

# A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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ABSTRACT. We construct a quadratic double Dirichlet series  $Z(s, w)$  built from single variable quadratic Dirichlet  $L$ -functions  $L(s, \chi)$  over  $\mathbb{Q}$ . We prove that  $Z(s, w)$  admits meromorphic continuation to the  $(s, w)$ -plane and satisfies a group of functional equations.

## 1. PRELIMINARIES

We present an overview of quadratic Dirichlet  $L$ -functions over  $\mathbb{Q}$ . We begin with the Riemann zeta-function. The zeta function  $\zeta(s)$  is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for  $\operatorname{Re}(s) > 1$ . The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Now we recall the characters on  $\mathbb{Z}$ . They are multiplicative functions  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ . Their image always lands in the roots of unity. The two flavors of characters of interest to use are:

- Dirichlet characters: multiplicative functions  $\chi_m : \mathbb{Z} \rightarrow \mathbb{C}$  modulo  $m \geq 1$  (in that they are  $m$ -periodic) and such that  $\chi_m(n) = 0$  if  $(m, n) > 1$ .
- Hilbert symbols: Dirichlet characters modulo 1.

If  $\chi$  is a character then its conjugate  $\bar{\chi}$  is also a character. Moreover,  $\bar{\chi}$  is the multiplicative inverse to  $\chi$  and the characters modulo  $m$  form a group under multiplication. This group is always finite and its order is  $\phi(m)$ . Characters also satisfy orthogonality properties:

**Theorem 1.1** (Orthogonality relations). *Let  $\chi$  and  $\psi$  be any two Dirichlets character modulo  $m$  and let  $a$  and  $b$  be any two integers modulo  $m$ . Then*

(i)

$$\frac{1}{\phi(m)} \sum_{a \pmod{m}} \chi(a) \bar{\psi}(a) = \delta_{\chi, \psi}.$$

(ii)

$$\frac{1}{\phi(m)} \sum_{\chi \pmod{m}} \chi(a) \bar{\chi}(b) = \delta_{a, b}.$$

**Todo:** [continue here] The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on  $\mathbb{F}_q[t]$ . If  $f \in \mathbb{F}_q[t]$  is a monic non-constant irreducible, define the quadratic residue symbol  $\chi_f$  by

$$\chi_f(g) = \left(\frac{f}{g}\right) = g^{\frac{|f|-1}{2}} \pmod{f},$$

for any  $g \in \mathbb{F}_q[t]$ . Then  $\chi_f(g) \in \{\pm 1\}$  provided  $f$  and  $g$  are relatively prime and  $\chi_f(g) = 0$  if  $(f, g) > 1$ . If  $b \in \mathbb{F}^\times$ , then we define the quadratic residue symbol  $\chi_b$  by

$$\chi_b(g) = \left( \frac{b}{m} \right) = \text{sgn}(b)^{\deg(f)},$$

where  $\text{sgn}(b) = \pm 1$  depending on if  $b \in (\mathbb{F}^\times)^2$  or not. Moreover, if  $d \in \mathbb{F}_q[t]$  then we set  $\text{sgn}(d) = \text{sgn}(b_n)$  if  $d(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_0$  (with  $b_n \neq 0$ ). Extending  $\chi_f$  multiplicativity in  $f$ ,  $\chi_f$  is defined for any  $f$  not necessarily monic. The quadratic residue symbol also has the following reciprocity property:

**Theorem 1.2** (Quadratic reciprocity). *If  $f, g \in \mathbb{F}_q[t]$  are monic, square-free, and relatively prime, then*

$$\left( \frac{f}{g} \right) = (-1)^{\frac{q-1}{2} \deg(f) \deg(g)} \left( \frac{g}{f} \right).$$

Note that if  $q \equiv 1 \pmod{4}$ , the sign in the statement of quadratic reciprocity is always 1 so that the reciprocity is perfect. We now describe the Hilbert symbols on  $\mathbb{F}_q[t]$ . In fact, there are only two Hilbert symbols, one nontrivial, and one trivial. The nontrivial Hilbert symbol is  $\chi_\theta$  where  $\theta \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$ :

$$\chi_\theta(f) = (-1)^{\deg(f)}.$$

Note that  $\overline{\chi_\theta} = \chi_\theta$ . The other Hilbert symbol is the trivial character  $\chi_\theta^2 = \chi_{\theta\theta} = \chi_1$ . In general, we denote a Hilbert symbol by  $\chi_a$  where  $a \in \{1, \theta\}$ .

We can now define the  $L$ -functions attached to the symbol  $\chi_f$  for not necessarily monic  $f$ . We define the  $L$ -series  $L(s, \chi_f)$  attached to  $\chi_f$  by a Dirichlet series or Euler product:

$$L(s, \chi_f) = \sum_{g \text{ monic}} \frac{\chi_f(g)}{|g|^s} = \prod_{P \text{ monic irr}} \left( 1 - \frac{\chi_f(P)}{|P|^s} \right)^{-1}.$$

By definition of the quadratic residue symbol,  $L(s, \chi_f) \ll \zeta(s)$  for  $\text{Re}(s) > 1$  so that  $L(s, \chi_f)$  is absolutely uniformly convergent on compacta in this region.  $L(s, \chi_f)$  also admits meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  if  $f$  is square-free and is analytic otherwise (see [1] for a proof). Moreover,  $L(s, \chi_f)$  is a polynomial in  $q^{-s}$  of degree at most  $\deg(f) - 1$ . The completed  $L$ -function is defined as follows:

$$L^*(s, \chi_f) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_f) & \text{if } \deg(f) \text{ is even,} \\ L(s, \chi_f) & \text{if } \deg(f) \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_f) = \begin{cases} q^{2s-1} |f|^{\frac{1}{2}-s} L^*(1-s, \chi_f) & \text{if } \deg(f) \text{ is even,} \\ q^{2s-1} (q|f|)^{\frac{1}{2}-s} L^*(1-s, \chi_f) & \text{if } \deg(f) \text{ is odd.} \end{cases}$$

Note that in the case  $\deg(f)$  is even, the conductor is  $|f|$  and in the case  $\deg(f)$  is odd, the conductor is  $q|f|$ . In other words, the gamma factors depend upon the degree of  $f$ . This will cause a small but important technical issue later when we want to derive functional equations for the quadratic double Dirichlet series.

## THE QUADRATIC DOUBLE DIRICHLET SERIES

## THE INTERCHANGE

## WEIGHTING TERMS

## FUNCTIONAL EQUATIONS

## MEROMORPHIC CONTINUATION

## POLES AND RESIDUES

## REFERENCES

- [1] Rosen, M. (2002). Number theory in function fields (Vol. 210). Springer Science & Business Media.
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- [3] Chinta, G., & Gunnells, P. E. (2007). Weyl group multiple Dirichlet series constructed from quadratic characters. *Inventiones mathematicae*, 167, 327-353.
- [4] Stanley, R. (2023). Enumerative Combinatorics: Volume 2. Cambridge University Press.