Analytic Number Theory

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Contents

Part I

Background Material

Chapter 1

Preliminaries

There is quite a bit of knowledge that most authors assume one is fluent in when writing any text on analytic number theory that is not necessarily standard material every reader knowns. A good selection is the following:

- Asymptotics,
- Dirichlet Characters,
- Special Sums,
- Integration Techniques & Transforms,
- The Gamma Function.

This is not an extensive list (depending on whom is writing), but it is a decent one for sure. In the interest of keeping this text mostly self-contained, this chapter is dedicated to the basics of these topics as they are the gadgets that will take center stage in our later investigations. A well-versed reader is encouraged to skim these sections for completeness. On the other hand, readers who are not completely comfortable these topics are encouraged to read this chapter in full and accept the material mentioned without proof as black box. The mathematics presented in this chapter belongs to an analytic number theorist's tool box rather than being pure analytic number theory. In order to improve the readability of the remainder of the text we will use the results presented here without reference unless it is a matter of clarity. As for standard knowledge, we assume familiarity with basic number theory, complex analysis, real analysis, functional analysis, topology, and algebra. We have also outsourced specific subtopics to the appendix and we will reference them when necessary.

1.1 Notational Conventions

Here we make some notational conventions throughout the rest of the text unless specified otherwise:

- The symbol ε denotes a small positive constant ($\varepsilon > 0$) that is not necessarily the same from line to line.
- If $a \in (\mathbb{Z}/m\mathbb{Z})^*$, we will always let $\overline{a}$ denote the multiplicative inverse. That is, $a\overline{a} \equiv 1 \pmod{m}$.

- For the complex variables z, s, and u, we write
$$z = x + iy, \quad s = \sigma + it, \quad \text{and} \quad u = \tau + ir,$$
for the real and imaginary parts of these variables respectively. In some cases we make exceptions and this will always be clear from context. Moreover, in certain expressions we often write $\mathrm{Im}(z)$ for clarity.

- If $r \in \mathbb{Z}$ denotes the order of a possible pole of a complex function, $r \geq 0$ if it is a pole and $r \leq 0$ if it is a zero.

- The nontrivial zeros of an L-function will be denoted by $\rho = \beta + i\gamma$ unless specified otherwise.

- By log we will always mean the principal branch of the logarithm.

- For a sum $\sum$ over integers satisfying a congruence condition, $\sum'$ will denote the sum restricted to relatively prime integers satisfying the same congruence.

- We will write $\int_{(a)}$ for the complex integral over the line whose real part is a and with positive orientation.

- $\delta_{a,b}$ will denote the indicator function for $a = b$. That is, $\delta_{a,b} = 1, 0$ according to if $a = b$ or not.

1.2 Asymptotics

Much of the language of analytic number theory is given in terms of asymptotics as it allows us to discuss approximate growth and dispense with superfluous constants. For this reason, asymptotics will be the first material that we will present. The asymptotics that we will cover are listed in the following table:

Asymptotic	Notation
Big O	$f(z) = O(g(z))$
Vinogradov's symbol	$f(z) \ll g(z)$
Order of magnitude symbol	$f(z) \asymp g(z)$
Little o	$f(z) = o(g(z))$
Asymptotic equivalence	$f(z) \sim g(z)$
Omega symbol	$f(z) = \Omega(g(z))$

Implicit in all of these asymptotics is some limiting process $z \to z_0$ where z_0 is some complex number or ∞ (and $\pm\infty$ for the real case respectively). If z_0 is finite, then it is understood that the asymptotic is assumed to hold for all z such that $|z - z_0| < \delta$ for some real $\delta > 0$. If z_0 is infinite, then the asymptotic is assumed to hold for all sufficiently large values of z. That is, $|z| > z_0$ for some z_0 (and $z > z_0$ or $z < z_0$ in the real case for $\pm\infty$ respectively). If the limiting process is not explicitly mentioned, it is assumed to be as $z \to \infty$ (or as $z \to \pm\infty$ for the real case). Often times, the asymptotics will hold for all admissible values of z and this will be clear from context although we still might suppress the specific limiting process.

Remark 1.2.1. *Suppose $f, g : \mathbb{Z}_{\geq 0} \to \mathbb{C}$. Extending $f(x)$ and $g(x)$ by making them piecewise linear so that they are piecewise continuous, we can consider estimates with n in place of z. All of the following theory still holds. Moreover, if we further take $f(x)$ or $g(x)$ to be a constant function, the following theory will still hold.*

O-estimates & Symbols

We say $f(z)$ **is of order** $g(z)$ or $f(z)$ is $O(g(z))$ as $z \to z_0$ and write $f(z) = O(g(z))$ if there is some positive constant c such that

$$|f(z)| \leq c|g(z)|,$$

holds as $z \to z_0$. We call this a **O-estimate**. We also say $f(z)$ has **growth at most** $g(z)$. The O-estimate says that for z close to z_0, the size of $f(z)$ grows like $g(z)$.

Remark 1.2.2. *Many authors assume that $g(z)$ is a nonnegative function so that the absolute values on $g(z)$ can be dropped. As we require asymptotics that will be used more generally, we do not make this assumption since one could very well replace $O(g(z))$ with $O(|g(z)|)$. In practice this deviation causes no issue.*

As a symbol, let $O(g(z))$ stand for a function $f(z)$ that is $O(g(z))$. Then we may use the O-estimates in algebraic equations and inequalities. Note that this extends the definition of the symbol because $f(z) = O(g(z))$ means $f(z)$ is $O(g(z))$. The constant c is not unique as any $c' > c$ also works. Any such constant is called the **implicit constant** of the O-estimate. The implicit constant may depend on one or more parameters, ε, σ, etc. If we wish to make these dependences known, we use subscripts O_ε, O_σ, $O_{\varepsilon,\sigma}$, etc. If it is possible to choose the implicit constant independent of a certain parameter then we say that the estimate is **uniform** with respect to that parameter. Moreover, we say that an implicit constant is **effective** if the constant is numerically computable and **ineffective** otherwise. Moreover, if we are interested in the dependence of the estimate on a certain parameter, say p, we will refer to the **p-aspect** to mean the part of the estimate that is dependent upon p. Lastly, in algebraic equations and inequalities involving O-estimates, it is often customary to refer to such an O-estimate as an **error term**. The symbol $\ll$ is known as **Vinogradov's symbol** and it is an alternative way to express O-estimates. We write $f(z) \ll g(z)$ as $z \to z_0$ if $f(z) = O(g(z))$ as $z \to z_0$. We also write $f(z) \gg g(z)$ as $z \to z_0$ to mean $g(z) \ll f(z)$ as $z \to z_0$. If there is a dependence of the implicit constant on parameters, we use subscripts to denote dependence on these parameters. If both $f(z) \ll g(z)$ and $g(z) \ll f(z)$ as $z \to z_0$, then we say $f(z)$ and $g(z)$ have the **same order of magnitude** and write $f(z) \asymp g(z)$ as $z \to z_0$. We also say $f(z)$ has **growth** $g(z)$. If there is a dependence of the implicit constant on parameters, we use subscripts to denote dependence on these parameters. From the definition of the O-estimate, this is equivalent to the existence of positive constants c_1 and c_2 such that

$$c_1|g(z)| \leq |f(z)| \leq c_2|g(z)|.$$

Equivalently, we can interchange $f(z)$ and $g(z)$ in the above equation.

o-estimates & Symbols

We say $f(z)$ **is of smaller order than** $g(z)$ or $f(z)$ is $o(g(z))$ as $z \to z_0$ and write $f(z) = o(g(z))$ if

$$\lim_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| = 0,$$

provided $g(z) \neq 0$ for all z sufficiently close to z_0. We call this a **o-estimate**. The o-estimate says that for z close to z_0, $g(z)$ dominates $f(z)$. If $f(z) = o(g(z))$ as $z \to z_0$, then $f(z) = O(g(z))$ as $z \to z_0$ where the implicit constant can be taken arbitrarily small by definition of the o-estimate. Therefore, o-estimates are stronger than O-estimates. As a symbol, let $o(g(z))$ stand for a function $f(z)$ that is $o(g(z))$. Then we may use the o-estimates in algebraic equations and inequalities. Note that this extends the definition of the symbol because $f(z) = o(g(z))$ means $f(z)$ is $o(g(z))$.

We say $f(z)$ **is asymptotic to** $g(z)$ or $f(z)$ and $g(z)$ are **asymptotically equivalent** as $z \to z_0$ and write $f(z) \sim g(z)$ if

$$\lim_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| = 1,$$

provided $g(z) \neq 0$ for all z sufficiently close to z_0. It is useful to think of asymptotic equivalence as $f(z)$ and $g(z)$ being the same size in the limit as $z \to z_0$. Immediately from the definition, we see that this is an equivalence relation on functions. In particular, if $f(z) \sim g(z)$ and $g(z) \sim h(z)$ then $f(z) \sim h(z)$. Also, if $f(z) \sim g(z)$ as $z \to z_0$, then $f(z) \asymp g(z)$ as $z \to z_0$ with $c_1 \le 1 \le c_2$. So asymptotic equivalence is stronger than being of the same order of magnitude. Also note that $f(x) \sim g(x)$ is equivalent to $f(x) = g(x)(1 + o(1))$ and hence implies $f(x) = g(x)(1 + O(1))$. We write $f(z) = \Omega(g(z))$ as $z \to z_0$ if

$$\limsup_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| > 0.$$

We also say $f(z)$ has **growth at least** $g(z)$. Observe that $f(z) = \Omega(g(z))$ is precisely the negation of $f(z) = o(g(z))$, so that $f(z) = \Omega(g(z))$ means $f(z) = o(g(z))$ is false. This is weaker than $f(z) \gg g(z)$ because $f(z) = \Omega(g(z))$ means $|f(z)| \ge c|g(z)|$ for values of z arbitrarily close to z_0 whereas $f(z) \gg g(z)$ means $|f(z)| \ge c|g(z)|$ for all values of z sufficiently close to z_0.

Algebraic Manipulation for O-estimates and o-estimates

Asymptotic estimates become increasingly more useful when we can use them in equations to represent approximations. We catalogue some of the most useful algebraic manipulations for O-estimates and o-estimates. Most importantly, if an algebraic equation involves a O-estimate or o-estimate then it is understood that the equation is not symmetric and is interpreted to be read from left to right. That is, any function of the form satisfying the estimate on the left-hand side also satisfies the estimate on the right-hand side too. We begin with O-estimates. The trivial algebraic manipulations are collected in the proposition below:

Proposition 1.2.1. *The following O-estimates hold as $z \to z_0$:*

(i) If $f(z) = O(g(z))$ and $g(z) = O(h(z))$, then $f(z) = O(h(z))$. Equivalently, $O(O(h(z))) = O(h(z))$.

(ii) If $f_i(z) = O(g_i(z))$ for $i = 1, 2$, then $f_1(z)f_2(z) = O(g_1(z)g_2(z))$.

(iii) If $f(z) = O(g(z)h(z))$, then $f(z) = g(z)O(h(z))$.

(iv) If $f_i(z) = O(g_i(z))$ for $i = 1, 2, \ldots, n$, then $\sum_{1 \le i \le n} f_i(z) = O\left(\sum_{1 \le i \le n} |g_i(z)|\right)$.

(v) If $f_n(z) = O(g_n(z))$ for $n \ge 1$, then $\sum_{n \ge 1} f_n(z) = O\left(\sum_{n \ge 1} |g_n(z)|\right)$ provided both $\sum_{n \ge 1} f_n(z)$ and $\sum_{n \ge 1} |g_n(z)|$ converge.

(vi) If $f(z) = O(g(z))$ as $z \to z_0$ and $h(z)$ is such that $h(z) \to z_0$ as $z \to z_0$, then $(f \circ h)(z) = O((g \circ h)(z))$.

(vii) If $f(z) = O(g(z))$, then $\mathrm{Re}(f(z)) = O(g(z))$ and $\mathrm{Im}(f(z)) = O(g(z))$.

Proof. Statements (i)-(iii) and (vi) follow immediately from the definition of the O-estimate. Statement (iv) follows from the definition and the triangle inequality. Statement (v) follows in the same way as (iv) given that both sums converge. Statement (vii) follows from the definition and the bounds $|x| \le |z|$ and $|y| \le |z|$. $\square$

We will also often use Proposition 1.2.1 (vi) in the case of the input for a function. The most common instances will be when $z \asymp w$ or $z \sim w$ (the latter case implying the former) where w is a function of z (usually one that is more simple than z itself). Taking $h(z) = w$, Proposition 1.2.1 (vi) says that if $f(z) = O(g(z))$ then $f(w) = O(g(w))$. In terms of Vinogradov's symbol, $f(z) \ll g(z)$ implies $f(w) \ll g(w)$. O-estimates also behave well with respect to integrals provided the functions involved are of a real variable:

Proposition 1.2.2. *Suppose $f(z)$ and $g(z)$ are functions of a real variable, $f(z) = O(g(z))$ as $z \to \infty$, $f(z)$ and $g(z)$ are integrable on the region where this estimate holds, and let $[z_1, z_2]$ belong to this region. Then*

$$\int_{z_1}^{z_2} f(z)\,dz = O\left(\int_{z_1}^{z_2} |g(z)|\,dz\right).$$

Proof. This follows immediately from the definition of the O-estimate. □

The next proposition is a collection of some useful expressions for simplifying equations involving O-estimates:

Proposition 1.2.3. *Let $f(z)$ be a function such that $f(z) \to 0$ as $z \to 0$. The following O-estimates hold as $z \to 0$:*

(i) $\frac{1}{1+O(f(z))} = 1 + O(f(z))$.

(ii) $(1 + O(f(z)))^p = 1 + O(f(z))$ *for any complex number* p.

(iii) $\log(1 + O(f(z))) = O(f(z))$.

(iv) $e^{1+O(f(z))} = 1 + O(f(z))$.

Proof. Taking the Taylor series truncated after the first term and applying Taylor's theorem, we have the O-estimates

(i) $\frac{1}{1+z} = 1 + O(z)$.

(ii) $(1 + z)^p = 1 + O(z)$.

(iii) $\log(1 + z) = O(z)$.

(iv) $e^z = 1 + O(z)$.

Now apply Proposition 1.2.1 (v) to each of these estimates, and use Proposition 1.2.1 (i). □

For o-estimates, the following properties are useful:

Proposition 1.2.4. *The following o-estimates hold as $z \to z_0$:*

(i) *If* $f(z) = o(g(z))$ *and* $g(z) = o(h(z))$, *then* $f(z) = o(h(z))$. *Equivalently,* $o(o(h(z))) = o(h(z))$.

(ii) *If* $f_i(z) = o(g_i(z))$ *for* $i = 1, 2$, *then* $f_1(z)f_2(z) = o(g_1(z)g_2(z))$.

(iii) *If* $f(z) = o(g(z)h(z))$, *then* $f(z) = g(z)o(h(z))$.

(iv) *If* $f_i(z) = o(g_i(z))$ *for* $i = 1, 2, \ldots, n$, *then* $\sum_{1 \le i \le n} f_i(z) = o\left(\sum_{1 \le i \le n} |g_i(z)|\right)$.

(v) *If* $f(z) = o(g(z))$ *as* $z \to z_0$ *and* $h(z)$ *is such that* $h(z) \to z_0$ *as* $z \to z_0$, *then* $(f \circ h)(z) = o((g \circ h)(z))$.

Proof. Statements (i)-(iii) and (v) follow immediately from the definition of the o-estimate. Statement (iv) follows from the definition and that $\sum_{1 \le i \le n} |g_i(z)| \ge |g_i(z)|$. □

Growth & Decay of Functions

We will also be interested in the growth rate of functions. We can compactly express these types of growth using asymptotics. There are many types of growth rates, but we will only recall the ones that are standard. Throughout let $c \geq 1$. First suppose $z \to z_0$. If $f(z) \asymp \log^c(z)$, we say that $f(z)$ is of **logarithmic growth**. If $f(z) \asymp z^c$, we say that $f(z)$ is of **polynomial growth**. If $f(z) \asymp e^{cz}$, we say that $f(z)$ is of **exponential growth**. Now suppose $z \to \infty$ or $z \to \pm\infty$ in the real case. If $f(z) \asymp \log^{-c}(z)$, we say that $f(z)$ is of **logarithmic decay**. If $f(z) \asymp x^{-c}$ for some $c \geq 1$, we say that $f(z)$ is of **polynomial decay**. If $f(z) \asymp e^{-cz}$, we say that $f(z)$ is of **exponential decay**. In all of these cases, we refer to the constant c as the **order** of growth or decay respectively. Our last two growth rates will be the most common ones we will use. Suppose $z \to \infty$ or $z \to \pm\infty$ in the real case. If $f(z) = \Omega(z^n)$ for all $n \geq 0$, then we say $f(z)$ is of **rapid growth**. Alternatively, if $f(z) = o(z^{-n})$ for all $n \geq 0$, then we say $f(z)$ is of **rapid decay**.

1.3 Dirichlet Characters

The most important multiplicative periodic functions for an analytic number theorist are the Dirichlet characters. A **Dirichlet character** χ modulo $m \geq 1$ (or of modulus $m \geq 1$) is an m-periodic completely multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ such that $\chi(a) = 0$ if and only if $(a, m) > 1$. Sometimes we will also write χ_m to denote a Dirichlet character modulo m if we need to express the dependence upon the modulus. For any $m \geq 1$, there is always the **principal Dirichlet character** modulo m which we denote by $\chi_{m,0}$ (sometimes also seen as $\chi_{0,m}$ or the ever more confusing χ_0) and is defined by

$$\chi_{m,0}(a) = \begin{cases} 1 & (a,m) = 1, \\ 0 & (a,m) > 1. \end{cases}$$

When $m = 1$, the principal Dirichlet character is identically 1 and we call this the **trivial Dirichlet character**. This is also the only Dirichlet character modulo 1, so $\chi_1 = \chi_{1,0}$. In general, we say a Dirichlet character χ is **principal** if it only takes values 0 or 1. We now discuss some basic facts of Dirichlet characters. Since $a^{\varphi(m)} \equiv 1 \pmod{m}$ by Euler's little theorem, where φ is Euler's totient function, the multiplicativity of χ implies $\chi(a)^{\varphi(m)} = 1$. Therefore the nonzero values of χ_m are $\varphi(m)$-th roots of unity. In particular, there are only finitely many Dirichlet characters of any fixed modulus m. Given two Dirichlet character χ and ψ modulo m, we define $\chi\psi$ by $\chi\psi(a) = \chi(a)\psi(a)$. This is also a Dirichlet character modulo m, so the Dirichlet characters modulo m form an abelian group denoted by X_m. If we have a Dirichlet character χ modulo m, then $\overline{\chi}$ defined by $\overline{\chi}(a) = \overline{\chi(a)}$ is also a Dirichlet character modulo m and is called the **conjugate Dirichlet character** of χ. Since the nonzero values of χ are roots of unity, if $(a, m) = 1$ then $\overline{\chi}(a) = \chi(a)^{-1}$. So $\overline{\chi}$ is the inverse of χ. This is all strikingly similar to characters on $(\mathbb{Z}/m\mathbb{Z})^*$ (see Appendix C.1), and there is a connection. To see it, by the periodicity of χ, it's nonzero values are uniquely determined by $(\mathbb{Z}/m\mathbb{Z})^*$. Then since χ is multiplicative, it descends to a character χ of $(\mathbb{Z}/m\mathbb{Z})^*$ (we abuse notation here). Conversely, if we are given a character χ of $(\mathbb{Z}/m\mathbb{Z})^*$ we can extend it to a Dirichlet character by defining it to be m-periodic and declaring $\chi(a) = 0$ if $(a, m) > 1$. We call this extension the **zero extension**. So in other words, Dirichlet characters modulo m are the zero extensions of group characters on $(\mathbb{Z}/m\mathbb{Z})^*$. Clearly zero-extension respects multiplication of characters. As groups are isomorphic to their character groups (see Proposition C.1.1), we deduce that the group of Dirichlet characters modulo m is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^*$. That is, $X_m \cong \widehat{(\mathbb{Z}/m\mathbb{Z})}^* \cong (\mathbb{Z}/m\mathbb{Z})^*$. In particular, there are $\varphi(m)$ Dirichlet characters modulo m. From now on we identify Dirichlet characters modulo m with their corresponding group characters of $(\mathbb{Z}/m\mathbb{Z})^*$. We now state two very useful relations called **orthogonality relations** for Dirichlet characters (this follows from the more general orthogonality relations in Appendix C.1 but we wish to give a direct proof):

Proposition 1.3.1.

(i) For any two Dirichlet characters χ and ψ modulo m,

$$\frac{1}{\varphi(m)} \sideset{}{'}\sum_{a \pmod m} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

(ii) For any $a, b \in (\mathbb{Z}/m\mathbb{Z})^*$,

$$\frac{1}{\varphi(m)} \sum_{\chi \pmod m} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

Proof. We will prove the statements separately.

(i) Denote the left-hand side by S and let b be such that $(b, m) = 1$. Then $a \to ab^{-1}$ is a bijection on $(\mathbb{Z}/m\mathbb{Z})^*$ so that

$$\frac{\chi(b)\overline{\psi}(b)}{\varphi(m)} \sideset{}{'}\sum_{a \pmod m} \chi(a)\overline{\psi}(a) = \frac{1}{\varphi(m)} \sideset{}{'}\sum_{a \pmod m} \chi(ab)\overline{\psi}(ab) = \frac{1}{\varphi(m)} \sideset{}{'}\sum_{a \pmod m} \chi(a)\overline{\psi}(a).$$

Consequently $\chi(b)\overline{\psi}(b)S = S$ so that $S = 0$ unless $\chi(b)\overline{\psi}(b) = 1$ for all b such that $(b, m) = 1$. This happens if and only if $\psi = \chi$ in which case $S = 1$. This proves (i).

(ii) Denote the left-hand side by S. Let ψ be any Dirichlet character modulo m. As $\chi \to \chi\overline{\psi}$ is a bijection on X_m, we have

$$\frac{\psi(a)\overline{\psi}(b)}{\varphi(m)} \sum_{\chi \pmod m} \chi(a)\overline{\chi}(b) = \frac{1}{\varphi(m)} \sum_{\chi \pmod m} \psi\chi(a)\overline{\psi\chi}(b) = \frac{1}{\varphi(m)} \sum_{\chi \pmod m} \chi(a)\overline{\chi}(b).$$

Therefore $\psi(a)\overline{\psi}(b)S = S$ so that $S = 0$ unless $\psi(a)\overline{\psi}(b) = \psi(a\overline{b}) = 1$ for all Dirichlet characters ψ modulo m. If this happens, then $a\overline{b} = 1 \pmod m$, or equivalently, $a \equiv b \pmod m$. Indeed, let $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ be the prime factorization of m. By the classification theorem for finite abelian groups,

$$(\mathbb{Z}/m\mathbb{Z})^* \cong (\mathbb{Z}/p_1^{r_1}\mathbb{Z})^* \times (\mathbb{Z}/p_2^{r_2}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_k^{r_k}\mathbb{Z})^*.$$

Now let n_i be a generator for the cyclic group $(\mathbb{Z}/p_i^{r_i}\mathbb{Z})^*$ and let ω_i be a primitive $p_i^{k_i}$-th root of unity for $1 \le i \le k$. Writing $a\overline{b} = n_1^{f_1} n_2^{f_2} \cdots n_k^{f_k}$, consider the Dirichlet character ψ modulo m defined by

$$\psi(n_1^{e_1} n_2^{e_2} \cdots n_k^{e_k}) = \omega_1^{e_1 f_1} \omega_2^{e_2 f_2} \cdots \omega_r^{e_r f_r}.$$

We have

$$\psi(1) = \omega_1^{f_1} \omega_2^{f_2} \cdots \omega_r^{f_r}.$$

As w_i has order $p_i^{k_i}$ and $0 \le f_i < p_i^{k_i} - 1$ for all i, the only way $\psi(1) = 1$ is if $f_i = 0$ for all i. Therefore $a\overline{b} = 1 \pmod m$. In this case $S = 1$. This proves (ii). ∎

In many practical settings, the orthogonality relations are often used in the following form:

Corollary 1.3.1.

(i) For any Dirichlet character χ modulo m,

$$\frac{1}{\varphi(m)}\sum_{a \pmod m}{}' \chi(a) = \delta_{\chi,\chi_{m,0}}.$$

(ii) For any $a \in (\mathbb{Z}/m\mathbb{Z})^*$,

$$\frac{1}{\varphi(m)}\sum_{\chi \pmod m} \chi(a) = \delta_{a,1}.$$

Proof. For (i), take $\psi = \chi_{m,0}$ in Proposition 1.3.1 (i). For (ii), take $b \equiv 1 \pmod m$ in Proposition 1.3.1 (ii). □

Now that we understand the basics of Dirichlet characters, we might be interested in computing them. This is not hard to do by hand for small m. For example, the table below gives the Dirichlet characters modulo 5 where i in $\chi_{5,i}$ is an indexing variable:

	0	1	2	3	4
$\chi_{5,0}$	0	1	1	1	1
$\chi_{5,1}$	0	1	i	$-i$	-1
$\chi_{5,2}$	0	1	$-i$	i	-1
$\chi_{5,3}$	0	1	-1	-1	1

If the modulus is large this is of course more difficult. However, there is a way to build Dirichlet characters modulo m_2 from those modulo m_1. Let χ_{m_1} be a Dirichlet character modulo m_1. If $m_1 \mid m_2$ then $(a, m_2) = 1$ implies $(a, m_1) = 1$. Therefore we can define a Dirichlet character χ_{m_2} by

$$\chi_{m_2}(a) = \begin{cases} \chi_{m_1}(a) & \text{if } (a, m_2) = 1, \\ 0 & \text{if } (a, m_2) > 1. \end{cases}$$

In this case, we say χ_{m_2} is **induced** from χ_{m_1} or that χ_{m_1} **lifts** to χ_{m_2}. All that is happening is χ_{m_2} is a Dirichlet character modulo m_2 whose values are given by those that χ_{m_1} takes. Clearly every Dirichlet character is induced from itself. On the other hand, provided there is a prime p dividing m_2 and not m_1 (so m_2 is a larger modulus), χ_{m_2} will be different from χ_{m_1}. For instance, $\chi_{m_2}(p) = 0$ but $\chi_{m_1}(p) \neq 0$. In general, we say a Dirichlet character is **primitive** if it is not induced by any character other than itself. Notice that the principal Dirichlet characters are precisely those Dirichlet characters induced from the trivial Dirichlet character, and the only primitive one is the trivial Dirichlet character. In any case, we can determine when Dirichlet characters are induced:

Proposition 1.3.2. *A Dirichlet character χ_{m_2} is induced from a Dirichlet character χ_{m_1} if and only if χ_{m_2} is constant on the residue classes in $(\mathbb{Z}/m_2\mathbb{Z})^*$ that are congruent modulo m_1. When this happens, χ_{m_1} is uniquely determined.*

Proof. For the forward implication, if χ_{m_2} is induced from χ_{m_1}, then χ_{m_2} is constant on the residue classes in $(\mathbb{Z}/m_2\mathbb{Z})^*$ that are congruent modulo m_1 because χ_{m_1} is. For the reverse implication, we first show that the reduction modulo m_1 map $\mathbb{Z}/m_2\mathbb{Z} \to \mathbb{Z}/m_1\mathbb{Z}$ induces a surjective homomorphism $\Phi : (\mathbb{Z}/m_2\mathbb{Z})^* \to (\mathbb{Z}/m_1\mathbb{Z})^*$. For $u_1 \in (\mathbb{Z}/m_1\mathbb{Z})^*$, let a be the product of all primes dividing $\frac{m_2}{m_1}$ but not u_1. Then $u_2 = u_1 + m_1 a$ is not divisible by any prime p dividing m_1 or $\frac{m_2}{m_1}$. Hence $(u_2, m_2) = 1$ so that $u_2 \in (\mathbb{Z}/m_2\mathbb{Z})^*$. Note that a is uniquely determined by u_1 so that u_2 is uniquely determined and hence Φ is unique. It's also a homomorphism because reduction modulo m_1 is. Now suppose χ_{m_2} is constant on the residue classes in $(\mathbb{Z}/m_2\mathbb{Z})^*$ that are congruent modulo m_1. Surjectivity of Φ implies χ_{m_2} induces a unique group character on $(\mathbb{Z}/m_1\mathbb{Z})^*$ and hence a unique Dirichlet character modulo m_1. By construction χ_{m_2} is induced from χ_{m_1}. $\square$

Why might we be interested in primitive Dirichlet characters? The reason is that the primitive Dirichlet character are the building blocks for all Dirichlet characters:

Theorem 1.3.1. *Every Dirichlet character χ is induced from a primitive Dirichlet character $\widetilde{\chi}$ that is uniquely determined by χ.*

Proof. Let the modulus of χ be m. Define a partial ordering on the set of Dirichlet characters where $\psi \le \chi$ if χ is induced from ψ. This ordering is clearly reflexive, and it is transitive by Proposition 1.3.2. Set $X = \{\psi : \psi \le \chi\}$. This set is nonempty, and is finite by Proposition 1.3.2. Now suppose $\chi_{m_1}, \chi_{m_2} \in X$. Setting $m_3 = (m_1, m_2)$, and we have a commuting square

$$\begin{tikzcd}
(\mathbb{Z}/m\mathbb{Z})^* \arrow[r, "\Phi"] \arrow[d, "\Phi"'] & (\mathbb{Z}/m_1\mathbb{Z})^* \arrow[d] \\
(\mathbb{Z}/m_2\mathbb{Z})^* \arrow[r] & (\mathbb{Z}/m_3\mathbb{Z})^*
\end{tikzcd}$$

where Φ is as in Proposition 1.3.2. Also from Proposition 1.3.2, χ is constant on the residue classes of $(\mathbb{Z}/m\mathbb{Z})^*$ that are congruent modulo m_1 or m_2 and hence also m_3. Therefore Proposition 1.3.2 implies there is a unique Dirichlet character χ_{m_3} modulo m_3 that lifts to χ_{m_1} and χ_{m_2}. We have now shown that every pair $\chi_{m_1}, \chi_{m_2} \in X$ has a lower bound χ_{m_3}. Hence X contains a primitive Dirichlet character $\widetilde{\chi}$ that is minimal with respect to this partial ordering. There is only one such element. Indeed, since $m_3 \le m_1, m_2$ the partial ordering is compatible with the total ordering by period. Thus $\widetilde{\chi}$ is unique. $\square$

In light of Theorem 1.3.1, we define **conductor** q of a Dirichlet character χ modulo m to be the period of the unique primitive character $\widetilde{\chi}$ that induces χ. This is the most important data of a Dirichlet character since it tells us how χ is built. Note that χ is primitive if and only if its conductor and modulus are equal. Also observe that if χ has conductor q, then χ is q-periodic (necessarily $q \mid m$), and the nonzero values of χ are all q-th roots of unity because those are the nonzero values of $\widetilde{\chi}$. Moreover, $\chi = \widetilde{\chi}\chi_{\frac{m}{q},0}$ by the definition of induced Dirichlet characters. We would also like to distinguish Dirichlet characters whose nonzero values are real or imaginary. We say χ is **real** if it is real-valued. Hence the nonzero values of χ are 1 or -1 since they must be roots of unity. We say χ is an **complex** if it is not real. More commonly, we distinguish Dirichlet characters modulo m by their order as an element of $(\mathbb{Z}/m\mathbb{Z})^*$. If χ is of order 2, 3, etc in $(\mathbb{Z}/m\mathbb{Z})^*$ then we say it is **quadratic**, **cubic**, etc. In particular, a Dirichlet character is quadratic if and only if it is real. For example, if m is odd then the Jacobi symbol $\left(\frac{\cdot}{m}\right)$ is a quadratic Dirichlet character. We will often let χ_m denote this quadratic Dirichlet character but we will always mention this explicitly when we do so. For any Dirichlet character χ, $\chi(-1) = \pm 1$ because $\chi(-1)^2 = 1$. We would like to distinguish this parity. Accordingly, we say χ is **even** if $\chi(-1) = 1$ and **odd** if $\chi(-1) = -1$. Clearly even Dirichlet characters are even functions and odd Dirichlet characters are odd functions. Moreover, χ and $\overline{\chi}$ have the same parity and any lift of χ has the same parity as χ.

1.4 Special Sums

Analytic number theory does not come without its class of special sums that appear naturally. They play the role of discrete counterparts to continuous objects (there is a rich underpinning here). Without a sufficient understanding of these sums, they would cause a discrete obstruction to an analytic problem that we wish to solve.

Ramanujan & Gauss Sums

Let's begin with the Ramanujan sum. For $m \geq 1$ and $b \in \mathbb{Z}$, the **Ramanujan sum** $r(b,m)$ is defined by

$$r(b,m) = \sideset{}{'}\sum_{a \pmod m} e^{\frac{2\pi i ab}{m}}.$$

Note that the Ramanujan sum is a finite sum of m-th roots of unity on the unit circle. Clearly $r(0,m) = \varphi(m)$. Ramanujan sums can be computed explicitly by means of the Möbius function (see Appendix A.1):

Proposition 1.4.1. *For any $m \geq 1$ and any nonzero $b \in \mathbb{Z}$,*

$$r(b,m) = \sum_{\ell \mid (b,m)} \ell \mu\left(\frac{m}{\ell}\right).$$

Proof. This is a computation:

$$\begin{aligned}
r(b,m) &= \sideset{}{'}\sum_{a \pmod m} e^{\frac{2\pi i ab}{m}} \\
&= \sum_{a \pmod m} e^{\frac{2\pi i ab}{m}} \sum_{d \mid (a,m)} \mu(d) && \text{Proposition A.2.1} \\
&= \sum_{d \mid m} \mu(d) \sum_{\substack{a \pmod m \\ d \mid a}} e^{\frac{2\pi i ab}{m}} \\
&= \sum_{d \mid m} \mu(d) \sum_{kd \pmod m} e^{\frac{2\pi i kdb}{m}} && a \to kd \\
&= \sum_{d \mid m} \mu(d) \sum_{k \left(\mathrm{mod}\ \frac{m}{d}\right)} e^{\frac{2\pi i kb}{\frac{m}{d}}}.
\end{aligned}$$

Now if $\frac{m}{d} \mid b$ the inner sum is $\frac{m}{d}$, and otherwise it is zero because $k \to k\bar{b}$ is a bijection on $\mathbb{Z}/\frac{m}{d}\mathbb{Z}$ and thus we are summing over all $\left(\frac{m}{d}\right)$-th roots of unity. So the double sum above reduces to

$$\sum_{\substack{\frac{m}{d} \mid b \\ d \mid m}} \frac{m}{d} \mu(d) = \sum_{\ell \mid (b,m)} \ell \mu\left(\frac{m}{\ell}\right),$$

upon performing the change of variables $\frac{m}{d} \to \ell$. $\square$

We can also define a Ramanujan sum associated to Dirichlet characters. Let χ be a Dirichlet character modulo m. For any $b \in \mathbb{Z}$, the **Ramanujan sum** $\tau(b,\chi)$ associated to χ is given by

$$\tau(b,\chi) = \sum_{a \pmod m} \chi(a) e^{\frac{2\pi i ab}{m}} = \sideset{}{'}\sum_{a \pmod m} \chi(a) e^{\frac{2\pi i ab}{m}},$$

where the last equality follows because $\chi(a) = 0$ unless $(a, m) = 1$. If $b = 1$ we will write $\tau(\chi)$ instead. That is, $\tau(\chi) = \tau(1, \chi)$. We call $\tau(\chi)$ the **Gauss sum** associated to *chi*. Ramanujan and Gauss sums associated to Dirichlet characters are interesting because the Dirichlet character is a multiplicative character while the exponential is an additive one. So these sums are a convolution between a multiplicative and additive character. This is the fundamental reason that makes them difficult to study as one needs to separate the additive and multiplicative structures. Observe that if $m = 1$ then χ is the trivial character and $\tau(b, \chi) = 1$. So the interesting cases are when $m \geq 2$. There are some basic properties of these sums which are very useful:

Proposition 1.4.2. *Let χ and ψ be nontrivial Dirichlet characters modulo m and n respectively and let $b \in \mathbb{Z}$. Then the following hold:*

(i) $\overline{\tau(b, \overline{\chi})} = \chi(-1)\tau(b, \chi)$.

(ii) If $(b, m) = 1$, then $\tau(b, \chi) = \overline{\chi}(b)\tau(\chi)$.

(iii) If $(b, m) > 1$ and χ is primitive, then $\tau(b, \chi) = 0$.

(iv) If $(m, n) = 1$, then $\tau(b, \chi\psi) = \chi(n)\psi(m)\tau(b, \chi)\tau(b, \psi)$.

(v) Let q be the conductor of χ and let $\widetilde{\chi}$ be the primitive Dirichlet character that lifts to χ. Then

$$\tau(\chi) = \mu\left(\frac{m}{q}\right)\widetilde{\chi}\left(\frac{m}{q}\right)\tau(\widetilde{\chi}).$$

Proof. We will prove the statements separately.

(i) Observe that $a \to -a$ is an isomorphism of $\mathbb{Z}/m\mathbb{Z}$. Thus

$$\begin{aligned}
\overline{\tau(b, \overline{\chi})} &= \overline{\sum_{a \pmod m} \overline{\chi}(a)e^{\frac{2\pi iab}{m}}} \\
&= \sum_{a \pmod m} \chi(a)e^{-\frac{2\pi iab}{m}} \\
&= \sum_{a \pmod m} \chi(-a)e^{\frac{2\pi iab}{m}} \\
&= \chi(-1)\sum_{a \pmod m} \chi(a)e^{\frac{2\pi iab}{m}} \\
&= \chi(-1)\tau(b, \chi),
\end{aligned}$$

and (i) follows.

(ii) The map $a \to a\overline{b}$ is an isomorphism of $\mathbb{Z}/m\mathbb{Z}$ since $(b, m) = 1$. Therefore

$$\tau(b, \chi) = \sum_{a \pmod m} \chi(a)e^{\frac{2\pi iab}{m}} = \sum_{a \pmod m} \chi(a\overline{b})e^{\frac{2\pi ia}{m}} = \overline{\chi}(b)\sum_{a \pmod m} \chi(a)e^{\frac{2\pi ia}{m}} = \overline{\chi}(b)\tau(\chi),$$

and (ii) is proven.

(iii) Now fix a divisor $d < m$ of m and choose an integer c such that $c \equiv 1 \pmod{m}$. Then necessarily $(c, m) = 1$. As $d \mid m$, $c \equiv 1 \pmod{d}$ and $(c, d) = 1$. Moreover, there is such a c with the additional property that $\chi(c) \neq 1$. For if not, χ is induced from $\chi_{d,0}$ which contradicts χ being primitive. Now set $d = \frac{m}{(b,m)} < m$ and choose c as above. Since $(c, m) = 1$, $a \to a\overline{c}$ is a bijection on $\mathbb{Z}/m\mathbb{Z}$, so that

$$\chi(c)\tau(b,\chi) = \sum_{a \pmod{m}} \chi(ac) e^{\frac{2\pi i ab}{m}} = \sum_{a \pmod{m}} \chi(a) e^{\frac{2\pi i ab\overline{c}}{m}}.$$

As $e^{\frac{2\pi i b}{m}}$ is a d-th root of unity, and $\overline{c} \equiv 1 \pmod{d}$ (because c is and $d \mid m$) we have $e^{\frac{2\pi i ab\overline{c}}{m}} = e^{\frac{2\pi i ab}{m}}$. Thus the last sum above is $\tau(b, \chi)$. So altogether $\chi(c)\tau(b, \chi) = \tau(b, \chi)$. Since $\chi(c) \neq 1$, we conclude $\tau(b, \chi) = 0$ proving (iii).

(iv) Since $(m, n) = 1$, the Chinese remainder theorem implies that $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/mn\mathbb{Z})$ via the isomorphism $(a, b) \to an + a'm$ with a taken modulo m and a' taken modulo n. Therefore

$$\begin{aligned}
\tau(b,\chi\psi) &= \sum_{an+a'm \pmod{mn}} \chi\psi(an+a'm) e^{\frac{2\pi i(an+a'm)b}{mn}} \\
&= \sum_{a \pmod{m}} \sum_{a' \pmod{n}} \chi\psi(an+a'm) e^{\frac{2\pi i(an+a'm)b}{mn}} \\
&= \sum_{a \pmod{m}} \sum_{a' \pmod{n}} \chi(an+a'm)\psi(an+a'm) e^{\frac{2\pi i(an+a'm)b}{mn}} \\
&= \sum_{a \pmod{m}} \sum_{a' \pmod{n}} \chi(an)\psi(a'm) e^{\frac{2\pi i(an+a'm)b}{mn}} \\
&= \chi(n)\psi(m) \sum_{a \pmod{m}} \sum_{a' \pmod{n}} \chi(a)\psi(a') e^{\frac{2\pi i ab}{m}} e^{\frac{2\pi i a'b}{n}} \\
&= \chi(n)\psi(m) \sum_{a \pmod{m}} \chi(a) e^{\frac{2\pi i ab}{m}} \sum_{a' \pmod{n}} \psi(a') e^{\frac{2\pi i a'b}{n}} \\
&= \chi(n)\psi(m)\tau(b,\chi)\tau(b,\psi).
\end{aligned}$$

This proves (iv).

(v) If $\left(\frac{m}{q}, q\right) > 1$, then $\widetilde{\chi}\left(\frac{m}{q}\right) = 0$ so we need to show $\tau(\chi) = 0$. As $\left(\frac{m}{q}, q\right) > 1$, there exists a prime p such that $p \mid \frac{m}{q}$ and $p \mid q$. By Euclidean division we may write any a modulo m in the form $a = a'\frac{m}{p} + a''$ with a' taken modulo p and a'' taken modulo $\frac{m}{p}$. Then

$$\tau(\chi) = \sum_{a \pmod{m}} \chi(a) e^{\frac{2\pi i a}{m}} = \sum_{\substack{a' \pmod{p} \\ a'' \left(\mathrm{mod}\ \frac{m}{p}\right)}} \chi\left(a'\frac{m}{p} + a''\right) e^{\frac{2\pi i\left(a'\frac{m}{p}+a''\right)}{m}}. \tag{1.1}$$

Since $p \mid \left(\frac{m}{q}, q\right)$, we have $p^2 \mid m$. Therefore $\left(a'\frac{m}{p} + a'', m\right) = 1$ if and only if $\left(a'\frac{m}{p} + a'', \frac{m}{p}\right) = 1$ and this latter condition is equivalent to $\left(a'', \frac{m}{p}\right) = 1$. Thus the last sum in Equation (1.1) is

$$\sum_{\substack{a' \pmod{p} \\ a'' \left(\mathrm{mod}\ \frac{m}{p}\right) \\ \left(a'', \frac{m}{p}\right)=1}} \chi\left(a'\frac{m}{p} + a''\right) e^{\frac{2\pi i\left(a'\frac{m}{p}+a''\right)}{m}}.$$

As $p \mid \frac{m}{q}$, we know $q \mid \frac{m}{p}$ so that $a'\frac{m}{p} + a'' \equiv a'' \pmod{q}$. Then Proposition 1.3.2 implies $\chi\left(a'\frac{m}{p} + a''\right) = \widetilde{\chi}(a'')$ and this sum is further reduced to

$$\sideset{}{'}\sum_{a'' \pmod{\frac{m}{p}}} \widetilde{\chi}(a'')e^{\frac{2\pi i a''}{m}} \sum_{a' \pmod{p}} e^{\frac{2\pi i a'}{p}}. \tag{1.2}$$

The inner sum in Equation (1.2) vanishes since it is the sum over all p-th roots of unity and thus $\tau(\chi) = 0$. Now suppose $\left(\frac{m}{q}, q\right) = 1$. Then (iv) implies

$$\tau(\chi) = \tau(\widetilde{\chi}\chi_{\frac{m}{q},0}) = \widetilde{\chi}\left(\frac{m}{q}\right)\chi_{\frac{m}{q},0}(q)\tau(\widetilde{\chi})\tau(\chi_{\frac{m}{q},0}) = \tau(\chi_{\frac{m}{q},0})\widetilde{\chi}\left(\frac{m}{q}\right)\tau(\widetilde{\chi}).$$

Now observe that $\tau(\chi_{\frac{m}{q},0}) = r\left(1, \frac{m}{q}\right)$. By Proposition 1.4.1 we see that $r\left(1, \frac{m}{q}\right) = \mu\left(\frac{m}{q}\right)$ and (v) follows. □

Notice that Proposition 1.4.2 reduces the evaluation of the Ramanujan sum $\tau(b, \chi)$ to that of the Gauss sum $\tau(\chi)$ at least when χ is primitive. When χ is not primitive and $(b, m) > 1$ we need to appeal to evaluating $\tau(b, \chi)$ by more direct means. Evaluating $\tau(\chi)$ for general characters χ turns out to be a very difficult problem and is still open. However, it is not difficult to determine the modulus of $\tau(\chi)$ when χ is primitive:

Theorem 1.4.1. *Let χ be a primitive Dirichlet character of conductor q. Then*

$$|\tau(\chi)| = \sqrt{q}.$$

Proof. If χ is the trivial character this is obvious since $\tau(\chi) = 1$. So we may assume χ is nontrivial. Now this is just a computation:

$$\begin{aligned}
|\tau(\chi)|^2 &= \tau(\chi)\overline{\tau(\chi)} \\
&= \sum_{a \pmod{q}} \tau(\chi)\overline{\chi}(a)e^{-\frac{2\pi i a}{q}} \\
&= \sum_{a \pmod{q}} \tau(a, \chi)e^{-\frac{2\pi i a}{q}} && \text{Proposition 1.4.2 (i) and (ii)} \\
&= \sum_{a \pmod{q}} \left(\sum_{a' \pmod{q}} \chi(a')e^{\frac{2\pi i a a'}{q}}\right) e^{-\frac{2\pi i a}{q}} \\
&= \sum_{a,a' \pmod{q}} \chi(a')e^{\frac{2\pi i a(a'-1)}{q}} \\
&= \sum_{a' \pmod{q}} \chi(a')\left(\sum_{a \pmod{q}} e^{\frac{2\pi i a(a'-1)}{q}}\right).
\end{aligned}$$

Let $S(a')$ denote the inner sum. For the a' such that $a'-1 \equiv 0 \pmod{q}$, $S(a') = q$. Otherwise $a \to \overline{a(a'-1)}$ is a bijection on $\mathbb{Z}/q\mathbb{Z}$ ($q \neq 1$ because χ is nontrivial) so that $S(a') = 0$ because it is the sum of all q-th roots of unity. It follows that the double sum is $\chi(1)q = q$. So altogether $|\tau(\chi)|^2 = q$ and hence $|\tau(\chi)| = \sqrt{q}$. □

As an almost immediate corollary to Theorem 1.4.1, we deduce a useful expression for primitive Dirichlet characters of conductor q in terms of additive characters on $(\mathbb{Z}/q\mathbb{Z})$:

Corollary 1.4.1. *Let χ be a primitive Dirichlet character of conductor q. Then*

$$\tau(n,\chi) = \overline{\chi}(n)\tau(\chi),$$

for all $n \in \mathbb{Z}$. In particular,

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{a \pmod q} \overline{\chi}(a) e^{\frac{2\pi i a n}{q}},$$

for all $n \in \mathbb{Z}$.

Proof. If χ is the trivial character this is obvious since $\tau(n,\chi) = 1$. So assume χ is nontrivial. If $(n,q) = 1$, then the first identity is Proposition 1.4.2 (ii). If $(n,q) > 1$, then the first identity follows from Proposition 1.4.2 (iii) and that $\overline{\chi}(n) = 0$. This proves the first identity in full. For the second identity, first note that $\tau(\chi) \neq 0$ by Theorem 1.4.1. Replacing χ with $\overline{\chi}$, dividing the first identity by $\tau(\chi)$, and expanding the Ramanujan sum, gives the second identity. □

In light of Theorem 1.4.1 we define the **epsilon factor** ε_χ for a Dirichlet character χ modulo m by

$$\varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{m}}.$$

Theorem 1.4.1 says that this value lies on the unit circle when χ is primitive and not the trivial character. In any case, the question of the evaluation of Gauss sums further boils down to determining what value the epsilon factor is. This is the real difficultly as the epsilon factor is very hard to calculate and its value is not known for general Dirichlet characters. When χ is primitive, there is a simple relationship between ε_χ and $\varepsilon_{\overline{\chi}}$:

Proposition 1.4.3. *Let χ be a primitive Dirichlet character of conductor q. Then*

$$\varepsilon_\chi \varepsilon_{\overline{\chi}} = \chi(-1).$$

Proof. If χ is trivial this is obvious since $\varepsilon_\chi = \varepsilon_{\overline{\chi}} = 1$. So assume χ is nontrivial. By Proposition 1.4.2 (iii) and that ε_χ lies on the unit circle,

$$\varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{q}} = \chi(-1)\frac{\overline{\tau(\chi)}}{\sqrt{q}} = \chi(-1)\varepsilon_{\overline{\chi}}^{-1},$$

from whence the statement follows. □

Quadratic Gauss Sums

Another important sum is the quadratic Gauss sum. For any $m \geq 1$ and any $b \in \mathbb{Z}$, the **quadratic Gauss sum** $g(b,m)$ is defined by

$$g(b,m) = \sum_{a \ (\mathrm{mod}\ m)} e^{\frac{2\pi i a^2 b}{m}}.$$

If $b = 1$ we write $g(m)$ instead. That is, $g(m) = g(1,m)$. It turns out that if χ_m is the quadratic Dirichlet character given by the Jacobi symbol, then $\tau(b,\chi_m) = g(b,m)$ provided m is square-free. This will take a little work to prove. We first reduce to the case when $(b,m) = 1$:

Proposition 1.4.4. *Let $m \geq 1$ be odd and let $b \in \mathbb{Z}$. Then*

$$g(b,m) = (b,m)g\left(\frac{b}{(b,m)}, \frac{m}{(b,m)}\right).$$

Proof. By Euclidean division write any a modulo m in the form $a = a'\frac{m}{(b,m)} + a''$ with a' take modulo (b,m) and a'' take modulo $\frac{m}{(b,m)}$. Then

$$\begin{aligned}
g(b,m) &= \sum_{a \pmod m} e^{\frac{2\pi i a^2 b}{m}} \\
&= \sum_{\substack{a' \pmod{(b,m)} \\ a'' \left(\text{mod } \frac{m}{(b,m)}\right)}} e^{\frac{2\pi i\left(a'\frac{m}{(b,m)}+a''\right)^2 b}{m}} \\
&= \sum_{a'' \left(\text{mod } \frac{m}{(b,m)}\right)} e^{\frac{2\pi i (a'')^2 b}{m}} \sum_{a' \pmod{(b,m)}} e^{\frac{2\pi i\left(2a''a'\frac{m}{(b,m)}+\left(a'\frac{m}{(b,m)}\right)^2\right)b}{m}} \\
&= \sum_{a'' \left(\text{mod } \frac{m}{(b,m)}\right)} e^{\frac{2\pi i (a'')^2 \frac{b}{(b,m)}}{\frac{m}{(b,m)}}} \sum_{a' \pmod{(b,m)}} e^{\frac{2\pi i\left(2a''a'\frac{m}{(b,m)}+\left(a'\frac{m}{(b,m)}\right)^2\right)\frac{b}{(b,m)}}{\frac{m}{(b,m)}}} \\
&= (b,m) \sum_{a'' \left(\text{mod } \frac{m}{(b,m)}\right)} e^{\frac{2\pi i (a'')^2 \frac{b}{(b,m)}}{\frac{m}{(b,m)}}},
\end{aligned}$$

where the last line follows because $\left(2a''a'\frac{m}{(b,m)} + \left(a'\frac{m}{(b,m)}\right)^2\right) \equiv 0 \left(\text{mod } \frac{m}{(b,m)}\right)$ and thus the second sum is (b,m). The remaining sum is $g\left(\frac{b}{(b,m)}, \frac{m}{(b,m)}\right)$ which finishes the proof. □

As a consequence of Proposition 1.4.4, we may always assume $(b,m) = 1$. Now we give an equivalent formulation of the Ramanujan sum associated to quadratic characters given by Jacobi symbols and show that in the case $m = p$ an odd prime, the Ramanujan and quadratic Gauss sums agree:

Proposition 1.4.5. *Let $m \geq 1$ and $b \in \mathbb{Z}$ be such that $(b,m) = 1$. Also let χ_m be the quadratic Dirichlet character given by the Jacobi symbol. Then*

$$\tau(b,\chi_m) = \sum_{a \pmod m} \left(1 + \left(\frac{a}{m}\right)\right) e^{\frac{2\pi i ab}{m}}.$$

Moreover, when $m = p$ is prime,

$$\tau(b,\chi_p) = g(b,p).$$

Proof. If $m = 1$ the claim is obvious since $\tau(b,\chi_1) = 1$ so assume $m > 1$. To prove the first statement, observe

$$\sum_{a \pmod m} \left(1 + \left(\frac{a}{m}\right)\right) e^{\frac{2\pi i ab}{m}} = \sum_{a \pmod m} e^{\frac{2\pi i ab}{m}} + \sum_{a \pmod m} \left(\frac{a}{m}\right) e^{\frac{2\pi i ab}{m}}.$$

The first sum on the right-hand side is zero as it is the sum over all m-th roots of unity since $(b,m) = 1$. This proves the first claim. Now let $m = p$ be an odd prime. From the definition of the Jacobi symbol we see that $1+\left(\frac{a}{p}\right) = 2, 0$ depending on if a is a quadratic residue modulo p or not provided $a \not\equiv 0 \pmod{p}$. If $a \equiv 0 \pmod{p}$, then $1+\left(\frac{a}{p}\right) = 1$. Moreover, if a is a quadratic residue modulo p, then $a \equiv (a')^2 \pmod{p}$ for some a'. So one the one hand,

$$\tau(b,\chi_p) = \sum_{a \pmod{p}} \left(1+\left(\frac{a}{p}\right)\right) e^{\frac{2\pi i ab}{p}} = 1 + 2 \sum_{\substack{a \pmod{p} \\ a \equiv (a')^2 \pmod{p} \\ a \not\equiv 0 \pmod{p}}} e^{\frac{2\pi i (a')^2 b}{p}}.$$

On the other hand,

$$g(b,p) = 1 + \sum_{\substack{a \pmod{p} \\ a \not\equiv 0 \pmod{p}}} e^{\frac{2\pi i a^2 b}{p}},$$

but this last sum counts every quadratic residue twice because $(-a)^2 = a^2$. Hence the two previous sums are equal. □

We would like to generalize the second statement in Proposition 1.4.5 to when m is square-free. In this direction, a series of reduction properties will be helpful:

Proposition 1.4.6. *Let $m, n \geq 1$, p be an odd prime, and $b \in \mathbb{Z}$. Then the following hold:*

(i) If $(b,p) = 1$, then $g(b,p^r) = pg(b,p^{r-2})$ for all $r \in \mathbb{Z}$ with $r \geq 2$.

(ii) If $(m,n) = 1$ and $(b,mn) = 1$, then $g(b,mn) = g(bn,m)g(bm,n)$.

(iii) If m is odd and $(b,m) = 1$, then $g(b,m) = \left(\frac{b}{m}\right)g(m)$ where $\left(\frac{b}{m}\right)$ is the Jacobi symbol.

Proof. We will prove the statements separately.

(i) First notice that

$$g(b,p^r) = \sum_{a \pmod{p^r}} e^{\frac{2\pi i a^2 b}{p^r}} = \sideset{}{'}\sum_{a \pmod{p^r}} e^{\frac{2\pi i a^2 b}{p^r}} + \sum_{a \pmod{p^{r-1}}} e^{\frac{2\pi i a^2 b}{p^{r-2}}},$$

since every a modulo p satisfies $(a,p) = 1$ or not. By Euclidean division every element a modulo p^{r-1} is of the form $a = a'p^{r-2} + a''$ with a' taken modulo p and a'' taken modulo p^{r-2}. Since $(a'p^{r-2} + a'') \equiv a'' \pmod{p^{r-2}}$, every a'' is counted p times modulo p^{r-2}. Along with the fact that $(a'p^{r-2} + a'')^2 \equiv (a'')^2 \pmod{p^{r-2}}$, these facts give the middle equality in the following chain:

$$\sum_{a \pmod{p^{r-1}}} e^{\frac{2\pi i a^2 b}{p^{r-2}}} = \sum_{\substack{a' \pmod{p} \\ a'' \left(\bmod p^{r-2}\right)}} e^{\frac{2\pi i \left(a'p^{r-2}+a''\right)^2 b}{p^{r-2}}} = p \sum_{a'' \pmod{p}} e^{\frac{2\pi i (a'')^2 b}{p^{r-2}}} = pg(b,p^{r-2}).$$

It remains to show

$$\sideset{}{'}\sum_{a \pmod{p^r}} e^{\frac{2\pi i a^2 b}{p^r}},$$

is zero. This sum is exactly $r(b,p^r)$ so by Proposition 1.4.1, and that $(b,p) = 1$, we conclude

$$\sideset{}{'}\sum_{a \pmod{p^r}} e^{\frac{2\pi i a^2 b}{p^r}} = \mu(p^r) = 0,$$

because $r \geq 2$. This proves (i).

(ii) Observe

$$g(bn,m)g(bm,n)=\left(\sum_{a \pmod m} e^{\frac{2\pi i a^2 bn}{m}}\right)\left(\sum_{a' \pmod n} e^{\frac{2\pi i (a')^2 bm}{n}}\right)=\sum_{\substack{a \pmod m\\ a' \pmod n}} e^{\frac{2\pi i\left((an)^2+(a'm)^2\right)b}{mn}}.$$

Note that $e^{\frac{2\pi i\left((an)^2+(a'm)^2\right)b}{mn}}$ only depends upon $(an)^2+(a'm)^2$ modulo mn. Clearly $(an+a'm)^2 \equiv (an)^2+(a'm)^2 \pmod{mn}$, so set $a''=an+a'm$ taken modulo mn. Since $(m,n)=1$, the Chinese remainder theorem implies that $(\mathbb{Z}/m\mathbb{Z})\times(\mathbb{Z}/n\mathbb{Z})\cong(\mathbb{Z}/mn\mathbb{Z})$ via the isomorphism $(a,a')\to an+a'm$. Thus the last sum above is equal to

$$\sum_{a'' \pmod{mn}} e^{\frac{2\pi i (a'')^2 b}{mn}},$$

which is precisely $g(b,mn)$. So (ii) is proven.

(iii) The claim is obvious if $m=1$ because $g(b,1)=1$ so assume $m>1$. If $m=p$, then Proposition 1.4.5, Proposition 1.4.2 (ii), and that quadratic characters are their own conjugate altogether imply the claim. Now let $r\geq 1$ and assume by strong induction that the claim holds when $m=p^{r'}$ for all positive integers r' such that $r'<r$. Then by (i), we have

$$g(b,p^r)=pg(b,p^{r-2})=\left(\frac{b}{p^{r-2}}\right)pg(p^{r-2})=\left(\frac{b}{p^{r-2}}\right)g(p^r)=\left(\frac{b}{p^r}\right)g(p^r). \tag{1.3}$$

It now suffices to prove the claim when $m=p^rq^s$ where q is another odd prime and $s\geq 1$. Then by (ii) and Equation (1.3), we compute

$$\begin{aligned}
g(b,p^rq^s)&=g(bq^s,p^r)g(bp^r,q^s)\\
&=\left(\frac{bq^s}{p^r}\right)\left(\frac{bp^r}{q^s}\right)g(p^r)g(q^s)\\
&=\left(\frac{b}{p^r}\right)\left(\frac{q^s}{p^r}\right)\left(\frac{b}{q^s}\right)\left(\frac{p^r}{q^s}\right)g(p^r)g(q^s)\\
&=\left(\frac{b}{p^rq^s}\right)\left(\frac{q^s}{p^r}\right)\left(\frac{p^r}{q^s}\right)g(p^r)g(q^s)\\
&=\left(\frac{b}{p^rq^s}\right)g(q^s,p^r)g(p^r,q^s)\\
&=\left(\frac{b}{p^rq^s}\right)g(p^rq^s).
\end{aligned}$$

This proves (iii). □

At last we can prove that our Ramanujan and quadratic Gauss sums agree for square-free m:

Theorem 1.4.2. *Suppose $m\geq 1$ be square-free and odd and let χ_m be the quadratic Dirichlet character given by the Jacobi symbol. Let $b\in\mathbb{Z}$ such that $(b,m)=1$. Then*

$$\tau(b,\chi_m)=g(b,m).$$

Proof. The claim is obvious if $m = 1$ because $\tau(b, \chi_1) = 1$ and $g(b, 1) = 1$ so assume $m > 1$. Since χ_m is quadratic, it suffices to prove the claim when $b = 1$ by Proposition 1.4.2 (ii) and Proposition 1.4.6 (iii). Now let $m = p_1 p_2 \cdots p_k$ be the prime decomposition of m. Repeated application of Proposition 1.4.2 (iv) gives the first equality in the chain

$$\begin{aligned}\tau(\chi) &= \prod_{1 \le i < j \le k} \chi_{p_i}(p_j)\chi_{p_j}(p_i)\tau(\chi_{p_i})\tau(\chi_{p_j}) \\ &= \prod_{1 \le i < j \le k} \chi_{p_i}(p_j)\chi_{p_j}(p_i)g(p_i)g(p_j) \\ &= \prod_{1 \le i < j \le k} g(p_j, p_i)g(p_i, p_j) \\ &= g(q).\end{aligned}$$

This proves the claim. □

Now let's turn to Proposition 1.4.6 and the evaluation of the quadratic Gauss sum. Proposition 1.4.6 (ii) and (iii) reduce the evaluation of $g(b, m)$ for odd m and $(b, m) = 1$ to computing $g(p)$ for p an odd prime. As with the Gauss sum, it is not difficult to compute the modulus of the quadratic Gauss sum:

Theorem 1.4.3. *Let $m \ge 1$ be odd. Then*

$$|g(m)| = \sqrt{m}.$$

Proof. By Proposition 1.4.6 (ii), it suffices to prove this when $m = p^r$ is a power of an odd prime. By Euclidean division write $r = 2n + r'$ for some positive integer n and with $r' = 0, 1$ depending on if r is even or odd respectively. Then Proposition 1.4.6 (i) implies

$$|g(p^r)|^2 = p^{2n}|g(p^{r'})|^2.$$

If $r' = 0$, then $2n = r$ so that $p^{2n} = p^r$. Thus $|g(p^r)| = \sqrt{p^r}$. If $r' = 1$, then Theorem 1.4.1 and Proposition 1.4.5 together imply $|g(p^{r'})|^2 = p$ so that the right-hand side above is $p^{2n+1} = p^r$ and again we have $|g(p^r)| = \sqrt{p^r}$. □

Accordingly, we define the **epsilon factor** ε_m for any $m \ge 1$ by

$$\varepsilon_m = \frac{g(m)}{\sqrt{m}}.$$

Theorem 1.4.3 says that this value lies on the unit circle when m is odd. Thus the question of the evaluation of quadratic Gauss sums reduces to determining what the epsilon factor is. This was completely resolved and the original proof is due to Gauss in 1808 (see [Gau08]). He actually treated the case m is even as well. We have avoided discussing this because we will not need it in the following and many of the previous proofs need to be augmented when m is even (see [Lan94] for a treatment of the even case). As for the evaluation, one of the cleanest proofs uses analytic techniques (see [Lan94]) and the precise statement is the following:

Theorem 1.4.4. *Let $m \ge 1$. Then*

$$\varepsilon_m = \begin{cases} (1+i) & \text{if } m \equiv 0 \pmod 4, \\ 1 & \text{if } m \equiv 1 \pmod 4, \\ 0 & \text{if } m \equiv 2 \pmod 4, \\ i & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

As an immediate corollary, this implies the evaluation of the epsilon factor ε_{χ_p} where χ_p is the quadratic Dirichlet character given by the Jacobi symbol for an odd prime p:

Corollary 1.4.2. *Let p be an odd prime and χ_p be the quadratic Dirichlet character given by the Jacobi symbol. Then*

$$\varepsilon_{\chi_p} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ i & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. The statement follows immediately from Theorem 1.4.4 and Proposition 1.4.5. □

Kloosterman & Salié Sums

Our last class of sums generalize both types of Ramanujan sums. For any $c \geq 1$ and $n, m \in \mathbb{Z}$, the **Kloosterman sum** $K(n, m, c)$ is defined by

$$K(n,m,c) = \sum_{\substack{a \pmod c \\ (a,c)=1}} e^{\frac{2\pi i(an+\overline{a}m)}{c}} = \sum_{a \pmod c}{}' e^{\frac{2\pi i(an+\overline{a}m)}{c}}.$$

Notice that if either $n = 0$ or $m = 0$ then the Kloosterman sum reduces to a Ramanujan sum. Kloosterman sums have similar properties to those of Ramanujan sums, but we will not need them. The only result we will need is a famous bound, often called the **Weil bound** for Kloosterman sums, proved by Weil (see [Wei48] for a proof):

Theorem 1.4.5 (Weil bound). *Let $c \geq 1$ and $n, m \in \mathbb{Z}$. Then*

$$|K(n,m,c)| \leq \sigma_0(c)\sqrt{(n,m,c)}\sqrt{c}.$$

Lastly, Salié sums are Kloosterman sums with Dirichlet characters. To be precise, for any $c \geq 1$, $n, m \in \mathbb{Z}$, and a Dirichlet character χ with conductor $q \mid c$, the **Salié sum** $S_\chi(n, m, c)$ is defined by

$$S_\chi(n,m,c) = \sum_{\substack{a \pmod c \\ (a,c)=1}} \chi(a) e^{\frac{2\pi i(an+\overline{a}m)}{c}} = \sum_{a \pmod c}{}' \chi(a) e^{\frac{2\pi i(an+\overline{a}m)}{c}}.$$

If either $n = 0$ or $m = 0$ then the Salié sum reduces to the Ramanujan sum associated to χ.

1.5 Integration Techniques & Transforms

Integration Techniques

Complex integrals are a core backbone of analytic number theory and all too often the domain we are integrating over is unbounded. Accordingly, the functions we would like to work with should be integrable over these domains. One way to accomplish this would be to require that our functions are bounded and our domains be of finite volume:

Proposition 1.5.1. *Suppose $f(z, w)$ is analytic on $\Omega \times D$ for some regions Ω and D with D of finite volume with respect to a measure $d\mu$. Also suppose that for any $z \in \Omega$, $f(z, w)$ is bounded on D. Then the integral*

$$\int_D f(z,w)\, d\mu,$$

is locally absolutely uniformly convergent for any $z \in W$. In particular, is holomorphic in z.

Proof. The second statement follows from Theorem B.1.2. So it suffices to show that the integral is locally absolutely uniformly convergent. Let K be a compact subset of Ω. As $f(z,w)$ is bounded on D for any z, $f(z,w)$ is locally absolutely uniformly convergent on $K \times D$. Then for $z \in K$, we have

$$\int_D f(z,w)\,d\mu \ll \int_D d\mu \ll 1,$$

because D has finite volume with respect to $d\mu$. Therefore the integral is locally absolutely uniformly convergent. □

We will also need a version of Proposition 1.5.1 when the region of integration is an unbounded interval but the integrand has sufficient decay:

Proposition 1.5.2. *Suppose $f(z,w)$ is analytic on $\Omega \times I$ for some region Ω and unbounded interval I with measure $d\mu$. Also suppose that for any $z \in \Omega$, $f(z,w)$ has at least polynomial decay of order $c > 1$. Then the integral*

$$\int_I f(z,w)\,d\mu,$$

is locally absolutely uniformly convergent for any $z \in W$. In particular, is holomorphic in z.

Proof. The second statement follows from Theorem B.1.2. So it suffices to show that the integral is locally absolutely uniformly convergent. Let K be a compact subset of Ω. As $f(z,w)$ has at least polynomial decay of order $c > 1$, $f(z,w) = O(w^{-c})$. Therefore $f(z,x)$ is absolutely uniformly bounded on $K \times I$. Then for $z \in K$, we have

$$\int_I f(z,w)\,d\mu \ll \int_I w^{-c}\,d\mu \ll 1,$$

because $c > 1$. Therefore the integral is locally absolutely uniformly convergent. □

There is also a useful analytic technique called **shifting the line of integration**:

Theorem 1.5.1 (Shifting the line of integration). *Suppose we are given an integral*

$$\int_{\mathrm{Re}(z)=a} f(z)\,dz \quad \text{or} \quad \int_{\mathrm{Im}(z)=a} f(z)\,dz,$$

and some real b with $b < a$ in the first case and $b > a$ in the second case. Suppose $f(z)$ is meromorphic on a strip bounded by the lines $\mathrm{Re}(z) = a, b$ or $\mathrm{Im}(z) = a, b$ and is holomorphic about the lines $\mathrm{Re}(z) = a, b$ or $\mathrm{Im}(z) = a, b$ respectively. Moreover, suppose $f(z) \to 0$ as $y \to \infty$ or $x \to \infty$ respectively. Then

$$\int_{(a)} f(z)\,dz = \int_{(b)} f(z)\,dz + 2\pi i \sum_{\rho \in P} \operatorname*{Res}_{z=\rho} f(z),$$

where P is the set of poles inside of the strip bounded by the lines $\mathrm{Re}(z) = a, b$ or $\mathrm{Im}(z) = a, b$ respectively.

Proof. To collect these cases, let (a) stand for the line $\mathrm{Re}(z) = a$ or $\mathrm{Im}(z) = a$ respectively with positive orientation. Let R_T be a rectangle, of height or width T respectively, given positive orientation, and with its edges on (a) and (b) respectively. Consider the limit

$$\lim_{T\to\infty} \int_{R_T} f(z)\,dz.$$

On the one hand, the residue theorem implies the integral is a sum of a $2\pi i$ multiple of the residues r_i in the rectangle R_T and hence the limit is a $2\pi i$ multiple of the sum of the residues in the strip bounded by (a) and (b). On the other hand, the integral can be decomposed into a sum of four integrals along the edges of R_T and by taking the limit, the edges other than (a) and (b) will tend to zero because $f(z) \to 0$ as $y \to \infty$ or $x \to \infty$ respectively. What remains in the limit is the difference between the integral along (a) and (b). So in total,

$$\int_{(a)} f(z)\, dz = \int_{(b)} f(z)\, dz + 2\pi i \sum_{\rho \in P} \operatorname*{Res}_{z=\rho} f(z).$$

□

A particular application of interest is when the integral in question is real and over the entire real line, the integrand is entire as a complex function, and one is trying to shift the line of integration of the complexified integral to $\mathrm{Im}(z) = a$. In this case, shifting the line of integration amounts to making the change of variables $x \to x - ia$ without affecting the initial line of integration. The last integral technique we will use is when we are summing integrals over a group and is called the **unfolding/folding method**:

Theorem 1.5.2 (Unfolding/folding method). *Suppose $f(z)$ is holomorphic on some region Ω. Moreover, suppose G is a countable group acting by automorphisms on Ω and let D and F be regions such that*

$$D = \bigcup_{g \in G} gF,$$

where the intersections $gF \cap hF$ are measure zero for all $g, h \in G$ with respect to some G-invariant measure $d\mu$. Then

$$\int_F \sum_{g \in G} f(gz)\, d\mu = \int_D f(z)\, d\mu,$$

provided the sum and or integral on either side are absolutely convergent and absolutely bounded.

Proof. First suppose $\int_F \sum_{g \in G} f(gz)\, d\mu$ converges absolutely. By the dominated convergence theorem, we may interchange the sum and integral. Upon making the change of variables $z \to g^{-1}z$, the invariance of $d\mu$ implies that the integral takes the form

$$\sum_{g \in G} \int_{gF} f(z)\, d\mu.$$

As G is countable and the intersections $gF \cap hF$ are measure zero, the overlap in $\bigcup_{g \in G} gF$ is also measure zero. As $D = \bigcup_{g \in G} gF$, the result follows. Of course, there is equality everywhere so we can also run the procedure in reverse provided $\int_D f(z)\, d\mu$ converges absolutely. □

In the unfolding/folding method, we refer to the going from the left-hand side to right-hand as **unfolding** and going from the right-hand to left-hand side as **folding**.

The Fourier Transform

The first type of integral transform we will need is the Fourier transform. Suppose $f(x)$ is absolutely integrable on $\mathbb{R}$. The **Fourier transform** $\hat{f}(t)$ of $f(x)$ is defined by

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x}\, dx.$$

This integral is absolutely convergent precisely because $f(x)$ is absolutely integrable on $\mathbb{R}$. Under some mild conditions on $f(x)$ the **Poisson summation formula** applies:

Theorem 1.5.3 (Poisson summation formula). *Suppose $f(x)$ is absolutely integrable on $\mathbb{R}$ and the function $\sum_{n\in\mathbb{Z}} f(x+n)$ is locally absolutely uniformly convergent and smooth. Then*

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{t\in\mathbb{Z}} \hat{f}(t)e^{2\pi itx}.$$

In particular,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{t\in\mathbb{Z}} \hat{f}(t).$$

Proof. Set

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n).$$

Then $F(x)$ is smooth by assumption and clearly 1-periodic. Therefore it admits a Fourier series (see Appendix B.4). We compute the t-th Fourier coefficient of $F(x)$ as follows:

$$\begin{aligned}
\hat{F}(t) &= \int_0^1 F(x)e^{-2\pi itx}\,dx \\
&= \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi itx}\,dx \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)e^{-2\pi itx}\,dx && \text{DCT} \\
&= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x)e^{-2\pi itx}\,dx \\
&= \int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx && \text{DCT} \\
&= \hat{f}(t).
\end{aligned}$$

Therefore the Fourier series of $F(x)$ is

$$F(x) = \sum_{t\in\mathbb{Z}} \hat{f}(t)e^{2\pi itx},$$

and the first statement follows by the definition of $F(x)$. Setting $x = 0$ proves the second statement. $\square$

There are two practical settings of interest. The first is when $f(x)$ is of **Schwarz class** which means that $f \in C^\infty(\mathbb{R})$ and f along with all of its derivative have rapid decay. For example, e^{-x^2} is of Schwarz class. For if $f(x)$ is of Schwarz class, the rapid decay implies that $f(x)$ is absolutely integrable over $\mathbb{R}$. Moreover, this also implies that $F(x)$ and all of its derivatives are locally absolutely uniformly convergent by the Weierstrass M-test. The uniform limit theorem then shows that $F(x)$ is smooth and thus the conditions of the Poisson summation formula are satisfied. The second application is when $f(z)$ is some holomorphic function that has at least polynomial decay of order $c > 1$ for any fixed y. Then $f(z)$ is absolutely integrable over $\mathbb{R}$ for any fixed y. Moreover, $F(z)$ is holomorphic. Then the conditions of the Poisson summation formula are satisfied for $z = iy$ with $y > 0$ and the identity theorem shows that this holds for all z. Our primary use of the Poisson summation formula will be to derive very striking transformation laws which are the key ingredient in analytically continuing many important functions.

The Mellin Transform

Like the Fourier transform, the Mellin transform is another type of integral transform. If $f(x)$ is a continuous function, then the **Mellin transform** $F(s)$ of $f(x)$ is given by

$$F(s) = \int_0^\infty f(x)x^s \, \frac{dx}{x}.$$

If $f(x)$ is a sufficiently nice function then the integral will be bounded in some half-plane in s. For example, this happens if $f(x)$ exhibits rapid decay and remains bounded as $x \to 0$. In this case, the integral is locally absolutely uniformly convergent for $\sigma > 0$. If the Mellin transform $F(s)$ is sufficiently nice, then the initial function can be recovered via means of the **inverse Mellin transform** $(\mathcal{M}^{-1}F)(x)$:

$$(\mathcal{M}^{-1}F)(x) = \frac{1}{2\pi i}\int_{(c)} F(s)x^{-s} \, ds.$$

It is not immediately clear that this integral exists and is independent of c. The following theorem makes precise what properties $F(s)$ needs to satisfy and for which c the inverse Mellin transform recovers the original function $f(x)$ (see [DB15] for a proof):

Theorem 1.5.4 (Mellin inversion formula). *Let a and b be reals such that $a < b$. Suppose $F(s)$ is analytic in the strip vertical $a < \sigma < b$, tends to zero uniformly as $t \to \infty$ along any line $\sigma = c$ for $a < c < b$, and that the integral of $F(s)$ along this line is locally absolutely uniformly convergent. Then if*

$$f(x) = \frac{1}{2\pi i}\int_{(c)} F(s)x^{-s} \, ds,$$

this integral is independent of c and moreover $F(s) = (\mathcal{M}f)(s)$. Conversely, suppose $f(x)$ is piecewise continuous such that its value is halfway between the limit values at any jump discontinuity and

$$F(s) = \int_0^\infty f(x)x^s \, \frac{dx}{x},$$

is locally absolutely uniformly convergent in the vertical strip $a < \sigma < b$. Then $f(x) = (\mathcal{M}^{-1}F)(x)$.

1.6 The Gamma Function

The gamma function is ubiquitous in analytic number theory and the better one understands it the better one will be at seeing the forest for the trees in any problem involving analytic number theory. The **gamma function** $\Gamma(s)$ is defined to be the Mellin transform of e^{-x}:

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1} \, dx,$$

for $\sigma > 0$. The integral is locally absolutely uniformly convergent in this region. Indeed, if K is a compact subset in the region $\sigma > 0$, then upon splitting the integral we have

$$\Gamma(s) = \int_0^1 e^{-x}x^{s-1} \, dx + \int_1^\infty e^{-x}x^{s-1} \, dx. \tag{1.4}$$

By Proposition 1.5.2, the second integral in Equation (1.4) is locally absolutely uniformly convergent in this region. As for the first integral, let $\beta = \min_{s\in K}\{\sigma\}$. Then

$$\int_0^1 e^{-x}x^{s-1}\,dx \ll \int_0^1 x^{\sigma-1}\,dx \ll_\beta 1. \tag{1.5}$$

Equation (1.5) implies that the first integral in Equation (1.4) is locally absolutely uniformly convergent too. Altogether, this means $\Gamma(s)$ is as well. Also note that for $s > 0$, $\Gamma(s)$ is real. The most basic properties of $\Gamma(s)$ are the following:

Proposition 1.6.1. $\Gamma(s)$ *satisfies the following properties:*

(i) $\Gamma(1) = 1$.

(ii) $\Gamma(s+1) = s\Gamma(s)$.

(iii) $\Gamma(\overline{s}) = \overline{\Gamma(s)}$.

Proof. We obtain (i) by direct computation:

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1.$$

An application of integration by parts gives (ii):

$$\Gamma(s+1) = \int_0^\infty e^{-x}x^s\,dx = -e^{-x}x^s\Big|_0^\infty + s\int_0^\infty e^{-x}x^{s-1}\,dx = s\int_0^\infty e^{-x}x^{s-1}\,dx = s\Gamma(s).$$

For (iii), since $\Gamma(s)$ is real for $s > 0$ we have $\Gamma(\overline{s}) = \overline{\Gamma(s)}$ on this half-line and then the identity theorem implies that this holds everywhere. □

From Proposition 1.6.1 we see that for $s = n$ a positive integer, $\Gamma(n) = (n-1)!$. So $\Gamma(s)$ can be thought of as a holomorphic extension of the factorial function. We can use property (ii) of Proposition 1.6.1 to extended $\Gamma(s)$ to a meromorphic function on all of $\mathbb{C}$:

Theorem 1.6.1. $\Gamma(s)$ *admits meromorphic continuation to* $\mathbb{C}$ *with poles at* $s = -n$ *for* $n \geq 0$. *All of these poles are simple and with residue* $\frac{(-1)^n}{n!}$ *at* $s = -n$.

Proof. Using Proposition 1.6.1, (ii) repeatedly, for any integer $n \geq 0$ we have

$$\Gamma(s) = \frac{\Gamma(s+1+n)}{s(s+1)\cdots(s+n)}.$$

The right-hand side defines an meromorphic function in the region $\sigma > -n$ and away from the points $0, -1, \ldots, -n$. Letting n be arbitrary, we see that $\Gamma(s)$ has meromorphic continuation to $\mathbb{C}$ with poles at $0, -1, -2, \ldots$. We now compute the residue at $s = -n$. Around this point $\Gamma(s)$ admits meromorphic continuation with representation

$$\frac{\Gamma(s+1+n)}{s(s+1)\cdots(s+n)},$$

where all of the factors except for $s+n$ are holomorphic at $s = -n$. Thus the pole is simple, and

$$\operatorname*{Res}_{s=-n}\Gamma(s) = \lim_{z\to -n}\frac{\Gamma(s+1+n)(s+n)}{s(s+1)\cdots(s+n)} = \frac{\Gamma(1)}{(-n)(1-n)\cdots(-1)} = \frac{(-1)^n}{n!}.$$ □

In particular, Theorem 1.6.1 implies $\mathrm{Res}_{s=0}\,\Gamma(s) = 1$ and $\mathrm{Res}_{s=1}\,\Gamma(s) = -1$. There are a few other properties of the gamma function that are famous and which we will use frequently. The first of which is the **Legendre duplication formula** (see [Rem98] for a proof):

Theorem 1.6.2 (Legendre duplication formula). *For any $s \in \mathbb{C} - \{0, -1, -2, \ldots\}$,*

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s).$$

As a first application, we can use this formula to compute $\Gamma\left(\frac{1}{2}\right)$. Letting $z = \frac{1}{2}$ in the Legendre duplication formula and recalling $\Gamma(1) = 1$, we see that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. There is also the important Hadamard factorization of the reciprocal of $\Gamma(s)$ (see [SSS03] for a proof):

Proposition 1.6.2. *For all $s \in \mathbb{C}$,*

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n\geq 1}\left(1 + \frac{s}{n}\right)e^{-\frac{s}{n}},$$

where γ is the Euler-Mascheroni constant.

In particular, $\frac{1}{\Gamma(s)}$ is entire so that $\Gamma(s)$ is nowhere vanishing on $\mathbb{C}$. Also, $\frac{1}{\Gamma(s)}$ is of order 1 (see Appendix B.5). In particular, this means that $\Gamma(s)$ is also order 1 for $\sigma > 0$. We call $\frac{\Gamma'}{\Gamma}(s)$ the **digamma function**. Equivalently, the digamma function is the logarithmic derivative of the gamma function. If we take the logarithmic derivative of the Hadamard factorization for $\frac{1}{s\Gamma(s)}$, we obtain a useful expression for the digamma function:

Corollary 1.6.1. *For all $s \in \mathbb{C}$,*

$$\frac{\Gamma'}{\Gamma}(s+1) = -\gamma + \sum_{n\geq 1}\left(\frac{1}{n} - \frac{1}{s+n}\right),$$

where γ is the Euler-Mascheroni constant. In particular, the digamma function has simple poles of residue -1 at the poles of the gamma function.

Proof. By Proposition 1.6.1 (ii), $\frac{1}{\Gamma(s+1)} = \frac{1}{s\Gamma(s)}$. Taking the logarithmic derivative using Proposition 1.6.2 we obtain

$$-\frac{\Gamma'}{\Gamma}(s+1) = \gamma + \sum_{n\geq 1}\left(\frac{1}{s+n} - \frac{1}{n}\right),$$

provided s is distance ε away from the poles of $\Gamma(s)$. This is the desired formula and the statement regarding the poles follows immediately. $\square$

We will also require a well-known approximation for the gamma function known as **Stirling's formula** (see [Rem98] for a proof):

Theorem 1.6.3 (Stirling's formula).

$$\Gamma(s) \sim_{\varepsilon,\delta} \sqrt{2\pi}s^{s-\frac{1}{2}}e^{-s},$$

provided $|\arg(s)| < \pi - \varepsilon$ and $|s| > \delta$ for some $\varepsilon, \delta > 0$.

If σ is bounded, Stirling's formula gives a useful estimate showing that $\Gamma(s)$ decays as $s \to \infty$:

Corollary 1.6.2. *Let $|\arg(s)| < \pi - \varepsilon$ and $|s| > \delta$ for some $\varepsilon, \delta > 0$. Then if σ is bounded, we have*

$$\Gamma(s) \sim_{\varepsilon,\delta} \sqrt{2\pi} t^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|}.$$

Proof. Stirling's formula can be equivalently expressed as

$$\Gamma(s) \sim_{\varepsilon,\delta} \sqrt{2\pi}(\sigma+it)^{\sigma-\frac{1}{2}+it} e^{-\sigma-it}.$$

Since σ is bounded, $e^{-\sigma-it} \ll 1$ and we obtain the the simplified asymptotic

$$\Gamma(s) \sim_{\varepsilon,\delta} \sqrt{2\pi}(it)^{\sigma-\frac{1}{2}+it}.$$

Similarly, x being bounded implies $i^{\sigma-\frac{1}{2}} \ll 1$ and we compute

$$(it)^{it} = e^{i|t|\log(i|t|)} = e^{i|t|(\log(i)+\log(|t|))} = e^{-\frac{\pi}{2}|t|+i|t|\log(|t|)} \sim e^{-\frac{\pi}{2}|t|},$$

where we have used the fact that $\log(i) = i\frac{\pi}{2}$. Together, we obtain the further simplified asymptotic

$$\Gamma(s) \sim_{\varepsilon,\delta} \sqrt{2\pi} t^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|},$$

which is the desired result $\square$

Strictly weaker than the asymptotic equivalence in Stirling's formula is the estimate

$$\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s}(1 + O_{\varepsilon,\delta}(1)). \tag{1.6}$$

Taking the logarithm (since $|\arg(s)| < \pi - \varepsilon$ the logarithm is defined) of this estimate gives

$$\log\Gamma(s) = \frac{1}{2}\log(2\pi) + \left(s - \frac{1}{2}\right)\log(s) - s + O_{\varepsilon,\delta}(1), \tag{1.7}$$

which will be useful. Using Equation (1.7) we can obtain a useful asymptotic formula for the digamma function:

Proposition 1.6.3.

$$\frac{\Gamma'}{\Gamma}(s) = \log(s) + O_{\varepsilon,\delta}(1),$$

provided $|\arg(s)| < \pi - \varepsilon$ and $|s| > \delta$ for some $\varepsilon, \delta > 0$.

Proof. Equation (1.7) give the simplified asymptotic

$$\log\Gamma(s) = \frac{1}{2}\log(2\pi) + s\log(s) - s + O_{\varepsilon,\delta}(1).$$

Set $g(s) = \frac{1}{2}\log(2\pi) + s\log(s) - s$ so that $\log\Gamma(s) = g(s) + O_{\varepsilon,\delta}(1)$. Then $\log\Gamma(s) - g(s) = O_{\varepsilon,\delta}(1)$, and by Cauchy's integral formula, we have

$$\begin{aligned}
\frac{\Gamma'}{\Gamma}(s) &= \frac{d}{ds}\left(g(s) + O_{\varepsilon,\delta}(1)\right) \\
&= g'(s) + \frac{d}{ds}(\log\Gamma(s) - g(s)) \\
&= \log(s) + \frac{1}{2\pi i}\int_{|u-s|=\eta} \frac{\log\Gamma(u) - g(u)}{(u-s)^2}\,du,
\end{aligned}$$

for some sufficiently small radius $\eta > 0$ depending upon ε and δ. Therefore

$$\left|\frac{\Gamma'}{\Gamma}(s) - \log(s)\right| \leq \frac{1}{2\pi}\int_{|u-s|=\eta} \frac{|\log\Gamma(u) - g(u)|}{\eta^2}\,|du| \ll_{\varepsilon,d} 1,$$

where the last estimate follows because $\log\Gamma(s) - g(s) = O_{\varepsilon,d}(1)$. □

The last result we will need is an explicit representation for $\log\Gamma(s)$ known as **Binet's log gamma formula**:

Proposition 1.6.4 (Binet's log gamma formula).

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\Gamma(s) - s + \frac{1}{2}\log(2\pi) + 2\int_0^\infty \frac{\tan^{-1}\left(\frac{x}{s}\right)}{e^{2\pi x} - 1}\,dx.$$

Part II

An Introduction to Holomorphic & Maass Forms

Chapter 2

Congruence Subgroups & Modular Curves

Every holomorphic or Maass form is a special type of function depending on certain subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. These are the congruence subgroups. The associated modular curve is the quotient of the upper half-space $\mathbb{H}$ by an action of this subgroup. We introduce these topics first as they are the foundation for discussing holomorphic and Maass forms in complete generality.

2.1 Congruence Subgroups

The **modular group** is $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. That is, the modular group is the set of matrices with integer entries and of determinant 1 determined up to sign. The reason we are only interested in these matrices up to sign is because the modular group has a natural action on the upper half-space $\mathbb{H}$ and this action will be invariant under a change in sign. The first result usually proved about the modular group is that it is generated by two matrices:

Proposition 2.1.1.

$$\mathrm{PSL}_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Proof. Set S and T to be the first and second generators respectively. Clearly they belong to $\mathrm{PSL}_2(\mathbb{Z})$. Also, S and T^n for $n \in \mathbb{Z}$ acts on $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$ by

$$S\gamma = S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \quad \text{and} \quad T^n\gamma = T^n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}.$$

In particular, S interchanges the upper left and lower left entries of γ up to sign and T^n adds an n multiple of the lower left entry to the upper left entry. We have to show $\gamma \in \langle S, T \rangle$ and we will accomplish this by showing that the inverse is in $\langle S, T \rangle$. If $|c| = 0$ then γ is the identity since $\det(\gamma) = 1$ so suppose $|c| \neq 0$. By Euclidean division we can write $a = qc + r$ for some $q \in \mathbb{Z}$ and $|r| < |c|$. Then

$$T^{-q}\gamma = \begin{pmatrix} a-qc & b-qd \\ c & d \end{pmatrix} = \begin{pmatrix} r & b-qd \\ c & d \end{pmatrix}.$$

Multiplying by S yields

$$ST^{-q}\gamma = S\begin{pmatrix} r & b-qd \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ r & b-qd \end{pmatrix},$$

and this matrix has the upper left entry at least as large as the lower left entry in norm. Actually the upper left entry is strictly larger since $|c| > |r|$ by Euclidean division. Therefore if we repeatedly apply this procedure, it must terminate with the lower left entry vanishing. But then we have reached the identity matrix. Therefore we have show γ has an inverse in $\langle S, T\rangle$. $\square$

We will also be interested in special subgroups of the modular group defined by congruence conditions on their entries. For $N \geq 1$, set

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

Then $\Gamma(N)$ is the kernel of the natural homomorphism $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$ so it is a normal subgroup with finite index. We call $\Gamma(N)$ the **principal congruence subgroup** of level N. For $\Gamma \leq \mathrm{PSL}_2(\mathbb{Z})$, we say Γ is a **congruence subgroup** if $\Gamma(N) \leq \Gamma$ for some N and the minimal such N is called the **level** of Γ. Note that if $M \mid N$, then $\Gamma(N) \leq \Gamma(M)$. Thus if Γ is a congruence subgroup of level N, then $\Gamma(kN) \leq \Gamma$ for all $k \geq 1$. This implies that congruence subgroups are closed under intersection. Also, it turns out that the natural homomorphism $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective:

Proposition 2.1.2. *The natural homomorphism* $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$ *is surjective.*

Proof. Suppose $\left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Then $\overline{a}\overline{d} - \overline{b}\overline{c} \equiv 1 \pmod N$ so by Bézout's identity (generalized to three integers) $(\overline{c}, \overline{d}, N) = 1$. We claim that there exists s and t such that $c = \overline{c} + sN$, $d = \overline{d} + tN$ with $(c, d) = 1$. Set $g = (\overline{c}, \overline{d})$. Then $(g, N) = 1$ because $(\overline{c}, \overline{d}, N) = 1$. If $\overline{c} = 0$ then set $s = 0$ so $c = 0$ and choose t such that $t \equiv 1 \pmod p$ for any prime $p \mid g$ and $t \equiv 0 \pmod p$ for any prime $p \nmid g$ and $p \mid \overline{d}$. Such a t exists by the Chinese remainder theorem. Now if $p \mid (c, d)$, then either $p \mid g$ or $p \nmid g$. If $p \mid g$, then $p \mid d - \overline{d} = tN$ which is absurd since $t \equiv 1 \pmod p$ and $(t, N) = 1$. If $p \nmid g$, then $p \nmid d - \overline{d} = tN$ but this is also absurd since $t \equiv 0 \pmod p$. Therefore $(c, d) = 1$ as claimed. If $\overline{c} = 0$ then $\overline{d} \neq 0$, and we can proceed similarly. Since $(c, d) = 1$ there exists a and b such that $ad - bc = 1$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$ and maps onto $\left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right)$. This proves surjectivity. $\square$

By Proposition 2.1.2, $[\mathrm{PSL}_2(Z) : \Gamma(N)] = |\mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})|$. Since $\Gamma(N) \leq \Gamma$ and $\Gamma(N)$ has finite index in $\mathrm{PSL}_2(\mathbb{Z})$ so does Γ. The subgroups

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\},$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\},$$

are particularly important and are congruence subgroups of level N. The latter subgroup is called the **Hecke congruence subgroup** of level N. Note that $\Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N)$. If Γ is a general congruence subgroup, it is useful to find a generating set for Γ in order to reduce results about Γ to that of the generators. This is usually achieved by performing some sort of Euclidean division argument on the entries of a matrix $\gamma \in \Gamma$ using the supposed generating set to construct the inverse for γ. We will also require a useful lemma which says that congruence subgroups are preserved under conjugation by elements of $\mathrm{GL}_2^+(\mathbb{Q})$ provided we restrict to those elements in $\mathrm{PSL}_2(\mathbb{Z})$:

Lemma 2.1.1. *Let* Γ *be a congruence subgroup and let* $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. *Then* $\alpha^{-1}\Gamma\alpha \cap \mathrm{PSL}_2(\mathbb{Z})$ *is a congruence subgroup.*

Proof. Recall that if Γ is of level M, then $\Gamma(kM) \le \Gamma$ for every $k \ge 1$. Thus there is an integer $\widetilde{N}$ such that $\Gamma(\widetilde{N}) \le \Gamma$, $\widetilde{N}\alpha \in \mathrm{GL}_2^+(\mathbb{Z})$, and $\widetilde{N}\alpha \in \mathrm{GL}_2^+(\mathbb{Z})$. Now let $N = \widetilde{N}^3$ and notice that any $\gamma \in \Gamma(N)$ is of the form

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix},$$

for $k_1, \ldots, k_4 \in \mathbb{Z}$. Therefore $\Gamma(N) \subseteq I + N\mathrm{Mat}_2(\mathbb{Z})$. Thus

$$\alpha\Gamma(N)\alpha^{-1} \le \alpha(I + N\mathrm{Mat}_2(\mathbb{Z}))\alpha^{-1} = I + \widetilde{N}\mathrm{Mat}_2(\mathbb{Z}).$$

As every matrix in $\alpha\Gamma(N)\alpha^{-1}$ has determinant 1 and $\Gamma(\widetilde{N}) \subseteq I + \widetilde{N}\mathrm{Mat}_2(\mathbb{Z})$, it follows that $\alpha\Gamma(N)\alpha^{-1} \le \Gamma(\widetilde{N})$. As $\Gamma(\widetilde{N}) \le \Gamma$, we conclude

$$\Gamma(N) \le \alpha^{-1}\Gamma\alpha,$$

and intersecting with $\mathrm{PSL}_2(\mathbb{Z})$ completes the proof. $\square$

Note that by Lemma 2.1.1, if $\alpha^{-1}\Gamma\alpha \subset \mathrm{PSL}_2(\mathbb{Z})$ then $\alpha^{-1}\Gamma\alpha$ is a congruence subgroup if Γ is. Moreover, since congruence subgroups are closed under intersection, Lemma 2.1.1 further implies that $\alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$ is a congruence subgroup for and two congruence subgroups Γ_1 and Γ_2 any any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$.

2.2 Modular Curves

Recall that $\mathrm{GL}_2^+(\mathbb{Q})$ naturally acts on the Riemann sphere $\hat{\mathbb{C}}$ by Möbius transformations. Explicitly, any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ acts on $z \in \hat{\mathbb{C}}$ by

$$\gamma z = \frac{az+b}{cz+d},$$

where $\gamma\infty = \frac{a}{c}$ and $\gamma\left(-\frac{d}{c}\right) = \infty$. Moreover, recall that this action is a group action, is invariant under scalar multiplication, and acts as automorphisms of $\hat{\mathbb{C}}$. Now observe

$$\mathrm{Im}(\gamma z) = \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{az+b}{cz+d}\frac{c\overline{z}+d}{c\overline{z}+d}\right) = \mathrm{Im}\left(\frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2}\right) = \det(\gamma)\frac{\mathrm{Im}(z)}{|cz+d|^2},$$

where the last equality follows because $\mathrm{Im}(\overline{z}) = -\mathrm{Im}(z)$ and $\det(\gamma) = ad - bc$. Since $\deg(\gamma) > 0$ and $|cz+d|^2 > 0$, γ preserves the sign of the imaginary part of z. So γ preserves the upper half-space $\mathbb{H}$, the lower half-space $\overline{\mathbb{H}}$, and the extended real line $\hat{\mathbb{R}}$ respectively. Moreover, γ restricts to an automorphism on these subspaces since Möbius transformations are automorphisms. In particular, $\mathrm{PSL}_2(\mathbb{Z})$ naturally acts on $\hat{\mathbb{C}}$ by Möbius transformations and preserves the upper half-space. Certain actions of subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ also play important roles. A **Fuchsian group** is any subgroup of $\mathrm{PSL}_2(\mathbb{R})$ that acts properly discontinuously on $\mathbb{H}$. It turns out that the modular group is a Fuchsian group (see [DS05] for a proof):

Proposition 2.2.1. *The modular group is a Fuchsian group.*

Note that Proposition 2.2.1 immediately implies that any subgroup of the modular group is also Fuchsian. In particular, all congruence subgroups are Fuchsian. This lets us say a little more about the action of a congruence subgroup Γ on $\mathbb{H}$. Indeed, points in $\mathbb{H}$ are closed since $\mathbb{H}$ is Hausdorff. Now Γ is a Fuchsian group so it acts properly discontinuously on $\mathbb{H}$. These two facts together imply that the Γ-orbit of any point in $\mathbb{H}$ is a discrete set. We will make use of this property later on, but for now we are ready to introduce modular curves. A **modular curve** is a quotient $\Gamma\backslash\mathbb{H}$ of the upper half-space $\mathbb{H}$ by a congruence subgroup Γ. We give $\Gamma\backslash\mathbb{H}$ the quotient topology induced from $\mathbb{H}$ as a subset of the Riemann sphere $\hat{\mathbb{C}}$. This gives $\Gamma\backslash\mathbb{H}$ some nice topological properties (see [DS05] for a proof):

Proposition 2.2.2. *For any congruence subgroup* Γ, *the modular curve* $\Gamma\backslash\mathbb{H}$ *is connected and Hausdorff.*

A **fundamental domain** for $\Gamma\backslash\mathbb{H}$ is a closed set $\mathcal{F}_\Gamma \subseteq \mathbb{H}$ satisfying the following conditions:

(i) Any point in $\mathbb{H}$ is Γ-equivalent to a point in $\mathcal{F}_\Gamma$.

(ii) If two points in $\mathcal{F}_\Gamma$ are Γ-equivalent via a non-identity element, then they lie on the boundary of $\mathcal{F}_\Gamma$.

(iii) The interior of $\mathcal{F}_\Gamma$ is a domain.

In other words, $\mathcal{F}_\Gamma$ is a complete set of representatives (possibly with overlap on the boundary) for $\Gamma\backslash\mathbb{H}$ that has a nice topological structure with respect to $\mathbb{H}$. Note that if $\mathcal{F}_\Gamma$ is a fundamental domain then so is $\gamma\mathcal{F}_\Gamma$ for any $\gamma \in \Gamma$ and moreover $\mathbb{H} = \bigcup_{\gamma\in\Gamma} \gamma\mathcal{F}_\Gamma$. So the choice of $\mathcal{F}_\Gamma$ is not unique. Intuitively, a fundamental domain is a geometric realization of $\Gamma\backslash\mathbb{H}$ which is often more fruitful than thinking of $\Gamma\backslash\mathbb{H}$ as an abstract set of equivalence classes. Moreover, it's suggestive that we can give $\Gamma\backslash\mathbb{H}$ a geometric structure and indeed we can (see [DS05] for more). Property (iii) is usually not included in the definition of a fundamental domain for many authors. The reason that we impose this additional property is because we will integrate over $\mathcal{F}_\Gamma$ and so we want $\mathcal{F}_\Gamma$ to genuinely represent a domain as a subset of $\mathbb{H}$.

Proposition 2.2.3.

$$\mathcal{F} = \left\{ z \in \mathbb{H} : |\mathrm{Re}(z)| \le \frac{1}{2} \text{ and } |z| \ge 1 \right\},$$

is a fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$.

Proof. Set $\mathrm{PSL}_2(\mathbb{Z}) = \langle S, T\rangle$ where S and T are as in Proposition 2.1.1. We first show any point in $\mathbb{H}$ is $\mathrm{PSL}_2(\mathbb{Z})$-equivalent to a point in $\mathcal{F}$. Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$, we have

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2} = \frac{y}{(cx+d)^2 + (cy)^2}.$$

Since $\det(\gamma) = 1$ we cannot have $c = d = 0$. Then as $y \neq 0$, $|cz+d|^2$ is bounded away from zero and moreover there are finitely many pairs (c,d) such that $|cz+d|^2$ is less than any given upper bound. Therefore there exists $\gamma_0 \in \mathrm{PSL}_2(\mathbb{Z})$ that minimizes $|cz+d|^2$ and hence maximizes $\mathrm{Im}(\gamma_0 z)$. In particular,

$$\mathrm{Im}(S\gamma_0 z) = \frac{\mathrm{Im}(\gamma_0 z)}{|\gamma_0 z|^2} \le \mathrm{Im}(\gamma_0 z).$$

The inequality above implies $|\gamma_0 z| \ge 1$. Since $\mathrm{Im}(T^n\gamma_0 z) = \mathrm{Im}(\gamma_0 z)$ for all $n \in \mathbb{Z}$, repeating the argument above with $T^n\gamma_0$ in place of γ_0, we see that $|T^n\gamma_0 z| \ge 1$. But T shifts the real part by 1 so we can choose n such that $|\mathrm{Re}(T^n\gamma_0 z)| \le \frac{1}{2}$. Therefore $T^n\gamma_0 \in \mathrm{PSL}_2(\mathbb{Z})$ sends z into $\mathcal{F}$ as desired. We will now show that if two points in $\mathcal{F}$ are $\mathrm{PSL}_2(\mathbb{Z})$-equivalent via a non-identity element, then they lie on the boundary of $\mathcal{F}$. Since $\mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by automorphisms, by Proposition 2.1.1 it suffices to show that S and T map $\mathcal{F}$ outside of $\mathcal{F}$ except for possibly the boundary. This is clear for T since it maps the left boundary line $\left\{z \in \mathbb{H} : \mathrm{Re}(z) = -\frac{1}{2} \text{ and } |z| \ge 1\right\}$ to the right boundary line $\left\{z \in \mathbb{H} : \mathrm{Re}(z) = \frac{1}{2} \text{ and } |z| \ge 1\right\}$ and every other point of $\mathcal{F}$ is mapped to the right of this line. For S, note that it maps the semicircle $\{z \in \mathbb{H} : |z| = 1\}$ to itself (although not identically) and maps ∞ to zero. Since Möbius transformations send circles to circles and lines to lines it follows that every other point of $\mathcal{F}$ is taken to a point enclosed by the semicircle $\{z \in \mathbb{H} : |z| = 1\}$. Lastly, the interior of $\mathcal{F}$ is a domain since it is open and path-connected. This finishes the proof. $\square$

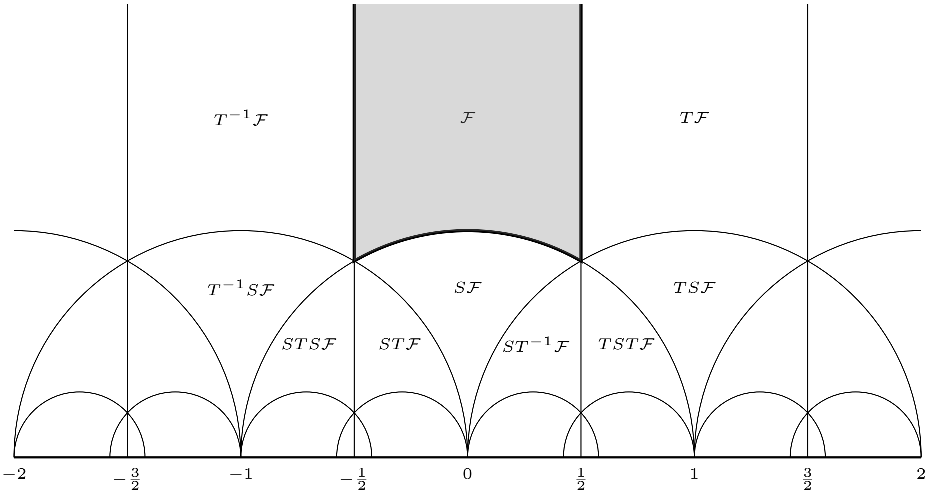

Figure 2.1: The standard fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$.

The shaded region in Figure 2.1 is the fundamental domain in Proposition 2.2.3 and we call it the **standard fundamental domain**. Figure 2.1 also displays how this fundamental domain changes under the actions of the generators of $\mathrm{PSL}_2(\mathbb{Z})$ as in Proposition 2.1.1. A fundamental domain for any other modular curve can be built from the standard fundamental domain as the following proposition shows (see [Kil15] for a proof):

Proposition 2.2.4. *Let Γ be any congruence subgroup. Then*

$$\mathcal{F}_\Gamma = \bigcup_{\gamma \in \Gamma\backslash \mathrm{PSL}_2(\mathbb{Z})} \gamma\mathcal{F},$$

is a fundamental domain for $\Gamma\backslash\mathbb{H}$.

We might notice that $\mathcal{F}$ in Figure 2.1 is unbounded as it doesn't contain the point ∞. However, if we consider $\mathcal{F} \cup \{\infty\}$ then it would appear that this space is compact. The point ∞ is an example of a cusp and we now make this idea precise. Since any $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ preserves $\hat{\mathbb{R}}$ and γ has integer entries, γ also preserves $\mathbb{Q} \cup \{\infty\}$. A **cusp** of $\Gamma\backslash\mathbb{H}$ is an element of of $\Gamma\backslash(\mathbb{Q} \cup \{\infty\})$. As Γ has finite index in the modular group, there can only be finitely many cusps and the number of cusps is at most the index of Γ. In particular, the Γ-orbit of ∞ is a cusp of $\Gamma\backslash\mathbb{H}$. We denote cusps by gothic characters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ or by representatives of their equivalence classes. For example, we let ∞ denote the cusp $\Gamma\infty$.

Remark 2.2.1. *It turns out that the cusps can be represented as the points needed to make a fundamental domain $\mathcal{F}_\Gamma$ compact as a subset of $\hat{\mathbb{C}}$. To see this, suppose $\mathfrak{a}$ is a limit point of $\mathcal{F}_\Gamma$ that does not belong to $\mathcal{F}_\Gamma$. Then $\mathfrak{a} \in \hat{\mathbb{R}}$. In the case of the standard fundamental domain $\mathcal{F}$, $\mathfrak{a} = \infty$ which is a cusp. Otherwise, $\mathcal{F}_\Gamma$ is a union of images of $\mathcal{F}$ by Proposition 2.2.4 and since* $\mathrm{PSL}_2(\mathbb{Z})\infty = \mathbb{Q} \cup \{\infty\}$, *we find that $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$.*

Let $\Gamma_\mathfrak{a} \leq \Gamma$ denote the stabilizer subgroup of the cusp $\mathfrak{a}$. For the ∞ cusp, we can describe Γ_∞ explicitly. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ stabilizes ∞, then necessarily $c = 0$ and since $\det(\gamma) = 1$ we must have $a = d = 1$. Therefore $\gamma = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some $b \in \mathbb{Z}$ and γ acts on $\mathbb{H}$ by translation by b. Of course, not every translation is guaranteed to belong to Γ. Letting t be the smallest positive integer such that $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, we have $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \rangle$. In particular, Γ_∞ is an infinite cyclic group. We say that Γ is **reduced at infinity** if $t = 1$ so that $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$. In particular, $\Gamma_1(N)$ and $\Gamma_0(N)$ are reduced at infinity.

Remark 2.2.2. *If Γ is of level N, then N is the smallest positive integer such that $\Gamma(N) \leq \Gamma$ so that $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$ is the minimal translation guaranteed to belong to Γ. However, there may be smaller translations so in general $t \leq N$.*

Moreover, for any cusp $\mathfrak{a}$ we have that $\Gamma_{\mathfrak{a}}$ is also an infinite cyclic group and we denote its generator by $\gamma_{\mathfrak{a}}$. To see this, if $\mathfrak{a} = \frac{a}{c}$ with $(a,c) = 1$ is a cusp of $\Gamma\backslash\mathbb{H}$ not equivalent to ∞, then there exists an $\sigma_{\mathfrak{a}} \in \mathrm{PSL}_2(\mathbb{Z})$ such that $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$. Indeed, there exists integers d and b such that $ad - bc = 1$ by Bézout's identity and then $\sigma_{\mathfrak{a}} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is such a matrix. It follows that $\Gamma_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}$ and since Γ_∞ is infinite cyclic so is $\Gamma_{\mathfrak{a}}$. We call any matrix $\sigma_{\mathfrak{a}} \in \mathrm{PSL}_2(\mathbb{Z})$ satisfying

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a} \quad \text{and} \quad \sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

a **scaling matrix** for the cusp $\mathfrak{a}$. Note that $\sigma_{\mathfrak{a}}$ is determined up to composition on the right by an element of Γ_∞. Scaling matrices are useful because they allow us to transfer information at the cusp $\mathfrak{a}$ to the cusp at ∞. Let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$ with scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ respectively. When investigating holomorphic forms, it will be useful to have a double coset decomposition for sets of the form $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$. This is referred to as the **Bruhat decomposition** for Γ:

Theorem 2.2.1 (Bruhat decomposition). *Let Γ be any congruence subgroup and let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$ with scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ respectively. Then we have the disjoint decomposition*

$$\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a},\mathfrak{b}}\Omega_\infty \bigcup_{\substack{c \geq 1 \\ d \pmod c}} \Omega_{d/c},$$

where

$$\Omega_\infty = \Gamma_\infty\omega_\infty = \omega_\infty\Gamma_\infty = \Gamma_\infty\omega_\infty\Gamma_\infty \quad \text{and} \quad \Omega_{d/c} = \Gamma_\infty\omega_{d/c}\Gamma_\infty,$$

for some $\omega_\infty = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ and $\omega_{d/c} = \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ with $c \geq 1$ if such a matrix exists otherwise $\Omega_{d/c}$ is empty. Moreover, the entries a and d of $\omega_{d/c}$ are determined modulo c.

Proof. We first show that Ω_∞ is nonempty if and only if $\mathfrak{a} = \mathfrak{b}$. Indeed, if $\omega \in \Omega_\infty$ then $\omega = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}$ for some $\gamma \in \Gamma$. Then

$$\gamma\mathfrak{b} = \sigma_{\mathfrak{a}}\omega\sigma_{\mathfrak{b}}^{-1}\mathfrak{b} = \sigma_{\mathfrak{a}}\omega\infty = \sigma_{\mathfrak{a}}\infty = \mathfrak{a}.$$

This shows that $\mathfrak{a} = \mathfrak{b}$. Conversely, suppose $\mathfrak{a} = \mathfrak{b}$. Then $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ contains $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$ so that Ω_∞ is nonempty. So Ω_∞ is nonempty if and only if $\mathfrak{a} = \mathfrak{b}$. In this case, for any two elements $\omega = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}$ and $\omega' = \sigma_{\mathfrak{a}}^{-1}\gamma'\sigma_{\mathfrak{a}}$ of Ω_∞, we have

$$\gamma'\gamma^{-1}\mathfrak{a} = \sigma_{\mathfrak{a}}\omega'\omega^{-1}\sigma_{\mathfrak{a}}^{-1}\mathfrak{a} = \sigma_{\mathfrak{a}}\omega'\omega^{-1}\infty = \sigma_{\mathfrak{a}}\mathfrak{a}.$$

Hence $\gamma'\gamma^{-1} \in \Gamma_{\mathfrak{a}}$ which implies $\omega'\omega^{-1} = \sigma_{\mathfrak{a}}^{-1}\gamma'\gamma^{-1}\sigma_{\mathfrak{a}} \in \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$. Therefore

$$\Omega_\infty = \Gamma_\infty\omega = \omega\Gamma_\infty = \Gamma_\infty\omega\Gamma_\infty,$$

where the latter two equalities hold because ω is a translation and translations commute. Every other element of $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ belongs to one of the double cosets $\Omega_{d/c}$ with $c \geq 1$ (since we are working in $\mathrm{PSL}_2(\mathbb{Z})$). The relation

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & * \\ c & d \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + cn & * \\ c & d + cm \end{pmatrix},$$

shows that $\Omega_{d/c}$ is determined uniquely by c and $d \pmod c$. Moreover, this relation shows that a and d are determined modulo c. This completes the proof of the theorem. □

Notice that the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ implies

$$\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a},\mathfrak{b}}\omega_\infty \bigcup_{\substack{c\geq 1 \\ d \pmod c}} \omega_{d/c}\Gamma_\infty,$$

where it is understood that the coset $\omega_{d/c}\Gamma_\infty$ is empty if the double coset $\Omega_{d/c}$ is too. This shows that every element of $\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ corresponds to a unique $(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}$ with $c \geq 1$, $d \in \mathbb{Z}$, and $(c,d) = 1$, and additionally the pair $(0,1)$ if and only if $\mathfrak{a} = \mathfrak{b}$ (this pair corresponds to ω_∞). Of course, this correspondence need not be surjective since many of the double cosets $\Omega_{d/c}$ may be empty. To track such c and d for which $\Omega_{d/c}$ is nonempty, let $\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ and $\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ be the sets given by

$$\mathcal{C}_{\mathfrak{a},\mathfrak{b}} = \left\{c \geq 1 : \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\right\} \quad \text{and} \quad \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c) = \left\{d \pmod c : \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\right\}.$$

Then $\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ and $\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ are precisely the sets of c and d take modulo c such that $\Omega_{d/c}$ is nonempty.

Remark 2.2.3. *The Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ implies*

$$\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a},\mathfrak{b}}\omega_\infty \bigcup_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ d\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \omega_{d/c}\Gamma_\infty,$$

where none of the cosets $\omega_{d/c}\Gamma_\infty$ are empty. In particular, $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ if and only if (c,d) is a pair with $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $d \in \mathbb{Z}$, and $d \pmod c \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$, or additionally $(0,1)$ if $\mathfrak{a} = \mathfrak{b}$.

We will now introduce Kloosterman & Salié sums associated to cusps. We being with the Kloosterman sums. Let $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ be scaling matrices for the cusps $\mathfrak{a}$ and $\mathfrak{b}$ respectively. Then for any $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ and $n,m \in \mathbb{Z}$, the **generalized Kloosterman sum** $K_{\mathfrak{a},\mathfrak{b}}(n,m,c)$ relative to $\mathfrak{a}$ and $\mathfrak{b}$ is defined by

$$K_{\mathfrak{a},\mathfrak{b}}(n,m,c) = \sum_{d\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)} e^{\frac{2\pi i(an+\overline{a}m)}{c}},$$

where a has been determined by $ad - bc = 1$. This sum is well-defined by the Bruhat decomposition because a is determined modulo c. In general, $K_{\mathfrak{a},\mathfrak{b}}(n,m,c)$ is not independent of the scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$. However, if $n = m = 0$ we trivially see by the Bruhat decomposition for Γ that

$$K_{\mathfrak{a},\mathfrak{b}}(0,0,c) = \left|\left\{d \pmod c : \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\right\}\right|,$$

which is independent of the scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$. Moreover, for $\Gamma = \Gamma_0(1)$ we have $\mathfrak{a} = \mathfrak{b} = \infty$ and the Bruhat decomposition for $\Gamma_0(1)$ implies

$$K_{\infty,\infty}(n,m,c) = K(n,m,c),$$

is the usual Kloosterman sum. Therefore if $\mathfrak{a} = \mathfrak{b} = \infty$, we will suppress these dependencies accordingly. The Salié sums are defined in a similar manner. Let $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ be scaling matrices for the cusps $\mathfrak{a}$ and $\mathfrak{b}$ respectively. Then for any $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, and $n,m \in \mathbb{Z}$, and Dirichlet character χ with conductor $q \mid c$, the **generalized Salié sum** $S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c)$ relative to $\mathfrak{a}$ and $\mathfrak{b}$ is defined by

$$S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c) = \sum_{d\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)} \chi(a)e^{\frac{2\pi i(an+\overline{a}m)}{c}},$$

where a has been determined by $ad-bc=1$. This sum is well-defined by the Bruhat decomposition because a is determined modulo c. Like the generalized Kloosterman sum, $S_{\mathfrak{a},\mathfrak{b}}(n,m,c)$ need not independent of the scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$. Moreover, for $\Gamma=\Gamma_0(1)$ we have $\mathfrak{a}=\mathfrak{b}=\infty$ and the Bruhat decomposition for $\Gamma_0(1)$ implies

$$S_{\chi,\infty,\infty}(n,m,c)=S_{\chi}(n,m,c),$$

is the usual Salié sum. Therefore if $\mathfrak{a}=\mathfrak{b}=\infty$, we will suppress these dependencies accordingly.

2.3 The Hyperbolic Measure

We will also need to integrate over $\Gamma\backslash\mathbb{H}$. In order to do this, we require a measure on $\mathbb{H}$. Our choice of measure will be the **hyperbolic measure** $d\mu$ given by

$$d\mu=d\mu(z)=\frac{dx\,dy}{y^2}.$$

The most important property about the hyperbolic measure is that it is $\mathrm{GL}_2^+(\mathbb{R})$-invariant (see [DS05] for a proof):

Proposition 2.3.1. *The hyperbolic measure $d\mu$ is $\mathrm{GL}_2^+(\mathbb{R})$-invariant.*

As this fact will be used so frequently any time we integrate, we will not mention it explicitly. The particular use is that if Γ is a congruence subgroup, $d\mu$ is Γ-invariant. One of the reasons that this is useful is because we can apply the unfolding/folding method to many integrals. The most common instance is when we are integrating the sum $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}f(\gamma z)$ of some holomorphic function $f(z)$ over a fundamental domain $\mathcal{F}_\Gamma$ for $\Gamma\backslash\mathbb{H}$. Indeed, $\mathbb{H}=\bigcup_{\gamma\in\Gamma}\gamma\mathcal{F}_\Gamma$ and so $\Gamma_\infty\backslash\mathbb{H}=\bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma\mathcal{F}_\Gamma$. Since $\mathcal{F}_\Gamma$ is a fundamental domain, the conditions of the unfolding/folding method are satisfied and it follows that

$$\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}f(\gamma z)\,d\mu=\int_{\Gamma_\infty\backslash\mathbb{H}}f(z)\,d\mu,$$

provided the sum and or integral on either side are absolutely convergent and absolutely bounded. It is also worth highlighting another fact. Any $\alpha\in\mathrm{GL}_2^+(\mathbb{Q})$ acts as an automorphism of $\mathbb{H}$ which implies that it induces a bijection between $\alpha^{-1}\Gamma\alpha\backslash\mathbb{H}$ and $\Gamma\backslash\mathbb{H}$ and hence between the fundamental domains $\mathcal{F}_{\alpha^{-1}\Gamma\alpha}$ and $\mathcal{F}_\Gamma$. Thus the change of variables $z\to\alpha z$ transforms the fundamental domain $\mathcal{F}_\Gamma$ into $\mathcal{F}_{\alpha^{-1}\Gamma\alpha}$. Therefore

$$\int_{\mathcal{F}_\Gamma}f(z)\,d\mu=\int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}}f(\alpha z)\,d\mu,$$

provided either side is bounded. Now let us discuss the volume of $\Gamma\backslash\mathbb{H}$. We let

$$V_\Gamma=\mathrm{Vol}(V_\Gamma)=\int_{\mathcal{F}_\Gamma}d\mu,$$

so that V_Γ is the volume of $\Gamma\backslash\mathbb{H}$. Also, if $\mathcal{F}_\Gamma=\mathcal{F}$ we write $V_\Gamma=V$. In other words, V is the volume of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. Since the integrand is Γ-invariant, V_Γ is independent of the choice of fundamental domain. Using Proposition 2.2.3, we have

$$V=\int_{\mathcal{F}}d\mu=\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\sqrt{1-x^2}}^{\infty}\frac{dy\,dx}{y^2}=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\sqrt{1-x^2}}\,dx=\arcsin(x)\Big|_{-\frac{1}{2}}^{\frac{1}{2}}=\frac{\pi}{3}.$$

Therefore V is finite. There is also a simple relation between V_Γ and the index of Γ in $\mathrm{PSL}_2(\mathbb{Z})$:

$$V_\Gamma = [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma]V, \tag{2.1}$$

which follows immediately from Proposition 2.2.4. Moreover, V_Γ is finite for every congruence subgroup Γ by Equation (2.1) and that congruence subgroups have finite index in the modular group. A particularly nice application of this fact is that any integral of the form

$$\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\, d\mu,$$

is bounded by Proposition 1.5.1 provided $f(z)$ is holomorphic and bounded. That is, bounded functions are absolutely convergent over $\mathcal{F}_\Gamma$ with respect to $d\mu$. Moreover, we have a useful lemma:

Lemma 2.3.1. *Let Γ be a congruence subgroup and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. If $\alpha^{-1}\Gamma\alpha \subseteq \mathrm{PSL}_2(\mathbb{Z})$, then $V_{\alpha^{-1}\Gamma\alpha} = V_\Gamma$ and $[\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma\alpha] = [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma]$.*

Proof. The first statement follows from the chain

$$V_{\alpha^{-1}\Gamma\alpha} = \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} d\mu = \int_{\mathcal{F}_\Gamma} d\mu = V_\Gamma,$$

where the middle equality is justified by making the change of variables $z \to \alpha^{-1}z$. The second statement is now immediate from Equation (2.1). $\square$

Chapter 3

The Theory of Holomorphic Forms

Holomorphic forms are special classes of functions on the upper half-space $\mathbb{H}$ of the complex plane. They are holomorphic, have a transformation law with respect to a congruence subgroup, and satisfy a growth condition. We will introduce these forms in a general context.

3.1 Holomorphic Forms

Define $j(\gamma, z)$ by

$$j(\gamma, z) = (cz + d),$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ and $z \in \mathbb{H}$. There is a very useful property that $j(\gamma, z)$ satisfies. To state it, let $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$. Then

$$\gamma\gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a'a + b'c & a'b + b'b \\ c'a + d'c & c'b + d'd \end{pmatrix},$$

and we have

$$\begin{aligned} j(\gamma', \gamma z) j(\gamma, z) &= \left(c' \frac{az + b}{cz + d} + d' \right) (cz + d) \\ &= (c'(az + b) + d'(cz + d)) \\ &= (c'a + d'c)z + c'b + d'd) \\ &= j(\gamma'\gamma, z). \end{aligned}$$

In short,

$$j(\gamma'\gamma, z) = j(\gamma', \gamma z) j(\gamma, z),$$

and this is called the **cocycle condition** for $j(\gamma, z)$. For any integer $k \geq 1$ and any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ we define the **slash operator** $|_{j,k}\gamma : C(\mathbb{H}) \to C(\mathbb{H})$ to be the linear operator given by

$$(f|_{j,k}\gamma)(z) = j(\gamma, z)^{-k} f(\gamma z).$$

If j and k are clear from content we will suppress this dependencies accordingly. If an operator commutes with the slash operators $|_{j,k}\gamma$ for every $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, we say that it is **invariant**. Now let Γ be a congruence subgroup of level N that is reduced at infinity and let χ be a Dirichlet character of conductor $q \mid N$. Set $\chi(\gamma) = \chi(d)$ for all for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is **holomorphic form** (or **modular form**) on $\Gamma\backslash\mathbb{H}$ of **weight** k, **level** N, and **character** χ, if the following properties are satisfied:

(i) f is holomorphic on $\mathbb{H}$.

(ii) $(f|_{j,k}\gamma)(z) = \chi(\gamma)f(z)$ for all $\gamma \in \Gamma$.

(iii) $(f|_{j,k}\alpha)(z) = O(1)$ for all $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$ (or equivalently $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$).

We say f is a **(holomorphic) cusp form** if the additional property is satisfied:

(iv) For all cusps $\mathfrak{a}$ and any $y > 0$, we have

$$\int_{iy}^{1+iy} (f|\sigma_{\mathfrak{a}})(z)\,dz = 0.$$

Property (ii) is called the **modularity condition** and we say f is **modular**. In particular, f is a function on $\mathcal{F}_\Gamma$. The modularity condition can equivalently be expressed as

$$f(\gamma z) = \chi(\gamma)j(\gamma, z)^k f(z).$$

Property (iii) is called the **growth condition** for holomorphic forms and we say f is **holomorphic at the cusps**. Clearly we only need to verify the growth condition on a set of scaling matrices for the cusps. To see the equivalence in the growth condition, every $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ is of the form $\alpha = \gamma\eta$ for some $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ and $\eta \in \mathrm{GL}_2^+(\mathbb{Q})$ of the form $\eta = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. To see this, if $c = 0$ the claim is obvious. For $c \neq 0$, let $r \geq 1$ be such that $r\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Z})$ and set $a' = \frac{a}{(a,c)}$ and $c' = \frac{c'}{(a,c)}$ so that $a', c' \in \mathbb{Z}$ with $(a', c') = 1$. Then there exists $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ with $\gamma^{-1} = \left(\begin{smallmatrix} * & * \\ -c' & a' \end{smallmatrix}\right)$. Moreover, $\gamma^{-1}r\alpha = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Z})$. Upon setting $\eta = \gamma^{-1}\alpha$, the claim is complete. From the decomposition $\alpha = \gamma\eta$, the cocycle condition gives

$$j(\alpha, z) = j(\gamma, \eta z),$$

and it follows that $(f|_{j,k}\alpha)(z) = O(1)$ for all $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ which proves the forward implication. The reverse implication is trivial since $\mathrm{PSL}_2(\mathbb{Z}) \subset \mathrm{GL}_2^+(\mathbb{Q})$. Holomorphic forms also admit Fourier series. Indeed, modularity implies

$$f(z+1) = f\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z\right) = f(z),$$

so that f is 1-periodic. Let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the $\mathfrak{a}$ cusp. As Lemma 2.1.1 implies $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$ is a congruence subgroup, it follows by the cocycle condition that $f|\sigma_{\mathfrak{a}}$ is a holomorphic form on $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}\backslash\mathbb{H}$ of the same weight and character as f. In particular, $f|\sigma_{\mathfrak{a}}$ is 1-periodic. Note that this means we only need to verify the growth condition as $y \to \infty$. Moreover, as $f|\sigma_{\mathfrak{a}}$ is holomorphic it has a Fourier series

$$(f|\sigma_{\mathfrak{a}})(z) = \sum_{n \geq 0} a_{\mathfrak{a}}(n, y)e^{2\pi inx},$$

where the sum is only over $n \geq 0$ because holomorphy at the cusps implies that $f|\sigma_{\mathfrak{a}}$ is bounded. We can simplify the Fourier coefficients $a_{\mathfrak{a}}(n, y)$. To see this, since $f|\sigma_{\mathfrak{a}}$ is holomorphic it satisfies the first order Cauchy-Riemann equations so that

$$\frac{1}{2}\left(\frac{\partial f|\sigma_{\mathfrak{a}}}{\partial x} + i\frac{\partial f|\sigma_{\mathfrak{a}}}{\partial y}\right) = 0.$$

Substituting in the Fourier series and equating coefficients we obtain the ODE

$$2\pi n a_{\mathfrak{a}}(n, y) + a_{\mathfrak{a},y}(n, y) = 0,$$

Solving this ODE by separation of variables, we see that there exists an $a_{\mathfrak{a}}(n)$ such that

$$a_{\mathfrak{a}}(n,y) = a_{\mathfrak{a}}(n)e^{-2\pi ny}.$$

The coefficients $a_{\mathfrak{a}}(n)$ are the only part of the Fourier series depending on the implicit congruence subgroup Γ. Using these coefficients instead, f admits a **Fourier series at the $\mathfrak{a}$ cusp**:

$$(f|\sigma_{\mathfrak{a}})(z) = \sum_{n\geq 0} a_{\mathfrak{a}}(n)e^{2\pi inz}.$$

If $\mathfrak{a} = \infty$, we will drop this dependence and in this case $f|\sigma_{\mathfrak{a}} = f$. Moreover, property (iv) implies that f is a cusp form if and only if $a_{\mathfrak{a}}(n) = 0$ for every cusp $\mathfrak{a}$. We can also easily derive a bound for the size of the Fourier coefficients of cusp forms. To see this, note that $\left|(f|\sigma_{\mathfrak{a}})(z)\mathrm{Im}(z)^{\frac{k}{2}}\right|$ is $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$-invariant by the modularity of $f|\sigma_{\mathfrak{a}}$, the cocycle condition, the identity $\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}z)^{\frac{k}{2}} = \frac{\mathrm{Im}(z)^{\frac{k}{2}}}{|j(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}},z)|^k}$, and that $|\chi(\gamma)| = 1$. Moreover, this function is bounded on $\mathcal{F}_{\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}}$ because f is a cusp form. Then $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$-invariance implies $\left|(f|\sigma_{\mathfrak{a}})(z)\mathrm{Im}(z)^{\frac{k}{2}}\right|$ is bounded on $\mathbb{H}$. From the definition of Fourier series, it follows that

$$a_{\mathfrak{a}}(n)\mathrm{Im}(z)^{\frac{k}{2}} = \int_{iy}^{1+iy} (f|\sigma_{\mathfrak{a}})(z)\mathrm{Im}(z)^{\frac{k}{2}}e^{-2\pi inz}\,dz \ll \int_0^1 e^{2\pi ny}\,dx \ll e^{2\pi ny}.$$

Upon setting $y = \frac{1}{n}$, the last expression is absolutely bounded and we obtain

$$a_{\mathfrak{a}}(n) \ll n^{\frac{k}{2}}.$$

This bound is known as the **Hecke bound** for holomorphic forms. It follows from the Hecke bound and the Taylor series of $\frac{1}{1-e^y}$ along with its derivatives, that

$$(f|\sigma_{\mathfrak{a}})(z) = O\left(\sum_{n\geq 1} n^{\frac{k}{2}}e^{-2\pi ny}\right) = O\left(\sum_{n\geq 1} n^{k}e^{-2\pi ny}\right) = O\left(\frac{e^{-2\pi y}}{(1-e^{-2\pi y})^2}\right) = O(e^{-2\pi y}).$$

This implies $(f|\sigma_{\mathfrak{a}})(z)$ exhibits rapid decay. Accordingly, we say that f exhibits **rapid decay at the cusps**. Observe that $(f|\sigma_{\mathfrak{a}})$ is then bounded on $\mathbb{H}$ and, in particular, f is bounded on $\mathbb{H}$.

3.2 Poincaré & Eisenstein Series

Let Γ be a congruence subgroup of level N. We will introduce two important classes of holomorphic forms on $\Gamma\backslash\mathbb{H}$ namely the Poincaré and Eisenstein series. Let $m \geq 0$, $k \geq 4$, χ be a Dirichlet character with conductor $q \mid N$, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. We define the m-th **(holomorphic) Poincaré series** $P_{m,k,\chi,\mathfrak{a}}(z)$ of weight k with character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp by

$$P_{m,k,\chi,\mathfrak{a}}(z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-k}e^{2\pi im\sigma_{\mathfrak{a}}^{-1}\gamma z}.$$

We call m the **index** of $P_{m,k,\chi,\mathfrak{a}}(z)$. If χ is the trivial character or $\mathfrak{a} = \infty$, we will drop these dependencies accordingly.

Remark 3.2.1. *The reason why we restrict to $k \geq 4$ is because for $k = 0, 2$ the Poincaré series need not converge (see Proposition B.9.1).*

We first verify that $P_{m,k,\chi,\mathfrak{a}}(z)$ is well-defined. It suffices to show that the summands are independent of the representatives γ and $\sigma_{\mathfrak{a}}$. To see that $\overline{\chi}(\gamma)$ is independent of γ, recall that $\Gamma_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}$ and let $\gamma' = \sigma_{\mathfrak{a}}\eta_\infty\sigma_{\mathfrak{a}}^{-1}\gamma$ with $\eta_\infty = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_\infty$. Then

$$\overline{\chi}(\gamma') = \overline{\chi}(\sigma_{\mathfrak{a}}\eta_\infty\sigma_{\mathfrak{a}}^{-1}\gamma) = \overline{\chi}(\sigma_{\mathfrak{a}})\chi(\eta_\infty)\overline{\chi}(\sigma_{\mathfrak{a}})^{-1}\overline{\chi}(\gamma) = \overline{\chi}(\gamma),$$

verifying that $\overline{\chi}(\gamma)$ is independent of the representative γ. As the set of representatives for the scaling matrix $\sigma_{\mathfrak{a}}$ is $\sigma_{\mathfrak{a}}\Gamma_\infty$ and the set of representatives for γ is $\Gamma_{\mathfrak{a}}\gamma$, the set of representatives for $\sigma_{\mathfrak{a}}^{-1}\gamma$ is $\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\gamma$. But as $\Gamma_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}$, this set of representatives is $\Gamma_\infty\sigma_{\mathfrak{a}}^{-1}\gamma$ and therefore it remains to verify independence from multiplication on the left by an element of Γ_∞ namely η_∞. The cocycle relation gives

$$j(\eta_\infty\sigma_{\mathfrak{a}}^{-1}\gamma, z) = j(\eta_\infty, \sigma_{\mathfrak{a}}^{-1}\gamma z)j(\sigma_{\mathfrak{a}}^{-1}\gamma, z) = j(\sigma_{\mathfrak{a}}^{-1}\gamma, z),$$

where the last equality follows because $j(\eta_\infty, \sigma_{\mathfrak{a}}^{-1}\gamma z) = 1$. This verifies that $j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)$ is independent of the representatives γ and $\sigma_{\mathfrak{a}}$. Moreover, we have

$$e^{2\pi i m\eta_\infty\sigma_{\mathfrak{a}}^{-1}\gamma z} = e^{2\pi i m(\sigma_{\mathfrak{a}}^{-1}\gamma z+n)} = e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma}e^{2\pi i mnz} = e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma},$$

which verifies that $e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}$ is independent of the representatives γ and $\sigma_{\mathfrak{a}}$. Therefore $P_{m,k,\chi,\mathfrak{a}}(z)$ is well-defined. To see that $P_{m,k,\chi,\mathfrak{a}}(z)$ is holomorphic on $\mathbb{H}$, first note that $|e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}| = e^{-2\pi m\mathrm{Im}(\mathfrak{a}^{-1}\gamma z)} < 1$. Then the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma$ gives

$$P_{m,k,\chi,\mathfrak{a}}(z) \ll \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{|cz+d|^k}.$$

As $k \geq 4$, this latter series is locally absolutely uniformly convergent for $z \in \mathbb{H}$ by Proposition B.9.1. Hence $P_{m,k,\chi,\mathfrak{a}}(z)$ does too and so it is holomorphic on $\mathbb{H}$. We now verify modularity for $P_{m,k,\chi,\mathfrak{a}}(z)$. This is just a computation:

$$\begin{aligned}
P_{m,k,\chi,\mathfrak{a}}(\gamma z) &= \sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma')j(\sigma_{\mathfrak{a}}^{-1}\gamma', \gamma z)^{-k}j(\gamma, z)^{-k}e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma')\left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma, z)}{j(\gamma, z)}\right)^{-k} j(\gamma, z)^{-k}e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= j(\gamma, z)^k \sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma')j(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma, z)^{-k}e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \chi(\gamma)j(\gamma, z)^k \sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma'\gamma)j(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma, z)^{-k}e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \chi(\gamma)j(\gamma, z)^k \sum_{\gamma'\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma')j(\sigma_{\mathfrak{a}}^{-1}\gamma', z)^{-k}e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma' z} \\
&= \chi(\gamma)P_{m,k,\chi,\mathfrak{a}}(z),
\end{aligned}$$

where in the second line we have used the cocycle condition and in the second to last line we have used that $\gamma' \to \gamma'\gamma^{-1}$ is a bijection on Γ. To verify the growth condition, we will need a technical lemma:

Lemma 3.2.1. *Let $a, b > 0$ be reals and consider the half-strip*

$$S_{a,b} = \{z \in \mathbb{H} : |x| \leq a \text{ and } y \geq b\}.$$

Then there is a $\delta \in (0,1)$ such that

$$|nz+m| \geq \delta|ni+m|,$$

for all $n, m \in \mathbb{Z}$ and all $z \in S_{a,b}$.

Proof. If $n=0$ then any δ is sufficient and this δ is independent of z. If $n \neq 0$, then the desired inequality is equivalent to

$$\left|\frac{z+\frac{m}{n}}{i+\frac{n}{m}}\right| \geq \delta.$$

So consider the function

$$f(z,r) = \left|\frac{z+r}{i+r}\right|,$$

for $z \in S_{a,b}$ and $r \in \mathbb{R}$. It suffices to show $f(z,r) \geq \delta$. As $z \in \mathbb{H}$, $z-r \neq 0$ so that $f(z,r)$ is continuous and positive on $S_{a,b} \times \mathbb{R}$. Now let $Y > b$ and consider the region

$$S_{a,b}^{Y} = \{z \in \mathbb{H} : |x| \leq a \text{ and } b \leq y \leq Y\}.$$

We claim that there exists a Y such that if $y > Y$ and $|x| > Y$ then $f(z,r)^2 > \frac{1}{4}$. Indeed, we compute

$$f(z,r)^2 = \frac{(z+r)(\overline{z}+r)}{(i+r)(-i+r)} = \frac{|z|^2+2xr+r^2}{1+r^2} \geq \frac{y+r^2}{1+r^2},$$

where in the inequality we have used the bound $|z|^2 \geq y$ and that x is bounded. Now $\frac{r^2}{1+r^2} \to 1$ as $r \to \pm\infty$ so there exists a Y such that $|r| > Y$ implies $\frac{r^2}{1+r^2} \geq \frac{1}{4}$. Then

$$\frac{y+r^2}{1+r^2} \geq \frac{y}{1+r^2} + \frac{r^2}{1+r^2} \geq \frac{y}{1+r^2} + \frac{1}{4} > \frac{1}{4}.$$

It follows that $f(z,r) > \frac{1}{2}$ outside of $S_{a,b}^{Y} \times [-Y, Y]$. But this latter region is compact and so $f(z,r)$ obtains a minimum δ' on it. Setting $\delta = \min\{\frac{1}{2}, \delta'\}$ completes the proof. $\square$

We can now verify the growth condition for $P_{m,k,\chi,\mathfrak{a}}(z)$. Let $\sigma_{\mathfrak{b}}$ be a scaling matrix for the cusp $\mathfrak{b}$. Then the bound $|e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}} z}| = e^{-2\pi m \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}} z)} < 1$, cocycle condition, and Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ together give

$$j(\sigma_{\mathfrak{b}}, z)^{-k} P_{m,k,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}} z) \ll \sum_{(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \frac{1}{|cz+d|^k}.$$

Now decompose this last sum as

$$\sum_{(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \frac{1}{|cz+d|^k} = \sum_{d \neq 0} \frac{1}{d^k} + \sum_{c \neq 0} \sum_{d \in \mathbb{Z}} \frac{1}{|cz+d|^k} = 2\sum_{d \geq 1} \frac{1}{d^k} + 2\sum_{c \geq 1} \sum_{d \in \mathbb{Z}} \frac{1}{|cz+d|^k}.$$

Since the first sum is absolutely uniformly bounded, it suffices to show that the double sum is too. To see this, let $y \geq 1$ and δ be as in Lemma 3.2.1. Then for any integer $N \geq 1$ we can write

$$\begin{aligned}
\sum_{c \geq 1} \sum_{d \in \mathbb{Z}} \frac{1}{|cz+d|^k} &= \sum_{c+|d| \leq N} \frac{1}{|cz+d|^k} + \sum_{c+|d| > N} \frac{1}{|cz+d|^k} \\
&\leq \sum_{c+|d| \leq N} \frac{1}{|cz+d|^k} + \sum_{c+|d| > N} \frac{1}{(\delta|ci+d|)^k} \\
&\leq \sum_{c+|d| \leq N} \frac{1}{|cz+d|^k} + \frac{1}{\delta^k} \sum_{c+|d| > N} \frac{1}{|ci+d|^k}.
\end{aligned}$$

As $\sum_{c \geq 1} \sum_{d \in \mathbb{Z}} \frac{1}{|ci+d|^k}$ converges by Proposition B.9.1, the second sum tends to zero as $N \to \infty$. As for the first sum, it is finite and each term is bounded. Thus the double sum is absolutely uniformly bounded. This verifies the growth condition. We collect this work as a theorem:

Theorem 3.2.1. *Let $m \geq 0$, $k \geq 4$, χ be a Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. The Poincaré series*

$$P_{m,k,\chi,\mathfrak{a}}(z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k} e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z},$$

is a weight k holomorphic form with character χ on $\Gamma\backslash\mathbb{H}$.

For $m = 0$, we write $E_{k,\chi,\mathfrak{a}}(z) = P_{0,k,\chi,\mathfrak{a}}(z)$ and call $E_{k,\chi,\mathfrak{a}}(z)$ the **(holomorphic) Eisenstein series** of weight k and character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp. It is defined by

$$E_{k,\chi,\mathfrak{a}}(z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma) j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}.$$

If χ is the trivial character or $\mathfrak{a} = \infty$, we will drop these dependencies accordingly. In particular, we have already verified the following theorem:

Theorem 3.2.2. *Let $k \geq 4$, χ be Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. The Eisenstein series*

$$E_{k,\chi,\mathfrak{a}}(z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma) j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k},$$

is a weight k holomorphic form with character χ on $\Gamma\backslash\mathbb{H}$.

We will now compute the Fourier series of the Poincaré series with positive index:

Proposition 3.2.1. *Let $m \geq 1$, $k \geq 4$, χ be Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$. The Fourier series of $P_{m,k,\chi,\mathfrak{a}}(z)$ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{b}$ cusp is given by*

$$(P_{m,k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t\geq 1}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t} + \left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1}\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{2\pi i^{-k}}{c}J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right)S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c)\right)e^{2\pi i t z}.$$

Proof. From the cocycle condition, the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$, and Remark 2.2.3, we have

$$(P_{m,k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \delta_{\mathfrak{a},\mathfrak{b}}e^{2\pi i m z} + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}, d\in\mathbb{Z} \\ d \pmod c\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(d)\frac{e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+cd}\right)}}{(cz+d)^k},$$

where a has been determined modulo c by $ad - bc = 1$ and we have used the fact that

$$\frac{a}{c} - \frac{1}{c^2z+cd} = \frac{az+b}{cz+d}.$$

Summing over all pairs (c, d) with $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $d \in \mathbb{Z}$, and $d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is the same as summing over all triples (c, ℓ, r) with $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $\ell \in \mathbb{Z}$, and r taken modulo c with $r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$. Indeed, this is seen by writing $d = c\ell + r$. Moreover, since $ad - bc = 1$ we have $a(c\ell + r) - bc = 1$ which further implies that $ar \equiv 1 \pmod{c}$. So we may take a to be the inverse for r modulo c. Then

$$\begin{aligned}
\sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}},d\in\mathbb{Z}\\ d \pmod{c}\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(d)\frac{e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+cd}\right)}}{(cz+d)^k} &= \sum_{(c,\ell,r)} \overline{\chi}(c\ell+r)\frac{e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k} \\
&= \sum_{(c,\ell,r)} \overline{\chi}(r)\frac{e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k} \\
&= \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \sum_{\ell\in\mathbb{Z}} \overline{\chi}(r)\frac{e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k} \\
&= \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \overline{\chi}(r) \sum_{\ell\in\mathbb{Z}} \frac{e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k},
\end{aligned}$$

where on the right-hand side it is understood that we are summing over all triples (c,ℓ,r) with the prescribed properties and the second line holds since χ has conductor diving the level and $d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is determined modulo c. Now let

$$I_{c,r}(z) = \sum_{\ell\in\mathbb{Z}} \frac{e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+c^2\ell+cr}\right)}}{(cz+c\ell+r)^k}.$$

We will apply the Poisson summation formula to $I_{c,r}(z)$. This is possible since the summands are absolutely integrable by Proposition 1.5.2 because they exhibit polynomial decay of order k and $I_{c,r}(z)$ is holomorphic because $(P_{m,k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z)$ is. By the identity theorem it suffices to apply the Poisson summation formula for $z = iy$ with $y > 0$. Accordingly, let $f(x)$ be given by

$$f(x) = \frac{e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2x+cr+ic^2y}\right)}}{(cx+r+icy)^k}.$$

As we have just noted, $f(x)$ is absolutely integrable on $\mathbb{R}$. We compute the Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i t x}\,dx = \int_{-\infty}^{\infty} \frac{e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2x+cr+ic^2y}\right)}}{(cx+r+icy)^k}e^{-2\pi i t x}\,dx.$$

Complexify the integral to get

$$\int_{\mathrm{Im}(z)=0} \frac{e^{2\pi i m\left(\frac{a}{c}-\frac{1}{c^2z+cr+ic^2y}\right)}}{(cz+r+icy)^k}e^{-2\pi i t z}\,dz.$$

Now make the change of variables $z \to z - \frac{r}{c} - icy$ to obtain

$$e^{2\pi i m\frac{a}{c}+2\pi i t\frac{r}{c}-2\pi t y}\int_{\mathrm{Im}(z)=y} \frac{e^{-\frac{2\pi i m}{c^2z}}}{(cz)^k}e^{-2\pi i t z}\,dz.$$

The integrand is meromorphic with a pole at $z = 0$. Moreover, we have

$$\frac{1}{(cz)^k} \ll \frac{1}{|cz|^k}, \quad e^{-\frac{2\pi i m}{c^2z}} \ll e^{-\frac{2\pi m\mathrm{Im}(z)}{|cz|^2}}, \quad \text{and} \quad e^{-2\pi i t z} \ll e^{2\pi t y}.$$

The first expression has polynomial decay, the second expression is bounded, and the third expression exhibits rapid decay if and only if $t < 0$ and when $t = 0$ it is bounded. So when $t \le 0$ we may take the limit as $\mathrm{Im}(z) \to \infty$ by shifting the line of integration and conclude that the integral vanishes. It remains to compute the integral for $t \ge 1$. To do this, make the change of variables $z \to -\frac{z}{2\pi it}$ to the last integral to rewrite it as

$$\begin{aligned}
-\frac{1}{2\pi it}\int_{(2\pi ty)} \frac{e^{-\frac{4\pi^2 mt}{c^2 z}}}{\left(-\frac{cz}{2\pi it}\right)^k} e^z\,dz &= -\frac{1}{2\pi it}\int_{(2\pi ty)} \left(-\frac{2\pi it}{cz}\right)^k e^{z-\frac{4\pi^2 mt}{c^2 z}}\,dz \\
&= \frac{(-2\pi it)^{k-1}}{c^k}\int_{(2\pi ty)} z^{-k} e^{z-\frac{4\pi^2 mt}{c^2 z}}\,dz \\
&= \frac{(-2\pi it)^{k-1}}{c^k}\int_{-\infty}^{(0^+)} z^{-k} e^{z-\frac{4\pi^2 mt}{c^2 z}}\,dz \\
&= \frac{2\pi i^{-k}}{c}\left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right),
\end{aligned}$$

where in the second to last line we have homotoped the line of integration to a Hankel contour about the negative real axis and in the last line we have used the Schläflin integral representation for the J-Bessel function (see Appendix B.7). So in total we obtain

$$\hat{f}(t) = \begin{cases} \left(\frac{2\pi i^{-k}}{c}\left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right) e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right) e^{-2\pi ty} & \text{if } t \ge 1, \\ 0 & \text{if } t \le 0. \end{cases}$$

By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z) = \sum_{t\ge1}\left(\frac{2\pi i^{-k}}{c}\left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right) e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right) e^{2\pi itz},$$

for all $z \in \mathbb{H}$. Plugging this back into the Poincaré series gives a form of the Fourier series:

$$\begin{aligned}
(P_{m,k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) &= \delta_{\mathfrak{a},\mathfrak{b}}e^{2\pi imz} + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r)\sum_{t\ge1}\left(\frac{2\pi i^{-k}}{c}\left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1} J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right) e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right) e^{2\pi imz} \\
&= \sum_{t\ge1}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t} + \left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1}\sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r)\frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right) e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right) e^{2\pi itz} \\
&= \sum_{t\ge1}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t} + \left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1}\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right)\sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(r)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}}\right) e^{2\pi itz}.
\end{aligned}$$

We will simplify the innermost sum. Since a is the inverse for r modulo c, the innermost sum above becomes

$$\sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(r)e^{2\pi im\frac{a}{c}+2\pi it\frac{r}{c}} = \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(\overline{a})e^{2\pi im\frac{a}{c}+2\pi it\frac{\overline{a}}{c}} = \sum_{r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \chi(a)e^{\frac{2\pi i(am+\overline{a}t)}{c}} = S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c).$$

So at last, we obtain our desired Fourier series:

$$(P_{m,k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t\geq 1}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{m,t} + \left(\frac{\sqrt{t}}{\sqrt{m}}\right)^{k-1}\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{2\pi i^{-k}}{c}J_{k-1}\left(\frac{4\pi\sqrt{mt}}{c}\right)S_{\chi,\mathfrak{a},\mathfrak{b}}(m,t,c)\right)e^{2\pi itz}. \qquad \square$$

An immediate consequence of Proposition 3.2.1 is that the Poincaré series $P_{m,k,\chi,\mathfrak{a}}(z)$ with positive index are cusp forms. In a similar manner, we can obtain the Fourier series of the Eisenstein series too:

Proposition 3.2.2. *Let $k\geq 4$, χ be Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$. The Fourier series of $E_{k,\chi,\mathfrak{a}}(z)$ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{b}$ cusp is given by*

$$(E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t\geq 0}\left(\delta_{\mathfrak{a},\mathfrak{b}} + \sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{(-2\pi it)^k}{(k-1)!c^k}S_{\chi,\mathfrak{a},\mathfrak{b}}(0,t,c)\right)e^{2\pi itz}.$$

Proof. From the cocycle condition, the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$, and Remark 2.2.3, we have

$$(E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \delta_{\mathfrak{a},\mathfrak{b}} + \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}},d\in\mathbb{Z}\\ d \pmod c\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\overline{\chi}(d)\frac{1}{(cz+d)^k},$$

where a has been determined modulo c by $ad-bc=1$ and we have used the fact that

$$\frac{a}{c}-\frac{1}{c^2z+cd} = \frac{az+b}{cz+d}.$$

Summing over all pairs (c,d) with $c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $d\in\mathbb{Z}$, and $d\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is the same as summing over all triples (c,ℓ,r) with $c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}$, $\ell\in\mathbb{Z}$, and r taken modulo c with $r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$. Indeed, this is seen by writing $d=c\ell+r$. Moreover, since $ad-bc=1$ we have $a(c\ell+r)-bc=1$ which further implies that $ar\equiv 1 \pmod c$. So we may take a to be the inverse for r modulo c. Then

$$\begin{aligned}
\sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}},d\in\mathbb{Z}\\ d \pmod c\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\overline{\chi}(d)\frac{1}{(cz+d)^k} &= \sum_{(c,\ell,r)}\overline{\chi}(c\ell+r)\frac{1}{(cz+c\ell+r)^k}\\
&= \sum_{(c,\ell,r)}\overline{\chi}(r)\frac{1}{(cz+c\ell+r)^k}\\
&= \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\sum_{\ell\in\mathbb{Z}}\overline{\chi}(r)\frac{1}{(cz+c\ell+r)^k}\\
&= \sum_{\substack{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}\\ r\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}}\overline{\chi}(r)\sum_{\ell\in\mathbb{Z}}\frac{1}{(cz+c\ell+r)^k},
\end{aligned}$$

where on the right-hand side it is understood that we are summing over all triples (c,ℓ,r) with the prescribed properties and the second line holds since χ has conductor diving the level and $d\in\mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ is determined modulo c. Now let

$$I_{c,r}(z) = \sum_{\ell\in\mathbb{Z}}\frac{1}{(cz+c\ell+r)^k}.$$

We apply the Poisson summation formula to $I_{c,r}(z)$. This is allowed since the summands are absolutely integrable by Proposition 1.5.2, as they exhibit polynomial decay of order k, and $I_{c,r}(z)$ is holomorphic because $(E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z)$ is. By the identity theorem it suffices to apply the Poisson summation formula for $z = iy$ with $y > 0$. So let $f(x)$ be given by

$$f(x) = \frac{1}{(cx + r + icy)^k}.$$

As we have just noted, $f(x)$ is absolutely integrable on $\mathbb{R}$. We compute the Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx = \int_{-\infty}^{\infty} \frac{e^{-2\pi itx}}{(cx + r + icy)^k}\,dx.$$

Complexify the integral to get

$$\int_{\mathrm{Im}(z)=0} \frac{e^{-2\pi itz}}{(cz + r + icy)^k}\,dz.$$

Now make the change of variables $z \to z - \frac{r}{c} - icy$ to obtain

$$e^{2\pi it\frac{r}{c} - 2\pi ty} \int_{\mathrm{Im}(z)=y} \frac{e^{-2\pi itz}}{(cz)^k}\,dz.$$

The integrand is meromorphic with a pole at $z = 0$. Moreover, we have

$$\frac{1}{(cz)^k} \ll \frac{1}{|cz|^k} \quad \text{and} \quad e^{-2\pi itz} \ll e^{2\pi ty}.$$

The first expression has polynomial decay while the expression exhibits rapid decay if and only if $t < 0$ and when $t = 0$ it is bounded. So when $t \le 0$ we may take the limit as $\mathrm{Im}(z) \to \infty$ by shifting the line of integration and conclude that the integral vanishes. It remains to compute the integral for $t \ge 1$. To do this, make the change of variables $z \to -\frac{z}{2\pi it}$ to the last integral to rewrite it as

$$-\frac{1}{2\pi it}\int_{(2\pi ty)} \frac{e^z}{\left(-\frac{cz}{2\pi it}\right)^k}\,dz = -\frac{1}{2\pi it}\int_{(2\pi ty)} \left(-\frac{2\pi it}{cz}\right)^k e^z\,dz = \frac{(-2\pi it)^{k-1}}{c^k}\int_{(2\pi ty)} \frac{e^z}{z^k}\,dz.$$

The integrand of the last integral has a pole of order k at $z = 0$. To find the residue, the Laurent series of $\frac{e^z}{z^k}$ is

$$\frac{e^z}{z^k} = \sum_{n\ge 0} \frac{z^{n-k}}{n!},$$

and thus the residue of the integrand is $\frac{1}{(k-1)!}$. In shifting the line of integration to $(-x)$, for some $x > 0$, we pass by this pole and obtain

$$\frac{(-2\pi it)^k}{(k-1)!c^k} + \int_{(-x)} \frac{e^z}{z^k}\,dz.$$

Moreover, we have

$$\frac{1}{z^k} \ll \frac{1}{|z|^k} \quad \text{and} \quad e^z \ll e^x.$$

The first expression has polynomial decay while the second expression exhibits rapid decay provided $x < 0$. Therefore we make take the limit as $x \to \infty$ by shifting the line of integration again and conclude that the latter integral vanishes. Altogether, we have show that

$$\hat{f}(t) = \begin{cases} \left(\frac{(-2\pi it)^k}{(k-1)!c^k}e^{2\pi it\frac{r}{c}}\right)e^{-2\pi ty} & \text{if } t \ge 1, \\ 0 & \text{if } t \le 0. \end{cases}$$

By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z) = \sum_{t \geq 1} \left(\frac{(-2\pi it)^k}{(k-1)!c^k} e^{2\pi it\frac{r}{c}} \right) e^{2\pi itz},$$

for all $z \in \mathbb{H}$. Substituting this back into the Eisenstein series gives a form of the Fourier series:

$$\begin{aligned}
(E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) &= \delta_{\mathfrak{a},\mathfrak{b}} + \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r) \sum_{t \geq 1} \left(\frac{(-2\pi it)^k}{(k-1)!c^k} e^{2\pi it\frac{r}{c}} \right) e^{2\pi imz} \\
&= \sum_{t \geq 0} \left(\delta_{\mathfrak{a},\mathfrak{b}} + \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}} \\ r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}}} \overline{\chi}(r) \frac{(-2\pi it)^k}{(k-1)!c^k} e^{2\pi it\frac{r}{c}} \right) e^{2\pi itz} \\
&= \sum_{t \geq 0} \left(\delta_{\mathfrak{a},\mathfrak{b}} + \sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{(-2\pi it)^k}{(k-1)!c^k} \sum_{r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(r) e^{2\pi it\frac{r}{c}} \right) e^{2\pi itz}.
\end{aligned}$$

We will simplify the innermost sum. Since a is the inverse for r modulo c, the innermost sum above becomes

$$\sum_{r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(r) e^{2\pi it\frac{r}{c}} = \sum_{r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \overline{\chi}(\overline{a}) e^{2\pi it\frac{\overline{a}}{c}} = \sum_{r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}} \chi(a) e^{\frac{2\pi i\overline{a}t}{c}} = S_{\chi,\mathfrak{a},\mathfrak{b}}(0,t,c).$$

So at last, we obtain our desired Fourier series:

$$(E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z) = \sum_{t \geq 0} \left(\delta_{\mathfrak{a},\mathfrak{b}} + \sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{(-2\pi it)^k}{(k-1)!c^k} S_{\chi,\mathfrak{a},\mathfrak{b}}(0,t,c) \right) e^{2\pi itz}.$$

□

An interesting observation from Proposition 3.2.2 is that $E_{k,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}}$ is necessarily a cusp form unless $\mathfrak{a} = \mathfrak{b}$.

3.3 Inner Product Spaces of Holomorphic Forms

Let Γ be a congruence subgroup of level N. Let $\mathcal{M}_k(\Gamma,\chi)$ denote the space of all weight k holomorphic forms with character χ on $\Gamma\backslash\mathbb{H}$. Let $\mathcal{S}_k(\Gamma,\chi)$ denote the associated subspace of cusp forms. Moreover, if the character χ is the trivial character, we will suppress the dependence upon χ. If Γ_1 and Γ_2 are two congruence subgroups such that $\Gamma_1 \leq \Gamma_2$, then we have the inclusion

$$\mathcal{M}_k(\Gamma_2,\chi) \subseteq \mathcal{M}_k(\Gamma_1,\chi).$$

So in general, the smaller the congruence subgroup the more holomorphic forms there are. We will need a dimensionality result regarding the space of holomorphic forms of a fixed weight. However, it will suffice to only require the result for forms with trivial character. The result is that $\mathcal{M}_k(\Gamma,\chi)$ is never too large (see [DS05] for a proof):

Theorem 3.3.1. $\mathcal{M}_k(\Gamma,\chi)$ *is finite dimensional.*

Since $\mathcal{S}_k(\Gamma,\chi)$ is a subspace of $\mathcal{M}_k(\Gamma,\chi)$, Theorem 3.3.1 implies that $\mathcal{S}_k(\Gamma,\chi)$ is also finite dimensional. It turns out that $\mathcal{S}_k(\Gamma,\chi)$ is naturally an inner product space. For $f,g\in\mathcal{S}_k(\Gamma,\chi)$, define their **Petersson inner product** by

$$\langle f,g\rangle_\Gamma=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}f(z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu.$$

If the congruence subgroup is clear from context we will suppress the dependence upon Γ. Since f and g have rapid decay at the cusps, the integral is locally absolutely uniformly convergent by Proposition 1.5.1. The integrand is also Γ-invariant so that the integral independent of the choice of fundamental domain. These two facts together imply that the Petersson inner product is well-defined. We will continue to use this notation even if f and g do not belong to $\mathcal{S}_k(\Gamma,\chi)$ provided the integral is locally absolutely uniformly convergent. A basic property of the Petersson inner product is that it is invariant with respect to the slash operator:

Proposition 3.3.1. *For any $f,g\in\mathcal{S}_k(\Gamma,\chi)$ and $\alpha\in\mathrm{PSL}_2(\mathbb{Z})$, we have*

$$\langle f|\alpha,g|\alpha\rangle_{\alpha^{-1}\Gamma\alpha}=\langle f,g\rangle_\Gamma.$$

Proof. This is just a computation:

$$\begin{aligned}
\langle f|\alpha,g|\alpha\rangle_{\alpha^{-1}\Gamma\alpha}&=\frac{1}{V_{\alpha^{-1}\Gamma\alpha}}\int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}}(f|\alpha)(z)\overline{(g|\alpha)(z)}\mathrm{Im}(z)^k\,d\mu\\
&=\frac{1}{V_\Gamma}\int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}}(f|\alpha)(z)\overline{(g|\alpha)(z)}\mathrm{Im}(z)^k\,d\mu && \text{Lemma 2.3.1}\\
&=\frac{1}{V_\Gamma}\int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}}|j(\alpha,z)|^{-2k}f(\alpha z)\overline{g(\alpha z)}\mathrm{Im}(z)^k\,d\mu\\
&=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}|j(\alpha,z)|^{-2k}f(z)\overline{g(z)}\mathrm{Im}(\alpha z)^k\,d\mu && z\to\alpha^{-1}z\\
&=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}f(z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu\\
&=\langle f,g\rangle_\Gamma.
\end{aligned}$$

□

More importantly, the Petersson inner product turns $\mathcal{S}_k(\Gamma,\chi)$ into a Hilbert space:

Proposition 3.3.2. *$\mathcal{S}_k(\Gamma,\chi)$ is a Hilbert space with respect to Petersson inner product.*

Proof. Let $f,g\in\mathcal{S}_k(\Gamma,\chi)$. Linearity of the integral immediately implies that the Petersson inner product is linear on $\mathcal{S}_k(\Gamma,\chi)$. It is also positive definite since

$$\langle f,f\rangle=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}f(z)\overline{f(z)}\mathrm{Im}(z)^k\,d\mu=\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}|f(z)|^2\mathrm{Im}(z)^k\,d\mu\geq 0,$$

with equality if and only if f is identically zero because $|f(z)|^2\mathrm{Im}(z)^k\geq 0$. To see that it is conjugate symmetric, observe

$$\begin{aligned}
\overline{\langle g, f\rangle} &= \overline{\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} g(z)\overline{f(z)}\mathrm{Im}(z)^k\,d\mu} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \overline{g(z)}f(z)\mathrm{Im}(z)^k\,\overline{d\mu} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \overline{g(z)}f(z)\mathrm{Im}(z)^k\,d\mu \qquad\qquad d\mu = \frac{dx\,dy}{y^2} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}\mathrm{Im}(z)^k\,d\mu \\
&= \langle f, g\rangle.
\end{aligned}$$

So the Petersson inner product is a Hermitian inner product on $\mathcal{S}_k(\Gamma,\chi)$. Since $\mathcal{S}_k(\Gamma,\chi)$ is finite dimensional by Theorem 3.3.1, it follows immediately that $\mathcal{S}_k(\Gamma,\chi)$ is a Hilbert space. □

Remark 3.3.1. *As a consequence of Proposition 3.3.2, the Petersson inner product is also non-degenerate on $\mathcal{S}_k(\Gamma,\chi)$. Actually, for the exact same reasoning this holds on $\mathcal{S}_k(\Gamma,\chi)\times\mathcal{M}_k(\Gamma,\chi)$ wherever the Petersson inner product is defined.*

Now suppose $f\in\mathcal{S}_k(\Gamma,\chi)$ with Fourier coefficients $a_{n,\mathfrak{a}}(f)$ at the $\mathfrak{a}$ cusp. Define linear functionals $\phi_{m,k,\chi,\mathfrak{a}}:\mathcal{S}_k(\Gamma,\chi)\to\mathbb{C}$ by

$$\phi_{m,k,\chi,\mathfrak{a}}(f) = a_{m,\mathfrak{a}}(f).$$

Since $\mathcal{S}_k(\Gamma,\chi)$ is a finite dimensional Hilbert space, the Riesz representation theorem implies that there exists unique $v_{m,k,\chi,\mathfrak{a}}(z)\in\mathcal{S}_k(\Gamma,\chi)$ such that

$$\langle f, v_{m,k,\chi,\mathfrak{a}}\rangle = \phi_{m,k,\chi,\mathfrak{a}}(f) = a_{m,\mathfrak{a}}(f),$$

We would like to know what these cusp forms are. It turns out that $v_{m,k,\chi,\mathfrak{a}}(z)$ will be the Poincaré series $P_{m,k,\chi,\mathfrak{a}}(z)$ of positive index up to a normalization factor. To see this, we compute the following inner product:

$$\begin{aligned}
\langle f, P_{m,k,\chi,\mathfrak{a}}\rangle &= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{P_{m,k,\chi,\mathfrak{a}}(z)}\mathrm{Im}(z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\chi(\gamma)\overline{j(\sigma_\mathfrak{a}^{-1}\gamma,z)^{-k}}f(z)e^{-2\pi i m\overline{\sigma_\mathfrak{a}^{-1}\gamma z}}\mathrm{Im}(z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\chi(\gamma)j(\sigma_\mathfrak{a}^{-1}\gamma,z)^{k}f(z)e^{-2\pi i m\overline{\sigma_\mathfrak{a}^{-1}\gamma z}}\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\left(\frac{j(\sigma_\mathfrak{a}^{-1}\gamma,z)}{j(\gamma,z)}\right)^{k}f(\gamma z)e^{-2\pi i m\overline{\sigma_\mathfrak{a}^{-1}\gamma z}}\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)^k\,d\mu && \text{modularity} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}j(\sigma_\mathfrak{a},\sigma_\mathfrak{a}^{-1}\gamma z)^{-k}f(\gamma z)e^{-2\pi i m\overline{\sigma_\mathfrak{a}^{-1}\gamma z}}\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)^k\,d\mu && \text{cocycle condition} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}j(\sigma_\mathfrak{a},\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a} z)^{-k}f(\gamma\sigma_\mathfrak{a} z)e^{-2\pi i m\overline{\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a} z}}\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a} z)^k\,d\mu && z\to\sigma_\mathfrak{a}z \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\infty\backslash\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}j(\sigma_\mathfrak{a},\gamma z)^{-k}f(\sigma_\mathfrak{a}\gamma z)e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(\gamma z)^k\,d\mu && \gamma\to\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{a}^{-1} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\infty\backslash\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}(f|\sigma_\mathfrak{a})(\gamma z)e^{-2\pi i m\overline{\gamma z}}\mathrm{Im}(\gamma z)^k\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\Gamma_\infty\backslash\mathbb{H}}(f|\sigma_\mathfrak{a})(z)e^{-2\pi i m\overline{z}}\mathrm{Im}(z)^k\,d\mu && \text{unfolding.}
\end{aligned}$$

Substituting in the Fourier series of f at the $\mathfrak{a}$ cusp, we can finish the computation:

$$\begin{aligned}
\frac{1}{V_\Gamma}\int_{\Gamma_\infty\backslash\mathbb{H}}(f|\sigma_\mathfrak{a})(z)e^{-2\pi i m\overline{z}}\mathrm{Im}(z)^k\,d\mu &= \frac{1}{V_\Gamma}\int_0^\infty\int_0^1\sum_{n\geq1}a_{n,\mathfrak{a}}(f)e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^k\,\frac{dx\,dy}{y^2} \\
&= \frac{1}{V_\Gamma}\int_0^\infty\sum_{n\geq1}\int_0^1 a_{n,\mathfrak{a}}(f)e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^k\,\frac{dx\,dy}{y^2} && \text{DCT} \\
&= \frac{1}{V_\Gamma}\int_0^\infty a_{m,\mathfrak{a}}(f)e^{-4\pi m y}y^k\,\frac{dy}{y^2},
\end{aligned}$$

where the last line follows because

$$\int_0^1 e^{2\pi i(n-m)x}\,dx = \delta_{n-m,0}, \tag{3.1}$$

so that the inner integral cuts off all the terms except the diagonal $n=m$. Then

$$\begin{aligned}
\frac{1}{V_\Gamma}\int_0^\infty a_{m,\mathfrak{a}}(f)e^{-4\pi m y}y^k\,\frac{dy}{y^2} &= \frac{a_{m,\mathfrak{a}}(f)}{V_\Gamma}\int_0^\infty e^{-4\pi m y}y^{k-1}\,\frac{dy}{y} \\
&= \frac{a_{m,\mathfrak{a}}(f)}{V_\Gamma}\int_0^\infty e^{-4\pi m y}y^{k-1}\,\frac{dy}{y} && y\to\frac{y}{4\pi m} \\
&= \frac{a_{m,\mathfrak{a}}(f)}{V_\Gamma(4\pi m)^{k-1}}\int_0^\infty e^{-y}y^{k-1}\,\frac{dy}{y} \\
&= \frac{\Gamma(k-1)}{V_\Gamma(4\pi m)^{k-1}}a_{m,\mathfrak{a}}(f) && \text{definition of }\Gamma(k-1).
\end{aligned}$$

In conclusion,

$$\langle f, P_{m,k,\chi,\mathfrak{a}} \rangle = \frac{\Gamma(k-1)}{V_\Gamma(4\pi m)^{k-1}} a_{m,\mathfrak{a}}(f), \tag{3.2}$$

and it follows that

$$v_{m,k,\chi,\mathfrak{a}}(z) = \frac{V_\Gamma(4\pi m)^{k-1}}{\Gamma(k-1)} P_{m,k,\chi,\mathfrak{a}}(z).$$

We will work with the Poincaré series $P_{m,k,\chi,\mathfrak{a}}(z)$ instead of their normalized counterparts $v_{m,k,\chi,\mathfrak{a}}(z)$. In any case, with Equation (3.2) in hand we can prove the following result:

Theorem 3.3.2. *The Poincaré series of positive index span $\mathcal{S}_k(\Gamma, \chi)$.*

Proof. Let $f \in \mathcal{S}_k(\Gamma, \chi)$ with Fourier coefficients $a_{n,\mathfrak{a}}(f)$ at the $\mathfrak{a}$ cusp. Since $\Gamma(k-1) \neq 0$, Equation (3.2) implies $\langle f, P_{m,k,\chi,\mathfrak{a}} \rangle = 0$ if and only if $a_{m,\mathfrak{a}}(f) = 0$. So f is orthogonal to all the Poincaré series $P_{m,k,\chi,\mathfrak{a}}$ of positive index if and only if every Fourier coefficient $a_{m,\mathfrak{a}}(f)$ is zero. But this happens if and only if f is identically zero. $\square$

3.4 Double Coset Operators

We are ready to introduce a class of general operators, depending upon double cosets, on a congruence subgroup Γ of level N. We will use these operators to define the diamond and Hecke operators. For $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, consider the double coset

$$\Gamma_1\alpha\Gamma_2 = \{\gamma_1\alpha\gamma_2 : \gamma_1 \in \Gamma_1 \text{ and } \gamma_2 \in \Gamma_2\}.$$

Then Γ_1 acts on the set $\Gamma_1\alpha\Gamma_2$ by left multiplication so that it decomposes into a disjoin union of orbit spaces. Thus

$$\Gamma_1\alpha\Gamma_2 = \bigcup_{\beta \in \Gamma_1\backslash\Gamma_1\alpha\Gamma_2} \Gamma_1\beta,$$

where the sum is over the orbit representatives β. However, in order for these operators to be well-defined it is necessary that the orbit decomposition above is a finite union. This is indeed the case and we will require a lemma which gives a way to describe the orbit representatives for $\Gamma_1\alpha\Gamma_2$ in terms of coset representatives:

Lemma 3.4.1. *Let Γ_1 and Γ_2 be congruence subgroups and let let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Set $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$. Then left multiplication map*

$$\Gamma_2 \to \Gamma_1\alpha\Gamma_2 \qquad \gamma_2 \mapsto \alpha\gamma_2,$$

induces a bijection from the coset space $\Gamma_3\backslash\Gamma_2$ to the orbit space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$.

Proof. We will show that the induced map is both surjective and injective. For surjectivity, the orbit representatives β of $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ are of the form $\beta = \gamma_1\alpha\gamma_2$ for some $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. Since Γ_1 is acting on $\Gamma_1\alpha\Gamma_2$ by left multiplication, β can be written as $\beta = \alpha\gamma_2'$ for some $\gamma_2' \in \Gamma_2$. This shows that the induced map is a surjection. To prove injectivity, let $\gamma_2, \gamma_2' \in \Gamma_2$ be such that the orbit space representatives $\alpha\gamma_2$ and $\alpha\gamma_2'$ are equivalent. That is,

$$\Gamma_1\alpha\gamma_2 = \Gamma_1\alpha\gamma_2'.$$

This implies $\alpha\gamma_2(\gamma_2')^{-1} \in \Gamma_1\alpha$ and so $\gamma_2(\gamma_2')^{-1} \in \alpha^{-1}\Gamma_1\alpha$. But we also have $\gamma_2(\gamma_2')^{-1} \in \Gamma_2$ and these two facts together imply $\gamma_2(\gamma_2')^{-1} \in \Gamma_3$. Hence

$$\Gamma_3\gamma_2 = \Gamma_3\gamma_2',$$

which shows that the induced map is also an injection. $\square$

With this lemma in hand, we can prove that the orbit decomposition of $\Gamma_1\alpha\Gamma_2$ is finite:

Proposition 3.4.1. *Let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then the orbit decomposition*

$$\Gamma_1\alpha\Gamma_2 = \bigcup_j \Gamma_1\beta_j,$$

with respect to the action of Γ_1 by left multiplication, is a finite union.

Proof. Let $\Gamma_3 = \alpha\Gamma_1\alpha^{-1} \cap \Gamma_2$. Then Γ_3 acts on Γ_2 by left multiplication. By Lemma 3.4.1, the number of orbits of $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ is the same as the number of cosets of $\Gamma_3\backslash\Gamma_2$ which is $[\Gamma_2 : \Gamma_3]$. By Lemma 2.1.1, $\alpha^{-1}\Gamma_1\alpha \cap \mathrm{PSL}_2(\mathbb{Z})$ is a congruence subgroup and hence $[\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma_1\alpha \cap \mathrm{PSL}_2(\mathbb{Z})]$ is finite. As $\Gamma_2 = \mathrm{PSL}_2(\mathbb{Z}) \cap \Gamma_2$ and $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \mathrm{PSL}_2(\mathbb{Z}) \cap \Gamma_2$, it follows that $[\Gamma_2 : \Gamma_3] \le [\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma_1\alpha \cap \mathrm{PSL}_2(\mathbb{Z})]$ completing the proof. $\square$

In light of Proposition 3.4.1, we will denote the orbit representatives by β_j to make it clear that there are finitely many. We can now introduce our operators. Fix some congruence subgroup Γ and consider $\mathcal{M}_k(\Gamma)$. Then for $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we define the operator $[\alpha]_k$ on $\mathcal{M}_k(\Gamma)$ to be the linear operator given by

$$(f[\alpha]_k)(z) = \det(\alpha)^{k-1} j(\alpha, z)^{-k} f(\alpha z),$$

Moreover, $[\alpha]_k$ is multiplicative. Indeed, if $\alpha, \alpha' \in \mathrm{GL}_2^+(\mathbb{Q})$, then

$$\begin{aligned}
((f[\alpha']_k)[\alpha]_k)(z) &= \det(\alpha)^{k-1} j(\alpha, z)^{-k} (f[\alpha']_k)(\alpha z) \\
&= \det(\alpha')^{k-1} \det(\alpha)^{k-1} j(\alpha', \alpha z)^{-k} j(\alpha, z)^{-k} f(\alpha'\alpha z) \\
&= \det(\alpha'\alpha)^{k-1} j(\alpha'\alpha, z)^{-k} f(\alpha'\alpha z) && \text{cocycle condition} \\
&= (f[\alpha'\alpha]_k)(z).
\end{aligned}$$

Also, if $\gamma \in \Gamma$ and we choose the representative with $\det(\gamma) = 1$, then the chain of equalities

$$(f[\gamma]_k)(z) = j(\gamma, z)^{-k} f(\gamma z) = \chi(\gamma) f(z),$$

is equivalent to the modularity of f with character χ on $\Gamma\backslash\mathbb{H}$. Thus f is holomorphic form with trivial character on $\Gamma\backslash\mathbb{H}$ if and only if $f[\gamma]_k = f$ for all $\gamma \in \Gamma$ where γ is chosen to be the representative with positive determinant. Now let Γ_1 and Γ_2 be two congruence subgroups and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. We define the **double coset operator** $[\Gamma_1\alpha\Gamma_2]_k$ on $\mathcal{M}_k(\Gamma_1)$ to be the linear operator given by

$$(f[\Gamma_1\alpha\Gamma_2]_k)(z) = \sum_j (f[\beta_j]_k)(z) = \sum_j \det(\beta_j)^{k-1} j(\beta_j, z)^{-k} f(\beta_j z),$$

By Proposition 3.4.1 this sum is finite. It remains to check that $f[\Gamma_1\alpha\Gamma_2]_k$ is well-defined. Indeed, if β_j and β_j' belong to the same orbit, then $\beta_j'\beta_j^{-1} \in \Gamma_1$. But then as $f \in \mathcal{M}_k(\Gamma_1)$, is it invariant under the $[\beta_j'\beta_j^{-1}]_k$ operator so that

$$(f[\beta_j]_k)(z) = ((f[\beta_j'\beta_j^{-1}]_k)[\beta_j]_k)(z) = (f[\beta_j']_k)(z),$$

and therefore the $[\Gamma_1\alpha\Gamma_2]_k$ operator is well-defined. Actually, the map $[\Gamma_1\alpha\Gamma_2]_k$ preserves holomorphic forms:

Proposition 3.4.2. *For any two congruence subgroups Γ_1 and Γ_2, $[\Gamma_1\alpha\Gamma_2]_k$ maps $\mathcal{M}_k(\Gamma_1)$ into $\mathcal{M}_k(\Gamma_2)$. Moreover, $[\Gamma_1\alpha\Gamma_2]_k$ preserves the subspace of cusp forms.*

Proof. Holomorphy is immediate since the sum in the definition of $f[\Gamma_1\alpha\Gamma_2]_k$ is finite by Proposition 3.4.1. For modularity, let $\gamma \in \Gamma_2$. Then

$$\begin{aligned}
(f[\Gamma_1\alpha\Gamma_2]_k)(\gamma z) &= \sum_j \det(\beta_j)^{k-1} j(\beta_j, \gamma z)^{-k} f(\beta_j \gamma z) \\
&= \sum_j \det(\beta_j\gamma)^{k-1} j(\beta_j, \gamma z)^{-k} f(\beta_j \gamma z) && \det(\gamma) = 1 \\
&= \sum_j \det(\beta_j\gamma)^{k-1} \left(\frac{j(\gamma, z)}{j(\beta_j\gamma, z)}\right)^k f(\beta_j \gamma z) && \text{cocycle condition} \\
&= j(\gamma, z)^k \sum_j \det(\beta_j\gamma)^{k-1} j(\beta_j\gamma, z)^{-k} f(\beta_j \gamma z) \\
&= j(\gamma, z)^k \sum_j \det(\beta_j)^{k-1} j(\beta_j, z)^{-k} f(\beta_j z) && \beta_j \to \beta_j\gamma^{-1} \\
&= j(\gamma, z)^k \sum_j (f[\beta_j]_k)(z) \\
&= j(\gamma, z)^k (f[\Gamma_1\alpha\Gamma_2]_k)(z).
\end{aligned}$$

This proves the modularity of $f[\Gamma_1\alpha\Gamma_2]_k$. For the growth condition, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$ of $\Gamma_2\backslash\mathbb{H}$. For any orbit representative β_j, $\beta_j\sigma_{\mathfrak{a}}$ takes ∞ to an element of $\mathbb{Q} \cup \{\infty\}$ since $\beta_j \in \mathrm{GL}_2^+(\mathbb{Q})$. In other words, $\beta_j\sigma_{\mathfrak{a}}\infty = \mathfrak{b}$ for some cusp $\mathfrak{b}$ of $\Gamma_1\backslash\mathbb{H}$. Then by the cocycle condition, we have

$$j(\sigma_{\mathfrak{a}}, z)^{-k}(f[\Gamma_1\alpha\Gamma_2]_k)(\sigma_{\mathfrak{a}} z) = \sum_j \det(\beta_j)^{k-1} j(\beta_j\sigma_{\mathfrak{a}}, z)^{-k} f(\beta_j\sigma_{\mathfrak{a}} z),$$

and the growth condition follows from that of f. Therefore $f[\Gamma_1\alpha\Gamma_2]_k \in \mathcal{M}_k(\Gamma_2)$. Lastly, it is clear that $f[\Gamma_1\alpha\Gamma_2]_k$ is a cusp form if f is. Therefore $[\Gamma_1\alpha\Gamma_2]_k$ preserves the subspace of cusp forms. □

The double coset operators are the most basic types of operators on holomorphic forms. They are the building blocks needed to define the more important diamond and Hecke operators.

3.5 Diamond & Hecke Operators

The diamond and Hecke operators are special linear operators that are used to construct a linear theory of holomorphic forms. They will also help us understand the Fourier coefficients. Throughout this discussion, we will obtain corresponding results for holomorphic forms with nontrivial characters. We will discuss the diamond operator first. To define them, we need to consider both the congruence subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$. Recall that $\Gamma_1(N) \le \Gamma_0(N)$ and consider the map

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^* \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to d \pmod{N},$$

(d is invertible modulo N since $c \equiv 0 \pmod{N}$ and $ad-bc = 1$). This is a surjective homomorphism and its kernel is exactly $\Gamma_1(N)$ so that $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ and $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$. Letting $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $f \in \mathcal{M}_k(\Gamma_1)$, consider $\left(f\left[\Gamma_1(N)\alpha\Gamma_1(N)\right]_k\right)(z)$. This is only dependent upon the lower-right entry d of α taken modulo N. To see this, since $\Gamma_1(N)$ is normal in $\Gamma_0(N)$, $\Gamma_1(N)\alpha = \alpha\Gamma_1(N)$

so that $\Gamma_1(N)\alpha\Gamma_1(N) = \alpha\Gamma_1(N)$ and hence there is only one representative for the orbit decomposition. Therefore

$$(f\,[\Gamma_1(N)\alpha\Gamma_1(N)]_k)\,(z) = \sum_j (f[\beta]_k)(z) = (f[\alpha]_k)(z).$$

This induces an action of $\Gamma_0(N)$ on $\mathcal{M}_k(\Gamma_1)$ and since $\Gamma_1(N)$ acts trivially, this is really an action of $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$. In other words, we have an induced action that depends only upon the lower-right entry d of α taken modulo N. So for any d modulo N, we define the **diamond operator** $\langle d\rangle : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$ to be the linear operator given by

$$(\langle d\rangle f)(z) = (f[\alpha]_k)(z),$$

for any $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Our discussion above has already shown that the diamond operators $\langle d\rangle$ are well-defined. Moreover, the diamond operators are also invertible with $\langle \overline{d}\rangle$ serving as an inverse and α^{-1} as a representative for the definition. Also, since the operator $[\alpha]_k$ is multiplicative and

$$\begin{pmatrix} * & * \\ 0 & d \end{pmatrix}\begin{pmatrix} * & * \\ 0 & e \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & de \end{pmatrix} \pmod{N},$$

the diamond operators are multiplicative. One reason the diamond operators are useful is that they decompose $\mathcal{M}_k(\Gamma_1(N))$ into eigenspaces. For any Dirichlet character χ modulo N, we let

$$\mathcal{M}_k(N,\chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) : \langle d\rangle f = \chi(d)f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^*\},$$

be the χ-eigenspace. Also let $\mathcal{S}_k(N,\chi)$ be the corresponding subspace of cusp forms. Then $\mathcal{M}_k(\Gamma_1(N))$ admits a decomposition into these eigenspaces:

Proposition 3.5.1. *We have a direct sum decomposition*

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi \pmod N} \mathcal{M}_k(N,\chi).$$

Moreover, this decomposition respects the subspace of cusp forms.

Proof. The diamond operators give a representation of $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$ over $\mathcal{M}_k(\Gamma_1(N))$. Explicitly,

$$\Phi : (\mathbb{Z}/N\mathbb{Z})^* \times \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N)) \qquad (d,f) \to \langle d\rangle f.$$

But any representation of a finite abelian group over $\mathbb{C}$ is completely reducible with respect to the characters of the group and every irreducible subrepresentation is 1-dimensional (see Theorem C.2.1). Since the characters of $(\mathbb{Z}/N\mathbb{Z})^*$ are given by Dirichlet characters, the decomposition as a direct sum follows. The decomposition respects the subspace of cusp forms because the double coset operators do. $\square$

If $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and we choose the representative with positive determinant, then $\chi(\gamma) = \chi(d)$ and the chain of equalities

$$(\langle d\rangle f)(z) = (f[\gamma]_k)(z) = j(\gamma,z)^{-k} f(\gamma z) = \chi(d)f(z),$$

is equivalent to the modularity of f with character χ on $\Gamma_0(N)\backslash\mathbb{H}$. Thus f is a holomorphic form with character χ on $\Gamma_0(N)\backslash\mathbb{H}$ if and only if $f[\gamma]_k = \chi(\gamma)f$ for all $\gamma \in \Gamma_0(N)$ where γ is chosen to be the representative with positive determinant. It follows that the diamond operators sieve holomorphic forms on $\Gamma_1(N)\backslash\mathbb{H}$ with trivial character in terms of holomorphic forms on $\Gamma_0(N)\backslash\mathbb{H}$ with nontrivial characters. In particular, $\mathcal{M}_k(N,\chi) = \mathcal{M}_k(\Gamma_0(N),\chi)$ and $\mathcal{S}_k(N,\chi) = \mathcal{S}_k(\Gamma_0(N),\chi)$. So by Proposition 3.5.1, we have

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi \pmod N} \mathcal{M}_k(\Gamma_0(N),\chi),$$

and this decomposition respects the subspace of cusp forms. This fact clarifies why it is necessary to consider holomorphic forms with nontrivial characters. Now it is time to define the Hecke operators. For a prime p, we define the p-th **Hecke operator** $T_p : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$ to be the linear operator given by

$$(T_pf)(z) = \left(\left[\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1(N)\right]_k f\right)(z).$$

By Proposition 3.4.2, T_p preserves the subspace of cusp forms. We will start discussing properties of the diamond and Hecke operators, but we first state an important lemma that will be used throughout (see [DS05] for a proof):

Lemma 3.5.1. *Let p be a prime. As sets,*

$$\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1(N) = \left\{\gamma \in \mathrm{Mat}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix}1 & *\\ 0 & p\end{pmatrix} \pmod N \text{ and } \det(\gamma) = p\right\}.$$

With Lemma 3.5.1, it is not too hard to see that the diamond and Hecke operators commute:

Proposition 3.5.2. *For every $d \in (\mathbb{Z}/N\mathbb{Z})^*$ and prime p, the diamond operator $\langle d\rangle$ and the Hecke operator T_p on $\mathcal{M}_k(\Gamma_1(N))$ commute:*

$$\langle d\rangle T_p = T_p\langle d\rangle$$

Proof. Let $\alpha = \left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right)$. For $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$, we have

$$\gamma\alpha\gamma^{-1} \equiv \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix} = \begin{pmatrix}1 & (p-1)ab\\ 0 & p\end{pmatrix} \pmod N,$$

because $c \equiv 0 \pmod N$, $ad - bc = 1$, and $ad \equiv 1 \pmod N$. By Lemma 3.5.1, $\gamma\alpha\gamma^{-1} \in \Gamma_1(N)\alpha\Gamma_1(N)$ and so we can use this representative in place of α. On the one hand,

$$\Gamma_1(N)\alpha\Gamma_1(N) = \bigcup_j \Gamma_1(N)\beta_j.$$

On the other hand, using $\gamma\alpha\gamma^{-1}$ in place of α and the normality of $\Gamma_1(N)$ in $\Gamma_0(N)$, we have

$$\Gamma_1(N)\alpha\Gamma_1(N) = \Gamma_1(N)\gamma\alpha\gamma^{-1}\Gamma_1(N) = \gamma\Gamma_1(N)\alpha\Gamma_1(N)\gamma^{-1} = \gamma\bigcup_j\Gamma_1(N)\beta_j\gamma^{-1} = \bigcup_j\Gamma_1(N)\gamma\beta_j\gamma^{-1}.$$

Upon comparing these two decompositions of $\Gamma_1(N)\alpha\Gamma_1(N)$ gives

$$\bigcup_j\Gamma_1(N)\beta_j = \bigcup_j\Gamma_1(N)\gamma\beta_j\gamma^{-1}.$$

Now let $f \in \mathcal{M}_k(\Gamma_1(N))$. Then this equivalence of unions implies

$$\langle d\rangle T_pf = \sum_j f[\beta_j\gamma]_k = \sum_j f[\gamma\beta_j]_k = T_p\langle d\rangle f.$$

□

Using Lemma 3.5.1 we can obtain an explicit description of the Hecke operator T_p:

Proposition 3.5.3. *Let $f \in \mathcal{M}_k(\Gamma_1(N))$. Then the Hecke operator T_p acts on f as follows:*

$$(T_p f)(z) = \begin{cases} \sum_{j \pmod p} \left(f \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \right)(z) + \left(f \left[\begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right]_k \right)(z) & \text{if } p \nmid N, \\ \sum_{j \pmod p} \left(f \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \right)(z) & \text{if } p \mid N, \end{cases}$$

where m and n are chosen such that $\det\left(\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)\right) = 1$.

Proof. Set $\Gamma_3 = \alpha^1 \Gamma_1(N)\alpha \cap \Gamma_1(N)$ where $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. Define

$$\beta_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \quad \text{and} \quad \beta_\infty = \begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} pm & n \\ Np & p \end{pmatrix},$$

for j taken modulo p and where where m and n are chosen such that $\det\left(\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)\right) = 1$. It suffices to show $\{\beta_1, \ldots, \beta_{p-1}\}$ and $\{\beta_1, \ldots, \beta_{p-1}, \beta_\infty\}$ are complete sets of orbit representatives for $\Gamma_1(N)\backslash\Gamma_1(N)\alpha\Gamma_1(N)$ depending on if $p \nmid N$ or not. To accomplish this, we will find a complete set of coset representatives for $\Gamma_3\backslash\Gamma_1(N)$ and then use Lemma 3.4.1. First we require an explicit description of Γ_3. Let

$$\Gamma^0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod p \right\},$$

and define

$$\Gamma_1^0(N, p) = \Gamma_1(N) \cap \Gamma^0(p).$$

We claim $\Gamma_3 = \Gamma_1^0(N, p)$. For the forward inclusion, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ and observe that

$$\alpha^{-1}\gamma\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} a & pd \\ p^{-1}c & d \end{pmatrix}.$$

If $\alpha^{-1}\gamma\alpha \in \Gamma_3$, then $\alpha^{-1}\gamma\alpha \in \Gamma_1(N)$ and thus $p \mid c$ so that $\alpha^{-1}\gamma\alpha \in \mathrm{PSL}_2(\mathbb{Z})$. Moreover, the previous computation implies $\alpha^{-1}\gamma\alpha \in \Gamma_1^0(N, p)$ and so $\Gamma_3 \subseteq \Gamma_1^0(N, p)$. For the reverse inclusion, suppose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^1(N, p)$. Then $b = pk$ for some $k \in \mathbb{Z}$. Now observe

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} a & k \\ pc & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \alpha^{-1}\gamma\alpha,$$

where $\gamma = \left(\begin{smallmatrix} a & k \\ pc & d \end{smallmatrix}\right)$. As $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ we conclude $\gamma \in \Gamma_1(N)$ as well. Now let

$$\alpha_j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_\infty = \begin{pmatrix} pm & n \\ N & 1 \end{pmatrix},$$

for j taken modulo p and where m and n are as before. Clearly $\alpha_j \in \Gamma_1(N)$ for all j. As $pm - Nn = 1$, we have $pm \equiv 1 \pmod N$ so that $\alpha_\infty \in \Gamma_1(N)$ as well. We claim that $\{\alpha_1, \ldots, \alpha_{p-1}\}$ and $\{\alpha_1, \ldots, \alpha_{p-1}, \alpha_\infty\}$ are sets of coset representatives for $\Gamma_3\backslash\Gamma_1(N)$ depending on if $p \nmid N$ or not. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$ and consider

$$\gamma\alpha_j^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b - aj \\ c & d - cj \end{pmatrix}.$$

As $\gamma\alpha_j^{-1} \in \Gamma_1(N)$ because both γ and γ_i are, $\gamma\alpha_j^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$ for some i if and only if

$$b \equiv aj \pmod{p}.$$

First suppose $p \nmid a$. Then a is invertible modulo p so we may take $j = \overline{a}b \pmod{p}$. Now suppose $p \mid a$. If there is some i satisfying $b \equiv ai \pmod{p}$, then we also have $p \mid b$. But as $ad - bc = 1$, this is impossible and so no such i exists. As $a \equiv 1 \pmod{N}$, $p \mid a$ if and only if $p \nmid N$. In this case consider instead

$$\gamma\alpha_\infty^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -n \\ -N & pm \end{pmatrix} = \begin{pmatrix} a - Nb & pmb - na \\ c - Nd & pmd - nc \end{pmatrix}.$$

Since $p \mid a$, we have $pmb - na \equiv 0 \mod p$ so that $\gamma\alpha_\infty^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$. Altogether, we have shown that $\{\alpha_1, \ldots, \alpha_{p-1}\}$ and $\{\alpha_1, \ldots, \alpha_{p-1}, \alpha_\infty\}$ are sets of coset representatives for $\Gamma_3\backslash\Gamma_1(N)$ depending on if $p \nmid N$ or not. To show they are complete sets, we need to show that no two representatives belong to the same coset. To this end, suppose j and j' are distinct, taken modulo p, and consider

$$\alpha_j\alpha_{j'}^{-1} = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & j' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & j - j' \\ 0 & 1 \end{pmatrix}.$$

Then $\alpha_j\alpha_{j'}^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$ if and only if $j - j' \equiv 0 \pmod{p}$. This is impossible since j and j' are distinct. Hence α_j and $\alpha_{j'}$ represent distinct cosets. Now consider

$$\alpha_j\alpha_\infty^{-1} = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -n \\ -N & pm \end{pmatrix} = \begin{pmatrix} 1 - Nj & pmj - n \\ -N & pm \end{pmatrix}.$$

Now $\alpha_j\alpha_\infty^{-1} \in \Gamma_3 = \Gamma_1^0(N,p)$ if and only if $pmj - n \equiv 0 \pmod{p}$. This is impossible since $pm - Nn = 1$ implies $p \nmid n$. Therefore α_j and α_∞ represent distinct cosets. It follows that $\{\alpha_1, \ldots, \alpha_{p-1}\}$ and $\{\alpha_1, \ldots, \alpha_{p-1}, \alpha_\infty\}$ are complete sets of coset representatives completing the proof. As

$$\alpha\alpha_j = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} = \beta_j \quad \text{and} \quad \alpha\alpha_\infty = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} pm & n \\ N & 1 \end{pmatrix} = \begin{pmatrix} pm & n \\ Np & p \end{pmatrix} = \beta_\infty,$$

Lemma 3.4.1 finishes the proof. $\square$

This explicit definition of T_p can be used to compute how the Hecke operators act on the Fourier coefficients of a holomorphic form:

Proposition 3.5.4. *Let $f \in \mathcal{M}_k(\Gamma_1(N))$ with Fourier coefficients $a_n(f)$. Then for primes p with $(p,N) = 1$,*

$$(T_pf)(z) = \sum_{n \geq 0} \left(a_{np}(f) + \chi_{N,0}(p)p^{k-1}a_{\frac{n}{p}}(\langle p \rangle f)\right)e^{2\pi inz},$$

is the Fourier series of T_pf where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid n$. Moreover, if $f \in \mathcal{M}_k(N,\chi)$, then $T_pf \in \mathcal{M}_k(N,\chi)$ and

$$(T_pf)(z) = \sum_{n \geq 0} \left(a_{np}(f) + \chi(p)p^{k-1}a_{\frac{n}{p}}(f)\right)e^{2\pi inz},$$

where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid n$.

Proof. In view of Proposition 3.5.1 and the linearity of the Hecke operators, it suffices to assume $f \in \mathcal{M}_k(N,\chi)$ and thus only the second formula needs to be verified. Observe that

$$\left(f\left[\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}\right]_k\right)(z) = \frac{1}{p}f\left(\frac{z+j}{p}\right) = \frac{1}{p}\sum_{n\geq 0} a_n(f)e^{\frac{2\pi in(z+j)}{p}}.$$

Summing over all j modulo p gives

$$\sum_{j \pmod p}\left(f\left[\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}\right]_k\right)(z) = \sum_{j \pmod p}\frac{1}{p}\sum_{n\geq 0} a_n(f)e^{\frac{2\pi in(z+j)}{p}} = \sum_{n\geq 0} a_n(f)e^{\frac{2\pi inz}{p}}\frac{1}{p}\sum_{j \pmod p} e^{\frac{2\pi nij}{p}}.$$

If $p \nmid N$ then the inner sum vanishes because it is the sum over all p-th roots of unity. If $p \mid N$ then the sum is p. Therefore

$$\sum_{j \pmod p}\left(f\left[\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}\right]_k\right)(z) = \sum_{n\geq 0} a_{np}(f)e^{2\pi inz}.$$

If $p \mid N$, then Proposition 3.5.3 implies

$$(T_pf)(z) = \sum_{n\geq 0} a_{np}(f)e^{2\pi inz}, \tag{3.3}$$

which is the claimed Fourier series of T_pf. $p \nmid N$, then we have the additional term

$$\begin{aligned}\left(f\left[\begin{pmatrix}m & n\\ N & p\end{pmatrix}\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\right]_k\right)(z) &= \left(\langle p\rangle f\left[\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\right]_k\right)(z)\\ &= p^{k-1}(\langle p\rangle f)(pz)\\ &= \sum_{n\geq 0} p^{k-1}a_n(\langle p\rangle f)e^{2\pi inpz}\\ &= \sum_{n\geq 0}\chi(p)p^{k-1}a_n(f)e^{2\pi inpz},\end{aligned}$$

where the first equality holds because $\left(\begin{smallmatrix}m & n\\ N & p\end{smallmatrix}\right) \in \Gamma_0(N)$ and the last equality holds because $\langle p\rangle f = \chi(p)f$. In this case, Proposition 3.5.3 gives

$$(T_pf)(z) = \sum_{n\geq 0} a_{np}(f) + \chi(p)p^{k-1}a_{\frac{n}{p}}(f)e^{2\pi inz}.$$

Since $\chi(p) = 0$ if $p \mid N$, these two cases can be expressed together as

$$(T_pf)(z) = \sum_{n\geq 0}\left(a_{np}(f) + \chi(p)p^{k-1}a_{\frac{n}{p}}(f)\right)e^{2\pi inz}.$$

□

We now mention the crucial result about Hecke operators which is that they form a simultaneously commuting family with the diamond operators:

Proposition 3.5.5. *Let p and q be primes and $d, e \in (\mathbb{Z}/N\mathbb{Z})^*$. The Hecke operators T_p and T_q and diamond operators $\langle d\rangle$ and $\langle e\rangle$ on $\mathcal{M}_k(\Gamma_1(N))$ form a simultaneously commuting family:*

$$T_pT_q = T_qT_p, \quad \langle d\rangle T_p = T_p\langle d\rangle, \quad \textit{and} \quad \langle d\rangle\langle e\rangle = \langle e\rangle\langle d\rangle.$$

Proof. Showing the diamond and Hecke operators commute was Proposition 3.5.2. To show commutativity of the diamond operators, let $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\eta = \left(\begin{smallmatrix} * & * \\ * & e \end{smallmatrix}\right) \in \Gamma_0(N)$. Then

$$\gamma\eta \equiv \begin{pmatrix} * & * \\ 0 & de \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & ed \end{pmatrix} \equiv \eta\gamma \pmod{N}.$$

Therefore $[\gamma\eta]_k = [\eta\gamma]_k$ and so for any $f \in \mathcal{M}_k(\Gamma_1(N))$, we have

$$\langle d\rangle\langle e\rangle f = f[\gamma\eta]_k = f[\eta\gamma]_k = \langle e\rangle\langle d\rangle f.$$

We now show that the Hecke operators commute. In view of Proposition 3.5.1 and linearity of the Hecke operators, it suffices to prove this for $f \in \mathcal{M}_k(N, \chi)$. Applying Proposition 3.5.4 twice, for any $n \geq 1$ we compute

$$\begin{aligned} a_n(T_pT_qf) &= a_{np}(T_qf) + \chi(p)p^{k-1}a_{\frac{n}{p}}(T_qf) \\ &= a_{npq}(f) + \chi(q)q^{k-1}a_{\frac{np}{q}}(f) + \chi(p)p^{k-1}(a_{\frac{nq}{p}}(f) + \chi(q)q^{k-1}a_{\frac{n}{pq}}(f)) \\ &= a_{npq}(f) + \chi(q)q^{k-1}a_{\frac{np}{q}}(f) + \chi(p)p^{k-1}a_{\frac{nq}{p}}(f) + \chi(pq)(pq)^{k-1}a_{\frac{n}{pq}}(f). \end{aligned}$$

The last expression is symmetric in p and q so that $a_n(T_pT_qf) = a_n(T_qT_pf)$ for all $n \geq 1$. Since all of the Fourier coefficients are equal, we get

$$T_pT_qf = T_qT_pf.$$

□

We can use Proposition 3.5.5 to construct diamond operators $\langle m\rangle$ and Hecke operators T_m for all $m \geq 1$. The **diamond operator** $\langle m\rangle : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$ is defined to be the linear operator given by

$$\langle m\rangle = \begin{cases} \langle m\rangle \text{ with } m \text{ taken modulo } N & \text{if } (m,N) = 1, \\ 0 & \text{if } (m,N) > 1. \end{cases}$$

Now for the Hecke operators. If $m = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ is the prime decomposition of m, then we define the m-th **Hecke operator** $T_m : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$ to be the linear operator given by

$$T_m = \prod_{1\leq i\leq k} T_{p_i^{r_i}},$$

where T_{p^r} is defined inductively by

$$T_{p^r} = \begin{cases} T_pT_{p^{r-1}} - p^{k-1}\langle p\rangle T_{p^{r-2}} & \text{if } p \nmid N, \\ T_p^r & \text{if } p \mid N, \end{cases}$$

for all $r \geq 2$. Then by Proposition 3.5.5, the Hecke operators T_m are multiplicative but not completely multiplicative in m. Moreover, they commute with the diamond operators $\langle m\rangle$. Using these definitions, Propositions 3.5.4 and 3.5.5, a more general formula for how the Hecke operators T_m act on Fourier coefficients can be derived:

Proposition 3.5.6. *Let $f \in \mathcal{M}_k(\Gamma_1(N))$ with Fourier coefficients $a_n(f)$. Then for $m \geq 1$ with $(m,N) = 1$,*

$$(T_mf)(z) = \sum_{n\geq 0}\left(\sum_{d\mid(n,m)} d^{k-1}a_{\frac{nm}{d^2}}(\langle d\rangle f)\right)e^{2\pi inz},$$

is the Fourier series of T_mf. Moreover, if $f \in \mathcal{M}_k(N,\chi)$, then

$$(T_mf)(z) = \sum_{n\geq 0}\left(\sum_{d\mid(n,m)} \chi(d)d^{k-1}a_{\frac{nm}{d^2}}(f)\right)e^{2\pi inz}.$$

Proof. In view of Proposition 3.5.1 and linearity of the Hecke operators, we may assume $f \in \mathcal{M}_k(N,\chi)$. Therefore we only need to verify the second formula. When $m = 1$ the result is obvious and when $m = p$, we have

$$\sum_{d \mid (n,p)} \chi(d)d^{k-1}a_{\frac{np}{d^2}}(f) = a_{np}(f) + \chi(p)p^{k-1}a_{\frac{n}{p}}(f),$$

which is the result obtained from Proposition 3.5.4. By induction assume that the desired formula holds for $m = 1, p, \ldots, p^{r-1}$. Using the definition of T_{p^r} and Proposition 3.5.4, for any $n \geq 1$ we compute

$$\begin{aligned} a_n(T_{p^r}f) &= a_n(T_pT_{p^{r-1}}f) - \chi(p)p^{k-1}a_n(T_{p^{r-2}}f) \\ &= a_{np}(T_{p^{r-1}}f) + \chi(p)p^{k-1}a_{\frac{n}{p}}(T_{p^{r-1}}f) - \chi(p)p^{k-1}a_n(T_{p^{r-2}}f). \end{aligned}$$

By our induction hypothesis, this last expression is

$$\sum_{d \mid (np,p^{r-1})} \chi(d)d^{k-1}a_{\frac{np^r}{d^2}}(f) + \chi(p)p^{k-1} \sum_{d \mid \left(\frac{n}{p},p^{r-1}\right)} \chi(d)d^{k-1}a_{\frac{np^{r-2}}{d^2}}(f) - \chi(p)p^{k-1} \sum_{d \mid (n,p^{r-2})} \chi(d)d^{k-1}a_{\frac{np^{r-2}}{d^2}}(f).$$

Write the first sum as

$$\sum_{d \mid (np,p^{r-1})} \chi(d)d^{k-1}a_{\frac{np^r}{d^2}}(f) = a_{np^r}(f) + \sum_{d \mid (n,p^{r-2})} \chi(d)d^{k-1}a_{\frac{np^{r-2}}{d^2}}(f),$$

and observe that the sum on the right-hand side cancels the entire third term above. Therefore our expression reduces to

$$\begin{aligned} a_{np^r}(f) + \chi(p)p^{k-1} \sum_{d \mid \left(\frac{n}{p},p^{r-1}\right)} \chi(d)d^{k-1}a_{\frac{np^{r-2}}{d^2}}(f) &= a_{np^r}(f) + \sum_{d \mid \left(\frac{n}{p},p^{r-1}\right)} \chi(dp)(dp)^{k-1}a_{\frac{np^{r-2}}{d^2}}(f) \\ &= a_{np^r}(f) + \sum_{\substack{d \mid (n,p^r) \\ d \neq 1}} \chi(d)d^{k-1}a_{\frac{np^r}{d^2}}(f) \\ &= \sum_{d \mid (n,p^r)} \chi(d)d^{k-1}a_{\frac{np^r}{d^2}}(f), \end{aligned}$$

where in the second line we have performed the change of variables $dp \to d$ in the sum. This proves the claim when $m = p^r$ for all $r \geq 0$. By multiplicativity of the Hecke operators, it suffices to prove the claim when $m = p^rq^s$ for another prime q and some $s \geq 0$. We compute

$$\begin{aligned} a_n(T_{p^rq^s}f) &= a_n(T_{p^r}T_{q^s}f) \\ &= \sum_{d_1 \mid (n,p^r)} \chi(d_1)d_1^{k-1}a_{\frac{np^r}{d_1^2}}(T_{q^s}f) \\ &= \sum_{d_1 \mid (n,p^r)} \chi(d_1)d_1^{k-1} \sum_{d_2 \mid \left(\frac{np^r}{d_1^2},q^s\right)} \chi(d_2)d_2^{k-1}a_{\frac{np^rq^s}{(d_1d_2)^2}}(f) \\ &= \sum_{d_1 \mid (n,p^r)} \sum_{d_2 \mid \left(\frac{np^r}{d_1^2},q^s\right)} \chi(d_1d_2)(d_1d_2)^{k-1}a_{\frac{np^rq^s}{(d_1d_2)^2}}(f). \end{aligned}$$

Summing over pairs (d_1, d_2) of divisors of (n, p^r) and $\left(\frac{np^r}{d^2}, q^s\right)$ respectively is the same as summing over divisors d of $(n, p^r q^s)$. Indeed, because p and q are relative prime, any such d is of the form $d = d_1 d_2$ where $d_1 \mid (n, p^r)$ and $d_2 \mid \left(\frac{np^r}{d_2^2}, q^s\right)$. Therefore the double sum becomes

$$\sum_{d \mid (n, p^r q^s)} \chi(d) d^{k-1} a_{\frac{np^r q^s}{d^2}}(f).$$

This completes the proof. $\square$

The diamond and Hecke operators turn out to be normal with respect to the Petersson inner product on the subspace of cusp forms. To prove this fact, we will require a lemma:

Lemma 3.5.2. *Let Γ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then there exist $\beta_1, \ldots, \beta_n \in \mathrm{GL}_2^+(\mathbb{Q})$, where $n = [\Gamma : \alpha^{-1}\Gamma\alpha \cap \Gamma] = [\Gamma : \alpha\Gamma\alpha^{-1} \cap \Gamma]$, and such that*

$$\Gamma\alpha\Gamma = \bigcup_j \Gamma\beta_j = \bigcup_j \beta_j\Gamma.$$

Proof. Apply Lemma 2.3.1 with the congruence subgroup $\alpha\Gamma\alpha^{-1} \cap \Gamma$ in place of Γ to get

$$[\mathrm{PSL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma\alpha \cap \Gamma] = [\mathrm{PSL}_2(\mathbb{Z}) : \alpha\Gamma\alpha^{-1} \cap \Gamma].$$

Dividing both sides by $[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma]$ gives

$$[\Gamma : \alpha^{-1}\Gamma\alpha \cap \Gamma] = [\Gamma : \alpha\Gamma\alpha^{-1} \cap \Gamma].$$

Therefore we can find coset representatives $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n \in \Gamma$ such that

$$\Gamma = \bigcup_j (\alpha^{-1}\Gamma\alpha \cap \Gamma)\gamma_j = \bigcup_j (\alpha^{-1}\Gamma\alpha \cap \Gamma)\widetilde{\gamma}_j^{-1}.$$

Invoking Lemma 3.4.1 twice, we can express each of these coset decompositions as an orbit decomposition:

$$\bigcup_j (\alpha^{-1}\Gamma\alpha \cap \Gamma)\gamma_j = \bigcup_j \Gamma\alpha\gamma_j \quad \text{and} \quad \bigcup_j (\alpha^{-1}\Gamma\alpha \cap \Gamma)\widetilde{\gamma}_j^{-1} = \bigcup_j \Gamma\alpha^{-1}\widetilde{\gamma}_j^{-1}.$$

It follows that

$$\Gamma = \bigcup_j \Gamma\alpha\gamma_j = \bigcup_j \widetilde{\gamma}_j\alpha\Gamma.$$

For each j the orbit spaces $\Gamma\alpha\gamma_j$ and $\widetilde{\gamma}_j\alpha\Gamma$ have nonempty intersection. For if they did we would have $\Gamma\alpha\gamma_j \subseteq \bigcup_{i \neq j} \widetilde{\gamma}_i\alpha\Gamma$ and thus $\Gamma\alpha\Gamma \subseteq \bigcup_{i \neq j} \widetilde{\gamma}_i\alpha\Gamma$. This contradicts the previous decomposition of Γ. Therefore we can find representatives $\beta_j \in \Gamma\alpha\gamma_j \cap \widetilde{\gamma}_j\alpha\Gamma$ for every j. Then β_j

$$\Gamma = \bigcup_j \Gamma\beta_j = \bigcup_j \beta_j\Gamma.$$

$\square$

We can use Lemma 3.5.2 to compute adjoints:

Proposition 3.5.7. *Let Γ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Set $\alpha' = \det(\alpha)\alpha^{-1}$. Then the following are true:*

(i) If $\alpha^{-1}\Gamma\alpha \subseteq \mathrm{PSL}_2(\mathbb{Z})$, then for all $f \in \mathcal{S}_k(\Gamma)$ and $g \in \mathcal{S}_k(\alpha^{-1}\Gamma\alpha)$, we have

$$\langle f[\alpha]_k, g\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g[\alpha']_k\rangle_\Gamma.$$

(ii) For all $f, g \in \mathcal{S}_k(\Gamma)$, we have

$$\langle f[\Gamma\alpha\Gamma]_k, g\rangle = \langle f, g[\Gamma\alpha'\Gamma]_k\rangle.$$

In particular, if $\alpha^{-1}\Gamma\alpha = \Gamma$ then $[\alpha]_k^* = [\alpha']_k$ and $[\Gamma\alpha\Gamma]_k^* = [\Gamma\alpha'\Gamma]_k$.

Proof. To prove (i) we first compute

$$\begin{aligned}
\langle f[\alpha]_k, g\rangle_{\alpha^{-1}\Gamma\alpha} &= \frac{1}{V_{\alpha^{-1}\Gamma\alpha}} \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} (f[\alpha]_k)(z)\overline{g(z)}\mathrm{Im}(z)^k\, d\mu \\
&= \frac{1}{V_{\alpha^{-1}\Gamma\alpha}} \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} \det(\alpha)^{k-1} j(\alpha, z)^{-k} f(\alpha z)\overline{g(z)}\mathrm{Im}(z)^k\, d\mu \\
&= \frac{1}{V_\Gamma} \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} \det(\alpha)^{k-1} j(\alpha, z)^{-k} f(\alpha z)\overline{g(z)}\mathrm{Im}(z)^k\, d\mu && \text{Lemma 2.3.1} \\
&= \frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} \det(\alpha)^{k-1} j(\alpha, \alpha^{-1}z)^{-k} f(z)\overline{g(\alpha^{-1}z)}\mathrm{Im}(\alpha^{-1}z)^k\, d\mu && z \to \alpha^{-1}z.
\end{aligned}$$

As α' acts as α^{-1} on $\mathbb{H}$, this latter integral is equivalent to

$$\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} \det(\alpha)^{k-1} j(\alpha, \alpha' z)^{-k} f(z)\overline{g(\alpha' z)}\mathrm{Im}(\alpha' z)^k\, d\mu.$$

Moreover, applying the cocycle condition and the identities $\mathrm{Im}(\alpha' z) = \det(\alpha')\frac{\mathrm{Im}(z)}{|j(\alpha', z)|^2}$, $j(\alpha\alpha', z) = \det(\alpha)$, and $\det(\alpha') = \det(\alpha)$ together, we can further rewrite the integral as

$$\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} \det(\alpha')^{k-1} \overline{j(\alpha', z)^{-k}} f(z)\overline{g(\alpha' z)}\mathrm{Im}(z)^k\, d\mu.$$

Reversing the first computation in the start of the proof but applied to this integral shows that that

$$\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} \det(\alpha')^{k-1} \overline{j(\alpha', z)^{-k}} f(z)\overline{g(\alpha' z)}\mathrm{Im}(z)^k\, d\mu = \langle f, g[\alpha']_k\rangle_\Gamma,$$

which completes the proof of (i). To prove (ii), the coset decomposition $\Gamma\alpha\Gamma = \bigcup_j \Gamma\beta_j$ from Lemma 3.5.2 implies that we can use the β_j as representatives in the definition of the $[\Gamma\alpha\Gamma]_k$ operator. As the β_j also satisfy $\Gamma\alpha\Gamma = \bigcup_j \beta_j\Gamma$, upon inverting β_j and noting that $\beta_j \in \Gamma\alpha$, we obtain $\Gamma\alpha^{-1}\Gamma = \bigcup_j \Gamma\beta_j^{-1}$. Since scalar multiplication commutes with matrices and the matrices in Γ have determinant 1, we conclude that $\Gamma\alpha'\Gamma = \bigcup_j \Gamma\beta_j'$ where $\beta_j' = \det(\beta_j)\beta_j^{-1}$ (also $\det(\beta_j) = \det(\alpha)$). So we can use the β_j' as representatives in the definition of the $[\Gamma\alpha'\Gamma]_k$ operator. The statement now follows from (i). The last statement is now obvious. □

We can now prove that the diamond and Hecke operators are normal:

Proposition 3.5.8. *On $\mathcal{S}_k(\Gamma_1(N))$, the diamond operators $\langle m\rangle$ and Hecke operators T_m are normal with respect to the Petersson inner product for all $m \geq 1$ with $(m, N) = 1$. Moreover, their adjoints are given by*

$$\langle m\rangle^* = \langle \overline{m}\rangle \quad \textit{and} \quad T_p^* = \langle \overline{p}\rangle T_p.$$

Proof. As taking the adjoint is a linear operator, the definition of the diamond and Hecke operators and Proposition 3.5.5 imply that it suffices to prove the two adjoint formulas for the when $m = p$ is prime. We will first prove the adjoint formula for $\langle p\rangle$. Let $\alpha = \left(\begin{smallmatrix} * & * \\ * & p \end{smallmatrix}\right) \in \Gamma_0(N)$. As $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ and $\det(\alpha) = 1$, Proposition 3.5.7 gives

$$\langle p\rangle^* = [\alpha]_k^* = [\alpha']_k = [\alpha^{-1}]_k = \langle \overline{p}\rangle.$$

This proves the adjoint formula for the diamond operators and normality follows from multiplicativity. For the Hecke operator T_p, let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ and note that $\alpha' = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Now Proposition 3.5.7 implies

$$T_p^* = [\Gamma_1(N)\alpha\Gamma_1(N)]_k^* = [\Gamma_1(N)\alpha'\Gamma_1(N)]_k.$$

Let m and n be such that $pm - Nn = 1$. Then

$$\begin{pmatrix} 1 & n \\ N & pm \end{pmatrix}\alpha' = \begin{pmatrix} 1 & n \\ N & pm \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & n \\ Np & pm \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} p & n \\ N & m \end{pmatrix} = \alpha\begin{pmatrix} p & n \\ N & m \end{pmatrix},$$

which implies that $\alpha' = \left(\begin{smallmatrix} 1 & n \\ N & pm \end{smallmatrix}\right)^{-1}\alpha\left(\begin{smallmatrix} p & n \\ N & m \end{smallmatrix}\right)$. As $\left(\begin{smallmatrix} 1 & n \\ N & pm \end{smallmatrix}\right) \in \Gamma_1(N)$ (note that $pm \equiv 1 \pmod N$ since $pm - Nn = 1$) and $\left(\begin{smallmatrix} p & n \\ N & m \end{smallmatrix}\right) \in \Gamma_0(N)$, substituting the triple product expression for α' into $[\Gamma_1(N)\alpha'\Gamma_1(N)]_k$ and noting that $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ yields

$$\Gamma_1(N)\alpha'\Gamma_1(N) = \Gamma_1(N)\alpha\Gamma_1(N)\begin{pmatrix} p & n \\ N & m \end{pmatrix}.$$

Now if the β_j are representatives for $[\Gamma_1\alpha\Gamma_1(N)]_k$, then $\Gamma_1\alpha\Gamma_1(N) = \bigcup_j \Gamma_1(N)\beta_j$. Therefore the formula above implies $\Gamma_1\alpha'\Gamma_1(N) = \bigcup_j \Gamma_1(N)\beta_j\left(\begin{smallmatrix} p & n \\ N & m \end{smallmatrix}\right)$ and so the $\beta_j\left(\begin{smallmatrix} p & n \\ N & m \end{smallmatrix}\right)$ can be used as representatives for $[\Gamma_1\alpha'\Gamma_1(N)]_k$. As $pm - Nn = 1$, $m \equiv \overline{p} \pmod N$ and so Proposition 3.5.5 implies that $T_p^* = \langle\overline{p}\rangle T_p$. This proves the adjoint formula for the Hecke operators and normality follows from multiplicativity. □

In the case of the modular group, Proposition 3.5.8 says that all of the diamond and Hecke operators are normal. Now suppose f is a non-constant holomorphic form with Fourier coefficients $a_n(f)$. Let the eigenvalue of T_m for f be $\lambda_f(m)$. We say that the $\lambda_f(m)$ are the **Hecke eigenvalues** of f. If f is a simultaneous eigenfunction for all diamond operators $\langle m\rangle$ and Hecke operators T_m with $(m, N) = 1$, we call f an **eigenform**. If the condition $(m, N) = 1$ can be dropped, so that f is a simultaneous eigenfunction for all diamond and Hecke operators, then we say f is a **Hecke eigenform**. In particular, on $\Gamma_1(1)\backslash\mathbb{H}$ all eigenforms are Hecke eigenforms. If f is an eigenform, then Proposition 3.5.6 immediately implies that the first Fourier coefficient of $T_m f$ is $a_m(f)$ and so $a_m(f) = \lambda_f(m)a_1(f)$ for all $m \geq 1$ with $(m, N) = 1$. Therefore we cannot have $a_1(f) = 0$ for this would mean f is constant. We can normalize f by dividing by $a_1(f)$ so that the Fourier series has constant term 1. It follows that the m-th Fourier coefficient of f, when $(m, N) = 1$, is precisely the Hecke eigenvalue $\lambda_f(m)$. This normalization is called the **Hecke normalization** of f. The **Petersson normalization** of f is where we normalize so that $\langle f, f\rangle = 1$. From the spectral theorem we derive an important corollary:

Theorem 3.5.1. *$\mathcal{S}_k(\Gamma_1(N))$ admits an orthonormal basis of eigenforms.*

Proof. This follows from the spectral theorem along with Propositions 3.5.5 and 3.5.8. □

As a near immediate consequence of Theorem 3.5.1, we can show that Hecke eigenvalues of eigenforms satisfy certain relations known as the **Hecke relations** for holomorphic forms:

Proposition 3.5.9 (Hecke relations, holomorphic version). *Let $f \in \mathcal{S}_k(N,\chi)$ be a Hecke eigenform with Hecke eigenvalues $\lambda_f(n)$. Then the Hecke eigenvalues are multiplicative and satisfy*

$$\lambda_f(n)\lambda_f(m) = \sum_{d \mid (n,m)} \chi(d)d^{k-1}\lambda_f\left(\frac{nm}{d^2}\right) \quad \text{and} \quad \lambda_f(nm) = \sum_{d \mid (n,m)} \mu(d)\chi(d)d^{k-1}\lambda_f\left(\frac{n}{d}\right)\lambda_f\left(\frac{m}{d}\right),$$

for all $n, m \geq 1$ with $(nm, N) = 1$.

Proof. By Theorem 3.5.1 it suffices to verify this when f is a Hecke eigenform. We may also assume f is Hecke normalized without any harm. The multiplicativity of the Hecke eigenvalues now follows from the multiplicity of the Hecke operators. Moreover, the first identity follows immediately from computing the n-th Fourier coefficient of $T_m f$ in two different ways. On the one hand, use that f is a Hecke eigenform. On the other hand, use Proposition 3.5.6. For the second identity, we have

$$\chi(p)p^{k-1} = \lambda_f(p)^2 - \lambda_f(p^2),$$

provided $(p, N) = 1$, by computing the p-th Fourier coefficient of $T_p f$ in two different ways as we just did above. The second identity now follows from the first because $\lambda_f(n)$ is a specially multiplicative function in n (see Theorem A.2.2). □

As an immediate consequence of the Hecke relations, the Hecke operators satisfy analogous relations:

Corollary 3.5.1. *The Hecke operators are multiplicative and satisfy*

$$T_n T_m = \sum_{d \mid (n,m)} \chi(d)d^{k-1}T_{\frac{nm}{d^2}} \quad \text{and} \quad T_{nm} = \sum_{d \mid (n,m)} \mu(d)\chi(d)d^{k-1}T_{\frac{n}{d}}T_{\frac{m}{d}},$$

for all $n, m \geq 1$ with $(nm, N) = 1$.

Proof. This is immediate from Theorem 3.5.1 and the Hecke relations. □

The identities in Corollary 3.5.1 can also be established directly. Moreover, the first identity is symmetric in n and m so it can be used to show that the Hecke operators commute.

3.6 Atkin–Lehner Theory

So far, our entire theory of holomorphic forms has started with a fixed congruence subgroup of some level. Atkin–Lehner theory, or the theory of oldforms & newforms, allows us to discuss holomorphic forms in the context of moving between levels. In this setting, we will only deal with congruence subgroups of the form $\Gamma_1(N)$ and cusp forms on $\Gamma_1(N)\backslash\mathbb{H}$. The easiest way lift a holomorphic form from a smaller level to a larger level is to observe that if $M \mid N$, then $\Gamma_1(N) \leq \Gamma_1(M)$ so there is a natural inclusion $\mathcal{S}_k(\Gamma_1(M)) \subseteq \mathcal{S}_k(\Gamma_1(N))$. There is a less trivial way of lifting from $\mathcal{S}_k(\Gamma_1(M))$ to $\mathcal{S}_k(\Gamma_1(N))$. For any $d \mid \frac{N}{M}$, let $\alpha_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $f \in \mathcal{S}_k(\Gamma_1(M))$, we consider

$$(f[\alpha_d]_k)(z) = \det(\alpha_d)^{k-1}j(\alpha_d, z)^{-k}f(\alpha_d z) = d^{k-1}f(dz).$$

It turns out that, $[\alpha_d]_k$ maps $\mathcal{S}_k(\Gamma_1(M))$ into $\mathcal{S}_k(\Gamma_1(N))$ and preserves the subspace of cusp forms:

Proposition 3.6.1. *Let M and N be positive integers such that $M \mid N$. For any $d \mid \frac{N}{M}$, $[\alpha_d]_k$ maps $\mathcal{S}_k(\Gamma_1(M))$ into $\mathcal{S}_k(\Gamma_1(N))$.*

Proof. It is clear that holomorphy is satisfied for $f[\alpha_d]_k$. To verify modularity, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d' \end{smallmatrix}\right) \in \Gamma_1(N)$. Then

$$\alpha_d\gamma\alpha_d^{-1} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d' \end{pmatrix}\begin{pmatrix} d^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & bd \\ d^{-1}c & d' \end{pmatrix} = \gamma',$$

where $\gamma' = \left(\begin{smallmatrix} a & bd \\ d^{-1}c & d' \end{smallmatrix}\right)$. Since $c \equiv 0 \pmod N$ and $d \mid \frac{N}{M}$, we deduce that $d^{-1}c \equiv 0 \pmod M$. So $\gamma' \in \Gamma_1(M)$ and therefore $\alpha_d\Gamma_1(N)\alpha_d^{-1} \subseteq \Gamma_1(M)$, or equivalently, $\Gamma_1(N) \subseteq \alpha_d^{-1}\Gamma_1(M)\alpha_d$. Writing $\gamma = \alpha_d^{-1}\gamma'\alpha_d$, we see that $j(\gamma', \alpha_d z) = j(\gamma, z)$ and

$$\begin{aligned}
(f[\alpha_d]_k)(\gamma z) &= d^{k-1}f(d\gamma z) \\
&= d^{k-1}f(d\alpha_d^{-1}\gamma'\alpha_d z) \\
&= d^{k-1}f(\gamma'\alpha_d z) && d\alpha_d^{-1}z = z \\
&= j(\gamma', \alpha_d z)d^{k-1}f(\alpha_d z) \\
&= j(\gamma, z)d^{k-1}f(dz) && j(\gamma', \alpha_d z) = j(\gamma, z) \text{ and } \alpha_d z = dz \\
&= j(\gamma, z)(f[\alpha_d]_k)(z).
\end{aligned}$$

This verifies the modularity of $f[\alpha_d]_k$. For the growth condition, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$ of $\Gamma_1(M)\backslash\mathbb{H}$. Then $\alpha_d\sigma_{\mathfrak{a}}$ takes ∞ to an element of $\mathbb{Q} \cup \{\infty\}$ since $\alpha_d \in \mathrm{GL}_2^+(\mathbb{Q})$. In other words, $\alpha_d\sigma_{\mathfrak{a}}\infty = \mathfrak{b}$ for some cusp $\mathfrak{b}$ of $\Gamma_1(N)\backslash\mathbb{H}$. Then the cocycle condition implies

$$j(\sigma_{\mathfrak{a}}, z)^{-k}(f[\alpha_d]_k)(\sigma_{\mathfrak{a}}z) = \det(\alpha_d)^{k-1}j(\alpha_d\sigma_{\mathfrak{a}}, z)^{-k}f(\alpha_d\sigma_{\mathfrak{a}}z),$$

and the growth condition follows from that of f. Lastly, it is also clear that $f[\alpha_d]_k$ is a cusp form if f is and so $f[\alpha_d]_k \in \mathcal{S}_k(\Gamma_1(N))$. $\square$

We can now define oldforms and newforms. For each divisor d of N, set

$$i_d : \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{d}\right)\right) \times \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{d}\right)\right) \to \mathcal{S}_k(\Gamma_1(N)) \qquad (f, g) \mapsto f + g[\alpha_d]_k.$$

This map is well-defined by Proposition 3.6.1. The subspace of **oldforms of level** N is

$$\mathcal{S}_k(\Gamma_1(N))^{\mathrm{old}} = \bigoplus_{p \mid N} \mathrm{Im}(i_p),$$

and the subspace of **newforms of level** N is

$$\mathcal{S}_k(\Gamma_1(N))^{\mathrm{new}} = \left(\mathcal{S}_k(\Gamma_1(N))^{\mathrm{old}}\right)^{\perp},$$

where the orthogonal complement is taken with respect to the Petersson inner product. The elements of such subspaces are called **oldforms** and **newforms** respectively. Both subspaces are invariant under the diamond and Hecke operators (see [DS05] for a proof):

Proposition 3.6.2. *The spaces $\mathcal{S}_k(\Gamma_1(N))^{\mathrm{old}}$ and $\mathcal{S}_k(\Gamma_1(N))^{\mathrm{new}}$ are invariant under the diamond operators $\langle m \rangle$ and Hecke operators T_m for all $m \geq 1$.*

As a corollary, we deduce that these subspaces admit orthogonal bases of eigenforms:

Corollary 3.6.1. *$\mathcal{S}_k(\Gamma_1(N))^{\mathrm{old}}$ and $\mathcal{S}_k(\Gamma_1(N))^{\mathrm{new}}$ admit orthonormal bases of eigenforms.*

Proof. This follows immediately from Theorem 3.5.1 and Proposition 3.6.2 □

Something quite amazing happens for the subspace in $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$; the condition $(m, N) = 1$ for eigenforms in a base can be removed. Therefore the eigenforms are actually eigenfunctions for all of the diamond and Hecke operators. We require a preliminary result whose proof is quite involved but it is not beyond the scope of this text (see [DS05] for a proof):

Lemma 3.6.1. *If $f \in \mathcal{S}_k(\Gamma_1(N))$ with Fourier coefficients $a_n(f)$ and such that $a_n(f) = 0$ whenever $(n, N) = 1$, then*

$$f = \sum_{p|N} p^{k-1} f_p[\alpha_p],$$

for some $f_p \in \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{p}\right)\right)$.

The important observation to make about Lemma 3.6.1 is that if $f \in \mathcal{S}_k(\Gamma_1(N))$ is such that its n-th Fourier coefficients vanish when n is relatively prime to the level, then f must be an oldform. With this lemma we can prove the main theorem about $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$. The introduction of some language will be useful for the statement and its proof. We say that f is a **primitive Hecke eigenform** if it is a nonzero Hecke normalized Hecke eigenform in $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$. We can now prove the main result about newforms:

Theorem 3.6.1. *Let $f \in \mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ be an eigenform. Then the following hold:*

(i) f is a Hecke eigenform.

(ii) If $\widetilde{f}$ satisfies the same conditions as f and has the same eigenvalues for the Hecke operators, then $\widetilde{f} = cf$ for some nonzero constant c.

Moreover, the primitive Hecke eigenforms in $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ form an orthogonal basis with respect to the Petersson inner product and each such primitive Hecke eigenforms f lies in an eigenspace $\mathcal{S}_k(N, \chi)$.

Proof. First suppose $f \in \mathcal{S}_k(\Gamma_1(N))$ is an eigenform with Fourier coefficients $a_n(f)$. For $m \geq 1$ with $(m, N) = 1$, there exists $\lambda_f(m), \mu_f(m) \in \mathbb{C}$ such that $T_m f = \lambda_f(m) f$ and $\langle m \rangle f = \mu_f(m) f$. Actually, $\langle m \rangle f = \mu_f(m) f$ holds for all $m \geq 1$ because $\langle m \rangle$ is the zero operator if $(m, N) > 1$ and in this case we can take $\mu_f(m) = 0$. If we set $\chi(n) = \mu_f(m)$, then χ is a Dirichlet character modulo N. This follows because multiplicativity of $\langle m \rangle$ implies the same for χ and χ is N-periodic since $\langle m \rangle$ is N-periodic ($\langle m \rangle$ is defined by m taken modulo N). But then $\langle m \rangle f = \chi(m) f$ so that $f \in \mathcal{S}_k(N, \chi)$. As f is an eigenform, we also have $a_m(f) = \lambda_f(m) a_1(f)$ provided $(m, N) = 1$. So if $a_1(f) = 0$, Lemma 3.6.1 implies $f \in \mathcal{S}_k(\Gamma_1(N))^{\text{old}}$. With this fact in hand, we can prove the statements.

(i) The claim is trivial if f is zero, so assume otherwise. If $f \in \mathcal{S}_k(\Gamma_1(N))^{\text{new}}$, then $f \notin \mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ and so by what we have shown $a_1(f) \neq 0$. Therefore we may Hecke normalize f so that $a_1(f) = 1$ and $a_m(f) = \lambda_f(m)$. Now set $g_m = T_m f - \lambda_f(m) f$ for any $m \geq 1$. By Proposition 3.6.2, $g_m \in \mathcal{S}_k(\Gamma_1(N))^{\text{new}}$. Moreover, g_m is an eigenform and its first Fourier coefficient is zero. But then $g_m \in \mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ too and so $g_m = 0$ because $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ and $\mathcal{S}_k(\Gamma_1(N))^{\text{old}}$ are orthogonal subspaces. This means $T_m f = \lambda_f(m) f$ for any $m \geq 1$. Therefore f is a primitive Hecke eigenform and so is a Hecke eigenform before Hecke normalization.

(ii) Suppose $\widetilde{f}$ satisfies the same conditions as f with the same eigenvalues for the Hecke operators. By (i), f and $\widetilde{f}$ are Hecke eigenforms. After Hecke normalization, f and $\widetilde{f}$ have the same Fourier coefficients and so are identical. It follows that before Hecke normalization $f = c\widetilde{f}$ for some nonzero constant c.

Note that our initial remarks together with (i) show that each primitive Hecke eigenform f belongs to some eigenspace $\mathcal{S}_k(N,\chi)$. By Corollary 3.6.1, $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ admits an orthogonal basis of eigenforms which by (i) are Hecke eigenforms. As $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ is finite dimensional (because $\mathcal{S}_k(\Gamma_1(N))$ is), it follows that all of the primitive Hecke eigenforms form an orthogonal basis for $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ if we can show that they are linearly independent. So suppose, to the contrary, that we have a nontrivial linear relation

$$\sum_{1\le i\le r} c_i f_i = 0,$$

for some primitive Hecke eigenforms f_i, nonzero constants c_i, and with r minimal. Note that $r \ge 2$ for else we do not have a nontrivial linear relation. Letting $m \ge 1$ applying the operator $T_m - \lambda_{f_1}(m)$ to our nontrivial linear relation gives

$$\sum_{2\le i\le r} c_i(\lambda_{f_i}(m) - \lambda_{f_1}(m))f_i = 0,$$

which has one less term. Since r was chosen to be minimal, this implies $\lambda_{f_i}(m) - \lambda_{f_1}(m)$ for all i. But m was arbitrary, so $f_i = f_1$ for all i by (ii). Hence $r = 1$ which is a contradiction. □

Statement (i) in Theorem 3.6.1 implies that primitive Hecke eigenforms satisfy the Hecke relations for all $n, m \ge 1$. Statement (ii) can be interpreted as saying that a basis of newforms for $\mathcal{S}_k(\Gamma_1(N))^{\text{new}}$ contains one element per "eigenvalue" where we mean a set of eigenvalues one for each Hecke operator T_m. Actually, we can obtain a slightly better result called **multiplicity one** for holomorphic forms:

Theorem 3.6.2 (Multiplicity one, holomorphic version). *Let f and g be primitive Hecke eigenforms of the same eigenvalue and level. Denote the Hecke eigenvalues by $\lambda_f(n)$ and $\lambda_g(n)$ respectively. If $\lambda_f(p) = \lambda_g(p)$ for all primes p, then $f = g$.*

Proof. The Hecke relations imply that $\lambda_f(n) = \lambda_g(n)$ for all $n \ge 1$. By Theorem 3.6.1 (ii), $f = g$. □

With a stronger version of multiplicity one, one can prove that the Fourier coefficients of primitive Hecke eigenforms are real. This result takes a lot of work to show (see [DS05] for a note and appropriate references) and while we will not need it in the following, we state it for convenience:

Theorem 3.6.3. *The Fourier coefficients of primitive Hecke eigenforms are real.*

We now require one last linear operator. Let

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

and note that $\det(W_N) = N$. We define the **Atkin–Lehner involution** ω_N to be the linear operator on $\mathcal{S}_k(\Gamma_1(N))$ given by

$$(\omega_N f)(z) = N^{\frac{k}{2}} j(W_n, z)^{-k} f(W_n z) = (\sqrt{N}z)^{-k} f\left(-\frac{1}{Nz}\right).$$

It is not too difficult to see that ω_N is an involution on $\mathcal{S}_k(\Gamma_1(N))$:

Proposition 3.6.3. *ω_N is an involution on $\mathcal{S}_k(\Gamma_1(N))$.*

Proof. We first need to show that ω_N maps $\mathcal{S}_k(\Gamma_1(N))$ into itself. Holomorphy is obvious. For modularity, note that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$, we have

$$W_N\gamma = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ Na & Nb \end{pmatrix} = \begin{pmatrix} d & -N^{-1}c \\ -Nb & a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \gamma' W_N,$$

where $\gamma' = \left(\begin{smallmatrix} d & -N^{-1}c \\ -Nb & a \end{smallmatrix}\right) \in \Gamma_1(N)$. It follows that

$$\begin{aligned}
(\omega_N f)(\gamma z) &= (\sqrt{N}\gamma z)^{-k} f(W_N\gamma z) \\
&= (\sqrt{N}\gamma z)^{-k} f(\gamma' W_N z) \\
&= \left(\sqrt{N}\frac{az+b}{cz+d}\right)^{-k}\left(\frac{b}{z}+a\right)^{k} f\left(-\frac{1}{Nz}\right) \\
&= \left(\sqrt{N}\frac{az+b}{cz+d}\right)^{-k}\left(\frac{z}{az+b}\right)^{-k} f\left(-\frac{1}{Nz}\right) \\
&= (cz+d)^k(\sqrt{N}z)^{-k} f\left(-\frac{1}{Nz}\right) \\
&= j(\gamma,z)^k(\omega_N f)(z).
\end{aligned}$$

This verifies modularity of $\omega_N f$. As for the growth condition, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$. Then $W_N\sigma_{\mathfrak{a}}$ takes ∞ to an element of $\mathbb{Q}\cup\{\infty\}$ since $W_N \in \mathrm{GL}_2^+(\mathbb{Q})$. In other words, $W_N\sigma_{\mathfrak{a}}\infty = \mathfrak{b}$ for some cusp $\mathfrak{b}$. Then the cocycle condition implies

$$j(\sigma_{\mathfrak{a}}, z)^{-k}(\omega_N f)(\sigma_{\mathfrak{a}} z) = N^{\frac{k}{2}} j(W_n\sigma_{\mathfrak{a}}, z)^{-k} f(W_n\sigma_{\mathfrak{a}} z),$$

and the growth condition follows from that of f. It is also clear that $\omega_N f$ is a cusp form because f is. Altogether, this shows $\omega_N f \in \mathcal{S}_k(\Gamma_1(N))$. Moreover, ω_N is also an involution because $\omega_N^2 f = f$. □

As $\omega_N f$ is an involution its only possible eigenvalues are ± 1. The important fact we need is how ω_N acts on $\mathcal{S}_k(N,\chi)$. To state the result, for $f \in \mathcal{S}_k(N,\chi)$ define

$$\overline{f}(z) = \overline{f(-\overline{z})}.$$

Then we have the following (see [CS17] for a proof):

Proposition 3.6.4. *If $f \in \mathcal{S}_k(N,\chi)$ is a primitive Hecke eigenform, then*

$$\omega_N f = \omega_N(f)\overline{f},$$

where $\overline{f} \in \mathcal{S}_k(N,\overline{\chi})$ is a primitive Hecke eigenform and $\omega_N(f) \in \mathbb{C}$ is nonzero with $|\omega_N(f)| = 1$.

3.7 The Ramanujan-Petersson Conjecture

We will now discuss a famous conjecture about the size of the Hecke eigenvalues of primitive Hecke eigenforms. Historically the conjecture was born from conjectures made about the **modular discriminant** Δ given by

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2),$$

which is a weight 12 primitive Hecke eigenform on $\Gamma_1(1)\backslash\mathbb{H}$ (see [DS05]). Therefore it is natural to begin our discussion here. It can be shown that the Fourier series of the modular discriminant is

$$\Delta(z) = \sum_{n\geq 1} \tau(n)e^{2\pi i n z},$$

where the $\tau(n)$ are integers with $\tau(1) = 1$ and $\tau(2) = -24$ (see [Apo76] for a proof). The function $\tau : \mathbb{N} \to \mathbb{Z}$ is called **Ramanujan's τ function**. Ramanujan himself studied this function in his 1916 paper (see [Ram16]), and computed $\tau(n)$ for $1 \leq n \leq 30$. From these computations he conjectured the following three properties τ should satisfy:

(i) If $(n, m) = 1$, then $\tau(nm) = \tau(n)\tau(m)$.

(ii) $\tau(p^n) = \tau(p^{n-1})\tau(p) - p^{11}\tau(p^{n-2})$ for all prime p.

(iii) $|\tau(p)| \leq 2p^{\frac{11}{2}}$ for all prime p.

Note that (i) and (ii) are strikingly similar to the properties satisfied by the Hecke operators. In fact, (i) and (ii) are special cases of the properties of Hecke operators. This ends our commentary on properties (i) and (ii). Property (iii) turned out to be drastically more difficult to prove and is known as the classical **Ramanujan-Petersson conjecture**. To state the Ramanujan-Petersson conjecture for holomorphic forms, suppose $f \in \mathcal{S}_k(N, \chi)$ is a primitive Hecke eigenform with Hecke eigenvalues $\lambda_f(n)$. For each prime p, consider the polynomial

$$1 - \lambda_f(p)p^{-\frac{k-1}{2}}p^{-s} + \chi(p)p^{-2s}.$$

We call this the p-th **Hecke polynomial** of f. Let $\alpha_1(p)$ and $\alpha_2(p)$ denote the roots. From this quadratic, we have

$$\alpha_1(p) + \alpha_2(p) = \lambda_f(p)p^{-\frac{k-1}{2}} \quad \text{and} \quad \alpha_1(p)\alpha_2(p) = \chi(p).$$

The more general **Ramanujan-Petersson conjecture** for holomorphic forms is following statement:

Theorem 3.7.1 (Ramanujan-Petersson conjecture, holomorphic version). *Suppose $f \in \mathcal{S}_k(N, \chi)$ is a primitive Hecke eigenform with Hecke eigenvalues $\lambda_f(n)$. Let $\alpha_1(p)$ and $\alpha_2(p)$ be the roots of the p-th Hecke polynomial. Then for all primes p,*

$$|\lambda_f(p)| \leq 2p^{\frac{k-1}{2}}.$$

Moreover, if $p \nmid N$, then

$$|\alpha_1(p)| = |\alpha_2(p)| = 1.$$

In the 1970's Deligne proved the Ramanujan-Petersson conjecture (see [Del71] and [Del74] for the full proof). The argument is significantly beyond the scope of this text, and in actuality follows from Deligne's work on the Weil conjectures (except in the case $k = 1$ which requires a modified argument). This requires understanding classical algebraic topology and ℓ-acid cohomology in addition to the basic analytic number theory. As such, the proof of the Ramanujan-Petersson conjecture has been one of the biggest advances in analytic number theory in recent decades. Note that the Ramanujan-Petersson conjecture and the Hecke relations together give the bound $\lambda_f(n) \ll \sigma_0(n)n^{\frac{k-1}{2}} \ll_\varepsilon n^{\frac{k-1}{2}+\varepsilon}$ (recall Proposition A.3.1).

3.8 Twists of Holomorphic Forms

We can also twist of holomorphic forms by Dirichlet characters. Let $f \in \mathcal{M}_k(N,\chi)$ with Fourier series

$$f(z) = \sum_{n \ge 0} a_n(f) e^{2\pi i n z},$$

and let ψ be a Dirichlet character modulo M. We define the **twisted holomorphic form** $f \otimes \psi$ of f twisted by ψ by the Fourier series

$$(f \otimes \psi)(z) = \sum_{n \ge 0} a_n(f)\psi(n) e^{2\pi i n z}.$$

In order for $f \otimes \psi$ to be well-defined, we need to prove that it is a holomorphic form. The following proposition proves this and more when ψ is primitive:

Proposition 3.8.1. *Suppose $f \in \mathcal{M}_k(N,\chi)$ and ψ is a primitive Dirichlet character of conductor q. Then $f \otimes \psi \in \mathcal{M}_k(Nq^2, \chi\psi^2)$. Moreover, if f is a cusp form then so is $f \otimes \psi$.*

Proof. By Corollary 1.4.1, we can write

$$\begin{aligned}
(f \otimes \psi)(z) &= \sum_{n \ge 0} a_n(f)\psi(n) e^{2\pi i n z} \\
&= \sum_{n \ge 0} a_n(f) \left(\frac{1}{\tau(\overline{\psi})} \sum_{r \pmod q} \overline{\psi}(r) e^{\frac{2\pi i r n}{q}} \right) e^{2\pi i n z} \\
&= \frac{1}{\tau(\overline{\psi})} \sum_{r \pmod q} \overline{\psi}(r) \sum_{n \ge 0} a_n(f) e^{2\pi i n \left(z + \frac{r}{q}\right)} \\
&= \frac{1}{\tau(\overline{\psi})} \sum_{r \pmod q} \overline{\psi}(r) f\left(z + \frac{r}{q}\right).
\end{aligned}$$

From this last expression, holomorphy is immediate since the sum is finite. For modularity, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(Nq^2)$ and set $\gamma_r = \left(\begin{smallmatrix} 1 & \frac{r}{q} \\ 0 & 1 \end{smallmatrix}\right)$ for every r modulo q. Then for r and r' modulo q, we compute

$$\gamma_r \gamma \gamma_{r'}^{-1} = \begin{pmatrix} 1 & \frac{r}{q} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -\frac{r'}{q} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + \frac{cr}{q} & b - \frac{ar'-dr}{q} - \frac{crr'}{q^2} \\ c & d - \frac{cr'}{q} \end{pmatrix}.$$

Since $c \equiv 0 \pmod{Nq^2}$, if we choose r' (for each r) such that $ar' \equiv dr \pmod q$, then $\gamma_r\gamma\gamma_r^{-1} \in \Gamma_0(N)$. Such a choice exists and is unique by Bézout's identity because a and d are relatively prime to q as $ad \equiv 1 \pmod{Nq^2}$. Making this choice and setting $\eta_r = \gamma_r\gamma\gamma_{r'}^{-1}$, we compute

$$f\left(\gamma z + \frac{r}{q}\right) = f(\gamma_r \gamma z) = f(\eta_r \gamma_{r'} z) = \chi(\eta_r) j(\eta_r, \gamma_{r'} z)^k f(\gamma_{r'} z) = \chi(\eta_r) j(\eta_r, \gamma_{r'} z)^k f\left(z + \frac{r'}{q}\right).$$

Moreover,

$$\chi(\eta_r) j(\eta_r, \gamma_{r'} z) = \chi\left(d - \frac{cr'}{q}\right)\left(c\gamma_{r'} z + d - \frac{cr'}{q}\right) = \chi(d)(cz+d) = \chi(\gamma) j(\gamma, z).$$

Together these two computations imply

$$f\left(\gamma z+\frac{r}{q}\right)=\chi(\gamma)j(\gamma,z)^k f\left(z+\frac{r'}{q}\right).$$

Now, as $ar'\equiv dr \pmod q$ and $ad\equiv 1 \pmod q$, we have

$$\overline{\psi}(r)=\overline{\psi}(a\overline{d}r')=\psi^2(d)\overline{\psi}(r')=\psi^2(\gamma)\overline{\psi}(r').$$

Putting everything together,

$$\begin{aligned}(f\otimes\psi)(\gamma z)&=\frac{1}{\tau(\overline{\psi})}\sum_{r \pmod q}\overline{\psi}(r)f\left(\gamma z+\frac{r}{q}\right)\\
&=\chi(\gamma)j(\gamma,z)^k\frac{1}{\tau(\overline{\psi})}\sum_{r \pmod q}\overline{\psi}(r)f\left(z+\frac{r'}{q}\right)\\
&=\chi\psi^2(\gamma)j(\gamma,z)^k\frac{1}{\tau(\overline{\psi})}\sum_{r \pmod q}\overline{\psi}(r')f\left(z+\frac{r'}{q}\right)\\
&=\chi\psi^2(\gamma)j(\gamma,z)^k(f\otimes\psi)(z).\end{aligned}$$

from which the modularity of $f\otimes\psi$ follows. For the growth condition, let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the cusp $\mathfrak{a}$ of $\Gamma_0(Nq^2)\backslash\mathbb{H}$. As $\gamma_r\in\mathrm{GL}_2^+(\mathbb{Q})$, $\gamma_r\sigma_{\mathfrak{a}}$ takes ∞ to an element of $\mathbb{Q}\cup\{\infty\}$. Thus $\gamma_r\sigma_{\mathfrak{a}}\infty=\mathfrak{b}$ for some cusp $\mathfrak{b}$ of $\Gamma_0(N)\backslash\mathbb{H}$. Then as $j(\gamma_r,\sigma_{\mathfrak{a}}z)=1$, our previous work and the cocycle condition together imply

$$j(\sigma_{\mathfrak{a}},z)^{-k}(f\otimes\psi)(\sigma_{\mathfrak{a}}z)=\frac{1}{\tau(\overline{\psi})}\sum_{r \pmod q}\overline{\psi}(r)j(\gamma_r\sigma_{\mathfrak{a}},z)^{-k}f(\gamma_r\sigma_{\mathfrak{a}}z),$$

and the growth condition follows from that of f. It is also clear that $f\otimes\psi$ is a cusp form if f is. This proves the claim. $\square$

The generalization of Proposition 3.8.1 to all characters is slightly more involved. To this end, define operators U_p and V_p on $\mathcal{M}_k(\Gamma_1(N))$ to be the linear operators given by

$$(U_pf)(z)=\sum_{n\ge 0}a_{np}(f)e^{2\pi inz},$$

and

$$(V_pf)(z)=\sum_{n\ge 0}a_n(f)e^{2\pi inpz},$$

if f has Fourier series

$$f(z)=\sum_{n\ge 0}a_n(f)e^{2\pi inz}.$$

We will show that both U_p and V_p map $\mathcal{M}_k(\Gamma_1(N))$ into $\mathcal{M}_k(\Gamma_1(Np))$ and more:

Lemma 3.8.1. *For any prime p, U_p and V_p map $\mathcal{M}_k(\Gamma_1(N))$ into $\mathcal{M}_k(\Gamma_1(Np))$. In particular, U_p and V_p map $\mathcal{M}_k(N,\chi)$ into $\mathcal{M}_k(Np,\chi\chi_{p,0})$. Moreover, U_p and V_p preserve the subspace of cusp forms.*

Proof. In light of Proposition 3.5.1, the first statement follows from the second. As $N \mid Np$, $\Gamma_1(Np) \leq \Gamma_1(N)$ so that $f \in \mathcal{M}_k(\Gamma_1(Np))$ if $f \in \mathcal{M}_k(\Gamma_1(N))$. Now suppose $f \in \mathcal{M}_k(N,\chi)$. Similarly, $N \mid Np$ implies $\Gamma_0(Np) \leq \Gamma_0(N)$ so that $f \in \mathcal{M}_k(Np,\chi\chi_{p,0})$ for the modulus Np character $\chi\chi_{p,0}$. Therefore we may assume $f \in \mathcal{M}_k(Np,\chi\chi_{p,0})$. Now consider U_p. As $p \mid Np$, Equation (3.3) implies $U_p = T_p$ for the p-th Hecke operator on $\mathcal{M}_k(\Gamma_1(Np))$ and the claim follows from the definition of the Hecke operators and Proposition 3.5.4. Now consider V_p. We have

$$(V_p f)(z) = f(pz),$$

and the claim follows by regarding $f \in \mathcal{M}_k(Np,\chi\chi_{p,0})$ and that $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ lies in the center of $\mathrm{PSL}_2(\mathbb{Z})$. $\square$

We can now generalize Proposition 3.8.1 to all characters:

Proposition 3.8.2. *Suppose $f \in \mathcal{M}_k(N,\chi)$ and ψ is a Dirichlet character modulo M. Then $f \otimes \psi \in \mathcal{M}_k(NM^2,\chi\psi^2)$. Moreover, if f is a cusp form then so is $f \otimes \psi$.*

Proof. Let $\widetilde{\psi}$ be the primitive character of conductor q inducing ψ. Then $\psi = \widetilde{\psi}\psi_{\frac{M}{q},0}$. As $\psi_{\frac{M}{q},0} = \prod_{p \mid \frac{M}{q}} \psi_{p,0}$, it suffices to prove the claim when ψ is primitive and when $\psi = \psi_{p,0}$. The primitive case follows from Proposition 3.8.1. So suppose $\psi = \psi_{p,0}$. Then

$$f \otimes \psi_{p,0} = f - V_p U_p f.$$

Now by Lemma 3.8.1, $V_p U_p f \in \mathcal{M}_k(Np^2,\chi\psi_{p,0}^2)$ and is a cusp form if f is (where we have written $\psi_{p,0}^2$ in place of $\chi_{p,0}$). Since we also have $f \in \mathcal{M}_k(Np^2,\chi\psi_{p,0}^2)$ (because $N \mid Np^2$ so that $\Gamma_1(Np^2) \leq \Gamma_1(N)$ and $\Gamma_0(Np^2) \leq \Gamma_0(N)$ and again writing $\psi_{p,0}^2$ in place of $\chi_{p,0}$), it follows that $f \otimes \psi_{p,0} \in \mathcal{M}_k(Np^2,\chi\psi_{p,0}^2)$ and is a cusp form if f is. This proves the claim in the case $\psi = \psi_{p,0}$ and thus completes the proof. $\square$

Chapter 4

The Theory of Maass Forms

Maass forms are the non-holomorphic analog to holomorphic forms. They are real-analytic, eigenfunctions for a differential operator, invariant with respect to a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, and satisfy a growth condition. We introduce both Maass forms and their general theory.

4.1 Maass Forms

Define $\varepsilon(\gamma, z)$ by

$$\varepsilon(\gamma, z) = \left(\frac{cz+d}{|cz+d|}\right),$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ and $z \in \mathbb{H}$. Note that $|\varepsilon(\gamma, z)| = 1$. Moreover, we have the relation

$$\varepsilon(\gamma, z) = \left(\frac{j(\gamma, z)}{|j(\gamma, z)|}\right).$$

As a consequence, the cocycle condition for $j(\gamma, z)$ implies

$$\varepsilon(\gamma'\gamma, z) = \varepsilon(\gamma', \gamma z)\varepsilon(\gamma, z),$$

and this is called the **cocycle condition** for $\varepsilon(\gamma, z)$. For any $k \in \mathbb{Z}$ and any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ we define the **slash operator** $|_{\varepsilon,k} : C(\mathbb{H}) \to C(\mathbb{H})$ to be the linear operator given by

$$(f|_{\varepsilon,k}\gamma)(z) = \varepsilon(\gamma, z)^{-k} f(\gamma z).$$

If ε and k are clear from content we will suppress this dependencies accordingly. Any operator that commutes with the slash operators $|_{\varepsilon,k}\gamma$ for every $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ is said to be **invariant**. We define differential operators $R_k : C^\infty(\mathbb{H}) \to C^\infty(\mathbb{H})$ and $L_k : C^\infty(\mathbb{H}) \to C^\infty(\mathbb{H})$ to be the linear operators given by

$$R_k = \frac{k}{2} + y\left(i\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \quad \text{and} \quad L_k = \frac{k}{2} + y\left(i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right).$$

We call these operators the **Maass differential operators**. In particular, R_k is the **Maass raising operator** and L_k is the **Mass lowering operator**. The **Laplace operator** $\Delta_k : C^\infty(\mathbb{H}) \to C^\infty(\mathbb{H})$ is the linear operator given by

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}.$$

When $k=0$, we will write $\Delta_0 = \Delta$ which is the usual Laplace operator on $\mathbb{H}$. Expanding the products $R_{k-2}L_k$ and $L_{k+2}R_k$ and invoking the Cauchy-Riemann equation

$$\frac{\partial}{\partial y} = i\frac{\partial}{\partial x},$$

we arrive at the identities

$$\Delta_k = -R_{k-2}L_k + \frac{k}{2}\left(1-\frac{k}{2}\right) \quad \text{and} \quad \Delta_k = -L_{k+2}R_k - \frac{k}{2}\left(1-\frac{k}{2}\right).$$

The Mass differential operators and the Laplace operator satisfy important relations (see [Bum97] for a proof):

Proposition 4.1.1. *The Laplace operator Δ_k is invariant. That is,*

$$\Delta_k(f|_{\varepsilon,k}\gamma) = \Delta_k(f)|_{\varepsilon,k}\gamma,$$

for all $f \in C^\infty(\mathbb{H})$ and $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. Moreover, the Maass differential operators R_k and L_k satisfy

$$(R_kf)|_{\varepsilon,k+2}\gamma = R_k(f|_{\varepsilon,k}\gamma) \quad \text{and} \quad (R_kf)|_{\varepsilon,k-2}\gamma = L_k(f|_{\varepsilon,k}\gamma).$$

Let Γ be a congruence subgroup of level N that is reduced at infinity and let χ be a Dirichlet character of conductor $q \mid N$. Set $\chi(\gamma) = \chi(d)$ for all for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is a **Maass form** on $\Gamma\backslash\mathbb{H}$ of **weight** k, **eigenvalue** λ, **level** N, and **character** χ, if the following properties are satisfied:

(i) f is smooth on $\mathbb{H}$.

(ii) $(f|_{\varepsilon,k}\gamma)(z) = \chi(\gamma)f(z)$ for all $\gamma \in \Gamma$.

(iii) f is an eigenfunction for Δ_k with eigenvalue λ.

(iv) $(f|_{\varepsilon,k}\alpha)(z) = o(e^{2\pi y})$ for all $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$ (or equivalently $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$).

We say f is a **(Maass) cusp form** if the additional property is satisfied:

(v) For all cusps $\mathfrak{a}$ and any $y > 0$, we have

$$\int_{iy}^{1+iy} (f|\sigma_{\mathfrak{a}})(z)\,dz = 0.$$

Property (ii) is called the **automorphy condition** and we say that f is **automorphic**. In particular, f admits a Fourier series and is a function on $\mathcal{F}_\Gamma$. The automorphy condition can equivalently be expressed as

$$f(\gamma z) = \chi(\gamma)\varepsilon(\gamma,z)^k f(z).$$

In property (iii), we will often let $s \in \mathbb{C}$ be such that $\lambda = s(1-s)$ and write $\lambda = \lambda(s)$ so that the eigenvalue can be determined by s. Property (iv) is called the **growth condition** for Maass forms and we say f has **moderate growth at the cusps**. Clearly we only need to verify the growth condition on a set of scaling matrices for the cusps. Moreover, the equivalence in the growth condition follows exactly in the same way as for holomorphic forms. Indeed, the decomposition $\alpha = \gamma\eta$ for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ with $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ and $\eta \in \mathrm{GL}_2^+(\mathbb{Q})$ of the form $\eta = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ along with the cocycle condition together imply

$$\varepsilon(\alpha, z) = \varepsilon(\gamma, \eta z),$$

and it follows that $(f|_{\varepsilon,k}\alpha)(z) = o(e^{2\pi y})$ for all $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ which proves the forward implication. The reverse implication is trivial since $\mathrm{PSL}_2(\mathbb{Z}) \subset \mathrm{GL}_2^+(\mathbb{Q})$. It turns turns out that Property (i) is implied by (iii). This is because Δ_k is an elliptic operator and any eigenfunction of an elliptic operator is automatically real-analytic and hence smooth (see [Eva22] for a proof in the weight zero case and [CS17] for notes on the general case).

Remark 4.1.1. *Holomorphic forms embed into Maass forms. If $f(z)$ is a weight k holomorphic form on $\Gamma\backslash\mathbb{H}$, then $F(z) = \mathrm{Im}(z)^{\frac{k}{2}} f(z)$ is a weight k Maass form on $\Gamma\backslash\mathbb{H}$. This is because $\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|j(\gamma,z)|^2}$ and $F(z)$ clearly has polynomial growth in $\mathrm{Im}(z)$. Moreover, as $f(z)$ is holomorphic, it satisfies the Cauchy-Riemann equations so that*

$$L_k(F(z)) = \left(\frac{k}{2} + y\left(i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\right)(F(z)) = \frac{k}{2}F(z) - \frac{k}{2}F(z) = 0.$$

Therefore

$$\Delta_k(F(z)) = \left(-R_{k-2}L_k + \frac{k}{2}\left(1 - \frac{k}{2}\right)\right)(F(z)) = \frac{k}{2}\left(1 - \frac{k}{2}\right)F(z).$$

This means $F(z)$ is an eigenfunction for Δ_k with eigenvalue $\lambda\left(\frac{k}{2}\right)$.

Maass forms also admit Fourier series. Indeed, automorphy implies

$$f(z+1) = f\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z\right) = f(z),$$

so that f is 1-periodic. Let $\sigma_{\mathfrak{a}}$ be a scaling matrix for the $\mathfrak{a}$ cusp. As Lemma 2.1.1 implies $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$ is a congruence subgroup, it follows by the cocycle condition that $f|\sigma_{\mathfrak{a}}$ is a Maass form on $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}\backslash\mathbb{H}$ of the same weight, eigenvalue, and character as f. In particular, $f|\sigma_{\mathfrak{a}}$ is 1-periodic. Note that this means we only need to verify the growth condition as $y \to \infty$. As $f|\sigma_{\mathfrak{a}}$ is smooth, it admits a Fourier series and thus f admits a **Fourier series at the $\mathfrak{a}$ cusp** given by

$$(f|\sigma_{\mathfrak{a}})(z) = \sum_{n \in \mathbb{Z}} a_{\mathfrak{a}}(n, y, s)e^{2\pi i n x},$$

and the sum is over all $n \in \mathbb{Z}$ since f may be unbounded as $z \to \infty$. The Fourier coefficients $a_{\mathfrak{a}}(n, y, s)$ are mostly determined by Δ_k. To see this, since $f|\sigma_{\mathfrak{a}}$ is smooth we may differentiate the Fourier series of $f|\sigma_{\mathfrak{a}}$ termwise. The fact that $f|\sigma_{\mathfrak{a}}$ is an eigenfunction for Δ_k with eigenvalue $\lambda(s)$ gives the ODE

$$(4\pi^2 n^2 y^2 - 2\pi n k y)a_{\mathfrak{a}}(n, y, s) - y^2 a_{\mathfrak{a},yy}(n, y, s) = \lambda(s)a_{\mathfrak{a}}(n, y, s).$$

If $n \neq 0$, this is a Whittaker equation. To see this, first put the ODE in homogeneous form

$$y^2 a_{\mathfrak{a},yy}(n, y, s) - (4\pi^2 n^2 y^2 - 2\pi n k y - \lambda(s))a_{\mathfrak{a}}(n, y, s) = 0.$$

Now make the change of variables $y \to \frac{y}{4\pi|n|}$ to get

$$y^2 a_{\mathfrak{a},yy}(n, 4\pi|n|y, s) - \left(\frac{y^2}{4} - \mathrm{sgn}(n)\frac{k}{2}y - \lambda(s)\right)a_{\mathfrak{a}}(n, 4\pi|n|y, s) = 0,$$

where $\mathrm{sgn}(n) = \pm 1$ if n is positive or negative respectively. Diving by y^2 results in

$$a_{\mathfrak{a},yy}(n, 4\pi|n|y, s) + \left(\frac{1}{4} - \frac{\mathrm{sgn}(n)\frac{k}{2}}{y} - \frac{\lambda(s)}{y^2}\right) a_{\mathfrak{a}}(n, 4\pi|n|y, s) = 0.$$

As $\lambda(s) = s(1-s) = \frac{1}{4} - (s - \frac{1}{2})^2$, the above equation becomes

$$a_{\mathfrak{a},yy}(n, 4\pi|n|y, s) + \left(\frac{1}{4} - \frac{\mathrm{sgn}(n)\frac{k}{2}}{y} - \frac{\frac{1}{4} - \left(s - \frac{1}{2}\right)^2}{y^2}\right) a_{\mathfrak{a}}(n, 4\pi|n|y, s) = 0.$$

This is the Whittaker equation (see Appendix B.8). Since f has moderate growth at the cusps, the general solution is the Whittaker function $W_{\mathrm{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y)$. Therefore

$$a_{\mathfrak{a}}(n, y, s) = a_{\mathfrak{a}}(n, s) W_{\mathrm{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y),$$

for some coefficients $a_{\mathfrak{a}}(n, s)$. If $n = 0$, then the differential equation is a second order linear ODE which is

$$-y^2 a_{\mathfrak{a},yy}(0, y, s) = \lambda(s) a_{\mathfrak{a}}(0, y, s).$$

This is a Cauchy-Euler equation, and since s and $1-s$ are the two roots of $z^2 - z + \lambda$, the general solution is

$$a_{\mathfrak{a}}(0, y, s) = a_{\mathfrak{a}}^{+}(0, s) y^s + a_{\mathfrak{a}}^{-}(0, s) y^{1-s},$$

The coefficients $a_{\mathfrak{a}}(n, s)$, $a_{\mathfrak{a}}^{+}(0, s)$, and $a_{\mathfrak{a}}^{-}(0, s)$ are the only part of the Fourier series that actually depend on the implicit congruence subgroup Γ. Using these coefficients, f admits **Fourier series at the $\mathfrak{a}$ cusp** given by

$$(f|\sigma_{\mathfrak{a}})(z) = a_{\mathfrak{a}}^{+}(0, s) y^s + a_{\mathfrak{a}}^{-}(0, s) y^{1-s} + \sum_{n \neq 0} a_{\mathfrak{a}}(n, s) W_{\mathrm{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y) e^{2\pi i n x}.$$

If $\mathfrak{a} = \infty$ or s is fixed, we will drop these dependencies accordingly. Moreover, property (v) implies that f is a cusp form if and only if $a_{\mathfrak{a}}^{+}(0, s) = 0$ and $a_{\mathfrak{a}}^{-}(0, s) = 0$ for every cusp $\mathfrak{a}$. We can also easily derive a bound for the size of the Fourier coefficients of cusp forms. Fix some $Y > 0$ and consider

$$\int_{\Gamma_\infty \backslash \mathbb{H}_Y} |(f|\sigma_{\mathfrak{a}})(z)|^2 \, d\mu,$$

where $\mathbb{H}_Y$ is the half-plane defined by $Y \leq \mathrm{Im}(z) \leq 2Y$. Since $\Gamma_\infty \backslash \mathbb{H}_Y$ is compact, this integral is absolutely bounded. Substituting in the Fourier series of f at the $\mathfrak{a}$ cusp, this integral can be expressed as

$$\int_Y^{2Y} \int_0^1 \sum_{n,m \geq 1} a_{\mathfrak{a}}(n, s) \overline{a_{\mathfrak{a}}(m, s)} W_{\mathrm{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y) \overline{W_{\mathrm{sgn}(m)\frac{k}{2}, s-\frac{1}{2}}(4\pi|m|y)} e^{2\pi i (n-m)x} \, \frac{dy}{y^2}.$$

Appealing to the dominated convergence theorem, we can interchange the sum and the two integrals. Upon making this interchange, the identity Equation (3.1) implies that the inner integral cuts off all of the terms except the diagonal $n = m$, resulting in

$$\sum_{n \geq 1} \int_Y^{2Y} |a_{\mathfrak{a}}(n, s)|^2 |W_{\mathrm{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y)|^2 \, \frac{dy}{y^2}.$$

In particular, we see that this is a sum of nonnegative terms and each integral is bounded. Retaining only a single term in the sum, we have

$$|a_{\mathfrak{a}}(n,s)|^2 \int_Y^{2Y} |W_{\mathrm{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)|^2 \frac{dy}{y^2} \ll \int_{\Gamma_\infty\backslash\mathbb{H}_Y} |(f|\sigma_{\mathfrak{a}})(z)|^2\,d\mu.$$

Moreover, $|(f|\sigma_{\mathfrak{a}})(z)|^2$ is bounded on $\Gamma_\infty\backslash\mathbb{H}_Y$, because this space is compact, so that

$$\int_{\Gamma_\infty\backslash\mathbb{H}_Y} |(f|\sigma_{\mathfrak{a}})(z)|^2\,d\mu \ll \int_Y^{2Y}\int_0^1 \frac{dx\,dy}{y^2} \ll \frac{1}{Y}.$$

Putting these two estimates together gives

$$|a_{\mathfrak{a}}(n,s)|^2 \int_Y^{2Y} |W_{\mathrm{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)|^2 \frac{dy}{y^2} \ll \frac{1}{Y}.$$

Taking $Y = \frac{1}{|n|}$ and making the change of variables $y \to \frac{y}{|n|}$, we obtain

$$|a_{\mathfrak{a}}(n,s)| \ll 1.$$

This bound is known as the **Hecke bound** for Maass forms. Using Lemma B.8.1 again, we have the estimate $W_{\mathrm{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y) = O((|n|y)^{\frac{k}{2}}e^{-2\pi|n|y})$. This estimate together with the Hecke bound and the Taylor series of $\frac{1}{1-e^y}$ along with its derivatives, it follows that

$$(f|\sigma_{\mathfrak{a}})(z) = O\left(y^{\frac{k}{2}}\sum_{n\neq 0}|n|^{\frac{k+1}{2}}e^{-2\pi|n|y}\right) = O\left(y^{\frac{k}{2}}\sum_{n\geq 1}n^k e^{-2\pi ny}\right) = O\left(\frac{y^{\frac{k}{2}}e^{-2\pi y}}{(1-e^{-2\pi y})^2}\right) = O(y^{\frac{k}{2}}e^{-2\pi y}).$$

Observe that $f|\sigma_{\mathfrak{a}}$ is then bounded on $\mathbb{H}$ and, in particular, f is bounded on $\mathbb{H}$. It is useful to have alternative Fourier series in the case of weight zero Maass forms that specify the Whittaker function explicitly. When $k=0$, Theorem B.8.1 implies that f admits a **Fourier series at the $\mathfrak{a}$ cusp** given by

$$(f|\sigma_{\mathfrak{a}})(z) = a_{\mathfrak{a}}^+(0,s)y^s + a_{\mathfrak{a}}^-(0,s)y^{1-s} + \sum_{n\neq 0} a_{\mathfrak{a}}(n,s)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x},$$

for some different Fourier coefficients $a_{\mathfrak{a}}(n,s)$ (they are $\sqrt{4|n|}$ multiples). If $\mathfrak{a}=\infty$ or s is fixed, we will drop these dependencies accordingly. The Fourier series can be further simplified. We say f is **even** if $f(-\overline{z}) = f(z)$ and is **odd** if $f(-\overline{z}) = -f(z)$. Since $-\overline{z} = -x+iy$, this means that f is even or odd with respect to x. Note that if f is odd it must be a cusp form. As $e^{2\pi i n x} = \cos(nx) + i\sin(nx)$, if f has eigenvalue $\lambda = s(1-s)$, then the Fourier series of f at the $\mathfrak{a}$ cusp takes the form

$$(f|\sigma_{\mathfrak{a}})(z) = a_{\mathfrak{a}}^+(0,s)y^s + a_{\mathfrak{a}}^-(0,s)y^{1-s} + \sum_{n\neq 0} a_{\mathfrak{a}}(n,s)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)\cos\left(\frac{2\pi nx}{t}\right),$$

if f is even, and

$$f(z) = \sum_{n\neq 0} a_{\mathfrak{a}}(n,s)i\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)\sin\left(\frac{2\pi nx}{t}\right),$$

if f is odd. Therefore, in even case $a_{\mathfrak{a}}(-n,s) = a_{\mathfrak{a}}(n,s)$ and in the odd case $-a_{\mathfrak{a}}(-n,s) = a_{\mathfrak{a}}(n,s)$ for all $n\geq 1$. Either way, f admits a **Fourier series at the $\mathfrak{a}$ cusp** given by

$$(f|\sigma_{\mathfrak{a}})(z) = a_{\mathfrak{a}}^+(0,s)y^s + a_{\mathfrak{a}}^-(0,s)y^{1-s} + \sum_{n\geq 1} a_{\mathfrak{a}}(n,s)\sqrt{y}K_{s-\frac{1}{2}}(2\pi ny)\,\mathrm{SC}\left(\frac{2\pi nx}{t}\right),$$

for some potentially different Fourier coefficients $a_{\mathfrak{a}}(n,s)$ (they are i multiples in the case f is odd) and where $\mathrm{SC}(x) = \cos(x)$ if f is even and $\mathrm{SC}(x) = \sin(x)$ if f is odd. As we have said, if f is odd or more generally a cusp form, then necessarily $a_{\mathfrak{a}}^{+}(0,s) = 0$ and $a_{\mathfrak{a}}^{-}(0,s) = 0$. From now on, if we are discussing an even or odd Maass form we will always mean this last Fourier series (and so the implicit weight is zero).

4.2 Poincaré & Eisenstein Series

Throughout, let Γ be a congruence subgroup of level N. We will introduce two classes of automorphic functions on $\Gamma\backslash\mathbb{H}$. The latter such functions will be Maass forms. Actually, all of these automorphic functions are defined on a large space $\mathbb{H} \times \{s \in \mathbb{C} : \sigma > 1\}$ and hence are functions of two variables. The special case functions will be Maass forms for fixed s. Let $m \geq 0$, $k \geq 0$, χ be a Dirichlet character with conductor $q \mid N$, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. Then the m-th **(automorphic) Poincaré series** $P_{m,k,\chi,\mathfrak{a}}(z,s)$ of weight k and character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp is defined by

$$P_{m,k,\chi,\mathfrak{a}}(z,s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s}e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}.$$

We call m the **index** of $P_{m,k,\chi,\mathfrak{a}}(z,s)$. If $k=0$, χ is the trivial character, or $\mathfrak{a} = \infty$, we will drop these dependencies accordingly. We first show that $P_{m,k,\chi,\mathfrak{a}}(z,s)$ is well-defined. It suffices to show that the summands are independent of the representatives γ and $\sigma_{\mathfrak{a}}$. This has already been accomplished when we introduced the holomorphic Poincaré series for $\overline{\chi}(\gamma)$ and $e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}$. Now just as with the holomorphic Poincaré series, the set of representatives of $\sigma_{\mathfrak{a}}^{-1}\gamma$ is $\Gamma_{\infty}\sigma_{\mathfrak{a}}^{-1}\gamma$ and it remains to verify independence from multiplication on the left by an element of Γ_{∞} namely η_{∞}. The cocycle relation implies

$$\varepsilon(\eta_{\infty}\sigma_{\mathfrak{a}}^{-1}\gamma, z) = \varepsilon(\eta_{\infty}, \sigma_{\mathfrak{a}}^{-1}\gamma z)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z) = \varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z),$$

where the last equality follows because $\varepsilon(\eta_{\infty}, \sigma_{\mathfrak{a}}^{-1}\gamma z) = 1$ as $j(\eta_{\infty}, \sigma_{\mathfrak{a}}^{-1}\gamma z) = 1$. Thus $\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)$ is independent of the representatives γ and $\sigma_{\mathfrak{a}}$. Lastly, we have

$$\mathrm{Im}(\eta_{\infty}\sigma_{\mathfrak{a}}^{-1}\gamma z) = \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z),$$

because η_{∞} does not affect the imaginary part as it acts by translation. Therefore $\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)$ is independent of the representatives γ and $\sigma_{\mathfrak{a}}$ as well. We conclude that $P_{m,k,\chi,\mathfrak{a}}(z,s)$ is well-defined. We claim $P_{m,k,\chi,\mathfrak{a}}(z,s)$ is also locally absolutely uniformly convergent for $z \in \mathbb{H}$ and $\sigma > 1$. To see this, first recall that $|e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}| = e^{-2\pi m\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)} < 1$. Then the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma$ yields

$$P_{m,k,\chi,\mathfrak{a}}(z,s) \ll \sum_{(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \frac{\mathrm{Im}(z)^{\sigma}}{|cz+d|^{2\sigma}},$$

and this latter series is locally absolutely uniformly convergent for $z \in \mathbb{H}$ and $\sigma > 1$ by Proposition B.9.1. Hence the same holds for $P_{m,k,\chi,\mathfrak{a}}(z,s)$. Verifying automorphy amounts to a computation:

$$\begin{aligned}
P_{m,k,\chi,\mathfrak{a}}(\gamma z,s) &= \sum_{\gamma' \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma')\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma',\gamma z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \sum_{\gamma' \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma')\left(\frac{\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma,z)}{\varepsilon(\gamma,z)}\right)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \varepsilon(\gamma,z)^k \sum_{\gamma' \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma')\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma,z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \chi(\gamma)\varepsilon(\gamma,z)^k \sum_{\gamma' \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma')\overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma,z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \chi(\gamma)\varepsilon(\gamma,z)^k \sum_{\gamma' \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma'\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma,z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \chi(\gamma)\varepsilon(\gamma,z)^k \sum_{\gamma' \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma')\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma',z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma' z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma'\gamma z} \\
&= \chi(\gamma)\varepsilon(\gamma,z)^k P_{m,\chi,\mathfrak{a}}(z,s),
\end{aligned}$$

where in the second line we have used the cocycle condition and in the second to last line we have used that $\gamma' \to \gamma'\gamma^{-1}$ is a bijection on Γ. As for the growth condition, let $\sigma_{\mathfrak{b}}$ be a scaling matrix for the cusp $\mathfrak{b}$. Then the bound $|e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z}| = e^{-2\pi m\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)} < 1$, cocycle condition, and the Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ together give

$$\varepsilon(\sigma_{\mathfrak{b}},z)^{-k}P_{m,k,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \ll \mathrm{Im}(z)^\sigma \sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{|cz+d|^{2\sigma}}.$$

Now decompose this sum as

$$\sum_{(c,d)\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{|cz+d|^{2\sigma}} = \sum_{d\neq 0}\frac{1}{d^{2\sigma}} + \sum_{c\neq 0}\sum_{d\in\mathbb{Z}}\frac{1}{|cz+d|^{2\sigma}} = 2\sum_{d\geq 1}\frac{1}{d^{2\sigma}} + 2\sum_{c\geq 1}\sum_{d\in\mathbb{Z}}\frac{1}{|cz+d|^{2\sigma}}.$$

Notice that the first sum is absolutely uniformly bounded provided $\sigma > 1$. Moreover, the exact same argument as for holomorphic Eisenstein series shows that the second sum is too. So for all $\mathrm{Im}(z) \geq 1$ and $\sigma > 1$, we have

$$\varepsilon(\sigma_{\mathfrak{b}},z)^{-k}P_{m,k,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \ll \mathrm{Im}(z)^\sigma = o(e^{2\pi\mathrm{Im}(z)}),$$

provided $\mathrm{Im}(z) \geq 1$ and $\sigma > 1$. This verifies the growth condition. We collect this work as a theorem:

Theorem 4.2.1. *Let $m \geq 0$, $k \geq 0$, χ be a Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. For $\sigma > 1$, the Poincaré series*

$$P_{m,k,\chi,\mathfrak{a}}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z},$$

is a smooth automorphic function on $\Gamma\backslash\mathbb{H}$.

For $m = 0$, we write $E_{k,\chi,\mathfrak{a}}(z,s) = P_{0,k,\chi,\mathfrak{a}}(z,s)$ and call $E_{k,\chi}(z)$ the **(Maass) Eisenstein series** of weight k and character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp. It is defined by

$$E_{k,\chi,\mathfrak{a}}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s.$$

If $k=0$, χ is the trivial character, or $\mathfrak{a}=\infty$, we will drop these dependencies accordingly. It turns out that $E_{k,\chi,\mathfrak{a}}(z,s)$ is actually a Maass form. The only thing left to verify is that $E_{k,\chi,\mathfrak{a}}(z,s)$ is an eigenfunction for Δ_k. To see this, first observe that

$$\Delta_k(y^s) = \left(-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\frac{\partial}{\partial x}\right)(y^s) = \lambda(s)y^s.$$

Therefore $\mathrm{Im}(z)^s$ is an eigenfunction for Δ_k with eigenvalue $\lambda(s)$. Since Δ_k is invariant,

$$\Delta_k((\mathrm{Im}(\cdot)^s|_{\varepsilon,k}\gamma)(z)) = ((\Delta_k\mathrm{Im}(\cdot)^s)|_{\varepsilon,k}\gamma)(z) = \lambda(s)(\mathrm{Im}(\cdot)^s|_{\varepsilon,k}\gamma)(z),$$

and so $(\mathrm{Im}(\cdot)^s|_{\varepsilon,k}\gamma)(z) = \varepsilon(\gamma,z)^{-k}\mathrm{Im}(\gamma z)^s$ is also an eigenfunction for Δ_k with eigenvalue $\lambda(s)$ for all $\gamma\in\mathrm{PSL}_2(\mathbb{Z})$. We immediately conclude that

$$\Delta_k(E_{k,\chi,\mathfrak{a}}(z,s)) = \lambda(s)E_{k,\chi,\mathfrak{a}}(z,s),$$

which shows $E_{k,\chi,\mathfrak{a}}(z,s)$ is also an eigenfunction for Δ_k with eigenvalue $\lambda(s)$. We collect this work as a theorem:

Theorem 4.2.2. *Let $k\geq 0$, χ be a Dirichlet character with conductor dividing the level, and $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$. For $\sigma>1$, the Eisenstein series*

$$E_{k,\chi,\mathfrak{a}}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-k}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s,$$

is a weight k Maass form with eigenvalue $\lambda(s)$ and character χ on $\Gamma\backslash\mathbb{H}$.

4.3 Inner Product Spaces of Automorphic Functions

We will describe the space of automorphic functions and appropriate subspaces of interest. Let Γ be a congruence subgroup of level N and let χ be a Dirichlet character of conductor $q\mid N$. We will let $\mathcal{A}_k(\Gamma,\chi)$ denote the space of all automorphic functions of weight k and character χ on $\Gamma\backslash\mathbb{H}$. Let $\mathcal{L}_k(\Gamma,\chi)$ be the subspace of $\mathcal{A}_k(\Gamma,\chi)$ consisting of those functions with bounded norm where the norm is given by

$$||f|| = \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}|f(z)|^2\,d\mu\right)^{\frac{1}{2}}.$$

Moreover, if χ is the trivial character of if $k=0$, we will suppress the dependencies accordingly. We will also write $\mathcal{A}_k(N,\chi)$ and $\mathcal{L}_k(N,\chi)$ respectively when $\Gamma=\Gamma_0(N)$. Since f is automorphic, the integral Γ-invariant and hence is independent of the choice of fundamental domain. Since this is an L^2-space, $\mathcal{L}_k(\Gamma,\chi)$ is an induced inner product space (because the parallelogram law is satisfied). In particular, for any $f,g\in\mathcal{L}_k(\Gamma,\chi)$ we define their **Petersson inner product** to be

$$\langle f,g\rangle_\Gamma = \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}f(z)\overline{g(z)}\,d\mu.$$

If the congruence subgroup is clear from context we will suppress the dependence upon Γ. The integral is locally absolutely uniformly convergent by the Cauchy–Schwarz inequality and that $f,g\in\mathcal{L}_k(\Gamma,\chi)$. As f and g are automorphic, the integral is independent of the choice of fundamental domain. These two facts imply that the Petersson inner product is well-defined. We will continue to use this notion even if f and g do not belong to $\mathcal{L}_k(\Gamma,\chi)$ provided the integral is locally absolutely uniformly convergent. Just as was the case for holomorphic forms, the Petersson inner product is invariant with respect to the slash operator:

Proposition 4.3.1. *For any $f, g \in \mathcal{L}_k(\Gamma, \chi)$ and $\alpha \in \mathrm{PSL}_2(\mathbb{Z})$, we have*

$$\langle f|\alpha, g|\alpha\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g\rangle_\Gamma.$$

Proof. This is just a computation:

$$\begin{aligned}
\langle f|\alpha, g|\alpha\rangle_{\alpha^{-1}\Gamma\alpha} &= \frac{1}{V_{\alpha^{-1}\Gamma\alpha}} \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} (f|\alpha)(z)\overline{(g|\alpha)(z)}\, d\mu \\
&= \frac{1}{V_\Gamma} \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} (f|\alpha)(z)\overline{(g|\alpha)(z)}\, d\mu && \text{Lemma 2.3.1} \\
&= \frac{1}{V_\Gamma} \int_{\mathcal{F}_{\alpha^{-1}\Gamma\alpha}} f(\alpha z)\overline{g(\alpha z)}\, d\mu && \frac{\overline{\varepsilon(\alpha, z)}}{\varepsilon(\alpha, z)} = 1 \\
&= \frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} f(z)\overline{g(z)}\, d\mu && z \to \alpha^{-1} z \\
&= \langle f, g\rangle_\Gamma.
\end{aligned}$$

□

More importantly, the Petersson inner product turns $\mathcal{L}_k(\Gamma, \chi)$ into a Hilbert space:

Proposition 4.3.2. *$\mathcal{L}_k(\Gamma, \chi)$ is a Hilbert space with respect to the Petersson inner product.*

Proof. To show that the Petersson inner product is a Hermitian inner product on $\mathcal{L}_k(\Gamma, \chi)$, just mimic the corresponding part of proof of Proposition 3.3.2 with $k = 0$. We now show that $\mathcal{L}_k(\Gamma, \chi)$ is complete. Let $(f_n)_{n\ge1}$ be a Cauchy sequence in $\mathcal{L}_k(\Gamma, \chi)$. Then $||f_n - f_m|| \to 0$ as $n, m \to \infty$. But

$$||f_n - f_m|| = \left(\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} |f_n(z) - f_m(z)|^2\, d\mu\right)^{\frac{1}{2}},$$

and this integral tends to zero if and only if $|f_n(z) - f_m(z)| \to 0$ as $n, m \to \infty$. Therefore $\lim_{n\to\infty} f_n(z)$ exists and we define the limiting function f by $f(z) = \lim_{n\to\infty} f_n(z)$. We claim that f is automorphic. Indeed, as the f_n are automorphic, we have

$$f(\gamma z) = \lim_{n\to\infty} f_n(\gamma z) = \lim_{n\to\infty} \chi(\gamma)\varepsilon(\gamma, z)^k f_n(z) = \chi(\gamma)\varepsilon(\gamma, z)^k \lim_{n\to\infty} f_n(z) = \chi(\gamma)\varepsilon(\gamma, z)^k f(z),$$

for any $\gamma \in \Gamma$. Also, $||f|| < \infty$. To see this, since $(f_n)_{n\ge1}$ is Cauchy we know $(||f_n||)_{n\ge1}$ converges. In particular, $\lim_{n\to\infty} ||f_n|| < \infty$. But

$$\lim_{n\to\infty} ||f_n|| = \lim_{n\to\infty} \left(\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} |f_n(z)|^2\, d\mu\right)^{\frac{1}{2}} = \left(\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} \left|\lim_{n\to\infty} f_n(z)\right|^2\, d\mu\right)^{\frac{1}{2}} = \left(\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} |f(z)|^2\, d\mu\right)^{\frac{1}{2}} = ||f||,$$

where the second equality holds by the dominated convergence theorem. Hence $||f|| < \infty$ as desired and so $f \in \mathcal{L}_k(\Gamma, \chi)$. We now show that $f_n \to f$ in the L^2-norm. Indeed,

$$||f(z) - f_n(z)|| = \left(\frac{1}{V_\Gamma} \int_{\mathcal{F}_\Gamma} |f(z) - f_n(z)|^2\, d\mu\right)^{\frac{1}{2}},$$

and it follows that $||f(z) - f_n(z)|| \to 0$ as $n \to \infty$ so that the Cauchy sequence $(f_n)_{n\ge1}$ converges. □

We will need two more subspaces. Let $\mathcal{B}_k(\Gamma, \chi)$ be the subspace of $\mathcal{A}_k(\Gamma, \chi)$ such that f is smooth and bounded and let $\mathcal{D}_k(\Gamma, \chi)$ be the subspace of $\mathcal{A}_k(\Gamma, \chi)$ such that f and $\Delta_k f$ are smooth and bounded. Again, if χ is the trivial character of if $k = 0$, we will suppress the dependencies accordingly. Since boundedness on $\mathbb{H}$ implies square-integrability over $\mathcal{F}_\Gamma$, we have the following chain of inclusions:

$$\mathcal{D}_k(\Gamma, \chi) \subseteq \mathcal{B}_k(\Gamma, \chi) \subseteq \mathcal{L}_k(\Gamma, \chi) \subseteq \mathcal{A}_k(\Gamma, \chi).$$

Moreover, $\mathcal{D}_k(\Gamma, \chi)$ is almost all of $\mathcal{L}_k(\Gamma, \chi)$ as the following proposition shows:

Proposition 4.3.3. *$\mathcal{D}_k(\Gamma, \chi)$ is dense in $\mathcal{L}_k(\Gamma, \chi)$.*

Proof. Note that $\mathcal{D}_k(\Gamma, \chi)$ is an algebra of functions that vanish at infinity. We will show that $\mathcal{D}_k(\Gamma, \chi)$ is nowhere vanishing, separates points, and is self-adjoint. For nowhere vanishing fix a $z \in \mathbb{H}$. Let φ_z be a bump function defined on some sufficiently small neighborhood U_z of z. Then

$$\Phi(v) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \overline{\chi}(\gamma)\varepsilon(\gamma, v)^{-k}\varphi_z(\gamma v),$$

belongs to $\mathcal{D}_k(\Gamma, \chi)$ and is nonzero at z (the automorphy follows exactly as in the case of Eisenstein series). We now show $\mathcal{D}_k(\Gamma, \chi)$ also separates points. To see this consider two distinct points $z, w \in \mathbb{H}$. Let $U_{z,w}$ be a small neighborhood of z not containing w. Then $\Phi_z\, |_{U_{z,w}}$ belongs to $\mathcal{D}_k(\Gamma, \chi)$ with $\Phi_z\, |_{U_{z,w}}\, (z) \neq 0$ and $\varphi_z\, |_{U_{z,w}}\, (w) = 0$. To see why $\mathcal{D}_k(\Gamma, \chi)$ is self-adjoint, recall that complex conjugation is smooth and commutes with partial derivatives so that if f belongs to $\mathcal{D}_k(\Gamma, \chi)$ then so does $\overline{f}$. Therefore the Stone–Weierstrass theorem for complex functions defined on locally compact Hausdorff spaces (as $\mathbb{H}$ is a locally compact Hausdorff space) implies that $\mathcal{D}_k(\Gamma, \chi)$ is dense in $C_0(\mathbb{H})$ with the supremum norm. Note that $\mathcal{L}_k(\Gamma, \chi) \subseteq C_0(\mathbb{H})$ on the level of sets. Now we show $\mathcal{D}_k(\Gamma, \chi)$ is dense in $\mathcal{L}_k(\Gamma, \chi)$. Let $f \in \mathcal{L}_k(\Gamma, \chi)$. By what we have just show, there exists a sequence $(f_n)_{n\geq 1}$ in $\mathcal{D}_k(\Gamma, \chi)$ converging to f in the supremum norm. But

$$||f - f_n|| = \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} |f(z) - f_n(z)|^2\, d\mu\right)^{\frac{1}{2}} \leq \left(\frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} \sup_{z\in\mathcal{F}_\Gamma} |f(z) - f_n(z)|^2\, d\mu\right)^{\frac{1}{2}},$$

and the last expression tends to zero as $n \to \infty$ because $f_n \to f$ in the supremum norm. ☐

As $\mathcal{D}_k(\Gamma, \chi) \subseteq \mathcal{B}_k(\Gamma, \chi)$, Proposition 4.3.3 implies that $\mathcal{B}_k(\Gamma, \chi)$ is dense in $\mathcal{L}_k(\Gamma, \chi)$ too. It can be shown that the Laplace operator Δ_k is positive and symmetric on $\mathcal{D}_k(\Gamma, \chi)$ and hence admits a self-adjoint extension to $\mathcal{L}_k(\Gamma, \chi)$ (see [Iwa02] for a proof in the weight zero case and [CS17] for notes on the general case):

Theorem 4.3.1. *On $\mathcal{L}_k(\Gamma, \chi)$, the Laplace operator Δ_k is positive semi-definite and self-adjoint.*

If we suppose $f \in \mathcal{L}_k(\Gamma, \chi)$ is an eigenfunction for Δ_k with eigenvalue λ, then Theorem 4.3.1 implies λ is real and positive. Since $\lambda = s(1-s)$ and λ is real, s and $1-s$ are either conjugates or real. In the former case, $s = 1 - \overline{s}$ and we find that

$$\sigma = 1 - \sigma \quad \text{and} \quad t = t.$$

Therefore $s = \frac{1}{2} + it$. In the later case, since s is real and λ is positive we must have $s \in (0, 1)$. It follows that in either case, we may write $\lambda = \frac{1}{4} + r^2$ and $s = \frac{1}{2} + \nu$ for unique r and ν satisfying $r \in \mathbb{R}$ or $ir \in \left[0, \frac{1}{2}\right)$ and $\nu \in i\mathbb{R}$ or $\nu \in \left[0, \frac{1}{2}\right)$ corresponding to the two cases respectively. In particular, we also have $\lambda = \frac{1}{4} - \nu^2$

and $\nu = ir$. We refer to r as the **spectral parameter** of f and ν as the **type** of f. We collect the ways of expressing λ below:

$$\lambda = s(1-s) = \frac{1}{4} - \nu^2 = \frac{1}{4} + r^2.$$

We now introduce variations of the Poincaré and Eisenstein series. Let $m \geq 0$, $k \geq 0$, χ be a Dirichlet character with conductor $q \mid N$, $\sigma_{\mathfrak{a}}$ be a scaling matrix for the $\mathfrak{a}$ cusp, and $\psi(y)$ be a smooth function such that $\psi(y) \ll_{\varepsilon} y^{1+\varepsilon}$ as $y \to 0$ Then the m-th **(automorphic) Poincaré series** $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ of weight k and character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp and with respect to $\psi(y)$ is defined by

$$P_{m,k,\chi,\mathfrak{a}}(z,\psi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}.$$

If $k = 0$, χ is the trivial character, or $\mathfrak{a} = \infty$, we will drop these dependencies accordingly. Moreover, if $\psi(y)$ is a bump function, we say that $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is **incomplete**. We claim that $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is well-defined. This is easy to see as we have already showed $\overline{\chi}(\gamma)$, $\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}$, $\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)$, and $e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}$, are all independent of representatives for γ and $\sigma_{\mathfrak{a}}$ when discussing the automorphic Poincaré series. So $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is well-defined. We claim $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is also locally absolutely uniformly convergent for $z \in \mathbb{H}$. To see this, we require a technical lemma:

Lemma 4.3.1. *For any compact subset K of $\mathbb{H}$, there are finitely many pairs $(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}$, with $c \neq 0$, for which*

$$\frac{\mathrm{Im}(z)}{|cz+d|^2} > 1,$$

for all $z \in K$.

Proof. Let $\beta = \sup_{z \in K} |z|$. As $|cz+d| \geq |cz| > 0$ and $\mathrm{Im}(z) < |z|$, we have

$$\frac{\mathrm{Im}(z)}{|cz+d|^2} \leq \frac{1}{|c^2 z|} \leq \frac{1}{|c|^2\beta}.$$

So if $\frac{\mathrm{Im}(z)}{|cz+d|^2} > 1$, then $\frac{1}{|c|^2\beta} > 1$ which is to say $|c| < \frac{1}{\sqrt{\beta}}$ and therefore $|c|$ is bounded. On the other hand, $|cz+d| \geq |d| \geq 0$. Excluding the finitely many terms $(c,0)$, we may assume $|d| > 0$. In this case, similarly

$$\frac{\mathrm{Im}(z)}{|cz+d|^2} \leq \left|\frac{z}{d^2}\right| \leq \frac{\beta}{|d|^2}.$$

So if $\frac{\mathrm{Im}(z)}{|cz+d|^2} > 1$, then $\frac{\beta}{|d|^2} > 1$ which is to say $|d| < \sqrt{\beta}$. So $|d|$ is also bounded. Since both $|c|$ and $|d|$ are bounded, the claim follows. $\square$

Now we are ready to show that $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is locally absolutely uniformly convergent for $z \in \mathbb{H}$. Let K be a compact subset of $\mathbb{H}$. Then it suffices to show $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is uniformly convergent on K. The bound $|e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z}| = e^{-2\pi m \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)} < 1$ and the Bruhat decomposition applied to $\sigma_{\mathfrak{a}}^{-1}\Gamma$ together give

$$P_{k,\chi,\mathfrak{a}}(z,\psi) \ll \psi(\mathrm{Im}(z)) + \sum_{(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right).$$

It now further suffices to show that the latter series above is uniformly convergent on K. By Lemma 4.3.1, there are all but finitely many terms in the sum with $\psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right) \ll_{\varepsilon} \left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)^{1+\varepsilon}$. But the finitely many other terms are all uniformly bounded on K because $\psi(y)$ is continuous (as it is smooth). Therefore

$$\sum_{(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \psi\left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right) \ll \sum_{(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)^{1+\varepsilon} \ll \sum_{(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}} \left(\frac{\mathrm{Im}(z)}{|cz+d|^2}\right)^{1+\varepsilon},$$

and this last series is locally absolutely uniformly convergent for $z \in \mathbb{H}$ by Proposition B.9.1. It follows that $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is too. Actually, we can do better if $\psi(y)$ is a bump function since finitely many terms will be nonzero. Indeed, $\sigma_{\mathfrak{a}}^{-1}\Gamma$ is a Fuchsian group because it is a subset of the modular group. So from $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\backslash\Gamma = \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma$ we see that $\{\sigma_{\mathfrak{a}}^{-1}\gamma z : \gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma\}$ is discrete. Since $\mathrm{Im}(z)$ is an open map it takes discrete sets to discrete sets so that $\{\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z) : \gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma\}$ is also discrete. Now $\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))$ is nonzero if and only if $\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z) \in \mathrm{Supp}(\psi)$ and $\{\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z) : \gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma\} \cap \mathrm{Supp}(\psi)$ is finite as it is a discrete subset of a compact set (since $\psi(y)$ has compact support). Hence finitely many of the terms are nonzero. Moreover, the compact support of $\psi(y)$ then implies that $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is also compactly supported (since the function $\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))$ is continuous and $\mathbb{C}$ is Hausdorff) and hence bounded on $\mathbb{H}$. As a consequence, $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is L^2-integrable. We collect this work as a theorem:

Theorem 4.3.2. *Let $m \geq 0$, $k \geq 0$, χ be a Dirichlet character with conductor dividing the level, $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$, and $\psi(y)$ be a smooth function such that $\psi(y) \ll_\varepsilon y^{1+\varepsilon}$ as $y \to 0$. The Poincaré series*

$$P_{m,k,\chi,\mathfrak{a}}(z,\psi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))e^{2\pi i m\sigma_{\mathfrak{a}}^{-1}\gamma z},$$

is a smooth automorphic function on $\Gamma\backslash\mathbb{H}$. If $\psi(y)$ is a bump function, $P_{m,k,\chi,\mathfrak{a}}(z,\psi)$ is L^2-integrable.

For $m = 0$, we write $E_{k,\chi,\mathfrak{a}}(z,\psi) = P_{0,k,\chi,\mathfrak{a}}(z,\psi)$ and call $E_{k,\chi,\mathfrak{a}}(z,\psi)$ the **(automorphic) Eisenstein series** of weight k and character χ on $\Gamma\backslash\mathbb{H}$ at the $\mathfrak{a}$ cusp and with respect to $\psi(y)$. It is defined by

$$E_{k,\chi,\mathfrak{a}}(z,\psi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)).$$

If $k = 0$, χ is the trivial character, or $\mathfrak{a} = \infty$, we will drop these dependencies accordingly. Moreover, if $\psi(y)$ is a bump function, we say that $E_{k,\chi,\mathfrak{a}}(z,\psi)$ is **incomplete**. We have already verified the following theorem:

Theorem 4.3.3. *Let $k \geq 0$, χ be a Dirichlet character with conductor dividing the level, $\mathfrak{a}$ be a cusp of $\Gamma\backslash\mathbb{H}$, and $\psi(y)$ be a smooth function such that $\psi(y) \ll_\varepsilon y^{1+\varepsilon}$ as $y \to 0$. The Eisenstein series*

$$E_{k,\chi,\mathfrak{a}}(z,\psi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\chi}(\gamma)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)),$$

is a smooth automorphic function on $\Gamma\backslash\mathbb{H}$. If $\psi(y)$ is a bump function, $E_{k,\chi,\mathfrak{a}}(z,\psi)$ is also L^2-integrable.

Unfortunately, the Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,\psi)$ fail to be Maass forms because they are not eigenfunctions for the Laplace operator. This is because compactly supported functions cannot be real-analytic (which as we have already mentioned is implied for any eigenfunction of the Laplace operator). However, the incomplete Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,\psi)$ are L^2-integrable where as the Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ are not. This is the advantage in working with incomplete Eisenstein series. We will compute their inner product against an arbitrary element of $\mathcal{L}_k(\Gamma,\chi)$. Let $f \in \mathcal{L}_k(\Gamma,\chi)$ with eigenvalue $\lambda(s)$ and consider

$E_{k,\chi,\mathfrak{a}}(\cdot,\psi)$. We compute their inner product as follows:

$$\begin{aligned}
\langle f, E_{k,\chi,\mathfrak{a}}(\cdot,\psi)\rangle &= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma} f(z)\overline{E_{k,\chi,\mathfrak{a}}(z,\psi)}\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\chi(\gamma)\overline{\varepsilon(\sigma_\mathfrak{a}^{-1}\gamma,z)^{-k}}f(z)\overline{\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))}\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\chi(\gamma)\varepsilon(\sigma_\mathfrak{a}^{-1}\gamma,z)^{k}f(z)\overline{\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))}\,d\mu && \frac{\overline{\varepsilon(\sigma_\mathfrak{a}^{-1}\gamma,z)}}{\varepsilon(\sigma_\mathfrak{a}^{-1}\gamma,z)}=1 \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\left(\frac{\varepsilon(\sigma_\mathfrak{a}^{-1}\gamma,z)}{\varepsilon(\gamma,z)}\right)^{k}f(\gamma z)\overline{\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))}\,d\mu && \text{automorphy} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\varepsilon(\sigma_\mathfrak{a},\sigma_\mathfrak{a}^{-1}\gamma z)^{-k}f(\gamma z)\overline{\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))}\,d\mu && \text{cocycle condition} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\varepsilon(\sigma_\mathfrak{a},\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a} z)^{-k}f(\gamma\sigma_\mathfrak{a} z)\overline{\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma\sigma_\mathfrak{a} z))}\,d\mu && z\to\sigma_\mathfrak{a}z \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\infty\backslash\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}\varepsilon(\sigma_\mathfrak{a},\gamma z)^{-k}f(\sigma_\mathfrak{a}\gamma z)\overline{\psi(\mathrm{Im}(\gamma z))}\,d\mu && \gamma\to\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{a}^{-1} \\
&= \frac{1}{V_\Gamma}\int_{\mathcal{F}_{\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}}\sum_{\gamma\in\Gamma_\infty\backslash\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}}(f|\sigma_\mathfrak{a})(\gamma z)\overline{\psi(\mathrm{Im}(\gamma z))}\,d\mu \\
&= \frac{1}{V_\Gamma}\int_{\Gamma_\infty\backslash\mathbb{H}}(f|\sigma_\mathfrak{a})(z)\overline{\psi(\mathrm{Im}(z))}\,d\mu && \text{unfolding.}
\end{aligned}$$

Substituting in the Fourier series of f at the $\mathfrak{a}$ cusp, we obtain

$$= \frac{1}{V_\Gamma}\int_0^\infty\int_0^1\left(a_\mathfrak{a}^+(0,s)y^s + a_\mathfrak{a}^-(0,s)y^{1-s} + \sum_{n\neq 0}a_\mathfrak{a}(n,s)W_{\mathrm{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)e^{2\pi inx}\right)\overline{\psi(y)}\,\frac{dx\,dy}{y^2}.$$

By the dominated convergence theorem, we can interchange the sum and the two integrals. Upon making this interchange, the identity Equation (3.1) implies that the inner integral cuts off all of the terms in the sum, resulting in

$$\frac{1}{V_\Gamma}\int_0^\infty(a_\mathfrak{a}^+(0,s)y^s + a_\mathfrak{a}^-(0,s)y^{1-s})\overline{\psi(y)}\,\frac{dx\,dy}{y^2}.$$

This latter integral is precisely the constant term in the Fourier series of f at the $\mathfrak{a}$ cusp. It follows that f is orthogonal to $\mathcal{E}_k(\Gamma,\chi)$ if and only if f is a cusp form. This illustrates an important interaction between incomplete Eisenstein series and Maass cusp forms. To state it in another way, let $\mathcal{E}_k(\Gamma,\chi)$ and $\mathcal{C}_k(\Gamma,\chi)$ denote the subspaces of $\mathcal{A}_k(\Gamma,\chi)$ generated by such forms respectively. If χ is the trivial character or if the weight k is zero, we will suppress these dependencies. Note that they are also subspaces of $\mathcal{B}_k(\Gamma,\chi)$. Moreover, let $\mathcal{C}_{k,\nu}(\Gamma,\chi)$ and $\mathcal{A}_{k,\nu}(\Gamma,\chi)$ denote the corresponding subspaces of $\mathcal{C}_k(\Gamma,\chi)$ and $\mathcal{A}_k(\Gamma,\chi)$ whose type is ν. For all of these spaces, if χ is the trivial character or if the weight k is zero, we will suppress these dependencies as well. So if $f\in\mathcal{B}_k(\Gamma,\chi)$, then

$$\mathcal{B}_k(\Gamma,\chi) = \mathcal{E}_k(\Gamma,\chi)\oplus\mathcal{C}_k(\Gamma,\chi),$$

by what we have just shown. Moreover, as $\mathcal{B}_k(\Gamma,\chi)$ is dense in $\mathcal{L}_k(\Gamma,\chi)$, we have

$$\mathcal{L}_k(\Gamma,\chi) = \overline{\mathcal{E}_k(\Gamma,\chi)}\oplus\overline{\mathcal{C}_k(\Gamma,\chi)},$$

where the closure is with respect to the topology induced by the L^2-norm.

4.4 Spectral Theory of the Laplace Operator

We are now ready to discuss the spectral theory of the Laplace operator Δ_k. What we want to do is to decompose $\mathcal{L}_k(\Gamma,\chi)$ into subspaces invariant under Δ_k such that on each subspace Δ_k has either pure point spectrum, absolutely continuous spectrum, or residual spectrum. Although the proof is beyond the scope of this text, the spectral resolution of the Laplace operator on $\mathcal{C}_k(\Gamma,\chi)$ is as follows (see [Iwa02] for a proof in the weight zero case and [CS17] for notes on the general case):

Theorem 4.4.1. *The Laplace operator Δ_k has pure point spectrum on $\mathcal{C}_k(\Gamma,\chi)$. The corresponding subspaces $\mathcal{C}_{k,\nu}(\Gamma,\chi)$ are finite dimensional and mutually orthogonal. Letting $\{u_j\}_{j\geq 1}$ be an orthonormal basis of cusp forms for $\mathcal{C}_k(\Gamma,\chi)$, every $f\in\mathcal{C}_k(\Gamma,\chi)$ admits a series of the form*

$$f(z)=\sum_{j\geq 1}\langle f,u_j\rangle u_j(z),$$

which is locally absolutely uniformly convergent if $f\in\mathcal{D}_k(\Gamma,\chi)$ and convergent in the L^2-norm otherwise.

We will now discuss the spectrum of the Laplace operator on $\mathcal{E}_k(\Gamma,\chi)$. Essential is the meromorphic continuation of the Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ (see [Iwa02] for a proof in the weight zero case and [CS17] for notes on the general case):

Theorem 4.4.2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be cusps of $\Gamma\backslash\mathbb{H}$. The Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ admits meromorphic continuation to $\mathbb{C}$ in the s-plane, via a Fourier series at the $\mathfrak{b}$ cusp given by*

$$E_{k,\chi,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s)=\delta_{\mathfrak{a},\mathfrak{b}}y^s+\tau_{\mathfrak{a},\mathfrak{b}}(0,s)y^{1-s}+\sum_{n\neq 0}\tau_{\mathfrak{a},\mathfrak{b}}(n,s)W_{\mathrm{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)e^{2\pi inx},$$

where $\tau_{\mathfrak{a},\mathfrak{b}}(0,s)$ and $\tau_{\mathfrak{a},\mathfrak{b}}(n,s)$ are meromorphic functions.

The Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ also satisfy a functional equation. To state it we need some notation. Fix an ordering of the cusps $\mathfrak{a}$ of $\Gamma\backslash\mathbb{H}$ and define

$$\mathcal{E}(z,s)=(E_{k,\chi,\mathfrak{a}}(z,s))_{\mathfrak{a}}^t \quad\text{and}\quad \Phi(s)=(\tau_{\mathfrak{a},\mathfrak{b}}(0,s))_{\mathfrak{a},\mathfrak{b}}.$$

In other words, $\mathcal{E}(z,s)$ is the column vector of the Eisenstein series and $\Phi(s)$ is the square matrix of meromorphic functions $\tau_{\mathfrak{a},\mathfrak{b}}(0,s)$ described in Theorem 4.4.2. Then we have the following (see [Iwa02] for a proof in the weight zero case and [CS17] for notes on the general case):

Theorem 4.4.3. *The Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ of weight k and character χ on $\Gamma\backslash\mathbb{H}$ satisfy the functional equation*

$$\mathcal{E}(z,s)=\Phi(s)\mathcal{E}(z,1-s).$$

The matrix $\Phi(s)$ is symmetric and satisfies the functional equation

$$\Phi(s)\Phi(1-s)=I.$$

Moreover, it is unitary on the line $\sigma=\frac{1}{2}$ and hermitian if s is real.

As $\Phi(s)$ is symmetric by Theorem 4.4.3, if $\mathfrak{a} = \infty$ or $\mathfrak{b} = \infty$, we will suppress these dependencies for $\tau_{\mathfrak{a},\mathfrak{b}}$. Understanding the poles of $\tau_{\mathfrak{a},\mathfrak{b}}$ are also important for understanding the poles of the Eisenstein series $E_{k,\chi,\mathfrak{a}}(z,s)$ (see [Iwa02] for a proof in the weight zero case and [CS17] for notes on the general case):

Theorem 4.4.4. *The functions $\tau_{\mathfrak{a},\mathfrak{b}}(0,s)$ are meromorphic for $\sigma \geq \frac{1}{2}$ with a finite number of simple poles in the segment $(\frac{1}{2},1]$. A pole of $\tau_{\mathfrak{a},\mathfrak{b}}(0,s)$ is also a pole of $\tau_{\mathfrak{a},\mathfrak{a}}(0,s)$. Moreover, the poles of $E_{k,\chi,\mathfrak{a}}(z,s)$ are among the poles of $\tau_{\mathfrak{a},\mathfrak{a}}(0,s)$, $E_{k,\chi,\mathfrak{a}}(z,s)$ has no poles on the line $\sigma = \frac{1}{2}$, and the residues of $E_{k,\chi,\mathfrak{a}}(z,s)$ are Maass forms in $\mathcal{E}_k(\Gamma,\chi)$.*

To begin decomposing the space $\mathcal{E}_k(\Gamma,\chi)$, consider the subspace $C_0^\infty(\mathbb{R}_{>0})$ of $\mathcal{L}^2(\mathbb{R}_{>0})$ with the normalized standard inner product

$$\langle f,g\rangle = \frac{1}{2\pi}\int_0^\infty f(r)\overline{g(r)}\,dr,$$

for any $f,g \in C_0^\infty(\mathbb{R}_{>0})$. For each cusps $\mathfrak{a}$ of $\Gamma\backslash\mathbb{H}$ we associate the **Eisenstein transform** $E_{k,\chi,\mathfrak{a}} : C_0^\infty(\mathbb{R}_{>0}) \to \mathcal{A}_k(\Gamma,\chi)$ defined by

$$(E_{k,\chi,\mathfrak{a}}f)(z) = \frac{1}{4\pi}\int_0^\infty f(r)E_{k,\chi,\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)dr.$$

Clearly $E_{k,\chi,\mathfrak{a}}f$ is automorphic because $E_{k,\chi,\mathfrak{a}}(z,s)$ is. It is not too hard to show the following (see [Iwa02] for a proof in the weight zero case and [CS17] for notes on the general case):

Proposition 4.4.1. *If $f \in C_0^\infty(\mathbb{R}_{>0})$, then $E_{\mathfrak{a}}f$ is L^2-integrable over $\mathcal{F}_\Gamma$. That is, $E_{k,\chi,\mathfrak{a}}$ maps $C_0^\infty(\mathbb{R}_{>0})$ into $\mathcal{L}_k(\Gamma,\chi)$. Moreover,*

$$\langle E_{k,\chi,\mathfrak{a}}f, E_{k,\chi,\mathfrak{b}}g\rangle = \delta_{\mathfrak{a},\mathfrak{b}}\langle f,g\rangle,$$

for any $f,g \in C_0^\infty(\mathbb{R}_{>0})$ and any two cusps $\mathfrak{a}$ and $\mathfrak{b}$.

We let $\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ denote the image of the Eisenstein transform $E_{k,\chi,\mathfrak{a}}$. We call $\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ the **Eisenstein space** of $E_{k,\chi,\mathfrak{a}}(z,s)$. An immediate consequence of Proposition 4.4.1 is that the Eisenstein spaces for distinct cusps are orthogonal. Moreover, since $E_{k,\chi,\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)$ is an eigenfunction for the Laplace operator with eigenvalue $\lambda = \frac{1}{4}+r^2$, and f and $E_{k,\chi,\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)$ are smooth, the Leibniz integral rule implies

$$\Delta E_{k,\chi,\mathfrak{a}} = E_{k,\chi,\mathfrak{a}}M,$$

where $M : C_0^\infty(\mathbb{R}_{>0}) \to C_0^\infty(\mathbb{R}_{>0})$ is the multiplication operator given by

$$(Mf)(r) = \left(\frac{1}{4}+r^2\right)f(r),$$

for all $f \in C_0^\infty(\mathbb{R}_{>0})$. Therefore if $E_{k,\chi,\mathfrak{a}}f$ belongs to $\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ then so does $E_{k,\chi,\mathfrak{a}}(Mf)$. But as $f, Mf \in C_0^\infty(\mathbb{R}_{>0})$, this means $\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ is invariant under the Laplace operator. While the Eisenstein spaces are invariant, they do not make up all of $\mathcal{E}_k(\Gamma,\chi)$. By Theorem 4.4.4, the residues of the Eisenstein series belong to $\mathcal{E}_k(\Gamma,\chi)$. Let $\mathcal{R}_k(\Gamma,\chi)$ denote the subspace generated by the residues of these Eisenstein series. We call any element of $\mathcal{R}_k(\Gamma,\chi)$ a **(residual) Maass form** (by Theorem 4.4.4 they are Maass forms). Also let $\mathcal{R}_{k,s_j}(\Gamma,\chi)$ denote the subspace generated by those residues taken at $s = s_j$. For both of these subspaces, if χ is the trivial character of if $k = 0$, we will suppress the dependencies accordingly. Since there are finitely many cusps of $\Gamma\backslash\mathbb{H}$, each $\mathcal{R}_{k,s_j}(\Gamma,\chi)$ is finite dimensional. As the number of residues in $(\frac{1}{2},1]$ is finite by Theorem 4.4.4, it follows that $\mathcal{R}_k(\Gamma,\chi)$ is finite dimensional too. So $\mathcal{R}_k(\Gamma,\chi)$ decomposes as

$$\mathcal{R}_k(\Gamma,\chi) = \bigoplus_{\frac{1}{2}<s_j\leq 1}\mathcal{R}_{k,s_j}(\Gamma,\chi).$$

This decomposition is orthogonal because the Maass forms belonging to distinct subspaces $\mathcal{R}_{k,s_j}(\Gamma,\chi)$ have distinct eigenvalues and eigenfunctions of self-adjoint operators are orthogonal (recall that Δ_k is self-adjoint by Theorem 4.3.1). Also, each subspace $\mathcal{R}_{k,s_j}(\Gamma,\chi)$ is clearly invariant under the Laplace operator because its elements are Maass forms. The residual forms are particularly simple in the weight zero case (see [Iwa02] for a proof):

Proposition 4.4.2. *There is only one residual form in $\mathcal{R}(\Gamma,\chi)$. It is obtained from the residue at $s=1$ and it is the constant function.*

We are now ready for the spectral resolution. Although the proof is beyond the scope of this text, the spectral resolution of the Laplace operator on $\mathcal{E}_k(\Gamma,\chi)$ is as follows (see [Iwa02] for a proof in the weight zero case and [CS17] for notes on the general case):

Theorem 4.4.5. *$\mathcal{E}_k(\Gamma,\chi)$ admits the orthogonal decomposition*

$$\mathcal{E}_k(\Gamma,\chi)=\mathcal{R}_k(\Gamma,\chi)\bigoplus_{\mathfrak{a}}\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi),$$

where the direct sum is over the cusps of $\Gamma\backslash\mathbb{H}$. The Laplace operator Δ_k has discrete spectrum on $\mathcal{R}_k(\Gamma,\chi)$ in the segment $[0,\frac{1}{4})$ and has pure continuous spectrum on each Eisenstein space $\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ covering the segment $\left[\frac{1}{4},\infty\right)$ uniformly with multiplicity 1. Letting $\{u_j\}_{j\geq1}$ be an orthonormal basis residual Maass forms for $\mathcal{R}_k(\Gamma,\chi)$, every $f\in\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)$ admits a decomposition of the form

$$f(z)=\sum_{j\geq1}\langle f,u_j\rangle u_j(z)+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\left\langle f,E_{k,\chi,\mathfrak{a}}\left(\cdot,\frac{1}{2}+\nu\right)\right\rangle E_{k,\chi,\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)\,dr.$$

The series and integrals are locally absolutely uniformly convergent if $f\in\mathcal{D}_k(\Gamma,\chi)$ and convergent in the L^2-norm otherwise.

Combining Theorems 4.4.1 and 4.4.5 gives the full spectral resolution of $\mathcal{L}_k(\Gamma,\chi)$:

Theorem 4.4.6. *$\mathcal{B}_k(\Gamma,\chi)$ admits the orthogonal decomposition*

$$\mathcal{B}_k(\Gamma,\chi)=\mathcal{C}_k(\Gamma,\chi)\oplus\mathcal{R}_k(\Gamma,\chi)\bigoplus_{\mathfrak{a}}\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi),$$

where the sum is over all cusps of $\Gamma\backslash\mathbb{H}$. The Laplace operator has pure point spectrum on $\mathcal{C}_k(\Gamma,\chi)$, discrete spectrum on $\mathcal{R}_k(\Gamma,\chi)$, and absolutely continuous spectrum on $\mathcal{E}_k(\Gamma,\chi)$. Letting $\{u_j\}_{j\geq1}$ be an orthonormal basis of Maass forms for $\mathcal{C}_k(\Gamma,\chi)\oplus\mathcal{R}_k(\Gamma,\chi)$, any $f\in\mathcal{L}_k(\Gamma,\chi)$ has a series of the form

$$f(z)=\sum_{j\geq1}\langle f,u_j\rangle u_j(z)+\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\left\langle f,E_{k,\chi,\mathfrak{a}}\left(\cdot,\frac{1}{2}+\nu\right)\right\rangle E_{k,\chi,\mathfrak{a}}\left(z,\frac{1}{2}+ir\right)\,dr,$$

which is locally absolutely uniformly convergent if $f\in\mathcal{D}_k(\Gamma,\chi)$ and convergent in the L^2-norm otherwise. Moreover,

$$\mathcal{L}_k(\Gamma,\chi)=\overline{\mathcal{C}_k(\Gamma,\chi)}\oplus\overline{\mathcal{R}_k(\Gamma,\chi)}\bigoplus_{\mathfrak{a}}\overline{\mathcal{E}_{k,\mathfrak{a}}(\Gamma,\chi)},$$

where the closure is with respect to the topology induced by the L^2-norm.

Proof. Combine Theorems 4.4.1 and 4.4.5 and use the fact that $\mathcal{B}_k(\Gamma,\chi)=\mathcal{E}_k(\Gamma,\chi)\oplus\mathcal{C}_k(\Gamma,\chi)$ for the first statement. The last statement holds because $\mathcal{B}_k(\Gamma,\chi)$ is dense in $\mathcal{L}_k(\Gamma,\chi)$. $\square$

4.5 Double Coset Operators

We can extend the theory of double coset operators to Maass form just as we did for holomorphic forms. Indeed, for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we define the operator $[\alpha]_k$ on $\mathcal{A}_{k,\nu}(\Gamma)$ to be the linear operator given by

$$(f[\alpha]_k)(z) = \det(\alpha)^{k-1}\varepsilon(\alpha, z)^{-k}f(\alpha z),$$

Moreover, $[\alpha]_k$ is multiplicative by mimicking the same argument as for holomorphic forms. Also, if $\gamma \in \Gamma$ and we choose the representative with positive determinant, then the chain of equalities

$$(f[\gamma]_k)(z) = \varepsilon(\gamma, z)^{-k}f(\gamma z) = \chi(\gamma)f(z),$$

is equivalent to the automorphy of f on $\Gamma\backslash\mathbb{H}$ with character χ. For any two congruence subgroups Γ_1 and Γ_2 (not necessarily of the same level) and any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we define the **double coset operator** $[\Gamma_1\alpha\Gamma_2]_k$ to be the linear operator on $\mathcal{A}_{k,\nu}(\Gamma_1)$ given by

$$(f[\Gamma_1\alpha\Gamma_2]_k)(z) = \sum_j (f[\beta_j]_k)(z) = \sum_j \det(\beta_j)^{k-1}\varepsilon(\beta_j, z)^{-k}f(\beta_j z).$$

As was the case for holomorphic forms, Proposition 3.4.1 implies that this sum is finite. Mimicking the same argument for $[\Gamma_1\alpha\Gamma_2]_k$, we see that $[\Gamma_1\alpha\Gamma_2]$ is also well-defined. There is also an analogous statement about the double coset operators for Maass forms:

Proposition 4.5.1. *For any two congruence subgroups Γ_1 and Γ_2, $[\Gamma_1\alpha\Gamma_2]_k$ maps $\mathcal{A}_{k,\nu}(\Gamma_1)$ into $\mathcal{A}_{k,\nu}(\Gamma_2)$. Moreover, $[\Gamma_1\alpha\Gamma_2]_k$ preserves the subspace of cusp forms.*

Proof. Mimicking the proof of Proposition 3.4.2 with smoothness replacing holomorphy, automorphy replacing modularity, and the analogous growth condition for Maass forms, the only piece left to verify is that $f[\Gamma_1\alpha\Gamma_2]_k$ is an eigenfunction for Δ with eigenvalue λ if f is. This is easy since the invariance of Δ implies

$$\Delta(f[\Gamma_1\alpha\Gamma_2]_k)(z) = \sum_j \Delta(f[\beta_j]_k)(z) = \lambda\sum_j \det(\beta_j)^{k-1}\varepsilon(\beta_j, z)^{-k}f(\beta_j z) = \lambda(f[\Gamma_1\alpha\Gamma_2]_k)(z).$$

Thus $f[\Gamma_1\alpha\Gamma_2]_k$ is an eigenfunction for Δ with eigenvalue λ. This completes the proof. $\square$

4.6 Diamond & Hecke Operators

Extending the theory of diamond operators and Hecke operators is also fairly straightforward. To see this, we have already shown that $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ so

$$\left(f\left[\Gamma_1(N)\alpha\Gamma_1(N)\right]_k\right)(z) = \sum_j (f[\beta_j]_k)(z) = (f[\alpha]_k)(z),$$

for any $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Therefore, for any d taken modulo N, we define the **diamond operator** $\langle d\rangle : \mathcal{A}_{k,\nu}(\Gamma_1(N)) \to \mathcal{A}_{k,\nu}(\Gamma_1(N))$ to be the linear operative given by

$$(\langle d\rangle f)(z) = (f[\alpha]_k)(z),$$

for any $\alpha = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$. As for holomorphic forms, the diamond operators are multiplicative and invertible. They also decompose $\mathcal{A}_{k,\nu}(\Gamma_1(N))$ into eigenspaces. For any Dirichlet character modulo N, let

$$\mathcal{A}_{k,\nu}(N, \chi) = \{f \in \mathcal{A}_{k,\nu}(\Gamma_1(N)) : \langle d\rangle f = \chi(d)f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^*\},$$

be the χ-eigenspace. Also let $\mathcal{C}_{k,\nu}(N, \chi)$ be the corresponding subspace of cusp forms. Then $\mathcal{A}_{k,\nu}(\Gamma_1(N))$ admits a decomposition into these eigenspaces:

Proposition 4.6.1. *We have a direct sum decomposition*

$$\mathcal{A}_{k,\nu}(\Gamma_1(N)) = \bigoplus_{\chi \ (mod\ N)} \mathcal{A}_{k,\nu}(N,\chi).$$

Moreover, this decomposition respects the subspace of cusp forms.

Proof. Mimic the proof of Proposition 3.5.1. □

If $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and we choose the representative with positive determinant, then $\chi(\gamma) = \chi(d)$ and the chain of equalities

$$(\langle d \rangle f)(z) = (f[\gamma]_k)(z) = f(\gamma z) = \chi(d) f(z),$$

is equivalent to the automorphy of f with character χ on $\Gamma_0(N)\backslash\mathbb{H}$. So f is a Maass form with character χ on $\Gamma_0(N)\backslash\mathbb{H}$ if and only if $f[\gamma]_k = \chi(\gamma)f$ for all $\gamma \in \Gamma_0(N)$ where γ is chosen to be the representative with positive determinant. It follows that the diamond operators sieve Maass forms on $\Gamma_1(N)\backslash\mathbb{H}$ with trivial character in terms of Maass forms on $\Gamma_0(N)\backslash\mathbb{H}$ with nontrivial characters. Precisely, $\mathcal{A}_{k,\nu}(N,\chi) = \mathcal{A}_{k,\nu}(\Gamma_0(N),\chi)$ and $\mathcal{C}_{k,\nu}(N,\chi) = \mathcal{C}_{k,\nu}(\Gamma_0(N),\chi)$. So by Proposition 4.6.1, we have

$$\mathcal{A}_{k,\nu}(\Gamma_1(N)) = \bigoplus_{\chi \ (\mathrm{mod}\ N)} \mathcal{A}_{k,\nu}(\Gamma_0(N),\chi),$$

and this decomposition respects the subspace of cusp forms. As for holomorphic forms, this decomposition helps clarify why we consider Maass forms with nontrivial characters. We define the Hecke operators in the same way as for holomorphic forms. For a prime p, we define the p-th **Hecke operator** $T_p : \mathcal{A}_{k,\nu}(\Gamma_1(N)) \to \mathcal{A}_{k,\nu}(\Gamma_1(N))$ to be the linear operator given by

$$(T_p f)(z) = \left(f \left[\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) \right]_k \right)(z).$$

By Proposition 4.5.1, T_p preserves the subspaces of Maass forms and cusp forms. The diamond and Hecke operators commute:

Proposition 4.6.2. *For every $d \in (\mathbb{Z}/N\mathbb{Z})^*$ and prime p, the diamond operator $\langle d \rangle$ and the Hecke operator T_p on $\mathcal{A}_{k,\nu}(\Gamma_1(N))$ commute:*

$$\langle d \rangle T_p = T_p \langle d \rangle$$

Proof. Mimic the proof of Proposition 4.6.2. □

Exactly as for holomorphic forms, Lemma 3.5.1 will give an explicit description of the Hecke operator T_p:

Proposition 4.6.3. *Let $f \in \mathcal{A}_{k,\nu}(\Gamma_1(N))$. Then the Hecke operator T_p acts on f as follows:*

$$(T_p f)(z) = \begin{cases} \displaystyle\sum_{j \ (mod\ p)} \left(f \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \right)(z) + \left(f \left[\begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right]_k \right)(z) & \text{if } p \nmid N, \\ \displaystyle\sum_{j \ (mod\ p)} \left(f \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \right)(z) & \text{if } p \mid N, \end{cases}$$

where m and n are chosen such that $\det\left(\left(\begin{smallmatrix} m & n \\ N & p \end{smallmatrix}\right)\right) = 1$.

Proof. Mimic the proof of Proposition 3.5.3. $\square$

We use Proposition 4.6.3 to understand how the Hecke operators act on the Fourier coefficients of Maass forms:

Proposition 4.6.4. *Let $f \in \mathcal{A}_{k,\nu}(\Gamma_1(N))$ Fourier coefficients $a_n(f)$, $a_0^+(f)$, and $a_0^-(f)$. Then for primes p with $(p, N) = 1$,*

$$(T_pf)(z) = (a_0^+(f) + \chi_{N,0}(p)a_0^+(\langle p\rangle f))y^{\frac{1}{2}+\nu} + (a_0^-(f) + \chi_{N,0}(p)a_0^-(\langle p\rangle f))y^{\frac{1}{2}-\nu} + \sum_{n\neq 0}\left(a_{np}(f) + \chi_{N,0}(p)a_{\frac{n}{p}}(\langle p\rangle f)\right)W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx},$$

is the Fourier series of T_pf where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid |n|$. Moreover, if $f \in \mathcal{A}_{k,\nu}(N,\chi)$, then $T_pf \in \mathcal{A}_{k,\nu}(N,\chi)$ and

$$(T_pf)(z) = (a_0^+(f) + \chi(p)a_0^+(f))y^{\frac{1}{2}+\nu} + (a_0^-(f) + \chi(p)a_0^-(f))y^{\frac{1}{2}-\nu} + \sum_{n\neq 0}\left(a_{np}(f) + \chi(p)a_{\frac{n}{p}}(f)\right)W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx},$$

where it is understood that $a_{\frac{n}{p}}(f) = 0$ if $p \nmid |n|$.

Proof. Mimic the proof of Proposition 3.5.4. $\square$

As for holomorphic forms, the Hecke operators form a simultaneously commuting family with the diamond operators:

Proposition 4.6.5. *Let p and q be primes and $d, e \in (\mathbb{Z}/N\mathbb{Z})^*$. Then the Hecke operators T_p and T_q and diamond operators $\langle d\rangle$ and $\langle e\rangle$ on $\mathcal{A}_{k,\nu}(\Gamma_1(N))$ form a simultaneously commuting family:*

$$T_pT_q = T_qT_p, \quad \langle d\rangle T_p = T_p\langle d\rangle, \quad \text{and} \quad \langle d\rangle\langle e\rangle = \langle e\rangle\langle d\rangle.$$

Proof. Mimic the proof of Proposition 3.5.5. $\square$

We use Proposition 4.6.5 to construct diamond operators $\langle m\rangle$ and Hecke operators T_m for all $m \geq 1$ exactly as for holomorphic forms. Explicitly, the **diamond operator** $\langle m\rangle : \mathcal{A}_{k,\nu}(\Gamma_1(N)) \to \mathcal{A}_{k,\nu}(\Gamma_1(N))$ is defined to be the linear operator given by

$$\langle m\rangle = \begin{cases} \langle m\rangle \text{ with } m \text{ taken modulo } N & \text{if } (m,N) = 1, \\ 0 & \text{if } (m,N) > 1. \end{cases}$$

For the Hecke operators, if $m = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ is the prime decomposition of m, then the m-th **Hecke operator** $T_m : \mathcal{A}_{k,\nu}(\Gamma,\chi) \to \mathcal{A}_{k,\nu}(\Gamma,\chi)$ is the linear operator given by

$$T_m = \prod_{1\leq i\leq k} T_{p_i^{r_i}},$$

where T_{p^r} is defined inductively by

$$T_{p^r} = \begin{cases} T_pT_{p^{r-1}} - \langle p\rangle T_{p^{r-2}} & \text{if } p \nmid N, \\ T_p^r & \text{if } p \mid N, \end{cases}$$

for all $r \geq 2$. By Proposition 4.6.5, the Hecke operators T_m are multiplicative but not completely multiplicative in m and they commute with the diamond operators $\langle m\rangle$. Moreover, a more general formula for how the Hecke operators T_m act on the Fourier coefficients can be derived:

Proposition 4.6.6. *Let $f \in \mathcal{A}_{k,\nu}(\Gamma_1(N))$ with Fourier coefficients $a_n(f)$, $a_0^+(f)$, and $a_0^-(f)$. Then for $m \geq 1$ with $(m, N) = 1$,*

$$(T_m f)(z) = \left(\sum_{d|m} a_0^+(\langle d\rangle f)\right) y^{\frac{1}{2}+\nu} + \left(\sum_{d|m} a_0^-(\langle d\rangle f)\right) y^{\frac{1}{2}-\nu} + \sum_{n\neq 0}\left(\sum_{d|(|n|,m)} a_{\frac{nm}{d^2}}(\langle d\rangle f)\right) W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi i n x},$$

is the Fourier series of $T_m f$. Moreover, if $f \in \mathcal{A}_{k,\nu}(N,\chi)$, then

$$(T_m f)(z) = \left(\sum_{d|m} \chi(d) a_0^+(f)\right) y^{\frac{1}{2}+\nu} + \left(\sum_{d|m} \chi(d) a_0^-(f)\right) y^{\frac{1}{2}-\nu} + \sum_{n\neq 0}\left(\sum_{d|(|n|,m)} \chi(d) a_{\frac{nm}{d^2}}(f)\right) W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi i n x}.$$

Proof. Mimic the proof of Proposition 3.5.6. ☐

The diamond and Hecke operators turn out to be normal with respect to the Petersson inner product on the subspace of cusp forms. Just as with holomorphic forms, we can use Lemma 3.5.2 to compute adjoints:

Proposition 4.6.7. *Let Γ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then the following are true:*

(i) If $\alpha^{-1}\Gamma\alpha \subseteq \mathrm{PSL}_2(\mathbb{Z})$, then for all $f \in \mathcal{C}_{k,\nu}(\Gamma,\chi)$ and $g \in \mathcal{C}_{k,\nu}(\alpha^{-1}\Gamma\alpha)$, we have

$$\langle f[\alpha]_k, g\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g[\alpha^{-1}]_k\rangle_\Gamma.$$

(ii) For all $f, g \in \mathcal{C}_{k,\nu}(\Gamma,\chi)$, we have

$$\langle f[\Gamma\alpha\Gamma]_k, g\rangle = \langle f, g[\Gamma\alpha^{-1}\Gamma]_k\rangle.$$

In particular, if $\alpha^{-1}\Gamma\alpha = \Gamma$ then $[\alpha]_k^* = [\alpha^{-1}]_k$ and $[\Gamma\alpha\Gamma]_k^* = [\Gamma\alpha^{-1}\Gamma]_k$.

Proof. Mimic the proof of Proposition 3.5.7. ☐

We can now prove that the diamond and Hecke operators are normal:

Proposition 4.6.8. *On $\mathcal{C}_{k,\nu}(\Gamma_1(N))$, the diamond operators $\langle m\rangle$ and Hecke operators T_m are normal with respect to the Petersson inner product for all $m \geq 1$ with $(m, N) = 1$. Moreover, their adjoints are given by*

$$\langle m\rangle^* = \langle \overline{m}\rangle \quad \text{and} \quad T_p^* = \langle \overline{p}\rangle T_p.$$

Proof. Mimic the proof of Proposition 3.5.8. ☐

When we are considering Maass forms on the full modular group, Proposition 4.6.8 says that all of the diamond and Hecke operators are normal. Before discussing the spectral theory, we need one last operator. Define $T_{-1} : \mathcal{A}_{k,\nu}(\Gamma_1(N)) \to \mathcal{A}_{k,\nu}(\Gamma_1(N))$ to be the linear operator given by

$$(T_{-1}f)(z) = f(-\overline{z}).$$

Clearly T_{-1} preserves the subspace of cusp forms. Moreover, we also have the following proposition encompassing important properties of the T_{-1} operator:

Proposition 4.6.9.

1. *On $\mathcal{A}_{k,\nu}(\Gamma_1(N))$, the operator T_{-1} commutes with the diamond operators $\langle m\rangle$ and Hecke operators T_n for all $m, n \geq 1$.*
2. *On $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ the operator T_{-1} is normal with respect to the Petersson inner product and the adjoint is given by*
$$T_{-1}^* = -T_{-1}.$$

Proof. To prove (i), for any matrix $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ observe that $-\overline{\alpha z} = \alpha(-\overline{z})$. Hence T_{-1} commutes with the $[\alpha]_k$ operator. Since T_{-1} is linear, it follows that it commutes with the double coset operators and hence the diamond operators and Hecke operators as well. For (ii), $z \to -\overline{z}$ is an automorphism of $\mathbb{H}$ with $d\mu \to -d\mu$. So for any $f, g \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$, we have

$$\begin{aligned}
\langle T_{-1}f, g\rangle &= \frac{1}{V_{\Gamma_1(N)}} \int_{\mathcal{F}_{\Gamma_1(N)}} (T_{-1}f)(z)\overline{g(z)}\, d\mu \\
&= \frac{1}{V_{\Gamma_1(N)}} \int_{\mathcal{F}_{\Gamma_1(N)}} f(-\overline{z})\overline{g(z)}\, d\mu \\
&= -\frac{1}{V_{\Gamma_1(N)}} \int_{\mathcal{F}_{\Gamma_1(N)}} f(z)\overline{g(-\overline{z})}\, d\mu \\
&= \langle f, -T_{-1}g\rangle.
\end{aligned}$$

This proves the adjoint formula $T_{-1}^* = -T_{-1}$ and normality is now clear. □

In the case of the modular group, Proposition 4.6.8 says that all of the diamond and Hecke operators are normal. Suppose f is a non-constant Maass form with Fourier coefficients $a_n(f)$. Let the eigenvalue of T_m for f be $\lambda_f(m)$. We say that the $\lambda_f(m)$ are the **Hecke eigenvalues** of f. Similar to the case for holomorphic forms, if f is a Maass form on $\Gamma_1(N)\backslash\mathbb{H}$ that is a simultaneous eigenfunction for the operator T_{-1} and all diamond operators $\langle m\rangle$ and Hecke operators T_m with $(m, N) = 1$, we call f an **eigenform**. Necessarily, f is even or odd because $T_{-1}^2 f = f$ so that the only possible eigenvalues are ± 1. If the condition $(m, N) = 1$ can be dropped, so that f is a simultaneous eigenfunction for all diamond and Hecke operators, then we say f is a **Hecke-Maass eigenform**. In particular, on $\Gamma_1(1)\backslash\mathbb{H}$ all eigenforms are Hecke-Maass eigenforms. As for holomorphic forms, if f an eigenform Proposition 4.6.6 implies that the first Fourier coefficient of $T_m f$ is $a_m(f)$ and so $a_m(f) = \lambda_f(m)a_1(f)$ for all $m \geq 1$ with $(m, N) = 1$. Therefore we cannot have $a_1(f) = 0$ and so we can normalize f by dividing by $a_1(f)$ which guarantees that the Fourier series has constant term 1. Then the m-th Fourier coefficient of f, when $(m, N) = 1$, is precisely the Hecke eigenvalue $\lambda_f(m)$. This normalization is called the **Hecke normalization** of f. The **Petersson normalization** of f is where we normalize so that $\langle f, f\rangle = 1$. From the spectral theorem we have an analogous corollary as for holomorphic forms:

Theorem 4.6.1. *$\mathcal{C}_{k,\nu}(\Gamma_1(N))$ admits an orthonormal basis of eigenforms.*

Proof. This follows from the spectral theorem along with Propositions 4.6.5, 4.6.8 and 4.6.9. □

In particular, Theorem 4.6.1 implies that the orthonormal basis in Theorem 4.4.1 can be taken to be an orthonormal basis of eigenforms since it is clear that the T_{-1} operator, diamond operators $\langle m\rangle$, and Hecke operators T_m all commute with Δ. Also, just as in the holomorphic setting, we have **Hecke relations** for Maass forms:

Proposition 4.6.10 (Hecke relations, Maass version). *Let $f \in \mathcal{C}_{k,\nu}(N,\chi)$ be a Hecke-Maass eigenform with Fourier coefficients $\lambda_f(n)$. Then the Fourier coefficients are multiplicative and satisfy*

$$\lambda_f(n)\lambda_f(m) = \sum_{d\mid(n,m)} \chi(d)\lambda_f\left(\frac{nm}{d^2}\right) \quad \text{and} \quad \lambda_f(nm) = \sum_{d\mid(n,m)} \mu(d)\chi(d)\lambda_f\left(\frac{n}{d}\right)\lambda_f\left(\frac{m}{d}\right),$$

for all $n, m \geq 1$ with $(nm, N) = 1$.

Proof. Mimic the proof of the Hecke relations for holomorphic forms. □

As an immediate consequence of the Hecke relations, the Hecke operators satisfy analogous relations:

Corollary 4.6.1. *The Hecke operators are multiplicative and satisfy*

$$T_nT_m = \sum_{d\mid(n,m)} \chi(d)T_{\frac{nm}{d^2}} \quad \text{and} \quad T_{nm} = \sum_{d\mid(n,m)} \mu(d)\chi(d)T_{\frac{n}{d}}T_{\frac{m}{d}},$$

for all $n, m \geq 1$ with $(nm, N) = 1$.

Proof. Mimic the proof of Corollary 3.5.1. □

Just as for holomorphic forms, the identities in Corollary 4.6.1 can also be established directly and the first identity can be used to show that the Hecke operators commute.

4.7 Atkin–Lehner Theory

There is also an Atkin–Lehner theory for Maass form. As with holomorphic forms, we will only deal with congruence subgroups of the form $\Gamma_1(N)$ and cusp forms on $\Gamma_1(N)\backslash\mathbb{H}$. The trivial way to lift Maass forms from a smaller level to a larger level is via the natural inclusion $\mathcal{C}_{k,\nu}(\Gamma_1(M)) \subseteq \mathcal{C}_{k,\nu}(\Gamma_1(N))$ provided $M \mid N$ which follows from $\Gamma_1(N) \leq \Gamma_1(M)$. Alternatively, for any $d \mid \frac{N}{M}$, let $\alpha_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $f \in \mathcal{C}_{k,\nu}(\Gamma_1(M))$, then

$$(f[\alpha_d]_k)(z) = d^{k-1}\varepsilon(\alpha_d, z)^{-k} f(\alpha_d z) = d^{k-1}f(dz).$$

Similar to holomorphic forms, $[\alpha_d]_k$ maps $\mathcal{C}_{k,\nu}(\Gamma_1(M))$ into $\mathcal{C}_{k,\nu}(\Gamma_1(N))$:

Proposition 4.7.1. *Let M and N be positive integers such that $M \mid N$. For any $d \mid \frac{N}{M}$, $[\alpha_d]_k$ maps $\mathcal{C}_{k,\nu}(\Gamma_1(M))$ into $\mathcal{C}_{k,\nu}(\Gamma_1(N))$.*

Proof. Mimicking the proof of Proposition 3.6.1 with smoothness replacing holomorphy, automorphy replacing modularity, and the analogous growth condition for Maass forms, the only piece left to verify is that $f[\alpha_d]$ is an eigenfunction for Δ with eigenvalue λ if f is. This is easy since the invariance of Δ implies

$$\Delta(f[\alpha_d]_k)(z) = \lambda d^{k-1}f(dz) = \lambda(f[\alpha_d]_k)(z).$$

Therefore $f[\alpha_d]_k$ is an eigenfunction for Δ with eigenvalue λ. This completes the proof. □

We can now define oldforms and newforms. For each divisor d of N, set

$$i_d : \mathcal{C}_{k,\nu}\left(\Gamma_1\left(\frac{N}{d}\right)\right) \times \mathcal{C}_{k,\nu}\left(\Gamma_1\left(\frac{N}{d}\right)\right) \to \mathcal{C}_{k,\nu}(\Gamma_1(N)) \qquad (f,g) \mapsto f + g[\alpha_d]_k.$$

This map is well-defined by Proposition 4.7.1. The subspace of **oldforms of level** N is

$$\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{old}} = \bigoplus_{p \mid N} \mathrm{Im}(i_p),$$

and the subspace of **newforms of level** N is

$$\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{new}} = \left(\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{old}}\right)^{\perp},$$

where the orthogonal complement is taken with respect to the Petersson inner product. The elements of such subspaces are called **oldforms** and **newforms** respectively. Both subspaces are invariant under the diamond and Hecke operators (mimic the proof of Proposition 3.6.2 given in [DS05]):

Proposition 4.7.2. *The spaces $\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{old}}$ and $\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{new}}$ are invariant under the diamond operators $\langle m \rangle$ and Hecke operators T_m for all $m \geq 1$.*

As a corollary, these subspaces admit orthogonal bases of eigenforms:

Corollary 4.7.1. *$\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{old}}$ and $\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{new}}$ admit orthonormal bases of eigenforms.*

Proof. This follows immediately from Theorem 4.6.1 and Proposition 4.7.2 □

We can remove the condition $(m,N) = 1$ for eigenforms in a basis of $\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{new}}$ so that the eigenforms are eigenfunctions for all of the diamond and Hecke operators. As for holomorphic forms, we need a preliminary result (mimic the proof of Lemma 3.6.1 as given in [DS05]):

Lemma 4.7.1. *If $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$ with Fourier coefficients $a_n(f)$ and such that $a_n(f) = 0$ whenever $(n,N) = 1$, then*

$$f = \sum_{p \mid N} f_p[\alpha_p]_k,$$

for some $f_p \in \mathcal{C}_{k,\nu}\left(\Gamma_1\left(\frac{N}{p}\right)\right)$.

As was the case for holomorphic forms, we observe from Lemma 4.7.1 that if $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))$ is such that its n-th Fourier coefficients vanish when n is relatively prime to the level, then f must be an oldform. The main theorem about $\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{new}}$ can now be proved. We say that f is a **primitive Hecke-Maass eigenform** if it is a nonzero Hecke normalized Hecke-Maass eigenform in $\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{new}}$. We can now prove the main result about newforms:

Theorem 4.7.1. *Let $f \in \mathcal{C}_{k,\nu}(\Gamma_1(N))^{\mathrm{new}}$ be an eigenform. Then the following hold:*

(i) f is a Hecke-Maass eigenform.

(ii) If $\widetilde{f}$ satisfies the same conditions as f and has the same eigenvalues for the Hecke operators, then $\widetilde{f} = cf$ for some nonzero constant c.

Moreover, the primitive Hecke-Maass eigenforms in $\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\text{new}}$ form an orthogonal basis with respect to the Petersson inner product and each such primitive Hecke-Maass eigenform f lies in an eigenspace $\mathcal{C}_{k,\nu}(N,\chi)$.

Proof. Mimic the proof of Theorem 3.6.1. □

Statement (i) in Theorem 4.7.1 implies that primitive Hecke-Maass eigenforms satisfy the Hecke relations for all $n, m \geq 1$. Statement (ii) in Theorem 4.7.1 is referred to as the **multiplicity one theorem** for Maass forms. So as is the case for holomorphic forms, $\mathcal{C}_{k,\nu}(\Gamma_1(N))^{\text{new}}$ contains one element per "eigenvalue" where we mean a set of eigenvalues one for each Hecke operator T_m. Since the Fourier coefficients are multiplicative, the Hecke relations imply that f is actually determined by its Fourier coefficients at primes.

Theorem 4.7.2 (Multiplicity one, Maass version). *Let f and g be primitive Hecke-Maass eigenforms of the same eigenvalue and level. Denote the Hecke eigenvalues by $\lambda_f(n)$ and $\lambda_g(n)$ respectively. If $\lambda_f(p) = \lambda_g(p)$ for all primes p, then $f = g$.*

Proof. Mimic the proof of multiplicity one for holomorphic forms. □

Interestingly, unlike holomorphic forms it is unknown if the Fourier coefficients of Maass forms are real or even algebraic in general. We require one last piece of machinery. As for holomorphic forms, we have an involution on the space $\mathcal{C}_{k,\nu}(N,\chi)$. Recall the matrix

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

with $\det(W_N) = N$. We define the **Atkin–Lehner involution** ω_N to be the linear operator on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$ given by

$$(\omega_N f)(z) = N^{\frac{k}{2}}\varepsilon(W_N, z)^{-k} f(W_N z) = \left(\frac{\sqrt{N}z}{|z|}\right)^{-k} f\left(-\frac{1}{Nz}\right).$$

It is not too difficult to see that ω_N is an involution on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$:

Proposition 4.7.3. *ω_N is an involution on $\mathcal{C}_{k,\nu}(\Gamma_1(N))$.*

Proof. Mimicking the proof of Proposition 3.6.3 with smoothness replacing holomorphy, automorphy replacing modularity, and the analogous growth condition for Maass forms, the only piece left to verify is that $\omega_N f$ is an eigenfunction for Δ with eigenvalue λ if f is. This is easy since the invariance of Δ implies

$$\Delta(\omega_N f)(z) = \lambda\left(\frac{\sqrt{N}z}{|z|}\right)^{-k} f(W_N z) = \lambda(\omega_N f)(z).$$

Thus $\omega_N f$ is an eigenfunction for Δ with eigenvalue λ. This completes the proof. □

As $\omega_N f$ is an involution its only possible eigenvalues are ± 1. The crucial fact we need is how ω_N acts on $\mathcal{C}_{k,\nu}(N,\chi)$. To state the result, for $f \in \mathcal{C}_{k,\nu}(N,\chi)$ define

$$\overline{f}(z) = \overline{f(-\overline{z})}.$$

Then we have the following (mimic the proof for holomorphic forms given in [CS17]):

Proposition 4.7.4. *If $f \in \mathcal{C}_{k,\nu}(N,\chi)$ is a primitive Hecke-Maass eigenform, then*

$$\omega_N f = \omega_N(f)\overline{f},$$

where $\overline{f} \in \mathcal{C}_{k,\nu}(N,\overline{\chi})$ is a primitive Hecke-Maass eigenform and $\omega_N(f) \in \mathbb{C}$ is nonzero with $|\omega_N(f)| = 1$.

4.8 The Ramanujan-Petersson Conjecture

As for the size of the Fourier coefficients of Maass form, much is currently unknown. But there is an analogous conjecture to the one for holomorphic forms. To state it, suppose $f \in \mathcal{C}_{k,\nu}(N, \chi)$ is a primitive Hecke-Maass eigenform with Fourier coefficients $\lambda_f(n)$. For each prime p, consider the polynomial

$$1 - \lambda_f(p)p^{-s} + \chi(p)p^{-2s}.$$

We call this the p-th **Hecke polynomial** of f. Let $\alpha_1(p)$ and $\alpha_2(p)$ denote the roots. Then

$$\alpha_1(p) + \alpha_2(p) = \lambda_f(p) \quad \text{and} \quad \alpha_1(p)\alpha_2(p) = \chi(p).$$

The **Ramanujan-Petersson conjecture** for Maass forms is following statement:

Conjecture 4.8.1 (Ramanujan-Petersson conjecture, Maass version). *Suppose $f \in \mathcal{C}_{k,\nu}(N, \chi)$ is a primitive Hecke-Maass eigenform with Hecke eigenvalues $\lambda_f(n)$. Let $\alpha_1(p)$ and $\alpha_2(p)$ be the roots of the p-th Hecke polynomial. Then for all primes p,*

$$|\lambda_f(p)| \le 2p^{-\frac{1}{2}}.$$

Moreover, if $p \nmid N$, then

$$|\alpha_1(p)| = |\alpha_2(p)| = 1.$$

The Ramanujan-Petersson conjecture has not yet been proved, but there has been partial progress toward the conjecture. The current best bound is $|\lambda_f(p)| \le 2p^{\frac{7}{64}-\frac{1}{2}}$ due to Kim and Sarnak (see [KRS03] for the proof). Under the Ramanujan-Petersson conjecture, the Hecke relations give the improved bound $\lambda_f(n) \ll \sigma_0(n)n^{-1} \ll_\varepsilon n^{\varepsilon-1}$ (recall Proposition A.3.1). It turns out that the Ramanujan-Petersson conjecture is tightly connected to another conjecture of Selberg about the smallest possible eigenvalue of Maass form on $\Gamma\backslash\mathbb{H}$. Note that the possible eigenvalues are discrete by Theorem 4.4.6 and so there exists a smallest eigenvalue. To state it, recall that if f is a Maass form with eigenvalue λ and spectral parameter r on $\Gamma\backslash\mathbb{H}$, then $\lambda = \frac{1}{4} + r^2$ with either $r \in \mathbb{R}$ or $ir \in [0, \frac{1}{2})$. The **Selberg conjecture** claims that the second case never occurs in weight zero:

Conjecture 4.8.2 (Selberg conjecture). *If λ is the smallest eigenvalue for Maass forms on $\Gamma\backslash\mathbb{H}$, then*

$$\lambda \ge \frac{1}{4}.$$

Selberg was able to achieve a remarkable lower bound using the analytic continuation of a certain Dirichlet series and the Weil bound for Kloosterman sums (see [Iwa02] for a proof):

Theorem 4.8.1. *If λ is the smallest eigenvalue for Maass forms on $\Gamma\backslash\mathbb{H}$, then*

$$\lambda \ge \frac{3}{16}.$$

In the language of automorphic representations, these two conjectures are a consequence of a much larger conjecture (see [BB13] for details).

4.9 Twists of Maass Forms

We can also twist of Maass forms by Dirichlet characters. Let $f \in \mathcal{A}_{k,\nu}(N,\chi)$ with Fourier series

$$f(z) = a_0^+(f)y^{\frac{1}{2}+\nu} + a_0^-(f)y^{\frac{1}{2}-\nu} + \sum_{n\neq 0} a_n(f)W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx}.$$

and let ψ be a primitive Dirichlet character modulo M. We define the **twisted Maass form** $f \otimes \psi$ of f twisted by ψ by the Fourier series

$$(f\otimes\psi)(z) = a_0^+(f)\psi(0)y^{\frac{1}{2}+\nu} + a_0^-(f)\psi(0)y^{\frac{1}{2}-\nu} + \sum_{n\neq 0} a_n(f)\psi(n)W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx}.$$

In order for $f \otimes \psi$ to be well-defined, we need to prove that it is a Maass form. The following proposition proves this and more when ψ is primitive:

Proposition 4.9.1. *Suppose $f \in \mathcal{A}_{k,\nu}(N,\chi)$ and ψ is a primitive Dirichlet character of conductor q. Then $f \otimes \psi \in \mathcal{A}_{k,\nu}(Nq^2, \chi\psi^2)$. Moreover, if f is a cusp form then so is $f \otimes \psi$.*

Proof. Mimicking the proof of Proposition 3.8.2 with $k = 0$, smoothness replacing holomorphy, automorphy replacing modularity, and the analogous growth condition for Maass forms, the only piece left to verify is that $f \otimes \psi$ is an eigenfunction for Δ with eigenvalue λ if f is in the case ψ is primitive. This is easy since the invariance of Δ implies

$$\Delta(f\otimes\chi)(z) = \frac{1}{\tau(\overline{\psi})}\sum_{r \pmod q}\overline{\psi}(r)\Delta(f)\left(z+\frac{r}{q}\right) = \frac{\lambda}{\tau(\overline{\psi})}\sum_{r \pmod q}\overline{\psi}(r)f\left(z+\frac{r}{q}\right) = \lambda(f\otimes\chi)(z).$$

□

The generalization of Proposition 4.9.1 to all characters is slightly more involved. Define operators U_p and V_p on $\mathcal{A}_{k,\nu}(\Gamma_1(N))$ to be the linear operators given by

$$(U_pf)(z) = a_0^+(f)y^{\frac{1}{2}+\nu} + a_0^-(f)y^{\frac{1}{2}-\nu} + \sum_{n\neq 0} a_{np}(f)W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx},$$

and

$$(V_pf)(z) = a_0^+(f)(py)^{\frac{1}{2}+\nu} + a_0^-(f)(py)^{\frac{1}{2}-\nu} + \sum_{n\neq 0} a_n(f)W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|py)e^{2\pi inpx},$$

if f has Fourier series

$$f(z) = a_0^+(f)y^{\frac{1}{2}+\nu} + a_0^-(f)y^{\frac{1}{2}-\nu} + \sum_{n\neq 0} a_n(f)W_{\mathrm{sgn}(n)\frac{k}{2},\nu}(4\pi|n|y)e^{2\pi inx}.$$

We will show that both U_p and V_p map $\mathcal{A}_{k,\nu}(\Gamma_1(N))$ into $\mathcal{A}_{k,\nu}(\Gamma_1(Np))$ and more:

Lemma 4.9.1. *For any prime p, U_p and V_p map $\mathcal{A}_{k,\nu}(\Gamma_1(N))$ into $\mathcal{A}_{k,\nu}(\Gamma_1(Np))$. In particular, U_p and V_p map $\mathcal{A}_{k,\nu}(N,\chi)$ into $\mathcal{A}_{k,\nu}(Np,\chi\chi_{p,0})$. Moreover, U_p and V_p preserve the subspace of cusp forms.*

Proof. Mimic the proof of Proposition 3.5.1. □

We can now generalize Proposition 4.9.1 to all characters:

Proposition 4.9.2. *Suppose $f \in \mathcal{A}_{k,\nu}(N,\chi)$ and ψ is a Dirichlet character modulo M. Then $f \otimes \psi \in \mathcal{A}_{k,\nu}(NM^2, \chi\psi^2)$. Moreover, if f is a cusp form then so is $f \otimes \psi$.*

Proof. Mimicking the proof of Proposition 3.8.2, it remains to show that $f \otimes \psi_{p,0}$ is an eigenfunction with eigenvalue λ if f is. As $U_p = T_p$ for the p-th Hecke operator on $\mathcal{A}_{k,\nu}(\Gamma_1(Np))$, U_p commutes with Δ. It is also clear that V_p commutes with Δ. These facts together with

$$f \otimes \psi_{p,0} = f - V_p U_p f,$$

show that $f \otimes \psi_{p,0}$ is an eigenfunction with eigenvalue λ if f is. $\square$

Chapter 5

Trace Formulas

There are various types of formulas that relate the Fourier coefficients of holomorphic and Maass forms to various other types of functions. Two of the most important such formulas are the Petersson and Kuznetsov trace formulas.

5.1 The Petersson Trace Formula

From Theorem 3.6.1, $\mathcal{S}_k(N,\chi)$ admits an orthonormal basis of Hecke eigenforms (note that generally they are not Hecke normalized). Denote this basis by $\{u_j\}_{1\le j\le r}$ where r is the dimension of $\mathcal{S}_k(N,\chi)$. Each of these forms admits a Fourier series at the $\mathfrak{a}$ cusp given by

$$(u_j|\sigma_{\mathfrak{a}})(z) = \sum_{n\ge 1} a_{j,\mathfrak{a}}(n)e^{2\pi i nz}.$$

The Petersson trace formula is an equation relating the Fourier coefficients $a_{j,\mathfrak{a}}(n)$ and $a_{j,\mathfrak{b}}(n)$ of the basis $\{u_j\}_{1\le j\le r}$ for two cusps $\mathfrak{a}$ and $\mathfrak{b}$ of $\Gamma_0(N)\backslash\mathbb{H}$ to a sum of J-Bessel functions and Salié sums. To prove the Petersson trace formula we compute the inner product of two Poincaré series $P_{n,k,\chi,\mathfrak{a}}(z)$ and $P_{m,k,\chi,\mathfrak{b}}(z)$ in two different ways. One way is geometric in nature while the other is spectral. Since Equation (3.2) says that $\langle P_{n,k,\chi,\mathfrak{a}}, P_{m,k,\chi,\mathfrak{b}}\rangle$ essentially extracts the m-th Fourier coefficient of $P_{n,k,\chi,\mathfrak{a}}$, the Petersson trace formula amounts to computing the m-th Fourier coefficient of $P_{n,k,\chi,\mathfrak{a}}$ in two different ways. We will begin with the geometric method first. This is easy as we have already computed the Fourier series of the Poincaré series. Applying Equation (3.2) to the Fourier series in Proposition 3.2.1 gives

$$\langle P_{n,k,\chi,\mathfrak{a}}, P_{m,k,\chi,\mathfrak{b}}\rangle = \frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}}\left(\delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m} + \left(\frac{\sqrt{m}}{\sqrt{n}}\right)^{k-1}\sum_{c\in\mathcal{C}_{\mathfrak{a},\mathfrak{b}}}\frac{2\pi i^{-k}}{c}J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right)S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c)\right).$$

This is the first half of the Petersson trace formula. To obtain the second half, we use the fact that $\{u_j\}_{1\le j\le r}$ is an orthonormal basis for $\mathcal{S}_k(N,\chi)$ and Equation (3.2) to write

$$P_{n,k,\chi,\mathfrak{a}}(z) = \sum_{1\le j\le r}\langle P_{n,k,\chi,\mathfrak{a}}, u_j\rangle u_j(z) = \sum_{1\le j\le r}\overline{\langle u_j, P_{n,k,\chi,\mathfrak{a}}\rangle}u_j(z) = \frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi n)^{k-1}}\sum_{1\le j\le r}\overline{a_{j,\mathfrak{a}}(n)}u_j(z).$$

This last expression is the spectral decomposition of $P_{n,k,\chi,\mathfrak{a}}(z)$ in terms of the basis $\{u_j\}_{1\le j\le r}$. So if we apply Equation (3.2) to this last expression, we obtain

$$\langle P_{n,k,\chi,\mathfrak{a}}, P_{m,k,\chi,\mathfrak{b}}\rangle = \left(\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi\sqrt{nm})^{k-1}}\right)^2\sum_{1\le j\le r}\overline{a_{j,\mathfrak{a}}(n)}a_{j,\mathfrak{b}}(m),$$

which is the second half of the Petersson trace formula. Equating the first and second halves and canceling the common $\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi m)^{k-1}}$ factor gives

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi n)^{k-1}} \sum_{1 \le j \le r} \overline{a_{j,\mathfrak{a}}(n)} a_{j,\mathfrak{b}}(m) = \delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m} + \left(\frac{\sqrt{m}}{\sqrt{n}}\right)^{k-1} \sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c).$$

Since $\left(\frac{\sqrt{m}}{\sqrt{n}}\right)^{k-1} = 1$ when $n = m$, we can factor this term out of the entire right-hand side and cancel it resulting in the **Petersson trace formula** relative to the $\mathfrak{a}$ and $\mathfrak{b}$ cusps:

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi\sqrt{nm})^{k-1}} \sum_{1 \le j \le r} \overline{a_{j,\mathfrak{a}}(n)} a_{j,\mathfrak{b}}(m) = \delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m} + \sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c).$$

We refer to the left-hand side side as the **spectral side** and the right-hand side as the **geometric side**. We collect our work as a theorem:

Theorem 5.1.1 (Petersson trace formula). *Let $\{u_j\}_{1 \le j \le r}$ be an orthonormal basis of Hecke eigenforms for $\mathcal{S}_k(N,\chi)$ with Fourier coefficients $a_{j,mfa}(n)$ at the $\mathfrak{a}$ cusp. Then for any positive integers $n, m \ge 1$ and any two cusps $\mathfrak{a}$ and $\mathfrak{b}$, we have*

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi\sqrt{nm})^{k-1}} \sum_{1 \le j \le r} \overline{a_{j,\mathfrak{a}}(n)} a_{j,\mathfrak{b}}(m) = \delta_{\mathfrak{a},\mathfrak{b}}\delta_{n,m} + \sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_{\chi,\mathfrak{a},\mathfrak{b}}(n,m,c).$$

A particularly important case is when $\mathfrak{a} = \mathfrak{b} = \infty$. For then $\mathcal{C}_{\infty,\infty} = \{c \ge 1 : c \equiv 0 \pmod{N}\}$, the Salié sum reduces to the usual one, and the Petersson trace formula takes the form

$$\frac{\Gamma(k-1)}{V_{\Gamma_0(N)}(4\pi\sqrt{nm})^{k-1}} \sum_{1 \le j \le r} \overline{a_j(n)} a_j(m) = \delta_{n,m} + \sum_{\substack{c \ge 1 \\ c \equiv 0 \pmod{N}}} \frac{2\pi i^{-k}}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) S_\chi(n,m,c).$$

5.2 Todo: [The Kuznetsov Trace Formula]

Part III

An Introduction to L-functions

Chapter 6

The Theory of L-functions

We start our discussion of L-functions with Dirichlet series. Dirichlet series are essential tools in analytic number theory because they are a way of analytically encoding arithmetic information. If the Dirichlet series possesses special properties we call it an L-function. From the analytic properties of L-functions we can extract number theoretic results. After discussing Dirichlet series we will define L-functions and their associated data. The material following the data of L-functions consists of many important discussions about the analytic properties of L-functions: the approximate functional equation, the Riemann hypothesis and Lindelöf hypothesis, the central value, logarithmic derivatives, zero density, zero-free regions, and explicit formulas.

6.1 Dirichlet Series

A **Dirichlet series** $D(s)$ is a sum of the form

$$D(s) = \sum_{n \geq 1} \frac{a(n)}{n^s},$$

with $a(n) \in \mathbb{C}$. We exclude the case $a(n) = 0$ for all $n \geq 1$ so that $D(s)$ is not identically zero. We would first like to understand where this series converges. It does not take much for $D(s)$ to converge uniformly in a sector:

Theorem 6.1.1. *Suppose $D(s)$ is a Dirichlet series that converges at $s_0 = \sigma_0 + it_0$. Then for any $H > 0$, $D(s)$ converges uniformly in the sector*

$$\{s \in \mathbb{C} : \sigma \geq \sigma_0 \text{ and } |t - t_0| \leq H(\sigma - \sigma_0)\}.$$

Proof. Set $R(u) = \sum_{n \geq u} \frac{a(n)}{n^{-s_0}}$ so that $a(n) = (R(n) - R(n+1))n^{s_0}$. Then for any two positive integers N and M with $1 \leq M < N$, partial summation (see Appendix B.3) implies

$$\sum_{M \leq n \leq N} \frac{a(n)}{n^s} = R(M)M^{s_0 - s} - R(N)N^{s_0 - s} - \sum_{M+1 \leq n \leq N} R(n)((n-1)^{s_0 - s} - n^{s_0 - s}). \tag{6.1}$$

We will now express the sum on the right-hand side as an integral. To do this, observe that

$$(n-1)^{s_0 - s} - n^{s_0 - s} = -(s_0 - s) \int_{n-1}^{n} u^{s_0 - s - 1} \, du.$$

Therefore

$$\begin{aligned}\sum_{M+1\le n\le N} R(n)((n-1)^{s_0-s}-n^{s_0-s}) &= -(s_0-s)\sum_{M+1\le n\le N} R(n)\int_{n-1}^{n} u^{s_0-s-1}\,du \\ &= -(s_0-s)\sum_{M+1\le n\le N}\int_{n-1}^{n} R(u)u^{s_0-s-1}\,du \\ &= -(s_0-s)\int_{M}^{N} R(u)u^{s_0-s-1}\,du,\end{aligned} \tag{6.2}$$

where the second to last line follows because $R(u)$ is constant on the interval $[u, u+1)$. Combining Equations (6.1) and (6.2) gives

$$\sum_{M\le n\le N}\frac{a(n)}{n^s} = R(M)M^{s_0-s} - R(N)N^{s_0-s} + (s_0-s)\int_M^N R(u)u^{s_0-s-1}\,du. \tag{6.3}$$

Now there exists an M such that $|R(u)| < \varepsilon$ for all $u \ge M$ because $D(s)$ is convergent at s_0. In particular, $|R(u)u^{s_0-s}| < \varepsilon$ for all $u \ge M$ because $\sigma \ge \sigma_0$. Moreover for s in the prescribed sector,

$$|s-s_0| \le (\sigma-\sigma_0) + |t-t_0| \le (H+1)(\sigma-\sigma_0).$$

These estimates and Equation (6.3) together imply

$$\left|\sum_{M\le n\le N}\frac{a(n)}{n^s}\right| \le 2\varepsilon + \varepsilon|s-s_0|\int_M^N u^{\sigma_0-\sigma-1}\,du \le 2\varepsilon + \varepsilon(H+1)(\sigma-\sigma_0)\int_M^N u^{\sigma_0-\sigma-1}\,du.$$

Since the integral is finite, $\sum_{M\le n\le N}\frac{a(n)}{n^s}$ can be made arbitrarily small uniformly for s in the desired sector. The claim now follows by the uniform version of Cauchy's criterion. □

By taking $H\to\infty$ in Theorem 6.1.1 we see that $D(s)$ converges in the region $\sigma > \sigma_0$. Let σ_c be the infimum of all σ for which $D(s)$ converges. We call σ_c the **abscissa of convergence** of $D(s)$. Similarly, let σ_a be the infimum of all σ for which $D(s)$ converges absolutely. Since the terms of $D(s)$ are holomorphic, the convergence is locally absolutely uniform (actually uniform in sectors) for $\sigma > \sigma_a$. It follows that $D(s)$ is holomorphic in the region $\sigma > \sigma_a$. We call σ_a the **abscissa of absolute convergence** of $D(s)$. One should think of σ_c and σ_a as the boundaries of convergence and absolute convergence respectively. Of course, anything can happen at $\sigma = \sigma_c$ and $\sigma = \sigma_a$, but to the right of these lines we have convergence and absolute convergence of $D(s)$ respectively. It turns out that σ_a is never far from σ_c provided σ_c is finite:

Theorem 6.1.2. *If $D(s)$ is a Dirichlet series with finite abscissa of convergence σ_c, then*

$$\sigma_c \le \sigma_a \le \sigma_c + 1.$$

Proof. The first inequality is trivial since absolute convergence implies convergence. For the second inequality, since $D(s)$ converges at $\sigma_c+\varepsilon$, the terms $a(n)n^{-(\sigma_c+\varepsilon)}$ tend to zero as $n\to\infty$. Therefore $a(n) \ll_\varepsilon n^{\sigma_c+\varepsilon}$ where the implicit constant is independent of n. But then $a(n)n^{-(\sigma_c+\varepsilon)} \ll_\varepsilon 1$ which implies $\sum_{n\ge1} a(n)n^{-(\sigma_c+1+2\varepsilon)}$ is absolutely convergent by the comparison test with respect to $\sum_{n\ge1} n^{-(1+\varepsilon)}$. In terms of $D(s)$, this means $\sigma_a \le \sigma_c + 1 + 2\varepsilon$ and taking $\varepsilon \to 0$ gives the second inequality. □

We will now introduce several convergence theorems for Dirichlet series. It will be useful to setup some notation first. If $D(s)$ is a Dirichlet series with coefficients $a(n)$, then for $X > 0$, we set

$$A(X) = \sum_{n \le X} a(n) \quad \text{and} \quad |A|(X) = \sum_{n \le X} |a(n)|.$$

These are the partial sums of the coefficients $a(n)$ and $|a(n)|$ up to X respectively. Our first convergence theorem relates boundedness of $A(X)$ to the value of σ_c:

Proposition 6.1.1. *Suppose $D(s)$ is a Dirichlet series and that $A(X) \ll 1$. Then $\sigma_c \le 0$.*

Proof. Let s be such that $\sigma > 0$. Since $A(X) \ll 1$, $A(X)X^{-s} \to 0$ as $X \to \infty$. Abel's summation formula (see Appendix B.3) then implies

$$D(s) = s\int_1^\infty A(u)u^{-(s+1)}\,du.$$

But because $A(u) \ll 1$, we have

$$s\int_1^\infty A(u)u^{-(s+1)}\,du \ll s\int_1^\infty u^{-(\sigma+1)}\,du = -\frac{s}{\sigma}u^{-\sigma}\Big|_1^\infty = \frac{s}{\sigma},$$

so that the integral exists for $\sigma > 0$. Thus $D(s)$ converges for $\sigma > 0$ and so $\sigma_c \le 0$. □

Our next theorem states that if the coefficients of $D(s)$ are of polynomial growth, we can obtain an upper bound for the abscissa of absolute convergence:

Proposition 6.1.2. *Suppose $D(s)$ is a Dirichlet series whose coefficients satisfy $a(n) \ll_\alpha n^\alpha$ for some real α. Then the abscissa of absolute convergence satisfies $\sigma_a \le 1 + \alpha$.*

Proof. It suffices to show that $D(s)$ is absolutely convergent in the region $\sigma > 1+\alpha$. For s is in this region, the polynomial bound gives

$$D(s) \ll \sum_{n \ge 1} \left|\frac{a(n)}{n^s}\right| \ll_\alpha \sum_{n \ge 1} \frac{1}{n^{\sigma-\alpha}}.$$

The latter series converges by the integral test because $\sigma - \alpha > 1$. Therefore $D(s)$ is absolutely convergent. □

Obtaining polynomial bounds on coefficients of Dirichlet series are, in most cases, not hard to establish. So the assumption in Proposition 6.1.2 is mild. Actually, there is a partial converse to Proposition 6.1.2 which gives an approximate size to $A(X)$:

Proposition 6.1.3. *Suppose $D(s)$ is a Dirichlet series with finite and nonnegative abscissa of absolute convergence σ_a. Then*

$$A(X) \ll_\varepsilon X^{\sigma_a+\varepsilon}.$$

Proof. By Abel's summation formula (see Appendix B.3),

$$\sum_{n \le X} \frac{a(n)}{n^{\sigma_a+\varepsilon}} = A(X)X^{-(\sigma_a+\varepsilon)} + (\sigma_a+\varepsilon)\int_0^X A(u)u^{-(\sigma_a+\varepsilon+1)}\,du. \tag{6.4}$$

If we set $R(u) = \sum_{n \ge u} \frac{a(n)}{n^{\sigma_a+\varepsilon}}$, then $a(n) = (R(n) - R(n+1))n^{\sigma_a+\varepsilon}$ and it follows that

$$A(u) = \sum_{n \le u}(R(n) - R(n+1))n^{\sigma_a+\varepsilon}.$$

Substituting this into Equation (6.4), we obtain

$$\int_0^X \sum_{n\le u}(R(n)-R(n+1))n^{\sigma_a+\varepsilon}u^{-(\sigma_a+\varepsilon+1)}\,du.$$

As $R(n)$ is constant on the interval $[n,n+1)$, linearity of the integral implies

$$\int_0^X \sum_{n\le u}(R(n)-R(n+1))n^{\sigma_a+\varepsilon}u^{-(\sigma_a+\varepsilon+1)}\,du = \sum_{0\le n\le X}(R(n)-R(n+1))n^{\sigma_a+\varepsilon}\int_n^{n+1}u^{-(\sigma_a+\varepsilon+1)}+O_\varepsilon(1),$$

where the O-estimate is present since X may not be an integer. Now $R(n)\ll_\varepsilon 1$ since it is the tail of $D(\sigma_a+\varepsilon)$ and moreover,

$$\int_n^{n+1}u^{-(\sigma_a+\varepsilon+1)} = -\frac{u^{-(\sigma_a+\varepsilon)}}{\sigma_a+\varepsilon}\bigg|_n^{n+1} = \frac{n^{-(\sigma_a+\varepsilon)}}{\sigma_a+\varepsilon}-\frac{(n+1)^{-(\sigma_a+\varepsilon)}}{\sigma_a+\varepsilon}\ll_\varepsilon 1,$$

because $\sigma_a+\varepsilon>0$. So

$$\int_0^X A(u)u^{-(\sigma_a+\varepsilon+1)}\,du = \int_0^X \sum_{n\le u}(R(n)-R(n+1))n^{\sigma_a+\varepsilon}u^{-(\sigma_a+\varepsilon+1)}\,du \ll_\varepsilon 1.$$

Also, $\sum_{n\le X}\frac{a(n)}{n^{\sigma_a+\varepsilon}}\ll_\varepsilon 1$ because $D(\sigma_a+\varepsilon)$ converges. We conclude

$$A(X)X^{-(\sigma_a+\varepsilon)} = \sum_{n\le X}\frac{a(n)}{n^{\sigma_a+\varepsilon}}-(\sigma_a+\varepsilon)\int_0^X A(u)u^{-(\sigma_a+\varepsilon+1)}\,du. \ll_\varepsilon 1,$$

which is equivalent to the desired estimate. □

A way to think about Proposition 6.1.3 is that if the abscissa of absolute convergence is $\sigma_a\ge 0$ then the size of the coefficients $a(n)$ is at most $n^{\sigma_a+\varepsilon}$ on average. Of corse, if $a(n)\ll_\alpha n^\alpha$ then Proposition 6.1.2 implies that $\sigma_a\le 1+\alpha$ and so Proposition 6.1.3 gives the significantly weaker estimate $A(X)\ll_\varepsilon X^{1+\alpha+\varepsilon}$. However, if we have a bound of the form $|A|(X)\ll_\alpha X^\alpha$ we can still obtain an upper estimate for the abscissa of absolute convergence:

Proposition 6.1.4. *Suppose $D(s)$ is a Dirichlet series such that $|A|(X)\ll_\alpha X^\alpha$ for some real $\alpha\ge 0$. Then the abscissa of absolute convergence satisfies $\sigma_a\le\alpha$.*

Proof. It suffices to show that $D(s)$ is absolutely convergent in the region $\sigma>\alpha$. Let s be in this region. Then

$$D(s)\ll\sum_{n\ge1}\left|\frac{a(n)}{n^s}\right| = \sum_{n\ge1}\frac{|a(n)|}{n^\sigma}.$$

By Abel's summation formula (see Appendix B.3),

$$\sum_{n\le N}\frac{|a(n)|}{n^\sigma} = |a(N)|N^{-\sigma}-|a(1)|+\sigma\int_1^N |A|(u)u^{-(\sigma+1)}\,du,$$

By assumption $|A|(u)\ll_\alpha u^\alpha$ and so $a(N)\ll_\alpha N^\alpha$. We then estimate as follows:

$$\sum_{n\le N}\frac{|a(n)|}{n^\sigma} = |a(N)|N^{-\sigma}-|a(1)|+\sigma\int_1^N |A|(u)u^{-(\sigma+1)}\,du \ll_\alpha |a(N)|N^{-\sigma}+|a(1)|+\sigma\int_1^N u^{\alpha-(\sigma+1)}\,du.$$

As $N \to \infty$, the left-hand side tends towards $\sum_{n\ge1}\frac{|a(n)|}{n^\sigma}$. As for the right-hand side, the first term tends to zero since $\sigma > \alpha$ and the second term remains bounded as they are independent of N. For the third term, we compute

$$\int_1^N u^{\alpha-(\sigma+1)}\,du = \frac{u^{\alpha-\sigma}}{\alpha-\sigma}\bigg|_1^N = \frac{N^{\alpha-\sigma}}{\alpha-\sigma} - \frac{1}{\alpha-\sigma},$$

which is also bounded as $N \to \infty$. This finishes the proof. □

Do not be fooled; Proposition 6.1.4 is in general weaker than Proposition 6.1.2. For example, from our comments following Proposition 6.1.3, if $D(s)$ is a Dirichlet series with coefficients $a(n)$ and we have the estimate $|A|(X) \ll_\beta X^\beta$ for some real β then Proposition 6.1.4 only says that $\sigma_a \le \beta$. If α is very small compared to β, this is a significantly worse upper bound for the abscissa of absolute convergence than what Proposition 6.1.2 would imply if $a(n) \ll_\alpha n^\alpha$. Actually, the question of sharp polynomial bounds for these coefficients can be very deep. However, if the coefficients $a(n)$ are always nonnegative, then **Landau's theorem** provides a way of obtaining a lower bound for their growth as well as describing a singularity of $D(s)$:

Theorem 6.1.3 (Landau's theorem). *Suppose $D(s)$ is a Dirichlet series with nonnegative coefficients $a(n)$ and finite abscissa of absolute convergence σ_a. Then σ_a is a singularity of $D(s)$.*

Proof. If we replace $a(n)$ by $a(n)n^{-\sigma_a}$ then we may assume $\sigma_a = 0$. Now suppose to the contrary that $D(s)$ was holomorphic at $s = 0$. Therefore for some $\delta > 0$, $D(s)$ is holomorphic in the domain

$$\mathcal{D} = \{s \in \mathbb{C} : \sigma_a > 0\} \cup \{s \in \mathbb{C} : |s| < \delta\}.$$

Write $D(s)$ as a power series at $s = 1$:

$$P(s) = \sum_{k\ge0} c_k(s-1)^k,$$

where

$$c_k = \frac{D^{(k)}(1)}{k!} = \frac{1}{k!}\sum_{n\ge1}\frac{a(n)(-\log(n))^k}{n},$$

because $D(s)$ is holomorphic and so we can differentiate termwise. The radius of convergence of $P(s)$ is the distance from $s = 1$ to the nearest singularity of $P(s)$. Since $P(s)$ is holomorphic on $\mathcal{D}$, the closest points are $\pm i\delta$. Therefore, the radius of convergence is at least $|1 \pm \delta| = \sqrt{1+\delta^2}$. We can write $\sqrt{1+\delta^2} = 1 + \delta'$ for some $\delta' > 0$. Then for $|s-1| < 1 + \delta'$, write $P(s)$ as

$$P(s) = \sum_{k\ge0}\frac{(s-1)^k}{k!}\sum_{n\ge1}\frac{a(n)(-\log(n))^k}{n} = \sum_{k\ge0}\frac{(1-s)^k}{k!}\sum_{n\ge1}\frac{a(n)(\log(n))^k}{n}.$$

If s is real with $s < 1$, then this last double sum is a sum of positive terms because $a(n) \ge 0$. Moreover, since $D(s)$ is convergent here the two sums can be interchanged by the dominated convergence theorem. Interchanging sums we see that

$$P(s) = \sum_{n\ge1}\frac{a(n)}{n}\sum_{k\ge0}\frac{(1-s)^k(\log(n))^k}{k!} = \sum_{n\ge1}\frac{a(n)}{n}e^{(1-s)\log(n)} = \sum_{n\ge1}\frac{a(n)}{n^s} = D(s),$$

for $-\delta' < s < 1$. As $\delta' > 0$, this implies that $D(s)$ converges absolutely (since $a(n)$ is nonnegative) for some $s < 0$ (say $s = -\frac{\delta'}{2}$) which contradicts $\sigma_a = 0$. □

Landau's theorem is very useful as it implies that if $D(s)$ is a Dirichlet series with nonnegative coefficients then $a(n) \not\ll_{\varepsilon} n^{\sigma_a-(1+\varepsilon)}$ because otherwise Proposition 6.1.2 implies $\sigma_a \le \sigma_a - \varepsilon$. Actually, Landau's theorem also implies $|A|(X) \not\ll_{\varepsilon} X^{\sigma_a-\varepsilon}$ for otherwise Proposition 6.1.4 would similarly imply $\sigma_a \le \sigma_\alpha - \varepsilon$. When we come across Dirichlet series whose coefficients are of polynomial growth or of polynomial growth on average, we will invoke these results without mention, except for Landau's theorem, as this is also common practice in the literature. Generally speaking, if the coefficients $a(n)$ are chosen at random, $D(s)$ will not possess any good properties outside of convergence in some region (it might not even possess that). However, the Dirichlet series we will encounter, and most of interest in the wild, will have multiplicative coefficients. In this case, the Dirichlet series admits an infinite product expression:

Proposition 6.1.5. *Suppose the coefficients $a(n)$ of a Dirichlet series $D(s)$ are multiplicative and satisfy $a(n) \ll_\alpha n^\alpha$ for some real $\alpha \ge 0$. Then*

$$D(s) = \prod_p \left(\sum_{k \ge 0} \frac{a(p^k)}{p^{ks}} \right),$$

for $\sigma > 1 + \alpha$. Conversely, suppose that there are coefficients $a(n)$ such that

$$\prod_p \left(\sum_{k \ge 0} \left| \frac{a(p^k)}{p^{ks}} \right| \right),$$

converges for $\sigma > 1 + \alpha$. Then the equality above defines a Dirichlet series $D(s)$ that converges absolutely in this region too. Moreover, if the coefficients $a(n)$ are completely multiplicative, then

$$D(s) = \prod_p (1 - a(p)p^{-s})^{-1},$$

for $\sigma > 1 + \alpha$.

Proof. Since $a(n) \ll_\alpha n^\alpha$, Proposition 6.1.2 implies that $D(s)$ converges absolutely for $\sigma > 1 + \alpha$. Let s be such that $\sigma > 1 + \alpha$. Since

$$\sum_{k \ge 0} \left| \frac{a(p^k)}{p^{ks}} \right| < \sum_{n \ge 1} \left| \frac{a(n)}{n^s} \right|,$$

the infinite series on the left converges because the right does by the absolute convergence of $D(s)$. Now let $N > 0$ be an integer. Then by the fundamental theorem of arithmetic

$$\prod_{p<N} \left(\sum_{k \ge 0} \frac{a(p^k)}{p^{ks}} \right) = \sum_{n<N} \frac{a(n)}{n^s} + \sideset{}{^*}\sum_{n \ge N} \frac{a(n)}{n^s}, \tag{6.5}$$

where the $*$ denotes that we are summing over only those additional terms $\frac{a(n)}{n^s}$ that appear in the expanded product on the left-hand side with $n \ge N$. As $N \to \infty$, the first sum on the right-hand side tends to $D(s)$ and the second sum tends to zero because it is part of the tail of $D(s)$ (which tends to zero by convergence). This proves that the product converges, and is equal to $D(s)$. Equation (6.5) also holds absolutely in the sense that

$$\prod_{p<N} \left(\sum_{k \ge 0} \left| \frac{a(p^k)}{p^{ks}} \right| \right) = \sum_{n<N} \left| \frac{a(n)}{n^s} \right| + \sideset{}{^*}\sum_{n \ge N} \left| \frac{a(n)}{n^s} \right|, \tag{6.6}$$

since $D(s)$ converges absolutely. For the converse statement, since the product

$$\prod_p \left(\sum_{k \geq 0} \left| \frac{a(p^k)}{p^{ks}} \right| \right),$$

converges for $\sigma > 1 + \alpha$ each factor is necessarily finite. That is, for each prime p the series $\sum_{k \geq 0} \frac{a(p^k)}{p^{ks}}$ converges absolutely in this region. Now fix an integer $N > 0$. Then Equation (6.6) holds. Taking $N \to \infty$ in Equation (6.6), the left-hand side converges by assumption. Therefore the right-hand sides does too. But the first sum on the right-hand side tends to

$$\sum_{n \geq 1} \left| \frac{a(n)}{n^s} \right|,$$

and the second sum is part of its tail. So the first sum must convergence hence defining an absolutely convergent Dirichlet series in $\sigma > 1 + \alpha$, and the second sum must tend to zero. Lastly, if the $a(n)$ are completely multiplicative the formula for a geometric series gives

$$\prod_p \left(\sum_{k \geq 0} \frac{a(p^k)}{p^{ks}} \right) = \prod_p \left(\sum_{k \geq 0} \left(\frac{a(p)}{p^s} \right)^k \right) = \prod_p (1 - a(p)p^{-s})^{-1}.$$

$\square$

Note that in Proposition 6.1.5, the requirement for a product to define an absolutely convergent Dirichlet series is that the series defining the factors in the product must be absolutely convergent. Thankfully, this is always the case for geometric series. Now suppose $D(s)$ is a Dirichlet series that has the product expression

$$D(s) = \prod_p (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1} \cdots (1 - \alpha_d(p)p^{-s})^{-1}.$$

We call this product the **Euler product** of $D(s)$, and it is said to be of **degree** d. In Proposition 6.1.5, complete multiplicativity of the coefficients is enough to guarantee that $D(s)$ has an Euler product of degree 1, but in general $D(s)$ will admit an Euler product of degree $d > 1$ if the coefficients are only multiplicative but satisfy additional properties like a recurrence relation. When we come across Dirichlet series whose coefficients are multiplicative or we are given an Euler product we will use Proposition 6.1.5 without mention as this is common practice in the literature. Lastly, if $D(s)$ has an Euler product then for any $N \geq 1$ we let $D^{(N)}(s)$ denote the Dirichlet series with the factors $p \mid N$ in the Euler product removed. That is,

$$D^{(N)}(s) = D(s) \prod_{p \mid N} (1 - \alpha_1(p)p^{-s})(1 - \alpha_2(p)p^{-s}) \cdots (1 - \alpha_d(p)p^{-s}).$$

Dually, for any $N \geq 1$ we let $D_N(s)$ denote the Dirichlet series only consisting of the factors $p \mid N$ in the Euler product. That is,

$$D_N(s) = \prod_{p \mid N} (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1} \cdots (1 - \alpha_d(p)p^{-s})^{-1}.$$

6.2 Perron Formulas

With the Mellin inversion formula, it is not hard to prove a very useful integral expression for the sum of coefficients of a Dirichlet series. First, we setup some general notation. If $D(s)$ is a Dirichlet series with coefficients $a(n)$, then for $X > 0$, we set

$$A'(X) = {\sum_{n \leq X}}' a(n),$$

where the $'$ indicates that the last term is multiplied by $\frac{1}{2}$ if X is an integer. We would like to relate $A'(X)$ to an integral involving the entire Dirichlet series $D(s)$. In particular, this integral is a type of inverse Mellin transform. Any formula that relates a finite sum of coefficients of a Dirichlet series to an integral involving the entire Dirichlet series is called a **Perron-type formula**. We will see several of them, the first being **(classical) Perron's formula** which is a consequence of Abel's summation formula and the Mellin inversion formula applied to Dirichlet series:

Theorem 6.2.1 (Perron's formula, classical version). *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for any $c > \sigma_a$,*

$$A'(X) = \frac{1}{2\pi i} \int_{(c)} D(s) X^s \, \frac{ds}{s}.$$

Proof. Let s be such that $\sigma > \sigma_a$. By Abel's summation formula (see Appendix B.3),

$$\sum_{n \geq 1} \frac{a(n)}{n^s} = \lim_{Y \to \infty} A'(Y) Y^{-s} + s \int_1^\infty A'(u) u^{-(s+1)} \, du.$$

Now $A'(Y) \leq A(Y)$ and $A(Y) \ll_\varepsilon Y^{\sigma_a + \varepsilon}$ by Proposition 6.1.3 so that $A'(Y)Y^{-s} \ll_\varepsilon Y^{\sigma_a + \varepsilon - \sigma}$. Choosing $\varepsilon < \sigma - \sigma_a$, this latter term tends to zero as $Y \to \infty$, which implies that $A'(Y)Y^{-s} \to 0$ as $Y \to \infty$. Therefore we can write the equation above as

$$\frac{D(s)}{s} = \int_1^\infty A'(u) u^{-(s+1)} \, du = \int_0^\infty A'(u) u^{-(s+1)} \, du,$$

where the second equality follows because $A(u) = 0$ in the interval $[0, 1)$. The Mellin inversion formula immediately gives the result. □

We would like to relate this sum to an integral involving the entire Dirichlet series $D(s)$. In particular, this integral is a type of inverse Mellin transform. Any formula that resembles Perron's formula is particularly useful because it allows one examine a sum of coefficients of a Dirichlet series, a discrete object, by means of a complex integral where analytic techniques are at our disposal. There is also a truncated version of Perron's formula which is often more useful for estimates rather than abstract results. To state it, we need to setup some notation and will require a lemma. For any $c > 0$, consider the discontinuous integral (see [Dav80])

$$\delta(y) = \frac{1}{2\pi i} \int_{(c)} y^s \, \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$$

Also, for any $T > 0$, let

$$I(y, T) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} y^s \, \frac{ds}{s},$$

be $\delta(y)$ truncated outside of height T. The lemma we require gives an approximation for how close $I(y, T)$ is to $\delta(y)$ (see [Dav80] for a proof):

Lemma 6.2.1. *For any $c > 0$, $y > 0$, and $T > 0$, we have*

$$I(y,T) - \delta(y) = \begin{cases} O\left(y^c \min\left(1, \frac{1}{T|\log(y)|}\right)\right) & \text{if } y \neq 1, \\ O\left(\frac{c}{T}\right) & \text{if } y = 1. \end{cases}$$

We can now state and prove **(truncated) Perron's formula**:

Theorem 6.2.2 (Perron's formula, truncated version). *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for any $c > \sigma_a$ and $T > 0$,*

$$A'(X) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} D(s)X^s \frac{ds}{s} + O\left(X^c \sum_{\substack{n\geq 1 \\ n\neq X}} \frac{a(n)}{n^c} \min\left(1, \frac{1}{T\left|\log\left(\frac{X}{n}\right)\right|}\right) + \delta_X a(X)\frac{c}{T}\right),$$

where $\delta_X = 1, 0$ according to if X is an integer or not.

Proof. By Appendix D.1, we have

$$A'(X) = \sum_{n\geq 1} a(n)\delta\left(\frac{X}{n}\right).$$

Using Lemma 6.2.1, we may replace $\delta\left(\frac{X}{n}\right)$ and obtain

$$A'(X) = \sum_{n\geq 1} a(n)\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{X^s}{n^s}\frac{ds}{s} + \sum_{\substack{n\geq 1 \\ n\neq X}} a(n) O\left(\frac{X^c}{n^c}\min\left(1, \frac{1}{T\left|\log\left(\frac{X}{n}\right)\right|}\right)\right) + \delta_X a(X) O\left(\frac{c}{T}\right).$$

Since $D(s)$ converges absolutely the sum may be moved inside of the first O-estimate. Then combine the two O-estimates. The dominated convergence theorem implies we may interchange the sum and the integral. The statement of the lemma follows. □

There is a slightly weaker variant of truncated Perron's formula that follows as a corollary:

Corollary 6.2.1. *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Then for any $c > \sigma_a$ and $T > 0$,*

$$A'(X) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} D(s)X^s \frac{ds}{s} + O\left(\frac{X^c}{T}\right),$$

Proof. For sufficiently large X, we have

$$\min\left(1, \frac{1}{T\left|\log\left(\frac{X}{n}\right)\right|}\right) \ll \frac{X^c}{T}.$$

The statement now follows from truncated Perron's formula. □

There is also a version of Perron's formula where we add a smoothing function. For any $X > 0$, we set

$$A_\psi(X) = \sum_{n \ge 1} a(n)\psi\left(\frac{n}{X}\right),$$

where $\psi(y)$ is smooth function such that $\psi(y) \to 0$ as $y \to \infty$. This is most useful in two cases. The first is when we choose $\psi(y)$ to be a bump function. In this setting, the bump function can be chosen such that it assigns weight 1 or 0 to the coefficients $a(n)$ and we can estimate sums like

$$\sum_{\frac{X}{2} \le n < X} a(n) \quad \text{or} \quad \sum_{X \le n < X+Y} a(n),$$

for some X and Y with $Y < X$. Sums of this type are called **unweighted**. As an example of an unweighted sum, let $\psi(y)$ be a bump function that is identically 1 on $[0,1]$ and has compact support within the interval $\left[1, \frac{X+1}{X}\right]$. For example,

$$\psi(y) = \begin{cases} 1 & \text{if } 0 \le y \le 1, \\ e^{-\frac{1-y}{\frac{X+1}{X} - y}} & \text{if } 1 \le y < \frac{X+1}{X}, \\ 0 & \text{if } y \ge \frac{X+1}{X}. \end{cases}$$

Then

$$A_\psi(X) = \sum_{n \ge 1} a(n)\psi\left(\frac{n}{X}\right) = \sum_{n \le X} a(n).$$

In the second case, we want $\psi(y)$ to dampen the $a(n)$ with a weight other than 1 or 0. Sums of this type are called **weighted**. In any case, suppose the support of $\psi(y)$ is contained in $[0, M]$. These conditions will force the Mellin transform $\Psi(s)$ of $\psi(y)$ to exist and have nice properties. To see that $\Psi(s)$ exists, let K be a compact set in $\mathbb{C}$ and let $\alpha = \max_{s \in K}\{\sigma\}$ and $\beta = \min_{s \in K}\{\sigma\}$. Note that $\psi(y)$ is bounded because it is compactly supported. Then for $s \in K$,

$$\Psi(s) = \int_0^\infty \psi(y)y^s \,\frac{dy}{y} \ll \int_0^M y^{\sigma-1}\,dy \ll_{\alpha,\beta} 1.$$

Therefore $\Psi(s)$ is locally absolutely uniformly convergent for $s \in \mathbb{C}$. In particular, the Mellin inversion formula implies that $\psi(y)$ is the Mellin inverse of $\Psi(s)$. As for nice properties, $\Psi(s)$ does not grow too fast in vertical strips:

Proposition 6.2.1. *Suppose $\psi(y)$ is a bump function and let $\Psi(s)$ denote its Mellin transform. Then for any $N \ge 1$,*

$$\Psi(s) \ll (|s| + 1)^{-N},$$

provided s is contained in the vertical strip $a < \sigma < b$ for any a and b with $0 < a < b$.

Proof. Fix a and b with $a < b$. Also, let the support of $\psi(y)$ be contained in $[0, M]$. Now consider

$$\Psi(s) = \int_0^\infty \psi(y)y^s \,\frac{dy}{y}.$$

Since $\psi(y)$ is compactly supported, integrating by parts yields

$$\Psi(s) = \frac{1}{s}\int_0^\infty \psi'(y)y^{s+1}\,\frac{dy}{y}.$$

Repeatedly integrating by parts $N \geq 1$ times, we arrive at

$$\Psi(s) = \frac{1}{s(s+1)\cdots(s+N-1)} \int_0^\infty \psi^{(N)}(y) y^{s+N} \, \frac{dy}{y}.$$

Therefore

$$\Psi(s) \ll (|s|+1)^{-N} \int_0^\infty \psi^{(N)}(y) y^{\sigma+N} \, \frac{dy}{y}.$$

The claim will follow if we can show that the integral is bounded. Since $\psi(y)$ is compactly supported in $[0, M]$ so is $\psi^{(N)}(y)$. In particular, $\psi^{(N)}(y)$ is bounded. Therefore

$$\int_0^\infty \psi^{(N)}(y) y^{\sigma+N} \, \frac{dy}{y} \ll \int_0^M y^{\sigma+N} \, \frac{dy}{y} = \frac{y^{\sigma+N}}{\sigma+N}\bigg|_0^M = \frac{M^{\sigma+N}}{\sigma+N} \ll \frac{M^{b+N}}{N} \ll 1,$$

where the second to last estimate follows because $a < \sigma < b$ with $0 < a < b$. So the integral is bounded and the claim follows. $\square$

The following theorem is **(smoothed) Perron's formula**:

Theorem 6.2.3 (Perron's formula, smoothed version). *Let $D(s)$ be a Dirichlet series with coefficient $a(n)$ and finite and nonnegative abscissa of absolute convergence σ_a. Let $\psi(y)$ be a bump function and denote its Mellin transform by $\Psi(s)$. Then for any $c > \sigma_a$,*

$$A_\psi(X) = \frac{1}{2\pi i} \int_{(c)} D(s)\Psi(s) X^s \, ds.$$

In particular,

$$\sum_{n \geq 1} a(n)\psi(n) = \frac{1}{2\pi i} \int_{(c)} D(s)\Psi(s) \, ds.$$

Proof. The first statement is just a computation:

$$\begin{aligned}
A_\psi(X) &= \sum_{n \geq 1} a(n)\psi\left(\frac{n}{X}\right) \\
&= \sum_{n \geq 1} \frac{a(n)}{2\pi i} \int_{(c)} \Psi(s) \left(\frac{n}{X}\right)^{-s} ds && \text{Mellin inversion formula} \\
&= \frac{1}{2\pi i} \int_{(c)} \sum_{n \geq 1} a(n)\Psi(s) \left(\frac{n}{X}\right)^{-s} ds && \text{DCT} \\
&= \frac{1}{2\pi i} \int_{(c)} D(s)\Psi(s) X^s \, ds.
\end{aligned}$$

This proves the first statement. For the second statement, take $X = 1$. $\square$

Smoothed Perron's formula is useful because it is often more versatile as the convergence of the integral is improved if $\psi(y)$ is chosen appropriately.

6.3 Analytic Data of L-functions

We are now ready to discuss L-functions in some generality. In the following, we will denote an L-functions by $L(s,f)$, and for the moment, f will carry no formal meaning. It is only used to suggest that the L-function is attached to some interesting arithmetic object f. When we discuss specific L-functions, f will carry a formal meaning. An **L-series** $L(s,f)$ is a Dirichlet series

$$L(s,f) = \sum_{n\geq 1} \frac{a_f(n)}{n^s},$$

where the $a_f(n) \in \mathbb{C}$ are coefficients usually attached to some arithmetic object f. We call $L(s,f)$ an **L-function** if it satisfies the following properties:

(i) $L(s,f)$ is locally absolutely uniformly convergent for $\sigma > 1$ and admits a degree d_f Euler product:

$$L(s,f) = \sum_{n\geq 1} \frac{a_f(n)}{n^s} = \prod_p (1-\alpha_1(p)p^{-s})^{-1}(1-\alpha_2(p)p^{-s})^{-1}\cdots(1-\alpha_{d_f}(p)p^{-s})^{-1},$$

with $a_f(n), \alpha_j(p) \in \mathbb{C}$, $a_f(1) = 1$, and $|\alpha_j(p)| \leq p$ for all primes p. We call

$$L_p(s,f) = (1-\alpha_1(p)p^{-s})^{-1}(1-\alpha_2(p)p^{-s})^{-1}\cdots(1-\alpha_{d_f}(p)p^{-s})^{-1},$$

the **local factor** at p, and the $\alpha_j(p)$ are called the **local roots** (or **local parameters**) at p.

(ii) There exists a factor

$$\gamma(s,f) = \pi^{-\frac{d_f s}{2}} \prod_{1\leq j\leq d_f} \Gamma\left(\frac{s+\kappa_j}{2}\right),$$

with $\kappa_j \in \mathbb{C}$ that are either real or appear in conjugate pairs. We also require $\mathrm{Re}(\kappa_j) > -1$. The κ_j are called the **local roots** (or **local parameters**) at infinity.

(iii) There exists an integer $q(f) \geq 1$ called the **conductor** such that $\alpha_j(p) \neq 0$ for all prime p such that $p \nmid q(f)$. If $p \mid q(f)$, then we say p is **ramified** and is **unramified** otherwise. The **analytic conductor** $\mathfrak{q}(s,f)$ is defined to be

$$\mathfrak{q}(s,f) = q(f)\mathfrak{q}_\infty(s,f),$$

where we set

$$\mathfrak{q}_\infty(s,f) = \prod_{1\leq j\leq d_f} (|s+\kappa_j|+3).$$

For simplicity, we will also suppress the dependence upon s if $s=0$. That is, $\mathfrak{q}(f) = \mathfrak{q}(0,f)$ and $\mathfrak{q}_\infty(f) = \mathfrak{q}_\infty(0,f)$.

(iv) The **completed L-function**

$$\Lambda(s,f) = q(f)^{\frac{s}{2}}\gamma(s,f)L(s,f),$$

must admit meromorphic continuation to $\mathbb{C}$ with at most poles at $s=0$ and $s=1$. Moreover, it must satisfy the **functional equation** given by

$$\Lambda(s,f) = \varepsilon(f)\Lambda(1-s,\overline{f}),$$

where $\varepsilon(f)$ is a complex number with $|\varepsilon(f)| = 1$ called the **root number** of $L(s,f)$, and $\overline{f}$ is an object associated to f called the **dual** of f such that $L(s,\overline{f})$ satisfies $a_{\overline{f}}(n) = \overline{a_f(n)}$, $\gamma(s,\overline{f}) = \gamma(s,f)$, $q(\overline{f}) = q(f)$, and $\varepsilon(\overline{f}) = \overline{\varepsilon(f)}$. We call $L(s,\overline{f})$ the **dual** of $L(s,f)$. If $\overline{f} = f$, we say $L(s,f)$ is **self-dual**.

(v) $L(s,f)$ admits meromorphic continuation to $\mathbb{C}$ with at most a pole at $s=1$ of order $r_f \in \mathbb{Z}$, and must be of order 1 (see Appendix B.5) after clearing the possible polar divisor.

Property (ii) ensures that $\gamma(s,f)$ is holomorphic and nonzero for $\sigma \geq 1$. Then as $\gamma(1,f)$ is nonzero, r_f is also the order of possible poles of $\Lambda(s,f)$ at $s=0$ and $s=1$ which are equal by the functional equation. It follows that $r_{\overline{f}} = -r_f$. As $L(s,f)$ admits meromorphic continuation by property (v), we denote the continuation by $L(s,f)$ as well. It is also clear that the product of L-functions is also an L-function. Accordingly, we say that an L-function is **primitive** if is cannot be written as a product of two L-functions. Moreover, we say that an L-function $L(s,f)$ belongs to the **Selberg class** if the additional adjustments to (i), (ii), and (v) hold:

(i) The local roots at p satisfy $|\alpha_j(p)| = 1$ if $p \nmid q(f)$ and $|\alpha_j(p)| \leq 1$ if $p \mid q(f)$.

(ii) The local roots at infinity satisfy $\mathrm{Re}(\kappa_j) \geq 0$.

(v) $L(s,f)$ has at most a simple pole at $s=1$. That is, $r_f \leq 1$.

The adjustment for (i) is called the **(generalized) Ramanujan-Petersson conjecture** and it forces $a_f(n) \ll \sigma_0(n) \ll_\varepsilon n^\varepsilon$ (recall Proposition A.3.1). The adjustment for (ii) is called the **(generalized) Selberg conjecture** and it ensures that $\gamma(s,f)$ is holomorphic and nonzero for $\sigma > 0$. As for the adjustment for (v), it is expected that $L(s,f)$ is entire unless $\alpha_i(p) \geq 0$. The Selberg class constitutes a very special class of L-functions. Nevertheless, suppose we are given two L-functions $L(s,f)$ and $L(s,g)$ with local roots $\alpha_j(p)$ and $\beta_\ell(p)$ at p and local roots κ_j and ν_ℓ at infinity respectively. We say that an L-function $L(s,f\otimes g)$ of degree $d_{f\otimes g} = d_f d_g$ is the **Rankin-Selberg convolution** of $L(s,f)$ and $L(s,g)$ (or **Rankin-Selberg square** if $f=g$) if it satisfies the following adjustments:

(i) The Euler product of $L(s,f\otimes g)$ takes the form

$$L(s,f\otimes g) = \prod_{p \nmid q(f)q(g)} L_p(s,f\otimes g) \prod_{p \mid q(f)q(g)} H_p(p^{-s}),$$

with

$$L_p(s,f\otimes g) = \prod_{\substack{1\leq j\leq d_f \\ 1\leq \ell\leq d_g}} \left(1-\alpha_j(p)\overline{\beta_\ell(p)}p^{-s}\right)^{-1} \quad \text{and} \quad H_p(p^{-s}) = \prod_{1\leq j\leq d_{f\otimes g}} (1-\gamma_j(p)p^{-s}),$$

for some $\gamma_j(p) \in \mathbb{C}$ with $|\gamma_j(p)| \leq p$.

(ii) The gamma factor $\gamma(s,f\otimes g)$ takes the form

$$\gamma(s,f\otimes g) = \pi^{-\frac{d_{f\otimes g}s}{2}} \prod_{\substack{1\leq j\leq d_f \\ 1\leq \ell\leq d_g}} \Gamma\left(\frac{s+\mu_{j,\ell}}{2}\right),$$

with the local roots at infinity satisfying the additional bounds $\mathrm{Re}(\mu_{j,\ell}) \leq \mathrm{Re}(\kappa_j) + \mathrm{Re}(\nu_\ell)$ and $|\mu_{j,\ell}| \leq |\kappa_j| + |\nu_\ell|$.

(iii) The root number $q(f\otimes g)$ satisfies $q(f\otimes g) \mid q(f)^{d_f} q(g)^{d_g}$. If $q(f\otimes g)$ is a proper divisor of $q(f)^{d_f}q(g)^{d_g}$, we say that **conductor dropping** occurs.

(v) $L(s,f\otimes g)$ has a pole of order $r_{f\otimes g} \geq 1$ at $s=1$ if $g = \overline{f}$.

A few additional comments are in order for L-functions in general. The **critical strip** is the vertical strip left invariant by the transformation $s \to 1-s$, that is, the region defined by

$$\left\{ s \in \mathbb{C} : \left| \sigma - \frac{1}{2} \right| \leq \frac{1}{2} \right\}.$$

Moreover, it is also the region where we cannot determine the value of an L-function from its representation as a Dirichlet series or using the functional equation. It turns out that much of the important information about L-functions are contained inside of the critical strip. The **critical line** is the line left invariant by the transformation $s \to 1-s$ which is the line defined by $\sigma = \frac{1}{2}$. It is also the line that bisects the critical strip vertically. The **central point** is the fixed point of the transformation $s \to 1-s$, in other words, the point $s = \frac{1}{2}$. Clearly the central point is also the center of the critical line. The critical strip, critical line, and central point are all displayed in Figure 6.1:

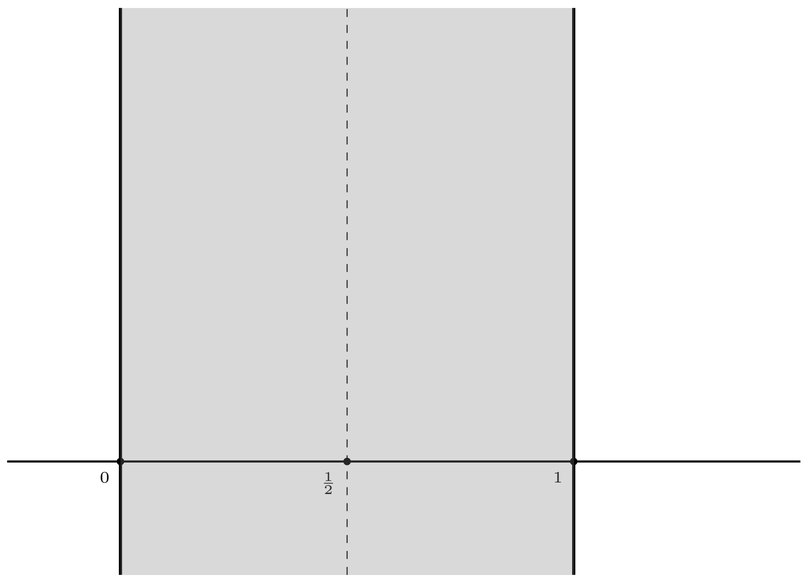

Figure 6.1: The critical strip, critical line, and central point.

6.4 The Approximate Functional Equation

If $L(s,f)$ is an L-function, then there is a formula which acts as a compromise between the functional equation for $L(s,f)$ and expressing $L(s,f)$ as a Dirichlet series. This formula is known as the approximate functional equation and it is important because it is valid inside of the critical strip and therefore can be used to obtain data about $L(s,f)$. First, we need to state an important asymptotic about the ratio of gamma factors that we will use later on. First suppose σ is bounded and $|t| > 1$. This guarantees that s is bounded away from zero so by Corollary 1.6.2, we have

$$\Gamma(s) \sim \sqrt{2\pi} t^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|}.$$

This gives the weaker estimates

$$\Gamma(s) \ll t^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|} \quad \text{and} \quad \frac{1}{\Gamma(s)} \ll t^{\frac{1}{2} - \sigma} e^{\frac{\pi}{2}|t|}.$$

It is not hard to obtain estimates that holds in vertical strips. Suppose s is in the vertical strip $a < \sigma < b$ and is distance ε away from the poles of $\Gamma(s)$. For $|t| > 1$, the estimates above are satisfied. Now recall that $\Gamma(s)$ is bounded on the compact region $a \le \sigma \le b$ with $|t| \le 1$ provided s is distance ε away from the poles of $\Gamma(s)$. It follows that the estimates

$$\Gamma(s) \ll_\varepsilon (|t|+1)^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|} \quad \text{and} \quad \frac{1}{\Gamma(s)} \ll (|t|+1)^{\frac{1}{2}-\sigma} e^{\frac{\pi}{2}|t|}, \tag{6.7}$$

are valid in the vertical strip $a < \sigma < b$ provided s is distance ε away from the poles of $\Gamma(s)$ in the former case. We immediately obtain the useful estimate

$$\frac{\Gamma(1-s)}{\Gamma(s)} \ll_\varepsilon (|t|+1)^{1-2\sigma},$$

valid in the vertical strip $a < \sigma < b$ provided s is distance ε away from the poles of $\Gamma(1-s)$. It easily follows from the definition of $\gamma(s,f)$ that the estimates

$$\frac{\gamma(1-s,f)}{\gamma(s,f)} \ll_\varepsilon \mathfrak{q}_\infty(s,f)^{\frac{1-2\sigma}{2}} \quad \text{and} \quad q(f)^{\frac{1-2s}{2}} \frac{\gamma(1-s,f)}{\gamma(s,f)} \ll_\varepsilon \mathfrak{q}(s,f)^{\frac{1-2\sigma}{2}}, \tag{6.8}$$

are valid in the vertical strip $a < \sigma < b$ provided s is distance ε away from the poles of $\gamma(1-s,f)$. We can use this last estimate to show that $L(s,f)$ is has polynomial growth in the t-aspect in vertical strips:

Proposition 6.4.1. *For any L-function $L(s,f)$ and $a < b$, $L(s,f)$ is of polynomial growth in the t-aspect in the vertical half-strip $a \le \sigma \le b$ with $|t| \ge 1$.*

Proof. On the one hand, for $\sigma > \max(1,b)$ we have $L(s,f) \ll 1$ on the line $\sigma = 1$ with $t \ge 1$. On the other hand, the functional equation and Equation (6.8) together imply

$$L(s,f) \ll \mathfrak{q}(s,f)^{\frac{1-2\sigma}{2}} L(1-s,f).$$

Clearly $\mathfrak{q}(s,f)^{\frac{1-2\sigma}{2}}$ is of polynomial growth in the t-aspect provided σ is bounded. Thus for bounded $\sigma < \min(0,a)$, we see that $L(s,f)$ is also of polynomial growth in the t-aspect for $|t| \ge 1$. Moreover, this estimate also implies $L(s,f)$ is bounded on the line $t = 1$ provided σ is bounded. As $L(s,f)$ is holomorphic for $|t| \ge 1$ and of order 1, we can apply the Phragmén-Lindelöf convexity principle in this region (see Appendix B.6) so that $L(s,f)$ is of polynomial growth in the t-aspect in the vertical half-strip $a \le \sigma \le b$ with $|t| \ge 1$. □

Proposition 6.4.1 is a very important property that L-functions possess. It is usually used to estimate Perron type formulas. We can also use it to deduce the **approximate function equation**:

Theorem 6.4.1 (Approximate functional equation). *Let $L(s,f)$ be an L-function, $\Phi(u)$ be an even holomorphic function bounded in the vertical strip $|\tau| < a+1$ for any $a > 1$ such that $\Phi(0) = 1$, and let $X > 0$. Then for s in the critical strip, we have*

$$L(s,f) = \sum_{n \ge 1} \frac{a_f(n)}{n^s} V_s\left(\frac{n}{\sqrt{q(f)}X}\right) + \varepsilon(s,f) \sum_{n \ge 1} \frac{\overline{a_f(n)}}{n^{1-s}} V_{1-s}\left(\frac{nX}{\sqrt{q(f)}}\right) + \frac{R}{q(f)^{\frac{s}{2}}\gamma(s,f)},$$

where $V_s(y)$ is the inverse Mellin transform defined by

$$V_s(y) = \frac{1}{2\pi i} \int_{(a)} \frac{\gamma(s+u,f)}{\gamma(s,f)} \Phi(u) y^{-u} \frac{du}{u},$$

and

$$\varepsilon(s,f)=\varepsilon(f)q(f)^{\frac{1}{2}-s}\frac{\gamma(1-s,f)}{\gamma(s,f)}.$$

Moreover, R is zero if $\Lambda(s,f)$ is entire, and otherwise

$$R=\left(\operatorname*{Res}_{u=1-s}+\operatorname*{Res}_{u=-s}\right)\frac{\Lambda(s+u,f)\Phi(u)X^u}{u}.$$

Proof. Let

$$I(X,s,f)=\frac{1}{2\pi i}\int_{(a)}\Lambda(s+u,f)\Phi(u)X^u\,\frac{du}{u}.$$

$L(s,f)$ has polynomial growth in the t-aspect by Proposition 6.4.1. From Equation (6.7) we see that $\gamma(s+u,f)$ exhibits rapid decay. Since $\Phi(u)$ is bounded, it follows that the integrand exhibits rapid decay in a vertical strip containing $|\tau|\le a$. Therefore the integral is locally absolutely uniformly convergent by Proposition 1.5.2. Moreover, we may shift the line of integration to $(-a)$. In doing so, we pass by a simple pole at $u=0$ and possible poles at $u=1-s$ and $u=-s$, giving

$$I(X,s,f)=\frac{1}{2\pi i}\int_{(-a)}\Lambda(s+u,f)\Phi(u)X^u\,\frac{du}{u}+\Lambda(s,f)+R.$$

Applying the functional equation to $\Lambda(s+u,f)$ and performing the change of variables $u\to -u$, we obtain

$$I(X,s,f)=-\varepsilon(f)I(X^{-1},1-s,\overline{f})+\Lambda(s,f)-R,$$

since $\Phi(u)$ is even. This equation is equivalent to

$$\Lambda(s,f)=I(X,s,f)+\varepsilon(f)I(X^{-1},1-s,\overline{f})+R.$$

Since $\mathrm{Re}(s+u)>1$, we can expand the L-function $L(s,f)$ inside of $I(X,s,f)$ as a Dirichlet series:

$$\begin{aligned}
I(X,s,f)&=\frac{1}{2\pi i}\int_{(a)}\Lambda(s+u,f)\Phi(u)X^u\,\frac{du}{u}\\
&=\frac{1}{2\pi i}\int_{(a)}q(f)^{\frac{s+u}{2}}\gamma(s+u,f)L(s+u,f)\Phi(u)X^u\,\frac{du}{u}\\
&=\frac{1}{2\pi i}\int_{(a)}\sum_{n\ge1}\frac{a_f(n)}{n^{s+u}}q(f)^{\frac{s+u}{2}}\gamma(s+u,f)\Phi(u)X^u\,\frac{du}{u}\\
&=\sum_{n\ge1}\frac{1}{2\pi i}\int_{(a)}\frac{a_f(n)}{n^{s+u}}q(f)^{\frac{s+u}{2}}\gamma(s+u,f)\Phi(u)X^u\,\frac{du}{u} && \text{DCT}\\
&=q(f)^{\frac{s}{2}}\gamma(s,f)\sum_{n\ge1}\frac{a_f(n)}{n^s}\frac{1}{2\pi i}\int_{(a)}\frac{\gamma(s+u,f)}{\gamma(s,f)}\Phi(u)\left(\frac{\sqrt{q(f)}X}{n}\right)^u\frac{du}{u}\\
&=q(f)^{\frac{s}{2}}\gamma(s,f)\sum_{n\ge1}\frac{a_f(n)}{n^s}V_s\left(\frac{n}{\sqrt{q(f)}X}\right).
\end{aligned}$$

Performing the same computation for $I(X^{-1},1-s,\overline{f})$ and substituting in the results, we arrive at

$$\Lambda(s,f)=q(f)^{\frac{s}{2}}\gamma(s,f)\sum_{n\ge1}\frac{a_f(n)}{n^s}V_s\left(\frac{n}{\sqrt{q(f)}X}\right)+\varepsilon(f)q(f)^{\frac{1-s}{2}}\gamma(1-s,f)\sum_{n\ge1}\frac{a_f(n)}{n^{1-s}}V_{1-s}\left(\frac{nX}{\sqrt{q(f)}}\right)+R.$$

Diving by $q(f)^{\frac{s}{2}}\gamma(s,f)$ completes the proof. □

The approximate functional equation was first developed by Hardy and Littlewood in the series [HL21, HL23, HL29]. The function $V_s(y)$ has the effect of smoothing out the two sums on the right-hand side of the approximate functional equation. In most cases, we will take

$$\Phi(u) = \cos^{-4d_f M}\left(\frac{\pi u}{4M}\right),$$

for an integer $M \geq 1$. Clearly $\Phi(u)$ holomorphic in the vertical strip $|\tau| < (2M-2)+1$, even, and satisfies $\Phi(0) = 1$. To see that it is bounded in this vertical strip, we write

$$\cos^{-4d_f M}\left(\frac{\pi u}{4M}\right) = \left(\frac{e^{i\frac{\pi u}{4M}} + e^{-i\frac{\pi u}{4M}}}{2}\right)^{-4d_f M} \ll e^{-d_f \pi |r|}, \tag{6.9}$$

where in the estimate we have used the reverse triangle equality. It follows that $\Phi(u)$ exhibits rapid decay. With this choice of $\Phi(u)$, we can prove a useful bound for $V_s(y)$:

Proposition 6.4.2. *Let $L(s,f)$ be an L-function, set $\Phi(u) = \cos^{-4d_f M}\left(\frac{\pi u}{4M}\right)$ for some integer $M \geq 1$, and let $V_s(y)$ be the inverse Mellin transform defined by*

$$V_s(y) = \frac{1}{2\pi i}\int_{(2M-2)} \frac{\gamma(s+u,f)}{\gamma(s,f)}\Phi(u)y^{-u}\frac{du}{u}.$$

Then for s in the critical strip, $V_s(y)$ satisfies the estimate

$$V_s(y) \ll \left(1 + \frac{y}{\sqrt{\mathfrak{q}_\infty(s,f)}}\right)^{-M}.$$

Proof. Suppose $0 \leq \sigma \leq \frac{1}{2}$ so that $\sigma - \frac{1}{2} \leq 0$. Then from Equation (6.7) and the reverse triangle inequality, we deduce that

$$\frac{\Gamma(s+u)}{\Gamma(s)} \ll \frac{(|t+r|+1)^{\sigma+\tau-\frac{1}{2}}}{(|t|+1)^{\sigma-\frac{1}{2}}}e^{\frac{\pi}{2}(|t|-|t+r|)} \ll (|t+r|+1)^{\tau}e^{\frac{\pi}{2}|r|}.$$

From the definition of $\mathfrak{q}(s,f)$, the above estimate implies

$$\frac{\gamma(s+u,f)}{\gamma(s,f)} \ll \mathfrak{q}_\infty(s,f)^{\frac{\tau}{2}}e^{d_f\frac{\pi}{2}|r|},$$

for s in the critical strip. By Equation (6.9), the integral defining $V_s(y)$ is locally absolutely uniformly convergent and we may shift the line of integration to (M). In doing so, we do not pass by any poles and obtain

$$V_s(y) = \frac{1}{2\pi i}\int_{(M)} \frac{\gamma(s+u,f)}{\gamma(s,f)}\Phi(u)y^{-u}\frac{du}{u}$$

The fact that u is bounded away from zero, Equation (6.9), and the estimate for the ratio of gamma factors gives the first estimate in the following chain:

$$
\begin{aligned}
V_s(y) &= \frac{1}{2\pi i}\int_{(M)} \frac{\gamma(s+u,f)}{\gamma(s,f)}\Phi(u)y^{-u}\frac{du}{u} \\
&\ll \int_{-\infty}^{\infty} \mathfrak{q}_\infty(s,f)^{\frac{M}{2}} e^{-d_f\frac{\pi}{2}|r|} y^{-M}\,dr \\
&\ll \int_{-\infty}^{\infty} \mathfrak{q}_\infty(s,f)^{\frac{M}{2}} e^{-d_f\frac{\pi}{2}|r|} (1+y)^{-M}\,dr \\
&\ll \int_{-\infty}^{\infty} e^{-\frac{\pi}{2}d_f|r|}\left(1+\frac{y}{\sqrt{\mathfrak{q}_\infty(s,f)}}\right)^{-M} dr \\
&\ll \left(1+\frac{y}{\sqrt{\mathfrak{q}_\infty(s,f)}}\right)^{-M}\int_{-\infty}^{\infty} e^{-\frac{\pi}{2}d_f|r|}\,dr \\
&\ll \left(1+\frac{y}{\sqrt{\mathfrak{q}_\infty(s,f)}}\right)^{-M},
\end{aligned}
$$

where in the third and fourth lines we have used that $cy \ll (1+cy)$ for all $y \geq 0$ and any c and the last line holds by Proposition 1.5.2 since the integrand exhibits rapid decay. This completes the proof. $\square$

From Proposition 6.4.2 we see that $V_s(y)$ is bounded for $y \ll_\varepsilon \mathfrak{q}_\infty(s,f)^{\frac{1}{2}+\varepsilon}$ and then starts to exhibit polynomial decay that can be taken arbitrarily large. In a similar spirit to the approximate functional equation, a useful summation formula can be derived from the functional equation of each L-function:

Theorem 6.4.2. *Let $\psi(y)$ be a bump function and let $\Psi(s)$ denote its Mellin transform. Then for any L-functions $L(s,f)$, we have*

$$
\sum_{n\geq 1} a_f(n)\psi(n) = \frac{\varepsilon(f)}{\sqrt{q(f)}}\sum_{n\geq 1} a_{\overline{f}}(n)\phi(n) + R\Psi(1),
$$

where $\phi(y)$ is the inverse Mellin transform defined by

$$
\phi(y) = \frac{1}{2\pi i}\int_{(a)} q(f)^s \frac{\gamma(s,f)}{\gamma(1-s,f)} y^{-s}\Psi(1-s)\,ds,
$$

for any $a>1$. Moreover, R is zero if $L(s,f)$ is entire, and otherwise

$$
R = \operatorname*{Res}_{s=1} L(s,f).
$$

Proof. By smoothed Perron's formula,

$$
\sum_{n\geq 1} a_f(n)\psi(n) = \frac{1}{2\pi i}\int_{(a)} L(s,f)\Psi(s)\,ds.
$$

By Propositions 6.2.1 and 6.4.1, the integrand has polynomial decay and therefore is locally absolutely uniformly convergent by Proposition 1.5.2. Shifting the line of integration to $(1-a)$, we pass by a potential pole at $s=1$ from $L(s,f)$ and obtain

$$
\sum_{n\geq 1} a_f(n)\psi(n) = \frac{1}{2\pi i}\int_{(1-a)} L(s,f)\Psi(s)\,ds + R\Psi(1).
$$

Applying the functional equation, we further have

$$\sum_{n\geq 1} a_f(n)\psi(n) = \frac{1}{2\pi i}\int_{(1-a)} \varepsilon(f)q(f)^{\frac{1}{2}-s}\frac{\gamma(1-s,f)}{\gamma(s,f)}L(1-s,\overline{f})\Psi(s)\,ds + R\Psi(1).$$

Performing the change of variables $s \to 1-s$ in this latter integral gives

$$\sum_{n\geq 1} a_f(n)\psi(n) = \frac{1}{2\pi i}\int_{(a)} \varepsilon(f)q(f)^{s-\frac{1}{2}}\frac{\gamma(s,f)}{\gamma(1-s,f)}L(s,\overline{f})\Psi(1-s)\,ds + R\Psi(1).$$

The proof is complete upon interchanging the sum over the Dirichlet series and the integral by the dominated convergence theorem and factoring out $\frac{\varepsilon(f)}{\sqrt{q(f)}}$. $\square$

6.5 The Riemann Hypothesis & Nontrivial Zeros

The zeros of L-functions $L(s,f)$ are closely tied to important arithmetic data of f. Recall that

$$L(s,f) = \prod_p (1-\alpha_1(p)p^{-s})^{-1}(1-\alpha_2(p)p^{-s})^{-1}\cdots(1-\alpha_{d_f}(p)p^{-s})^{-1},$$

for $\sigma > 1$. This product vanishes if and only if one of its factors are zero. As $\sigma > 1$, this is impossible so that $L(s,f)$ has no zeros in this region. The functional equation will allow us to understand more about the zeros of $L(s,f)$. Rewrite the functional equation for $L(s,f)$ as

$$L(s,f) = \varepsilon(f)q(f)^{\frac{1}{2}-s}\frac{\gamma(1-s,f)}{\gamma(s,f)}L(1-s,\overline{f}). \tag{6.10}$$

If $\sigma < 0$ then $L(1-s,\overline{f})$ is nonzero by our previous comments. Moreover, $\gamma(1-s,f)$ is holomorphic and nonzero in this region because $\mathrm{Re}(\kappa_j) > -1$. We conclude that poles of $\gamma(s,f)$ are zeros of $L(s,f)$ for $\sigma < 0$. Such a zero is called a **trivial zero**. From the definition of $\gamma(s,f)$, they are all simple and of the form $s = -(\kappa_j + 2n)$ for some local root at infinity κ_j and some integer $n \geq 0$. Any other zero of $L(s,f)$ is called a **nontrivial zero** and it lies inside of the critical strip (it may also be a pole of $\gamma(s,f)$). Now let ρ be a nontrivial zero of $L(s,f)$. Note that $L(\overline{s},\overline{f}) = \overline{L(s,f)}$ for $\sigma > 1$ where $L(s,f)$ is defined by a Dirichlet series and thus for all s by the identity theorem. It follows that $\overline{\rho}$ is a nontrivial zero of $L(s,\overline{f})$ an from the functional equation $1-\overline{\rho}$ is also a nontrivial zero of $L(s,f)$. In short, the nontrivial zeros occur in pairs:

$$\rho \quad \text{and} \quad 1-\overline{\rho}.$$

We can sometimes say more. If $L(s,f)$ takes real values for $s > 1$, the Schwarz reflection principle implies $L(\overline{s},f) = \overline{L(s,f)}$ and that $L(s,f)$ takes real values on the entire real axis save for the possible poles at $s = 0$ and $s = 1$. We find that $\overline{\rho}$ and $1-\overline{\rho}$ are nontrivial zeros too and therefore the nontrivial zeros of $L(s,f)$ come in sets of four and are displayed in Figure 6.2:

$$\rho, \quad \overline{\rho}, \quad 1-\rho, \quad \text{and} \quad 1-\overline{\rho}.$$

The **Riemann hypothesis** for $L(s,f)$ says that this symmetry should be as simple as possible:

Conjecture 6.5.1 (Riemann hypothesis, $L(s,f)$ version). *For the L-function $L(s,f)$, all of the nontrivial zeros lie on the line $\sigma = \frac{1}{2}$.*

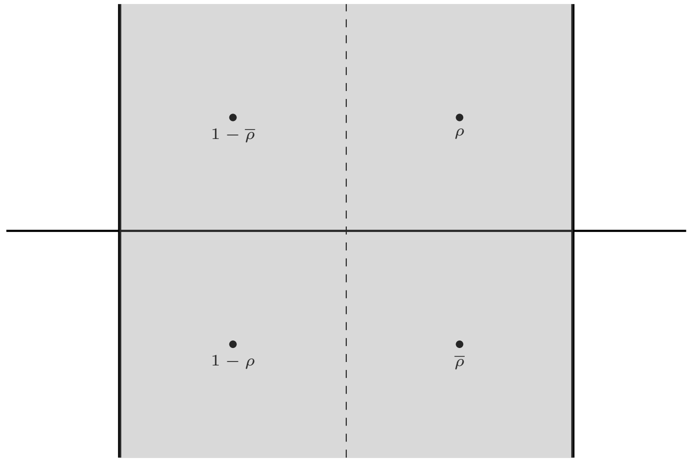

Figure 6.2: Symmetric nontrivial zeros.

Somewhat confusingly, do not expect the Riemann hypothesis to hold for just any L-function but we expect it to hold for many L-functions. In particular, the **grand Riemann hypothesis** says that this symmetry should hold for any L-function in the Selberg class:

Conjecture 6.5.2 (Grand Riemann hypothesis). *For any Selberg class L-function $L(s,f)$, all of the nontrivial zeros lie on the line $\sigma = \frac{1}{2}$.*

So far, the Riemann hypothesis remains completely out of reach for any L-function and thus the grand Riemann hypothesis does as well.

6.6 The Lindelöf Hypothesis & Convexity Arguments

Instead of asking about the zeros of an L-function $L(s,f)$ on the critical line, we can ask about the growth of $L(s,f)$ on the critical line. A **convexity argument** is one where estimates about the growth of an L-function on the critical line are deduced. Usually this is achieved by methods of complex analysis and Sirling's formula. The standard argument for any L-function is known **Lindelöf convexity argument**. It is essentially a refinement of the proof of Proposition 6.4.1. Let

$$p_{r_f}(s) = \left(\frac{s-1}{s+1}\right)^{r_f}.$$

Note that $p_{r_f}(s) \sim 1$. The first step is to guarantee the Phragmén-Lindelöf convexity principle for $p_{r_f}(s)L(s,f)$ in a region containing the critical strip. As $L(s,f)$ is of order 1, this is assured (see Appendix B.6). Therefore, we are reduced to estimating the growth of $p_{r_f}(s)L(s,f)$ for σ to the left of 0 and to the right of 1. That is, just outside the edges of the critical strip. The right edge is easily estimated by setting $\sigma = 1+\varepsilon$ so that

$$p_{r_f}(1+\varepsilon+it)L(1+\varepsilon+it,f) \ll_\varepsilon 1.$$

The left edge is only slightly more difficult. The functional equation and Equation (6.8) together imply

$$p_{r_f}(s)L(s,f) \ll p_{r_f}(s)\mathfrak{q}(s,f)^{\frac{1-2\sigma}{2}}L(1-s,f),$$

for s in any vertical strip with distance ε away from the poles of $\gamma(1-s,f)$. Setting $\sigma = -\varepsilon$, we obtain

$$p_{r_f}(-\varepsilon+it)L(-\varepsilon+it,f) \ll_\varepsilon \mathfrak{q}(s,f)^{\frac{1}{2}+\varepsilon}.$$

As $p_{r_f}(s)L(s,f)$ is holomorphic in a region containing the vertical strip $-\varepsilon \le \sigma \le 1+\varepsilon$, the Phragmén-Lindelöf convexity principle gives

$$L(s,f) \ll_\varepsilon \mathfrak{q}(s,f)^{\frac{1-\sigma}{2}+\varepsilon},$$

in the vertical strip $-\varepsilon \le \sigma \le 1+\varepsilon$. At the critical line, we have the **convexity bound**:

$$L\left(\frac{1}{2}+it,f\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it,f\right)^{\frac{1}{4}+\varepsilon}. \tag{6.11}$$

The **Lindelöf hypothesis** for $L(s,f)$ says that the exponent can be reduced to ε:

Conjecture 6.6.1 (Lindelöf hypothesis, $L(s,f)$ version). *For the L-function $L(s,f)$, we have*

$$L\left(\frac{1}{2}+it,f\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it,f\right)^{\varepsilon}.$$

Just as for the Riemann hypothesis, we do not expect the Lindelöf hypothesis to hold for just any L-function. Accordingly, the **grand Lindelöf hypothesis** says that the exponent can be reduced to ε for any L-function in the Selberg class and we expect this to hold:

Conjecture 6.6.2 (Grand Lindelöf hypothesis). *For any Selberg class L-function $L(s,f)$, we have*

$$L\left(\frac{1}{2}+it,f\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it,f\right)^{\varepsilon}.$$

Like the Riemann hypothesis, we have been unable to prove the Lindelöf hypothesis for any L-function. However, the Lindelöf hypothesis seems to be much more tractable. Generally speaking, any improvement upon the exponent in the convexity bound in any aspect of the analytic conductor is called a **subconvexity estimate** (or a **convexity breaking bound**). Any argument used to do so is called a **subconvexity argument**.

6.7 Estimating the Central Value

The Lindelöf hypothesis is concerned with the growth of the L-function $L(s,f)$ along the critical line, but sometimes we are only concerned with the size of $L(s,f)$ at the central point. The value of $L(s,f)$ at the central point is called the **central value** of $L(s,f)$. Many important properties about $L(s,f)$ can be connected to its central value. Any argument used to estimate the central value of an L-function is called a **central value estimate**. We will prove central value estimate which gives a very useful upper bound in the $q(f)$-aspect. To state it, let $\psi(y)$ be a bump function with compact support in $\left[\frac{1}{2},2\right]$. For example,

$$\psi(y) = \begin{cases} e^{-\frac{1}{9-(4y-5)^2}} & \text{if } |4y-5|<3, \\ 0 & \text{if } |4y-5| \ge 3. \end{cases}$$

The theorem is the following:

Theorem 6.7.1. *Let $L(s,f)$ be an L-function and let $\psi(y)$ be a bump function with compact support in $\left[\frac{1}{2},2\right]$. Then we have*

$$L\left(\frac{1}{2},f\right) \ll_\varepsilon \left|\max_{X \ll \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \frac{A_\psi(X)}{q(f)^{\frac{1}{4}}}\right| + \left|\frac{S}{q(f)^{\frac{1}{4}}}\right|,$$

where S is zero if $\Lambda(s,f)$ is entire, and otherwise

$$S = \left(\operatorname*{Res}_{u=\frac{1}{2}} + \operatorname*{Res}_{u=-\frac{1}{2}}\right)\Lambda\left(\frac{1}{2}+u, f\right).$$

Proof. Taking $s=\frac{1}{2}$, $X=1$, and $\Phi(u)=\cos^{-4dM}\left(\frac{\pi u}{4M}\right)$ with $M \gg 1$ in the approximate functional equation gives

$$L\left(\frac{1}{2},f\right) = \sum_{n\geq 1}\frac{a_f(n)}{\sqrt{n}}V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right) + \varepsilon(f)\sum_{n\geq 1}\frac{\overline{a_f(n)}}{\sqrt{n}}V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right) + \frac{R}{q(f)^{\frac{1}{4}}\gamma\left(\frac{1}{2},f\right)}.$$

This implies the bound

$$L\left(\frac{1}{2},f\right) \ll \left|\sum_{n\geq 1}\frac{a_f(n)}{\sqrt{n}}V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right)\right| + \left|\frac{S}{q(f)^{\frac{1}{4}}}\right|.$$

Now consider the set of functions $\left\{\psi\left(\frac{y}{2^k}\right)\right\}_{k\in\mathbb{Z}}$. Since $\psi\left(\frac{y}{2^k}\right)$ has support in $[2^{k-1},2^{k+1}]$, the sum $\sigma(y) = \sum_{k\in\mathbb{Z}}\psi\left(\frac{y}{2^k}\right)$, defined for $y>0$, is finite since at most finitely many terms are nonzero for every y. It is also bounded away from zero since for any $y>0$ there is some $k\in\mathbb{Z}$ for which $2^k \leq y \leq 3\cdot 2^{k-1}$ so that $\frac{y}{2^k}$ is at least distance $\frac{1}{2}$ from the endpoints of $\left[\frac{1}{2},2\right]$. Defining $\psi_k(y) = \psi\left(\frac{y}{2^k}\right)\sigma(y)^{-1}$, it follows that $\{\psi_k(y)\}_{k\in\mathbb{Z}}$ satisfies

$$\sum_{k\in\mathbb{Z}}\psi_k(y) = 1,$$

for any $y>0$. Then we can write

$$V_s(y) = \sum_{k\in\mathbb{Z}}\psi_k(y)V_s(y).$$

It follows that

$$\begin{aligned}
\sum_{n\geq 1}\frac{a_f(n)}{\sqrt{n}}V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right) &= \sum_{n\geq 1}\frac{a_f(n)}{\sqrt{n}}\sum_{k\in\mathbb{Z}}\psi_k\left(\frac{n}{\sqrt{q(f)}}\right)V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right) \\
&= \sum_{k\in\mathbb{Z}}\sum_{n\geq 1}\frac{a_f(n)}{\sqrt{n}}\psi_k\left(\frac{n}{\sqrt{q(f)}}\right)V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right) && \text{DCT} \\
&\ll_\varepsilon \sum_{k\in\mathbb{Z}}\left|\sum_{\substack{n\ll\mathfrak{q}(f)^{\frac{1}{2}+\varepsilon} \\ 2^{k-1}\sqrt{q(f)}\leq n\leq 2^{k+1}\sqrt{q(f)}}}\frac{a_f(n)}{\sqrt{n}}\psi_k\left(\frac{n}{\sqrt{q(f)}}\right)\right| \\
&\ll_\varepsilon \sum_{k\ll\log(q(f)^{\frac{\varepsilon}{2}})}\left|\sum_{\substack{n\ll\mathfrak{q}(f)^{\frac{1}{2}+\varepsilon} \\ 2^{k-1}\sqrt{q(f)}\leq n\leq 2^{k+1}\sqrt{q(f)}}}\frac{a_f(n)}{\sqrt{n}}\psi_k\left(\frac{n}{\sqrt{q(f)}}\right)\right|,
\end{aligned}$$

where in the second to last line we have used that $V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right)$ is bounded for $n \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2},f\right)^{\frac{1}{2}+\varepsilon} \ll_\varepsilon \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}$ and then exhibits polynomial decay thereafter by Proposition 6.4.2 and in the last line we have used that

$\psi_k(y)$ has compact support in $[2^{k-1}, 2^{k+1}]$ (recall $k \in \mathbb{Z}$). Since $\sigma(y)$ is bounded away from zero and $\log(y) \ll y$, we obtain the crude bound

$$\sum_{k \ll \log(q(f)^{\frac{\varepsilon}{2}})} \left| \sum_{\substack{n \ll \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon} \\ 2^{k-1}\sqrt{q(f)} \le n \le 2^{k+1}\sqrt{q(f)}}} \frac{a_f(n)}{\sqrt{n}} \psi_k\left(\frac{n}{\sqrt{q(f)}}\right) \right| \ll_\varepsilon q(f)^{\frac{\varepsilon}{2}} \max_{X \ll \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \left| \sum_{\frac{X}{2} \le n \le 2X} \frac{a_f(n)}{\sqrt{n}} \psi\left(\frac{n}{X}\right) \right|.$$

We will estimate this latter sum. Abel's summation formula (see Appendix B.3) gives

$$\sum_{\frac{X}{2} \le n \le 2X} \frac{a_f(n)}{\sqrt{n}} \psi\left(\frac{n}{X}\right) = \frac{A_\psi(2X)}{\sqrt{2X}} - \frac{A_\psi\left(\frac{X}{2}\right)}{\sqrt{\frac{X}{2}}} + \frac{1}{2}\int_{\frac{X}{2}}^{2X} A_\psi(u) u^{-\frac{3}{2}}\, du.$$

But as

$$\frac{1}{2}\int_{\frac{X}{2}}^{2X} A_\psi(u) u^{-\frac{3}{2}}\, du \ll X \max_{\frac{X}{2} \le u \le 2X} A_\psi(u) u^{-\frac{3}{2}} \ll \max_{\frac{X}{2} \le u \le 2X} \frac{A_\psi(u)}{\sqrt{u}},$$

we obtain the bound

$$\max_{X \ll \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \sum_{\frac{X}{2} \le n \le 2X} \frac{a_f(n)}{\sqrt{n}} \psi\left(\frac{n}{X}\right) \ll \max_{X \ll \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \frac{A_\psi(X)}{\sqrt{X}}.$$

Putting everything together gives

$$\sum_{n \ge 1} \frac{a_f(n)}{\sqrt{n}} V_{\frac{1}{2}}\left(\frac{n}{\sqrt{q(f)}}\right) \ll_\varepsilon q(f)^{\frac{\varepsilon}{2}} \max_{X \ll \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \frac{A_\psi(X)}{\sqrt{X}} \ll_\varepsilon \max_{X \ll \mathfrak{q}(f)^{\frac{1}{2}+\varepsilon}} \frac{A_\psi(X)}{q(f)^{\frac{1}{4}}},$$

where in the last estimate we may replace X with $q(f)^{\frac{1}{2}+\varepsilon}$ because $X \ge 1$ so that X is bounded away from zero. □

6.8 Logarithmic Derivatives

There is an incredibly useful formula for the logarithmic derivative of any L-function which is often the starting point for deeper analytic investigations. To deduce it, we will need a more complete understanding of $\Lambda(s, f)$. First observe that the zeros ρ of $\Lambda(s, f)$ are contained inside of the critical strip. Indeed, we have already remarked that $L(s, f)$ has no zeros for $\sigma > 0$ and clearly $\gamma(s, f)$ does not have zeros in this region as well. Therefore $\Lambda(s, f)$ is nonzero for $\sigma > 1$. By the functional equation, $\Lambda(s, f)$ is also nonzero for $\sigma < 0$ too. In other words, the zeros of $\Lambda(s, f)$ are the nontrivial zeros of $L(s, f)$. Before we state our result, we setup some notation. For an L-function $L(s, f)$ we define

$$\xi(s, f) = (s(1-s))^{r_f} \Lambda(s, f).$$

Note that $\xi(s, f)$ is essentially just $\Lambda(s, f)$ with the potential poles at $s = 0$ and $s = 1$ removed. From the functional equation, we also have

$$\xi(s, f) = \varepsilon(f)\xi(1-s, \overline{f}).$$

We now state our desired result:

Proposition 6.8.1. *For any L-function $L(s,f)$, there exist constants $A(f)$ and $B(f)$ such that*

$$\xi(s,f) = e^{A(f)+B(f)s} \prod_{\rho \neq 0,1} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

and hence the sum

$$\sum_{\rho \neq 0,1} \frac{1}{|\rho|^{1+\varepsilon}},$$

is convergent provided the product and sum are both counted with multiplicity and ordered with respect to the size of the ordinate. Moreover,

$$-\frac{L'}{L}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} + \frac{1}{2}\log q(f) + \frac{\gamma'}{\gamma}(s,f) - B(f) - \sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Proof. For the first statement, observe that $\xi(s,f)$ is entire since the only possible poles of $\Lambda(s,f)$ are at $s=0$ and $s=1$ and are of order r_f. We also claim that $\xi(s,f)$ is of order 1. By the functional equation, it suffices to show this for $\sigma \geq \frac{1}{2}$. This follows from $L(s,f)$ being of order 1 and Equation (6.7) (within ε of the poles of $\gamma(s,f)$ we know $\xi(s,f)$ is bounded because it is entire). By the Hadamard factorization theorem (see Appendix B.5),

$$\xi(s,f) = e^{A(f)+B(f)s} \prod_{\rho \neq 0,1} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

for some constants $A(f)$ and $B(f)$ and the desired sum converges. This proves the first statement. For the second, taking the logarithmic derivative of the definition of $\xi(s,f)$ yields

$$\frac{\xi'}{\xi}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} + \frac{1}{2}\log q(f) + \frac{\gamma'}{\gamma}(s,f) + \frac{L'}{L}(s,f). \tag{6.12}$$

On the other hand, taking the logarithmic derivative of the Hadamard factorization gives

$$\frac{\xi'}{\xi}(s,f) = B(f) + \sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{6.13}$$

Equating Equations (6.12) and (6.13), we arrive at

$$B(f) + \sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = \frac{r_f}{s} + \frac{r_f}{s-1} + \frac{1}{2}\log q(f) + \frac{\gamma'}{\gamma}(s,f) + \frac{L'}{L}(s,f).$$

Isolating $-\frac{L'}{L}(s,f)$ completes the proof. $\square$

We now need to make a few comments. Our first is regarding the constants $A(f)$ and $B(f)$. The explicit evaluation of these constants can be challenging and heavily depends upon the arithmetic object f. However, useful estimates are not too difficult to obtain. We also claim $A(\overline{f}) = \overline{A(f)}$ and $B(\overline{f}) = \overline{B(f)}$. To see this, recall that $L(\overline{s},\overline{f}) = \overline{L(s,f)}$. Then $\xi(\overline{s},\overline{f}) = \overline{\xi(s,f)}$ because $\gamma(s,\overline{f}) = \gamma(s,f)$, the κ_j are real or occur in conjugate pairs, and $\Gamma(\overline{s}) = \overline{\Gamma(s)}$. But then the functional equation and Proposition 6.8.1 together imply

$$e^{A(\overline{f})+B(\overline{f})s} = \frac{\xi'}{\xi}(0,\overline{f}) = \overline{\frac{\xi'}{\xi}(0,f)} = e^{\overline{A(f)}+\overline{B(f)}s},$$

and the claim follows. Our second comment concerns the negative logarithmic derivative of $L(s,f)$. In general, this function attracts much attention for analytic investigations. As $L(s,f)$ is holomorphic for $\sigma > 1$ and admits an Euler product there, we can take the logarithm of the Euler product (turning it into a sum) and differentiate termwise to obtain

$$-\frac{L'}{L}(s,f) = -\sum_p \sum_{1\le j\le d_f} \frac{d}{ds}\log(1-\alpha_j(p)p^{-s}) = \sum_p \sum_{1\le j\le d_f} \frac{\alpha_j(p)\log(p)}{(1-\alpha_j(p)p^{-s})p^s}. \tag{6.14}$$

From the Taylor series of $\frac{1}{1-s}$, it follows that $-\frac{L'}{L}(s,f)$ is a locally absolutely uniformly convergent Dirichlet series of the form

$$-\frac{L'}{L}(s,f) = \sum_{n\ge1}\frac{\Lambda_f(n)}{n^s},$$

for $\sigma > 1$, where

$$\Lambda_f(n) = \begin{cases} \sum_{1\le j\le d_f}\alpha_j(p)^k\log(p) & \text{if } n=p^k \text{ for some } k\ge1, \\ 0 & \text{otherwise.}\end{cases}$$

It is worth noting that $\Lambda_{\overline{f}}(n) = \overline{\Lambda_f(n)}$.

6.9 Zero Density

The deepest subject of the theory of L-functions is arguably the distribution of the zeros of L-functions. Here we introduce a method of counting zeros of L-functions and gaining a very simple understanding of their density as a result. We first require an immensely useful lemma:

Lemma 6.9.1. *Let $L(s,f)$ be an L-function. The following statements hold:*

(i) The constant $B(f)$ satisfies

$$\mathrm{Re}(B(f)) = -\sum_{\rho\ne0,1}\mathrm{Re}\left(\frac{1}{\rho}\right),$$

where the sum is counted with multiplicity and ordered with respect to the size of the ordinate.

(ii) For any $T\ge0$, the number of nontrivial zeros $\rho=\beta+i\gamma$ with $\rho\ne0,1$ and such that $|T-\gamma|\le1$ is $O(\log\mathfrak{q}(iT,f))$.

(iii) For $\sigma>1$, we have

$$\mathrm{Re}\left(\frac{1}{s-\rho}\right) > 0 \quad \textit{and} \quad \mathrm{Re}\left(\frac{1}{s+\kappa_j}\right) > 0.$$

(iv) We have

$$-\frac{L'}{L}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} - \sum_{|s+\kappa_j|<1}\frac{1}{s+\kappa_j} - \sum_{\substack{|s-\rho|<1\\ \rho\ne0,1}}\frac{1}{s-\rho} + O(\log\mathfrak{q}(s,f)),$$

for any s in the vertical strip $-\frac{1}{2}\le\sigma\le2$. Moreover,

$$\mathrm{Re}\left(-\frac{L'}{L}(s,f)\right) \le \mathrm{Re}\left(\frac{r_f}{s}\right)+\mathrm{Re}\left(\frac{r_f}{s-1}\right) - \sum_{|s+\kappa_j|<1}\mathrm{Re}\left(\frac{1}{s+\kappa_j}\right) - \sum_{\substack{|s-\rho|<1\\ \rho\ne0,1}}\mathrm{Re}\left(\frac{1}{s-\rho}\right) + O(\log\mathfrak{q}(s,f)),$$

and we may disregard any term in either sum provided $1<\sigma\le2$.

Proof. We will prove each statement separately.

(i) To prove (i), first recall that $B(\overline{f}) = \overline{B(f)}$. Then Equation (6.13) and the functional equation together imply

$$2\mathrm{Re}(B(f)) = B(f) + B(\overline{f}) = -\sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{1-s-\overline{\rho}} + \frac{1}{\rho} + \frac{1}{\overline{\rho}} \right),$$

where we have made use of the fact that the nontrivial zeros occur is pairs ρ and $1-\overline{\rho}$ where the latter is also a nontrivial zero of $L(1-s,\overline{f})$. Now fix s such that it does not coincide with the ordinate of a nontrivial zero. Then s is bounded away from all of the nontrivial zeros and it follows that $\frac{1}{(s-\rho)} + \frac{1}{(1-s-\overline{\rho})} \ll \frac{1}{\rho^2}$ and $\frac{1}{\rho} + \frac{1}{\overline{\rho}} \ll \frac{1}{\rho^2}$. Therefore the sums

$$\sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{1-s-\overline{\rho}} \right) \quad \text{and} \quad \sum_{\rho \neq 0,1} \left(\frac{1}{\rho} + \frac{1}{\overline{\rho}} \right),$$

converge absolutely by Proposition 6.8.1 and so we can sum them separately. The first sum vanishes by again using the fact that the nontrivial zeros occur is pairs ρ and $1-\overline{\rho}$. Thus

$$2\mathrm{Re}(B(f)) = \sum_{\rho \neq 0,1} \left(\frac{1}{\rho} + \frac{1}{\overline{\rho}} \right) = -\sum_{\rho \neq 0,1} \mathrm{Re}\left(\frac{1}{\rho} \right),$$

which gives (i).

(ii) For (ii), we first bound two important quantities. For the first quantity, the definition of $\Lambda_f(n)$ and that $|\alpha_j(p)| \leq p$ together imply the weak bound $|\Lambda_f(n)| \leq d_f n \log(n)$. Then

$$\frac{L'}{L}(s,f) \ll d_f \zeta'(s-1) \ll \log \mathfrak{q}(f), \tag{6.15}$$

provided $\sigma > 2$. For the second quantity, Proposition 1.6.3 provides the estimate $\frac{\Gamma'}{\Gamma}(s) \ll \log(|s|+1)$, given $\sigma > 0$, and from the definition of $\gamma(s,f)$ it follows that $\frac{\gamma'}{\gamma}(s,f) \ll \log \mathfrak{q}_\infty(s,f)$. Thus

$$\frac{1}{2}\log q(f) + \frac{\gamma'}{\gamma}(s,f) \ll \log \mathfrak{q}(s,f), \tag{6.16}$$

for $\sigma > 0$. Now fix $T \geq 0$ and let $s = 3 + iT$. Taking the real part of the formula for the negative logarithmic derivative in Proposition 6.8.1 and combining Equations (6.15) and (6.16) with (i) results in

$$\sum_{\rho \neq 0,1} \mathrm{Re}\left(\frac{1}{s-\rho} \right) \ll \log \mathfrak{q}(iT,f).$$

But as

$$\frac{2}{9+(T-\gamma)^2} \leq \mathrm{Re}\left(\frac{1}{s-\rho} \right) \leq \frac{3}{4+(T-\gamma)^2},$$

we obtain

$$\sum_{\rho \neq 0,1} \frac{1}{1+(T-\gamma)^2} \ll \log \mathfrak{q}(iT,f), \tag{6.17}$$

which is stronger than the first statement of (ii) since all of the terms in the sum are positive. The second statement is also clear.

(iii) For (iii), just observe that

$$\operatorname{Re}\left(\frac{1}{s-\rho}\right) = \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-g)^2} > 0 \quad \text{and} \quad \operatorname{Re}\left(\frac{1}{s+\kappa_j}\right) = \frac{\sigma+\operatorname{Re}(\kappa_j)}{(\sigma+\operatorname{Re}(\kappa_j))^2+(t+\operatorname{Im}(\kappa_j))^2} > 0,$$

where the first bound holds because $\beta \le 1$ and the second bound holds because $\operatorname{Re}(\kappa_j) > -1$.

(iv) To deduce (iv), let s belong to the vertical strip $-\frac{1}{2} \le \sigma \le 2$. Using that Equation (6.15), we can write

$$-\frac{L'}{L}(s,f) = -\frac{L'}{L}(s,f) + \frac{L'}{L}(3+it,f) + O(\log \mathfrak{q}(s,f)).$$

Applying the formula for the negative logarithmic derivative in Proposition 6.8.1 to the two terms on the right-hand side and using that $\frac{\gamma'}{\gamma}(s,f) \ll \log \mathfrak{q}_\infty(s,f)$ given $\sigma > 0$, we get

$$-\frac{L'}{L}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} + \frac{\gamma'}{\gamma}(s,f) - \sum_{\rho \neq 0,1}\left(\frac{1}{s-\rho} - \frac{1}{3+it-\rho}\right) + O(\log \mathfrak{q}(s,f)).$$

We now estimate the remaining sum. Retain the first part of the terms for which $|s-\rho| < 1$. The contribution from the second part of these terms is $O(\log \mathfrak{q}(it,f))$ by (ii). For those terms with $|s-\rho| \ge 1$, we have

$$\left|\frac{1}{s-\rho} - \frac{1}{3+it-\rho}\right| \le \frac{3-\sigma}{(3-\beta)^2+(t-\gamma)^2} \le \frac{3}{1+(t-\gamma)^2}.$$

Therefore from Equation (6.17), the contribution of these terms is $O(\log \mathfrak{q}(it,f))$ too. It follows that

$$-\frac{L'}{L}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} + \frac{\gamma'}{\gamma}(s,f) - \sum_{\substack{|s-\rho|<1 \\ \rho\neq 0,1}} \frac{1}{s-\rho} + O(\log \mathfrak{q}(s,f)).$$

Lastly, from the definition of $\gamma(s,f)$ and that $\Gamma(s) = s^{-1}\Gamma(s+1)$, we have

$$\frac{\gamma'}{\gamma}(s,f) = -\frac{d_f}{2}\log(\pi) - \sum_{1\le j\le d_f}\frac{1}{s+\kappa_j} + \sum_{1\le j\le d_f}\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1+\frac{s+\kappa_j}{2}\right) = -\sum_{|s+\kappa_j|<1}\frac{1}{s+\kappa_j} + O(\log \mathfrak{q}_\infty(s,f)),$$

where in the last line we have made use of the estimate $\frac{\Gamma'}{\Gamma}(s) \ll \log(|s|+1)$ provided by Proposition 1.6.3 given $\sigma > 0$. Then

$$-\frac{L'}{L}(s,f) = \frac{r_f}{s} + \frac{r_f}{s-1} - \sum_{|s+\kappa_j|<1}\frac{1}{s+\kappa_j} - \sum_{\substack{|s-\rho|<1 \\ \rho\neq 0,1}} \frac{1}{s-\rho} + O(\log \mathfrak{q}(s,f)),$$

which is the first statement of (iv). For second statement, take the real part of this estimate and rewrite it in the weaker form

$$\sum_{|s+\kappa_j|<1}\operatorname{Re}\left(\frac{1}{s+\kappa_j}\right) + \sum_{\substack{|s-\rho|<1 \\ \rho\neq 0,1}}\operatorname{Re}\left(\frac{1}{s-\rho}\right) \le \operatorname{Re}\left(-\frac{L'}{L}(s,f)\right) + \operatorname{Re}\left(\frac{r_f}{s}\right) + \operatorname{Re}\left(\frac{r_f}{s-1}\right) + O(\log \mathfrak{q}(s,f)).$$

We can disregard any of the terms in either sum provided $1 < \sigma \le 2$ by (iii). $\square$

With Lemma 6.9.1 in hand, we can deduce a result which estimates the number of nontrivial zeros in a box. Accordingly, for any $T \geq 0$ we define

$$N(T,f) = |\{\rho = \beta + i\gamma \in \mathbb{C} : L(\rho, f) = 0 \text{ with } 0 \leq \beta \leq 1 \text{ and } |\gamma| \leq T\}|.$$

In other words, $N(T,f)$ is the number of nontrivial zeros of $L(s,f)$ with ordinate in $[-T,T]$. We will prove the following:

Theorem 6.9.1. *For any L-function $L(s,f)$ and $T \geq 1$,*

$$N(T,f) = \frac{T}{\pi}\log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right) + O(\log \mathfrak{q}(iT,f)).$$

Proof. Let $T \geq 1$ and set

$$N'(T,f) = \left|\{\rho = \beta + i\gamma \in \mathbb{C} : L(\rho, f) = 0 \text{ with } 0 \leq \beta \leq 1 \text{ and } 0 < \gamma \leq T\}\right|.$$

As the nontrivial zeros occur in pairs ρ and $1-\overline{\rho}$ where the latter is also a nontrivial zero of $L(s,\overline{f})$, it follows that

$$N(T,f) = N'(T,f) + N'(T,\overline{f}) + O(\log \mathfrak{q}(f)),$$

where $O(\log \mathfrak{q}(f))$ accounts for the possible real nontrivial zeros. There are finitely many such nontrivial zeros because the interval $0 \leq s \leq 1$ is compact. We will estimate $N'(T,f)$ and in doing so we may assume $L(s,f)$ does not vanish on the line $t = T$ by varying T by a sufficiently small constant, if necessary, and observing that $N(T,f)$ is modified by a quantity of size $O(\log \mathfrak{q}(iT,f))$ by Lemma 6.9.1 (i). Since the nontrivial zeros are isolated, let $\delta > 0$ be small enough such that $\Lambda(s,f)$ has no nontrivial zeros for $-\delta \leq t < 0$. Then by our previous comments and the argument principle,

$$N'(T,f) = \frac{1}{2\pi i}\int_{\eta}\frac{\xi'}{\xi}(s,f)\,ds + O(\log \mathfrak{q}(iT,f)),$$

where $\eta = \sum_{1 \leq i \leq 6}\eta_i$ is the contour in Figure 6.3:
Since $\log(s) = \log|s| + i\arg(s)$, we have

$$\frac{1}{2\pi i}\int_{\eta}\frac{\xi'}{\xi}(s,f)\,ds = \frac{1}{2\pi i}\int_{\eta}\frac{d}{ds}\log|\xi(s,f)|\,ds + \frac{1}{2\pi}\int_{\eta}\frac{d}{ds}\arg\xi(s,f)\,ds = \frac{1}{2\pi}\Delta_{\eta}\arg(\xi(s,f)),$$

where the last equality holds by parameterizing the curve η and noting that η is closed so that the first integral vanishes. For convenience, set $\eta_L = \eta_1+\eta_2+\eta_3$ and $\eta_R = \eta_4+\eta_5+\eta_6$. Recall that we have already shown $\xi(\overline{s},\overline{f}) = \overline{(\xi(s,f))}$. Using this fact along with the functional equation and that $-\arg(s) = \arg(\overline{s})$, we compute

$$\begin{aligned}
\Delta_{\eta_L}\arg(\xi(s,f)) &= \Delta_{\eta_L}\arg\left(\varepsilon(f)\xi(1-s,\overline{f})\right) \\
&= -\Delta_{\overline{\eta_R}}\arg\left(\varepsilon(f)\xi(s,\overline{f})\right) \\
&= \Delta_{\overline{\eta_R}}\arg\left(\overline{\varepsilon(f)}\xi(\overline{s},f)\right) \\
&= \Delta_{\eta_R}\arg\left(\overline{\varepsilon(f)}\xi(s,f)\right) \\
&= \Delta_{\eta_R}\arg\left(\overline{\varepsilon(f)}\right) + \Delta_{\eta_R}\arg(\xi(s,f)) \\
&= \Delta_{\eta_R}\arg(\xi(s,f)).
\end{aligned}$$

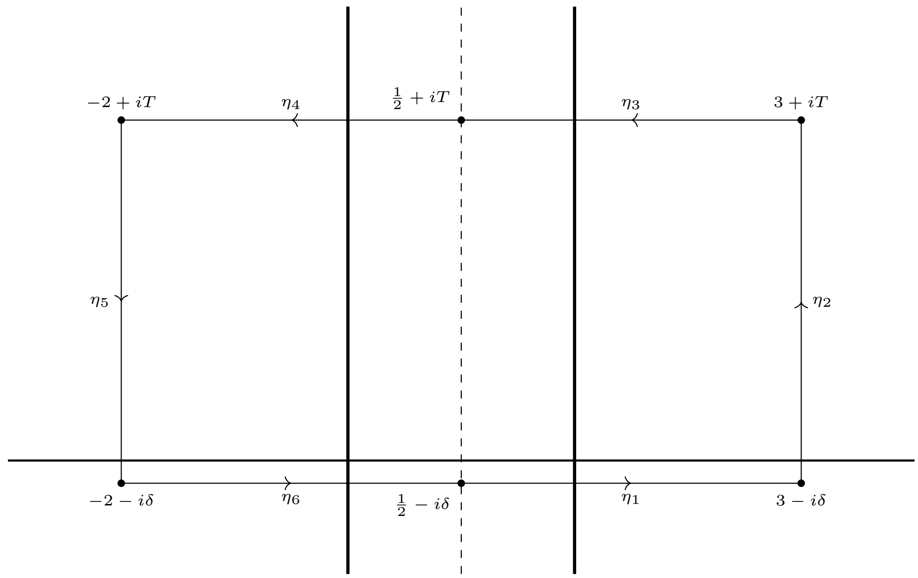

Figure 6.3: A zero counting contour.

In other words, the change in argument along η_L is equal to the change in argument along η_R and so

$$\frac{1}{2\pi i}\int_\eta \frac{\xi'}{\xi}(s,f)\,ds = \frac{1}{\pi}\Delta_{\eta_L}\arg\xi(s,f).$$

Thus to estimate the integral, we will now estimate the change in argument along η_L of each factor in

$$\xi(s,f) = (s(1-s))^{r_f} q(f)^{\frac{s}{2}}\pi^{-\frac{d_f s}{2}}\prod_{1\le j\le d_f}\Gamma\left(\frac{s+\kappa_j}{2}\right)L(s,f).$$

For the factor $(s(1-s))^{r_f}$, we first have

$$\Delta_{\eta_L}\arg(s) = \arg(s)\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} = \arg\left(\frac{1}{2}+iT\right) - \arg\left(\frac{1}{2}-i\delta\right) = O\left(\frac{1}{T}\right),$$

and

$$\Delta_{\eta_L}\arg(1-s) = \arg(1-s)\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} = \arg\left(\frac{1}{2}-iT\right) - \arg\left(\frac{1}{2}+i\delta\right) = O\left(\frac{1}{T}\right),$$

where in both computations we have used $\arg(s) = \tan^{-1}\left(\frac{t}{\sigma}\right) = \frac{\pi}{2} + O\left(\frac{1}{t}\right)$, provided $\sigma > 0$, which holds by the Laurent series of the inverse tangent. Combining these two bounds, we obtain

$$\Delta_{\eta_L}(s(1-s))^{r_f} = O\left(\frac{1}{T}\right). \tag{6.18}$$

For the factor $q(f)^{\frac{s}{2}}$, we use that $\arg(s) = \operatorname{Im}(\log(s))$ and compute

$$\Delta_{\eta_L}\arg q(f)^{\frac{s}{2}} = \operatorname{Im}(\log q(f)^{\frac{s}{2}})\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} = \log q(f)\left(\frac{T}{2}+\frac{\delta}{2}\right) = \frac{T}{2}\log q(f) + O(1). \tag{6.19}$$

For the factor $\pi^{-\frac{d_f s}{2}}$, we use that $\arg(s) = \operatorname{Im}(\log(s))$ and compute

$$\Delta_{\eta_L} \arg(\pi^{-\frac{d_f s}{2}}) = \operatorname{Im}(\log(\pi^{-\frac{d_f s}{2}}))\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} = \log\left(\frac{1}{\pi^{d_f}}\right)\left(\frac{T}{2}+\frac{\delta}{2}\right) = \frac{T}{2}\log\left(\frac{1}{\pi^{d_f}}\right) + O(1). \tag{6.20}$$

For the factor $\prod_{1\le j\le d_f}\Gamma\left(\frac{s+\kappa_j}{2}\right)$, we first use Equation (1.7) (valid since $T \ge 1$) and that $\arg(s) = \operatorname{Im}(\log(s))$ to obtain

$$\begin{aligned}
\Delta_{\eta_L} \arg\Gamma(s) &= \operatorname{Im}(\log\Gamma(s))\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} \\
&= T\log\left|\frac{1}{2}+iT\right| - T + \delta\log\left|\frac{1}{2}+i\delta\right| - \delta + O(1) \\
&= T\log(T) - T + O(1) \\
&= T\log\left(\frac{T}{e}\right) + O(1).
\end{aligned}$$

It follows that

$$\Delta_{\eta_L} \arg\left(\prod_{1\le j\le d_f}\Gamma\left(\frac{s+\kappa_j}{2}\right)\right) = \frac{T}{2}\log\left(\frac{T^{d_f}}{(2e)^{d_f}}\right) + O(\log\mathfrak{q}(f)). \tag{6.21}$$

For the factor $L(s,f)$, note that

$$\Delta_{\eta_L} \arg(L(s,f)) = \operatorname{Im}(\log L(s,f))\Big|_{\frac{1}{2}-i\delta}^{\frac{1}{2}+iT} = \operatorname{Im}\left(\int_{\eta_L}\frac{L'}{L}(s,f)\,ds\right).$$

By Equation (6.15), the integral is $O(\log q(f))$ on η_2. On η_1 and η_3, Lemma 6.9.1 (ii) and (iv) together imply that the integral is $O(\log\mathfrak{q}(iT,f))$. It follows that

$$\Delta_{\eta_L} \arg(L(s,f)) = O(\log\mathfrak{q}(iT,f)). \tag{6.22}$$

Combining Equations (6.18) to (6.22) results in

$$\frac{1}{\pi}\Delta_{\eta_L} \arg\xi(s,f) = \frac{T}{2\pi}\log\left(\frac{T^{d_f}}{(2\pi e)^{d_f}}\right) + O(\log\mathfrak{q}(iT,f)),$$

and therefore

$$N'(T,f) = \frac{T}{2\pi}\log\left(\frac{T^{d_f}}{(2\pi e)^{d_f}}\right) + O(\log\mathfrak{q}(iT,f)),$$

The claim follows immediately from this estimate and that the same exact estimate holds for $N'(T,\overline{f})$ by using the L-function $L(s,\overline{f})$. □

It is worth noting that the main term in the proof of Theorem 6.9.1 comes from the change in argument of $q(f)^{\frac{s}{2}}\gamma(s,f)$ along the vertical segment η_3 (equivalently η_4). Moreover, the contribution from $L(s,f)$ is only to the error term. This is a good example of how analytic information of an L-function is intrinsically connected to its gamma factor. Also, with Theorem 6.9.1 we can now derive our zero density estimate:

Corollary 6.9.1. *For and L-function $L(s,f)$ and $T \ge 1$,*

$$\frac{N(T,f)}{T} \sim \frac{1}{\pi}\log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right).$$

Proof. From Theorem 6.9.1,

$$\frac{N(T,f)}{T} = \frac{1}{\pi}\log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right)\left(1+O\left(\frac{\log \mathfrak{q}(iT,f)}{\log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right)T}\right)\right) = \frac{1}{\pi}\log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right)\left(1+O\left(\frac{1}{T}\right)\right).$$

Since $O\left(\frac{1}{T}\right) = o(1)$, the result follows. □

Corollary 6.9.1 can be interpreted as saying that for large T the density of $N(T,f)$ is approximately $\frac{1}{\pi}\log\left(\frac{q(f)T^{d_f}}{(2\pi e)^{d_f}}\right)$. Since this grows as $T \to \infty$, we see that the nontrivial zeros tend to accumulate farther up the critical strip with logarithmic growth. We can dispense with this accumulation. If $\rho = \beta + i\gamma$ is a nontrivial zero of $L(s,f)$, then we call $\rho_{\text{unf}} = \beta + i\omega$ the **unfolded nontrivial zero** corresponding to ρ where

$$\omega = \frac{\gamma}{\pi}\log\left(\frac{q(f)|\gamma|^{d_f}}{(2\pi e)^{d_f}}\right).$$

Now for any $W \geq 0$, define

$$N_{\text{unf}}(W,f) = |\{\rho_{\text{unf}} = \beta + i\omega \in \mathbb{C} : L(\rho,f) = 0 \text{ with } 0 \leq \beta \leq 1 \text{ and } |\omega| \leq W\}|.$$

In other words, $N_{\text{unf}}(T,f)$ is the number of unfolded nontrivial zeros of $L(s,f)$ with ordinate in $[-W,W]$. We then have the following well-known result:

Proposition 6.9.1. *For any L-function $L(s,f)$ and $W \geq \frac{1}{\pi}\log\left(\frac{q(f)}{(2\pi e)^{d_f}}\right)$,*

$$\frac{N_{\mathit{unf}}(W,f)}{W} \sim 1.$$

Proof. Consider the function $f(t)$ defined by

$$f(t) = \frac{t}{\pi}\log\left(\frac{q(f)|t|^{d_f}}{(2\pi e)^{d_f}}\right),$$

for $t \in \mathbb{R}$. Sine $f(t)$ is a strictly increasing continuous function, it has an inverse $g(w)$ for $w \in \mathbb{R}$. It follows that $|\omega| \leq W$ if and only if $|\rho| \leq g(W)$ and so $N_{\text{unf}}(W,f) = N(g(W),f)$. But by Corollary 6.9.1 and that $g(w)$ is the inverse of $f(t)$, we have $N(g(W),f) \sim W$. It follows that

$$N_{\text{unf}}(W,f) \sim W,$$

which is equivalent to the claim. □

We interpret Proposition 6.9.1 as saying that the unfolded nontrivial zeros are evenly spaced opposed to Corollary 6.9.1 which says that they tend to accumulate up the critical strip.

6.10 A Zero-free Region

Although the Riemann hypothesis remains out of reach, some progress has been made to understand regions inside of the critical strip for which L-functions are nonzero except for possibly one real exception. Such regions are known as **zero-free regions** and there is great interest in improving the breadth of such regions. We will derive a standard zero-free region for any L-function under some mild assumptions. First a useful lemma:

Lemma 6.10.1. *Let $L(s,f)$ be an L-function such that $\mathrm{Re}(\Lambda_f(n)) \ge 0$ provided $(n,q(f)) = 1$. Also suppose that $|\alpha_j(p)| \le \frac{p}{2}$ for ramified primes p. Then $L(1,f) \ne 0$ and hence $r_f \ge 0$. Moreover, there exists a constant $c > 0$ such that $L(s,f)$ has at most r_f real zeros in the region*

$$\sigma \ge 1 - \frac{c}{d_f(r_f+1)\log \mathfrak{q}(f)}.$$

Proof. Let β_j be a real nontrivial zero with $\frac{1}{2} \le \beta_j \le 1$. There are finitely many β_j since they belong to the compact interval $\frac{1}{2} \le s \le 1$ so we have $1 \le j \le n$ for some $n \ge 1$. Letting $1 < \sigma \le 2$, and applying Lemma 6.9.1 (iv) while discarding all the terms except those corresponding to the nontrivial zeros β_j, we obtain the inequality

$$\sum_{1 \le j \le n} \frac{1}{\sigma - \beta_j} < \frac{r_f}{\sigma - 1} + \mathrm{Re}\left(\frac{L'}{L}(\sigma,f)\right) + O(\log \mathfrak{q}(f)).$$

To estimate $\mathrm{Re}\left(\frac{L'}{L}(\sigma,f)\right)$, we first note that as $\mathrm{Re}(\Lambda_f(n)) \ge 0$ provided $(n,q(f)) = 1$ by assumption, the Dirichlet series of $\frac{L'^{(q(f))}}{L^{(q(f))}}(s,f)$ shows that

$$\mathrm{Re}\left(\frac{L^{(q(f))\prime}}{L^{(q(f))}}(\sigma,f)\right) \le 0.$$

This gives an estimate for the contribution of the local factors of $L(s,f)$ corresponding to unramified primes. For the contribution of the local factors corresponding to ramified primes, we use Equation (6.14) to compute

$$\mathrm{Re}\left(\frac{L'_{q(f)}}{L_{q(f)}}(\sigma,f)\right) \le \left|\frac{L'_{q(f)}}{L_{q(f)}}(\sigma,f)\right| = \left|\sum_{p \mid q(f)} \sum_{1 \le j \le d_f} \frac{\alpha_j(p)\log(p)}{(1-\alpha_j(p)p^{-\sigma})p^{\sigma}}\right| \le d_f \sum_{p \mid q(f)} \log(p) \le d_f \log(q(f)),$$

where in the second inequality we have made use of the assumption $|\alpha_j(p)| \le \frac{p}{2}$ for ramified primes p to conclude that $\left|\frac{\alpha_j(p)}{(1-\alpha_j(p)p^{-\sigma})p^{\sigma}}\right| \le 1$. These estimates together imply

$$\sum_{1 \le j \le n} \frac{1}{\sigma - \beta_j} < \frac{r_f}{\sigma - 1} + O(\log d_f \mathfrak{q}(f)).$$

From this inequality we see that $\beta_j \ne 1$. For if some $\beta_j = 1$, $r_f < 0$ and the right-hand side is negative for σ sufficiently close to 1 contradicting the positivity of the left-hand side. Thus $L(1,f) \ne 0$ and hence $r_f \ge 0$. As there are finitely many β_j, there exists a $c > 0$ such that the β_j satisfy

$$\beta_j \ge 1 - \frac{c}{d_f(r_f+1)\log \mathfrak{q}(f)}.$$

Setting $\sigma = 1 + \frac{2c}{d_f \log \mathfrak{q}(f)}$ and choosing c smaller, if necessary, we guarantee $1 < \sigma \le 2$. Then the two inequalities above together imply

$$\frac{n d_f \log \mathfrak{q}(f)}{2c + \frac{c}{(r_f+1)}} < \left(\frac{r_f}{2c} + O(1)\right) d_f \log \mathfrak{q}(f).$$

Isolating n, we see that

$$n < r_f + \frac{r_f}{2(r_f+1)} + O(c),$$

and taking c smaller, if necessary, we have $n \le r_f$. As $L(s,f)$ is nonzero for $\sigma > 1$, it follows that there are at most r_f real zeros satisfying

$$\sigma \ge 1 - \frac{c}{d_f(r_f+1)\log \mathfrak{q}(f)}.$$

This completes the proof. $\square$

We can now prove our zero-free region result:

Theorem 6.10.1. *Let $L(s,f)$ be an L-function with at most a simple pole at $s=1$, $\mathrm{Re}(\Lambda_f(n)) \ge 0$ provided $(n,q(f))=1$, and $|\alpha_j(p)| \le \frac{p}{2}$ for ramified primes p. Then there exists a constant $c>0$ such that $L(s,f)$ has no zeros in the region*

$$\sigma \ge 1 - \frac{c}{d_f^2 \log(\mathfrak{q}(f)(|t|+3))},$$

except for possibly one simple real zero β_f with $\beta_f < 1$ in the case $L(s,f)$ has a simple pole at $s=1$.

Proof. For $t \in \mathbb{R}$, let $L(s,g)$ be the L-function defined by

$$L(s,g) = L^3(s,f)L^3(s,\overline{f})L^4(s+it,f)L^4(s+it,\overline{f})L(s+2it,f)L(s+2it,\overline{f}).$$

Clearly $d_g = 16d_f$ and so $\mathfrak{q}(g)$ satisfies

$$\mathfrak{q}(g) \le \mathfrak{q}(f)^6\mathfrak{q}(it,f)^8\mathfrak{q}(2it,f)^2 \le \mathfrak{q}(f)^{16d_f}(|t|+3)^{10d_f} < (\mathfrak{q}(f)(|t|+3))^{16d_f}.$$

We claim that $\mathrm{Re}(\Lambda_g(n)) \ge 0$ for $(n,q(f))=1$. To see this, let p be an unramified prime. The local roots of $L(s,g)$ at p are $\alpha_j(p)$ and $\overline{\alpha_j(p)}$ both with multiplicity three, $\alpha_j(p)p^{-it}$ and $\overline{\alpha_j(p)}p^{-it}$ both with multiplicity four, and $\alpha_j(p)p^{-2it}$ and $\overline{\alpha_j(p)}p^{-2it}$ both with multiplicity one. So for any $k \ge 1$, the sum of k-th powers of these local roots is

$$\sum_{1\le j\le d_f}\left(6\mathrm{Re}(\alpha_j(p)^k) + 8\mathrm{Re}(\alpha_j(p)^k)p^{-kit} + 2\mathrm{Re}(\alpha_j(p)^k)p^{-2kit}\right).$$

The real part of this expression is

$$(6+8\cos\log(p^{kt}) + 2\cos\log(p^{2kt}))\mathrm{Re}(\Lambda_f(p^k)) = 4(1+\cos\log(p^{kt}))^2\mathrm{Re}(\Lambda_f(p^k)) \ge 0.$$

where we have made use of the identity $3+4\cos(\theta)+\cos(2\theta) = 2(1+\cos(\theta))^2$. It follows that $\mathrm{Re}(\Lambda_g(n)) \ge 0$ for $(n,q(f))=1$. Therefore the conditions of Lemma 6.10.1 are satisfied for $L(s,g)$. Now let $\rho = \beta + i\gamma$ be a complex nontrivial zero of $L(s,f)$. Setting $t=\gamma$, $L(s,g)$ has a real nontrivial zero at $s=\beta$ of order at least 8 and a pole at $s=1$ of order 6. That is, $r_g = 6$. But Lemma 6.10.1 implies that $L(s,g)$ can have at most 6 real nontrivial zeros in the given region. Letting the constant for the region in Lemma 6.10.1 be c', it follows that β must satisfy

$$\beta < 1 - \frac{c'}{d_g(r_g+1)\log \mathfrak{q}(g)} < 1 - \frac{c'}{1792 d_f^2 \log(\mathfrak{q}(f)(|\gamma|+3))},$$

Take $c = \frac{c'}{1792}$. Now let β be a real nontrivial zero of $L(s,f)$. Since $\mathrm{Re}(\Lambda_f(n)) \ge 0$ and $L(s,f)$ has at most a simple pole at $s=1$, Lemma 6.10.1 implies, upon shrinking c if necessary, that there is at most one simple real zero β_f in the desired region and it can only occur if $L(s,f)$ has a simple pole at $s=1$. Note that if b_f exists, then $\beta_f < 1$ because $L(s,f)$ is nonzero for $\sigma > 1$ and has a simple pole at $s=1$. This completes the proof. $\square$

Some comments are in order. Since the zeros of $L(s,f)$ occur in pairs ρ and $1-\overline{\rho}$, the zero-free region in Theorem 6.10.1 implies a symmetric zero-free region about the critical line with at most one real zero in each half as displayed in Figure 6.4. The possible simple zero b_f in Theorem 6.10.1 is referred to as a **Siegel zero** (or **exceptional zero**). If β_f is a Siegel zero of $L(s,\chi)$, then $1-\beta_f$ is also a real nontrivial zero of $L(s,\chi)$ since the nontrivial zeros occur in pairs (or sets of four in some cases we have discussed). This immediately implies that $\beta_f > \frac{1}{2}$ for we cannot have $\beta_f = \frac{1}{2}$ since the zero is simple and either β_f or $1-\beta_f$ must be at least $\frac{1}{2}$. Such zeros are conjectured to not exist. In full generality, Theorem 6.10.1 is the best result that one can hope for. It is possible to obtain better zero-free regions in certain cases but this heavily depends upon the particular L-function of interest and hence the arithmetic object f attached to $L(s,f)$. In many cases it is possible to augment the proof of Theorem 6.10.1 to make the constant c effective and such results have important applications. If $L(s,f)$ is of Selberg class, the conclusion of Theorem 6.10.1 is expected to hold without the possibly of a Siegel zero since $L(s,f)$ is expected to satisfy the Riemann hypothesis. Nevertheless, we can often satisfy the assumptions of Theorem 6.10.1. Indeed, if $\alpha_i(p) \geq 0$ then the assumptions are satisfied immediately. If not, since $|\alpha_i(p)| \leq 1$ the assumptions will hold for $\zeta(s)L(s,f)$ provided $L(s,f)$ does not have a pole at $s=1$. This is often possible to prove in practice since $L(s,f)$ is expected to be entire unless $\alpha_i(p) \geq 0$.

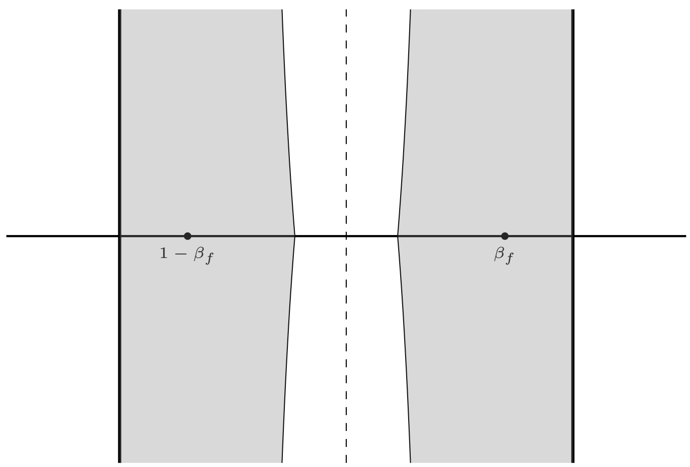

Figure 6.4: The symmetric zero-free region in Theorem 6.10.1.

6.11 The Explicit Formula

A formula somewhat analogous to the approximate functional equation can be derived for the negative logarithmic derivative of any L-function. This formula is called the the **explicit formula**:

Theorem 6.11.1 (Explict Formula). *Let $\psi(y)$ be a bump function with compact support and let $\Psi(s)$ denote its Mellin transform. Set $\phi(y) = y^{-1}\psi(y^{-1})$ so that its Mellin transform satisfies $\Phi(s) = \Psi(1-s)$. Then for any L-function $L(s,f)$, we have*

$$\sum_{n\geq 1}\big(\Lambda_f(n)\psi(n) + \Lambda_{\overline{f}}(n)\phi(n)\big) = \psi(1)\log q(f) + r_f\Psi(1) + \frac{1}{2\pi i}\int_{(\frac{1}{2})}\left(\frac{\gamma'}{\gamma}(s,f) + \frac{\gamma'}{\gamma}(1-s,\overline{f})\right)\Psi(s)\,ds - \sum_{\rho}\Psi(\rho),$$

where the sum is counted with multiplicity and ordered with respect to the size of the ordinate.

Proof. By smoothed Perron's formula, we have

$$\sum_{n \geq 1} \Lambda_f(n)\psi(n) = \frac{1}{2\pi i}\int_{(c)} -\frac{L'}{L}(s,f)\Phi(s)\,ds,$$

for any $c > 1$. Since the trivial zeros are isolated, let $\delta > 0$ be such that there are no trivial zeros in the vertical strip $-2\delta \leq \sigma < 0$. By Propositions 6.2.1 and 6.4.1, the integrand has polynomial decay and therefore is locally absolutely uniformly convergent by Proposition 1.5.2. Shifting the line of integration to $(-\delta)$, we pass by simple poles at $s = 1$ and $s = \rho$ for every nontrivial zero ρ obtaining

$$\sum_{n \geq 1} \Lambda_f(n)\psi(n) = r_f\Psi(1) - \sum_{\rho}\Psi(\rho) + \frac{1}{2\pi i}\int_{(-\delta)} -\frac{L'}{L}(s,f)\Psi(s)\,ds.$$

For the latter integral, the functional equation and that $q(\overline{f}) = q(f)$ together imply

$$-\frac{L'}{L}(s,f) = \log q(f) + \frac{\gamma'}{\gamma}(s,f) + \frac{\gamma'}{\gamma}(1-s,\overline{f}) + \frac{L'}{L}(1-s,\overline{f}).$$

Thus

$$\frac{1}{2\pi i}\int_{(-\delta)} -\frac{L'}{L}(s,f)\Phi(s)\,ds = \frac{1}{2\pi i}\int_{(-\delta)} \left(\log q(f) + \frac{\gamma'}{\gamma}(s,f) + \frac{\gamma'}{\gamma}(1-s,\overline{f}) + \frac{L'}{L}(1-s,\overline{f})\right)\Psi(s)\,ds.$$

The integral over the first term on the right-hand side is $\psi(1)\log q(f)$ by the Mellin inversion formula. As for the last term, smoothed Perron's formula and that $\Phi(s) = \Psi(1-s)$ together give

$$\frac{1}{2\pi i}\int_{(-\delta)} -\frac{L'}{L}(1-s,\overline{f})\Psi(s)\,ds = \sum_{n \geq 1}\Lambda_{\overline{f}}(n)\phi(n).$$

So altogether, we have

$$\begin{aligned}\sum_{n \geq 1}\big(\Lambda_f(n)\psi(n) + \Lambda_{\overline{f}}(n)\phi(n)\big) &= \psi(1)\log q(f) + r_f\Psi(1) \\ &\quad + \frac{1}{2\pi i}\int_{(-\delta)}\left(\frac{\gamma'}{\gamma}(s,f) + \frac{\gamma'}{\gamma}(1-s,\overline{f})\right)\Psi(s)\,ds - \sum_{\rho}\Psi(\rho).\end{aligned}$$

Lastly, we note that by Propositions 1.6.3 and 6.2.1, the remaining integrand has polynomial decay and therefore is locally absolutely uniformly convergent by Proposition 1.5.2. Shifting the line of integration to $\left(\frac{1}{2}\right)$, we pass over no poles because the residues of $\frac{\gamma'}{\gamma}(1-s,\overline{f})$ are negative of those of $\frac{\gamma'}{\gamma}(s,f)$ for $\sigma \geq -\delta$ (the nontrivial zeros of $L(s,f)$ occur in pairs ρ and $1-\overline{\rho}$). This completes the proof. $\square$

The explicit formula is a very useful tool for analytic investigations. Since $\Lambda_f(n)$ is essentially a weighted sum over prime powers, the explicit formula can be thought of as expressing a smoothed weighted sum over primes for f in terms of the zeros of $L(s,f)$.

Chapter 7

Types of L-functions

We discuss a variety of L-functions: the Riemann zeta function, L-functions attached to Dirichlet characters, and Hecke L-functions. In the case of Hecke L-functions, we also describe a method of Rankin and Selberg for constructing new L-functions from old ones.

7.1 The Riemann Zeta Function

The Definition & Euler Product of $\zeta(s)$

The **Riemann zeta function** or simply the **zeta function** $\zeta(s)$ is defined as an L-series:

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

This is the prototypical example of a Dirichlet series as all the coefficients are 1. We will see that $\zeta(s)$ is a Selberg class L-function. As the coefficients are trivially polynomially bounded, $\zeta(s)$ is locally absolutely uniformly convergent for $\sigma > 1$. Also note that $\zeta(s)$ is necessarily nonzero in this region. Determining the Euler product is also an easy matter. As the coefficients are obviously completely multiplicative, we have the degree 1 Euler product:

$$\zeta(s) = \prod_p (1-p^{-s})^{-1},$$

in this region as well. The local factor at p is $\zeta_p(s) = (1-p^{-s})^{-1}$ with local root 1.

The Integral Representation of $\zeta(s)$: Part I

Riemann's ingenious insight was to analytically continue $\zeta(s)$. By this, he sought to find a representation of $\zeta(s)$ defined on a larger region than $\sigma > 1$. This is the approach we will take, and the argument follows the same line of reasoning as that of Riemann. We consider the gamma function $\Gamma\left(\frac{s}{2}\right)$:

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-x} x^{\frac{s}{2}} \frac{dx}{x}.$$

Remark 7.1.1. *We have chosen to express the gamma function in terms of the measure $\frac{dx}{x}$ instead of dx. This is a tactical change for two reasons. The first is that $\frac{dx}{x}$ is invariant under the change of variables $x \to Cx$ for any constant C. The second is that under the change of variables $x \to \frac{1}{x}$ we have $\frac{dx}{x} \to -\frac{dx}{x}$ but the bounds of integration are also flipped. So we may leave the measure invariant provided we don't* *flip the bounds of integration. These types of change of variables are essential in the study of L-functions which motivates the use of this measure.*

Performing the change of variables $x \to \pi n^2 x$ for fixed $n \geq 1$ yields

$$\Gamma\left(\frac{s}{2}\right) = \pi^{\frac{s}{2}} n^s \int_0^\infty e^{-\pi n^2 x} x^{\frac{s}{2}} \frac{dx}{x}. \tag{7.1}$$

Let $\sigma > 1$. Dividing by $\pi^{\frac{s}{2}} n^s$ and summing over $n \geq 1$, we see that

$$\begin{aligned}
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \sum_{n \geq 1} \int_0^\infty e^{-\pi n^2 x} x^{\frac{s}{2}} \frac{dx}{x} \\
&= \int_0^\infty \sum_{n \geq 1} e^{-\pi n^2 x} x^{\frac{s}{2}} \frac{dx}{x} && \text{DCT} \\
&= \int_0^\infty \omega(x) x^{\frac{s}{2}} \frac{dx}{x},
\end{aligned}$$

where we set

$$\omega(x) = \sum_{n \geq 1} e^{-\pi n^2 x}.$$

Therefore we have an integral representation

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty \omega(x) x^{\frac{s}{2}} \frac{dx}{x}. \tag{7.2}$$

This was essentially Riemann's insight: rewrite the Riemann zeta function in terms of a Mellin transform. Unfortunately, we cannot proceed until we understand $\omega(x)$. So we will make a slight detour and come back to the integral representation after.

Jacobi's Theta Function $\vartheta(s)$

Jacobi's theta function $\vartheta(s)$ is defined for $\sigma > 0$ by

$$\vartheta(s) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 s} = 1 + 2 \sum_{n \geq 1} e^{-\pi n^2 s}.$$

It is locally absolutely uniformly convergent in this region by the ratio test. Its relation to $\omega(s)$ is given by the identity

$$\omega(s) = \frac{\vartheta(s) - 1}{2}. \tag{7.3}$$

We see from the Taylor series of $\frac{1}{1-e^{\sigma}}$, that

$$\omega(s) = O\left(\sum_{n \geq 1} e^{-\pi n^2 \sigma}\right) = O\left(\sum_{n \geq 1} e^{-\pi n \sigma}\right) = O\left(\frac{1}{1 - e^{-\pi\sigma}}\right) = O(e^{-\pi\sigma}),$$

and so $\omega(s)$ exhibits rapid decay. The essential fact about Jacobi's theta function we will need is the **functional equation for Jacobi's theta function** that was known to Riemann:

Theorem 7.1.1 (Functional equation for Jacobi's theta function). *For $\sigma > 0$,*

$$\vartheta(s) = \frac{1}{\sqrt{s}}\vartheta\left(\frac{1}{s}\right).$$

Proof. By the identity theorem it suffices to verify this for $s = \sigma$ with $\sigma > 0$. Set $f(x) = e^{-\pi x^2 s}$. Then $f(x)$ is of Schwarz class. We compute its Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx = \int_{-\infty}^{\infty} e^{-\pi x^2 s}e^{-2\pi itx}\,dx = \int_{-\infty}^{\infty} e^{-\pi(x^2 s+2itx)}\,dx.$$

Making the change of variables $x \to \frac{x}{\sqrt{s}}$, the last integral above becomes

$$\frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} e^{-\pi\left(x^2+\frac{2itx}{\sqrt{s}}\right)}.$$

Complete the square in the exponent by noticing

$$-\pi\left(x^2+\frac{2itx}{\sqrt{s}}\right) = -\pi\left(\left(x+\frac{it}{\sqrt{s}}\right)^2+\frac{t^2}{s^2}\right).$$

Taking exponentials, this implies that the previous integral is equal to

$$\frac{e^{-\frac{\pi t^2}{s}}}{\sqrt{s}}\int_{-\infty}^{\infty} e^{-\pi\left(x+\frac{it}{\sqrt{s}}\right)^2}\,dx.$$

The change of variables $x \to \frac{x}{\sqrt{s}} - \frac{it}{\sqrt{s}}$ is permitted without affecting the line of integration by viewing the integral as a complex integral, noting that the integrand is entire as a complex function, and shifting the line of integration. This gives

$$\frac{e^{-\frac{\pi t^2}{s}}}{\sqrt{s}}\int_{-\infty}^{\infty} e^{-\pi\left(x+\frac{it}{\sqrt{s}}\right)^2}\,dx = \frac{e^{-\frac{\pi t^2}{s}}}{\sqrt{s}}\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = \frac{e^{-\frac{\pi t^2}{s}}}{\sqrt{s}},$$

where the last equality follows because the last integral above is 1 since it is the Gaussian integral (see Appendix D.1). Thus

$$\hat{f}(t) = \frac{e^{-\frac{\pi t^2}{s}}}{\sqrt{s}}.$$

By the Poisson summation formula, we have

$$\vartheta(s) = \sum_{t\in\mathbb{Z}}\frac{e^{-\frac{\pi t^2}{s}}}{\sqrt{s}} = \frac{1}{\sqrt{s}}\sum_{t\in\mathbb{Z}}e^{-\frac{\pi t^2}{s}} = \frac{1}{\sqrt{s}}\vartheta\left(\frac{1}{s}\right),$$

and the identity theorem finishes the proof. □

We will use this functional equation to analytically continue $\zeta(s)$.

The Integral Representation of $\zeta(s)$: Part II

Returning to the Riemann zeta function, we split the integral in Equation (7.2) into two pieces

$$\int_0^\infty \omega(x)x^{\frac{s}{2}}\,\frac{dx}{x} = \int_0^1 \omega(x)x^{\frac{s}{2}}\,\frac{dx}{x} + \int_1^\infty \omega(x)x^{\frac{s}{2}}\,\frac{dx}{x}. \tag{7.4}$$

Since $\omega(x)$ has rapid decay, the second piece is locally absolutely uniformly convergent for $\sigma > 1$ by Proposition 1.5.2. Hence it is analytic there. The idea now is to rewrite the first piece in the same form and symmetrize the result as much as possible. We being by performing a change of variables $x \to \frac{1}{x}$ to the first piece to obtain

$$\int_1^\infty \omega\left(\frac{1}{x}\right)x^{-\frac{s}{2}}\,\frac{dx}{x}$$

Now the functional equation for Jacobi's theta function $\vartheta(x)$ and Equation (7.3) together imply

$$\omega\left(\frac{1}{x}\right) = \frac{\vartheta\left(\frac{1}{x}\right)-1}{2} = \frac{\sqrt{x}\vartheta(x)-1}{2} = \frac{\sqrt{x}(2\omega(x)+1)-1}{2} = \sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}. \tag{7.5}$$

Equation (7.5) gives the first equality in the following chain:

$$\begin{aligned}\int_1^\infty \omega\left(\frac{1}{x}\right)x^{-\frac{s}{2}}\,\frac{dx}{x} &= \int_1^\infty \left(\sqrt{x}\omega(x) + \frac{\sqrt{x}}{2} - \frac{1}{2}\right)x^{-\frac{s}{2}}\,\frac{dx}{x}\\ &= \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\,\frac{dx}{x} + \int_1^\infty \frac{x^{\frac{1-s}{2}}}{2}\,\frac{dx}{x} - \int_1^\infty \frac{x^{-\frac{s}{2}}}{2}\,\frac{dx}{x}\\ &= \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\,\frac{dx}{x} + \frac{1}{1-s} - \frac{1}{s}\\ &= \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\,\frac{dx}{x} - \frac{1}{s(1-s)}.\end{aligned}$$

Substituting this result back into Equation (7.4) with Equation (7.2) yields the integral representation

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left[-\frac{1}{s(1-s)} + \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\,\frac{dx}{x} + \int_1^\infty \omega(x)x^{\frac{s}{2}}\,\frac{dx}{x}\right].$$

This integral representation will give analytic continuation. To see this, first observe that everything outside the brackets is entire. Moreover, the two integrals are locally absolutely uniformly convergent on $\mathbb{C}$ by Proposition 1.5.2. The fractional term is holomorphic except for simple poles at $s = 0$ and $s = 1$. The meromorphic continuation to $\mathbb{C}$ follows with possible simple poles at $s = 0$ and $s = 1$. There is no pole at $s = 0$. Indeed, $\gamma(s,\zeta)$ has a simple pole coming from the gamma factor there and so its reciprocal has a simple zero. This cancels the corresponding simple pole of $\frac{1}{s(1-s)}$ so that $\zeta(s)$ has a removable singularity and thus is holomorphic at $s = 0$. At $s = 1$, $\gamma(s,\zeta)$ is nonzero, and so $\zeta(s)$ has a simple pole. Therefore $\zeta(s)$ has meromorphic continuation to all of $\mathbb{C}$ with a simple pole at $s = 1$.

The Functional Equation, Critical Strip & Residue of $\zeta(s)$

An immediate consequence of applying the symmetry $s \to 1-s$ to the integral representation is the following functional equation:

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}}\zeta(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\pi^{\frac{1-s}{2}}}\zeta(1-s).$$

We identify the gamma factor as

$$\gamma(s,\zeta) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right),$$

with $\kappa = 0$ the only local root at infinity. Clearly it satisfies the required bounds. The conductor is $q(\zeta) = 1$ so no primes ramify. The completed zeta function is

$$\Lambda(s,\zeta) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

with functional equation

$$\Lambda(s,\zeta) = \Lambda(1-s,\zeta).$$

This is the functional equation of $\zeta(s)$ and in this case is just a reformulation of the previous functional equation. From it we find that the root number is $\varepsilon(\zeta) = 1$ and that $\zeta(s)$ is self-dual. We can now show that the order of $\zeta(s)$ is 1. As there is only a simple pole at $s = 1$, multiply by $(s-1)$ to clear the polar divisor. As the integrals in the integral representation are locally absolutely uniformly convergent, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s,\zeta)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

Thus the reciprocal of the gamma factor is also of order 1. It follows that

$$(s-1)\zeta(s) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

This shows $(s-1)\zeta(s)$ is of order 1, and thus $\zeta(s)$ is as well after removing the polar divisor. We now compute the residue of $\zeta(s)$ at $s = 1$:

Proposition 7.1.1.

$$\operatorname*{Res}_{s=1}\zeta(s) = 1.$$

Proof. The only term in the integral representation of $\zeta(s)$ contributing to the pole is $-\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\frac{1}{s(1-s)}$. Observe

$$\lim_{s\to1}\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} = 1,$$

because $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Therefore

$$\operatorname*{Res}_{s=1}\zeta(s) = \operatorname*{Res}_{s=1} -\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\frac{1}{s(1-s)} = \operatorname*{Res}_{s=1} -\frac{1}{s(1-s)} = \lim_{s\to1} -\frac{(s-1)}{s(1-s)} = 1.$$

□

We summarize all of our work into the following theorem:

Theorem 7.1.2. *$\zeta(s)$ is a Selberg class L-function. For $\sigma > 1$, it has a degree 1 Euler product given by*

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}.$$

Moreover, it admits meromorphic continuation to $\mathbb{C}$ via the integral representation

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left[-\frac{1}{s(1-s)} + \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\frac{dx}{x} + \int_1^\infty \omega(x)x^{\frac{s}{2}}\frac{dx}{x}\right],$$

with functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Lambda(s,\zeta) = \Lambda(1-s,\zeta),$$

and there is a simple pole at $s = 1$ of residue 1.

Lastly, we note that by virtue of the functional equation we can also compute $\zeta(0)$. Indeed, since $\mathrm{Res}_{s=1}\,\zeta(s) = 1$, we have

$$\lim_{s\to 1}(s-1)\Lambda(s,\zeta) = \operatorname*{Res}_{s=1}\zeta(s)\lim_{s\to 1}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) = 1.$$

In other words, $\Lambda(s,\zeta)$ has a simple pole at $s=1$ with residue 1 too. Since the completed zeta function is completely symmetric as $s \to 1-s$, it has a simple pole at $s=0$ with residue 1. Hence

$$1 = \lim_{s\to 1}(s-1)\Lambda(1-s,\zeta) = \operatorname*{Res}_{s=1}\Gamma\left(\frac{1-s}{2}\right)\lim_{s\to 1}\pi^{-\frac{1-s}{2}}\zeta(1-s) = -2\zeta(0),$$

because $\mathrm{Res}_{s=0}\,\Gamma(s) = 1$. Therefore $\zeta(0) = -\frac{1}{2}$.

7.2 Dirichlet L-functions

The Definition & Euler Product of $L(s,\chi)$

To every Dirichlet character χ there is an associated L-function. Throughout we will let m denote the modulus and q the conductor of χ respectively. The **Dirichlet L-function** $L(s,\chi)$ attached to the Dirichlet character χ is defined as an L-series:

$$L(s,\chi) = \sum_{n\geq 1}\frac{\chi(n)}{n^s}.$$

Since $\chi(n) = 0$ if $(n,m) > 1$, the above sum can be restricted to all positive integers relatively prime to m. We will see that $L(s,\chi)$ is a Selberg class L-function if χ is primitive and of conductor $q > 1$ (in the case $q = 1$, $L(s,\chi) = \zeta(s)$). From now we make this assumption about χ. As $|\chi(n)| \ll 1$, $L(s,\chi)$ is locally absolutely uniformly convergent for $\sigma > 1$. Because χ is completely multiplicative we also have the degree 1 Euler product:

$$L(s,\chi) = \prod_p (1-\chi(p)p^{-s})^{-1} = \prod_{p\nmid m}(1-\chi(p)p^{-s})^{-1},$$

in this region as well. The last equality holds because if $p \mid m$ we have $\chi(p) = 0$. So for $p \mid m$, the local factor at p is $L_p(s,\chi) = 1$ with local root 0. For $p \nmid m$ the local factor at p is $L_p(s,\chi) = (1-\chi(p)p^{-s})^{-1}$ with local root $\chi(p)$.

The Integral Representation of $L(s,\chi)$: Part I

The integral representation for $L(s,\chi)$ is deduced in a similar way as for $\zeta(s)$. However, it will depend on if χ is even or odd. To handle both cases simultaneously let $\mathfrak{a} = 0, 1$ according to whether χ is even or odd. In other words,

$$\mathfrak{a} = \frac{\chi(1)-\chi(-1)}{2}.$$

We also have $\chi(-1) = (-1)^{\mathfrak{a}}$. Note that $\mathfrak{a}$ takes the same value for both χ and $\overline{\chi}$. Making the substitution $s \to s+\mathfrak{a}$ in Equation (7.1) and multiplying by $\chi(n)$ yields

$$\chi(n)\Gamma\left(\frac{s+\mathfrak{a}}{2}\right) = \pi^{\frac{s+\mathfrak{a}}{2}}n^s\int_0^\infty \chi(n)n^{\mathfrak{a}}e^{-\pi n^2 x}x^{\frac{s+\mathfrak{a}}{2}}\,\frac{dx}{x},$$

after moving the $n^{\mathfrak{a}}$ on the inside of the integral. Let $\sigma > 1$. Dividing by $\pi^{\frac{s+\mathfrak{a}}{2}} n^s$ and summing over $n \geq 1$, we see that

$$\begin{aligned} \pi^{-\frac{s+\mathfrak{a}}{2}} \Gamma\left(\frac{s+\mathfrak{a}}{2}\right) L(s,\chi) &= \sum_{n\geq 1} \int_0^\infty \chi(n) n^{\mathfrak{a}} e^{-\pi n^2 x} x^{\frac{s+\mathfrak{a}}{2}} \frac{dx}{x} \\ &= \int_0^\infty \sum_{n\geq 1} \chi(n) n^{\mathfrak{a}} e^{-\pi n^2 x} x^{\frac{s+\mathfrak{a}}{2}} \frac{dx}{x} \qquad \text{DCT} \\ &= \int_0^\infty \omega_\chi(x) x^{\frac{s+\mathfrak{a}}{2}} \frac{dx}{x}, \end{aligned}$$

where we set

$$\omega_\chi(x) = \sum_{n\geq 1} \chi(n) n^{\mathfrak{a}} e^{-\pi n^2 x}.$$

Therefore we have an integral representation

$$L(s,\chi) = \frac{\pi^{\frac{s+\mathfrak{a}}{2}}}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)} \int_0^\infty \omega_\chi(x) x^{\frac{s+\mathfrak{a}}{2}} \frac{dx}{x}, \tag{7.6}$$

and just like $\zeta(s)$ we need to find a functional equation for $\omega_\chi(x)$ before we can proceed.

Dirichlet's Theta Function $\vartheta_\chi(s)$

Dirichlet's theta function $\vartheta_\chi(s)$ attached to the character χ, is defined for $\sigma > 0$ by

$$\vartheta_\chi(s) = \sum_{n\in\mathbb{Z}} \chi(n) n^{\mathfrak{a}} e^{-\pi n^2 s} = 2\sum_{n\geq 1} \chi(n) n^{\mathfrak{a}} e^{-\pi n^2 s}.$$

It is locally absolutely uniformly convergent in this region by the ratio test. Notice that the term corresponding to $n = 0$ vanishes because $\chi(0) = 0$, and $\chi(n)n^{\mathfrak{a}} = \chi(-n)(-n)^{\mathfrak{a}}$ so that the n and $-n$ terms agree. Therefore the relationship between the twisted theta function and $\omega_\chi(s)$ is

$$\omega_\chi(s) = \frac{\vartheta_\chi(s)}{2}. \tag{7.7}$$

Remark 7.2.1. *Equation* (7.7) *is a slightly less complex relationship than Equation* (7.3)*. This is because assuming $q > 1$ means $\chi(0) = 0$.*

We see from the Taylor series of $\frac{1}{1-e^{\sigma}}$ and its derivative, that

$$\omega_\chi(s) = O\left(\sum_{n\geq 1} n e^{-\pi n^2 \sigma}\right) = O\left(\sum_{n\geq 1} n e^{-\pi n \sigma}\right) = O\left(\frac{e^{-\pi\sigma}}{(1-e^{-\pi\sigma})^2}\right) = O(e^{-\pi\sigma}),$$

and so $\omega(s)$ exhibits rapid decay. The essential fact we will need is the **functional equation for Dirichlet's theta function**:

Theorem 7.2.1 (Functional equation for Dirichlet's theta function). *Let χ be a primitive Dirichlet character of conductor $q > 1$. For $\sigma > 0$,*

$$\vartheta_\chi(s) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}(qs)^{\frac{1}{2}+\mathfrak{a}}} \vartheta_{\overline{\chi}}\left(\frac{1}{q^2 s}\right).$$

Proof. By the identity theorem it suffices to verify this for $s=\sigma$ with $\sigma>0$. Since χ is q-periodic, we can write

$$\vartheta_\chi(s)=\sum_{a \pmod q}\chi(a)\sum_{m\in\mathbb{Z}}(mq+a)^{\mathfrak{a}}e^{-\pi(mq+a)^2 s}.$$

Set $f(x)=(xq+a)^{\mathfrak{a}}e^{-\pi(xq+a)^2 s}$. Then $f(x)$ is of Schwarz class. We compute its Fourier transform:

$$\hat{f}(t)=\int_{-\infty}^{\infty}f(x)e^{-2\pi itx}\,dx=\int_{-\infty}^{\infty}(xq+a)^{\mathfrak{a}}e^{-\pi(xq+a)^2 s}e^{-2\pi itx}\,dx=\int_{-\infty}^{\infty}(xq+a)^{\mathfrak{a}}e^{-\pi((xq+a)^2 s+2itx)}\,dx.$$

By performing the change of variables $x\to\frac{x}{q\sqrt{s}}-\frac{a}{q}$, the last integral above becomes

$$\frac{e^{\frac{2\pi iat}{q}}}{qs^{\frac{1+\mathfrak{a}}{2}}}\int_{-\infty}^{\infty}x^{\mathfrak{a}}e^{-\pi\left(x^2+\frac{2itx}{q\sqrt{s}}\right)}\,dx.$$

Complete the square in the exponent by observing

$$-\pi\left(x^2+\frac{2itx}{q\sqrt{s}}\right)=-\pi\left(\left(x+\frac{it}{q\sqrt{s}}\right)^2+\frac{t^2}{q^2 s}\right).$$

Taking exponentials, this implies that the previous integral is equal to

$$\frac{e^{\frac{2\pi iat}{q}-\frac{\pi t^2}{q^2 s}}}{qs^{\frac{1+\mathfrak{a}}{2}}}\int_{-\infty}^{\infty}x^{\mathfrak{a}}e^{-\pi\left(x+\frac{it}{q\sqrt{s}}\right)^2}\,dx.$$

The change of variables $x\to x-\frac{it}{q\sqrt{s}}$ is permitted without affecting the line of integration by viewing the integral as a complex integral, noting that the integrand is entire as a complex function, and shifting the line of integration. This gives

$$\frac{e^{\frac{2\pi iat}{q}-\frac{\pi t^2}{q^2 s}}}{qs^{\frac{1+\mathfrak{a}}{2}}}\int_{-\infty}^{\infty}\left(x-\frac{it}{q\sqrt{s}}\right)^{\mathfrak{a}}e^{-\pi x^2}\,dx=\frac{e^{\frac{2\pi iat}{q}-\frac{\pi t^2}{q^2 s}}}{qs^{\frac{1+\mathfrak{a}}{2}}}\int_{-\infty}^{\infty}\left(x+\frac{t}{iq\sqrt{s}}\right)^{\mathfrak{a}}e^{-\pi x^2}\,dx.$$

If $\mathfrak{a}=0$, we obtain

$$\frac{e^{\frac{2\pi iat}{q}-\frac{\pi t^2}{q^2 s}}}{qs^{\frac{1+\mathfrak{a}}{2}}}\int_{-\infty}^{\infty}e^{-\pi x^2}\,dx=\frac{e^{\frac{2\pi iat}{q}-\frac{\pi t^2}{q^2 s}}}{qs^{\frac{1+\mathfrak{a}}{2}}},\tag{7.8}$$

where the equality holds because the integral is 1 since it is the Gaussian integral (see Appendix D.1). If $\mathfrak{a}=1$, then by direct computation

$$\int_{-\infty}^{\infty}xe^{-\pi x^2}\,dx=-\frac{1}{2\pi}e^{-\pi x^2}\Big|_{-\infty}^{\infty}=0,$$

and thus

$$\frac{e^{\frac{2\pi iat}{q}-\frac{\pi t^2}{q^2 s}}}{qs^{\frac{1+\mathfrak{a}}{2}}}\int_{-\infty}^{\infty}\left(\frac{t}{iq\sqrt{s}}\right)e^{-\pi x^2}\,dx=\frac{e^{\frac{2\pi iat}{q}-\frac{\pi t^2}{q^2 s}}}{qs^{\frac{1+\mathfrak{a}}{2}}}\left(\frac{t}{iq\sqrt{s}}\right)\int_{-\infty}^{\infty}e^{-\pi x^2}\,dx=\frac{e^{\frac{2\pi iat}{q}-\frac{\pi t^2}{q^2 s}}}{qs^{\frac{1+\mathfrak{a}}{2}}}\left(\frac{t}{iq\sqrt{s}}\right),\tag{7.9}$$

where the last equality follows because the last integral is the Gaussian integral again. Since $\left(\frac{t}{iq\sqrt{s}}\right)^{\mathfrak{a}}=1$ if $\mathfrak{a}=0$, Equations (7.8) and (7.9) together imply

$$\hat{f}(t)=\frac{e^{\frac{2\pi iat}{q}-\frac{\pi t^2}{q^2 s}}}{qs^{\frac{1+\mathfrak{a}}{2}}}\left(\frac{t}{iq\sqrt{s}}\right)^{\mathfrak{a}}.$$

By the Poisson summation formula, we have

$$
\begin{aligned}
\vartheta_\chi(s) &= \sum_{a \pmod q} \chi(a) \sum_{t\in\mathbb{Z}} \frac{e^{\frac{2\pi i a t}{q} - \frac{\pi t^2}{q^2 s}}}{q s^{\frac{1+\mathfrak{a}}{2}}} \left(\frac{t}{iq\sqrt{s}}\right)^{\mathfrak{a}} \\
&= \frac{1}{i^{\mathfrak{a}} q^{1+\mathfrak{a}} s^{\frac{1}{2}+\mathfrak{a}}} \sum_{a \pmod q} \chi(a) \sum_{t\in\mathbb{Z}} t^{\mathfrak{a}} e^{\frac{2\pi i a t}{q} - \frac{\pi t^2}{q^2 s}} \\
&= \frac{1}{i^{\mathfrak{a}} q^{1+\mathfrak{a}} s^{\frac{1}{2}+\mathfrak{a}}} \sum_{t\in\mathbb{Z}} t^{\mathfrak{a}} e^{-\frac{\pi t^2}{q^2 s}} \sum_{a \pmod q} \chi(a) e^{\frac{2\pi i a t}{q}} \\
&= \frac{1}{i^{\mathfrak{a}} q^{1+\mathfrak{a}} s^{\frac{1}{2}+\mathfrak{a}}} \sum_{t\in\mathbb{Z}} t^{\mathfrak{a}} e^{-\frac{\pi t^2}{q^2 s}} \tau(t,\chi) && \text{definition of } \tau(t,\chi) \\
&= \frac{\tau(\chi)}{i^{\mathfrak{a}} q^{1+\mathfrak{a}} s^{\frac{1}{2}+\mathfrak{a}}} \sum_{t\in\mathbb{Z}} \overline{\chi}(t) t^{\mathfrak{a}} e^{-\frac{\pi t^2}{q^2 s}} && \text{Corollary 1.4.1} \\
&= \frac{\varepsilon_\chi}{i^{\mathfrak{a}} (qs)^{\frac{1}{2}+\mathfrak{a}}} \sum_{t\in\mathbb{Z}} \overline{\chi}(t) t^{\mathfrak{a}} e^{-\frac{\pi t^2}{q^2 s}} && \varepsilon_\chi = \frac{\tau(\chi)}{\sqrt{q}} \\
&= \frac{\varepsilon_\chi}{i^{\mathfrak{a}} (qs)^{\frac{1}{2}+\mathfrak{a}}} \vartheta_{\overline{\chi}}\left(\frac{1}{q^2 s}\right),
\end{aligned}
$$

and the identity theorem finishes the proof. □

Notice that the functional equation relates $\vartheta_\chi(s)$ to $\vartheta_{\overline{\chi}}(s)$. Regardless, we will use this functional equation to analytically continue $L(s,\chi)$.

The Integral Representation of $L(s,\chi)$: Part II

Returning to $L(s,\chi)$, split the integral in Equation (7.6) into two pieces

$$
\int_0^\infty \omega_\chi(x) x^{\frac{s+\mathfrak{a}}{2}} \frac{dx}{x} = \int_0^{\frac{1}{q}} \omega_\chi(x) x^{\frac{s+\mathfrak{a}}{2}} \frac{dx}{x} + \int_{\frac{1}{q}}^\infty \omega_\chi(x) x^{\frac{s+\mathfrak{a}}{2}} \frac{dx}{x}. \tag{7.10}
$$

Since $\omega_\chi(x)$ exhibits rapid decay, the second piece is locally absolutely uniformly convergent for $\sigma > 1$ by Proposition 1.5.2. Hence it is an analytic function. We now rewrite the first piece in the same form and symmetrize the result as much as possible. Start by performing a change of variables $x \to \frac{1}{q^2 x}$ to the first piece to obtain

$$
q^{-(s+\mathfrak{a})} \int_{\frac{1}{q}}^\infty \omega_\chi\left(\frac{1}{q^2 x}\right) x^{-\frac{s+\mathfrak{a}}{2}} \frac{dx}{x}.
$$

Now the functional equation for Dirichlet's theta function $\vartheta_\chi(x)$ and Equation (7.7) together imply

$$
\begin{aligned}
\omega_\chi\left(\frac{1}{q^2x}\right) &= \frac{\vartheta_\chi\left(\frac{1}{q^2x}\right)}{2} \\
&= \frac{i^{\mathfrak{a}}(qx)^{\frac{1}{2}+\mathfrak{a}}}{\varepsilon_{\overline{\chi}}}\frac{\vartheta_{\overline{\chi}}(x)}{2} \\
&= \varepsilon_\chi(-i)^{\mathfrak{a}}(qx)^{\frac{1}{2}+\mathfrak{a}}\frac{\vartheta_{\overline{\chi}}(x)}{2} \quad \text{Proposition 1.4.3 and } \chi(-1)=(-1)^{\mathfrak{a}} \\
&= \frac{\varepsilon_\chi(qx)^{\frac{1}{2}+\mathfrak{a}}}{i^{\mathfrak{a}}}\frac{\vartheta_{\overline{\chi}}(x)}{2} \\
&= \frac{\varepsilon_\chi(qx)^{\frac{1}{2}+\mathfrak{a}}}{i^{\mathfrak{a}}}\omega_{\overline{\chi}}(x).
\end{aligned}
\tag{7.11}
$$

Equation (7.11) gives the first equality in the following chain:

$$
\begin{aligned}
q^{-(s+\mathfrak{a})}\int_{\frac{1}{q}}^{\infty}\omega_\chi\left(\frac{1}{q^2x}\right)x^{-\frac{s+\mathfrak{a}}{2}}\,\frac{dx}{x} &= q^{-(s+\mathfrak{a})}\int_{\frac{1}{q}}^{\infty}\left(\frac{\varepsilon_\chi(qx)^{\frac{1}{2}+\mathfrak{a}}}{i^{\mathfrak{a}}}\omega_{\overline{\chi}}(x)\right)x^{-\frac{s+\mathfrak{a}}{2}}\,\frac{dx}{x} \\
&= \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1}{2}-s}\int_{\frac{1}{q}}^{\infty}\omega_{\overline{\chi}}(x)x^{\frac{(1-s)+\mathfrak{a}}{2}}\,\frac{dx}{x}.
\end{aligned}
$$

Substituting this last expression back into Equation (7.10) with Equation (7.6) gives the integral representation

$$
L(s,\chi) = \frac{\pi^{\frac{s+\mathfrak{a}}{2}}}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}\left[\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1}{2}-s}\int_{\frac{1}{q}}^{\infty}\omega_{\overline{\chi}}(x)x^{\frac{(1-s)+\mathfrak{a}}{2}}\,\frac{dx}{x} + \int_{\frac{1}{q}}^{\infty}\omega_\chi(x)x^{\frac{s+\mathfrak{a}}{2}}\,\frac{dx}{x}\right].
$$

This integral representation will give analytic continuation. Indeed, we know everything outside the brackets is entire. The two integrals are locally absolutely uniformly convergent on $\mathbb{C}$ by Proposition 1.5.2. This gives analytic continuation to all of $\mathbb{C}$. In particular, $L(s,\chi)$ has no poles.

The Functional Equation & Critical Strip of $L(s,\chi)$

An immediate consequence of applying the symmetry $s \to 1-s$ to the integral representation is the following functional equation:

$$
q^{\frac{s}{2}}\frac{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}{\pi^{\frac{s+\mathfrak{a}}{2}}}L(s,\chi) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1-s}{2}}\frac{\Gamma\left(\frac{(1-s)+\mathfrak{a}}{2}\right)}{\pi^{\frac{(1-s)+\mathfrak{a}}{2}}}L(1-s,\overline{\chi}).
$$

We identify the gamma factor as

$$
\gamma(s,\chi) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right),
$$

with $\kappa = \mathfrak{a}$ the only local root at infinity. Clearly it satisfies the required bounds. The conductor is $q(\chi) = q$ and if p is an unramified prime then the local root is $\chi(p) \neq 0$. The completed L-function is

$$
\Lambda(s,\chi) = q^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi),
$$

with functional equation

$$\Lambda(s,\chi) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}\Lambda(1-s,\overline{\chi}).$$

From it we see that the root number is $\varepsilon(\chi) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}$ and that $L(s,\chi)$ has dual $L(s,\overline{\chi})$. We now show that $L(s,\chi)$ is of order 1. Since $L(s,\chi)$ has no poles, we do not need to clear any polar divisors. As the integrals in the integral representation are locally absolutely uniformly convergent, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s,\chi)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So the reciprocal of the gamma factor is also of order 1. It follows that

$$L(s,\chi) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So $L(s,\chi)$ is of order 1. We summarize all of our work into the following theorem:

Theorem 7.2.2. *For any primitive Dirichlet character χ of conductor $q > 1$, $L(s,\chi)$ is a Selberg class L-function. For $\sigma > 1$, it has a degree 1 Euler product given by*

$$L(s,\chi) = \prod_p (1-\chi(p)p^{-s})^{-1} = \prod_{p \nmid q} (1-\chi(p)p^{-s})^{-1}.$$

Moreover, it admits analytic continuation to $\mathbb{C}$ via the integral representation

$$L(s,\chi) = \frac{\pi^{\frac{s+\mathfrak{a}}{2}}}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}\left[\frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1}{2}-s}\int_{\frac{1}{q}}^\infty \omega_{\overline{\chi}}(x)x^{\frac{(1-s)+\mathfrak{a}}{2}}\,\frac{dx}{x} + \int_{\frac{1}{q}}^\infty \omega_\chi(x)x^{\frac{s+\mathfrak{a}}{2}}\,\frac{dx}{x}\right],$$

and possesses the functional equation

$$q^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi) = \Lambda(s,\chi) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}\Lambda(1-s,\overline{\chi}).$$

Beyond Primitivity of of $L(s,\chi)$

We can still obtain meromorphic continuation of $L(s,\chi)$ if χ is not primitive. Indeed, if χ is induced by $\widetilde{\chi}$, then $\chi(p) = \widetilde{\chi}(p)$ if $p \nmid q$ and $\chi(p) = 0$ if $p \mid m$ so that

$$L(s,\chi) = \prod_{p\nmid m}(1-\widetilde{\chi}(p)p^{-s})^{-1} = \prod_p (1-\widetilde{\chi}(p)p^{-s})^{-1}\prod_{p\mid m}(1-\widetilde{\chi}(p)p^{-s}) = L(s,\widetilde{\chi})\prod_{p\mid m}(1-\widetilde{\chi}(p)p^{-s}). \tag{7.12}$$

From this relation, we can prove the following:

Theorem 7.2.3. *For any Dirichlet character χ modulo m of conductor $q > 1$, $L(s,\chi)$ admits meromorphic continuation to $\mathbb{C}$ and if χ is principal there is a simple pole at $s = 1$ of residue $\prod_{p\mid m}(1-\widetilde{\chi}(p)p^{-1})$ where $\widetilde{\chi}$ is the primitive character inducing χ.*

Proof. This follows from Theorems 7.1.2 and 7.2.2 and Equation (7.12). □

7.3 Hecke L-functions

The Definition & Euler Product of $L(s, f)$

We will investigate the L-functions of holomorphic cusp forms. Let $f \in \mathcal{S}_k(N, \chi)$ and denote its Fourier series by

$$f(z) = \sum_{n \geq 1} a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z},$$

so that the $a_f(n)$ are the Hecke eigenvalues of f. The **Hecke L-function** $L(s, f)$ of f is defined as an L-series:

$$L(s, f) = \sum_{n \geq 1} \frac{a_f(n)}{n^s}.$$

We will see that $L(s, f)$ is a Selberg class L-function if f is a primitive Hecke eigenform. From now on, we make this assumption about f. As we have noted, the Hecke relations and the Ramanujan-Petersson conjecture for holomorphic forms together imply $a_f(n) \ll_\varepsilon n^\varepsilon$. So $L(s, f)$ is locally absolutely uniformly convergent for $\sigma > 1 + \varepsilon$ and hence locally absolutely uniformly convergent for $\sigma > 1$. The L-function will also have an Euler product. Indeed, the Hecke relations imply that the coefficients $a_f(n)$ are multiplicative and satisfy

$$a_f(p^n) = \begin{cases} a_f(p^{n-1}) a_f(p) - \chi(p) a_f(p^{n-2}) & \text{if } p \nmid N, \\ (a_f(p))^n & \text{if } p \mid N, \end{cases} \tag{7.13}$$

for all primes p and $n \geq 2$. Because $L(s, f)$ converges absolutely in the region $\sigma > 1$, multiplicativity of the Hecke eigenvalues implies

$$L(s, f) = \prod_p \left(\sum_{n \geq 0} \frac{a_f(p^n)}{p^{ns}} \right),$$

in this region. We now simplify the factor inside the product using this Equation (7.13). On the one hand, if $p \nmid N$:

$$\begin{aligned} \sum_{n \geq 0} \frac{a_f(p^n)}{p^{ns}} &= 1 + \frac{a_f(p)}{p^s} + \sum_{n \geq 2} \frac{a_f(p^n)}{p^{ns}} \\ &= 1 + \frac{a_f(p)}{p^s} + \sum_{n \geq 2} \frac{a_f(p^{n-1}) a_f(p) - \chi(p) a_f(p^{n-2})}{p^{ns}} \\ &= 1 + \frac{a_f(p)}{p^s} + \frac{a_f(p)}{p^s} \sum_{n \geq 1} \frac{a_f(p^n)}{p^{ns}} - \frac{\chi(p)}{p^{2s}} \sum_{n \geq 0} \frac{a_f(p^n)}{p^{ns}} \\ &= 1 + \left(\frac{a_f(p)}{p^s} - \frac{\chi(p)}{p^{2s}} \right) \sum_{n \geq 0} \frac{a_f(p^n)}{p^{ns}}. \end{aligned}$$

By isolating the sum we find

$$\sum_{n \geq 0} \frac{a_f(p^n)}{p^{ns}} = \left(1 - \frac{a_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}} \right)^{-1}.$$

On the other hand, if $p \mid N$ we have

$$\sum_{n \geq 0} \frac{a_f(p^n)}{p^{ns}} = \sum_{n \geq 0} \frac{(a_f(p))^n}{p^{ns}} = \left(1 - a_f(p) p^{-s} \right)^{-1}.$$

Therefore

$$L(s,f)=\prod_{p\nmid N}(1-a_f(p)p^{-s}+\chi(p)p^{-2s})^{-1}\prod_{p\mid N}(1-a_f(p)p^{-s})^{-1}.$$

If $p\nmid N$, let $\alpha_1(p)$ and $\alpha_2(p)$ be the roots of $1-a_f(p)p^{-s}+\chi(p)p^{-2s}$. That is,

$$(1-\alpha_1(p)p^{-s})(1-\alpha_2(p)p^{-s})=(1-a_f(p)p^{-s}+\chi(p)p^{-2s}).$$

If $p\mid N$, let $\alpha_1(p)=a_f(p)$ and $\alpha_2(p)=0$. We can then express $L(s,f)$ as a degree 2 Euler product:

$$L(s,f)=\prod_p(1-\alpha_1(p)p^{-s})^{-1}(1-\alpha_2(p)p^{-s})^{-1}.$$

The local factor at p is $L_p(s,f)=(1-\alpha_1(p)p^{-s})^{-1}(1-\alpha_2(p)p^{-s})^{-1}$ with local roots $\alpha_1(p)$ and $\alpha_2(p)$. Upon applying partial fraction decomposition to the local factor, we find

$$\frac{1}{1-\alpha_1(p)p^{-s}}\frac{1}{1-\alpha_2(p)p^{-s}}=\frac{\frac{\alpha_1(p)}{\alpha_1(p)-\alpha_2(p)}}{1-\alpha_1(p)p^{-s}}+\frac{\frac{-\alpha_2(p)}{\alpha_1(p)-\alpha_2(p)}}{1-\alpha_2(p)p^{-s}}.$$

Expanding both sides as series in p^{-s}, and comparing coefficients gives

$$a_f(p^n)=\frac{\alpha_1(p)^{n+1}-\alpha_2(p)^{n+1}}{\alpha_1(p)-\alpha_2(p)}. \tag{7.14}$$

The Integral Representation of $L(s,f)$

We now want to find an integral representation for $L(s,f)$. Consider the following Mellin transform:

$$\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}.$$

We don't yet know that this integral defines an analytic function for $\sigma>1$. In any case, we compute

$$\begin{aligned}
\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}&=\int_0^\infty\sum_{n\geq1}a_f(n)n^{\frac{k-1}{2}}e^{-2\pi ny}y^{s+\frac{k-1}{2}}\,\frac{dy}{y}\\
&=\sum_{n\geq1}a_f(n)n^{\frac{k-1}{2}}\int_0^\infty e^{-2\pi ny}y^{s+\frac{k-1}{2}}\,\frac{dy}{y} && \text{DCT}\\
&=\sum_{n\geq1}\frac{a_f(n)}{(2\pi)^{s+\frac{k-1}{2}}n^s}\int_0^\infty e^{-y}y^{s+\frac{k-1}{2}}\,\frac{dy}{y} && y\to\frac{y}{2\pi n}\\
&=\frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}}\sum_{n\geq1}\frac{a_f(n)}{n^s}\\
&=\frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}}L(s,f).
\end{aligned}$$

As this last expression is analytic function for $\sigma>1$, the integral is too. Rewriting, we have an integral representation

$$L(s,f)=\frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma\left(s+\frac{k-1}{2}\right)}\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}. \tag{7.15}$$

Now split the integral on the right-hand side into two pieces

$$\int_0^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y} = \int_0^{\frac{1}{\sqrt{N}}} f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y} + \int_{\frac{1}{\sqrt{N}}}^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}. \tag{7.16}$$

Since $f(iy)$ exhibits rapid decay, the second piece is locally absolutely uniformly convergent for $\sigma > 1$ by Proposition 1.5.2. Hence it is an analytic there. Now we will rewrite the first piece in the same form and symmetrize the result as much as possible. Begin by performing the change of variables $y \to \frac{1}{Ny}$ to the first piece to obtain

$$\int_{\frac{1}{\sqrt{N}}}^\infty f\left(\frac{i}{Ny}\right)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y}.$$

Rewriting in terms of the Atkin–Lehner involution and recalling that $\omega_N f = \omega_N(f)\overline{f}$ by Proposition 3.6.4, we have

$$\begin{aligned}
\int_{\frac{1}{\sqrt{N}}}^\infty f\left(\frac{i}{Ny}\right)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y} &= \int_{\frac{1}{\sqrt{N}}}^\infty f\left(-\frac{1}{iNy}\right)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y} \\
&= \int_{\frac{1}{\sqrt{N}}}^\infty \left(\sqrt{N}iy\right)^k (\omega_N f)(iy)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y} \\
&= \int_{\frac{1}{\sqrt{N}}}^\infty \left(\sqrt{N}iy\right)^k \omega_N(f)\overline{f}(iy)(Ny)^{-s-\frac{k-1}{2}}\,\frac{dy}{y} \\
&= \omega_N(f)i^kN^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^\infty \overline{f}(iy)y^{(1-s)-\frac{k-1}{2}}\,\frac{dy}{y}.
\end{aligned}$$

Substituting this result back into Equation (7.16) with Equation (7.15) yields the integral representation

$$L(s,f) = \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma\left(s+\frac{k-1}{2}\right)}\left[\omega_N(f)i^kN^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^\infty \overline{f}(iy)y^{(1-s)+\frac{k-1}{2}}\,\frac{dy}{y} + \int_{\frac{1}{\sqrt{N}}}^\infty f(iy)y^{s+\frac{k-1}{2}}\,\frac{dy}{y}\right].$$

This integral representation will give analytic continuation. To see this, we know everything outside the brackets is entire. The two integrals are locally absolutely uniformly convergent on $\mathbb{C}$ by Proposition 1.5.2. Hence we have analytic continuation to all of $\mathbb{C}$. In particular, we have shown that $L(s,f)$ has no poles.

The Functional Equation & Critical Strip of $L(s,f)$

An immediate consequence of applying the symmetry $s \to 1-s$ to the integral representation is the following functional equation:

$$\frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}}L(s,f) = \omega_N(f)i^kN^{-\frac{s}{2}}\frac{\Gamma\left((1-s)+\frac{k-1}{2}\right)}{(2\pi)^{(1-s)+\frac{k-1}{2}}}L(1-s,\overline{f}).$$

Using the Legendre duplication formula for the gamma function we find that

$$\begin{aligned}
\frac{\Gamma\left(s+\frac{k-1}{2}\right)}{(2\pi)^{s+\frac{k-1}{2}}} &= \frac{1}{(2\pi)^{s+\frac{k-1}{2}}2^{1-\left(s+\frac{k-1}{2}\right)}\sqrt{\pi}}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right) \\
&= \frac{1}{2\pi^{s+\frac{1}{2}}}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right) \\
&= \frac{1}{\sqrt{4\pi}}\pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right).
\end{aligned}$$

The constant factor in front is independent of s and so can be canceled in the functional equation. Therefore we identify the gamma factor as

$$\gamma(s,f) = \pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right),$$

with $\kappa_1 = k-1$ and $\kappa_2 = k+1$ the local roots at infinity. The conductor is $q(f) = N$, so the primes dividing the level ramify, and by the Ramanujan-Petersson conjecture for holomorphic forms, $\alpha_1(p) \neq 0$ and $\alpha_2(p) \neq 0$ for all primes $p \nmid N$. The completed L-function is

$$\Lambda(s,f) = N^{-\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)L(s,f),$$

with functional equation

$$\Lambda(s,f) = \omega_N(f)i^k\Lambda(1-s,\overline{f}).$$

This is the functional equation of $L(s,f)$. From it, the root number is $\varepsilon(f) = \omega_N(f)i^k$ and we see that $L(s,f)$ has dual $L(s,\overline{f})$. We will now show that $L(s,f)$ is of order 1. Since $L(s,f)$ has no poles, we do not need to clear any polar divisors. As the integrals in the representation is locally absolutely uniformly convergent, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s,f)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So the reciprocal of the gamma factor is also of order 1. Then

$$L(s,f) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So $L(s,f)$ is of order 1. We summarize all of our work into the following theorem:

Theorem 7.3.1. *For any primitive Hecke eigenform $f \in \mathcal{S}_k(N,\chi)$, $L(s,f)$ is a Selberg class L-function. For $\sigma > 1$, it has a degree 2 Euler product given by*

$$L(s,f) = \prod_{p\nmid N}(1-a_f(p)p^{-s}+\chi(p)p^{-2s})^{-1}\prod_{p\mid N}(1-a_f(p)p^{-s})^{-1}.$$

Moreover, it admits analytic continuation to $\mathbb{C}$ via the integral representation

$$L(s,f) = \frac{(2\pi)^{s+\frac{k-1}{2}}}{\Gamma\left(s+\frac{k-1}{2}\right)}\left[\omega_N(f)i^kN^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^{\infty}\overline{f}(iy)y^{(1-s)+\frac{k-1}{2}}\frac{dy}{y} + \int_{\frac{1}{\sqrt{N}}}^{\infty}f(iy)y^{s+\frac{k-1}{2}}\frac{dy}{y}\right],$$

and possesses the functional equation

$$N^{-\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)L(s,f) = \Lambda(s,f) = \omega_N(f)i^k\Lambda(1-s,\overline{f}).$$

Beyond Primitivity of of $L(s,f)$

We can still obtain analytic continuation of $L(s,f)$ if f is not a primitive Hecke eigenform. Indeed, since the primitive Hecke eigenforms form a basis for the space of newforms, we can prove the following:

Theorem 7.3.2. *For any $f \in \mathcal{S}_k(\Gamma_1(N))$, $L(s,f)$ admits analytic continuation to $\mathbb{C}$.*

Proof. If f is a newform, this follows from Theorems 3.6.1 and 7.3.1. Now suppose f is an oldform. Then there is a divisor $d \mid N$ with $d > 1$ such that

$$f(z) = g(z) + d^{k-1}h(dz) = g(z) + \prod_{p^r || d}(V_p^r h)(z),$$

for some $g, h \in \mathcal{S}_k\left(\Gamma_1\left(\frac{N}{d}\right)\right)$. Note that $V_p h \in \mathcal{S}_k\left(\Gamma_1\left(\frac{Np}{d}\right)\right)$ by Lemma 3.8.1. The claim now follows by induction on the level of f and that V_p clearly preserves primitive Hecke eigenforms. $\square$

7.4 Hecke-Maass L-functions

The Definition & Euler Product of $L(s,f)$

We will investigate the L-functions of weight zero Maass cusp forms. Let $f \in \mathcal{C}_\nu(N,\chi)$ be even or odd and denote its Fourier series by

$$f(z) = \sum_{n \geq 1} a_f(n)\sqrt{y}K_\nu(2\pi n y)\,\mathrm{SC}(2\pi n x),$$

so that the $a_f(n)$ are the Hecke eigenvalues of f. The **Hecke-Maass L-function** $L(s,f)$ of f is defined as an L-series:

$$L(s,f) = \sum_{n \geq 1} \frac{a_f(n)}{n^s}.$$

We will see that $L(s,f)$ is a Selberg class L-function if f is a primitive Hecke-Maass eigenform. From now on, we make this assumption about f. The Ramanujan-Petersson conjecture for Maass forms is not know so $L(s,f)$ has not been proven to be a Selberg class L-function. Although, it is conjectured to be, so throughout we will make this additional assumption. As we have noted, the Hecke relations and the Ramanujan-Petersson conjecture for Maass forms together imply $a_f(n) \ll_\varepsilon n^\varepsilon$. So $L(s,f)$ is locally absolutely uniformly convergent for $\sigma > 1+\varepsilon$ and hence locally absolutely uniformly convergent for $\sigma > 1$. The L-function will have an Euler product. Indeed, the Hecke relations imply that the coefficients $a_f(n)$ are multiplicative and satisfy Equation (7.13). Mimicking the argument exactly as for Hecke L-functions,

$$L(s,f) = \prod_{p \nmid N}(1 - a_f(p)p^{-s} + \chi(p)p^{-2s})^{-1}\prod_{p \mid N}(1 - a_f(p)p^{-s})^{-1}.$$

If $p \nmid N$, let $\alpha_1(p)$ and $\alpha_2(p)$ be the roots of $1 - a_f(p)p^{-s} + \chi(p)p^{-2s}$. That is,

$$(1 - \alpha_1(p)p^{-s})(1 - \alpha_2(p)p^{-s}) = (1 - a_f(p)p^{-s} + \chi(p)p^{-2s}).$$

If $p \mid N$, let $\alpha_1(p) = a_f(p)$ and $\alpha_2(p) = 0$. We can then express $L(s,f)$ as a degree 2 Euler product:

$$L(s,f) = \prod_p (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1}.$$

The local factor at p is $L_p(s,f) = (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1}$ with local roots $\alpha_1(p)$ and $\alpha_2(p)$. Moreover, just as in the case for Hecke L-functions, the formula Equation (7.14) also holds.

The Integral Representation of $L(s,f)$

We want to find an integral representation for $L(s,f)$. Recall that f is an eigenfunction for T_{-1} with eigenvalue ± 1. Equivalently, f is even if the eigenvalue is 1 and odd if the eigenvalue is -1. The integral representation will depend upon this parity. To handle both cases simultaneously, let $\mathfrak{a} = 0, 1$ according to whether f is even or odd. In other words,

$$\mathfrak{a} = \frac{1 - a_f(-1)}{2}.$$

Now consider the following Mellin transform:

$$\int_0^\infty \left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)(iy) y^{s-\frac{1}{2}+\mathfrak{a}} \frac{dy}{y}.$$

The derivative operator is present because if f is odd, $\mathrm{SC}(x) = \sin(x)$. In any case, the smoothness of f implies that we may differentiate its Fourier series termwise to obtain

$$\left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)(z) = \sum_{n \ge 1} a_f(n)(2\pi n)^{\mathfrak{a}} \sqrt{y} K_\nu(2\pi n y)\cos(2\pi n x).$$

Therefore regardless if f is even or odd, the Fourier series of $\left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)(z)$ has $\mathrm{SC}(x) = \cos(x)$ and the integral does not vanish identically. Similar to Hecke L-functions, we don't yet know that this integral defines an analytic function for $\sigma > 1$. Nevertheless, we compute

$$\begin{aligned}
\int_0^\infty \left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)(iy) y^{s-\frac{1}{2}+\mathfrak{a}} \frac{dy}{y} &= \int_0^\infty \sum_{n \ge 1} a_f(n)(2\pi n)^{\mathfrak{a}} K_\nu(2\pi n y) y^{s+\mathfrak{a}} \frac{dy}{y} \\
&= \sum_{n \ge 1} a_f(n)(2\pi n)^{\mathfrak{a}} \int_0^\infty K_\nu(2\pi n y) y^{s+\mathfrak{a}} \frac{dy}{y} && \text{DCT} \\
&= \sum_{n \ge 1} \frac{a_f(n)}{(2\pi)^s n^s} \int_0^\infty K_\nu(y) y^{s+\mathfrak{a}} \frac{dy}{y} && y \to \frac{y}{2\pi n} \\
&= \frac{\Gamma\left(\frac{s+\mathfrak{a}+\nu}{2}\right)\Gamma\left(\frac{s+\mathfrak{a}-\nu}{2}\right)}{2^{2-\mathfrak{a}}\pi^s} \sum_{n \ge 1} \frac{a_f(n)}{n^s} && \text{Appendix D.1} \\
&= \frac{\Gamma\left(\frac{s+\mathfrak{a}+\nu}{2}\right)\Gamma\left(\frac{s+\mathfrak{a}-\nu}{2}\right)}{2^{2-\mathfrak{a}}\pi^s} L(s,f).
\end{aligned}$$

This last expression is analytic function for $\sigma > 1$ and so the integral is too. Rewriting, we have an integral representation

$$L(s,f) = \frac{2^{2-\mathfrak{a}}\pi^s}{\Gamma\left(\frac{s+\mathfrak{a}+\nu}{2}\right)\Gamma\left(\frac{s+\mathfrak{a}-\nu}{2}\right)} \int_0^\infty \left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)(iy) y^{s-\frac{1}{2}+\mathfrak{a}} \frac{dy}{y}. \tag{7.17}$$

Now split the integral on the right-hand side into two pieces

$$\int_0^\infty \left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)(iy) y^{s-\frac{1}{2}+\mathfrak{a}} \frac{dy}{y} = \int_0^{\frac{1}{\sqrt{N}}} \left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)(iy) y^{s-\frac{1}{2}+\mathfrak{a}} \frac{dy}{y} + \int_{\frac{1}{\sqrt{N}}}^\infty \left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)(iy) y^{s-\frac{1}{2}+\mathfrak{a}} \frac{dy}{y}. \tag{7.18}$$

Since $\left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)(iy)$ has moderate decay as $y \to \infty$ because $f(iy)$ does, the second piece is locally absolutely uniformly convergent for $\sigma > 1$ by Proposition 1.5.2. Hence it is analytic there. Now we will rewrite the first piece in the same form and symmetrize the result as much as possible. Performing the change of variables $y \to \frac{1}{Ny}$ to the first piece to obtain

$$\int_{\frac{1}{\sqrt{N}}}^{\infty} \left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)\left(\frac{i}{Ny}\right)(Ny)^{-s+\frac{1}{2}-\mathfrak{a}}\frac{dy}{y}.$$

We will rewrite this in terms of the Atkin–Lehner involution. But first we require an identity that relates $\frac{\partial}{\partial x}^{\mathfrak{a}}$ with the Atkin-Lehner involution ω_N. By the identity theorem it suffices verify this for $z \in \mathbb{H}$ with $|z|$ fixed. Observe that $-\frac{1}{Nz} = \frac{-x}{N|z|^2} + \frac{iy}{N|z|^2}$. Now differentiate termwise to see that

$$\begin{aligned}
\left(\frac{\partial}{\partial x}^{\mathfrak{a}} \omega_N f\right)(z) &= \left(\frac{\partial}{\partial x}^{\mathfrak{a}}\right) f\left(-\frac{1}{Nz}\right) \\
&= \left(\frac{\partial}{\partial x}^{\mathfrak{a}}\right)\sum_{n\geq 1} a_f(n)\sqrt{\frac{y}{N|z|^2}}K_\nu(2\pi ny)\,\mathrm{SC}\left(-2\pi n\frac{x}{N|z|^2}\right) \\
&= (-N|z|^2)^{-\mathfrak{a}}\sum_{n\geq 1} a_f(n)(2\pi n)^{\mathfrak{a}}\sqrt{\frac{y}{N|z|^2}}K_\nu\left(2\pi n\frac{y}{N|z|^2}\right)\cos\left(-2\pi n\frac{x}{N|z|^2}\right) \\
&= (-N|z|^2)^{-\mathfrak{a}}\left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)\left(-\frac{1}{Nz}\right).
\end{aligned}$$

By the identity theorem, we have

$$\left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)\left(-\frac{1}{Nz}\right) = (-N|z|^2)^{\mathfrak{a}}\left(\frac{\partial}{\partial x}^{\mathfrak{a}} \omega_N f\right)(z),$$

for all $z \in \mathbb{H}$. Rewriting in terms of the Atkin–Lehner involution and recalling that $\omega_N f = \omega_N(f)\overline{f}$ by Proposition 4.7.4, we find that

$$\begin{aligned}
\int_{\frac{1}{\sqrt{N}}}^{\infty} \left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)\left(\frac{i}{Ny}\right)(Ny)^{-s+\frac{1}{2}-\mathfrak{a}}\frac{dy}{y} &= \int_{\frac{1}{\sqrt{N}}}^{\infty} \left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)\left(-\frac{1}{iNy}\right)(Ny)^{-s+\frac{1}{2}-\mathfrak{a}}\frac{dy}{y} \\
&= \int_{\frac{1}{\sqrt{N}}}^{\infty} (-Ny^2)^{\mathfrak{a}}\left(\left(\frac{\partial}{\partial x}^{\mathfrak{a}}\right)\omega_N f\right)(iy)(Ny)^{-s+\frac{1}{2}-\mathfrak{a}}\frac{dy}{y} \\
&= \int_{\frac{1}{\sqrt{N}}}^{\infty} (-Ny^2)^{\mathfrak{a}}\omega_N(f)\left(\left(\frac{\partial}{\partial x}^{\mathfrak{a}}\right)\overline{f}\right)(iy)(Ny)^{-s+\frac{1}{2}-\mathfrak{a}}\frac{dy}{y} \\
&= w_N(f)(-1)^{\mathfrak{a}}N^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^{\infty} \left(\left(\frac{\partial}{\partial x}^{\mathfrak{a}}\right)\overline{f}\right)(iy)y^{(1-s)-\frac{1}{2}+\mathfrak{a}}\frac{dy}{y}.
\end{aligned}$$

Substituting this result back into Equation (7.18) with Equation (7.17) gives the integral representation

$$\begin{aligned}
L(s,f) &= \frac{2^{2-\mathfrak{a}}\pi^s}{\Gamma\left(\frac{s+\mathfrak{a}+\nu}{2}\right)\Gamma\left(\frac{s+\mathfrak{a}-\nu}{2}\right)} \\
&\quad\cdot\left[w_N(f)(-1)^{\mathfrak{a}}N^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^{\infty}\left(\left(\frac{\partial}{\partial x}^{\mathfrak{a}}\right)\overline{f}\right)(iy)y^{(1-s)-\frac{1}{2}+\mathfrak{a}}\frac{dy}{y} + \int_{\frac{1}{\sqrt{N}}}^{\infty}\left(\frac{\partial}{\partial x}^{\mathfrak{a}} f\right)(iy)y^{s-\frac{1}{2}+\mathfrak{a}}\frac{dy}{y}\right].
\end{aligned}$$

This integral representation will give analytic continuation. Indeed, everything outside the brackets is entire and the two integrals are locally absolutely uniformly convergent on $\mathbb{C}$ by Proposition 1.5.2. Hence we have analytic continuation to all of $\mathbb{C}$. In particular, $L(s,f)$ has no poles.

The Functional Equation & Critical Strip of $L(s,f)$

An immediate consequence of applying the symmetry $s \to 1-s$ to the integral representation is the following functional equation:

$$\frac{\Gamma\left(\frac{s+\mathfrak{a}+\nu}{2}\right)\Gamma\left(\frac{s+\mathfrak{a}-\nu}{2}\right)}{2^{2-\mathfrak{a}}\pi^s}L(s,f) = \omega_N(f)(-1)^{\mathfrak{a}}N^{-\frac{s}{2}}\frac{\Gamma\left(\frac{(1-s)+\mathfrak{a}+\nu}{2}\right)\Gamma\left(\frac{(1-s)+\mathfrak{a}-\nu}{2}\right)}{2^{2-\mathfrak{a}}\pi^{1-s}}L(1-s,\overline{f}).$$

The constant factor in the denominator is independent of s and so can be canceled in the functional equation. Therefore we identify the gamma factor as

$$\gamma(s,f) = \pi^{-s}\Gamma\left(\frac{s+\mathfrak{a}+\nu}{2}\right)\Gamma\left(\frac{s+\mathfrak{a}-\nu}{2}\right),$$

with $\kappa_1 = \mathfrak{a}+\nu$ and $\kappa_2 = \mathfrak{a}-\nu$ the local roots at infinity (these are complex conjugates because ν is either purely imaginary of real). The conductor is $q(f) = N$, so the primes dividing the level ramify, and by the Ramanujan-Petersson conjecture for Maass forms, $\alpha_1(p) \neq 0$ and $\alpha_2(p) \neq 0$ for all primes $p \nmid N$. The completed L-function is

$$\Lambda(s,f) = N^{-\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s+\mathfrak{a}+\nu}{2}\right)\Gamma\left(\frac{s+\mathfrak{a}-\nu}{2}\right)L(s,f),$$

with functional equation

$$\Lambda(s,f) = \omega_N(f)(-1)^{\mathfrak{a}}\Lambda(1-s,\overline{f}).$$

This is the functional equation of $L(s,f)$. From it, the root number is $\varepsilon(f) = \omega_N(f)(-1)^{\mathfrak{a}}$ and we see that $L(s,f)$ has dual $L(s,\overline{f})$. We will now show that $L(s,f)$ is of order 1. Since $L(s,f)$ has no poles, we do not need to clear any polar divisors. As the integrals in the representation is locally absolutely uniformly convergent, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s,f)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So the reciprocal of the gamma factor is also of order 1. Then

$$L(s,f) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So $L(s,f)$ is of order 1. We summarize all of our work into the following theorem:

Theorem 7.4.1. *For any even or odd primitive Hecke-Maass eigenform $f \in \mathcal{C}_\nu(N,\chi)$, $L(s,f)$ is a Selberg class L-function provided the Ramanujan-Petersson conjecture for Maass forms holds. For $\sigma > 1$, it has a degree 2 Euler product given by*

$$L(s,f) = \prod_{p\nmid N}(1-a_f(p)p^{-s}+\chi(p)p^{-2s})^{-1}\prod_{p\mid N}(1-a_f(p)p^{-s})^{-1}.$$

Moreover, it admits analytic continuation to $\mathbb{C}$ via the integral representation

$$\begin{aligned}L(s,f) = &\frac{2^{2-\mathfrak{a}}\pi^s}{\Gamma\left(\frac{s+\mathfrak{a}+\nu}{2}\right)\Gamma\left(\frac{s+\mathfrak{a}-\nu}{2}\right)} \\ &\cdot\left[w_N(f)(-1)^{\mathfrak{a}}N^{\frac{1}{2}-s}\int_{\frac{1}{\sqrt{N}}}^\infty\left(\left(\frac{\partial}{\partial x}^{\mathfrak{a}}\right)\overline{f}\right)(iy)y^{(1-s)-\frac{1}{2}+\mathfrak{a}}\frac{dy}{y}+\int_{\frac{1}{\sqrt{N}}}^\infty\left(\frac{\partial}{\partial x}^{\mathfrak{a}}f\right)(iy)y^{s-\frac{1}{2}+\mathfrak{a}}\frac{dy}{y}\right].\end{aligned}$$

and possesses the functional equation

$$N^{-\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s+\mathfrak{a}+\nu}{2}\right)\Gamma\left(\frac{s+\mathfrak{a}-\nu}{2}\right)L(s,f) = \Lambda(s,f) = \omega_N(f)(-1)^{\mathfrak{a}}\Lambda(1-s,\overline{f}).$$

Beyond Primitivity of of $L(s,f)$

We can still obtain analytic continuation of $L(s,f)$ if f is not a primitive Hecke-Maass eigenform. Similarly to the Hecke L-function case, this holds because the primitive Hecke-Maass eigenforms form a basis for the space of newforms:

Theorem 7.4.2. *For any even or odd $f \in \mathcal{C}_\nu(\Gamma_1(N))$, $L(s,f)$ admits analytic continuation to $\mathbb{C}$.*

Proof. Mimic the proof of Theorem 7.3.2. □

7.5 The Rankin-Selberg Method

The Definition & Euler Product of $L(s, f \otimes g)$

The Rankin-Selberg method is a process by which we can construct new L-functions from old ones. Instead of giving the general definition outright, we first provide a full discussion of the method only in the simplest case. Many technical difficulties arise in the fully general setting. Let $f, g \in \mathcal{S}_k(1)$ be primitive Hecke eigenforms with Fourier series

$$f(z) = \sum_{n \geq 1} a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z} \quad \text{and} \quad g(z) = \sum_{n \geq 1} a_g(n) n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

The L-function $L(s, f \times g)$ of f and g is given by the L-series

$$L(s, f \times g) = \sum_{n \geq 1} \frac{a_{f \times g}(n)}{n^s} = \sum_{n \geq 1} \frac{a_f(n)\overline{a_g(n)}}{n^s} = \sum_{n \geq 1} \frac{a_f(n)\overline{a_g(n)}}{n^s},$$

The **Rankin-Selberg convolution** $L(s, f \otimes g)$ of f and g is defined as an L-series:

$$L(s, f \otimes g) = \sum_{n \geq 1} \frac{a_{f \otimes g}(n)}{n^s} = \zeta(2s) L(s, f \times g),$$

where $a_{f \otimes g}(n) = \sum_{n = m\ell^2} a_f(m)\overline{a_g(m)}$. Since $a_f(n) \ll_\varepsilon n^\varepsilon$ and $a_g(n) \ll_\varepsilon n^\varepsilon$, $a_{f \times g}(n) \ll_\varepsilon n^\varepsilon$ as well. Hence $L(s, f \times g)$ is locally absolutely uniformly convergent for $\sigma > 1 + \varepsilon$ and hence locally absolutely uniformly convergent for $\sigma > 1$. Since $\zeta(2s)$ is also locally absolutely uniformly convergent in this region, the same follows for $L(s, f \times g)$ too. The L-function $L(s, f \times g)$ will also have an Euler product. To see this, let $\alpha_j(p)$ and $\beta_\ell(p)$ be the local roots at p of $L(s,f)$ and $L(s,g)$ respectively. Since $L(s, f \otimes g)$ converges absolutely in the region $\sigma > 1$, multiplicativity of the Hecke eigenvalues implies

$$L(s, f \otimes g) = \zeta(2s) L(s, f \times g) = \prod_{p \nmid NM} (1 - p^{-2s})^{-1} \prod_p \left(\sum_{n \geq 0} \frac{a_f(p^n)\overline{a_g(p^n)}}{p^{ns}} \right),$$

in this region. We now simplify the factor inside the latter product using Equation (7.14):

$$\begin{aligned} \sum_{n \geq 0} \frac{a_f(p^n)\overline{a_g(p^n)}}{p^{ns}} &= \sum_{n \geq 0} \left(\frac{\alpha_1(p)^{n+1} - \alpha_2(p)^{n+1}}{\alpha_1(p) - \alpha_2(p)} \right) \left(\frac{(\overline{\beta_1(p)})^{n+1} - (\overline{\beta_2(p)})^{n+1}}{\overline{\beta_1(p)} - \overline{\beta_2(p)}} \right) p^{-ns} \\ &= (\alpha_1(p) - \alpha_2(p))^{-1} \left(\overline{\beta_1(p)} - \overline{\beta_2(p)} \right)^{-1} \end{aligned}$$

$$\cdot\left[\sum_{n\geq 1}\frac{\alpha_1(p)^n(\overline{\beta_1(p)})^n}{p^{(n-1)s}}+\frac{\alpha_2(p)^n(\overline{\beta_2(p)})^n}{p^{(n-1)s}}-\frac{\alpha_1(p)^n(\overline{\beta_2(p)})^n}{p^{(n-1)s}}-\frac{\alpha_2(p)^n(\overline{\beta_1(p)})^n}{p^{(n-1)s}}\right]$$

$$= (\alpha_1(p)-\alpha_2(p))^{-1}\left(\overline{\beta_1(p)}-\overline{\beta_2(p)}\right)^{-1}\left[\alpha_1(p)\overline{\beta_1(p)}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)^{-1}\right.$$
$$+\alpha_2(p)\overline{\beta_2(p)}\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)^{-1}-\alpha_1(p)\overline{\beta_2(p)}\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)^{-1}$$
$$\left.-\alpha_2(p)\overline{\beta_1(p)}\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)^{-1}\right]$$
$$= (\alpha_1(p)-\alpha_2(p))^{-1}\left(\overline{\beta_1(p)}-\overline{\beta_2(p)}\right)^{-1}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)^{-1}$$
$$\cdot\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)^{-1}\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)^{-1}\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)^{-1}$$
$$\cdot\left[\alpha_1(p)\overline{\beta_1(p)}\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)\right.$$
$$+\alpha_2(p)\overline{\beta_2(p)}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)$$
$$-\alpha_1(p)\overline{\beta_2(p)}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_1(p)}p^{-s}\right)$$
$$\left.-\alpha_2(p)\overline{\beta_1(p)}\left(1-\alpha_1(p)\overline{\beta_1(p)}p^{-s}\right)\left(1-\alpha_2(p)\overline{\beta_2(p)}p^{-s}\right)\left(1-\alpha_1(p)\overline{\beta_2(p)}p^{-s}\right)\right].$$

The term in the brackets simplifies to

$$\left(1-\alpha_1(p)\alpha_2(p)\overline{\beta_1(p)\beta_2(p)}p^{-2s}\right)(\alpha_1(p)-\alpha_2(p))\left(\overline{\beta_1(p)}-\overline{\beta_2(p)}\right),$$

because all of the other terms are killed by symmetry in $\alpha_1(p), \alpha_2(p), \overline{\beta_1(p)}$, and $\overline{\beta_2(p)}$. The Ramanujan-Petersson conjecture for holomorphic forms implies $\alpha_1(p)\alpha_2(p)\overline{\beta_1(p)\beta_2(p)} = 1$. Therefore the corresponding factor above is $(1-p^{-2s})$. This factor cancels the local factor at p in the Euler product of $\zeta(2s)$, so that

$$\sum_{n\geq 0}\frac{a_f(p^n)\overline{a_g(p^n)}}{p^{ns}} = \prod_{1\leq j,\ell\leq 2}\left(1-\alpha_j(p)\overline{\beta_\ell(p)}p^{-s}\right)^{-1}.$$

Hence

$$L(s, f\otimes g) = \prod_p\left(\sum_{n\geq 0}\frac{a_f(p^n)\overline{a_g(p^n)}}{p^{ns}}\right).$$

In total we have a degree 4 Euler product:

$$L(s, f\otimes g) = \prod_p\prod_{1\leq j,\ell\leq 2}\left(1-\alpha_j(p)\overline{\beta_\ell(p)}p^{-s}\right)^{-1}.$$

The local factor at p is $L_p(s, f\otimes g) = \prod_{1\leq j,\ell\leq 2}\left(1-\alpha_j(p)\overline{\beta_\ell(p)}p^{-s}\right)^{-1}$.

The Integral Representation of $L(s, f\otimes g)$: Part I

We now look for an integral representation for $L(s, f\otimes g)$. Consider the following integral:

$$\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\,d\mu.$$

This will turn out to be a Mellin transform as we will soon see. The existence of this integral is not immediately clear because we cannot appeal to Proposition 1.5.1 or Proposition 1.5.2 directly. In any case, we have

$$
\begin{aligned}
\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\,d\mu &= \int_0^\infty\int_0^1 f(x+iy)\overline{g(x+iy)}y^{s+k}\,\frac{dx\,dy}{y^2} \\
&= \int_0^\infty\int_0^1 \sum_{n,m\geq 1} a_f(n)\overline{a_g(m)}(nm)^{\frac{k-1}{2}}e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^{s+k}\,\frac{dx\,dy}{y^2} \\
&= \int_0^\infty\sum_{n,m\geq 1}\int_0^1 a_f(n)\overline{a_g(m)}(nm)^{\frac{k-1}{2}}e^{2\pi i(n-m)x}e^{-2\pi(n+m)y}y^{s+k}\,\frac{dx\,dy}{y^2} \qquad \text{DCT} \\
&= \int_0^\infty\sum_{n\geq 1} a_f(n)\overline{a_g(n)}n^{k-1}e^{-4\pi ny}y^{s+k}\,\frac{dy}{y^2},
\end{aligned}
$$

where the last line follows by Equation (3.1). The rest is a computation:

$$
\begin{aligned}
\int_0^\infty\sum_{n\geq 1} a_f(n)\overline{a_g(n)}n^{k-1}e^{-4\pi ny}y^{s+k}\,\frac{dy}{y^2} &= \sum_{n\geq 1} a_f(n)\overline{a_g(n)}n^{k-1}\int_0^\infty e^{-4\pi ny}y^{s+k}\,\frac{dy}{y^2} && \text{DCT} \\
&= \sum_{n\geq 1}\frac{a_f(n)\overline{a_g(n)}}{(4\pi)^{s+k-1}n^s}\int_0^\infty e^{-y}y^{s+k-1}\,\frac{dy}{y} && y\to\frac{y}{4\pi n} \\
&= \frac{\Gamma\left(s+k-1\right)}{(4\pi)^{s+k-1}}\sum_{n\geq 1}\frac{a_f(n)\overline{a_g(n)}}{n^s} \\
&= \frac{\Gamma\left(s+k-1\right)}{(4\pi)^{s+k-1}}L(s,f\times g).
\end{aligned}
$$

This last expression is locally absolutely uniformly convergent for $\sigma > 1$ because the L-function is and the gamma factor is holomorphic in this region. Therefore the original integral is locally absolutely uniformly convergent in this region as well. At this point we have an integral representation

$$
L(s,f\times g) = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)}\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\,d\mu.
$$

We rewrite the integral as follows:

$$
\begin{aligned}
\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+k}\,d\mu &= \int_{\mathcal{F}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma z)\overline{g(\gamma z)}\mathrm{Im}(\gamma z)^{s+k}\,d\mu && \text{folding} \\
&= \int_{\mathcal{F}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^k\overline{j(\gamma,z)^k}f(z)\overline{g(z)}\mathrm{Im}(\gamma z)^{s+k}\,d\mu && \text{modularity} \\
&= \int_{\mathcal{F}} f(z)\overline{g(z)}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}|j(\gamma,z)|^{2k}\mathrm{Im}(\gamma z)^{s+k}\,d\mu \\
&= \int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\gamma z)^s\,d\mu \\
&= \int_{\mathcal{F}} f(z)\overline{g(z)}\mathrm{Im}(z)^k E(z,s)\,d\mu.
\end{aligned}
$$

Note that $E(z,s)$ is the weight zero Eisenstein series on $\Gamma_0(1)\backslash\mathbb{H}$ at the ∞ cusp. Altogether, this gives the integral representation

$$L(s,f\times g)=\frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)}\int_{\mathcal{F}}f(z)\overline{g(z)}\mathrm{Im}(z)^kE(z,s)\,d\mu, \tag{7.19}$$

which is valid for $\sigma>1$. We cannot investigate the integral any further until we understand the Fourier series of $E(z,s)$ and have a functional equation as $s\to 1-s$. Therefore we will take a necessary detour and return to the integral after.

The Fourier Series and Functional Equation of $E(z,s)$

We will compute the Fourier series of $E(z,s)$. To do this we will need the following technical lemma:

Lemma 7.5.1. *For $\sigma>1$ and $b\in\mathbb{Z}$,*

$$\sum_{m\geq 1}\frac{r(b,m)}{m^{2s}}=\begin{cases}\frac{\zeta(2s-1)}{\zeta(2s)} & \text{if } b=0,\\ \frac{\sigma_{1-2s}(|b|)}{\zeta(2s)} & \text{if } b\neq 0,\end{cases}$$

where $\sigma_s(b)$ is the generalized sum of divisors function.

Proof. If $\sigma>1$ then the desired evaluation of the sum is locally absolutely uniformly convergent because the Riemann zeta function is in that region. Hence the sum will be too provided we prove the identity. Suppose $b=0$. Then $r(0,m)=\varphi(m)$. Since $\varphi(m)$ is multiplicative we have

$$\sum_{m\geq 1}\frac{\varphi(m)}{m^{2s}}=\prod_p\left(\sum_{k\geq 0}\frac{\varphi(p^k)}{p^{k(2s)}}\right). \tag{7.20}$$

Recalling that $\varphi(p^k)=p^k-p^{k-1}$ for $k\geq 1$, make the following computation:

$$\begin{aligned}\sum_{k\geq 0}\frac{\varphi(p^k)}{p^{k(2s)}}&=1+\sum_{k\geq 1}\frac{p^k-p^{k-1}}{p^{k(2s)}}\\ &=\sum_{k\geq 0}\frac{1}{p^{k(2s-1)}}-\frac{1}{p}\sum_{k\geq 1}\frac{1}{p^{k(2s-1)}}\\ &=\sum_{k\geq 0}\frac{1}{p^{k(2s-1)}}-p^{-2s}\sum_{k\geq 0}\frac{1}{p^{k(2s-1)}}\\ &=(1-p^{-2s})\sum_{k\geq 0}\frac{1}{p^{k(2s-1)}}\\ &=\frac{1-p^{-2s}}{1-p^{-(2s-1)}}.\end{aligned} \tag{7.21}$$

Combining Equations (7.20) and (7.21) gives

$$\sum_{m\geq 1}\frac{\varphi(m)}{m^{2s}}=\frac{\zeta(2s-1)}{\zeta(2s)}.$$

Now suppose $b \neq 0$, Proposition 1.4.1 gives the first equality in the following chain:

$$\begin{aligned}
\sum_{m \geq 1} \frac{r(b,m)}{m^{2s}} &= \sum_{m \geq 1} m^{-2s} \sum_{\ell \mid (b,m)} \ell \mu\left(\frac{m}{\ell}\right) \\
&= \sum_{\ell \mid b} \ell \sum_{m \geq 1} \frac{\mu(m)}{(m\ell)^{2s}} \\
&= \left(\sum_{\ell \mid b} \ell^{1-2s}\right)\left(\sum_{m \geq 1} \frac{\mu(m)}{m^{2s}}\right) \\
&= \sigma_{1-2s}(b) \sum_{m \geq 1} \frac{\mu(m)}{m^{2s}} \\
&= \sigma_{1-2s}(|b|) \sum_{m \geq 1} \frac{\mu(m)}{m^{2s}} \\
&= \frac{\sigma_{1-2s}(|b|)}{\zeta(2s)} && \text{Proposition A.2.2.}
\end{aligned}$$

□

We can now compute the Fourier series of $E(z,s)$:

Proposition 7.5.1. *The Fourier series of $E(z,s)$ is given by*

$$E(z,s) = y^s + y^{1-s}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} + \sum_{t \geq 1}\left(\frac{2\pi^s |t|^{s-\frac{1}{2}\sigma_{1-2s}(|t|)}}{\Gamma(s)\zeta(2s)}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|t|y)\right)e^{2\pi i t x}.$$

Proof. Fix s with $\sigma > 1$. By the Bruhat decomposition for $\Gamma_0(1)$ and Remark 2.2.3, we have

$$E(z,s) = \operatorname{Im}(z)^s + \sum_{\substack{c \geq 1, d \in \mathbb{Z} \\ (c,d)=1}} \frac{\operatorname{Im}(z)^s}{|cz+d|^{2s}}.$$

Summing over all pairs $(c,d) \in \mathbb{Z}^2 - \{\mathbf{0}\}$ with $c \geq 1$, $d \in \mathbb{Z}$, and $(c,d) = 1$ is the same as summing over all triples (c,ℓ,r) with $c \geq 1$, $\ell \in \mathbb{Z}$, r taken modulo c, and $(r,c) = 1$. This is seen by writing $d = c\ell + r$. Therefore

$$\sum_{\substack{c \geq 1, d \in \mathbb{Z} \\ (c,d)=1}} \frac{\operatorname{Im}(z)^s}{|cx+icy+d|^{2s}} = \sum_{(c,\ell,r)} \frac{\operatorname{Im}(z)^s}{|cz+c\ell+r|^{2s}} = \sideset{}{'}\sum_{\substack{c \geq 1 \\ r \pmod{c}}} \sum_{\ell \in \mathbb{Z}} \frac{\operatorname{Im}(z)^s}{|cz+c\ell+r|^{2s}}.$$

where on the right-hand side it is understood that we are summing over all triples (c,ℓ,r) with the prescribed properties. Now let

$$I_{c,r}(z,s) = \sum_{\ell \in \mathbb{Z}} \frac{\operatorname{Im}(z)^s}{|cz+c\ell+r|^{2s}}.$$

We apply the Poisson summation formula to $I_{c,r}(z,s)$. This is allowed since the summands are absolutely integrable by Proposition 1.5.2, as they exhibit polynomial decay of order $\sigma > 1$, and $I_{c,r}(z,s)$ is holomorphic because $E(z,s)$ is. By the identity theorem it suffices to apply the Poisson summation formula for $z = iy$ with $y > 0$. So let $f(x)$ be given by

$$f(x) = \frac{y^s}{|cx+r+icy|^{2s}}.$$

Then $f(x)$ is absolutely integrable on $\mathbb{R}$ as we have just mentioned. We compute the Fourier transform:

$$\begin{aligned}
\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx &= \int_{-\infty}^{\infty} \frac{y^s}{|cx+r+icy|^{2s}}e^{-2\pi itx}\,dx \\
&= \int_{-\infty}^{\infty} \frac{y^s}{((cx+r)^2+(cy)^2)^s}e^{-2\pi itx}\,dx \\
&= e^{2\pi it\frac{r}{c}}\int_{-\infty}^{\infty} \frac{y^s}{((cx)^2+(cy)^2)^s}e^{-2\pi itx}\,dx && x \to x-\frac{r}{c} \\
&= \frac{e^{2\pi it\frac{r}{c}}}{c^{2s}}\int_{-\infty}^{\infty} \frac{y^s}{(x^2+y^2)^s}e^{-2\pi itx}\,dx \\
&= \frac{e^{2\pi it\frac{r}{c}}}{c^{2s}}\int_{-\infty}^{\infty} \frac{y^{s+1}}{((xy)^2+y^2)^s}e^{-2\pi itxy}\,dx && x \to xy \\
&= \frac{e^{2\pi it\frac{r}{c}}}{c^{2s}}\int_{-\infty}^{\infty} \frac{y^{1-s}}{(x^2+1)^s}e^{-2\pi itxy}\,dx.
\end{aligned}$$

Appealing to Appendix D.1 to compute this latter integral, we see that

$$\hat{f}(t) = \begin{cases} \frac{y^{1-s}}{c^{2s}}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} & \text{if } t=0, \\ \frac{e^{2\pi it\frac{r}{c}}}{c^{2s}}\frac{2\pi^s|t|^{s-\frac{1}{2}}}{\Gamma(s)}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|t|y) & \text{if } t\neq 0. \end{cases}$$

By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z,s) = \frac{y^{1-s}}{c^{2s}}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + \sum_{t\neq 0}\left(\frac{e^{2\pi it\frac{r}{c}}}{c^{2s}}\frac{2\pi^s|t|^{s-\frac{1}{2}}}{\Gamma(s)}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|t|y)\right)e^{2\pi itx},$$

for all $z \in \mathbb{H}$. Substituting this back into the Eisenstein series gives a form of the Fourier series:

$$\begin{aligned}
E_(z,s) &= y^s + \sideset{}{'}\sum_{\substack{c\geq 1 \\ r \pmod{c}}}\left(\frac{y^{1-s}}{c^{2s}}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + \sum_{t\geq 1}\left(\frac{e^{2\pi it\frac{r}{c}}}{c^{2s}}\frac{2\pi^s|t|^{s-\frac{1}{2}}}{\Gamma(s)}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|t|y)\right)e^{2\pi itx}\right) \\
&= y^s + y^{1-s}\sideset{}{'}\sum_{\substack{c\geq 1 \\ r \pmod{c}}}\frac{1}{c^{2s}}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + \sum_{t\geq 1}\left(\sideset{}{'}\sum_{\substack{c\geq 1 \\ r \pmod{c}}}\frac{e^{2\pi it\frac{r}{c}}}{c^{2s}}\frac{2\pi^s|t|^{s-\frac{1}{2}}}{\Gamma(s)}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|t|y)\right)e^{2\pi itx} \\
&= y^s + y^{1-s}\sum_{c\geq 1}\frac{r(0,c)}{c^{2s}}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + \sum_{t\geq 1}\left(\sum_{c\geq 1}\frac{r(t,c)}{c^{2s}}\frac{2\pi^s|t|^{s-\frac{1}{2}}}{\Gamma(s)}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|t|y)\right)e^{2\pi itx}.
\end{aligned}$$

By applying Lemma 7.5.1 to compute the Dirichlet series of Ramanujan sums, we obtain the desired Fourier series:

$$E(z,s) = y^s + y^{1-s}\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} + \sum_{t\geq 1}\left(\frac{2\pi^s|t|^{s-\frac{1}{2}\sigma_{1-2s}(|t|)}}{\Gamma(s)\zeta(2s)}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|t|y)\right)e^{2\pi itx}.$$

□

Having computed the Fourier series, we would like to obtain a functional equation for $E(z,s)$ as $s \to 1-s$ (this follows from Theorem 4.4.3 but we will demonstrate a full proof for the single Eisenstein series on $\Gamma_0(1)\backslash\mathbb{H}$). To this end, we define $E^*(z,s)$ by

$$E^*(z,s) = \Lambda(2s,\zeta)E(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s).$$

From Proposition 7.5.1, the Fourier coefficients $a^*(n,y,s)$ of $E^*(z,s)$ in the Fourier series

$$E^*(z,s) = a^*(0,y,s) + \sum_{n\neq 0} a^*(n,y,s)e^{2\pi i n x},$$

are given by

$$a^*(n,y,s) = \begin{cases} y^s\pi^{-s}\Gamma(s)\zeta(2s) + y^{1-s}\pi^{-\left(s-\frac{1}{2}\right)}\Gamma(s-\frac{1}{2})\zeta(2s-1) & \text{if } n=0, \\ 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y) & \text{if } n\neq 0. \end{cases}$$

We can now derive a functional equation for $E^*(z,s)$. Using the definition and functional equation for $\Lambda(2s-1,\zeta)$, we can rewrite the second term in the constant coefficient to get

$$a^*(n,y,s) = \begin{cases} y^s\Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta) & \text{if } n=0, \\ 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y) & \text{if } n\neq 0. \end{cases} \tag{7.22}$$

Now observe that the constant coefficient is invariant under $s\to 1-s$. Each $n\neq 0$ coefficient is also invariant under $s\to 1-s$. To see this we will use two facts. First, from Appendix B.7, $K_s(y)$ is invariant under $s\to -s$ and so $K_{s-\frac{1}{2}}(2\pi|n|y)$ is invariant as $s\to 1-s$. Second, for $n\geq 1$ we have

$$n^{s-\frac{1}{2}}\sigma_{1-2s}(n) = n^{\frac{1}{2}-s}n^{2s-1}\sigma_{1-2s}(n) = n^{\frac{1}{2}-s}n^{2s-1}\sum_{d\mid n} d^{1-2s} = n^{\frac{1}{2}-s}\sum_{d\mid n}\left(\frac{n}{d}\right)^{2s-1} = n^{\frac{1}{2}-s}\sigma_{2s-1}(n),$$

where the second to last equality follows by writing $n^{2s-1} = \left(\frac{n}{d}\right)^{2s-1}d^{2s-1}$ for each $d\mid n$. These two facts together give the invariance of the $n\neq 0$ coefficients under $s\to 1-s$. Altogether, we have shown the following functional equation for $E^*(z,s)$:

$$E^*(z,s) = E^*(z,1-s).$$

We already knew $E(z,s)$ possessed a functional equation as $s\to 1-s$ from Theorem 4.4.3 but we needed an explicit version in order to understand $L(s,f\otimes g)$. We can now obtain meromorphic continuation of $E^*(z,s)$ in s to all of $\mathbb{C}$ for any $z\in\mathbb{H}$. We first write $E^*(z,s)$ as a Fourier series using Equation (7.22):

$$E^*(z,s) = y^s\Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta) + \sum_{n\neq 0} 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}.$$

Since $\Lambda(2s,\zeta)$ is meromorphic on $\mathbb{C}$, the constant term of $E^*(z,s)$ is as well. To finish the meromorphic continuation of $E^*(z,s)$ it now suffices to show

$$\sum_{n\neq 0} 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x},$$

is meromorphic on $\mathbb{C}$. We will actually prove it is locally absolutely uniformly convergent on this region. So let K be a compact subset of $\mathbb{C}$. Then we have to show $E^*(z,s)$ is absolutely convergent on K for any $z\in\mathbb{H}$. To achieve this we need two bounds, one for $\sigma_{1-2s}(|n|)$ and one for $K_{s-\frac{1}{2}}(2\pi|n|y)$. For the first bound, we use the estimate $\sigma_0(|n|)\ll_\varepsilon |n|^\varepsilon$ (recall Proposition A.3.1). Therefore we have the crude bound

$$\sigma_{1-2s}(|n|) = \sum_{d\mid n} d^{1-2s} < \sigma_0(|n|)|n|^{1-2s} \ll_\varepsilon |n|^{1-2s+\varepsilon}.$$

For the second estimate, Lemma B.7.2 gives

$$K_{s-\frac{1}{2}}(2\pi|n|y) \ll e^{-2\pi|n|y}.$$

Using these two estimates, we have

$$\sum_{n\neq 0} 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx} \ll_{\varepsilon} \sum_{n\geq 1} 4n^{\frac{1}{2}-s+\varepsilon}\sqrt{y}e^{-2\pi ny}. \tag{7.23}$$

This latter series is absolutely uniformly convergent on K by the ratio test. Therefore $E^*(z,s)$ is absolutely convergent on K for any $z \in \mathbb{H}$ and the meromorphic continuation to $\mathbb{C}$ follows. It remains to investigate the poles and residues. We will accomplish this from direct inspection of the Fourier coefficients:

Proposition 7.5.2. *$E^*(z,s)$ has simple poles at $s = 0$ and $s = 1$, and*

$$\operatorname*{Res}_{s=0} E^*(z,s) = -\frac{1}{2} \quad \text{and} \quad \operatorname*{Res}_{s=1} E^*(z,s) = \frac{1}{2}.$$

Proof. Since the constant term in the Fourier series of $E^*(z,s)$ is the only non-holomorphic term, poles of $E^*(z,s)$ can only come from that term. So we are reduced to understanding the poles of

$$y^s\Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta). \tag{7.24}$$

Notice $\Lambda(2s,\zeta)$ has simple poles at $s = 0$, $s = \frac{1}{2}$ (one from the Riemann zeta function and one from the gamma factor) and no others. It follows that $E^*(z,s)$ has a simple pole at $s = 0$ coming from the y^s term in Equation (7.24), and by the functional equation there is also a pole at $s = 1$ coming from the y^{1-s} term. At $s = \frac{1}{2}$, both terms in Equation (7.24) have simple poles and we will show that the singularity there is removable. Recall $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Also, by Proposition 7.1.1, $\operatorname{Res}_{s=\frac{1}{2}} \zeta(2s) = \frac{1}{2}$ and $\operatorname{Res}_{s=\frac{1}{2}} \zeta(2(1-s)) = -\frac{1}{2}$. So altogether

$$\operatorname*{Res}_{s=\frac{1}{2}} E^*(z,s) = \operatorname*{Res}_{s=\frac{1}{2}}[y^s\Lambda(2s,\zeta) + y^{1-s}\Lambda(2(1-s),\zeta)] = \frac{1}{2}y^{\frac{1}{2}} - \frac{1}{2}y^{\frac{1}{2}} = 0.$$

Hence the singularity at $s = \frac{1}{2}$ is removable. As for the residues at $s = 0$ and $s = 1$, the functional equation implies that they are negatives of each other. So it suffices to compute the residue at $s = 0$. Recall $\zeta(0) = -\frac{1}{2}$ and $\operatorname{Res}_{s=0}\Gamma(s) = 1$. Then together we find

$$\operatorname*{Res}_{s=0} E^*(z,s) = \operatorname*{Res}_{s=0} y^s\Lambda(2s,\zeta) = -\frac{1}{2}.$$

□

This completes our study of $E(z,s)$.

The Integral Representation of $L(s, f\otimes g)$: Part II

We can now continue with the Rankin-Selberg convolution $L(s, f\otimes g)$. Writing Equation (7.19) in terms of $E^*(z,s)$ and $L(s, f\otimes g)$ results in the integral representation

$$L(s, f\otimes g) = \frac{(4\pi)^{s+k-1}\pi^s}{\Gamma(s+k-1)\Gamma(s)}\int_{\mathcal{F}} f(z)\overline{g(z)}\operatorname{Im}(z)^k E^*(z,s)\,d\mu.$$

This integral representation will give analytic continuation. To see this, note that the gamma factors are analytic for $\sigma < 0$. By the functional equation for $E^*(z,s)$, the integral is invariant as $s \to 1-s$. These two facts together give analytic continuation to $\mathbb{C}$ outside of the critical strip. The continuation inside of the critical strip will be meromorphic because of the poles of $E^*(z,s)$. To see this, taking the integral representation and substituting the Fourier series for $E^*(z,s)$ gives

$$\begin{aligned} L(s, f\otimes g) = \frac{(4\pi)^{s+k-1}\pi^s}{\Gamma(s+k-1)\Gamma(s)}\Bigg[\int_{\mathcal{F}} f(x+iy)\overline{g(x+iy)}y^k(y^s\Lambda(2s,\zeta)+y^{1-s}\Lambda(2(1-s),\zeta))\,\frac{dx\,dy}{y^2} \\ + \int_{\mathcal{F}} f(x+iy)\overline{g(x+iy)}y^k\sum_{n\neq 0}2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}\,\frac{dx\,dy}{y^2}\Bigg], \end{aligned} \tag{7.25}$$

and we are reduced to showing that both integrals are locally absolutely uniformly convergent in the critical strip and away from the poles of $E^*(z,s)$. Indeed, the first integral is locally absolutely uniformly convergent in this region by Proposition 1.5.1. As for the second integral, Equation (7.23) implies that it is

$$O_\varepsilon\left(\int_{\mathcal{F}} f(x+iy)\overline{g(x+iy)}y^k\sum_{n\geq 1}4n^{\frac{1}{2}-s+\varepsilon}\sqrt{y}e^{-2\pi ny}\,\frac{dx\,dy}{y^2}\right).$$

As the sum in the integrand is holomorphic, we can now appeal to Proposition 1.5.1. The meromorphic continuation to the critical strip and hence to all of $\mathbb{C}$ follows. In particular, $L(s, f\otimes g)$ has at most simple poles at $s=0$ and $s=1$. Actually, there is no pole at $s=0$. Indeed, $\gamma(s, f\otimes g)$ has a simple pole at $s=0$ coming from the gamma factors and therefore its reciprocal has a simple zero. This cancels the simple pole at $s=0$ coming from $E^*(z,s)$ and therefore $L(s, f\otimes g)$ is has a removable singularity at $s=0$. So there is at worst a simple pole at $s=1$.

The Functional Equation, Critical Strip & Residues of $L(s, f\otimes g)$

An immediate consequence of the symmetry of integral representation is the functional equation:

$$\frac{\Gamma(s+k-1)\Gamma(s)}{(4\pi)^{s+k-1}\pi^s}L(s, f\otimes g) = \frac{\Gamma((1-s)+k-1)\Gamma(1-s)}{(4\pi)^{(1-s)+k-1}\pi^{1-s}}L(1-s, f\otimes g).$$

Applying the Legendre duplication formula for the gamma function twice we see that

$$\begin{aligned} \frac{\Gamma(s+k-1)\Gamma(s)}{(4\pi)^{s+k-1}\pi^s} &= \frac{2^{2s+k-3}}{(4\pi)^{s+k-1}\pi^{s+1}}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) \\ &= \frac{1}{2^{k+1}\pi^k}\pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right). \end{aligned} \tag{7.26}$$

The factor in front is independent of s and can therefore be canceled in the functional equation. We identify the gamma factor as:

$$\gamma(s, f\otimes g) = \pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right),$$

with $\mu_{1,1}=k-1$, $\mu_{2,2}=k$, $\mu_{1,2}=0$, and $\mu_{2,1}=1$ the local roots at infinity. The completed L-function is

$$\Lambda(s, f\otimes g) = \pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L(s, f\otimes g),$$

so the conductor is $q(f\otimes g)=1$ and no primes ramify. Clearly $q(f\otimes g)\mid q(f)^2q(g)^2$ and conductor dropping does not occur. Then

$$\Lambda(s, f\otimes g) = \Lambda(1-s, f\otimes g),$$

is the functional equation of $L(s, f \otimes g)$. In particular, the root number $\varepsilon(f \otimes g) = 1$, and $L(s, f \otimes g)$ is self-dual. We can now show that $L(s, f \otimes g)$ is of order 1. Since the possible pole at $s = 1$ is simple, multiplying by $(s-1)$ clears the possible polar divisor. As the integrals in the integral representation are locally absolutely uniformly convergent, computing the order amounts to estimating the gamma factor. Since the reciprocal of the gamma function is of order 1, we have

$$\frac{1}{\gamma(s, f \otimes g)} \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

So the reciprocal of the gamma factor is also of order 1. Then we find that

$$(s-1)L(s, f \otimes g) \ll_\varepsilon e^{|s|^{1+\varepsilon}}.$$

Thus $(s-1)L(s, f \otimes g)$ is of order 1, and so $L(s, f \otimes g)$ is as well after removing the polar divisor. At last, we compute the residue of $L(s, f \otimes g)$ at $s = 1$:

Proposition 7.5.3. *Let $f, g \in \mathcal{S}_k(1)$ be primitive Hecke eigenforms. Then*

$$\operatorname*{Res}_{s=1} L(s, f \otimes g) = \frac{4^k \pi^{k+1} V}{2\Gamma(k)} \langle f, g \rangle,$$

where $\langle f, g \rangle$ is the Petersson inner product.

Proof. As $V = \frac{\pi}{3}$, Proposition 7.5.2 implies

$$\operatorname*{Res}_{s=1} L(s, f \otimes g) = \frac{4^k \pi^{k+1}}{\Gamma(k)} \operatorname*{Res}_{s=1} \int_{\mathcal{F}} f(z)\overline{g(z)} \operatorname{Im}(z)^k E^*(z, s)\, d\mu = \frac{4^k \pi^{k+1} V}{2\Gamma(k)} \langle f, g \rangle. \qquad \square$$

Notice that if $f = g$, then $\langle f, f \rangle \neq 0$ and therefore the residue at $s = 1$ is not zero and hence not a removable singularity. Actually, this is the only instance in which there is a pole since Theorem 3.6.1 implies that the primitive Hecke eigenforms are orthogonal so that $\langle f, g \rangle = 0$ unless $f = g$. We summarize all of our work into the following theorem:

Theorem 7.5.1. *For any two primitive Hecke eigenforms $f, g \in \mathcal{S}_k(1)$, $L(s, f \otimes g)$ is a Selberg class L-function. For $\sigma > 1$, it has a degree 4 Euler product given by*

$$L(s, f \otimes g) = \prod_p \left(\sum_{n \geq 0} \frac{a_f(p^n)\overline{a_g(p^n)}}{p^{ns}} \right).$$

Moreover, it admits meromorphic continuation to $\mathbb{C}$ via the integral representation

$$L(s, f \otimes g) = \frac{(4\pi)^{s+k-1}\pi^s}{\Gamma(s+k-1)\Gamma(s)} \int_{\mathcal{F}} f(z)\overline{g(z)} \operatorname{Im}(z)^k E^*(z, s)\, d\mu,$$

possesses the functional equation

$$\pi^{-2s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) L(s, f \otimes g) = \Lambda(s, f \otimes g) = \Lambda(1-s, f \otimes g),$$

and if $f = g$ there is simple pole at $s = 1$ of residue $\frac{4^k \pi^{k+1} V}{2\Gamma(k)} \langle f, g \rangle$.

The Rankin-Selberg Method

The Rankin-Selberg method is much more complicated in general, but the argument is essentially the same. Let f and g both be primitive Hecke or Heck-Maass eigenforms with Fourier coefficients $a_f(n)$ and $a_g(n)$ respectively. We suppose f has weight k/type ν, level N, and character χ, and g has weight ℓ/type η, level M, and character ψ. The L-function $L(s, f \times g)$ of f and g is given by the L-series

$$L(s, f \times g) = \sum_{n \geq 1} \frac{a_{f \times g}(n)}{n^s} = \sum_{n \geq 1} \frac{a_f(n)\overline{a_g(n)}}{n^s}.$$

The **Rankin-Selberg convolution** $L(s, f \otimes g)$ of f and g is defined as an L-series:

$$L(s, f \otimes g) = \sum_{n \geq 1} \frac{a_{f \otimes g}(n)}{n^s} = L(2s, \chi\overline{\psi})L(s, f \times g),$$

where $a_{f \otimes g}(n) = \sum_{n = m\ell^2} \chi\overline{\psi}(\ell^2)a_f(m)\overline{a_g(m)}$. The following argument is the **Rankin-Selberg method**:

Method 7.5.1 (Rankin-Selberg method). Let f and b both be primitive Hecke or Heck-Maass eigenforms. Also suppose the following:

(i) f has weight k/type ν, level N, and character χ.

(ii) g has weight ℓ/type η, level M, and character ψ.

(iii) The Ramanujan-Petersson conjecture for Maass forms holds if f or g are Heck-Maass eigenforms.

Then the Rankin-Selberg convolution $L(s, f \otimes g)$ is a Selberg class L-function.

We make a few remarks about the Rankin-Selberg method. Local absolute uniform convergence for $\sigma > 1$ are proved in the exactly the same way as we have described. The argument for the Euler product is also similar. However, if either $N > 1$ or $M > 1$, the computation becomes more difficult to compute since the local p factors for $p \mid NM$ change. Moreover, the situation is increasingly complicated if $(N, M) > 1$ since conductor dropping can occur. The integral representation has a similar argument, but if the weights/types are distinct the resulting Eisenstein series becomes more complicated. In particular, it is on $\Gamma_0(NM)\backslash\mathbb{H}$ and if $NM > 1$, then there is more than just the cusp at ∞. Therefore the functional equation of the Eisenstein series at the ∞ cusp then reflects into a linear combination of Eisenstein series at the other cusps. This results in the requirement to compute the Fourier coefficients of all of these Eisenstein series. Moreover, this procedure can be generalized to remove the primitive Hecke and/or primitive Hecke-Maass eigenform conditions by taking linear combinations, but we won't attempt discussing this further.

7.6 Applications of the Rankin-Selberg Method

The Ramanujan-Petersson Conjecture on Average

Let f be a Hecke or Hecke-Maass eigenform. Using Rankin-Selberg convolutions, it is possible to show the weaker result that $a_f(n) \ll_\varepsilon n^\varepsilon$ holds on average without assuming the corresponding Ramanujan-Petersson conjecture:

Proposition 7.6.1. *Let f be a primitive Hecke or Hecke-Maass eigenform. Then for any $X > 0$, we have*

$$\sum_{n \le X} |a_f(n)| \ll_\varepsilon X^{1+\varepsilon},$$

Proof. By the Cauchy-Schwarz inequality,

$$\left(\sum_{n \le X} |a_f(n)|\right)^2 \le X \sum_{n \le X} |a_f(n)|^2, \tag{7.27}$$

The Rankin-Selberg square $L(s, f \otimes f)$ is locally absolutely uniformly convergent for $\sigma > \frac{3}{2}$. Therefore it still admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$. By Landau's theorem, the abscissa of absolute convergence of $L(s, f \otimes f)$, and hence $L(s, f \times f)$ too, is 1 so by Proposition 6.1.3 we have

$$\sum_{n \le X} |a_f(n)|^2 \ll_\varepsilon X^{1+\varepsilon},$$

for any $\varepsilon > 0$. Substituting this bound into Equation (7.27), we obtain

$$\left(\sum_{n \le X} |a_f(n)|\right)^2 \ll_\varepsilon X^{2+\varepsilon},$$

and taking the square root yields

$$\sum_{n \le X} |a_f(n)| \ll_\varepsilon X^{1+\varepsilon}.$$

□

The bound in Proposition 7.6.1 should be compared with the implication $a_f(n) \ll_\varepsilon n^\varepsilon$ that follows from the corresponding Ramanujan-Petersson conjecture. While Proposition 7.6.1 is not useful in the holomorphic form case, it is in the Maass form case. Indeed, recall that if f is a primitive Hecke-Maass eigenform we needed to assume the Ramanujan-Petersson conjecture for Maass forms to ensure $a_f(n) \ll_\varepsilon n^\varepsilon$ so that $L(s, f)$ was locally absolutely uniformly convergent for $\sigma > 1$. However, Propositions 6.1.4 and 7.6.1 together imply that $L(s, f)$ is locally absolutely uniformly convergent for $\sigma > 1 + \varepsilon$ and hence for $\sigma > 1$.

Strong Multiplicity One

Let f be a primitive Hecke or Hecke-Maass eigenform. Then f is determined by Hecke eigenvalues at primes for fixed weight/type, and level. Using Rankin-Selberg convolutions, we can prove **strong multiplicity one** for holomorphic or Maass forms which says that f is determined by Hecke eigenvalues at all but finitely many primes for fixed weight/type, level, and character:

Theorem 7.6.1 (Strong multiplicity one, holomorphic and Maass versions). *Let f and g both be primitive Hecke or Hecke-Maass eigenforms of the same weight/type, level, and character. Denote the Hecke eigenvalues by $\lambda_f(n)$ and $\lambda_g(n)$ respectively. If $a_f(p) = a_g(p)$ for all but finitely many primes p, then $f = g$.*

Proof. Let S be the set the primes for which $\lambda_f(p) \neq \lambda_g(p)$ including the primes that ramify for $L(s, f)$ and $L(s, g)$. By assumption, S is finite. As the local factors of $L(s, f \otimes g)$ are holomorphic and nonzero at $s = 1$, the order of the pole of $L(s, f \otimes g)$ is the same as the order of the pole of

$$L(s, f \otimes g) \prod_{p \in S} L_p(s, f \otimes g)^{-1} = \prod_{p \notin S} L_p(s, f \otimes g).$$

But as f and g have the same the weight/type, level, and character, and $\lambda_f(p) = \lambda_g(p)$ for all $p \notin S$, we have

$$\prod_{p \notin S} L_p(s, f \otimes g) = \prod_{p \notin S} L_p(s, f \otimes f).$$

Since $L(s, f \otimes f)$ has a simple pole at $s = 1$, it follows that $L(s, f \otimes g)$ does too. But then $f = g$. $\square$

Chapter 8

Classical Applications

We will discuss some classical applications of L-functions. Our first result is a crowning gem of analytic number theory: Dirichlet's theorem. This result is a consequence of a non-vanishing result for Dirichlet L-functions at $s = 1$. Then we discuss Siegel zeros in the case of Dirichlet L-functions and as a result obtain a lower bound for Dirichlet L-functions at $s = 1$. Lastly, we prove the prime number theorem and its variant for primes restricted to a certain residue class, the Siegel–Walfisz theorem, in the classical manner.

8.1 Dirichlet's Theorem

One of the more well-known arithmetic results proved using L-functions is **Dirichlet's theorem**:

Theorem 8.1.1 (Dirichlet's theorem). *Let a and m be positive integers such that $(a, m) = 1$. Then the arithmetic progression $\{a + km \mid k \in \mathbb{N}\}$ contain infinitely many primes.*

We will delay the proof for the moment, for it is well-worth understanding the some of the motivation behind why this theorem is interesting and how exactly Dirichlet used the analytic techniques of L-functions to attack this purely arithmetic statement. We being by recalling Euclid's famous theorem on the infinitude of the primes. Euclid's proof is completely elementary and arithmetic in nature. He argues that if there were finitely many primes $p_1, p_2, \ldots, p_k$ then a short consideration of $(p_1p_2 \cdots p_k) + 1$ shows that this number must either be divisible by a prime not in our list or must be prime itself. As primes are the multiplicative building blocks of arithmetic, Euclid assures us that we have an ample amount of primes to work with. Now there is a slightly stronger result due to Euler (see [Eul44]) requiring analytic techniques:

Theorem 8.1.2. *The series*

$$\sum_p \frac{1}{p},$$

diverges.

Proof. For $\sigma > 1$, taking the logarithm of the Euler product of $\zeta(s)$, we get

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s}).$$

The Taylor series of the logarithm gives

$$\log(1 - p^{-s}) = \sum_{k \geq 1} (-1)^{k-1} \frac{(-p^{-s})^k}{k} = \sum_{k \geq 1} (-1)^{2k-1} \frac{1}{kp^{ks}},$$

so that

$$\log \zeta(s) = \sum_p \sum_{k\geq 1} \frac{1}{kp^{ks}}.$$

The double sum restricted to $k \geq 2$ is uniformly bounded for $\sigma > 1$. To see this, first observe

$$\sum_{k\geq 2} \frac{1}{kp^{ks}} \ll \sum_{k\geq 2} \frac{1}{p^k} = \frac{1}{p^2} \sum_{k\geq 0} \frac{1}{p^k} = \frac{1}{p^2}(1-p^{-1})^{-1} \leq \frac{2}{p^2},$$

where the last inequality follows because $p \geq 2$. Then

$$\sum_p \sum_{k\geq 2} \frac{1}{kp^{ks}} \ll 2\sum_p \frac{1}{p^2} < 2\sum_{n\geq 1} \frac{1}{n^2} = 2\zeta(2).$$

Therefore

$$\log \zeta(s) - \sum_p \frac{1}{p^s} = \sum_p \sum_{k\geq 2} \frac{1}{kp^{ks}},$$

and remains bounded as $s \to 1$. The claim now follows since $\zeta(s)$ has a simple pole at $s = 1$. □

Theorem 8.1.2 tells us that there are infinitely many primes but also that the primes are not too "sparse" in the integers for otherwise the series would converge. The idea Dirichlet used to prove his result on primes in arithmetic progressions was in a very similar spirit. He sought out to prove the divergence of the series

$$\sum_{p\equiv a \pmod m} \frac{1}{p},$$

for positive integers a and m with $(a,m) = 1$ as the divergence immediately implies there are infinitely many primes p of the form $p \equiv a \pmod m$. In the case $a = 1$ and $m = 2$ we recover Theorem 8.1.2 exactly since every prime is odd. Dirichlet's proof proceeds in a similar way to that of Theorem 8.1.2 and this is where Dirichlet used what are now known as Dirichlet characters and Dirichlet L-functions. The proof can be broken into three steps. The first is to proceed as Euler did, but with the Dirichlet L-function $L(s,\chi)$ where χ has modulus m. That is, write $L(s,\chi)$ as a sum over primes and a bounded term as $s \to 1$. The next step is to use the orthogonality relations of the characters to sieve out the correct sum. The last step is to show the non-vanishing result $L(1,\chi) \neq 0$ for all non-principal characters χ. This is the essential part of the proof as it is what assures us that the sum diverges. Luckily, we have done most of the hard work to prove this already:

Theorem 8.1.3. *Let χ be a non-principal Dirichlet character. Then $L(1,\chi)$ is finite and nonzero.*

Proof. This follows immediately by applying Lemma 6.10.1 to $\zeta(s)L(s,\chi)$ and noting that $L(s,\chi)$ is holomorphic. □

We now prove Dirichlet's theorem:

Proof of Dirichlet's theorem. Let χ be a Dirichlet character modulo m. Then for $\sigma > 1$, taking the logarithm of the Euler product of $L(s,\chi)$ gives

$$\log L(s,\chi) = -\sum_p \log(1-\chi(p)p^{-s}).$$

The Taylor series of the logarithm implies

$$\log(1-\chi(p)p^{-s}) = \sum_{k\geq 1}(-1)^{k-1}\frac{(-\chi(p)p^{-s})^k}{k} = \sum_{k\geq 1}(-1)^{2k-1}\frac{\chi(p^k)}{kp^{ks}},$$

so that

$$\log L(s,\chi) = \sum_p \sum_{k\geq 1}\frac{\chi(p^k)}{kp^{ks}}.$$

The double sum restricted to $k \geq 2$ is uniformly bounded for $\sigma > 1$. Indeed, first observe

$$\left|\sum_{k\geq 2}\frac{\chi(p^k)}{kp^{ks}}\right| \ll \sum_{k\geq 2}\frac{1}{p^k} = \frac{1}{p^2}\sum_{k\geq 0}\frac{1}{p^k} = \frac{1}{p^2}(1-p^{-1})^{-1} \leq \frac{2}{p^2},$$

where the last inequality follows because $p > 2$. Then

$$\left|\sum_p \sum_{k\geq 2}\frac{\chi(p^k)}{kp^{ks}}\right| \leq 2\sum_p \frac{1}{p^2} < 2\sum_{n\geq 1}\frac{1}{n^2} = 2\zeta(2),$$

as desired. Now write

$$\sum_{\chi \pmod m}\overline{\chi(a)}\log L(s,\chi) = \sum_{\chi \pmod m}\sum_p \frac{\overline{\chi(a)}\chi(p)}{p^s} + \sum_{\chi \pmod m}\overline{\chi(a)}\sum_p\sum_{k\geq 2}\frac{\chi(p^k)}{kp^{ks}}.$$

By the orthogonality relations (Proposition 1.3.1 (ii)), we find that

$$\sum_{\chi \pmod m}\sum_p \frac{\overline{\chi(a)}\chi(p)}{p^s} = \sum_p \frac{1}{p^s}\sum_{\chi \pmod m}\overline{\chi(a)}\chi(p) = \varphi(m)\sum_{p\equiv a \pmod m}\frac{1}{p^s},$$

and so

$$\sum_{\chi \pmod m}\overline{\chi(a)}\log L(s,\chi) - \sum_{\chi \pmod m}\overline{\chi(a)}\sum_p\sum_{k\geq 2}\frac{\chi(p^k)}{kp^{ks}} = \varphi(m)\sum_{p\equiv a \pmod m}\frac{1}{p^s}.$$

The triple sum is uniformly bounded for $\sigma > 1$ because the inner double sum is and there are finitely many Dirichlet characters modulo m. Therefore it suffices to show that the first sum on the left-hand side diverges as $s \to 1$. For $\chi = \chi_{m,0}$,

$$L(s,\chi_{m,0}) = \zeta(s)\prod_{p|m}(1-p^{-s}),$$

so the corresponding term in the sum is

$$\overline{\chi_{m,0}}(a)\log L(s,\chi_{m,0}) = \log\zeta(s) + \sum_{p|m}\log(1-p^{-s}),$$

which diverges as $s \to 1$ because $\zeta(s)$ has a simple pole at $s = 1$. We will be done if $\log L(s,\chi)$ remains bounded as $s \to 1$ for all $\chi \neq \chi_{m,0}$. So assume χ is non-principal. Then we must show $L(1,\chi)$ is finite and nonzero. This follows from Theorem 8.1.3. □

For primitive χ of conductor $q > 1$, we know from Theorem 8.1.3 that $L(1,\chi)$ is finite and nonzero. It is interesting to know whether or not this value is computable in general. Indeed it is. The computation is fairly straightforward and only requires some basic properties of Ramanujan and Gauss sums that we have already developed. The idea is to rewrite the character values $\chi(n)$ so that we can collapse the infinite series into a Taylor series. Our result is the following:

Theorem 8.1.4. *Let χ be a primitive Dirichlet character with conductor $q > 1$. Then*

$$L(1,\chi) = -\frac{\chi(-1)\tau(\chi)}{q}\sum_{a \pmod q}{}' \chi(a)\log\left(\sin\left(\frac{\pi a}{q}\right)\right) \quad \text{or} \quad L(1,\chi) = -\frac{\chi(-1)\tau(\chi)\pi i}{q^2}\sum_{a \pmod q}{}' \chi(a)a,$$

according to whether χ is even or odd.

Proof. Make the following computation:

$$\begin{aligned}
\chi(n) &= \frac{1}{\overline{\tau(\chi)}}\overline{\tau(n,\chi)} && \text{Corollary 1.4.1} \\
&= \frac{1}{\tau(\overline{\chi})}\tau(n,\chi) && \text{Proposition 1.4.2 (i)} \\
&= \frac{\tau(\chi)}{\tau(\chi)\tau(\overline{\chi})}\tau(n,\chi) \\
&= \frac{\chi(-1)\tau(\chi)}{q}\tau(n,\chi) && \text{Theorem 1.4.1 and Proposition 1.4.3} \\
&= \frac{\chi(-1)\tau(\chi)}{q}\sum_{a \pmod q}{}' \chi(a)e^{\frac{2\pi i a n}{q}}.
\end{aligned}$$

Substituting this result into the definition of $L(1,\chi)$, we find that

$$\begin{aligned}
L(1,\chi) &= \sum_{n\geq 1}\frac{1}{n}\left(\frac{\chi(-1)\tau(\chi)}{q}\sum_{a \pmod q}{}' \chi(a)e^{\frac{2\pi i a n}{q}}\right) \\
&= \frac{\chi(-1)\tau(\chi)}{q}\sum_{a \pmod q}{}' \chi(a)\sum_{n\geq 1}\frac{e^{\frac{2\pi i a n}{q}}}{n} \qquad (8.1) \\
&= \frac{\chi(-1)\tau(\chi)}{q}\sum_{a \pmod q}{}' \chi(a)\log\left(\left(1-e^{\frac{2\pi i a}{q}}\right)^{-1}\right),
\end{aligned}$$

where in the last line we have used the Taylor series of the logarithm (notice $a \neq q$ so that $e^{\frac{2\pi i a}{q}} \neq 1$ and hence the logarithm is defined). We have now expressed $L(1,\chi)$ as a finite sum. In order to simplify the last expression in Equation (8.1), we deal with the logarithm. Since $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, we have

$$1-e^{\frac{2\pi i a}{q}} = -2ie^{\frac{\pi i a}{q}}\left(\frac{e^{\frac{\pi i a}{q}}-e^{-\frac{\pi i a}{q}}}{2i}\right) = -2ie^{\frac{\pi i a}{q}}\sin\left(\frac{\pi a}{q}\right).$$

Therefore the last expression in Equation (8.1) becomes

$$\frac{\chi(-1)\tau(\chi)}{q}\sum_{a \pmod q}{}' \chi(a)\log\left(\left(-2ie^{\frac{\pi i a}{q}}\sin\left(\frac{\pi a}{q}\right)\right)^{-1}\right).$$

As $0 < a < q$, we have $0 < \frac{\pi a}{q} < \pi$ so that $\sin\left(\frac{\pi a}{q}\right)$ is never negative. Therefore we can split up the logarithm term and obtain

$$-\frac{\chi(-1)\tau(\chi)}{q}\left(\log(-2i)\sideset{}{'}\sum_{a \pmod q}\chi(a) + \frac{\pi i}{q}\sideset{}{'}\sum_{a \pmod q}\chi(a)a + \sideset{}{'}\sum_{a \pmod q}\chi(a)\log\left(\sin\left(\frac{\pi a}{q}\right)\right)\right).$$

By the orthogonality relations (Corollary 1.3.1 (i)), the first sum above vanishes. Therefore

$$L(1,\chi) = -\frac{\chi(-1)\tau(\chi)}{q}\left(\frac{\pi i}{q}\sideset{}{'}\sum_{a \pmod q}\chi(a)a + \sideset{}{'}\sum_{a \pmod q}\chi(a)\log\left(\sin\left(\frac{\pi a}{q}\right)\right)\right). \tag{8.2}$$

Equation (8.2) simplifies in that one of the two sums vanish depending on if χ is even or odd. For the first sum in Equation (8.2), observe that

$$\frac{\pi i}{q}\sideset{}{'}\sum_{a \pmod q}\chi(a)a = -\frac{\chi(-1)\pi i}{q}\sideset{}{'}\sum_{a \pmod q}\chi(-a)(-a),$$

which vanishes if χ is even. For the second sum in Equation (8.2), we have an analogous relation of the form

$$\sideset{}{'}\sum_{a \pmod q}\chi(a)\log\left(\sin\left(\frac{\pi a}{q}\right)\right) = \chi(-1)\sideset{}{'}\sum_{a \pmod q}\chi(-a)\log\left(\sin\left(\frac{\pi a}{q}\right)\right),$$

which vanishes if χ is odd. This finishes the proof. $\square$

Theorem 8.1.4 encodes some interesting identities. For example, if χ is the non-principal Dirichlet character modulo 4, then χ is uniquely defined by $\chi(1) = 1$ and $\chi(3) = \chi(-1) = -1$. In particular, χ is odd and its conductor is 4. Now

$$\tau(\chi) = \sideset{}{'}\sum_{a \pmod 4}\chi(a)e^{\frac{2\pi i a}{4}} = e^{\frac{2\pi i}{4}} - e^{\frac{6\pi i}{4}} = i - (-i) = 2i,$$

so by Theorem 8.1.4 we get

$$L(1,\chi) = -\frac{\chi(-1)\tau(\chi)\pi i}{16}(1-3) = \frac{\pi}{4}.$$

Expanding out $L(1,\chi)$ gives

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4},$$

which is the famous **Madhava–Leibniz formula** for π.

8.2 Siegel's Theorem

The discussion of Siegel zeros first arose during the study of zero-free regions for Dirichlet L-functions. Refining the argument used in Theorem 6.10.1, we can show that Siegel zeros only exist when the character χ is quadratic. But first we improve the zero-free region for the Riemann zeta function:

Theorem 8.2.1. *There exists a constant $c > 0$ such that $\zeta(s)$ has no zeros in the region*

$$\sigma \geq 1 - \frac{c}{\log(|t|+3)}.$$

Proof. By Theorem 6.10.1 applied to $\zeta(s)$, it suffices to show that $\zeta(s)$ has no real nontrivial zeros. To see this, let $\eta(s)$ be defined by

$$\eta(s) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s}.$$

Note that $\eta(s)$ converges for $\sigma > 0$ by Proposition 6.1.1. Now for $0 < s < 1$ and even n, $\frac{1}{n^s} - \frac{1}{(n+1)^s} > 0$ so that $\eta(s) > 0$. But for $\sigma > 0$, we have

$$(1 - 2^{1-s})\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} - 2\sum_{n \geq 1} \frac{1}{(2n)^s} = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s} = \eta(s).$$

Therefore $\zeta(s)$ cannot admit a zero for $0 < s < 1$ because then $\eta(s)$ would be zero too. This completes the proof. □

Theorem 8.2.1 shows that $\zeta(s)$ has no Siegel zeros. Moreover, since $1 - 2^{1-s} < 0$ for $0 < s < 1$, the proof shows that $\zeta(s) < 0$ in this interval as well. As for the height of the first zero, it occurs on the critical line (as predicted by the Riemann hypothesis) at height $t \approx 14.134$ (see [Dav80] for a further discussion). The first 15 zeros were computed by Gram in 1903 (see [Gra03]). Since then, billions of zeros have been computed and have all been verified to lie on the critical line. The analogous situation for Dirichlet L-functions is only slightly different but causes increasing complexity in further study. We first show that if a Siegel zero exists for the Dirichlet L-function of a primitive character, then the character is necessarily quadratic:

Theorem 8.2.2. *Let χ be a primitive Dirichlet character of conductor $q > 1$. Then there exists a constant $c > 0$ such that $L(s, \chi)$ has no zeros in the region*

$$\sigma \geq 1 - \frac{c}{\log(q(|t| + 3))},$$

except for possibly one simple real zero β_χ with $\beta_{chi} < 1$ in the case χ is quadratic.

Proof. By Theorem 6.10.1 applied to $\zeta(s)L(s, \chi)$, and shrinking c if necessary, it remains to show that there not a simple real zero β_χ if χ is not quadratic. For this, let $L(s, g)$ be the L-function defined by

$$L(s, g) = L^3(s, \chi_{q,0})L^4(s, \chi)L(s, \chi^2).$$

We have $d_g = 8$ and $\mathfrak{q}(g)$ satisfies

$$\mathfrak{q}(g) \leq \mathfrak{q}(\chi_{q,0})^3 \mathfrak{q}(\chi)^4 \mathfrak{q}(\chi^2) \leq q^8 3^5 < (3q)^8.$$

Moreover, $\mathrm{Re}(\Lambda_g(n)) \geq 0$ for $(n, q) = 1$. To see this, suppose p is an unramified prime. The local roots of $L(s, g)$ at p are 1 with multiplicity three, $\chi(p)$ with multiplicity four, and $\chi^2(p)$ with multiplicity one. So for any $k \geq 1$, the sum of k-th powers of these local roots is

$$3 + 4\chi^k(p) + \chi^{2k}(p).$$

Writing $\chi(p) = e^{i\theta}$, the real part of this expression is

$$3 + 4\cos(\theta) + \cos(2\theta) = 2(1 + \cos(\theta))^2 \geq 0,$$

where we have also made use of the identity $3 + 4\cos(\theta) + \cos(2\theta) = 2(1 + \cos(\theta))^2$. Thus $\mathrm{Re}(\Lambda_g(n)) \geq 0$ for $(n, q) = 1$, and the conditions of Lemma 6.10.1 are satisfied for $L(s, g)$ (recall Equation (7.12) for the L-functions $L(s, \chi_{q,0})$ and $L(s, \chi^2)$). On the one hand, if β be a real nontrivial zero of $L(s, \chi)$ then $L(s, g)$ has a real nontrivial zero at $s = \beta$ of order at least 4. On the other hand, using Equation (7.12) and that $\chi^2 \neq \chi_{q,0}$, $L(s, g)$ has a pole at $s = 1$ of order 3. Then, upon shrinking c if necessary, Lemma 6.10.1 gives a contradiction since $r_g = 3$. This completes the proof. □

Siegel zeros present an unfortunate obstruction to zero-free region results for Dirichlet L-functions when the primitive character χ is quadratic. However, if we no longer require the constant c in the zero-free region to be effective, we can obtain a much better result for how close the Siegel zero can be to 1. Ultimately, this improved bound results from a lower bound for $L(1,\chi)$ (recall that this is nonzero from our discussion about Dirichlet's theorem). **Siegel's theorem** refers to either this lower bound or to the improved zero-free region. In the lower bound version, Siegel's theorem is the following:

Theorem 8.2.3 (Siegel's theorem, lower bound version). *Let χ be a primitive quadratic Dirichlet character modulo $q > 1$. Then there exists a positive constant $c_1(\varepsilon)$ such that*

$$L(1,\chi) \geq \frac{c_1(\varepsilon)}{q^\varepsilon}.$$

In the zero-free region version, Siegel's theorem takes the following form:

Theorem 8.2.4 (Siegel's theorem, zero-free region version). *Let χ be a primitive quadratic Dirichlet character modulo $q > 1$. Then there exists a positive constant $c_2(\varepsilon)$ such that $L(s,\chi)$ has no real zeros in the segment*

$$\sigma \geq 1 - \frac{c_2(\varepsilon)}{q^\varepsilon}.$$

The largest defect of Siegel's theorem, in either version, is that the implicit constants $c_1(\varepsilon)$ and $c_2(\varepsilon)$ are ineffective (and not necessarily equal). Actually, the lower bound result is slightly stronger as it implies the zero-free region result. We will first prove the zero-free region given the lower bound, and then we will prove the lower bound. Before we begin, we need two small lemmas about the size of $L'(\sigma,\chi)$ and $L(\sigma,\chi)$ for σ close to 1:

Lemma 8.2.1. *Let χ be a non-principal Dirichlet character modulo $m > 1$. Then $L'(\sigma,\chi) = O(\log^2(m))$ for any σ such that $0 \leq 1-\sigma \leq \frac{1}{\log(m)}$.*

Proof. Setting $A(X) = \sum_{n \leq X} \chi(n)$ we have $A(X) \ll 1$ by Corollary 1.3.1 (i) and that χ is periodic. Therefore $\sigma_c \leq 0$ by Proposition 6.1.1. Hence for σ in the prescribed region, $L(\sigma,\chi)$ is holomorphic and its derivative is given by

$$L'(\sigma,\chi) = \sum_{n \geq 1} \frac{\chi(n)\log(n)}{n^\sigma} = \sum_{n<m} \frac{\chi(n)\log(n)}{n^\sigma} + \sum_{n \geq m} \frac{\chi(n)\log(n)}{n^\sigma}.$$

We will show that the last two sums are both $O(\log^2(m))$. For the first sum, if $n < m$, we have

$$\left|\frac{\chi(n)\log(n)}{n^\sigma}\right| \leq \frac{1}{n^\sigma}\log(n) = \frac{n^{1-\sigma}}{n}\log(n) < \frac{m^{1-\sigma}}{n}\log(n) < \frac{e}{n}\log(m),$$

where the last inequality follows because $1-\sigma \leq \frac{1}{\log(m)}$. Then

$$\left|\sum_{n \leq m} \frac{\chi(n)\log(n)}{n^s}\right| < e\log(m)\sum_{n<m}\frac{1}{n} < e\log(m)\int_1^m \frac{1}{n}\,dn \ll \log^2(m).$$

For the second sum, $A(Y) \ll 1$ so that $A(Y)\log(Y)Y^{-\sigma} \to 0$ as $Y \to \infty$. Then Abel's summation formula (see Appendix B.3) gives

$$\sum_{n \geq m} \frac{\chi(n)\log(n)}{n^\sigma} = -A(m)\log(m)m^{-\sigma} - \int_m^\infty A(u)(1-\sigma\log(u))u^{-(\sigma+1)}\,du. \tag{8.3}$$

Since $0 \leq 1-\sigma \leq \frac{1}{\log(m)}$, we have $1-\sigma\log(u) \leq \frac{\log(u)}{\log(m)}$. Also, we have the more precise estimate $|A(X)| \leq m$ because χ is m-periodic and $|\chi(n)| \leq 1$. With these estimates and Equation (8.3) we make the following computation:

$$\begin{aligned}
\left|\sum_{n\geq m}\frac{\chi(n)\log(n)}{n^\sigma}\right| &\leq |A(m)|\log(m)m^{-\sigma} + \int_m^\infty |A(u)|(1-\sigma\log(u))u^{-(\sigma+1)}\,du \\
&\leq |A(m)|\log(m)m^{-\sigma} + \log(m)\int_m^\infty |A(u)|\log(u)u^{-(\sigma+1)}\,du \\
&\leq m^{1-\sigma}\log(m) + m\int_m^\infty \log(u)u^{-(\sigma+1)}\,du \\
&= m^{1-\sigma}\log(m) + m\left(-\log(u)\frac{u^{-\sigma}}{\sigma}\bigg|_m^\infty + \int_m^\infty \frac{u^{-(\sigma+1)}}{s}\,du\right) \\
&= m^{1-\sigma}\log(m) + m\left(-\log(u)\frac{u^{-\sigma}}{\sigma} - \frac{u^{-\sigma}}{\sigma^2}\right)\bigg|_m^\infty \\
&= m^{1-\sigma}\log(m) + m\left(\log(m)\frac{m^{-\sigma}}{\sigma} + \frac{m^{-\sigma}}{\sigma^2}\right) \\
&\ll m^{1-\sigma}\log(m) \\
&\ll e\log(m),
\end{aligned}$$

where in the fourth line we have used integration by parts and the last line holds because $1-\sigma \leq \frac{1}{\log(m)}$. But $e\log(m) = O(\log^2(m))$ so the second sum is also $O(\log^2(m))$. Therefore we have shown $L'(\sigma,\chi) = O(\log^2(m))$ finishing the proof. □

The second lemma is even easier and is proved in exactly the same way:

Lemma 8.2.2. *Let χ be a non-principal Dirichlet character modulo $m > 1$. Then $L(\sigma,\chi) = O(\log(m))$ for any σ such that $0 \leq 1-\sigma \leq \frac{1}{\log(m)}$.*

Proof. Note that

$$L(\sigma,\chi) = \sum_{n\geq 1}\frac{\chi(n)}{n^\sigma} = \sum_{n<m}\frac{\chi(n)}{n^\sigma} + \sum_{n\geq m}\frac{\chi(n)}{n^\sigma}.$$

It suffices to show that the last two sums are both $O(\log^2(m))$. For the first sum, since $n < m$, we have

$$\left|\frac{\chi(n)}{n^\sigma}\right| \leq \frac{1}{n^\sigma} = \frac{n^{1-\sigma}}{n} < \frac{m^{1-\sigma}}{n} < \frac{e}{n},$$

where the last inequality follows because $1-\sigma \leq \frac{1}{\log(m)}$. Therefore

$$\left|\sum_{n\leq m}\frac{\chi(n)}{n^s}\right| < e\sum_{n<m}\frac{1}{n} < e\log(m)\int_1^m \frac{1}{n}\,dn \ll \log(m).$$

As for the second sum, setting $A(Y) = \sum_{n\leq Y}\chi(n)$ we have $A(Y) \ll 1$ by Corollary 1.3.1 (i) and that χ is periodic. Thus $A(Y)Y^{-\sigma} \to 0$ as $Y \to \infty$. Then Abel's summation formula (see Appendix B.3) gives

$$\sum_{n\geq m}\frac{\chi(n)}{n^\sigma} = -A(m)m^{-\sigma} - \int_m^\infty A(u)u^{-(\sigma+1)}\,du. \tag{8.4}$$

Using the more precise estimate $|A(X)| \leq m$, because χ is m-periodic and $|\chi(n)| \leq 1$, with Equation (8.4), we make the following computation:

$$
\begin{aligned}
\left|\sum_{n \geq m} \frac{\chi(n)\log(n)}{n^\sigma}\right| &\leq |A(m)|m^{-\sigma} + \int_m^\infty |A(u)|u^{-(\sigma+1)}\,du \\
&\leq |A(m)|m^{-\sigma} + \int_m^\infty |A(u)|\log(u)u^{-(\sigma+1)}\,du \\
&\leq m^{1-\sigma} + m\int_m^\infty \log(u)u^{-(\sigma+1)}\,du \\
&= m^{1-\sigma} + m\left(-\log(u)\frac{u^{-\sigma}}{\sigma}\Big|_m^\infty + \int_m^\infty \frac{u^{-(\sigma+1)}}{s}\,du\right) \\
&= m^{1-\sigma} + m\left(-\log(u)\frac{u^{-\sigma}}{\sigma} - \frac{u^{-\sigma}}{\sigma^2}\right)\Big|_m^\infty \\
&= m^{1-\sigma} + m\left(\log(m)\frac{m^{-\sigma}}{\sigma} + \frac{m^{-\sigma}}{\sigma^2}\right) \\
&\ll m^{1-\sigma} \\
&\ll e,
\end{aligned}
$$

where in the fourth line we have used integration by parts and the last line holds because $1-\sigma \leq \frac{1}{\log(m)}$. But $e = O(\log^2(m))$ so the second sum is also $O(\log^2(m))$. Therefore we have shown $L(\sigma,\chi) = O(\log(m))$ which completes the proof. □

We will now prove the zero-free region version of Siegel's theorem, assuming the lower bound version, and using Lemma 8.2.1:

Proof of Siegel's theorem, zero-free region version. We we will prove the theorem by contradiction. Clearly the result holds for a single q, and notice that the result also holds provided we bound q from above by taking the maximum of all the $c_2(\varepsilon)$. Therefore we may suppose q is arbitrarily large. In this case, if there was a real zero β with $\beta \geq 1 - \frac{c_2(\varepsilon)}{q^\varepsilon}$, equivalently $1-\beta \leq \frac{c_2(\varepsilon)}{q^\varepsilon}$, then for large enough q we have $0 \leq 1-\beta \leq \frac{1}{\log(q)}$ so that $L'(\sigma,\chi) = O(\log^2(q))$ for $\beta \leq \sigma \leq 1$ by Lemma 8.2.1. These two estimates and the mean value theorem together give

$$L(1,\chi) = L(1,\chi) - L(\beta,\chi) = L'(\sigma,\chi)(1-\beta) \ll \frac{\log^2(q)}{q^\varepsilon}.$$

Upon taking $\frac{\varepsilon}{2}$ in the lower bound version of Siegel's theorem, we obtain

$$\frac{1}{q^{\frac{\varepsilon}{2}}} \ll L(1,\chi) \ll \frac{\log^2(q)}{q^\varepsilon},$$

which is a contradiction for large q. □

It remains to prove the lower bound version of Siegel's theorem. The idea is to combine two Dirichlet L-functions attached to distinct characters with distinct moduli and use this new L-function to derivative a lower bound for a single Dirichlet L-function at $s = 1$. We first need a lemma:

Lemma 8.2.3. *Let χ_1 and χ_2 be two quadratic Dirichlet characters and let $L(s,g)$ be the L-function defined by*

$$L(s,g) = \zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2).$$

Then $\Lambda_g(n) \geq 0$. In particular, $a_g(n) \geq 0$ and $a_g(0) = 1$.

Proof. For any prime p, the local roots at p are 1 with multiplicity one, $\chi_1(p)$ with multiplicity one, $\chi_2(p)$ with multiplicity one, and $\chi_1\chi_2(p)$ with multiplicity one. So for any $k \geq 1$, the sum of k-th powers of these local roots is

$$(1 + \chi_1^k(p))(1 + \chi_2^k(p)) \geq 0.$$

Thus $\Lambda_g(n) \geq 0$. It follows immediately from the Euler product of $L(s,g)$ that $a_g(n) \geq 0$ too. Also, it is clear from the Euler product of $L(s,g)$ that $a_g(0) = 1$. □

The key ingredient in the proof of the lower bound version of Siegel's theorem is an estimate for the L-function $L(s,g)$ in Lemma 8.2.3 relative to the modulus q_1q_2 in a small interval on the real axis close to 1. We now prove the theorem:

Proof of Siegel's theorem, lower bound version. Let χ_1 and χ_2 be two distinct primitive quadratic and non-principal characters modulo q_1 and q_2 respectively. Let $L(s,g)$ be the L-function defined by

$$L(s,g) = \zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2).$$

Observe that $L(s,g)$ is holomorphic on $\mathbb{C}$ except for a simple pole at $s = 1$. Let λ be the residue at this pole so that

$$\lambda = L(1,\chi_1)L(1,\chi_2)L(1,\chi_1\chi_2).$$

Now $L(s,g)$ is represented as an absolutely convergent series for $\sigma > 1$ so that it has a power series about $s = 2$ with radius 1:

$$L(s,g) = \sum_{m\geq 0} \frac{L^{(m)}(2,g)}{m!}(s-2)^m,$$

for $|s-2| < 1$ (we are abusing notation with $L^{(m)}$). We can compute $L^{(m)}(2,g)$ using the Dirichlet series by differentiating termwise:

$$L^{(m)}(2,g) = \frac{d^m}{ds^m}\left(\sum_{n\geq 1} \frac{a_g(n)}{n^s}\right)\Bigg|_{s=2} = (-1)^m \sum_{n\geq 1} \frac{a_g(n)\log^m(n)}{n^s}\Bigg|_{s=2} = (-1)^m \sum_{n\geq 1} \frac{a_g(n)\log^m(n)}{n^2}. \tag{8.5}$$

Since the $a_g(n)$ are nonnegative by Lemma 8.2.3, it follows that $L^{(m)}(2,g)$ is nonnegative and therefore we may write

$$L(s,g) = \sum_{m\geq 0} b_g(m)(2-s)^m,$$

for $|s-2| < 1$ and with $b_g(m)$ nonnegative. Also, Equation (8.5) and the fact that the $a_g(n)$ are nonnegative with $a_g(0) = 1$ together imply that $b_g(0) > 1$. Then

$$L(s,g) - \frac{\lambda}{s-1} = L(s,g) - \lambda\sum_{m\geq 0}(2-s)^m = \sum_{m\geq 0}(b_g(m) - \lambda)(2-s)^m, \tag{8.6}$$

and the last series must be absolutely convergent for say $|s-2| < 2$ because $L(s,g) - \frac{\lambda}{s-1}$ is holomorphic as we have removed the pole at $s = 1$. We now wish to estimate $L(s,g)$ and $\frac{\lambda}{s-1}$ on the circle $|s-2| = \frac{3}{2}$. Let

χ be a non-principal Dirichlet character modulo m and let $A(X) = \sum_{n \le X} \chi(n)$. Then Abel's summation formula and that $A(X) \ll 1$ (by Corollary 1.3.1 (i) and that χ is periodic) together imply

$$L(s,\chi) = s \int_1^\infty A(u) u^{-(s+1)}\, du,$$

for $\sigma > 0$. Now suppose $\sigma \ge \frac{1}{2}$. As $|A(X)| \le m$, we obtain

$$|L(s,\chi)| \le m|s| \int_1^\infty u^{-(\sigma+1)}\, du = -m|s| \frac{u^{-\sigma}}{\sigma}\bigg|_1^\infty = \frac{m|s|}{\sigma} \le 2m|s|.$$

In particular, on the disk $|s-2| \le \frac{3}{2}$ we have the estimates

$$L(s,\chi_1) \ll q_1, \quad L(s,\chi_2) \ll q_2, \quad \text{and} \quad L(s,\chi_1\chi_2) \ll q_1 q_2.$$

Since $\zeta(s)$ is bounded on the circle $|s-2| = \frac{3}{2}$ (it's a compact set) and $\lambda = L(1,\chi_1)L(1,\chi_2)L(1,\chi_1\chi_2)$, we obtain the bounds

$$L(s,g) \ll q_1^2 q_2^2 \quad \text{and} \quad \frac{\lambda}{s-1} \ll q_1^2 q_2^2,$$

on this circle as well. Cauchy's inequality for the size of coefficients of a power series applied to Equation (8.6) on the circle $|s-2| = \frac{3}{2}$ gives

$$b_g(m) - \lambda \ll q_1^2 q_2^2 \left(\frac{2}{3}\right)^m. \tag{8.7}$$

Let M be a positive integer. For real s with $\frac{7}{8} < s < 1$ we have $2 - s < \frac{9}{8}$ and using Equations (8.6) and (8.7) together we can upper bound the tail of $L(s,g) - \frac{\lambda}{s-1}$:

$$\begin{aligned}
\left|\sum_{m \ge M} (b_g(m) - \lambda)(2-s)^m\right| &\le \sum_{m \ge M} |b_g(m) - \lambda|(2-s)^m \\
&\ll q_1^2 q_2^2 \sum_{m \ge M} \left(\frac{2}{3}(2-s)\right)^m \\
&\ll q_1^2 q_2^2 \sum_{m \ge M} \left(\frac{3}{4}\right)^m \\
&\ll q_1^2 q_2^2 \left(\frac{3}{4}\right)^M \\
&\ll q_1^2 q_2^2 e^{-\frac{M}{4}},
\end{aligned}$$

where the last estimate follows because $(\frac{3}{4})^M < e^{-\frac{M}{4}}$ (which is equivalent to $\log\left(\frac{3}{4}\right) < -\frac{1}{4}$). Let c be the implicit constant. Using that the $b_g(m)$ are nonnegative, $b(0) > 1$, and the previous estimate for the tail, we can estimate $L(s,g) - \frac{\lambda}{s-1}$ from below for $\frac{7}{8} < s < 1$. Indeed, discarding the $b_g(m)$ terms for $1 \le m \le M$, bounding the constant term below by 1, and use the tail estimate, gives

$$L(s,g) - \frac{\lambda}{s-1} \ge 1 - \lambda \sum_{0 \le m \le M-1} (2-s)^m - c q_1^2 q_2^2 e^{-\frac{M}{4}} = 1 - \lambda \frac{(2-s)^M - 1}{1-s} - c q_1^2 q_2^2 e^{-\frac{M}{4}}, \tag{8.8}$$

which is valid for any positive integer M. Now chose M such that

$$\frac{1}{2}e^{-\frac{1}{4}} \leq cq_1^2q_2^2e^{-\frac{M}{4}} < \frac{1}{2}. \tag{8.9}$$

Upon isolating $L(s,g)$ in Equation (8.8) and using the second estimate in Equation (8.9), we get

$$L(s,g) \geq \frac{1}{2} - \lambda\frac{(2-s)^M}{1-s}. \tag{8.10}$$

Taking the logarithm of the first estimate in Equation (8.9) and isolating M, we obtain

$$M \leq 8\log(q_1q_2) + c, \tag{8.11}$$

for some different constant c. It follows that

$$(2-s)^M = e^{M\log(2-s)} < e^{M(1-s)} \leq c(q_1q_2)^{8(1-s)}, \tag{8.12}$$

for some different constant c, where in the first estimate we have used the Taylor series of the logarithm truncated at the first term and in the second estimate we have used Equation (8.11). Since $1-s$ is positive for $\frac{7}{8} < s < 1$, we can combine Equations (8.10) and (8.12) which gives

$$L(s,g) \geq \frac{1}{2} - \lambda\frac{c}{1-s}(q_1q_2)^{8(1-s)}. \tag{8.13}$$

This is our desired lower bound for $L(s,g)$. We will now choose the character χ_1. If there exists a Siegel zero β_1 with $1-\frac{\varepsilon}{16} < \beta_1 < 1$, let χ_1 be the character corresponding to the Dirichlet L-function that admits this Siegel zero. Then $L(\beta_1,g) = 0$ independent of the choice of χ_2. If there is no such Siegel zero, choose χ_1 to be any quadratic primitive character and β_1 to be any number such that $1-\frac{\varepsilon}{16} < \beta_1 < 1$. Then $L(\beta_1,g) < 0$ independent of the choice of χ_2. Indeed, $\zeta(s)$ is negative in this segment (actually for $0 \leq s < 1$) and each of the Dirichlet L-function defining $L(s,g)$ is positive at $s=1$ (the Euler product implies Dirichlet L-function are positive for $s>1$ and they are in fact nonzero for $s=1$ by Theorem 8.1.3) and do not admit a zero for $\beta_1 < s \leq 1$ by our choice of β_1. In either case, $L(\beta_1,g) \leq 0$ so isolating λ and disregarding the constants in Equation (8.13) with $s=\beta_1$ gives the weaker estimate

$$\lambda \gg \frac{1-\beta_1}{(q_1q_2)^{8(1-\beta_1)}}. \tag{8.14}$$

We will now choose $\chi_2 = \chi$ and hence $q_2 = q$ as in the statement of the theorem. Notice that, independent of any work we have done, the theorem holds for a single q. Moreover, the theorem holds provided we bound q from above by taking the minimum of the $c_1(\varepsilon)$. Therefore we may assume q is arbitrarily large and in particular that $q > q_1$. Using Lemma 8.2.2 with $\sigma = 1$ applied to $L(\sigma,\chi_1)$ and $L(\sigma,\chi_1\chi)$, we obtain

$$\lambda \ll \log(q_1)\log(q_1q)L(1,\chi). \tag{8.15}$$

Combining Equations (8.14) and (8.15) yields

$$\frac{1-\beta_1}{(q_1q)^{8(1-\beta_1)}} \ll \log(q_1)\log(q_1q)L(1,\chi).$$

As β_1 and q_1 are fixed and $\log(q_1q) = O(\log(q))$, isolating $L(1,\chi)$ gives the weaker estimate

$$\frac{1}{q^{8(1-\beta_1)}\log(q)} \ll L(1,\chi). \tag{8.16}$$

But $1-\frac{\varepsilon}{16}<\beta_1<1$ so that $0<8(1-\beta_1)<\frac{\varepsilon}{2}$ which combined with Equation (8.16) yields

$$\frac{1}{q^{\varepsilon}} \ll_{\varepsilon} \frac{1}{q^{\frac{\varepsilon}{2}}\log(q)} \ll L(1,\chi),$$

where the first estimate follows because $\log(q) \ll_{\varepsilon} q^{\frac{\varepsilon}{2}}$ for sufficiently large q. This is equivalent to the statement in the theorem. □

The part of the proof of the lower bound version of Siegel's theorem which makes $c_1(\varepsilon)$ (and hence $c_2(\varepsilon)$) ineffective is the value of β_1. The choice of β_1 depends upon the existence of a Siegel zero near 1 and relative to the given ε. Since we don't know if Siegel zeros exist, this makes estimating β_1 relative to ε ineffective. Many results in analytic number theory make use of Siegel's theorem and hence are also ineffective. Many important problems investigate methods to get around using Siegel's theorem in favor of a weaker result that is effective. So far, no Siegel zero has been shown to exist or not exist for Dirichlet L-functions. But some progress has been made to showing that they are rare:

Proposition 8.2.1. *Let χ_1 and χ_2 be two distinct quadratic Dirichlet characters of conductor q_1 and q_2. If $L(s,\chi_1)$ and $L(s,\chi_2)$ have Siegel zeros β_1 and β_2 respectively and $\chi_1\chi_2$ is not principal, then there exists a positive constant c such that*

$$\min(\beta_1,\beta_2) < 1-\frac{c}{\log(q_1q_2)}.$$

Proof. We may assume χ_1 and χ_2 are primitive since if $\widetilde{\chi_i}$ is the primitive character inducing χ_i, for $i=1,2$, the only difference in zeros between $L(s,\chi_i)$ and $L(s,\widetilde{\chi_i})$ occur on the line $\sigma=0$. Now let $\widetilde{\chi}$ be the primitive character of conductor q inducing $\chi_1\chi_2$. From Equation (7.12) with $\chi_1\chi_2$ in place of χ, we find that

$$\left|\frac{L'}{L}(s,\chi_1\chi_2)-\frac{L'}{L}(s,\widetilde{\chi})\right| = \left|\sum_{p\mid q_1q_2}\frac{\widetilde{\chi}(p)\log(p)p^{-s}}{1-\widetilde{\chi}(p)p^{-s}}\right| \le \sum_{p\mid q_1q_2}\frac{\log(p)p^{-\sigma}}{1-p^{-\sigma}} \le \sum_{p\mid q_1q_2}\log(p) \le \log(q_1q_2). \tag{8.17}$$

Let $s=\sigma$ with $1<\sigma\le 2$. Using the reverse triangle inequality, we deduce from Equation (8.17) that

$$-\frac{L'}{L}(\sigma,\chi_1\chi_2) < c\log(q_1q_2), \tag{8.18}$$

for some positive constant c. By Lemma 6.9.1 (iv) applied to $\zeta(s)$ while discarding all of the terms in both sums, we have

$$-\frac{\zeta'}{\zeta}(\sigma) < A+\frac{1}{\sigma-1}, \tag{8.19}$$

for some positive constant A. By Lemma 6.9.1 (iv) applied to $L(s,\chi_i)$ and only retaining the term corresponding to β_i, we have

$$-\frac{L'}{L}(\sigma,\chi_i) < A\log(q_i)+\frac{1}{\sigma-\beta_i}, \tag{8.20}$$

for $i=1,2$ and some possibly larger constant A. Now by Lemma 8.2.3, $-\frac{L'}{L}(\sigma,g)\ge 0$. Combining Equations (8.17) to (8.20) with this fact implies

$$0 < A+\frac{1}{\sigma-1}+A\log(q_1)-\frac{1}{\sigma-\beta_1}+A\log(q_2)-\frac{1}{\sigma-\beta_2}+c\log(q_1q_2).$$

Taking c larger, if necessary, we arrive at the simplified estimate

$$0 < \frac{1}{\sigma - 1} - \frac{1}{\sigma - \beta_1} - \frac{1}{\sigma - \beta_2} + c\log(q_1 q_2),$$

which we rewrite as

$$\frac{1}{\sigma - \beta_1} + \frac{1}{\sigma - \beta_2} < \frac{1}{\sigma - 1} + c\log(q_1 q_2),$$

Now let $\sigma = 1 + \frac{\delta}{\log(q_1 q_2)}$ for some $\delta > 0$. Upon substituting, we have

$$\frac{1}{\sigma - \beta_1} + \frac{1}{\sigma - \beta_2} < \left(c + \frac{1}{\delta}\right)\log(q_1 q_2).$$

If $\min(\beta_1, \beta_2) \geq 1 - \frac{c}{\log(q_1 q_2)}$, then we arrive at

$$2(\delta + c) < c + \frac{1}{\delta},$$

which is a contradiction if we take δ small enough so that $2\delta^2 + c\delta < 1$. □

From Proposition 8.2.1 we immediately see that for every modulus $m > 1$ there is at most one primitive quadratic Dirichlet character that can admit a Siegel zero:

Proposition 8.2.2. *For every integer $m > 1$, there is at most one Dirichlet character χ modulo m such that $L(s, \chi)$ has a Siegel zero. If this Siegel zero exists, χ is necessarily quadratic.*

Proof. Let $\widetilde{\chi}$ be the primitive character inducing χ. As the zeros of $L(s, \chi)$ and $L(s, \widetilde{\chi})$ differ only on the line $\sigma = 0$, Theorem 8.2.2 implies that χ must be quadratic. Suppose χ_1 and χ_2 are two distinct character modulo m, of conductors q_1 and q_2, admitting Siegel zeros β_1 and β_2. Then $\chi_1\chi_2 \neq \chi_{m,0}$. Moreover, $\beta_1 \geq 1 - \frac{c_1}{\log(q_1)}$ and $\beta_2 \geq 1 - \frac{c_2}{\log(q_2)}$ for some positive constants c_1 and c_2. Taking c smaller, if necessary, we have $\min(\beta_1, \beta_2) \geq 1 - \frac{c}{\log(q_1 q_2)}$ which contradicts Proposition 8.2.1. □

8.3 The Prime Number Theorem

The function $\psi(x)$ is defined by

$$\psi(x) = \sum_{n \leq x} \Lambda(n),$$

for $x > 0$. We will obtain an explicit formula for $\psi(x)$ analogous to the explicit formula for the Riemann zeta function. The explicit formula for $\psi(x)$ will be obtained by applying truncated Perron's formula to the logarithmic derivative of $\zeta(s)$. Since $\psi(x)$ is discontinuous when x is a prime power, we need to work with a slightly modified function to apply the Mellin inversion formula. Define $\psi_0(x)$ by

$$\psi_0(x) = \begin{cases} \psi(x) & \text{if } x \text{ is not a prime power,} \\ \psi(x) - \frac{1}{2}\Lambda(x) & \text{if } x \text{ is a prime power.} \end{cases}$$

Equivalently, $\psi_0(x)$ is $\psi(x)$ except that its value is halfway between the limit values when x is a prime power. Stated another way, if x is a prime power the last term in the sum for $\psi_0(x)$ is multiplied by $\frac{1}{2}$. The **explicit formula** for $\psi(x)$ is the following:

Theorem 8.3.1 (Explicit formula for $\psi(x)$). *For $x \geq 2$,*

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1 - x^{-2}),$$

where the sum is counted with multiplicity and ordered with respect to the size of the ordinate.

A few comments are in order before we prove the explicit formula for $\psi(x)$. First, since ρ is conjectured to be of the form $\rho = \frac{1}{2} + i\gamma$ via the Riemann hypothesis for the Riemann zeta function, x is conjectured to be the main term in the explicit formula. The constant $\frac{\zeta'}{\zeta}(0)$ can be shown to be $\log(2\pi)$ (see [Dav80] for a proof). Also, using the Taylor series of the logarithm, the last term can be expressed as

$$\frac{1}{2}\log(1 - x^{-2}) = \frac{1}{2}\sum_{m \geq 1}(-1)^{m-1}\frac{(-x^{-2})^m}{m} = \sum_{m \geq 1}(-1)^{2m-1}\frac{x^{-2m}}{2m} = \sum_{m \geq 1}\frac{x^{-2m}}{-2m} = \sum_{\omega}\frac{x^\omega}{\omega},$$

where ω runs over the trivial zeros of $\zeta(s)$. We will now prove the explicit formula for $\psi(x)$:

Proof of the explicit formula for $\psi(x)$. Applying truncated Perron's formula to $-\frac{\zeta'}{\zeta}(s)$ gives

$$\psi_0(x) - J(x,T) \ll x^c \sum_{\substack{n \geq 1 \\ n \neq x}} \frac{\Lambda(n)}{n^c} \min\left(1, \frac{1}{T\left|\log\left(\frac{x}{n}\right)\right|}\right) + \delta_x \Lambda(x)\frac{c}{T}, \tag{8.21}$$

where

$$J(x,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{\zeta'}{\zeta}(s)x^s \frac{ds}{s},$$

$c > 1$, and it is understood that $\delta_x = 0$ unless x is a prime power. Take $T > 2$ not coinciding with the ordinate of a nontrivial zero and let $c = 1 + \frac{1}{\log(x^2)}$ so that $x^c = \sqrt{e}x$ and $1 < c < 2$. The first step is to estimate the right-hand side of Equation (8.21). We deal with the terms corresponding to n such that n is bounded away from x before anything else. So suppose $n \leq \frac{3}{4}x$ or $n \geq \frac{5}{4}x$. For these n, $\log\left(\frac{x}{n}\right)$ is bounded away from zero so that their contribution is

$$\ll \frac{x^c}{T}\sum_{n \geq 1}\frac{\Lambda(n)}{n^c} \ll \frac{x^c}{T}\left(-\frac{\zeta'}{\zeta}(c)\right) \ll \frac{x\log(x)}{T}, \tag{8.22}$$

where the last estimate follows from Lemma 6.9.1 (iv) applied to $\zeta(s)$ while discarding all of the terms in both sums and our choice of c (in particular $\log(c) \ll \log(x)$). Now we deal with the terms n close to x. Consider those n for which $\frac{3}{4}x < n < x$. Let x_1 be the largest prime power less than x. We may also suppose $\frac{3}{4}x < x_1 < x$ since otherwise $\Lambda(n) = 0$ and these terms do not contribute anything. Moreover, $\frac{x^c}{n^c} \ll 1$. For the term $n = x_1$, we have

$$\log\left(\frac{x}{n}\right) = -\log\left(1 - \frac{x - x_1}{x}\right) \geq \frac{x - x_1}{x},$$

where we have obtained the inequality by using Taylor series of the logarithm truncated after the first term. The contribution of this term is then

$$\ll \Lambda(x_1)\min\left(1, \frac{x}{T(x - x_1)}\right) \ll \log(x)\min\left(1, \frac{x}{T(x - x_1)}\right). \tag{8.23}$$

For the other such n, we can write $n = x_1 - v$, where v is an integer satisfying $0 < v < \frac{1}{4}x$, so that

$$\log\left(\frac{x}{n}\right) \geq \log\left(\frac{x_1}{n}\right) = -\log\left(1 - \frac{v}{x_1}\right) \geq \frac{v}{x_1},$$

where we have obtained the latter inequality by using Taylor series of the logarithm truncated after the first term. The contribution for these n is then

$$\ll \sum_{0<v<\frac{1}{4}x} \Lambda(x_1 - v)\frac{x_1}{Tv} \ll \frac{x}{T}\sum_{0<v<\frac{1}{4}x} \frac{\Lambda(x_1 - v)}{v} \ll \frac{x\log(x)}{T}\sum_{0<v<\frac{1}{4}x} \frac{1}{v} \ll \frac{x\log^2(x)}{T}. \tag{8.24}$$

The contribution for those n for which $x < n < \frac{5}{4}x$ is handled in exactly the same way with x_1 being the least prime power larger than x. Let $\langle x\rangle$ be the distance between x and the nearest prime power other than x if x itself is a prime power. Combining Equations (8.23) and (8.24) with our previous comment, the contribution for those n with $\frac{3}{4}x < n < \frac{5}{4}x$ is

$$\ll \frac{x\log^2(x)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right). \tag{8.25}$$

Putting Equations (8.22) and (8.25) together and noticing that the error term in Equation (8.22) is absorbed by the second error term in Equation (8.25), we obtain

$$\psi_0(x) - J(x,T) \ll \frac{x\log^2(x)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right). \tag{8.26}$$

This is the first part of the proof. Now we estimate $J(x,T)$ by appealing to the residue theorem. Let $U \geq 1$ be an odd integer. Let Ω be the region enclosed by the contours $\eta_1, \ldots, \eta_4$ in Figure 8.1 and set $\eta = \sum_{1\leq i\leq 4} \eta_i$ so that $\eta = \partial\Omega$.

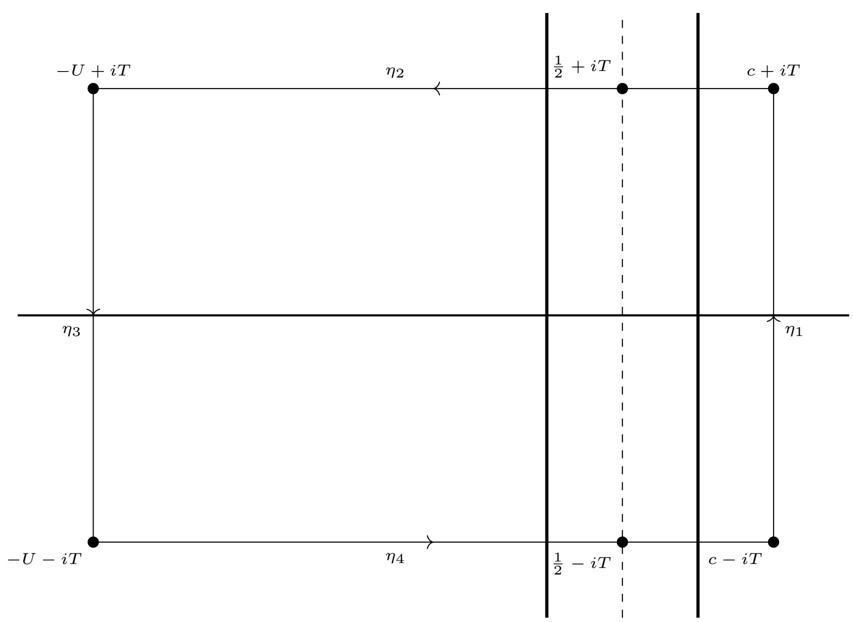

Figure 8.1: Contour for the explicit formula for $\psi(x)$

We may express $J(x,T)$ as

$$J(x,T) = \frac{1}{2\pi i}\int_{\eta_1} -\frac{\zeta'}{\zeta}(s)x^s\,\frac{ds}{s}.$$

The residue theorem together with the formula for the negative logarithmic derivative in Proposition 6.8.1 applied to $\zeta(s)$ and Corollary 1.6.1 imply

$$J(x,T) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \sum_{0<2m<U} \frac{x^{-2m}}{-2m} + \frac{1}{2\pi i}\int_{\eta_2+\eta_3+\eta_4} -\frac{\zeta'}{\zeta}(s)x^s\,\frac{ds}{s}, \tag{8.27}$$

where $\rho = \beta + i\gamma$ is a nontrivial zero of ζ. We will estimate $J(x,T)$ by estimating the remaining integral. By Lemma 6.9.1 (ii) applied to $\zeta(s)$, the number of nontrivial zeros satisfying $|\gamma - T| < 1$ is $O(\log(T))$. Among the ordinates of these nontrivial zeros, there must be a gap of size $\gg \frac{1}{\log(T)}$. Upon varying T by a bounded amount (we are varying in the interval $[T-1, T+1]$) so that it belongs to this gap, we can additionally ensure

$$\gamma - T \gg \frac{1}{\log(T)},$$

for all the nontrivial zeros of $\zeta(s)$. To estimate part of the horizontal integrals over η_2 and η_4, Lemma 6.9.1 (iv) applied to $\zeta(s)$ gives

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma-T|<1} \frac{1}{s-\rho} + O(\log(T)),$$

on the parts of these segments with $-1 \le \sigma \le 2$. By our choice of T, $|s-\rho| \ge |\gamma - T| \gg \frac{1}{\log(T)}$ so that each term in the sum is $O(\log(T))$. There are at most $O(\log(T))$ such terms by Lemma 6.9.1 (ii) applied to $\zeta(s)$, so we find that

$$\frac{\zeta'}{\zeta}(s) = O(\log^2(T)),$$

on the parts of these segments with $-1 \le \sigma \le 2$. It follows that the parts of the horizontal integrals over η_2 and η_4 with $-1 \le \sigma \le c$ (recall $c < 2$) contribute

$$\ll \frac{\log^2(T)}{T}\int_{-1}^{c} x^\sigma\,d\sigma \ll \frac{\log^2(T)}{T}\int_{-\infty}^{c} x^\sigma\,d\sigma \ll \frac{x\log^2(T)}{T\log(x)}, \tag{8.28}$$

where in the last estimate we have used the choice of c. To estimate the remainder of the horizontal integrals, we need a bound for $\frac{\zeta'}{\zeta}(s)$ when $\sigma < -1$ and away from the trivial zeros. To this end, write the functional equation for $\zeta(s)$ in the form

$$\zeta(s) = \pi^{s-1}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(1-s),$$

and take the logarithmic derivative to get

$$\frac{\zeta'}{\zeta}(s) = \log(\pi) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{\zeta'}{\zeta}(1-s).$$

Let s be such that $\sigma < -1$ and suppose s is distance $\frac{1}{2}$ away from the trivial zeros. We will estimate every term on the right-hand side of the previous identity. The first term is constant and the last term is bounded since it is an absolutely convergent Dirichlet series. As for the digamma terms, since s is away from the trivial zeros, Proposition 1.6.3 implies $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) = O(\log|1-s|)$ and $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) = O(\log|s|)$. However, as $\sigma < -1$ and s is away from the trivial zeros, s and $1-s$ are bounded away from zero so that $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) = O(\log|s|)$. Putting these estimates together gives

$$\frac{\zeta'}{\zeta}(s) \ll \log(|s|), \tag{8.29}$$

for $\sigma < -1$. Using Equation (8.29), the parts of the horizontal integrals over η_2 and η_4 with $-U \leq \sigma \leq -1$ contribute

$$\ll \frac{\log(T)}{T} \int_{-U}^{-1} x^\sigma \, d\sigma \ll \frac{\log(T)}{Tx\log(x)}. \tag{8.30}$$

Combining Equations (8.28) and (8.30) gives

$$\frac{1}{2\pi i} \int_{\eta_2+\eta_4} -\frac{\zeta'}{\zeta}(s)x^s \frac{ds}{s} \ll \frac{x\log^2(T)}{T\log(x)} + \frac{\log(T)}{Tx\log(x)} \ll \frac{x\log^2(T)}{T\log(x)}. \tag{8.31}$$

To estimate the vertical integral, we use Equation (8.29) again to conclude that

$$\frac{1}{2\pi i} \int_{\eta_3} -\frac{\zeta'}{\zeta}(s)x^s \frac{ds}{s} \ll \frac{\log(U)}{U} \int_{-T}^{T} x^{-U} \, dt \ll \frac{T\log(U)}{Ux^U}. \tag{8.32}$$

Combining Equations (8.27), (8.31) and (8.32) and taking the limit as $U \to \infty$, the error term in Equation (8.32) vanishes and the sum over m in Equation (8.27) evaluates to $\frac{1}{2}\log(1-x^{-2})$ (as we have already mentioned) giving

$$J(x,T) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1-x^{-2}) + \frac{x\log^2(T)}{T\log(x)}. \tag{8.33}$$

Substituting Equation (8.33) into Equation (8.26), we at last obtain

$$\psi_0(x) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1-x^{-2}) + \frac{x\log^2(xT)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right), \tag{8.34}$$

where the second to last term on the right-hand side is obtained by combining the error term in Equation (8.31) with the first error term in Equation (8.26). The theorem follows by taking the limit as $T \to \infty$. $\square$

Note that the convergence of the right-hand side in the explicit formula for $\psi(x)$ is uniform in any interval not containing a prime power since $\psi(x)$ is continuous there. Moreover, we have an approximate formula for $\psi(x)$ as a corollary:

Corollary 8.3.1. *For $x \geq 2$ and $T > 2$,*

$$\psi_0(x) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} + R(x,T),$$

where ρ runs over the nontrivial zeros of $\zeta(s)$ counted with multiplicity and ordered with respect to the size of the ordinate, and

$$R(x,T) \ll \frac{x\log^2(xT)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right),$$

where $\langle x\rangle$ is the distance between x and the nearest prime power other than x if x itself is a prime power. Moreover, if x is an integer, we have the simplified estimate

$$R(x,T) \ll \frac{x\log^2(xT)}{T}.$$

Proof. This follows from Equation (8.34) since $\frac{\zeta'}{\zeta}(0)$ is constant and $\frac{1}{2}\log(1-x^2)$ is bounded for $x \geq 2$. If x is an integer, then $\langle x \rangle \geq 1$ so that $\log(x)\min\left(1, \frac{x}{T\langle x \rangle}\right) \leq \frac{x\log(x)}{T}$ and this term can be absorbed into $O\left(\frac{x\log^2(xT)}{T}\right)$. □

With our refined explicit formula in hand, we are ready to discuss and prove the prime number theorem. The **prime counting function** $\pi(x)$ is defined by

$$\pi(x) = \sum_{p \leq x} 1,$$

for $x > 0$. Equivalently, $\pi(x)$ counts the number of primes that no larger than x. Euclid's infinitude of the primes is equivalent to $\pi(x) \to \infty$ as $x \to \infty$. A more interesting question is to ask how the primes are distributed among the integers. The classical **prime number theorem** answers this question and the precise statement is the following:

Theorem 8.3.2 (Prime number theorem, classical version). *For $x \geq 2$,*

$$\pi(x) \sim \frac{x}{\log(x)}.$$

We will delay the proof for the moment and give some intuition and historical context to the result. Intuitively, the prime number theorem is a result about how dense the primes are in the integers. To see this, notice that the result is equivalent to the asymptotic

$$\frac{\pi(x)}{x} \sim \frac{1}{\log(x)}.$$

Letting $x \geq 2$, the left-hand side is the probability that a randomly chosen positive integer no larger than x is prime. Thus the asymptotic result says that for large enough x, the probability that a randomly chosen integer no larger than x is prime is approximately $\frac{1}{\log(x)}$. We can also interpret this as saying that the average gap between primes no larger than x is approximately $\frac{1}{\log(x)}$. As a consequence, a positive integer with at most $2n$ digits is about half as likely to be prime than a positive integer with at most n digits. Indeed, there are $10^n - 1$ numbers with at most n digits, $10^{2n} - 1$ with at most $2n$ digits, and $\log(10^{2n} - 1)$ is approximately $2\log(10^n)$. Note that the prime number theorem says nothing about the exact error $\pi(x) - \frac{x}{\log(x)}$ as $x \to \infty$. The theorem only says that the relative error tends to zero:

$$\lim_{x \to \infty} \frac{\pi(x) - \frac{x}{\log(x)}}{\frac{x}{\log(x)}} = 0.$$

Now for some historical context. While Gauss was not the first to put forth a conjectural form of the prime number theorem, he was known for compiling extensive tables of primes and he suspected that the density of the primes up to x was roughly $\frac{1}{\log(x)}$. How might one suspect this is the correct density? Well, let $d\delta_p$ be the weighted point measure that assigns $\frac{1}{p}$ at the prime p and zero everywhere else. Then

$$\sum_{p \leq x} \frac{1}{p} = \int_1^x d\delta_p(u).$$

We can interpret the integral as integrating the density $d\delta_p$ over the volume $[1, x]$. Let's try and find a more explicit expression for the density $d\delta_p$. Euler (see [Eul44]), argued

$$\sum_{p \leq x} \frac{1}{p} \sim \log\log(x).$$

But notice that

$$\log\log(x) = \int_1^{\log(x)} \frac{du}{u} = \int_e^x \frac{1}{u} \frac{du}{\log u},$$

where in the second equality we have made the change of variables $u \to \log(u)$. So altogether,

$$\sum_{p \leq x} \frac{1}{p} \sim \int_e^x \frac{1}{u} \frac{du}{\log u}.$$

This is an asymptotic formula that gives a more explicit representation of the density $d\delta_p$. Notice that both sides of this asymptotic are weighted the same, the left-hand side by $\frac{1}{p}$, and the right-hand side by $\frac{1}{u}$. If we remove these weight (this is not strictly allowed), then we might hope

$$\pi(x) = \sum_{p \leq x} 1 \sim \int_e^x \frac{du}{\log(u)}.$$

Accordingly, we define the **logarithmic integral** $\mathrm{Li}(x)$ by

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)},$$

for $x \geq 2$. Notice that $\mathrm{Li}(x) \sim \frac{x}{\log x}$ because

$$\lim_{x \to \infty} \left| \frac{\mathrm{Li}(x)}{\frac{x}{\log x}} \right| = \lim_{x \to \infty} \left| \frac{\int_2^x \frac{dt}{\log(t)}}{\frac{x}{\log x}} \right| = \lim_{x \to \infty} \left| \frac{\frac{1}{\log(x)}}{\frac{\log(x)-1}{\log^2(x)}} \right| = \lim_{x \to \infty} \left| \frac{\log(x)}{\log(x) - 1} \right| = 1.$$

where in the second equality we have used L'Hôpital's rule. So an equivalent statement is the logarithmic integral version of the **prime number theorem**:

Theorem 8.3.3 (Prime number theorem, logarithmic integral version). *For $x \geq 2$,*

$$\pi(x) \sim \mathrm{Li}(x).$$

Interpreting the logarithmic integral as an integral of density over volume, then for large x the density of primes up to x is approximately $\frac{1}{\log(x)}$ which is what both versions of the prime number theorem claim. Legendre was the first to put forth a conjectural form of the prime number theorem. In 1798 (see [Leg98]) he claimed that $\pi(x)$ was of the form

$$\frac{x}{A\log(x) + B},$$

for some constants A and B. In 1808 (see [Leg08]) he refined his conjecture by claiming

$$\frac{x}{\log(x) + A(x)},$$

where $\lim_{x\to\infty} A(x) \approx 1.08366$. Riemann's 1859 manuscript (see [Rie59]) contains an outline for how to prove the prime number theorem, but it was not until 1896 that the prime number theorem was proved independently by Hadamard and de la Vallée Poussin (see [Had96] and [Pou97]). Their proofs, as well as every proof thereon out until 1949, used complex analytic methods in an essential way (there are now elementary proofs due to Erdös and Selberg). We are now ready to prove the prime number theorem. Strictly speaking, we will prove the absolute error version of the **prime number theorem**, due to de la Vallée Poussin, which bounds the absolute error between $\pi(x)$ and $\mathrm{Li}(x)$:

Theorem 8.3.4 (Prime number theorem, absolute error version). *For $x \ge 2$, there exists a positive constant c such that*

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-c\sqrt{\log(x)}}\right).$$

Proof. It suffices to assume x is an integer, because $\pi(x)$ can only change value at integers and the other functions in the statement are increasing. We will first prove

$$\psi(x) = x + O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{8.35}$$

for some positive constant c. To achieve this, we estimate the sum over the nontrivial zeros of $\zeta(s)$ in Corollary 8.3.1. So let $T > 2$ not coinciding with the ordinate of a nontrivial zero, and suppose $\rho = \beta + i\gamma$ is a nontrivial zero with $|\gamma| < T$. By Theorem 8.2.1, we know $\beta < 1 - \frac{c}{\log(T)}$ for some positive constant c. It follows that

$$|x^\rho| = x^\beta < x^{1-\frac{c}{\log(T)}} = xe^{-c\frac{\log(x)}{\log(T)}}. \tag{8.36}$$

As $|\rho| > |\gamma|$, letting $\gamma_1 > 0$ (which is bounded away from zero since the zeros of $\zeta(s)$ are discrete and we know that there is no real nontrivial zero) be the ordinate of the first nontrivial zero, applying integration by parts gives

$$\sum_{|\gamma|<T} \frac{1}{\rho} \ll \sum_{\gamma_1 \le |\gamma| < T} \frac{1}{\gamma} \ll \int_{\gamma_1}^{T} \frac{dN(t)}{t} = \frac{N(T)}{T} + \int_{\gamma_1}^{T} \frac{N(t)}{t^2}\, dt \ll \log^2(T), \tag{8.37}$$

where in the last estimate we have used that $N(t) \ll t\log(t)$ which follows from Corollary 6.9.1. Putting Equations (8.36) and (8.37) together gives

$$\sum_{|\gamma|<T} \frac{x^\rho}{\rho} \ll x\log^2(T)e^{-c\frac{\log(x)}{\log(T)}}. \tag{8.38}$$

As $\psi(x) \sim \psi_0(x)$ and x is an integer, Equation (8.38) with Corollary 8.3.1 together imply

$$\psi(x) - x \ll x\log^2(T)e^{-c\frac{\log(x)}{\log(T)}} + \frac{x\log^2(xT)}{T}. \tag{8.39}$$

We will now let T be determined by

$$\log^2(T) = \log(x),$$

or equivalently,

$$T = e^{\sqrt{\log(x)}}.$$

With this choice of T (note that if $x \ge 2$ then $T > 2$), we can estimate Equation (8.39) as follows:

$$\begin{aligned}
\psi(x) - x &\ll x\log(x)e^{-c\sqrt{\log(x)}} + x\left(\log^2(x) + \log(x)\right)e^{-\sqrt{\log(x)}} \\
&\ll x\log(x)e^{-c\sqrt{\log(x)}} + x\log^2(x)e^{-\sqrt{\log(x)}} \\
&\ll x\log^2(x)e^{-\min(1,c)\sqrt{\log(x)}}.
\end{aligned}$$

As $\log(x) \ll_\varepsilon e^{-\varepsilon\sqrt{\log(x)}}$, we conclude that

$$\psi(x) - x \ll xe^{-c\sqrt{\log(x)}},$$

for some smaller c with $c < 1$. This is equivalent to Equation (8.35). Now let

$$\pi_1(x) = \sum_{n \le x} \frac{\Lambda(n)}{\log(n)}.$$

We can write $\pi_1(x)$ in terms of $\psi(x)$ as follows:

$$\begin{aligned}
\pi_1(x) &= \sum_{n \le x} \frac{\Lambda(n)}{\log(n)} \\
&= \sum_{n \le x} \Lambda(n) \int_n^x \frac{dt}{t\log^2(t)} + \frac{1}{\log(x)} \sum_{n \le x} \Lambda(n) \\
&= \int_2^x \sum_{n \le t} \Lambda(n) \frac{dt}{t\log^2(t)} + \frac{1}{\log(x)} \sum_{n \le x} \Lambda(n) \\
&= \int_2^x \frac{\psi(t)}{t\log^2(t)}\, dt + \frac{\psi(x)}{\log(x)}.
\end{aligned}$$

Applying Equation (8.35) to the last expression yields

$$\pi_1(x) = \int_2^x \frac{t}{t\log^2(t)}\, dt + \frac{x}{\log(x)} + O\left(\int_2^x \frac{e^{-c\sqrt{\log(t)}}}{\log^2(t)}\, dt + \frac{xe^{-c\sqrt{\log(x)}}}{\log(x)}\right). \tag{8.40}$$

Upon applying integrating by parts to the main term in Equation (8.40), we obtain

$$\int_2^x \frac{t}{t\log^2(t)}\, dt + \frac{x}{\log(x)} = \int_2^x \frac{dt}{\log(t)} + \frac{2}{\log(2)} = \mathrm{Li}(x) + \frac{2}{\log(2)}. \tag{8.41}$$

As for the error term in Equation (8.40), $\log^2(t)$ and $\log(x)$ are both bounded away from zero so that

$$\int_2^x \frac{e^{-c\sqrt{\log(t)}}}{\log^2(t)}\, dt + \frac{xe^{-c\sqrt{\log(x)}}}{\log(x)} \ll \int_2^x e^{-c\sqrt{\log(t)}}\, dt + xe^{-c\sqrt{\log(x)}}.$$

For $t \le x^{\frac{1}{4}}$, we use the bound $e^{-c\sqrt{\log(t)}} < 1$ so that

$$\int_2^{x^{\frac{1}{4}}} e^{-c\sqrt{\log(t)}}\, dt < \int_2^{x^{\frac{1}{4}}} dt \ll x^{\frac{1}{4}}.$$

For $t > x^{\frac{1}{4}}$, $\sqrt{\log(t)} > \frac{1}{2}\sqrt{\log(x)}$ and thus

$$\int_2^{x^{\frac{1}{4}}} e^{-c\sqrt{\log(t)}}\, dt \le e^{-c\frac{1}{2}\sqrt{\log(x)}} \int_2^{x^{\frac{1}{4}}} dt \ll x^{\frac{1}{4}} e^{-c\frac{1}{2}\sqrt{\log(x)}}.$$

All of these estimates together imply

$$\int_2^x \frac{e^{-c\sqrt{\log(t)}}}{\log^2(t)}\, dt + \frac{xe^{-c\sqrt{\log(x)}}}{\log(x)} \ll xe^{-c\sqrt{\log(x)}}, \tag{8.42}$$

for some smaller c. Combining Equations (8.40) to (8.42) yields

$$\pi_1(x) = \mathrm{Li}(x) + O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{8.43}$$

where the constant in Equation (8.41) has been absorbed into the error term. We now pass from $\pi_1(x)$ to $\pi(x)$. If p is a prime such that $p^m < x$ for some $m \geq 1$, then $p < x^{\frac{1}{2}} < x^{\frac{1}{3}} < \cdots < x^{\frac{1}{m}}$. Therefore

$$\pi_1(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\log(n)} = \sum_{p^m \leq x} \frac{\log(p)}{m \log(p)} = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \cdots. \tag{8.44}$$

Moreover, as $\pi(x^{\frac{1}{n}}) < x^{\frac{1}{n}}$ for any $n \geq 1$, we see that $\pi(x) - \pi_1(x) = O(x^{\frac{1}{2}})$. This estimate together with Equations (8.43) and (8.44) gives

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-c\sqrt{\log(x)}}\right),$$

because $x^{\frac{1}{2}} \ll xe^{-c\sqrt{\log(x)}}$. This completes the proof. □

The proof of the logarithmic integral and classical versions of the prime number theorem are immediate:

Proof of prime number theorem, logarithmic integral and classical versions. By the absolute error version of the prime number theorem,

$$\pi(x) = \mathrm{Li}(x)\left(1 + O\left(\frac{xe^{-c\sqrt{\log(x)}}}{\mathrm{Li}(x)}\right)\right).$$

But we have shown $\mathrm{Li}(x) \sim \frac{x}{\log(x)}$ so that

$$\frac{xe^{-c\sqrt{\log(x)}}}{\mathrm{Li}(x)} \sim \log(x)e^{-c\sqrt{\log(x)}} = o(1),$$

where the equality holds since $\log(x) \ll_\varepsilon e^{-\varepsilon\sqrt{\log(x)}}$. The logarithm integral version follows. The classical version also holds using the asymptotic $\mathrm{Li}(x) \sim \frac{x}{\log(x)}$. □

In the proof of the logarithmic integral and classical versions of the prime number theorem, we saw that $xe^{-c\sqrt{\log(x)}} < \frac{x}{\log(x)}$ for sufficiently large x. Therefore the exact error $\pi(x) - \mathrm{Li}(x)$ grows slower than $\pi(x) - \frac{x}{\log x}$ for sufficiently large x. This means that $\mathrm{Li}(x)$ is a better numerical approximation to $\pi(x)$ than $\frac{x}{\log(x)}$. There is also the following result due to Hardy and Littlewood (see [HL16]) which gives us more information:

Proposition 8.3.1. *$\pi(x) - \mathrm{Li}(x)$ changes sign infinitely often as $x \to \infty$.*

So in addition, Proposition 8.3.1 implies that $\mathrm{Li}(x)$ never underestimates or overestimates $\pi(x)$ continuously. On the other hand, the exact error $\pi(x) - \frac{x}{\log(x)}$ is positive provided $x \geq 17$ (see [RS62]). It is also worthwhile to note that in 1901 Koch showed that the Riemann hypothesis improves the error term in the absolute error version of the prime number theorem (see [Koc01]):

Proposition 8.3.2. *For $x \geq 2$, we have*

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log(x)),$$

provided the Riemann hypothesis for the Riemann zeta function holds.

Proof. If ρ is a nontrivial zero of $\zeta(s)$, the Riemann hypothesis implies $|x^\rho| = \sqrt{x}$. Therefore as in the proof of the absolute error version of the prime number theorem,

$$\sum_{|\gamma|<T} \frac{x^\rho}{\rho} \ll \sqrt{x}\log^2(T),$$

for $T > 2$ not coinciding with the ordinate of a nontrivial zero. Repeating the same argument with T determined by

$$T^2 = x,$$

gives

$$\psi(x) = x + O(\sqrt{x}\log^2(x)),$$

and then transferring to $\pi_1(x)$ and finally $\pi(x)$ gives

$$\pi(x) = x + O(\sqrt{x}\log(x)).$$

□

8.4 The Siegel-Walfisz Theorem

Let χ be a Dirichlet character modulo $m > 1$. The function $\psi(x, \chi)$ is defined by

$$\psi(x, \chi) = \sum_{n \le x} \chi(n)\Lambda(n),$$

for $x > 0$. This function plays the analogous role of $\psi(x)$ but for Dirichlet L-functions. Accordingly, we will derive an explicit formula for $\psi(x, \chi)$ in a similar manner to that of $\psi(x)$. Because $\psi(x, \chi)$ is discontinuous when x is a prime power, we also introduce a slightly modified function. Define $\psi_0(x, \chi)$ by

$$\psi_0(x, \chi) = \begin{cases} \psi(x, \chi) & \text{if } x \text{ is not a prime power,} \\ \psi(x, \chi) - \frac{1}{2}\chi(x)\Lambda(x) & \text{if } x \text{ is a prime power.} \end{cases}$$

Thus $\psi_0(x, \chi)$ is $\psi(x, \chi)$ except that its value is halfway between the limit values when x is a prime power. We will also need to define a particular constant that will come up. For a character χ, define $b(\chi)$ to be $\frac{L'}{L}(0, \chi)$ if χ is odd and to be the constant term in the Laurent series of $\frac{L'}{L}(s, \chi)$ if χ is even (as in the even case $\frac{L'}{L}(s, \chi)$ has a pole at $s = 0$). The **explicit formula** for $\psi(x, \chi)$ is the following:

Theorem 8.4.1 (Explicit formula for $\psi(x, \chi)$). *Let χ be a primitive Dirichlet character of conductor $q > 1$. Then for $x \ge 2$,*

$$\psi_0(x, \chi) = -\sum_{\rho} \frac{x^\rho}{\rho} - b(\chi) + \tanh^{-1}(x^{-1}),$$

if χ is odd, and

$$\psi_0(x, \chi) = -\sum_{\rho} \frac{x^\rho}{\rho} - \log(x) - b(\chi) - \frac{1}{2}\log(1 - x^{-2}),$$

if χ is even, and where in both expressions, ρ runs over the nontrivial zeros of $L(s, \chi)$ counted with multiplicity and ordered with respect to the size of the ordinate.

As for $\psi(x)$, a few comments are in order. Unlike the explicit formula for $\psi(x)$, there is no main term x in the explicit formula for $\psi(x,\chi)$. This is because $L(s,\chi)$ does not have a pole at $s=1$. The constant $b(\chi)$ can be expressed in terms of $B(\chi)$ (see [Dav80] for a proof). Also, in the case χ is odd the Taylor series of the inverse hyperbolic tangent lets us write

$$\tanh^{-1}(x^{-1}) = \sum_{m\geq 1}\frac{x^{-(2m-1)}}{2m-1} = -\sum_{m\geq 1}\frac{x^{-(2m-1)}}{-(2m-1)} = -\sum_{\omega}\frac{x^{\omega}}{\omega},$$

where ω runs over the trivial zeros of $L(s,\chi)$. In the case χ is even, $\frac{1}{2}\log(1-x^{-2})$ accounts for the contribution of the trivial zeros just as for $\zeta(s)$. We will now prove the explicit formula for $\psi(x,\chi)$:

Proof of the explicit formula for $\psi(x,\chi)$. By truncated Perron's formula applied to $-\frac{L'}{L}(s,\chi)$, we get

$$\psi_0(x,\chi) - J(x,T,\chi) \ll x^c\sum_{\substack{n\geq 1\\ n\neq x}}\frac{\chi(n)\Lambda(n)}{n^c}\min\left(1,\frac{1}{T\left|\log\left(\frac{x}{n}\right)\right|}\right) + \delta_x\chi(x)\Lambda(x)\frac{c}{T}, \tag{8.45}$$

where

$$J(x,T,\chi) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{L'}{L}(s,\chi)x^s\,\frac{ds}{s},$$

$c>1$, and it is understood that $\delta_x = 0$ unless x is a prime power. Take $T>2$ not coinciding with the ordinate of a nontrivial zero and let $c = 1+\frac{1}{\log(x^2)}$ so that $x^c = \sqrt{e}x$. We will estimate the right-hand side of Equation (8.45). First, we estimate the terms corresponding to n such that n is bounded away from x. So suppose $n\leq\frac{3}{4}x$ or $n\geq\frac{5}{4}x$. For these n, $\log\left(\frac{x}{n}\right)$ is bounded away from zero so that their contribution is

$$\ll\frac{x^c}{T}\sum_{n\geq 1}\frac{\chi(n)\Lambda(n)}{n^c} \ll \frac{x^c}{T}\left(-\frac{L'}{L}(c,\chi)\right) \ll \frac{x\log(x)}{T}, \tag{8.46}$$

where the last estimate follows from Lemma 6.9.1 (iv) applied to $L(s,\chi)$ while discarding all of the terms in both sums and our choice of c (in particular $\log(c)\ll\log(x)$). Now we estimate the terms n close to x. So consider those n for which $\frac{3}{4}x<n<x$ and let x_1 be the largest prime power less than x. We may also assume $\frac{3}{4}x<x_1<x$ since otherwise $\Lambda(n)=0$ and these terms do not contribute anything. Moreover, $\frac{x^c}{n^c}\ll 1$. For the term $n=x_1$, we have the estimate

$$\log\left(\frac{x}{n}\right) = -\log\left(1-\frac{x-x_1}{x}\right) \geq \frac{x-x_1}{x},$$

where we have obtained the inequality by using Taylor series of the logarithm truncated after the first term. The contribution of this term is

$$\ll\chi(x_1)\Lambda(x_1)\min\left(1,\frac{x}{T(x-x_1)}\right) \ll \log(x)\min\left(1,\frac{x}{T(x-x_1)}\right). \tag{8.47}$$

For the other n, we write $n = x_1 - v$, where v is an integer satisfying $0<v<\frac{1}{4}x$, so that

$$\log\left(\frac{x}{n}\right) \geq \log\left(\frac{x_1}{n}\right) = -\log\left(1-\frac{v}{x_1}\right) \geq \frac{v}{x_1},$$

where we have obtained the latter inequality by using Taylor series of the logarithm truncated after the first term. The contribution for these n is

$$\ll\sum_{0<v<\frac{1}{4}x}\chi(x_1-v)\Lambda(x_1-v)\frac{x_1}{Tv} \ll \frac{x}{T}\sum_{0<v<\frac{1}{4}x}\frac{\Lambda(x_1-v)}{v} \ll \frac{x\log(x)}{T}\sum_{0<v<\frac{1}{4}x}\frac{1}{v} \ll \frac{x\log^2(x)}{T}. \tag{8.48}$$

The contribution for those n for which $x < n < \frac{5}{4}x$ is handled in exactly the same way with x_1 being the least prime power larger than x. Let $\langle x \rangle$ be the distance between x and the nearest prime power other than x if x itself is a prime power. Combining Equations (8.47) and (8.48) with our previous comment, the contribution for those n with $\frac{3}{4}x < n < \frac{5}{4}x$ is

$$\ll \frac{x\log^2(x)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x \rangle}\right). \tag{8.49}$$

Putting Equations (8.46) and (8.49), the error term in Equation (8.46) is absorbed by the second error term in Equation (8.49) and we obtain

$$\psi_0(x,\chi) - J(x,T,\chi) \ll \frac{x\log^2(x)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x \rangle}\right). \tag{8.50}$$

Now we estimate $J(x,T,\chi)$ by using the residue theorem. Let $U \geq 1$ be an integer with U is even if χ is odd and odd if χ is even. Let Ω be the region enclosed by the contours $\eta_1, \ldots, \eta_4$ in Figure 8.2 and set $\eta = \sum_{1 \leq i \leq 4} \eta_i$ so that $\eta = \partial\Omega$. We may write $J(x,T,\chi)$ as

$$J(x,T,\chi) = \frac{1}{2\pi i}\int_{\eta_1} -\frac{L'}{L}(s,\chi)x^s\,\frac{ds}{s}.$$

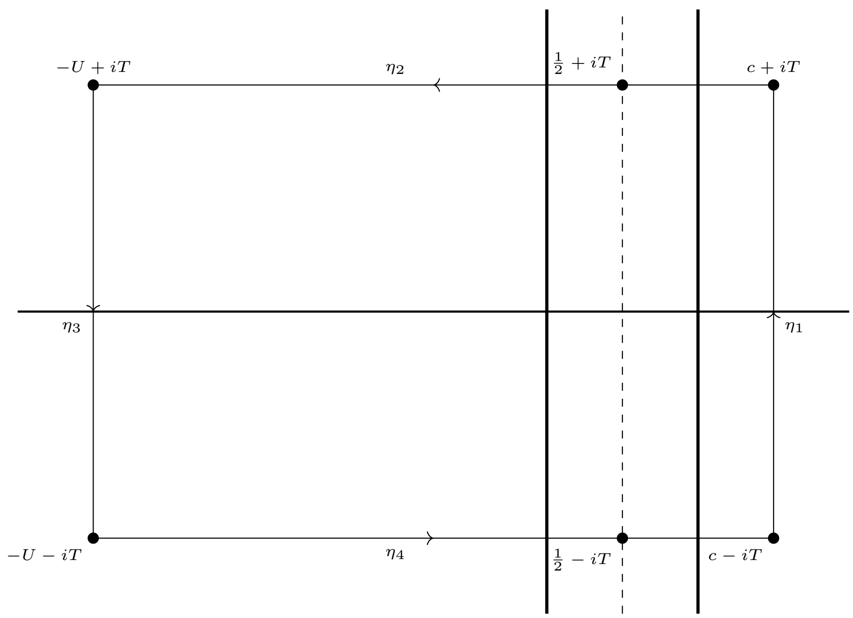

Figure 8.2: Contour for the explicit formula for $\psi(x,\chi)$

We now separate the cases that χ is even or odd. If χ is odd, then the residue theorem, the formula for the negative logarithmic derivative in Proposition 6.8.1 applied to $L(s,\chi)$, and Corollary 1.6.1 together give

$$J(x,T,\chi) = -\sum_{|\gamma|<T}\frac{x^\rho}{\rho} - b(\chi) - \sum_{0<2m+1<U}\frac{x^{-(2m-1)}}{-(2m-1)} + \frac{1}{2\pi i}\int_{\eta_2+\eta_3+\eta_4} -\frac{L'}{L}(s,\chi)x^s\,\frac{ds}{s}, \tag{8.51}$$

where $\rho = \beta + i\gamma$ is a nontrivial zero of $L(s,\chi)$. If χ is even, then there is a minor complication because $L(s,\chi)$ has a simple zero at $s=0$ and so the integrand has a double pole at $s=0$. To find the residue, the Laurent series are

$$\frac{L'}{L}(s,\chi) = \frac{1}{s} + b(\chi) + \cdots \quad \text{and} \quad \frac{x^s}{s} = \frac{1}{s} + \log(x) + \cdots,$$

and thus the residue of the integrand is $-(\log(x)+b(\chi))$. Now as before, the residue theorem, the formula for the negative logarithmic derivative in Proposition 6.8.1 applied to $L(s,\chi)$, and Corollary 1.6.1 together give

$$J(x,T,\chi) = -\sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \log(x) - b(\chi) - \sum_{0<2m<U}\frac{x^{-2m}}{-2m} + \frac{1}{2\pi i}\int_{\eta_2+\eta_3+\eta_4} -\frac{L'}{L}(s,\chi)x^s\,\frac{ds}{s}, \tag{8.52}$$

where $\rho=\beta+i\gamma$ is a nontrivial zero of $L(s,\chi)$. We now estimate the remaining integrals in Equations (8.51) and (8.52). For this estimate, the parity of χ does not matter so we make no such restriction. By Lemma 6.9.1 (ii) applied to $L(s,\chi)$, the number of nontrivial zeros satisfying $|\gamma-T|<1$ is $O(\log(qT))$. Among the ordinates of these nontrivial zeros, there must be a gap of size $\gg \frac{1}{\log(qT)}$. Upon varying T by a bounded amount (we are varying in the interval $[T-1,T+1]$) so that it belongs to this gap, we can additionally ensure

$$\gamma - T \gg \frac{1}{\log(qT)},$$

for all the nontrivial zeros of $L(s,\chi)$. To estimate part of the horizontal integrals over η_2 and η_4, Lemma 6.9.1 (iv) applied to $L(s,\chi)$ gives

$$\frac{L'}{L}(s,\chi) = \sum_{|\gamma-T|<1}\frac{1}{s-\rho} + O(\log(qT)),$$

on the parts of these segments with $-1\le\sigma\le 2$. Our choice of T implies $|s-\rho|\ge|\gamma-T|\gg\frac{1}{\log(qT)}$ so that each term in the sum is $O(\log(qT))$. As there are at most $O(\log(qT))$ such terms by Lemma 6.9.1 (ii) applied to $L(s,\chi)$, we have

$$\frac{L'}{L}(s,\chi) = O(\log^2(qT)),$$

on the parts of these segments with $-1\le\sigma\le 2$. It follows that the parts of the horizontal integrals over η_2 and η_4 with $-1\le\sigma\le c$ (recall $c<2$) contribute

$$\ll \frac{\log^2(qT)}{T}\int_{-1}^{c} x^\sigma\,d\sigma \ll \frac{\log^2(qT)}{T}\int_{-\infty}^{c} x^\sigma\,d\sigma \ll \frac{x\log^2(qT)}{T\log(x)}. \tag{8.53}$$

where in the last estimate we have used the choice of c. To estimate the remainder of the horizontal integrals, we require a bound for $\frac{L'}{L}(s,\chi)$ when $\sigma<-1$ and away from the trivial zeros. To find such a bound, write the functional equation for $L(s,\chi)$ in the form

$$L(s,\chi) = \frac{\varepsilon_\chi}{i^{\mathfrak{a}}}q^{\frac{1}{2}-s}\pi^{s-1}\frac{\Gamma\left(\frac{(1-s)+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}L(1-s,\chi),$$

and take the logarithmic derivative to get

$$\frac{L'}{L}(s,\chi) = -\log(q)+\log(\pi)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{(1-s)+\mathfrak{a}}{2}\right)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)+\frac{L'}{L}(1-s,\chi).$$

Now let s be such that $\sigma<-1$ and suppose s is distance $\frac{1}{2}$ away from the trivial zeros. We will estimate every term on the right-hand side of the identity above. The second term is constant and the last term is bounded since it is an absolutely convergent Dirichlet series. For the digamma terms,

s is away from the trivial zeros so Proposition 1.6.3 implies $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{(1-s)+\mathfrak{a}}{2}\right) = O(\log|(1-s)+\mathfrak{a}|)$ and $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) = O(\log|s+\mathfrak{a}|)$. However, as $\sigma < -1$ and s is away from the trivial zeros, $s+\mathfrak{a}$ and $(1-s)+\mathfrak{a}$ are bounded away from zero so that $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{(1-s)+\mathfrak{a}}{2}\right) = O(\log|s|)$ and $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) = O(\log|s|)$. Putting these estimates together with the first term yields

$$\frac{L'}{L}(s,\chi) \ll \log(q|s|), \tag{8.54}$$

for $\sigma < -1$. Using Equation (8.54), the parts of the horizontal integrals over η_2 and η_4 with $-U \le \sigma \le -1$ contribute

$$\ll \frac{\log(qT)}{T}\int_{-U}^{-1} x^{\sigma}\,d\sigma \ll \frac{\log(qT)}{Tx\log(x)}. \tag{8.55}$$

Combining Equations (8.53) and (8.55) gives

$$\frac{1}{2\pi i}\int_{\eta_2+\eta_4} -\frac{L'}{L}(s)x^s\frac{ds}{s} \ll \frac{x\log^2(qT)}{T\log(x)} + \frac{\log(qT)}{Tx\log(x)} \ll \frac{x\log^2(qT)}{T\log(x)}. \tag{8.56}$$

To estimate the vertical integral, we use Equation (8.54) again to conclude that

$$\frac{1}{2\pi i}\int_{\eta_3} -\frac{L'}{L}(s,\chi)x^s\frac{ds}{s} \ll \frac{\log(qU)}{U}\int_{-T}^{T} x^{-U}\,dt \ll \frac{T\log(qU)}{Ux^U}. \tag{8.57}$$

Combining Equations (8.51), (8.56) and (8.57) and taking the limit as $U \to \infty$, the error term in Equation (8.57) vanishes and the sum over m in Equations (8.51) and (8.52) evaluates to $-\tanh^{-1}(x^{-1})$ or $\frac{1}{2}\log(1-x^{-2})$ respectively (as we have already mentioned) giving

$$J(x,T,\chi) = -\sum_{|\gamma|<T}\frac{x^{\rho}}{\rho} - b(\chi) + \tanh^{-1}(x^{-1}) + \frac{x\log^2(qT)}{T\log(x)}, \tag{8.58}$$

if χ is odd, and

$$J(x,T,\chi) = -\sum_{|\gamma|<T}\frac{x^{\rho}}{\rho} - \log(x) - b(\chi) - \frac{1}{2}\log(1-x^{-2}) + \frac{x\log^2(qT)}{T\log(x)}, \tag{8.59}$$

if χ is even. Substituting Equations (8.58) and (8.59) into Equation (8.50) in the respective cases, we obtain

$$\psi_0(x,\chi) = -\sum_{|\gamma|<T}\frac{x^{\rho}}{\rho} - b(\chi) + \tanh^{-1}(x^{-1}) + \frac{x\log^2(xqT)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right), \tag{8.60}$$

if χ is odd, and

$$\psi_0(x,\chi) = -\sum_{|\gamma|<T}\frac{x^{\rho}}{\rho} - \log(x) - b(\chi) - \frac{1}{2}\log(1-x^{-2}) + \frac{x\log^2(xqT)}{T} + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right), \tag{8.61}$$

if χ is even, and where the second to last term on the right-hand side in both equations are obtained by combining the error term in Equation (8.56) with the first error term in Equation (8.50). The theorem follows by taking the limit as $T \to \infty$. □

As was the case for $\psi(x)$, the convergence of the right-hand side in the explicit formula for $\psi(x,\chi)$ is uniform in any interval not containing a prime power since $\psi(x,\chi)$ is continuous there. Moreover, there is an approximate formula for $\psi(x,\chi)$ as a corollary which holds for all Dirichlet characters:

Corollary 8.4.1. *Let χ be a Dirichlet character modulo $m>1$. Then for $2 \le T \le x$,*

$$\psi_0(x,\chi) = -\frac{x^{\beta_\chi}}{\beta_\chi} - \sideset{}{'}\sum_{|\gamma|<T} \frac{x^\rho}{\rho} + R(x,T,\chi),$$

where ρ runs over the nontrivial zeros of $L(s,\chi)$ counted with multiplicity and ordered with respect to the size of the ordinate, the $'$ in the sum indicates that we are excluding the terms corresponding to real zeros, the term corresponding to a Siegel zero β_χ is present only if $L(s,\chi)$ admits a Siegel zero, and

$$R(x,T,\chi) \ll \frac{x\log^2(xmT)}{T} + x^{1-\beta_\chi}\log(x) + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right),$$

where $\langle x\rangle$ is the distance between x and the nearest prime power other than x if x itself is a prime power and again the term corresponding to a Siegel zero β_χ is present only if $L(s,\chi)$ admits a Siegel zero. Moreover, if x is an integer, we have the simplified estimate

$$R(x,T,\chi) \ll \frac{x\log^2(xmT)}{T} + x^{1-\beta_\chi}\log(x).$$

Proof. We first reduced to the case that χ is primitive. Let $\widetilde{\chi}$ be the primitive character inducing χ and denote its conductor by q. We estimate

$$\begin{aligned}
|\psi_0(x,\chi) - \psi_0(x,\widetilde{\chi})| &\le \sum_{\substack{n\le x\\(n,m)>1}} \Lambda(n)\\
&= \sum_{p\mid m}\sum_{\substack{v\ge 1\\p^v\le x}} \log(p)\\
&\ll \log(x)\sum_{p\mid m}\log(p)\\
&\ll \log(x)\log(m)\\
&\ll \log^2(xm),
\end{aligned} \tag{8.62}$$

where the third line holds because $p^v \le x$ implies $v \le \frac{\log(x)}{\log(p)}$ so that there are $O(\log(x))$ many terms in the inner sum and in the last line we have used the simple estimates $\log(x) \ll \log(xm)$ and $\log(m) \ll \log(xm)$. Therefore the difference between $\psi_0(x,\chi)$ and $\psi_0(x,\widetilde{\chi})$ is $O(\log^2(xm))$. Now for $2 \le T \le x$, we have $\log^2(xm) \ll \frac{x\log^2(xmT)}{T}$, which implies that the difference is absorbed into $O\left(\frac{x\log^2(xmT)}{T}\right)$ which is the first term in the error for $R(x,T,\chi)$. As $R(x,T,\widetilde{\chi}) \ll R(x,T,\chi)$ because $q \le m$, and there are finitely many nontrivial zeros of $L(s,\chi)$ that are not nontrivial zeros of $L(s,\widetilde{\chi})$ (all occurring on the line $\sigma = 0$), it suffices to assume that χ is primitive. The claim will follow from estimating terms in Equations (8.60) and (8.61). We will estimate the constant $b(\chi)$ first. The formula for the negative logarithmic derivative in Proposition 6.8.1 applied to $L(s,\chi)$ at $s=2$ implies

$$0 = -\frac{L'}{L}(2,\chi) - \frac{1}{2}\log(q) + \frac{1}{2}\log(\pi) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{2+\mathfrak{a}}{2}\right) + B(\chi) + \sum_\rho\left(\frac{1}{2-\rho} + \frac{1}{\rho}\right). \tag{8.63}$$

Adding Equation (8.63) to the formula for the negative logarithmic derivative in Proposition 6.8.1 applied to $L(s,\chi)$ gives

$$-\frac{L'}{L}(s,\chi)=-\frac{L'}{L}(2,\chi)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{2+\mathfrak{a}}{2}\right)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)-\sum_{\rho}\left(\frac{1}{s-\rho}-\frac{1}{2-\rho}\right).$$

As the first two terms are constant, we obtain a weaker estimate

$$-\frac{L'}{L}(s,\chi)=\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)-\sum_{\rho}\left(\frac{1}{s-\rho}-\frac{1}{2-\rho}\right)+O(1).$$

If χ is odd, we set $s=0$ since $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)$ does not have a pole there. If χ is even, we compare constant terms in the Laurent series using the Laurent series

$$\frac{L'}{L}(s,\chi)=\frac{1}{s}+b(\chi)+\cdots\quad\text{and}\quad\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)=\frac{1}{s}+b+\cdots,$$

for some constant b. In either case, our previous estimate gives

$$b(\chi)=-\sum_{\rho}\left(\frac{1}{\rho}+\frac{1}{2-\rho}\right)+O(1). \tag{8.64}$$

Let $\rho=\beta+i\gamma$. For all the terms with $|\gamma|>1$, we estimate

$$\sum_{|\gamma|>1}\left(\frac{1}{\rho}+\frac{1}{2-\rho}\right)\le\sum_{|\gamma|>1}\left|\frac{1}{\rho}+\frac{1}{2-\rho}\right|=\sum_{|\gamma|>1}\frac{2}{|\rho(2-\rho)|}\ll\sum_{|\gamma|>1}\frac{1}{|\rho|^2}\ll\log(q), \tag{8.65}$$

where the second to last estimate holds since $2-\rho\gg\rho$ because β is bounded and the last estimate holds by the convergent sum in Proposition 6.8.1and Lemma 6.9.1 (ii) both applied to $L(s,\chi)$ (recall that the tail of a convergent series is bounded). For the terms corresponding to $2-\rho$ with $|\gamma|\le 1$, we have

$$\sum_{|\gamma|\le 1}\frac{1}{2-\rho}\le\sum_{|\gamma|\le 1}\frac{1}{|2-\rho|}\ll\log(q), \tag{8.66}$$

where the last estimate holds by using Lemma 6.9.1 (ii) applied to $L(s,\chi)$ and because the nontrivial zeros are bounded away from 2. Combining Equations (8.64) to (8.66) yields

$$b(\chi)=-\sum_{|\gamma|\le 1}\frac{1}{\rho}+O(\log(q)). \tag{8.67}$$

Inserting Equation (8.67) into Equations (8.60) and (8.61) and noting that $\tanh^{-1}(x^{-1})$ and $\frac{1}{2}\log(1-x^{-2})$ are both bounded for $x\ge 2$ gives

$$\psi_0(x,\chi)=-\sum_{|\gamma|<T}\frac{x^\rho}{\rho}+\sum_{|\gamma|\le 1}\frac{1}{\rho}+R'(x,T,\chi), \tag{8.68}$$

where

$$R'(x,T,\chi)\ll\frac{x\log^2(xqT)}{T}+\log(x)\min\left(1,\frac{x}{T\langle x\rangle}\right),$$

and we have absorbed the error in Equation (8.67) into $O\left(\frac{x\log^2(xqT)}{T}\right)$ because $2 \le T \le x$. Extracting the terms corresponding to the possible real zeros β_χ and $1-\beta_\chi$ in Equation (8.68), we obtain

$$\psi_0(x,\chi) = -\sum_{|\gamma|<T}{}' \frac{x^\rho}{\rho} + \sum_{|\gamma|\le 1}{}' \frac{1}{\rho} - \frac{x^{\beta_\chi}-1}{\beta_\chi} - \frac{x^{1-\beta_\chi}-1}{1-\beta_\chi} + R'(x,T,\chi). \tag{8.69}$$

We now estimate some of the terms in Equation (8.69). For the second sum, we have $\rho \gg \frac{1}{\log(q)}$ since γ is bounded and $\beta < 1 - \frac{c}{\log(q|\gamma|)}$ for some positive constant c by Theorem 8.2.2. Thus

$$\sum_{|\gamma|\le 1}{}' \frac{1}{\rho} \ll \sum_{|\gamma|\le 1}{}' \log(q) \ll \log^2(q), \tag{8.70}$$

where the last estimate holds by Lemma 6.9.1 (ii) applied to $L(s,\chi)$. Similarly,

$$\frac{x^{1-\beta_\chi}-1}{1-\beta_\chi} \ll x^{1-\beta_\chi}\log(x), \tag{8.71}$$

because $\rho \gg \frac{1}{\log(q)}$ implies $1-\beta_\chi \gg \frac{1}{\log(q)} \gg \frac{1}{\log(x)}$. Substituting Equations (8.70) and (8.71) into Equation (8.69) and noting that β_χ is bounded yields

$$\psi_0(x,\chi) = -\frac{x^{\beta_\chi}}{\beta_\chi} - \sum_{|\gamma|<T}{}' \frac{x^\rho}{\rho} + R(x,T,\chi), \tag{8.72}$$

where

$$R(x,T,\chi) \ll \frac{x\log^2(xqT)}{T} + x^{1-\beta_\chi}\log(x) + \log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right)$$

and we have absorbed the error in Equation (8.70) into $O\left(\frac{x\log^2(xqT)}{T}\right)$ because because $2 \le T \le x$. If x is an integer, then $\langle x\rangle \ge 1$ so that $\log(x)\min\left(1, \frac{x}{T\langle x\rangle}\right) \le \frac{x\log(x)}{T}$ and this term can be absorbed into $O\left(\frac{x\log^2(xqT)}{T}\right)$. □

We can now discuss the Siegel–Walfisz theorem. Let a and m be positive integers with $m > 1$ and $(a,m) = 1$. The **prime counting function** $\pi(x : a, m)$ is defined by

$$\pi(x;a,m) = \sum_{\substack{p\le x \\ p\equiv a \pmod{m}}} 1,$$

for $x > 0$. Equivalently, $\pi(x;a,m)$ counts the number of primes that no larger than x and are equivalent to a modulo m. This is the analog of $\pi(x)$ that is naturally associated to Dirichlet characters modulo m. Accordingly, there are asymptotics for $\pi(x;a,m)$ analogous to those for $\pi(x)$. To prove them, we will require an auxiliary function. The function $\psi(x;a,m)$ is defined by

$$\psi(x;a,m) = \sum_{\substack{n\le x \\ n\equiv a \pmod{m}}} \Lambda(n),$$

for $x \ge 1$. This is just $\psi(x)$ restricted to only those terms equivalent to a modulo m. As for the asymptotics, the classical **Siegel–Walfisz theorem** is the first of them and the precise statement is the following:

Theorem 8.4.2 (Siegel–Walfisz theorem, classical version). *Let a and m be positive integers with $m > 1$, $(a, m) = 1$, and let $N \geq 1$. For $x \geq 2$,*

$$\pi(x; a, m) \sim \frac{x}{\varphi(m)\log(x)},$$

provided $m \leq \log^N(x)$.

The logarithmic integral version of the **Siegel–Walfisz theorem** is equivalent and sometimes more useful:

Theorem 8.4.3 (Siegel–Walfisz theorem, logarithmic integral version). *Let a and m be positive integers with $m > 1$, $(a, m) = 1$, and let $N \geq 1$. For $x \geq 2$,*

$$\pi(x; a, m) \sim \frac{\mathrm{Li}(x)}{\varphi(m)},$$

provided $m \leq \log^N(x)$.

We will prove the absolute error version of the **Siegel–Walfisz theorem** which is slightly stronger as it bounds the absolute error between $\pi(x; a, m)$ and $\frac{\mathrm{Li}(x)}{\varphi(m)}$:

Theorem 8.4.4 (Siegel–Walfisz theorem, absolute error version). *Let a and m be positive integers with $m > 1$, $(a, m) = 1$, and let $N \geq 1$. For $x \geq 2$, there exists a positive constant c such that*

$$\pi(x; a, m) = \frac{\mathrm{Li}(x)}{\varphi(m)} + O\left(xe^{-c\sqrt{\log(x)}}\right),$$

provided $m \leq \log^N(x)$.

Proof. It suffices to assume x is an integer, because $\pi(x; a, m)$ can only change value at integers and the other functions in the statement are increasing. We being with the identity

$$\psi(x; a, m) = \frac{1}{\varphi(m)} \sum_{\chi \pmod m} \overline{\chi}(a)\psi(x, \chi), \tag{8.73}$$

which holds by the orthogonality relations (Proposition 1.3.1 (ii)). Let $\widetilde{\chi}$ be the primitive character inducing χ. Then Equation (8.62) implies

$$\psi(x, \chi) = \psi(x, \widetilde{\chi}) + O(\log^2(xm)). \tag{8.74}$$

When $\chi = \chi_{m,0}$ we have $\psi(x, \widetilde{\chi}) = \psi(x)$, and as $\psi(x, \chi) \sim \psi_0(x, \chi)$, substituting Equation (8.35) into Equation (8.74) gives

$$\psi(x, \chi_{m,0}) = \psi(x) + O\left(xe^{-c\sqrt{\log(x)}} + \log^2(xm)\right), \tag{8.75}$$

for some positive constant c. Upon combining Equations (8.73) and (8.75), we obtain

$$\psi(x; a, m) = \frac{x}{\varphi(m)} + \frac{1}{\varphi(m)} \sum_{\substack{\chi \pmod m \\ \chi \neq \chi_{m,0}}} \overline{\chi}(a)\psi(x, \chi) + O\left(\frac{1}{\varphi(m)}\left(xe^{-c\sqrt{\log(x)}} + \log^2(xm)\right)\right). \tag{8.76}$$

We now prove

$$\psi(x, \chi) = -\frac{x^{\beta_\chi}}{\beta_\chi} + O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{8.77}$$

for some potentially different constant c, where $\chi \neq \chi_{m,0}$, and the term corresponding to β_χ appears if and only if $L(s,\chi)$ admits a Siegel zero. To accomplish this, we estimate the sum over the nontrivial zeros of $L(s,\chi)$ in Corollary 8.4.1. So fix a non-principal χ modulo m and let $\widetilde{\chi}$ be the primitive character inducing χ. Let $2 \le T \le x$ not coinciding with the ordinate of a nontrivial zero and let $\rho = \beta + i\gamma$ be a complex nontrivial zero of $L(s,\chi)$ with $|\gamma| < T$. By Theorem 8.2.2, all of the zeros ρ satisfy $\beta < 1 - \frac{c}{\log(mT)}$ for some possibly smaller c (recall that the nontrivial zeros of $L(s,\chi)$ that are not nontrivial zeros of $L(s,\widetilde{\chi})$ lie on the line $\sigma = 0$). It follows that

$$|x^\rho| = x^\beta < x^{1-\frac{c}{\log(mT)}} = xe^{-c\frac{\log(x)}{\log(mT)}}. \tag{8.78}$$

As $|\rho| > |\gamma|$, for those terms with $|\gamma| > 1$ (unlike the Riemann zeta function we do not have a positive lower bound for the first ordinate γ_1 of a nontrivial zero that is not real since Siegel zeros may exist), applying integration by parts gives

$$\sum_{1<|\gamma|<T} \frac{1}{\rho} \ll \sum_{1<|\gamma|<T} \frac{1}{\gamma} \ll \int_1^T \frac{dN(t,\widetilde{\chi})}{t} = \frac{N(T,\widetilde{\chi})}{T} + \int_1^T \frac{N(t,\widetilde{\chi})}{t^2}\,dt \ll \log^2(mT) \ll \log^2(xm), \tag{8.79}$$

where in the second to last estimate we have used that $N(t,\widetilde{\chi}) \ll t\log(qt) \ll t\log(mt)$ which follows from Corollary 6.9.1 applied to $L(s,\widetilde{\chi})$ and that $q \le m$ and in the last estimate we have used the bound $T \le x$. For the remaining terms with $|\gamma| \le 1$ (that do not correspond to real nontrivial zeros), Equation (8.70) along with $q \le m$ gives

$$\sideset{}{'}\sum_{|\gamma|\le 1} \frac{1}{\rho} \ll \log^2(m). \tag{8.80}$$

Combining Equations (8.77) to (8.80) yields

$$\sideset{}{'}\sum_{|\gamma|<T} \frac{x^\rho}{\rho} \ll x\log^2(xm)e^{-c\frac{\log(x)}{\log(mT)}}, \tag{8.81}$$

where the $'$ in the sum indicates that we are excluding the terms corresponding to real zeros and the error in Equation (8.80) has been absorbed by that in Equation (8.79). As $\psi(x,\chi) \sim \psi_0(x,\chi)$ and x is an integer, inserting Equation (8.81) into Corollary 8.4.1 results in

$$\psi(x,\chi) + \frac{x^{\beta_\chi}}{\beta_\chi} \ll x\log^2(xm)e^{-c\frac{\log(x)}{\log(mT)}} + \frac{x\log^2(xmT)}{T} + x^{1-\beta_\chi}\log(x), \tag{8.82}$$

where the terms corresponding to real zeros are present if and only if $L(s,\chi)$ admits a Siegel zero. We now let T be determined by $T = x$ for $2 \le x < 3$ and

$$\log^2(T) = \log(x),$$

or equivalently,

$$T = e^{\sqrt{\log(x)}},$$

for $x \ge 3$. With this choice of T (note that if $x \ge 2$ then $2 \le T \le x$) and that $m \ll \log^N(x)$, we can estimate Equation (8.82) as follows:

$$\begin{aligned}
\psi(x)+\frac{x^{\beta_\chi}}{x} &\ll x(\log^2(x)+\log^2(m))e^{-c\frac{\log(x)}{\log(m)+\sqrt{\log(x)}}}+x(\log^2(x)+\log^2(m)+\log(x))e^{-\sqrt{\log(x)}}+x^{1-\beta_\chi}\log(x) \\
&\ll x(\log^2(x)+\log^2(m))e^{-c\frac{\log(x)}{\log(m)+\sqrt{\log(x)}}}+x(\log^2(x)+\log^2(m)+\log(x))e^{-\sqrt{\log(x)}}+x^{1-\beta_\chi}\log(x) \\
&\ll x(\log^2(x)+\log^2(m))e^{-c\sqrt{\log(x)}}+x(\log^2(x)+\log^2(m))e^{-\sqrt{\log(x)}}+x^{1-\beta_\chi}\log(x) \\
&\ll x\log^2(x)e^{-c\sqrt{\log(x)}}+x\log^2(x)e^{-\sqrt{\log(x)}}+x^{1-\beta_\chi}\log(x) \\
&\ll x\log^2(x)e^{-\min(1,c)\frac{\sqrt{\log(x)}}{\log(m)}},
\end{aligned}$$

where in the last estimate we have used that $x^{1-\beta_\chi}\le x^{\frac{1}{2}}$ because $\beta_\chi\ge\frac{1}{2}$. As $\log(x)\ll_\varepsilon e^{-\varepsilon\sqrt{\log(x)}}$, we conclude that

$$\psi(x)+\frac{x^{\beta_\chi}}{x}\ll xe^{-c\sqrt{\log(x)}},$$

for some smaller c with $c<1$. This is equivalent to Equation (8.77). Substituting Equation (8.77) into Equation (8.76) and noting that there is at most one Siegel zero for characters modulo m by Proposition 8.2.2, we arrive at

$$\psi(x;a,m)=\frac{x}{\varphi(m)}-\frac{\overline{\chi}_1(a)x^{\beta_\chi}}{\varphi(m)\beta_\chi}+O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{8.83}$$

where χ_1 is the single quadratic character modulo m such that $L(s,\chi_1)$ admits a Siegel zero if it exists and we have absorbed the second term in the error in Equation (8.76) into the first since $\log(x)\ll_\varepsilon e^{-\varepsilon\sqrt{\log(x)}}$ and $m\ll\log^N(x)$. Taking $\varepsilon=\frac{1}{2N}$ in the zero-free region version of Siegel's theorem, $m^{\frac{1}{2N}}\ll\sqrt{\log(x)}$ so that $\beta_\chi<1-\frac{c}{\sqrt{\log(x)}}$ for some potentially smaller constant c. It follows that m^{2N} Therefore

$$x^{\beta_\chi}<x^{1-\frac{c}{\sqrt{\log(x)}}}=xe^{-c\log(x)}. \tag{8.84}$$

Combining Equations (8.83) and (8.84) gives the simplified estimate

$$\psi(x;a,m)=\frac{x}{\varphi(m)}+O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{8.85}$$

for some potentially smaller constant c. Now let

$$\pi_1(x;a,m)=\sum_{\substack{n\le x\\ n\equiv a \pmod m}}\frac{\Lambda(n)}{\log(n)}.$$

We can write $\pi_1(x;a,m)$ in terms of $\psi(x;a,m)$ as follows:

$$\begin{aligned}\pi_1(x;a,m) &= \sum_{\substack{n\le x\\ n\equiv a \pmod m}} \frac{\Lambda(n)}{\log(n)}\\ &= \sum_{\substack{n\le x\\ n\equiv a \pmod m}} \Lambda(n)\int_n^x \frac{dt}{t\log^2(t)} + \frac{1}{\log(x)}\sum_{\substack{n\le x\\ n\equiv a \pmod m}} \Lambda(n)\\ &= \int_2^x \sum_{\substack{n\le x\\ n\equiv a \pmod m}} \Lambda(n)\frac{dt}{t\log^2(t)} + \frac{1}{\log(x)}\sum_{\substack{n\le x\\ n\equiv a \pmod m}} \Lambda(n)\\ &= \int_2^x \frac{\psi(t;a,m)}{t\log^2(t)}\,dt + \frac{\psi(x;a,m)}{\log(x)}.\end{aligned}$$

Applying Equation (8.85) to the last expression yields

$$\pi_1(x;a,m) = \int_2^x \frac{t}{\varphi(m)t\log^2(t)}\,dt + \frac{x}{\varphi(m)\log(x)} + O\left(\int_2^x \frac{e^{-c\sqrt{\log(t)}}}{\log^2(t)}\,dt + \frac{xe^{-c\sqrt{\log(x)}}}{\log(x)}\right). \tag{8.86}$$

Applying integrating by parts to the main term in Equation (8.86), we obtain

$$\int_2^x \frac{t}{\varphi(m)t\log^2(t)}\,dt + \frac{x}{\varphi(m)\log(x)} = \int_2^x \frac{dt}{\varphi(m)\log(t)} + \frac{2}{\varphi(m)\log(2)} = \frac{\mathrm{Li}(x)}{\varphi(m)} + \frac{2}{\varphi(m)\log(2)}. \tag{8.87}$$

As for the error term in Equation (8.86), we use Equation (8.42). Combining Equations (8.42), (8.86) and (8.87) yields

$$\pi_1(x;a,m) = \frac{\mathrm{Li}(x)}{\varphi(m)} + O\left(xe^{-c\sqrt{\log(x)}}\right), \tag{8.88}$$

for some smaller c and where the constant in Equation (8.87) has been absorbed into the error term. As last we pass from $\pi_1(x;a,m)$ to $\pi(x;a,m)$. If p is a prime such that $p^m < x$ for some $m \ge 1$, then $p < x^{\frac{1}{2}} < x^{\frac{1}{3}} < \cdots < x^{\frac{1}{m}}$. Therefore

$$\pi_1(x;a,m) = \sum_{\substack{n\le x\\ n\equiv a \pmod m}} \frac{\Lambda(n)}{\log(n)} = \sum_{\substack{p^m\le x\\ p^m\equiv a \pmod m}} \frac{\log(p)}{m\log(p)} = \pi(x;a,m) + \frac{1}{2}\pi(x^{\frac{1}{2}};a,m) + \cdots. \tag{8.89}$$

Moreover, as $\pi(x^{\frac{1}{n}};a,m) < x^{\frac{1}{n}}$ for any $n \ge 1$, we see that $\pi(x;a,m) - \pi_1(x;a,m) = O(x^{\frac{1}{2}})$. This estimate together with Equations (8.88) and (8.89) gives

$$\pi(x) = \frac{\mathrm{Li}(x)}{\varphi(m)} + O\left(xe^{-c\sqrt{\log(x)}}\right),$$

because $x^{\frac{1}{2}} \ll xe^{-c\sqrt{\log(x)}}$. This completes the proof. □

It is interesting to note that the constant c in the Siegel-Walfisz theorem is ineffective because of the use of Siegel's theorem (it also depends upon N). This is unlike the prime number theorem, where the constant c can be made to be effective. The proof of the logarithmic integral and classical versions of the Siegel-Walfisz theorem are immediate:

Proof of Siegel-Walfisz theorem, logarithmic integral and classical versions. By the absolute error version of the Siegel-Walfisz theorem,

$$\pi(x;a,m) = \frac{\mathrm{Li}(x)}{\varphi(m)}\left(1 + O\left(\frac{\varphi(m)xe^{-c\sqrt{\log(x)}}}{\mathrm{Li}(x)}\right)\right).$$

As $\mathrm{Li}(x) \sim \frac{x}{\log(x)}$, we have

$$\frac{\varphi(m)xe^{-c\sqrt{\log(x)}}}{\mathrm{Li}(x)} \sim \varphi(m)\log(x)e^{-c\sqrt{\log(x)}} = o(1),$$

where the equality holds since $m \ll \log^N(x)$ and $\log(x) \ll_\varepsilon e^{-\varepsilon\sqrt{\log(x)}}$. The logarithm integral version follows. The classical version also holds using the asymptotic $\mathrm{Li}(x) \sim \frac{x}{\log(x)}$. $\square$

Also, we have an optimal error term, in a much wider range of m, assuming the Riemann hypothesis for Dirichlet L-functions:

Proposition 8.4.1. *Let a and m be positive integers with $m > 1$ and $(a,m) = 1$. For $x \geq 2$, we have*

$$\pi(x;a,m) = \frac{\mathrm{Li}(x)}{\varphi(m)} + O(\sqrt{x}\log(x)),$$

provided $m \leq x$ and the Riemann hypothesis for Dirichlet L-functions holds.

Proof. Let χ be a Dirichlet character modulo m. If ρ is a nontrivial zero of $L(s,\chi)$, the Riemann hypothesis for Dirichlet L-functions implies $|x^\rho| = \sqrt{x}$ and that Siegel zeros do not exist so we may merely assume $m \leq x$. Therefore as in the proof of the absolute error version of the Siegel-Walfisz theorem,

$$\sum_{|\gamma|<T} \frac{x^\rho}{\rho} \ll \sqrt{x}\log^2(x),$$

for $2 \leq T \leq x$ not coinciding with the ordinate of a nontrivial zero. Repeating the same argument with T determined by $T = x$ for $2 \leq x < 3$ and

$$T^2 = x,$$

for $x \geq 3$ gives

$$\psi(x;a,m) = \frac{x}{\varphi(m)} + O(\sqrt{x}\log^2(x)),$$

and then transferring to $\pi_1(x)$ and finally $\pi(x)$ gives

$$\pi(x;a,m) = \frac{x}{\varphi(m)} + O(\sqrt{x}\log(x)).$$

$\square$

Part IV

An Introduction to Sieve Theory

Chapter 9

The Theory of Sieves

Sieves are an important tool in analytic number theory because they allow for the estimation of the size of of a sifted sequence of numbers from some initial sequence. In practice, one is usually sifting out only those indices relatively prime to some fixed integer and up to some prescribed size. In the following, we introduce sieves in some generality.

9.1 The Language of Sieves

Let $\mathcal{A} = (a_n)_{n\geq 1}$ be a nonnegative sequence. We call $\mathcal{A}$ the **sifting sequence**. For any integer $d \geq 1$, let $\mathcal{A}_d$ be the subsequence consisting of those terms a_n with $n \equiv 0 \pmod{d}$. Define the sum $S(x;\mathcal{A})$ by

$$S(x;\mathcal{A}) = \sum_{n\leq x} a_n,$$

for $x \geq 0$. We call a finite subset $\mathcal{P}$ of primes the **sifting range** and we define $P(z)$ by

$$P(z) = \prod_{\substack{p\in\mathcal{P}\\ p<z}} p,$$

for $z \geq 2$. We call z the **sifting level**. In other words, $P(z)$ is the product of the primes in the sifting range up to z. We define the corresponding **sifting function** $S(x,z;\mathcal{A},\mathcal{P})$ by

$$S(x,z;\mathcal{A},\mathcal{P}) = \sum_{\substack{n\leq x\\ (n,P(z))=1}} a_n,$$

for $x \geq 0$ and $z \geq 2$. Moreover, $S(x,z;\mathcal{A},\mathcal{P})$ is $S(x;\mathcal{A})$ except those terms a_n with $(n,P(z)) > 1$ have been removed. This last condition can also be expressed as removing those terms a_n with $n \equiv 0 \pmod{p}$ for $p \mid P(z)$. So $S(x,z;\mathcal{A},\mathcal{P})$ is obtained from $S(x;\mathcal{A})$ by removing terms whose indices correspond to certain residue classes. From this we immediately arrive at **Buchstab's identity**:

$$S(x,z;\mathcal{A},\mathcal{P}) = S(x;\mathcal{A}) - \sum_{p\mid P(z)} S(x,p;\mathcal{A}_p,\mathcal{P}).$$

This is a useful relation for the sifting function. There is also another equally useful relation. To derive it, applying Proposition A.2.1 to $S(x,z;\mathcal{A},\mathcal{P})$ gives

$$S(x,z;\mathcal{A},\mathcal{P}) = \sum_{n\leq x} \sum_{d\mid (n,P(z))} \mu(d) a_n,$$

and upon interchanging the two sums we obtain **Legendre's identity**:

$$S(x,z;\mathcal{A},\mathcal{P}) = \sum_{d \mid P(z)} \mu(d)S(x,\mathcal{A}_d).$$

The usefulness of sieves comes from replacing the Möbius function with weights that essentially serve as a truncation of the Möbius function. In particular, we replace $\mu(d)$ with a weighting factor λ_d coming from a real sequence $\Lambda = (\lambda_d)_{d \ge 1}$ satisfying $\lambda_d = 0$ unless d is square-free, $d \mid P(z)$, and $d < D$ for some $D \ge 1$. We will also let λ represent the arithmetic function $\lambda(d) = \lambda_d$. We call Λ a **sieve**, λ_d a **sieve weight**, and the minimal such D the **sieve level** of Λ. We also define the **sifting variable** s by

$$s = \frac{\log(D)}{\log(z)}.$$

That is, s measures the size of the sieve level relative to the sifting level on a logarithmic scale. We then define the **sieving function** $S^{\Lambda}(x,z;\mathcal{A},\mathcal{P})$ by

$$S^{\Lambda}(x,z;\mathcal{A},\mathcal{P}) = \sum_{d \mid P(z)} \lambda_d S(x,\mathcal{A}_d),$$

for $x > 0$ and $z \ge 2$. The sieving function $S^{\Lambda}(x,z;\mathcal{A},\mathcal{P})$ is finite because $\lambda_d = 0$ if $d \ge D$. Expanding $S(x,\mathcal{A}_d)$, we can write

$$S^{\Lambda}(x,z;\mathcal{A},\mathcal{P}) = \sum_{n \le x} a_n \sum_{d \mid (n,P(z))} \lambda_d.$$

Now set

$$\theta_n^0 = \sum_{d \mid (n,P(z))} \mu(d) \quad \text{and} \quad \theta_n = \sum_{d \mid (n,P(z))} \lambda_d.$$

We will also let θ^0 and θ represent the arithmetic functions $\theta^0(n) = \theta_n^0$ and $\theta(n) = \theta_n$ respectively. Equivalently, we can express θ_n^0 and θ_n in terms of the Dirichlet convolutions (see Appendix A.1) $\theta^0 = \mu * \mathbf{1}$ and $\theta = \lambda * \mathbf{1}$ respectively. In particular, θ_n^0 and θ_n (and hence θ^0 and θ as well) both depend on the sifting level z, but we suppress this dependence from the notation. In any case, we have

$$S^{\Lambda}(x,z;\mathcal{A},\mathcal{P}) = \sum_{n \le x} a_n \theta_n.$$

We say that Λ is an **upper sieve** if the sieving function $S^{\Lambda}(x,z;\mathcal{A},\mathcal{P})$ is an upper bound for the sifting function $S(x,z;\mathcal{A},\mathcal{P})$. From the definition of the sifting function, Λ will be an upper sieve if and only if $\theta_n \ge \theta_n^0$. Analogously, we say that Λ is a **lower sieve** if the sieving function $S^{\Lambda}(x,z;\mathcal{A},\mathcal{P})$ is a lower bound for the sifting function $S(x,z;\mathcal{A},\mathcal{P})$. Similarly, Λ will be a lower sieve if and only if $\theta_n \le \theta_n^0$. We will often optimize the choice of sieve weights such that the lower or upper bound is as tight as possible. When we wish to distinguish these cases, we denote upper and lower sieves by $\Lambda^{\pm} = (\lambda_d^{\pm})_{d \ge 1}$ respectively. Moreover, we write

$$S^{\pm}(x,z;\mathcal{A},\mathcal{P}) = \sum_{d \mid P(z)} \lambda_d^{\pm} S(x,\mathcal{A}_d) \quad \text{and} \quad \theta_n^{\pm} = \sum_{d \mid (n,P(z))} \lambda_d^{\pm},$$

respectively. Then we have the upper and lower bounds

$$S^{-}(x,z;\mathcal{A},\mathcal{P}) \le S(x,z;\mathcal{A},\mathcal{P}) \le S^{+}(x,z;\mathcal{A},\mathcal{P}),$$

provided

$$\theta_n^- \leq \theta_n^0 \leq \theta_n^+. \tag{9.1}$$

We can also compose sieves to form new sieves from old ones. To explain the construction, let $\Lambda' = (\lambda'_d)_{d\geq 1}$ and $\Lambda'' = (\lambda''_d)_{d\geq 1}$ be two sieves of levels D' and D'' respectively. Also let $\theta' = \lambda' * \mathbf{1}$ and $\theta'' = \lambda'' * \mathbf{1}$. Then we define the **composite sieve** $\Lambda = \Lambda'\Lambda''$ of Λ' and Λ'' by

$$\lambda_d = \sum_{\substack{d_1,d_2 \mid d \\ [d_1,d_2]=d}} \lambda'_{d_1}\lambda''_{d_2},$$

for $d \geq 1$. Clearly Λ is a sieve of level $D'D''$. Moreover, summing over pairs (d_1, d_2) of divisors of n is the same as summing over triples (d_1, d_2, d) of divisors of n with $[d_1, d_2] = d$, we see that

$$\theta = \lambda * \mathbf{1} = (\lambda' * \mathbf{1})(\lambda'' * \mathbf{1}) = \theta'\theta''.$$

From this chain and Equation (9.1) it follows that the composition of an upper and lower sieve is a lower sieve while the composition of two upper or two lower sieves is an upper sieve.

9.2 Estimating the Sifting Function

Ultimately our aim is to estimate the sifting function in terms of upper or lower sieving functions. In order to achieve this, we require an additional assumption about the sums $S(x, \mathcal{A}_d)$. In particular, we assume that there exists a multiplicative arithmetic function g with $g(1) = 1$, $0 \leq g(d) < 1$ for all $d > 1$ where $g(d) = 0$ unless d is square-free and the primes dividing d belong to $\mathcal{P}$, satisfies

$$\prod_{w\leq p<z} (1 - g(p))^{-1} \leq K \left(\frac{\log(z)}{\log(w)}\right)^{\kappa}, \tag{9.2}$$

for constants $K > 1$ and $\kappa \geq 0$ and all w and z with $z > w \geq 2$, and such that

$$S(x, \mathcal{A}_d) = g(d)M(x; \mathcal{A}) + r_d(x; \mathcal{A}), \tag{9.3}$$

for some smooth functions $M(x; \mathcal{A})$ and $r_d(x; \mathcal{A})$. We call the function g a **density function** and the constant κ as the **sieve dimension**. Using Legendre's identity, we can write

$$S(x, z; \mathcal{A}, \mathcal{P}) = M(x; \mathcal{A}) \sum_{d \mid P(z)} \mu(d)g(d) + \sum_{d \mid P(z)} \mu(d)r_d(x; \mathcal{A}).$$

Moreover, if we define functions $V(z; \mathcal{P})$ and $R(x, z; \mathcal{A}, \mathcal{P})$ by

$$V(z; \mathcal{P}) = \sum_{d \mid P(z)} \mu(d)g(d) \quad \text{and} \quad R(x, z; \mathcal{A}, \mathcal{P}) = \sum_{d \mid P(z)} \mu(d)r_d(x; \mathcal{A}),$$

for $x > 0$ and $z \geq 2$, then we can further write

$$S(x, z; \mathcal{A}, \mathcal{P}) = M(x; \mathcal{A})V(z; \mathcal{P}) + R(x, z; \mathcal{A}, \mathcal{P}).$$

As g is multiplicative, the definition of the Möbius function allows us to express $V(z; \mathcal{P})$ as a product:

$$V(z; \mathcal{P}) = \prod_{p \mid P(z)} (1 - g(p)). \tag{9.4}$$

In particular, since $g(p)=0$ if $p \notin \mathcal{P}$, Equation (9.2) can be expressed in the form

$$\frac{V(w;\mathcal{P})}{V(z;\mathcal{P})} \le K\left(\frac{\log(z)}{\log(w)}\right)^{\kappa}. \tag{9.5}$$

Since the Möbius function changes sign, it is difficult to estimate $R(x,z;\mathcal{A},\mathcal{P})$ beyond trivial bounds. We can do much better with a sieve Λ. In this case, the definition of the sieving function gives

$$S_d^{\Lambda}(x,z;\mathcal{A},\mathcal{P}) = M(x;\mathcal{A})\sum_{d|P(z)}\lambda_d g(d) + \sum_{d|P(z)}\lambda_d r_d(x;\mathcal{A}).$$

Defining functions $V^{\Lambda}(z;\mathcal{P})$ and $R^{\Lambda}(x,z;\mathcal{A},\mathcal{P})$ by

$$V^{\Lambda}(z;\mathcal{P}) = \sum_{d|P(z)}\lambda_d g(d) \quad \text{and} \quad R^{\Lambda}(x,z;\mathcal{A},\mathcal{P}) = \sum_{d|P(z)}\lambda_d r_d(x;\mathcal{A}),$$

for $x>0$ and $z \ge 2$, we can further write

$$S^{\Lambda}(x,z;\mathcal{A},\mathcal{P}) = M(x;\mathcal{A})V^{\Lambda}(z;\mathcal{P}) + R^{\Lambda}(x,z;\mathcal{A},\mathcal{P}).$$

For upper or lower sieves $\Lambda^{\pm}$, we instead write

$$V^{\pm}(z;\mathcal{P}) = \sum_{d|P(z)}\lambda_d^{\pm} g(d) \quad \text{and} \quad R^{\pm}(x,z;\mathcal{A},\mathcal{P}) = \sum_{d|P(z)}\lambda_d^{\pm} r_d(x;\mathcal{A}),$$

respectively. Therefore, estimates for the sieving function $S^{\Lambda}(x,z;\mathcal{A},\mathcal{P})$ reduce to estimates for $M(x;\mathcal{A})$, $V^{\Lambda}(z;\mathcal{P})$, and $R^{\Lambda}(x,z;\mathcal{A},\mathcal{P})$. In order for Equation (9.3) to be useful, the functions $M(x;\mathcal{A})$ and $r_d(x;\mathcal{A})$ should be such that we know the order of magnitude of $M(x;\mathcal{A})$ and that $r_d(x;\mathcal{A})$ is of smaller order of magnitude. In this case, it suffices to estimate $V^{\Lambda}(z;\mathcal{P})$, and $R^{\Lambda}(x,z;\mathcal{A},\mathcal{P})$. For $R^{\Lambda}(x,z;\mathcal{A},\mathcal{P})$, we will usually discard possible cancellation from the individual terms $\lambda_d r_d(x;\mathcal{A})$ and use the trivial bound

$$R^{\lambda}(x,z;\mathcal{A},\mathcal{P}) \le \sum_{d|P(z)}|\lambda_d r_d(x;\mathcal{A})|.$$

In order to obtain estimates for $V^{\Lambda}(z;\mathcal{P})$, we express it in a more useful form that involves $V(z;\mathcal{P})$. For this, we define multiplicative arithmetic functions h and j defined by

$$h(p^r) = \frac{g(p^r)}{(1-g(p^r))} \quad \text{and} \quad j(p^r) = \frac{1}{(1-g(p^r))},$$

for all primes p and $r \ge 0$. Moreover, these two formulas imply

$$j(p^r) = \frac{h(p^r)}{g(p^r)} = 1 + h(p^r).$$

We call h the **relative density function**. In particular, $j = h * \mathbf{1}$. Now as $\theta = \lambda * \mathbf{1}$, the Möbius inversion formula (see Appendix A.2) implies $\lambda = \theta * \mu$ so that

$$V^{\Lambda}(z;\mathcal{P}) = \sum_{d|P(z)}\sum_{e|d}\theta_e \mu\left(\frac{d}{e}\right) g(d).$$

Making the change of variables $d \to ed$ and noting that $(d,e)=1$ because $P(z)$ is square-free, we compute

$$\begin{aligned}
V^{\Lambda}(z;\mathcal{P}) &= \sum_{de|P(z)} \theta_e \mu(d) g(de) \\
&= \sum_{e|P(z)} \theta_e g(e) \prod_{p|\frac{P(z)}{e}} (1-g(p)) \\
&= \sum_{e|P(z)} \theta_e h(e) \prod_{p|P(z)} (1-g(p)),
\end{aligned}$$

where the second line holds by the multiplicativity of g and the definition of the Möbius function. Defining the function $G^{\Lambda}(z;\mathcal{P})$ by

$$G^{\Lambda}(z;\mathcal{P}) = \sum_{d|P(z)} \theta_d g(d),$$

for $z \geq 2$, Equation (9.4) further implies that

$$V^{\Lambda}(z;\mathcal{P}) = G^{\Lambda}(z;\mathcal{P}) V(z;\mathcal{P}).$$

Therefore estimates for $V^{\Lambda}(z;\mathcal{P})$ reduce to estimates for $G^{\Lambda}(z;\mathcal{P})$ and $V(z;\mathcal{P})$.

Chapter 10

Types of Sieves

10.1 Todo: [Combinatorial Sieves]

The idea behind the combinatorial sieve is to choose sieve weights such that the sieving function is a truncation of the inclusion-exclusion principle applied to the sifting function. To motivate this choice, we being by iteratively applying Buchstab's identity $r \geq 1$ times to obtain

$$S(x, z; \mathcal{A}, \mathcal{P}) = \sum_{\substack{d \mid P(z) \\ \omega(d) < r}} \mu(d) S(x; \mathcal{A}_d) + (-1)^r \sum_{\substack{d \mid P(z) \\ \omega(d) = r}} S(x, p_d; \mathcal{A}_d, \mathcal{P}),$$

where p_d is the smallest prime divisor of d. This identity can be thought of as an inclusion-exclusion principle for the sifting function. As $S(x, p_d; \mathcal{A}_d, \mathcal{P})$ is nonnegative, we obtain the upper bound

$$S(x, z; \mathcal{A}, \mathcal{P}) \leq \sum_{\substack{d \mid P(z) \\ \omega(d) < r}} \mu(d) S(x; \mathcal{A}_d),$$

if r is odd, and the lower bound

$$S(x, z; \mathcal{A}, \mathcal{P}) \geq \sum_{\substack{d \mid P(z) \\ \omega(d) < r}} \mu(d) S(x; \mathcal{A}_d),$$

is r is even. We now wish to consider sieves whose sieving functions replace these upper and lower bounds with sums that are closer approximations to the sifting function. Precisely, we will replace the condition $\omega(d) < r$ with $d \in \mathcal{D}$ for set of positive integers $\mathcal{D}$ with a small amount of small prime divisors. A **combinatorial sieve** $\Lambda = (\lambda_d)_{d \geq 1}$ is a sieve for which $\lambda_d = \mu(d)$ or $\lambda_d = 0$. We will be concerned with upper and lower combinatorial sieves $\Lambda^+ = (\lambda_d^+)_{d \geq 1}$ and $\Lambda^- = (\lambda_d^+)_{d \geq 1}$ where $\lambda_d^{\pm} = \mu(d)$ only for d belonging to the sets

$$\mathcal{D}^+ = \{d = p_1 p_2 \cdots p_r : d < D,\, p_r < p_{r-1} < \cdots < p_1, \text{ and } p_m < y_m \text{ for } m \text{ odd}\},$$

and

$$\mathcal{D}^- = \{d = p_1 p_2 \cdots p_r : d < D,\, p_r < p_{r-1} < \cdots < p_1 < y, \text{ and } p_m < y_m \text{ for } m \text{ even}\},$$

respectively and for some choice of integer $D \geq 1$ and parameters $y_m > 0$ for all $m \geq 1$. Clearly D is the level of $\Lambda^{\pm}$.

10.2 Todo: [Λ^2 Sieves]

10.3 Todo: [The Large Sieve]

Part V

An Introduction to Moments of L-functions

Chapter 11

Moments of L-functions

11.1 The Katz-Sarnak Philosophy

The Katz-Sarnak philosophy is an idea that certain statistics about families of L-functions should match statistics for random matrices coming from some particular compact matrix group. One starts with some class of zeros to look at, say zeros of an individual L-function high up the critical strip or zeros of for some collection of L-functions low down on the critical strip. Actually, one works with the corresponding unfolded nontrivial zeros since they are evenly spaced on average. Then some class of test functions are introduced to carry out the statistical calculations in order to reveal the similarity with some class of matrices. In the following, we give a loose introduction to the Katz-Sarnak philosophy.

The Work of Montgomery & Dyson

The beginning of the connection between random matrix theory and analytic number theory was at Princeton in the 1970s via discussions between Montgomery and Dyson. They found similarities between statistical information about the nontrivial distribution of the zeros of the Riemann zeta function and calculations in random matrix theory about unitary matrices. To do this, they considered the unfolded nontrivial zeros $\rho_{\text{unf}} = \beta + i\omega$ of $\zeta(s)$ with positive ordinate, that is $\omega > 0$, and indexed them according to the size of ordinate. So let $\Omega = (\omega_n)_{n \geq 1}$ denote the increasing sequence of positive ordinates of the unfolded nontrivial zeros of $\zeta(s)$. Montgomery and Dyson considered the **two-point correlation function** $F(\alpha, \beta; \zeta, W)$ for $\zeta(s)$, defined by

$$F(\alpha, \beta; \zeta, W) = \frac{1}{W} |\{(\omega_n, \omega_m) \in \Omega^2 : \omega_n, \omega_m \leq W \text{ and } \omega_n - \omega_m \in [\alpha, \beta]\}|,$$

for any real α and β with $\alpha < \beta$ and $W > 0$. What this function measures is the probability of how close pairs of zeros tend to be with respect to some fixed distance and up to some fixed height. In other words, the correlation between distances of zeros. They wanted to understand if the limiting distribution

$$F(\alpha, \beta; \zeta) = \lim_{W \to \infty} F(\alpha, \beta; \zeta, W),$$

exists and what can be said about it. The following conjecture made by Montgomery, known as Montgomery's pair correlation conjecture, answers this:

Conjecture 11.1.1 (Montgomery's pair correlation conjecture). *For any α and β with $\alpha < \beta$, $F(\alpha, \beta; \zeta)$ exists provided the Riemann hypothesis for the Riemann zeta function holds. Moreover,*

$$F(\alpha, \beta; \zeta) = \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 + \delta(x)\right) dx,$$

where $\delta(x)$ is the Dirac delta function.

Montgomery's pair correlation conjecture still remains out of reach, but there is very good numerical evidence supporting it from some unpublished work of Odlyzko (see [Odl92]). Dyson recognized that Montgomery's pair correlation conjecture models a similar situation in random matrix theory that he had investigated earlier. Consider an $N \times N$ unitary matrix $A \in U(N)$ with eigenphases θ_n for $1 \leq n \leq N$ denoted in increasing order. Clearly the average density of the eigenphases of A in $[0, 2\pi)$ is $\frac{N}{2\pi}$. For any eigenphase θ, let ϕ be the **unfolded eigenphase** corresponding to θ be defined by

$$\phi = \frac{N}{2\pi}\theta.$$

It follows that the average density of the unfolded eigenphases of A in $[0, N)$ is 1. Let $\Phi = (\phi_n)_{1 \leq n \leq N}$ denote the increasing sequence of unfolded eigenphases of A. We consider the **two-point correlation function** $F(\alpha, \beta; A, U(N))$ for A, defined by

$$F(\alpha, \beta; A, U(N)) = \frac{1}{N}|\{(\phi_n, \phi_m) \in \Phi^2 : \phi_n - \phi_m \in [\alpha, \beta]\}|,$$

for any real α and β with $\alpha < \beta$. Since $U(N)$ has a Haar measure dA, we can compute the global distribution of $F(\alpha, \beta; A, U(N))$ over $U(N)$, namely $F(\alpha, \beta; U(N))$, defined by

$$F(\alpha, \beta; U(N)) = \int_{U(N)} F(\alpha, \beta; A, U(N))\, dA.$$

Analogously, we want to understand if the limiting distribution

$$F(\alpha, \beta; U) = \lim_{N \to \infty} F(\alpha, \beta; U(N)),$$

exists and what can be said about it. Dyson showed the following (see [Dys62] for a proof):

Proposition 11.1.1. *For any real α and β with $\alpha < \beta$, $F(\alpha, \beta; U)$ exists. Moreover,*

$$F(\alpha, \beta; U) = \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 + \delta(x)\right) dx,$$

where $\delta(x)$ is the Dirac delta function.

The right-hand side of Proposition 11.1.1 is exactly the same formula given in Montgomery's pair correlation conjecture. In other words, if Montgomery's pair correlation conjecture is true then the two-point correlation of the unfolded nontrivial zeros of the Riemann zeta function in the limit as we move up the critical line exactly match the two-point correlation of the unfolded eigenphases of unitary matrices in the limit as the size of the matrices increase. In short, statistical information about the Riemann zeta function agrees with statistical information about the eigenvalues of unitary matrices. This is the origin of the Katz-Sarnak philosophy.

The Work of Katz & Sarnak

Katz and Sarnak generalized the work of Montgomery and Dyson by establishing a connection between families of L-functions and other compact matrix groups. For the ease of categorization, Katz and Sarnak associated a **symmetry type** to each compact matrix group that they studied. The underlying compact matrix group associated to each symmetric type is called a **matrix ensemble**. The symmetry types and associated matrix ensembles are described in the following table:

Symmetry Type	Matrix Ensemble
Unitary (U)	$\mathrm{U}(N)$
Orthogonal (O^+)	$\mathrm{SO}(2N)$
Orthogonal (O^-)	$\mathrm{SO}(2N+1)$
Symplectic (Sp)	$\mathrm{USp}(2N)$

Let $G(N)$ be a matrix ensemble, where G denotes the symmetry type, and let dA denote the Haar measure. For any $A \in G(N)$, let $\Phi = (\phi_n)_{1\leq n\leq N}$ denote the increasing sequence of unfolded eigenphases of A. Katz and Sarnak considered two local spacing distributions between the unfolded eigenphases of A. The first was the **two-point correlation function** $F(\alpha,\beta;A,G(N))$ for A, defined by

$$F(\alpha,\beta;A,G(N)) = \frac{1}{N}|\{(\phi_n,\phi_m)\in\Phi^2 : \phi_n-\phi_m\in[\alpha,\beta]\}|,$$

for any real α and β with $\alpha<\beta$. They computed the global distribution of $F(\alpha,\beta;A,G(N))$ over $G(N)$, namely $F(\alpha,\beta;G(N))$, defined by

$$F(\alpha,\beta;G(N)) = \int_{G(N)} F(\alpha,\beta;A,G(N))\,dA,$$

and sought to understand if the limiting distribution

$$F(\alpha,\beta;G) = \lim_{N\to\infty} F(\alpha,\beta;G(N)),$$

exists and what can be said about it. Note that in the case $G(N)=U(N)$, this is exactly the two-point correlation function considered by Dyson. Katz and Sarnak succeeded in generalizing Dyson's work (see [KS23] for a proof):

Proposition 11.1.2. *For symmetry type G and any real α and β with $\alpha<\beta$, $F(\alpha,\beta;G)$ exists. Moreover,*

$$F(\alpha,\beta;G) = \int_\alpha^\beta \left(1-\left(\frac{\sin(\pi x)}{\pi x}\right)^2+\delta(x)\right)dx,$$

where $\delta(x)$ is the Dirac delta function.

The second local spacing distribution was the **k-th consecutive spacing function** for A, defined by

$$\mu_k(\alpha,\beta;A,G(N)) = \frac{1}{N}|\{1\leq j\leq N : (\phi_{j+k},\phi_j)\in\Phi^2 \text{ and } \phi_{j+k}-\phi_j\in[\alpha,\beta]\}|,$$

for any real α and β with $\alpha<\beta$ and $k\geq 1$. Again, they proceeded to compute the global distribution over $G(N)$, namely $\mu_k(\alpha,\beta;G(N))$, defined by

$$\mu_k(\alpha,\beta;G(N)) = \int_{G(N)} \mu_k(\alpha,\beta;A,G(N))\,dA,$$

and asked if the limiting distribution

$$\mu_k(\alpha,\beta;G) = \lim_{N\to\infty} \mu_k(\alpha,\beta;G(N)),$$

exists and what can be said about it. They were able the show the following (see [KS23] for a proof):

Proposition 11.1.3. *For any symmetry type G and any real α and β with $\alpha < \beta$ and $k \geq 1$, $\mu_k(\alpha, \beta; G)$ exists. Moreover, it is independent of the particular symmetry type G.*

In particular, Propositions 11.1.2 and 11.1.3 show that the limiting distributions $F(\alpha, \beta; G)$ and $\mu_k(\alpha, \beta; G)$ are both independent of the symmetry type G. However, this symmetry independence is not true for all limiting distributions. Katz and Sarnak also considered a local and global distributions associated to single eigenphases. The local distribution they considered, associated to a single eigenphase, was the **one-level density function** $\Delta(\alpha, \beta; A, G(N))$ for A, defined by

$$\Delta(\alpha, \beta; A, G(N)) = |\{\phi \in \Phi : \phi \in [\alpha, \beta]\}|.$$

They computed the global distribution, namely $\Delta(\alpha, \beta; G(N))$, defined by

$$\Delta(\alpha, \beta; G(N)) = \int_{G(N)} \Delta(\alpha, \beta; A, G(N))\, dA,$$

and asked if the limiting distribution

$$\Delta(\alpha, \beta; G) = \lim_{N \to \infty} \Delta(\alpha, \beta; G(N)),$$

exists and what can be said about it. Precisely, they proved the following (see [KS23] for a proof):

Proposition 11.1.4. *For any symmetry type G and any real α and β with $\alpha < \beta$ and $k \geq 1$, $\Delta(\alpha, \beta; G)$ exists. Moreover, it depends upon the particular symmetry type G.*

The global distribution they considered, associated to a single eigenphase, was the **k-th eigenphase function** $\nu_k(\alpha, \beta, G(N))$ for $G(N)$, defined by

$$\nu_k(\alpha, \beta; G(N)) = dA\left(\{A \in G(N) : \phi_k \in [\alpha, \beta]\}\right),$$

for any real α and β with $\alpha < \beta$ and $k \geq 1$. Again, they asked if the limiting distribution

$$\nu_k(\alpha, \beta; G) = \lim_{N \to \infty} \nu_k(\alpha, \beta; G(N)),$$

exists and what can be said about it. They were able to show the following (see [KS23] for a proof):

Proposition 11.1.5. *For any symmetry type G and any real α and β with $\alpha < \beta$ and $k \geq 1$, $\nu_k(\alpha, \beta; G)$ exists. Moreover, it depends upon the particular symmetry type G.*

In short, Katz and Sarnak studied four limiting distributions $F(\alpha, \beta; G)$, $\mu_k(\alpha, \beta; G)$, $\Delta(\alpha, \beta; G)$, and $\nu_k(\alpha, \beta; G)$. The former two are distributions about collections of eigenphases of unitary matrices and are independent of the symmetry type G while the latter two are distributions about single eigenphases of unitary matrices and depend upon the symmetry type G. Analogous distributions can be defined for families of L-functions. We say that a collection of L-functions $(L(s_\alpha, f_\alpha))_{\alpha \in I}$, for some infinite indexing set $I \subset \mathbb{R}_{\geq 0}$, is a **family** if it is an ordered set with respect to the analytic conductor and if $\mathfrak{q}(s_\alpha, f_a) \to \infty$ as $\alpha \to \infty$. We say that a family $(L(s_\alpha, f_\alpha))_{\alpha \in I}$ is **continuous** if $f_\alpha = f_\beta$ for all $\alpha, \beta \in I$ and $s_\alpha = \sigma + it_\alpha$ where t_α is a continuous function of α. Necessarily, $t_\alpha \to \infty$ as $\alpha \to \infty$ and I is a half-open ray. We say that a family $(L(s_\alpha, f_\alpha))_{\alpha \in I}$ is **discrete** if $f_\alpha \neq f_\beta$ for all distinct $\alpha, \beta \in I$ and I is discrete. Necessarily, $q(f_\alpha) \to \infty$ as $\alpha \to \infty$ and, reindexing if necessary, $I \subseteq \mathbb{Z}_{\geq 0}$. Katz and Sarnak arrived at a heuristical conjecture known as the **Katz-Sarnak philosophy** in terms of families of L-functions:

Conjecture 11.1.2 (Katz-Sarnak Philosophy).

(i) The statistics about collections of eigenphases of a random matrix belonging to a matrix ensemble of symmetry type G, in the limit as the size of the matrix tends to infinity, should model statistics about the nontrivial zeros of a continuous family of L-functions as the heights of the nontrivial zeros tends to infinity.

(ii) The statistics about a single eigenphase of a random matrix belonging to a matrix ensemble of symmetry type G, in the limit as the size of the matrix tends to infinity, should model statistics about a discrete family of L-functions as the size of the conductor tends to infinity.

Determining the symmetry type of a family is generally a difficult task. Below are some well-studied families and their symmetry types (see [CFK^{+}05] for a determination of the symmetry type):

Symmetry Type	Family
Unitary	$\{L(\sigma+it,f) : L(s,f)$ is primitive and $t \geq 0\}$ ordered by t
	$\{L(s,\chi) : \chi$ is a primitive character modulo q with $q \geq 1\}$ ordered by q
Orthogonal	$\{L(s,f) : f \in \mathcal{S}_k(N,\chi)$ and $k \geq 1\}$ ordered by k
	$\{L(s,f) : f \in \mathcal{S}_k(N,\chi)$ and $N \geq 1\}$ ordered by N
Symplectic	$\{L(s,\chi_d) : d$ is a fundamental discriminant with $\chi_d(n) = \left(\frac{d}{n}\right)\}$ ordered by $\lvert d\rvert$

Characteristic Polynomials of Unitary Matrices

The Katz-Sarnak philosophy conjectures that the statistics about nontrivial zeros of L-functions are modeled by the statistics of eigenphases of random unitary matrices. However, there is a striking surface level connection between L-functions and the characteristic polynomials of unitary matrices which we now describe. For $A \in U(N)$, let

$$L(s,A) = \det(I - sA) = \prod_{1 \leq n \leq N} (1 - se^{i\theta_n}),$$

be the characteristic polynomial of A. It will turn out that $L(s,A)$ has strikingly similar properties to an L-function. The product expression for $L(s,A)$ is clearly analogous to the Euler product expression for an L-function. Upon expanding the product, we obtain

$$L(s,A) = \sum_{0 \leq n \leq N} a_n s^n,$$

for some coefficients a_n. This expression is the analogue to the Dirichlet series representation for an L-function. Of course, as $L(s,A)$ is a polynomial it is analytic on $\mathbb{C}$. Moreover, $L(s,A)$ possesses a functional equation of shape $s \to \frac{1}{s}$. To see this, first observe that multiplicativity of the determinant gives

$$L(s,A) = (-1)^N \det(A) s^N \det(I - s^{-1}A^{-1}).$$

As A is unitary, $L(s,A^{-1}) = L(s,A^*) = L(s,\overline{A})$. So the above equation can be expressed as

$$L(s,A) = (-1)^N \det(A) s^N L\left(\frac{1}{s}, \overline{A}\right).$$

This is the analogue of the functional equation for $L(s, A)$ and it is of shape $s \to \frac{1}{s}$. We identify the analogues of the gamma factor and conductor as 1 and N respectively. Letting $\Lambda(s, A)$ be defined by

$$\Lambda(s, A) = s^{-\frac{N}{2}} L(s, A),$$

the functional equation can be expressed as

$$\Lambda(s, A) = (-1)^N \det(A) \Lambda\left(\frac{1}{s}, \overline{A}\right).$$

From it, the analogue of root number is seen to be $(-1)^N \det(A)$ and $L(s, A)$ has dual $L(s, \overline{A})$. As the transformation $s \to \frac{1}{s}$ leaves the unit circle invariant, the unit circle is the analogue of the critical line. The fixed point of the transformation $s \to \frac{1}{s}$ is $s = 1$ which is the analogue of the central point. Moreover, as the zeros of $L(s, A)$ are precisely the eigenvalues of A which lie on the unit circle, because A is unitary, the analogue of the Riemann hypothesis is true for $L(s, A)$. We also have an analogue of the approximate functional equation. By substituting the polynomial representation of $L(s, A)$ into the functional equation, we obtain

$$\sum_{0 \le n \le N} a_n s^n = (-1)^N \det(A) s^N \sum_{0 \le n \le N} \overline{a_n} s^{-n} = (-1)^N \det(A) \sum_{0 \le n \le N} \overline{a_n} s^{N-n}.$$

Upon comparing coefficients, we find that

$$a_n = (-1)^N \det(A) \overline{a_{N-n}},$$

for $0 \le n \le N$. So for odd N,

$$L(s, A) = \sum_{0 \le n \le \frac{N-1}{2}} a_n s^n + (-1)^N \det(A) s^N \sum_{0 \le n \le \frac{N-1}{2}} \overline{a_n} s^{-n},$$

and for even N,

$$L(s, A) = a_{\frac{N}{2}} s^{\frac{N}{2}} + \sum_{0 \le n \le \frac{N}{2}-1} a_n s^n + (-1)^N \det(A) s^N \sum_{0 \le n \le \frac{N}{2}-1} \overline{a_n} s^{-n}.$$

These equations together are the analogue of the approximate functional equation. This similarity between L-functions and the characteristic polynomials of unitary matrices was heavily exploited by Conrey, Farmer, Keating, Rubinstein, and Snaith to make phenomenal conjectures about the moments of L-functions.

11.2 Todo: [Types of Moments]

11.3 Todo: [The CFKRS Conjectures]

Part VI

Appendices

Appendix A

Number Theory

A.1 Arithmetic Functions

An arithmetic function f is a function $f : \mathbb{N} \to \mathbb{C}$. That is, it takes the positive integers into the complex numbers. We say that f is **additive** if $f(nm) = f(n) + f(m)$ for all positive integers n and m such that $(n, m) = 1$. If this condition simply holds for all n and m then we say f is **completely additive**. Similarly, we say that f is **multiplicative** if $f(nm) = f(n)f(m)$ for all positive integers n and m such that $(n, m) = 1$. If this condition simply holds for all n and m then we say f is **completely multiplicative**. Many important arithmetic functions are either additive, completely additive, multiplicative, or completely multiplicative. Note that if a f is additive or multiplicative then f is uniquely determined by its values on prime powers and if f is completely additive or completely multiplicative then it is uniquely determined by its values on primes. Moreover, if f is additive or completely additive then $f(1) = 0$ and if f is multiplicative or completely multiplicative then $f(1) = 1$. Below is a list defining the most important arithmetic functions (some of these functions are restrictions of common functions but we define them here as arithmetic functions because their domain being $\mathbb{N}$ is important):

(i) The **constant function**: The function $\mathbf{1}(n)$ restricted to all $n \geq 1$. This function is neither additive or multiplicative.

(ii) The **unit function**: The function $e(n)$ defined by

$$e(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

This function is completely multiplicative.

(iii) The **identity function**: The function $\mathrm{id}(n)$ restricted to all $n \geq 1$. This function is completely multiplicative.

(iv) The **logarithm**: The function $\log(n)$ restricted to all $n \geq 1$. This function is completely additive.

(v) The **Möbius function**: The function $\mu(n)$ defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free with an even number of prime factors,} \\ -1 & \text{if } n \text{ is square-free with an odd number of prime factors,} \\ 0 & \text{if } n \text{ is not square-free,} \end{cases}$$

for all $n \geq 1$. This function is multiplicative.

(vi) The **characteristic function of square-free integers**: The square of the Möbius function $\mu^2(n)$ for all $n \geq 1$. This function is multiplicative.

(vii) **Liouville's function**: The function $\lambda(n)$ defined by

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is composed of } k \text{ not necessarily distinct prime factors,} \end{cases}$$

for all $n \geq 1$. This function is completely multiplicative.

(viii) **Euler's totient function**: The function $\varphi(n)$ defined by

$$\varphi(n) = \sideset{}{'}\sum_{m \pmod n} 1,$$

for all $n \geq 1$. This function is multiplicative.

(ix) The **divisor function**: The function $\sigma_0(n)$ defined by

$$\sigma_0(n) = \sum_{d|n} 1,$$

for all $n \geq 1$. This function is multiplicative.

(x) The **sum of divisors function**: The function $\sigma_1(n)$ defined by

$$\sigma_1(n) = \sum_{d|n} d,$$

for all $n \geq 1$. This function is multiplicative.

(xi) The **generalized sum of divisors function**: The function $\sigma_s(n)$ defined by

$$\sigma_s(n) = \sum_{d|n} d^s,$$

for all $n \geq 1$ and any complex number s. This function is multiplicative.

(xii) The **number of distinct prime factors function**: The function $\omega(n)$ defined by

$$\omega(n) = \sum_{p|n} 1,$$

for all $n \geq 1$. This function is additive.

(xiii) The **total number of prime divisors function**: The function $\Omega(n)$ defined by

$$\Omega(n) = \sum_{p^m|n} 1,$$

for all $n \geq 1$ and where $m \geq 1$. This function is completely additive.

(xiv) The **von Mangoldt function**: The function $\Lambda(n)$ defined by

$$\Lambda(n) = \begin{cases} 0 & \text{if } n \text{ is not a prime power,} \\ \log(p) & \text{if } n = p^m \text{ for some prime } p \text{ and integer } m \geq 1, \end{cases}$$

for all $n \geq 1$. This function is neither additive or multiplicative.

If f and g are two arithmetic functions, then we can define a new arithmetic function $f * g$ called the **Dirichlet convolution** of f and g defined by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right),$$

for all $n \geq 1$. This is especially useful when f and g are multiplicative:

Proposition A.1.1. *If f and g are multiplicative arithmetic functions, then so is their Dirichlet convolution $f * g$.*

A.2 The Möbius Function

Recall that the Möbius function is the arithmetic function μ defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free with an even number of prime factors,} \\ -1 & \text{if } n \text{ is square-free with an odd number of prime factors,} \\ 0 & \text{if } n \text{ is not square-free,} \end{cases}$$

and it is multiplicative. It also satisfies an important summation property:

Proposition A.2.1.

$$\sum_{d|n} \mu(d) = \delta_{n,1}.$$

From this property, the important **Möbius inversion formula** can be derived:

Theorem A.2.1 (Möbius inversion formula). *Suppose f and g are arithmetic functions. Then*

$$g(n) = \sum_{d|n} f(d),$$

for all $n \geq 1$, if and only if

$$f(n) = \sum_{d|n} g(d) \mu\left(\frac{n}{d}\right)$$

for all $n \geq 1$.

In terms of Dirichlet convolution, the Möbius inversion formula is equivalent to stating that $g = f * \mathbf{1}$ if and only if $f = g * \mu$. Using Möbius inversion, the following useful formula can also be derived:

Proposition A.2.2. *For $\sigma > 1$,*

$$\sum_{n \geq 1} \frac{\mu(n)}{n^s} = \zeta(s)^{-1} = \prod_p (1 - p^{-s}).$$

There is also an important similar statement to the Möbius inversion formula that we will need:

Theorem A.2.2. *Let f be an arithmetic function and let B be the completely multiplicative function defined on primes p by*

$$B(p) = f(p)^2 - f(p^2).$$

Then

$$f(n)f(m) = \sum_{d|(n,m)} B(d) f\left(\frac{nm}{d^2}\right),$$

for all $n, m \geq 1$, if and only if

$$f(nm) = \sum_{d|(n,m)} \mu(d)B(d) f\left(\frac{n}{d}\right) f\left(\frac{m}{d}\right),$$

for all $n, m \geq 1$.

Any arithmetic function f satisfying the conditions of Theorem A.2.2 is said to be **specially multiplicative**.

A.3 The Sum and Generalized Sum of Divisors Functions

It is very useful to know that $\sigma_0(n)$ grows slowly:

Proposition A.3.1.

$$\sigma_0(n) \ll_\varepsilon n^\varepsilon.$$

This is all we really need to know for the sum of divisors function. As for the generalized sum of divisors function, it has the remarkable property that it can be written as a product. To state it, recall that $\mathrm{ord}_p(n)$ is the positive integer satisfying $p^{\mathrm{ord}_p(n)} \mid\mid n$. Then we have the following statement:

Proposition A.3.2. *For $s \neq 0$,*

$$\sigma_s(n) = \prod_{p|n} \frac{p^{(\mathrm{ord}_p(n)+1)s} - 1}{p^s - 1}.$$

A.4 Quadratic Symbols

Let p be an odd prime. We are often interested in when the equation $x^2 = a \pmod p$ is solvable for some $a \in \mathbb{Z}$. The **Legendre symbol** $\left(\frac{a}{p}\right)$ keeps track of this:

$$\left(\frac{a}{q}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a \pmod p \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv a \pmod p \text{ is not solvable,} \\ 0 & \text{if } a \equiv 0 \pmod p. \end{cases}$$

Euler's criterion gives an alternative expression for Legendre symbol when a is coprime to p:

Proposition A.4.1 (Euler's criterion). *Let p be an odd prime and suppose $(a,p) = 1$. Then*

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} \pmod p.$$

From the definition and Euler's criterion is it not difficult to show that the Legendre symbol satisfies the following properties:

Proposition A.4.2. *Let p be an odd prime and let $a, b \in \mathbb{Z}$. Then the following hold:*

(i) If $a \equiv b \pmod p$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

(ii) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

From Proposition A.4.2, to compute the Legendre symbol in general it suffices to know how to compute $\left(\frac{-1}{p}\right)$, $\left(\frac{2}{p}\right)$, and $\left(\frac{q}{p}\right)$ where q is another odd prime. The **supplemental laws of quadratic reciprocity** are formulas for the first two symbols:

Proposition A.4.3 (Supplemental laws of quadratic reciprocity). *Let p be an odd prime.*

(i)

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ -1 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

(ii)

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if } p \equiv 1, 7 \pmod 8, \\ -1 & \text{if } p \equiv 3, 5 \pmod 8. \end{cases}$$

The **law of quadratic reciprocity** handles the last symbol by relating $\left(\frac{q}{p}\right)$ to $\left(\frac{p}{q}\right)$:

Theorem A.4.1 (Law of quadratic reciprocity). *Let p and q be distinct odd primes. Then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \text{ or } q \equiv 1 \pmod 4, \\ -1 & \text{if } p \equiv 3 \pmod 4 \text{ and } q \equiv 3 \pmod 4. \end{cases}$$

We can generalize the Jacobi symbol further by making it multiplicative in the denominator. Let n be a positive odd integer with prime factorization $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ and let $a \in \mathbb{Z}$. The **Jacobi symbol** $\left(\frac{a}{n}\right)$ is defined by

$$\left(\frac{a}{n}\right) = \prod_{1 \le i \le k} \left(\frac{a}{p_i}\right)^{r_i}.$$

When $n = p$ is prime, the Jacobi symbol reduces to the Legendre symbol and the Jacobi symbol is precisely the unique multiplicative extension of the Legendre symbol to all positive odd integers. Accordingly, the Jacobi symbol has the following properties:

Proposition A.4.4. *Let m and n be positive odd integers and let $a, b \in \mathbb{Z}$. Then the following hold:*

(i) If $a \equiv b \pmod p$, then $\left(\frac{a}{n}\right) = \left(\frac{b}{n}\right)$.

(ii) $\left(\frac{a}{mn}\right) = \left(\frac{a}{m}\right)\left(\frac{a}{n}\right)$.

(iii) $\left(\frac{ab}{n}\right) = \left(\frac{a}{n}\right)\left(\frac{b}{n}\right)$.

There is also an associated reciprocity law:

Proposition A.4.5. *Let m and n be distinct positive odd integers. Then*

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (-1)^{\frac{m-1}{2}\frac{n-1}{2}} = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4 \text{ or } n \equiv 1 \pmod 4, \\ -1 & \text{if } m \equiv 3 \pmod 4 \text{ and } n \equiv 3 \pmod 4. \end{cases}$$

We can further generalize the Jacobi symbol so that it is valid for all integers. Let $a, n \in \mathbb{Z}$ where $n = up_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ is the prime factorization of n with $u = \pm 1$. The **Kronecker symbol** $\left(\frac{a}{n}\right)$ is defined by

$$\left(\frac{a}{n}\right) = \left(\frac{a}{u}\right) \prod_{1 \le i \le k} \left(\frac{a}{p_i}\right)^{r_i},$$

where we set

$$\left(\frac{a}{1}\right) = 1, \quad \left(\frac{a}{-1}\right) = \begin{cases} 1 & \text{if } a \ge 0, \\ -1 & \text{if } a < 0, \end{cases} \quad \left(\frac{a}{0}\right) = \begin{cases} 1 & \text{if } a = \pm 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\left(\frac{a}{2}\right) = \begin{cases} 1 & \text{if } a \equiv 1, 7 \pmod 8, \\ -1 & \text{if } a \equiv 3, 5 \pmod 8, \\ 0 & \text{if } a \equiv 0 \pmod 2. \end{cases}$$

When n is a positive odd integer, the Kronecker symbol reduces to the Jacobi symbol. The Kronecker symbol also satisfies a reciprocity law:

Proposition A.4.6. *Let $m, n \in \mathbb{Z}$. Then*

$$\left(\frac{m}{n}\right)\left(\frac{n}{|m|}\right) = (-1)^{\frac{m^{(2)}-1}{2}\frac{n^{(2)}-1}{2}}.$$

where $m^{(2)}$ and $n^{(2)}$ are the parts of m and n relatively prime to 2 respectively.

Appendix B

Analysis

B.1 Local Absolute Uniform Convergence & Boundedness

Often, we are interested in some series

$$\sum_{n\geq 1} f_n(z),$$

where the $f_n(z)$ are analytic functions on some region Ω. We say that the series above is **locally absolutely uniformly convergent** if

$$\sum_{n\geq 1} |f_n(z)|,$$

converges uniformly on compact subsets of Ω. This mode of convergence is very useful because it is enough to guarantee the series is analytic on Ω:

Theorem B.1.1. *Suppose $(f_n(z))_{n\geq 1}$ is a sequence of analytic functions on a region Ω. Then if*

$$\sum_{n\geq 1} f_n(z),$$

is locally absolutely uniformly convergent, it is analytic on Ω.

We can also apply this idea in the case of integrals. Suppose we have an integral

$$\int_D f(z,x)\,dx,$$

where $f(z,x)$ is an analytic function on some region $\Omega \times D$. The integral is a function of z, and we say that the integral is **locally absolutely uniformly convergent** if

$$\int_D |f(z,x)|\,dx,$$

converges uniformly on compact subsets of Ω. Similar to the series case, this mode of convergence is very useful because it guarantees the integral is analytic on Ω:

Theorem B.1.2. *Suppose $f(z,x)$ is an analytic function on a region $\Omega \times D$. Then if*

$$\int_D f(z,x)\,dx,$$

is locally absolutely uniformly convergent, it is holomorphic on Ω.

B.2 Interchange of Integrals, Sums & Derivatives

Often, we would like to interchange a limit and a integral. This process is not always allowed, but in many instances it is. The **dominated convergence theorem** (DCT) covers the most well-known sufficient condition:

Theorem B.2.1 (Dominated convergence theorem). *Let $(f_n(z))_{n\geq 1}$ be a sequence of continuous real or complex integrable functions on some region Ω. Suppose that that the sequence converges pointwise to a function $f(z)$, and that there is some integrable function g on Ω such that*

$$|f_n(z)| \leq g(z)$$

for all $n \geq 1$ and all $z \in \Omega$. Then $f(z)$ is integrable on Ω and

$$\lim_{n\to\infty} \int_\Omega f_n(z)\,dz = \int_\Omega f(z)\,dz.$$

This theorem is often employed when the underlying sequence is a sequence of partial sums of an absolutely convergent series ($g(s)$ will be the absolute series). In this case we have the following result:

Corollary B.2.1. *Suppose $\sum_{n\geq 1} f_n(z)$ is an absolutely convergent series of real or complex continuous functions that are integrable on some region Ω. Then*

$$\sum_{n\geq 1} \int_\Omega f_n(z)\,dz = \int_\Omega \sum_{n\geq 1} f_n(z)\,dz,$$

provided the sum and integral on either side are absolutely convergent and absolutely bounded.

Of course, we can apply Corollary B.2.1 repeatedly to interchange a sum with multiple integrals provided that the partial sums are absolutely convergent in each variable.
Other times we would also like to interchange a derivative and an integral. The **Leibniz integral rule** tells us when this is allowed:

Theorem B.2.2 (Leibniz integral rule). *Suppose $f(\mathbf{x},t)$ is a function such that both $f(\mathbf{x},t)$ and its partial derivative $\frac{\partial}{\partial x_i} f(\mathbf{x},t)$ are continuous in $\mathbf{x}$ and t in some region including $\Omega \times [a(\mathbf{x}), b(\mathbf{x})]$ for some real-valued functions $a(\mathbf{x})$ and $b(\mathbf{x})$ and region Ω. Also suppose that $a(\mathbf{x})$ and $b(\mathbf{x})$ are continuous with continuous partial derivatives $\frac{\partial}{\partial x_i} a(\mathbf{x})$ and $\frac{\partial}{\partial x_i} b(\mathbf{x})$ for $\mathbf{x} \in \Omega$. Then for $\mathbf{x} \in \Omega$, we have*

$$\frac{\partial}{\partial x_i}\left(\int_{a(\mathbf{x})}^{b(\mathbf{x})} f(\mathbf{x},t)\,dt\right) = f(\mathbf{x}, b(\mathbf{x}))\frac{\partial}{\partial x_i} b(\mathbf{x}) - f(\mathbf{x}, a(\mathbf{x}))\frac{\partial}{\partial x_i} a(\mathbf{x}) + \int_{a(\mathbf{x})}^{b(\mathbf{x})} \frac{\partial}{\partial x_i} f(\mathbf{x},t)\,dt.$$

The Leibniz integral rule is sometimes applied in the case when $a(\mathbf{x}) = a$ and $b(\mathbf{x}) = b$ are constant. In this case, we get the following corollary:

Corollary B.2.2. *Suppose $f(\mathbf{x},t)$ is a function such that both $f(\mathbf{x},t)$ and its partial derivative $\frac{\partial}{\partial x_i} f(\mathbf{x},t)$ are continuous in $\mathbf{x}$ and t in some region including $\Omega \times [a,b]$ for some reals a and b and region Ω. Then for $\mathbf{x} \in \Omega$, we have*

$$\frac{\partial}{\partial x_i}\left(\int_a^b f(\mathbf{x},t)\,dt\right) = \int_a^b \frac{\partial}{\partial x_i} f(\mathbf{x},t)\,dt.$$

B.3 Summation Formulas

The most well-known summation formula is **partial summation**:

Theorem B.3.1 (Partial summation). *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of complex numbers. Then for any positive integers N and M with $1 \leq M < N$ we have*

$$\sum_{M \leq k \leq N} a_k(b_{k+1} - b_k) = (a_N b_{N+1} - a_M b_M) - \sum_{M+1 \leq k \leq N} b_k(a_k - a_{k-1}).$$

There is a more useful summation formula for analytic number theory as it lets one estimate discrete sums by integrals. For this we need some notation. If $(a_n)_{n\geq 1}$ is a sequence of complex numbers, for every $X > 0$ set

$$A(X) = \sum_{n \leq X} a_n.$$

Then **Abel's summation formula** is the following:

Theorem B.3.2 (Abel's summation formula). *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. For every X and Y with $0 \leq X < Y$ and continuously differentiable function $\phi : [X, Y] \to \mathbb{C}$, we have*

$$\sum_{X \leq n \leq Y} a_n \phi(n) = A(Y)\phi(Y) - A(X)\phi(X) - \int_X^Y A(u)\phi'(u)\, du.$$

There are also some useful corollaries. For example, if we take the limit as $Y \to \infty$ we obtain:

Corollary B.3.1. *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. For every $X \geq 0$ and continuously differentiable function $\phi(y)$, we have*

$$\sum_{n \geq X} a_n \phi(n) = \lim_{Y \to \infty} A(Y)\phi(Y) - A(X)\phi(X) - \int_X^\infty A(u)\phi'(u)\, du.$$

We can take this corollary further by letting $X < 1$ so that $A(X) = 0$ to get the following:

Corollary B.3.2. *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. For every continuously differentiable function $\phi(y)$, we have*

$$\sum_{n \geq 1} a_n \phi(n) = \lim_{Y \to \infty} A(Y)\phi(Y) - \int_1^\infty A(u)\phi'(u)\, du.$$

B.4 Fourier Series

Let $N \geq 1$ be an integer. If $f(x)$ is N-periodic and integrable on $[0, N]$, then we define the n-th **Fourier coefficient** $\hat{f}(n)$ of $f(x)$ to be

$$\hat{f}(n) = \int_0^N f(x)e^{-\frac{2\pi i n x}{N}}\, dx.$$

The **Fourier series** of $f(x)$ is defined by the series

$$\sum_{n \in \mathbb{Z}} \hat{f}(n)e^{\frac{2\pi i n x}{N}}.$$

There is the question of whether the Fourier series of $f(x)$ converges at all and if so does it even converge to $f(x)$ itself. Under reasonable conditions the answer is yes as is seen in the following proposition:

Proposition B.4.1. *If $f(x)$ is smooth and N-periodic it converges uniformly to its Fourier series everywhere.*

In particular, all holomorphic N-periodic functions $f(z)$ converge uniformly to their Fourier series everywhere because for fixed y, $f(z)$ restricts to an N-periodic smooth function on $\mathbb{R}$ that's integrable on $[0, N]$. Actually we can do a little better. If $f(z)$ is N-periodic and meromorphic on $\mathbb{C}$, then after clearing polar divisors we will have a holomorphic function on $\mathbb{C}$ and hence it will have a Fourier series converging uniformly everywhere. Therefore $f(z)$ will have such a Fourier series with meromorphic Fourier coefficients. In either situation, the case $N = 1$ is the most commonly seen. So for meromorphic (or holomorphic) 1-periodic functions $f(z)$,

$$f(z) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n z},$$

uniformly everywhere.

B.5 Factorizations, Order & Rank

The **elementary factors**, also referred to as **primary factors**, are the entire functions $E_n(z)$ defined by

$$E_n(z) = \begin{cases} 1 - z & \text{if } n = 0, \\ (1-z)e^{z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}} & \text{if } n \neq 0. \end{cases}$$

If $f(z)$ is an entire function, then it admits a factorization in terms of its zeros and the elementary factors. This is called the **Weierstrass factorization** of $f(z)$:

Theorem B.5.1 (Weierstrass factorization). *Let $f(z)$ be an entire function with $\{a_n\}_{n \geq 1}$ the nonzero zeros of $f(z)$ counted with multiplicity. Also suppose that $f(z)$ has a zero of order m at $z = 0$ where it is understood that if $m = 0$ we mean $f(0) \neq 0$ and if $m < 0$ we mean $f(z)$ has a pole of order $|m|$ at $z = 0$. Then there exists an entire function $g(z)$ and sequence of nonnegative integers $(p_n)_{n \geq 1}$ such that*

$$f(z) = z^m e^{g(z)} \prod_{n \geq 1} E_{p_n}\left(\frac{z}{a_n}\right).$$

The Weierstrass factorization of $f(z)$ can be strengthened if $f(z)$ does not grow too fast. We say $f(z)$ is of **finite order** if there exists a $\rho_0 > 0$ such that

$$f(z) \ll e^{|z|^{\rho_0}},$$

for all $z \in \mathbb{C}$. The **order** ρ of $f(z)$ is the infimum of the ρ_0. Let $q = \lfloor \rho \rfloor$. If there is no such ρ_0, $f(z)$ is said to be of **infinite order** and we set $\rho = q = \infty$. Let $\{a_n\}_{n \geq 1}$ be the nonzero zeros of $f(z)$ that are not zero and ordered such that $a_n \to \infty$ as $n \to \infty$ if there are infinitely many zeros. Then we define the **rank** of $f(z)$ to be the smallest positive integer p such that the series

$$\sum_{n \geq 1} \frac{1}{|a_n|^{p+1}},$$

converges. If there is no such integer we set $p = \infty$ and if there are finitely many zeros we set $p = 0$. We set $g = \max\{p, q\}$ and call g the **genus** of $f(z)$. We can now state the **Hadamard factorization** of $f(z)$:

Theorem B.5.2 (Hadamard factorization). *Let $f(z)$ be an entire function of finite order ρ. If p is the rank and g is the genus, then $g \le \rho$. Moreover, let $\{a_n\}_{n\ge1}$ be the nonzero zeros of $f(z)$ counted with multiplicity and suppose that $f(z)$ has a zero of order m at $z=0$ where it is understood that if $m=0$ we mean $f(0)\neq 0$ and if $m<0$ we mean $f(z)$ has a pole of order $|m|$ at $z=0$. Then there exists a polynomial $Q(z)$ of degree at most q such that*

$$f(z) = z^m e^{Q(z)} \prod_{n\ge1} E_p\left(\frac{z}{a_n}\right).$$

Moreover, the sum

$$\sum_{n\ge1} \frac{1}{|a_n|^{\rho+\varepsilon}},$$

converges.

B.6 The Phragmén-Lindelöf Convexity Principle

The **Phragmén-Lindelöf convexity principle** is a generic name for extending the maximum modulus principle to unbounded regions. The **Phragmén-Lindelöf convexity principle** for vertical strips is the case when the unbounded region is the vertical strip $a<\sigma<b$:

Theorem B.6.1 (Phragmén-Lindelöf convexity principle, vertical strip version). *Let $f(s)$ be a holomorphic function on an open neighborhood of the vertical strip $a<\sigma<b$ such that $f(s) \ll e^{|s|^A}$ for some $A \ge 0$. Then the following hold:*

(i) If $|f(s)| \le M$ for $\sigma = a, b$, that is on the boundary edges of the strip, then $|f(s)| \le M$ for all s in the strip.

(ii) Assume that there is a continuous function $g(t)$ such that

$$f(a+it) \ll g(t)^\alpha \quad \text{and} \quad f(b+it) \ll g(t)^\beta,$$

for all $t \in \mathbb{R}$. Then

$$f(s) \ll g(t)^{\alpha\ell(\sigma)+\beta(1-\ell(\sigma))},$$

where ℓ is the linear function such that $\ell(a)=1$ and $\ell(b)=0$.

We will also need a variant. The **Phragmén-Lindelöf convexity principle** for vertical half-strips is the case when the unbounded region is the vertical half-strip $a<\sigma<b$ with $t>c$:

Theorem B.6.2 (Phragmén-Lindelöf convexity principle, vertical half-strip version). *Let $f(s)$ be a holomorphic function on an open neighborhood of the vertical strip $a<\sigma<b$ with $t>c$ such that $f(s) \ll e^{|s|^A}$ for some $A \ge 0$. Then the following hold:*

(i) If $|f(s)| \le M$ for $\sigma = a, b$ with $t \ge c$ and $t = c$ with $a \le \sigma \le b$, that is on the boundary edges of the half-strip, then $|f(s)| \le M$ for all s in the strip.

(ii) Assume that there is a continuous function $g(t)$ such that

$$f(a+it) \ll g(t)^\alpha \quad \text{and} \quad f(b+it) \ll g(t)^\beta,$$

for all $t \ge c$. Then

$$f(s) \ll g(t)^{\alpha\ell(\sigma)+\beta(1-\ell(\sigma))},$$

where ℓ is the linear function such that $\ell(a)=1$ and $\ell(b)=0$.

B.7 Bessel Functions

For any $\nu \in \mathbb{C}$, the **Bessel equation** is the ODE

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0.$$

There are two linearly independent solutions to this equation. One solution is the **Bessel function of the first kind** $J_\nu(x)$ defined by

$$J_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu}.$$

For integers n, $J_n(x)$ is entire and we have

$$J_n(x) = (-1)^n J_{-n}(x).$$

Otherwise, $J_\nu(x)$ has a pole at $x = 0$ and $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent solutions to the Bessel equation. The other solution is the **Bessel function of the second kind** $Y_\nu(x)$ defined by

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)},$$

for non-integers ν, and for integers n is

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x).$$

For any integer n, we also have

$$Y_n(x) = (-1)^n Y_{-n}(x).$$

For the J-Bessel function there is also an important integral representation called the **Schläflin integral representation**:

Proposition B.7.1 (Schläflin integral representation for the J-Bessel function). *For any $\nu \in \mathbb{C}$ and* $\mathrm{Re}(x) > 0$*,*

$$J_\nu(x) = \frac{1}{2\pi i}\left(\frac{x}{2}\right)^\nu \int_{-\infty}^{(0^+)} t^{-(\nu+1)}e^{t-\frac{x^2}{4t}}\,dt.$$

The **modified Bessel equation** is the ODE

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \nu^2)y = 0.$$

Like the Bessel equation, there are two linearly independent solutions. One solution is the **modified Bessel function of the first kind** $I_\nu(x)$ given by

$$I_\nu(x) = i^{-\nu}J_\nu(ix) = \sum_{n\geq 0}\frac{1}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu}.$$

For integers n, this solution is symmetric in n. That is,

$$I_n(x) = I_{-n}(x).$$

We also have a useful integral representation in a half-plane:

Proposition B.7.2. *For any* $\nu \in \mathbb{C}$ *and* $\mathrm{Re}(x) > 0$,

$$I_\nu(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos(t)}\cos(\nu t)\,dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-x\cosh(t)-\nu t}\,dt,$$

where the integrals are understood to be complex integrals.

From this integral representation we can show the following asymptotic:

Lemma B.7.1. *For any* $\nu \in \mathbb{C}$ *and* $\mathrm{Re}(x) > 0$,

$$I_\nu(x) = O(e^x).$$

The other solution is the **modified Bessel function of the second kind** $K_\nu(x)$ defined by

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)},$$

for non-integers ν, and for integers n is

$$K_n(x) = \lim_{\nu\to n} K_\nu(x).$$

This is one of the more important types of Bessel functions as they appear in the Fourier coefficients of certain Eisenstein series. This function is symmetric in ν even when ν is an integer. That is,

$$K_\nu(x) = K_{-\nu}(x),$$

for all ν. We also have a very useful integral representation in a half-plane:

Proposition B.7.3. *For any* $\nu \in \mathbb{C}$ *and* $\mathrm{Re}(x) > 0$,

$$K_\nu(x) = \int_0^\infty e^{-x\cosh(t)}\cosh(\nu t)\,dt,$$

where the integral is understood to be a complex integral. In particular, $K_\nu(x)$ *is real provided* x *is positive.*

From this integral representation it does not take much to show the following asymptotic:

Lemma B.7.2. *For any* $\nu \in \mathbb{C}$ *and* $\mathrm{Re}(x) > 0$,

$$K_\nu(x) = o(e^{-x}).$$

B.8 Whittaker Functions

For $\kappa, \mu \in \mathbb{C}$, the **Whittaker equation** is the ODE

$$\frac{dw}{dz} + \left(\frac{1}{4} - \frac{\kappa}{z} - \frac{\frac{1}{4}-\mu^2}{z^2}\right)w = 0.$$

Throughout we assume $-2\mu \notin \mathbb{Z}_{\geq 1}$. There are two linearly independent solutions to this equation. If we additionally assume that $w(z) = o(e^{2\pi\mathrm{Im}(z)})$ as $\mathrm{Im}(z) \to \infty$, then there is only one linearly independent solution. This solution is the **Whittaker function** $W_{\kappa,\mu}(z)$. It can be expressed in the form

$$W_{\kappa,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-\frac{z}{2}}U\left(\mu-\kappa+\frac{1}{2}, 1+2\mu, z\right),$$

where $U(\alpha,\beta,z)$ is the **confluent hypergeometric function** initially defined by

$$U(\alpha,\beta,z)=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-zu}u^{\alpha-1}(1+u)^{\beta-\alpha-1}\,du,$$

for $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(z)>0$ and then analytically continued to $\mathbb{C}^3$. From this integral representation we can show the following asymptotic:

Lemma B.8.1. *Let $\kappa,\mu\in\mathbb{C}$ with $-2\mu\notin\mathbb{Z}_{\geq 1}$ and $-2(\mu-\kappa)\notin\mathbb{Z}_{\geq 1}$. Then for* $\mathrm{Re}(z)>0$,

$$W_{\kappa,\mu}(z)\sim z^{\kappa}e^{-\frac{z}{2}}.$$

The Whittaker function has a simplified form in special cases:

Theorem B.8.1. *Let $\nu,\alpha\in\mathbb{C}$ with $-2\nu\notin\mathbb{Z}_{\geq 1}$. Then for* $\mathrm{Re}(z)>0$,

$$W_{0,\nu}(z)=\left(\frac{z}{\pi}\right)^{\frac{1}{2}}K_\nu\left(\frac{z}{2}\right)\quad \textit{and}\quad W_{\alpha,\alpha-\frac{1}{2}}(z)=z^{\alpha}e^{-\frac{z}{2}}.$$

B.9 Sums Over Lattices

Let $\mathbf{a}=(a_1,a_2,\ldots,a_d)\in\mathbb{R}^d$ and let $||\mathbf{a}||=\sqrt{a_1^2+a_2^2+\cdots+a_d^2}$ be the usual norm. We are often interested in series that obtained by summing over the lattice $\mathbb{Z}^d\subset\mathbb{R}^d$. In particular, we have the following general result:

Theorem B.9.1. *Let $d\geq 1$ be an integer. Then*

$$\sum_{\mathbf{a}\in\mathbb{Z}^d-\{\mathbf{0}\}}\frac{1}{||a||^s},$$

is locally absolutely uniformly convergent in the region $\sigma>d$.

In a practical setting, we usually restrict to the case $d=2$. In this setting, with a little more work can show a more useful result:

Proposition B.9.1. *Let $z\in\mathbb{H}$. Then*

$$\sum_{(n,m)\in\mathbb{Z}^2-\{\mathbf{0}\}}\frac{1}{|nz+m|^s},$$

is locally absolutely uniformly convergent in the region $\sigma>2$. In addition, it is locally absolutely uniformly convergent as a function of z provided $\sigma>2$.

Appendix C

Algebra

C.1 Character Groups

For any finite abelian group G, a **character** φ is a homomorphism $\varphi : G \to \mathbb{C}$. They form a group, denoted $\widehat{G}$, under multiplication called the **character group** of G. If G is an additive group, we say that any $\varphi \in \Gamma$ is a **additive character**. Similarly, if G is a multiplicative group, we say that any $\varphi \in \Gamma$ is a **multiplicative character**. In any case, if $|G| = n$ then $\varphi(g)^n = \varphi(g^n) = 1$ so that φ takes values in the n-th roots of unity. Moreover, to every character φ there is its **conjugate character** $\overline{\varphi}$ defined by $\overline{\varphi}(g) = \overline{\varphi(a)}$. Clearly the conjugate character is also a character. Since φ takes its value in the roots of unity, $\overline{\varphi(a)} = \varphi(a)^{-1}$ so that $\overline{\varphi} = \varphi^{-1}$. One of the central theorems about characters is that the character group of G is isomorphic to G:

Proposition C.1.1. *Any finite abelian group G is isomorphic to its character group. That is,*

$$G \cong \widehat{G}.$$

The characters also satisfy certain **orthogonality relations**:

Proposition C.1.2 (Orthogonality relations). *Let G be a finite abelian group.*

(i) For any two characters χ and ψ of G,

$$\frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi}(g) = \delta_{\chi,\psi}.$$

(ii) For any $g, h \in G$,

$$\frac{1}{|G|} \sum_{\chi \in \widehat{G}} \chi(g)\overline{\chi}(h) = \delta_{g,h}.$$

C.2 Representation Theory

Let G be a group and V be a vector space over a field $\mathbb{F}$. A **representation** (ρ, V), or just ρ if the underlying vector space V is clear, of G on V is a map

$$\rho : G \times V \to V \qquad (g, v) \mapsto \rho(g, v) = g \cdot v,$$

such that the following properties are satisfied:

1. For any $g \in G$, the map
$$\rho : V \to V \qquad v \mapsto g \cdot v,$$
is linear.

2. For any $g, h \in G$ and $v \in V$,
$$1 \cdot v = v \quad \text{and} \quad g \cdot (h \cdot v) = (gh) \cdot v.$$

Therefore ρ defines an action of G on V. An equivalent definition of a representation of G on V is a homomorphism from G into $\mathrm{Aut}(V)$. By abuse of notation, we also denote this homomorphism by ρ. If the dimension of V is n, then (ρ, V) is sai to be an **n-dimensional**. We say that (ρ, W) is a **subrepresentation** of (V, ρ) if $W \subseteq V$ is a G-invariant subspace. In particular, (ρ, W) is a representation itself. Lastly, if (ρ_1, V_1) and (ρ_2, V_2) are two representations, we can form the **direct sum representation** $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$ where $\rho_1 \oplus \rho_2$ acts diagonally on $V_1 \oplus V_2$. A natural question to ask is how representation can be decomposed as a direct sum of other representations. We say (ρ, V) is **irreducible** if it contains no proper G-invariant subspaces and is **completely irreducible** if it decomposes as a direct sum of irreducible subrepresentations.
We will only need one very useful theorem about representations when G is a finite abelian group and V is a vector space over $\mathbb{C}$. In this case G has a group of characters $\widehat{G}$, and the underlying vector space V is completely reducible with respect to the characters of G:

Theorem C.2.1. *Let V be a vector space over $\mathbb{C}$ and let Φ be a representation of a group G on V. If G is a finite abelian group, then*
$$V = \bigoplus_{\chi \in \widehat{G}} V_\chi,$$
where
$$V_\chi = \{v \in V : g \cdot v = \chi(g)v \text{ for all } g \in G\}.$$
In particular, V is completely reducible and every irreducible subrepresentation is 1-dimensional.

Appendix D

Miscellaneous

D.1 Special Integrals

Below is a table of well-known integrals that are used throughout the text:

Reference	Assumptions	Integral
Gaussian		$\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$
[Gol06]	$s, \nu \in \mathbb{C}, \mathrm{Re}(s+\nu) > -1$	$\int_0^{\infty} K_\nu(y) y^s \frac{dy}{y} = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)$
[Gol06]	$n \in \mathbb{Z},\ s \in \mathbb{C},\ y > 0$	$\int_{-\infty}^{\infty} \frac{e^{-2\pi inxy}}{(x^2+1)^s}\, dx = \begin{cases} \frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} & \text{if } n = 0, \\ \frac{2\pi^s \|n\|^{s-\frac{1}{2}} y^{s-\frac{1}{2}}}{\Gamma(s)} K_{s-\frac{1}{2}}(2\pi\|n\|y) & \text{if } n \neq 0. \end{cases}$
[Dav80]	$c > 0$	$\frac{1}{2\pi i}\int_{(c)} y^s \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$

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