ANALYTIC CONTINUATION OF DIRICHLET L-FUNCTIONS & THE MELLIN TRANSFORM

ACHINTYA POLAVARAPU, HENRY TWISS

ABSTRACT. We describe the analytic continuation of Dirichlet L-functions $L(s,\chi)$ arising from primitive characters of modulus $q > 1$ by taking the Mellin transform of a theta function. This is preluded by a recount of the analytic continuation of the Riemann zeta function in a similar manner. After proving the analytic continuation of these Dirichlet series, we give a short discussion on the underlying technique of taking the Mellin transform of a theta function and discuss the case of L-functions corresponding to integral weight modular forms.

1. THE ζ-FUNCTION & HISTORICAL REMARKS

The **Riemann zeta function** $\zeta(s)$ is the Dirichlet series defined as

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

The zeta function is intimately connected to the distribution of primes (see [1, 2]) and has been one of the cornerstones of analytic number theory since its birth. It arose from Euler's study of sums of the form

$$\sum_{n\geq 1} \frac{1}{n^k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots ,$$

where $k \geq 2$ is an integer (see [3] for more). By standard series tests, $\zeta(s)$ is absolutely uniformly convergent in the half-plane $\Re(s) > 1$ and hence defines a holomorphic function there. While it has a singularity at $s = 1$ by Landau's theorem, in 1859 Riemann analytically continued $\zeta(s)$ to all of $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1 (see [4]). This was achieved by deriving the integral representation for $\Re(s) > 1$:

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \left[-\frac{1}{s(1-s)} + \int_1^\infty \theta(x) x^{(1-s)/2} \frac{dx}{x} + \int_1^\infty \theta(x) x^{s/2} \frac{dx}{x} \right], \tag{1.1}$$

where $\theta(x) = \sum_{n\geq 1} e^{-\pi n^2 x}$. This is "essentially" **Jacobi's theta function** $\vartheta(x)$ as

$$\vartheta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = 1 + 2\sum_{n\geq 1} e^{-\pi n^2 x} = 1 + 2\theta(x).$$

One derives 1.1 from the preliminary integral representation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty \theta(x) x^{s/2} \frac{dx}{x}. \tag{1.2}$$

This preliminary integral representation is achieved by taking the Mellin transform of the theta function $\theta(x)$. Unfortunately, while $\theta(x)$ admits exponential decay as $x \to \infty$, it does not converge

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as $x \to 1$ and so we cannot conclude that the integral in 1.2 is analytic on $\mathbb{C}$. To turn 1.2 into 1.1, Riemann used the following result known to Jacobi (see [4]), namely

$$\vartheta(s) = \frac{1}{\sqrt{s}}\vartheta\left(\frac{1}{s}\right),$$

to obtain 1.1. This is necessary because in 1.1, $\theta(x)$ converges at $s = 1$ and so the integrals are analytic on $\mathbb{C}$. Therefore the right-hand side is naturally defined for all $s \in \mathbb{C} - \{1\}$ and at $s = 1$ the polynomial term has a simple pole of residue 1. So taking the right-hand side as the definition of $\zeta(s)$, we see that $\zeta(s)$ is analytic on $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1 as previously mentioned. Moreover, by the natural symmetry of the two integral terms under $s \to 1 - s$ and invariance of the polynomial term, $\zeta(s)$ also possesses the symmetric functional equation

$$\frac{\Gamma\left(s/2\right)}{\pi^{s/2}}\zeta(s) = \frac{\Gamma\left((1-s)/2\right)}{\pi^{(1-s)/2}}\zeta(1-s).$$

This can be viewed as the Mellin transform lifting of the transformation law for $\vartheta(s)$ to $\zeta(s)$, and it is with this functional equation that we can use the Dirichlet series representation of $\zeta(s)$ for $\Re(s) > 1$ to determine information about $\zeta(s)$ in the region $\Re(s) < 0$.

2. The Functional Equation for Dirichlet L-functions

The Dirichlet L-function attached to the character χ is the series

$$L(s,\chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

Throughout let q be the conductor of χ. If $q = 1$ then we recover $\zeta(s)$. Since $\chi(n) \ll 1$, $L(s,\chi)$ converges absolutely uniformly for $\Re(s) > 1$. The series does not necessarily converge absolutely uniformly for $\Re(s) \leq 1$, but it does admit analytic continuation to this region analogous to the case for $\zeta(s)$. Precisely, we will show the following:

Theorem 2.1. *For a primitive Dirichlet character χ with conductor $q > 1$, $L(s,\chi)$ admits analytic continuation to $\mathbb{C}$.*

We will derive the analytic continuation of $L(s,\chi)$ by expressing the L-function as an integral which will converge on all of $\mathbb{C}$. Details in the argument depend on if χ is even or odd, so to treat both cases simultaneously we define $\delta_\chi \in \{0,1\}$ by $\chi(-1) = (-1)^{\delta_\chi}$.

Proof sketch. Upon substituting $s \to s + \delta_\chi$ into the definition of the gamma function, we obtain

$$\chi(n)\Gamma\left((s+\delta_\chi)/2\right) = \pi^{(s+\delta_\chi)/2} n^s \int_0^\infty \chi(n) n^{\delta_\chi} e^{-\pi n^2 x} x^{(s+\delta_\chi)/2}\,\frac{dx}{x}.$$

Proceeding exactly as for the zeta function (sum over $n \geq 1$ and apply some minor algebra), we arrive at the preliminary integral representation:

$$L(s,\chi) = \frac{\pi^{(s+\delta_\chi)/2}}{\Gamma\left((s+\delta_\chi)/2\right)} \int_0^\infty \theta_\chi(x) x^{(s+\delta_\chi)/2}\,\frac{dx}{x}. \tag{2.1}$$

where

$$\theta_\chi(x) = \sum_{n \geq 1} \chi(n) n^{\delta_\chi} e^{-\pi n^2 x} = \frac{1}{2}\sum_{n \in \mathbb{Z}} \chi(n) n^{\delta_\chi} e^{-\pi n^2 x}.$$

This is essentially a twisted version of Jacobi's theta function. The key insight is that it is a sum of Schwarz functions over a lattice and so we can expect that an application of Poisson summation will give a functional equation of shape $s \to \frac{1}{s}$ just as for Jacobi's theta function in the case of $\zeta(s)$. Since our theta function has a character attached, we first sieve out the character:

$$\theta_\chi(x) = \sum_{a \ (\mathrm{mod}\ q)} \chi(a) \sum_{m\in\mathbb{Z}} (mq+a)^{\delta_\chi} e^{-\pi(mq+a)^2 x}.$$

Now we apply Poisson summation to the inner sum. Set $f(y) = (yq+a)^{\delta_\chi} e^{-\pi(yq+a)^2 x}$. Making a change of variable and completing the square of $(yq+a)^2 x + 2\pi i t y$ in the exponent, the Fourier transform of f becomes

$$\hat{f}(t) = \int_{-\infty}^{\infty} (yq+a)^{\delta_\chi} e^{-\pi(yq+a)^2 x} e^{-2\pi i t y}\, dx = \frac{e^{\frac{2\pi i a t}{q}} e^{-\frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\delta_\chi}{2}}} \int_{-\infty}^{\infty} x^{\delta_\chi} e^{-\pi\left(x + \frac{it}{q\sqrt{s}}\right)^2} dx.$$

One now complexifies the integral and shifts the line of integration to to $\Im(z) = \frac{t}{q\sqrt{s}}$, with no addition of residues since the integrand is holomorphic, obtaining

$$\frac{e^{\frac{2\pi i a t}{q}}}{q} \frac{e^{-\frac{\pi t^2}{q^2 s}}}{\sqrt{s}} \int_{-\infty}^{\infty} \left(x - \frac{it}{qs}\right)^{\delta_\chi} dx = \frac{e^{\frac{2\pi i a t}{q}}}{q} \frac{e^{-\frac{\pi t^2}{q^2 s}}}{\sqrt{s}} \left(\frac{it}{qs}\right)^{\delta_\chi},$$

where the equality follows by realising the integral as essentially a Gaussian integral. Poisson summation then yields

$$\begin{aligned}
\theta_\chi(x) &= \sum_{a \ (\mathrm{mod}\ q)} \chi(a) \sum_{m\in\mathbb{Z}} (mq+a)^{\delta_\chi} e^{-\pi(mq+a)^2 x} \\
&= \sum_{a \ (\mathrm{mod}\ q)} \chi(a) \sum_{t\in\mathbb{Z}} \frac{e^{\frac{2\pi i a t}{q}}}{q} \frac{e^{-\frac{\pi t^2}{q^2 s}}}{\sqrt{s}} \left(\frac{it}{qs}\right)^{\delta_\chi} \\
&= \frac{1}{i^{\delta_\chi} q^{1+\delta_\chi} s^{\frac{1}{2}+\delta_\chi}} \sum_{t\in\mathbb{Z}} t^{\delta_\chi} e^{-\frac{\pi t^2}{q^2 s}} \tau(t,\chi) && \text{evaluation of } \tau(t,\chi) \\
&= \frac{\varepsilon_\chi}{i^{\delta_\chi} q^{1+\delta_\chi} s^{\frac{1}{2}+\delta_\chi}} \theta_{\overline{\chi}}\left(\frac{1}{q^2 x}\right).
\end{aligned}$$

This is the appropriate transformation law for the twisted theta function. We can now derive the symmetric integral representation for $L(s,\chi)$. Ignoring the gamma factor in 2.1 and splitting the integral at $x = 1/q$, the fixed point of the transformation law, we have

$$\int_0^\infty \theta_\chi(x) x^{(s+\delta_\chi)/2} \frac{dx}{x} = \int_0^{1/q} \theta_\chi(x) x^{(s+\delta_\chi)/2} \frac{dx}{x} + \int_{1/q}^\infty \theta_\chi(x) x^{(s+\delta_\chi)/2} \frac{dx}{x}. \tag{2.2}$$

Now change variables $x \to \frac{1}{q^2 x}$ in the first integral and apply the transformation law:

$$\int_0^{1/q} \theta_\chi(x) x^{(s+\delta_\chi)/2} \frac{dx}{x} = \int_{1/q}^\infty \theta_\chi\left(\frac{1}{q^2}\right) x^{-(s+\delta_\chi)/2} \frac{dx}{x} = \frac{\varepsilon_\chi}{i^{\delta_\chi}} \int_{1/q}^\infty \theta_{\overline{\chi}}(x) x^{((1-s)+\delta_\chi)/2} \frac{dx}{x}.$$

Substituting back into 2.2 and applying 2.1 yields

$$L(s,\chi) = \frac{\pi^{(s+\delta_\chi)/2}}{\Gamma\left((s+\delta_\chi)/2\right)} \left[\frac{\varepsilon_\chi}{i^{\delta_\chi}} \int_{1/q}^\infty \theta_{\overline{\chi}}(x) x^{((1-s)+\delta_\chi)/2} \frac{dx}{x} + \int_{1/q}^\infty \theta_\chi(x) x^{(s+\delta_\chi)/2} \frac{dx}{x}\right] \tag{2.3}$$

□

By virtue of the decay of $\theta_\chi(x)$ both integrals in 2.3 are holomorphic on $\mathbb{C}$ and thus the right-hand side gives analytic continuation to $\mathbb{C}$. Also, the symmetry of the right-hand side as $s \to 1-s$ immediately results in the following corollary which is better known as the functional equation for $L(s,\chi)$:

Corollary 2.2.

$$q^{s/2}\frac{\Gamma\left((s+\delta_\chi)/2\right)}{\pi^{(s+\delta_\chi)/2}}L(s,\chi) = \frac{\varepsilon_\chi}{i^{\delta_\chi}}q^{(1-s)/2}\frac{\Gamma\left(((1-s)+\delta_\chi)/2\right)}{\pi^{((1-s)+\delta_\chi)/2}}L(1-s,\chi).$$

3. The Mellin Transform & Theta Functions

The unifying idea underpinning functional equations of L-functions is to find an integral representation that is symmetric under $s \to 1-s$. The integral representation is obtained by taking the Mellin transform of a theta function, and the symmetry of the integral is lifted from a transformation law for the theta function. Let us being with the Mellin transform. If f is some continuous function, then the **Mellin transform** $\{\mathcal{M}f\}(s)$ of f is given by

$$\{\mathcal{M}f\}(s) = \int_0^\infty f(x)x^s\,\frac{dx}{x}.$$

If f is a sufficiently nice function, say bounded near 0 and of exponential decay near ∞, this integral converge in a half-plane. The classical example is when $f = e^{-x}$ so that $\{\mathcal{M}e^{-x}\}(s) = \Gamma(s)$. In our case, we want f to be a theta function. A **theta function** is a absolutely convergent series that is a sum of exponentials over $\mathbb{Z}$ that is symmetric in the sign of $\mathbb{Z}$. Both the zeta function and Dirichlet L-functions are associated to a theta function:

$$\zeta(s) \longleftrightarrow \theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = 1 + 2\sum_{n\geq 1} e^{-\pi n^2 x},$$
$$L(s,\chi) \longleftrightarrow \theta_\chi(x) = \sum_{n\in\mathbb{Z}} \chi(n)n^{\delta_\chi}e^{-\pi n^2 x} = 2\sum_{n\geq 1} \chi(n)n^{\delta_\chi}e^{-\pi n^2 x}.$$

By "associated" we mean that if one takes the Mellin transform over the subsum $n \geq 1$ on the left-hand side, then the corresponding L-functions on the right-hand side is obtained up to gamma factors. For example, this is 2.1. More generally, given some theta function $\theta(x)$ we can obtain an L-functions $L(s,\theta)$ by taking the Mellin transform. In order to obtain a functional equation for the L-functions, the theta function must admit a transformation law:

$$\theta(x) \sim \theta\left(1/cx\right),$$

for some $c > 0$. In this case, we can decompose the Mellin transform as

$$\{\mathcal{M}f\}(s) = \int_0^{1/\sqrt{c}} f(x)x^s\,\frac{dx}{x} + \int_{1/\sqrt{c}}^\infty f(x)x^s\,\frac{dx}{x}.$$

Making the change of variables $x \to 1/cx$ to the first integral, we can apply the transformation law and symmetrize the Mellin transformation to respect $s \to 1-s$ as much as possible. Roughly,

$$L(s,\theta) = \text{polar factor} + \int_{1/\sqrt{c}}^\infty \theta(x)x^{1-s}\,\frac{dx}{x} + \int_{1/\sqrt{c}}^\infty \theta(x)x^s\,\frac{dx}{x}$$

The resulting integrals will be analytic by virtue of the rapid decay of $\theta(x)$, and therefore give analytic continuation of the L-functions to $\mathbb{C}$. The functional equation then follows immediately from the symmetry of the integral representation.

Let's give an example. If f is a weight k cuspidal modular form on the full modular group $\mathrm{PSL}_2(\mathbb{Z})$, then f admits a Fourier expansion at the ∞ cusp:

$$f(z) = \sum_{n \geq 1} a(n) e^{2\pi i n z}.$$

We can package the Fourier coefficient of f into an L-functions $L(s, f)$ called the L-functions associated to f:

$$L(s, f) = \sum_{n \geq 1} \frac{a_f(n)}{n^s}.$$

where $a_f(n) = a(n) n^{-(k-1)/2}$. By the Ramanujan conjecture (see [5]), $a_f(n) \ll 1$ so that $L(s, f)$ converges absolutely uniformly on compact sets for $\Re(s) > 1$. We would like to analytically continue $L(s, f)$ in the same way as for the zeta function an Dirichlet L-functions. What's the underlying theta function? Well, it comes naturally equip to f as the Fourier series of f along the positive imaginary axis:

$$f(iy) = \sum_{n \geq 1} a_\infty(n) e^{-2\pi n y}.$$

Due to the negative sign in the exponent, it exhibits the required exponential decay and by modularity

$$f\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} iy\right) = (-iy)^k f\left(1/iy\right).$$

This transformation law is more geometric in nature since the modularity of f describes how f changes under a Möbius transformation.

References

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