{1}{4} - \mu^2}{z^2}\right) w = 0.$$

There are two linearly independent solutions to this equation provided $-2\mu \notin \mathbb{Z}_+$. If we additionally assume that $w(z) = o(e^{2\pi y})$ as $y \to \infty$ then there is only one linearly independent solution. This solution is the **Whittaker function** $W_{\kappa,\mu}(z)$. It can be expressed in the form

$$W_{\kappa,\mu}(z) = z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}} U\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu, z\right),$$

where $U(\alpha, \beta, z)$ is the **confluent hypergeometric function** initially defined by

$$U(\alpha, \beta, z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zu} u^{\alpha-1} (1+u)^{\beta-\alpha-1}\, du,$$

for $\mathrm{Re}(\alpha) > 0$ and $x > 0$, and then by analytic continuation to $\mathbb{C}^3$. From this integral representation we can show the following asymptotic:

Lemma B.7.1. *For any $\kappa, \mu \in \mathbb{C}$ with $-2\mu \notin \mathbb{Z}_+$,*

$$W_{\kappa,\mu}(z) \sim_\varepsilon z^\kappa e^{-\frac{z}{2}},$$

provided $|\arg(z)| \leq \frac{\pi}{2} - \varepsilon$.

The Whittaker function also satisfies the important symmetry properties

$$W_{\kappa,\mu}(z) = W_{\kappa,-\mu}(z) \quad \text{and} \quad \overline{W_{\kappa,\mu}(z)} = W_{\overline{\kappa},\overline{\mu}}(\overline{z}).$$

In particular, $W_{\kappa,\mu}(z)$ is conjugate symmetric if κ and z are both real and μ is either real or purely imaginary. The Whittaker function also has a simplified form in special cases:

Theorem B.7.1. *For any $\nu, \alpha \in \mathbb{C}$ with $-2\nu \notin \mathbb{Z}_+$,*

$$W_{0,\nu}(z) = \left(\frac{z}{\pi}\right)^{\frac{1}{2}} K_\nu\left(\frac{z}{2}\right) \quad \textit{and} \quad W_{\alpha,\alpha-\frac{1}{2}}(z) = z^\alpha e^{-\frac{z}{2}}.$$

B.8 Lattice Sums

Consider $\mathbb{R}^d$ with the standard inner product $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||$. We are often interested in series that are obtained by summing over the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. In particular, we have the following general result:

Theorem B.8.1. *Let $d \geq 1$ be an integer. Then*

$$\sum_{\mathbf{a} \in \mathbb{Z}^d - \{\mathbf{0}\}} \frac{1}{||\mathbf{a}||^s},$$

is locally absolutely uniformly convergent in the region $\sigma > d$.

In a practical setting, we usually restrict to the case $d = 2$. In this setting, with a little more work can show a more useful result:

Proposition B.8.1. *Let $z \in \mathbb{H}$. Then*

$$\sum_{(n,m) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \frac{1}{|nz + m|^s},$$

is locally absolutely uniformly convergent in the region $\sigma > 2$. In addition, it is locally absolutely uniformly convergent as a function of z provided $\sigma > 2$.

Appendix C

Algebra

C.1 Finitely Generated Modules Over Principal Ideal Domains

Let M be a finitely generated module over a principal ideal domain R. The following result is the **structure theorem for finitely generated modules over principal ideal domains**:

Theorem (Structure theorem for finitely generated modules over principal ideal domains). *Let M be a finitely generated module over a principal ideal domain R. Then*

$$M \cong R^n \oplus R/d_1M \oplus \cdots \oplus R/d_tR,$$

for a unique $n \geq 0$ and $d_i \in R$ for $1 \leq i \leq t$ such that

$$d_tR \subset d_{t-1}R \subset \cdots \subset d_1R.$$

In the special case that M is a finitely generated abelian group, we obtain the **structure theorem for finitely generated abelian groups**:

Theorem (Structure theorem for finitely generated abelian groups). *Let M be a finitely generated abelian group. Then*

$$M \cong \mathbb{Z}^n \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_t\mathbb{Z},$$

for a unique $n \geq 0$ and $k_i \in \mathbb{Z}$ such that $k_i \mid k_{i+1}$ for $1 \leq i \leq t-1$.

In particular, any subgroup of a free abelian group M of rank n is also free abelian and has rank at most n. If $N \leq M$ is a subgroup of rank n then the quotient is also finite:

Proposition C.1.1. *Let M be a free abelian group of rank n and suppose $N \leq M$ is a subgroup that is also a free abelian group of rank n. Then $|M/N|$ is finite and*

$$|M/N| = |\det(A)|,$$

for any base change matrix A from a basis of M to a basis of N.

C.2 Galois Theory

Let E/F be an algebraic extension. We say that E/F is **Galois** if it is both normal and separable. Actually, there are several equivalent conditions for E/F to be Galois provided E/F is finite:

Proposition C.2.1. *Let E/F be a finite extension. Then the following are equivalent:*

(i) E/F is Galois.

(ii) E is the splitting field of some polynomial in $F[x]$.

(iii) $|\operatorname{Aut}_F(E)| = |E/F|$.

If E/F is Galois, we define **Galois group** $\operatorname{Gal}(E/F)$ by

$$\operatorname{Gal}(E/F) = \operatorname{Aut}_F(E).$$

In other words, $\operatorname{Gal}(E/F)$ is the group of automorphism of E that fix F pointwise. Now if H is a subgroup of $\operatorname{Gal}(E/F)$, we define the **fixed field** E^H of E by H to be

$$E^H = \{\alpha \in E : \sigma(\alpha) = \alpha \text{ for all } \sigma \in H\}.$$

In other words, E^H is the intermediate field of E/F that consists of those elements that are fixed pointwise by every automorphism in H. If E/F is finite, say of degree n, then being separable means the primitive element theorem applies and so $E = F(\theta)$ for some $\theta \in E$. The minimal polynomial $m_\theta(x)$ of θ over F is necessarily of degree n and θ has n conjugates $\theta_1, \ldots, \theta_n$, with say $\theta_1 = \theta$, which are the roots of $m_\theta(x)$. The automorphisms of $\operatorname{Gal}(E/F)$ are then the induced by mapping θ to one of its conjugates $\theta_1, \ldots, \theta_n$. The most important result of Galois theory is the **fundamental theorem of Galois theory** which relates intermediate fields of E/F to subgroups of the Galois group provided E/F is finite:

Theorem C.2.1 (Fundamental theorem of Galois theory). *Let E/F be a finite Galois extension and let K be an intermediate field. Then E/K is Galois. Moreover, the maps*

$$K \to \operatorname{Gal}(E/K) \quad \textit{and} \quad H \to E^H,$$

are inclusion-reversing bijections between the intermediate fields of E/F and the subgroups of $\operatorname{Gal}(E/F)$. *We have*

$$\frac{|\operatorname{Gal}(E/F)|}{|H|} = |E^H/F|.$$

In addition, E^H/F is Galois if and only if H is a normal subgroup of $\operatorname{Gal}(E/F)$ *and in this case there is an isomorphism*

$$\operatorname{Gal}(E^H/F) \cong \operatorname{Gal}(E/F)/H.$$

Galois groups also behave nicely with respect to composite fields:

Proposition C.2.2. *Let E_1/F and E_2/F be finite Galois extensions and let E be the composite of E_1 and E_2. Then E/F is Galois. Moreover, E_1 and E_2 are linearly disjoint over F in $\overline{F}$ if and only if $E_1 \cap E_2 = F$.*

C.3 Character Groups

For any finite abelian group G, a **character** φ is a homomorphism $\varphi : G \to \mathbb{C}$. They form a group, denoted $\widehat{G}$, under multiplication called the **character group** of G. If G is an additive group, we say that any $\varphi \in \Gamma$ is a **additive character**. Similarly, if G is a multiplicative group, we say that any $\varphi \in \Gamma$ is a **multiplicative character**. In any case, if $|G| = n$ then $\varphi(g)^n = \varphi(g^n) = 1$ so that φ takes values in the n-th roots of unity. Moreover, to every character φ there is its **conjugate character** $\overline{\varphi}$ defined by $\overline{\varphi}(g) = \overline{\varphi(a)}$. Clearly the conjugate character is also a character. Since φ takes its value in the roots of unity, $\overline{\varphi(a)} = \varphi(a)^{-1}$ so that $\overline{\varphi} = \varphi^{-1}$. One of the central theorems about characters is that the character group of G is isomorphic to G:

Proposition C.3.1. *Any finite abelian group G is isomorphic to its character group. That is,*

$$G \cong \widehat{G}.$$

The characters also satisfy certain **orthogonality relations**:

Proposition (Orthogonality relations). *Let G be a finite abelian group.*

(i) For any two characters χ and ψ of G,

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi}(g) = \delta_{\chi,\psi}.$$

(ii) For any $g, h \in G$,

$$\frac{1}{|G|}\sum_{\chi\in\widehat{G}}\chi(g)\overline{\chi}(h) = \delta_{g,h}.$$

C.4 Representation Theory

Let G be a group, F be a field, and V be an F-vector space. A **representation** ρ of G on V is a map

$$\rho : G \times V \to V \qquad (g, v) \mapsto \rho(g, v) = g \cdot v,$$

such that the following properties are satisfied:

1. For any $g \in G$, the map
$$\rho : V \to V \qquad v \mapsto g \cdot v,$$
is linear.

2. For any $g, h \in G$ and $v \in V$,
$$1 \cdot v = v \quad \text{and} \quad g \cdot (h \cdot v) = (gh) \cdot v.$$

Therefore ρ defines an action of G on V. An equivalent definition of a representation of G on V is a homomorphism from G into $\mathrm{Aut}(V)$. We also denote this homomorphism by ρ. If the dimension of V is n then ρ is said to be **n-dimensional**. We say that (ρ, W) is a **subrepresentation** of (V, ρ) if $W \subseteq V$ is a G-invariant subspace. In particular, (ρ, W) is a representation itself. Lastly, if (ρ_1, V_1) and (ρ_2, V_2) are two representations, we can form the **direct sum representation** $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$ where $\rho_1 \oplus \rho_2$ acts

diagonally on $V_1 \oplus V_2$. A natural question to ask is how representation can be decomposed as a direct sum of other representations. We say ρ is **irreducible** if it contains no proper G-invariant subspaces and is **completely irreducible** if it decomposes as a direct sum of irreducible subrepresentations.
We will only need one very useful theorem about representations when G is a finite abelian group and V is a complex vector space. In this case G has a group of characters $\widehat{G}$, and the underlying complex vector space V is completely reducible with respect to the characters of G:

Theorem C.4.1. *Let V be a complex vector space and let Φ be a representation of a group G on V. If G is a finite abelian group then*

$$V = \bigoplus_{\chi \in \widehat{G}} V_\chi,$$

where

$$V_\chi = \{v \in V : g \cdot v = \chi(g)v \text{ for all } g \in G\}.$$

In particular, V is completely reducible and every irreducible subrepresentation is 1-dimensional.

Appendix D

Topology

D.1 Fundamental Domains

Let G be a group acting on a connected Hausdorff space X by automorphisms. Then we can consider the quotient space X/G of X by the action of G. We would like the quotient X/G to inherit properties of G. For this, we require G to satisfy two properties. The first is that G is a discrete group. For the second, we say G acts **properly discontinuously** if for every $x \in X$ there exists a neighborhood U_x of x such that the intersection $gU_x \cap U_x$ is empty for all $g \in G$ unless g is the identity. In the case G is discrete and acts properly discontinuously, the quotient X/G inherits nice topological properties:

Proposition D.1.1. *Let G be a group acting on a a connected Hausdorff space X by automorphisms. If G is discrete and acts properly discontinuously then X/G is connected Hausdorff.*

In the case of Proposition D.1.1, we often want to consider a useful set of representatives of X/G. A **fundamental domain** for X/G is a closed subset $D \subseteq X$ satisfying the following conditions:

(i) Any point in X is G-equivalent to a point in D.

(ii) If two points in D are G-equivalent via a non-identity element then they lie on the boundary of D.

(iii) The interior of D is a domain.

In other words, D is a complete set of representatives (possibly with overlap on the boundary) for X/G that has a nice topological structure with respect to X. Note that if D is a fundamental domain then so is gD for any $g \in G$ and moreover

$$X = \bigcup_{g \in G} gD.$$

In particular, the choice of fundamental domain is not unique. Intuitively, a fundamental domain is a geometric realization of X/G which is often more fruitful than thinking of X/G as an abstract set of equivalence classes. Indeed, if X is a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ then property (iii) ensures that we can integrate over D.

Appendix E

Miscellaneous

E.1 Special Integrals

Below is a table of well-known integrals that are used throughout the text:

Reference	Assumptions	Integral
Gaussian		$\displaystyle\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$
[Dav80]	$c > 0$	$\displaystyle\frac{1}{2\pi i}\int_{(c)} y^s\,\frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$
[Gol06]	$s, \nu \in \mathbb{C}, \mathrm{Re}(s+\nu) > -1$	$\displaystyle\int_0^{\infty} K_\nu(y) y^s\,\frac{dy}{y} = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)$
[Gol06]	$n \in \mathbb{Z}$, $s \in \mathbb{C}$, $y > 0$	$\displaystyle\int_{-\infty}^{\infty} \frac{e^{-2\pi i n x y}}{(x^2+1)^s}\,dx = \begin{cases} \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} & \text{if } n = 0, \\ \frac{2\pi^s \lvert n\rvert^{s-\frac{1}{2}} y^{s-\frac{1}{2}}}{\Gamma(s)} K_{s-\frac{1}{2}}(2\pi \lvert n\rvert y) & \text{if } n \neq 0. \end{cases}$

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