

SUBCONVEXITY FOR GL_2 L -FUNCTIONS COMPUTATIONS

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1. SETUP: SUMS & FORMS

Let $q \geq 1$ and let ψ be a Dirichlet character modulo q . For $m \in \mathbb{Z}$, let

$$c_q(m) = \sum_{a \pmod{q}} e^{\frac{2\pi i m a}{q}} \quad \text{and} \quad c_\psi(m) = \sum_{a \pmod{q}} \psi(a) e^{\frac{2\pi i m a}{q}},$$

be the Ramanujan and Gauss sums respectively. For ℓ such that $(\ell, q) = 1$, we have

$$c_\psi(\ell m) = \overline{\psi(\ell)} c_\psi(m),$$

and moreover

$$c_\psi(m) = \overline{\psi(m)} c_\psi(1),$$

provided ψ is primitive. Throughout we will let f and g be two weight k and level 1 holomorphic cusp forms. They admit Fourier series

$$f(z) = \sum_{m \geq 1} a(m) e^{2\pi i m z} = \sum_{m \geq 1} A(m) m^{\frac{k-1}{2}} e^{2\pi i m z} \quad \text{and} \quad g(z) = \sum_{m \geq 1} b(m) e^{2\pi i m z} = \sum_{m \geq 1} B(m) m^{\frac{k-1}{2}} e^{2\pi i m z},$$

and are normalized so that

$$a(1) = b(1) = 1.$$

In particular, $a(m)$ and $b(m)$ are the m -th Hecke eigenvalues of f and g respectively. We define the L -functions

$$L(s, f \otimes \psi) = \sum_{m \geq 1} \frac{A(m) \psi(m)}{m^s} \quad \text{and} \quad L(s, f \times c_\psi) = \sum_{m \geq 1} \frac{A(m) c_\psi(m)}{m^s}.$$

These two L -functions are most related when ψ is primitive since we have the asymptotic

$$L(s, f \times c_\psi) \sim \sqrt{q} L(s, f \otimes \bar{\psi}).$$

They are least related when $\psi = \psi_{q,0}$ is the trivial character modulo q as

$$L(s, f \times c_\psi) = L^{(q)}(s, f).$$

Morally, one should think of $L(s, f \times c_\psi)$ as an arithmetically smoothed version of $L(s, f \otimes \psi)$. This will allow for some additional saving when studying the second moment of $L(s, f \times c_\psi)$. We will also require Maass cusp forms so let $\{\mu_j\}$ represent an orthonormal basis of Maass cusp forms on $\Gamma_0(\ell_1 \ell_2) \backslash \mathbb{H}$ with spectral parameter t_j for μ_j . They admit Fourier series

$$\mu_j(z) = \sum_{m \neq 0} \rho_j(m) \sqrt{y} K_{it_j}(2\pi|m|y) e^{2\pi i m x},$$

and the Fourier coefficients are normalized so that

$$\rho_j(m) = \rho_j(\text{sgn}(m)) \lambda_j(|m|),$$

where $\lambda_j(m)$ is the m -th Hecke eigenvalue of μ_j . We will also need the Petersson inner product on $\Gamma_0(\ell_1\ell_2)\backslash\mathbb{H}$, defined by

$$\langle F, G \rangle = \frac{1}{\mathcal{V}} \int_{\Gamma_0(\ell_1\ell_2)\backslash\mathbb{H}} F(z) \overline{G(z)} d\mu,$$

where

$$\mathcal{V} = \text{vol}(\Gamma_0(\ell_1\ell_2)\backslash\mathbb{H}) = \frac{\pi}{3} \ell_1 \ell_2 \prod_{p|\ell_1\ell_2} (1 + p^{-1}).$$

Also, define the functions

$$V_{f,g}^{\ell_1,\ell_2} = V_{f,g}^{\ell_1,\ell_2}(z) = \overline{f(\ell_1 z)} g(\ell_2 z) \text{Im}(z)^k \quad \text{and} \quad V_{f,v}^{\ell_1} = V_{f,v}^{\ell_1}(z) = \overline{f(\ell_1 z)} E(z, \text{Todo} : [\mathbf{s}]; k) \text{Im}(z)^{\frac{k}{2}}.$$

2. SHIFTED DIRICHLET SERIES

The Dirichlet Series $D_{f,g}(s; h, \ell_1, \ell_2)$. Let $h \geq 1$. Our first Dirichlet series $D_{f,g}(s; h, \ell_1, \ell_2)$ is given by

$$D_{f,g}(s; h, \ell_1, \ell_2) = \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m)b(n)}{(n\ell_2)^{s+k-1}}.$$

This series is absolutely convergent for $\text{Re}(s) > 1$ and admits meromorphic continuation to $\frac{1-k}{2} - C_1 < \text{Re}(s)$, for any $C_1 > 0$, and in these two regions it satisfies the bounds

$$D_{f,g}(s; h, \ell_1, \ell_2) \ll_{\text{Todo}:[\ell_1,\ell_2]} h^{\frac{k-1}{2}+\varepsilon} \quad \text{and} \quad D_{f,g}(s; h, \ell_1, \ell_2) \ll_{\text{Todo}:[\ell_1,\ell_2]} h^{k+2C_1+\varepsilon},$$

respectively. In the region $\text{Re}(s) < \frac{1-k}{2}$ and ε away from the poles, the meromorphic continuation is given by the absolutely convergent spectral expansion

$$D_{f,g}(s; h, \ell_1, \ell_2) = \sum_{t_j} \overline{\rho_j(-h) \langle V_{f,g}^{\ell_1,\ell_2}, \mu_j \rangle} h^{\frac{1}{2}-s} \frac{\Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j) \Gamma(1-s)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(s+k-1)}.$$

In short, $D_{f,g}(s; h, \ell_1, \ell_2)$ has meromorphic continuation to \mathbb{C} but we do not have a representation in the strip $\frac{1-k}{2} \leq \text{Re}(s) \leq 1$. The poles occur at $s = \frac{1}{2} - \ell + it_j$ for $\ell \geq 0$ and the residue at this pole is

$$\text{Res}_{s=\frac{1}{2}-\ell+it_j} D_{f,g}(s; h, \ell_1, \ell_2) = \overline{\rho_j(-h) \langle V_{f,g}^{\ell_1,\ell_2}, \mu_j \rangle} h^{\ell-it_j} \frac{(-1)^\ell}{\ell!} \frac{\Gamma(-\ell + 2it_j) \Gamma(\frac{1}{2} + \ell - it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(k - \ell - \frac{1}{2} + it_j)}$$

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The Dirichlet Series $D_{f,v}(w; n, \ell_1, \ell_2)$. Let $n \geq 1$. Our second Dirichlet series $D_{f,v}(w; n, \ell_1, \ell_2)$ is given by

$$D_{f,v}(w; n, \ell_1, \ell_2) = \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m) \sigma_{1-2v}(h) h^{v-\frac{1}{2}}}{h^{w+\frac{k-1}{2}}}.$$

This series is absolutely convergent for $\text{Re}(w) > \frac{1}{2} + \text{Re}(v)$ and admits meromorphic continuation. To state the meromorphic continuation, let $c > 0$ be such that if v satisfies $\zeta(2v) \neq 0$, then $\text{Re}(v) \geq \frac{1}{2} - \frac{8c}{\log(2+\text{Im}(v))}$. For such a c , we set

$$\delta(s, v, u) = \frac{c}{\log(3 + |\text{Im}(s+u)| + |\text{Im}(v)|)} \quad \text{and} \quad \delta_v = \delta(0, 0, v).$$

Then we have meromorphic continuation to $\text{Re}(v) \geq \frac{1}{2} - \delta(w, v, u)$ with $\text{Re}(w) > 1 - \frac{k}{2} - \text{Re}(v) - C_2$, for any $C_2 > 0$, and in these two regions satisfies the bounds

Todo : [xxx]

In the region $\operatorname{Re}(w) < \frac{1-k}{2}$ and ε away from the poles, the meromorphic continuation is given by the absolutely convergent spectral expansion

$$D_{f,v}(w; n, \ell_1, \ell_2) = \sum_{t_j} \overline{\rho_j(-\ell_2 n) \langle V_{f,v}^{\ell_1}, \mu_j \rangle} (\ell_2 n)^{\frac{1}{2}-w} \frac{\Gamma(w - \frac{1}{2} + it_j) \Gamma(w - \frac{1}{2} - it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j)} \\ \cdot \frac{\Gamma(1-w) \Gamma(w)}{\Gamma(w + v + \frac{k}{2} - 1) \Gamma(w - v + \frac{k}{2})},$$

In short, $D_{f,v}(w; n, \ell_1, \ell_2)$ admits meromorphic continuation to \mathbb{C} but we do not have a representation in the strip $\frac{1-k}{2} \leq \operatorname{Re}(w) \leq \frac{1}{2} + \operatorname{Re}(v)$. The poles occur at $w = \frac{1}{2} - \ell + it_j$ for $\ell \geq 0$ and the residue at this pole is

$$\operatorname{Res}_{w=\frac{1}{2}-\ell+it_j} D_{f,v}(w; n, \ell_1, \ell_2) = \overline{\rho_j(-\ell_2 n) \langle V_{f,v}^{\ell_1}, \mu_j \rangle} (\ell_2 n)^{\ell-it_j} \frac{(-1)^\ell}{\ell!} \frac{\Gamma(\frac{1}{2} + \ell - it_j) \Gamma(\frac{1}{2} - \ell + it_j)}{\Gamma(\frac{k-1}{2} - \ell + v + it_j) \Gamma(\frac{k+1}{2} - \ell - v + it_j)} \\ \cdot \frac{\Gamma(-\ell + 2it_j)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j)}.$$

The Multiple Dirichlet Series $Z_{f,g}(s, v, u, \ell_1, \ell_2)$. We now wish to construct a multiple Dirichlet series from $D_{f,g}(s; h, \ell_1, \ell_2)$ and $D_{f,v}(w; n, \ell_1, \ell_2)$. To do this we will suppose

$$\operatorname{Re}(s) > 1, \quad \operatorname{Re}(w) > \frac{1}{2} + \operatorname{Re}(v), \quad \text{and} \quad \operatorname{Re}(v) \geq \frac{1}{2} - \delta(s, v, u).$$

Letting ε be such that $\operatorname{Re}(w) > \frac{1}{2} + \operatorname{Re}(v) + \varepsilon$, both Dirichlet series $D_{f,g}(s; h, \ell_1, \ell_2)$ and $D_{f,v}(w; n, \ell_1, \ell_2)$ converge absolutely and satisfy the estimates

$$D_{f,g}(s; h, \ell_1, \ell_2) \ll_{\ell_1, \ell_2} h^{\frac{k-1}{2}+\varepsilon} \quad \text{and} \quad D_{f,v}(w; n, \ell_1, \ell_2) \ll_{\ell_1, \ell_2} n^{\frac{k-1}{2}+\varepsilon}.$$

Thus for

$$\operatorname{Re}(s) > 1, \quad \operatorname{Re}(u) > \frac{k+1}{2}, \quad \text{and} \quad \operatorname{Re}(v) \geq \frac{1}{2} - \delta(s, v, u),$$

we may define the multiple Dirichlet series $Z_{f,g}(s, v, u; \ell_1, \ell_2)$ by

$$Z_{f,g}(s, v, u; \ell_1, \ell_2) = (\ell_1 \ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m) b(n) \sigma_{1-2v}(h)}{(\ell_2 n)^{s+k-1} h^u}.$$

It is absolutely convergent in this region. Moreover, $Z_{f,g}(s, v, u; \ell_1, \ell_2)$ satisfies the interchange

$$Z_{f,g}(s, v, u; \ell_1, \ell_2) = \sum_{h \geq 1} \frac{D_{f,g}(s; h, \ell_1, \ell_2) \sigma_{1-2v}(h)}{h^u} = \sum_{n \geq 1} \frac{D_{f,v}(u + v - \frac{k}{2}; n, \ell_1, \ell_2) b(n)}{(\ell_2 n)^{s+k-1}},$$

where both representations converge absolutely. In the region where $D_{f,g}(s; h, \ell_1, \ell_2)$ admits a spectral expansion, we have an absolutely convergent spectral expansion for $Z_{f,g}(s, v, u; \ell_1, \ell_2)$ given by

$$Z_{f,g}(s, v, u; \ell_1, \ell_2) = \sum_{t_j} \overline{\rho_j(-1) \langle V_{f,g}^{\ell_1, \ell_2}, \mu_j \rangle} (\ell_1 \ell_2)^{\frac{k-1}{2}} \frac{\Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j) \Gamma(1-s)}{\Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j) \Gamma(s+k-1)} \\ \cdot \frac{L(s + u - \frac{1}{2}, \mu_j) L(s + u + 2v - \frac{3}{2}, \mu_j)}{\zeta(2s + 2u + 2v - 2)}.$$

Similarly, in the region where $D_{f,v}(u+v-\frac{k}{2}; n, \ell_1, \ell_2)$ admits a spectral expansion, we have an absolutely convergent spectral expansion for $Z_{f,g}(s, v, u; \ell_1, \ell_2)$ given by

$$Z_{f,g}(s, v, u; \ell_1, \ell_2) = \sum_{t_j} \overline{\rho_j(-1) \langle V_{f,v}^{\ell_1}, \mu_j \rangle} \ell_1^{\frac{k-1}{2}} \ell_2^{1-s-v-u} \frac{\Gamma(u+v-\frac{k+1}{2}+it_j) \Gamma(u+v-\frac{k+1}{2}-it_j)}{\Gamma(\frac{1}{2}+it_j) \Gamma(\frac{1}{2}-it_j)} \\ \cdot \frac{\Gamma(\frac{k}{2}+1-u-v) \Gamma(u+v-\frac{k}{2})}{\Gamma(u+2v-1)\Gamma(u)} \frac{L^{(\ell_2)}(s+u+v-1, g \otimes \mu_j)}{\zeta^{(\ell_2)}(2s+2u+2v-2)} \sum_{\alpha \geq 0} \frac{b(\ell_2^\alpha) \lambda_j(\ell_2^{\alpha+1})}{(\ell_2^\alpha)^{s+u+v-1+\frac{k-1}{2}}}.$$

The poles of $Z_{f,g}(s, v, u; \ell_1, \ell_2)$ are inherited from the poles of $D_{f,g}(s; h, \ell_1, \ell_2)$ of $D_{f,v}(u+v-\frac{k}{2}; n, \ell_1, \ell_2)$ with corresponding residues.

3. SUBCONVEXITY

Setup. Let $G(x)$ be a smooth function with compact support in the interval $[1, 2]$ and let $g(s)$ be the Mellin transform. For a Dirichlet character χ modulo Q , we define

$$B_\chi(x) = \sum_{m \geq 1} A(m) \chi(m) G\left(\frac{m}{x}\right) \quad \text{and} \quad B_{c_\chi}(x) = \sum_{m \geq 1} A(m) \overline{c_\chi}(m) G\left(\frac{m}{x}\right).$$

Using a smooth dyadic partition of unity and summation by parts, we have the bounds

$$L\left(\frac{1}{2}, f \otimes \chi\right) \ll Q^{-\frac{1}{2}} \max_{x \ll Q^{1+\varepsilon}} B_\chi(x) \quad \text{and} \quad L\left(\frac{1}{2}, f \otimes c_\chi\right) \ll Q^{-\frac{1}{2}} \max_{x \ll Q^{1+\varepsilon}} B_{c_\chi}(x).$$

Since $L(s, f \otimes \chi) \ll Q^{-\frac{1}{2}} L(s, f \otimes c_\chi)$, we have

$$\left| L\left(\frac{1}{2}, f \otimes \chi\right) \right|^2 \ll Q^{-2} \max_{x \ll Q^{1+\varepsilon}} |B_{c_\chi}(x)|^2.$$

So to obtain a subconvexity estimate for $L(s, f \otimes \chi)$ at $s = \frac{1}{2}$, it suffices to estimate $B_{c_\chi}(x)$ for $x \ll Q^{1+\varepsilon}$. Now let $q \geq 1$ and ψ be a Dirichlet character modulo q . We define

$$S_\chi(x, q) = \frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} |B_{c_\psi}(x)|^2 \left| \sum_{\ell \sim L} \chi(\ell) \overline{\psi}(\ell) \right|^2,$$

where $\ell \sim L$ means that $\ell \in [L, 2L]$ and is prime. As all of the terms in the sum are nonnegative, retaining only the term corresponding to $\psi = \chi$, the prime number theorem gives the lower bound

$$\frac{L^2}{Q \log^2(L)} |B_{c_\chi}(x)|^2 \ll S_\chi(x, q).$$

It follows that

$$\frac{L^2 Q}{\log^2(L)} \left| L\left(\frac{1}{2}, f \otimes \chi\right) \right|^2 \ll \frac{L^2}{Q \log^2(L)} \max_{x \ll Q^{1+\varepsilon}} |B_{c_\chi}(x)|^2 \ll \max_{x \ll Q^{1+\varepsilon}} \sum_{|q-Q| \ll Q^\varepsilon} S_\chi(x, q),$$

Hence

$$\left| L\left(\frac{1}{2}, f \otimes \chi\right) \right|^2 \ll \frac{1}{L^{2+\varepsilon} Q} \max_{x \ll Q^{1+\varepsilon}} \sum_{|q-Q| \ll Q^\varepsilon} S_\chi(x, q).$$

Now recall the Mellin inverse

$$\frac{1}{2\pi i} \int_{(2)} \frac{e^{\frac{\pi v^2}{y^2}} Q^{2v}}{y} dv = e^{-\frac{y^2 \log^2(Q)}{\pi}} \ll \begin{cases} 1 & \text{if } |q-Q| \ll \frac{Q^{1+\varepsilon}}{y}, \\ Q^{-A} & \text{if } |q-Q| \gg \frac{Q^{1+\varepsilon}}{y}, \end{cases}$$

for any $A \gg 1$. From this integral transform, we conclude that

$$\sum_{|q-Q| \ll Q^\varepsilon} S_\chi(x, q) \ll \frac{1}{2\pi i} \int_{(2)} \sum_{q \geq 1} \frac{S_\chi(x, q)}{q^{2v}} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} dv.$$

To estimate the right-hand side, we will rewrite the Dirichlet series over q . To do this, we first expand $S_\chi(x, q)$:

$$\begin{aligned} S_\chi(x, q) &= \frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} |B_{c_\psi}(x)|^2 \left| \sum_{\ell \sim L} \chi(\ell) \bar{\psi}(\ell) \right|^2 \\ &= \frac{1}{\varphi(q)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\psi \pmod{q}} \sum_{m, n \geq 1} A(m) A(n) G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_\psi(m) c_{\bar{\psi}}(n) \chi(\ell_1) \bar{\psi}(\ell_1) \bar{\chi}(\ell_2) \psi(\ell_2) \\ &= \frac{1}{\varphi(q)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\psi \pmod{q}} \sum_{m, n \geq 1} A(m) A(n) G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_\psi(\ell_1 m) c_{\bar{\psi}}(\ell_2 n) \chi(\ell_1) \bar{\chi}(\ell_2) \\ &= \sum_{\ell_1, \ell_2 \sim L} \sum_{m, n \geq 1} A(m) A(n) G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_q(\ell_1 m - \ell_2 n) \chi(\ell_1) \bar{\chi}(\ell_2), \end{aligned}$$

where in the last line we have used the identity

$$\frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} c_\psi(\ell_1 m) c_{\bar{\psi}}(\ell_2 n) = c_q(\ell_1 m - \ell_2 n).$$

Using the relation

$$\sum_{q \geq 1} \frac{c_q(\ell_1 m - \ell_2 n)}{q^{2v}} = \begin{cases} \frac{\zeta(2v-1)}{\zeta(2v)} & \text{if } \ell_1 m = \ell_2 n, \\ \frac{\sigma_{1-2v}(h)}{\zeta^{2v}} & \text{if } \ell_1 m = \ell_2 n + h, \end{cases}$$

we can express the Dirichlet series over q as a diagonal and off-diagonal term:

$$\begin{aligned} \sum_{q \geq 1} \frac{S_\chi(x, q)}{q^{2v}} &= \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n} \frac{\zeta(2v-1)}{\zeta(2v)} A(m) A(n) G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \\ &\quad + \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n + h} \frac{\sigma_{1-2v}(h)}{\zeta(2v)} A(m) A(n) G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \end{aligned}$$

Thus

$$\begin{aligned} &\frac{1}{2\pi i} \int_{(2)} \sum_{q \geq 1} \frac{S_\chi(x, q)}{q^{2v}} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} dv \\ &= \frac{1}{2\pi i} \int_{(2)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n} \frac{\zeta(2v-1)}{\zeta(2v)} A(m) A(n) G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} dv \\ &\quad + \frac{1}{2\pi i} \int_{(2)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n + h} \frac{\sigma_{1-2v}(h)}{\zeta(2v)} A(m) A(n) G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} dv. \end{aligned}$$

The Diagonal Contribution. We will estimate

$$\frac{1}{2\pi i} \int_{(2)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n} \frac{\zeta(2v-1)}{\zeta(2v)} A(m) A(n) G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} dv.$$

The integral over v is

$$\frac{1}{2\pi i} \int_{(2)} \frac{\zeta(2v-1)}{\zeta(2v)} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} dv \ll \sum_{|q-Q| \ll Q^\varepsilon} \varphi(q) \ll Q^{1+\varepsilon}.$$

Therefore the diagonal contribution is

$$\begin{aligned} &\ll Q^{1+\varepsilon} \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n} A(m) A(n) G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \\ &\ll Q^{1+\varepsilon} \sum_{\ell_1, \ell_2 \sim L} \sum_{\substack{\ell_1 m = \ell_2 n \\ m, n \ll Q^{1+\varepsilon}}} A(m) A(n) \chi(\ell_1) \bar{\chi}(\ell_2) \\ &\ll Q^{1+\varepsilon} \sum_{\ell_1, \ell_2 \sim L} \sum_{\substack{d \geq 1 \\ d \ll \frac{Q^{1+\varepsilon}}{L}}} A(\ell_1 \ell_2 d) A(\ell_1 \ell_2 d) \chi(\ell_1) \bar{\chi}(\ell_2) \\ &\ll L Q^{2+\varepsilon}. \end{aligned}$$

The Off-diagonal Contribution. We will estimate

$$\frac{1}{2\pi i} \int_{(2)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n + h} \frac{\sigma_{1-2v}(h)}{\zeta(2v)} A(m) A(n) G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} dv.$$

Applying the Mellin inversion formula to $G\left(\frac{m}{x}\right)$ and $G\left(\frac{n}{x}\right)$, we can express the off-diagonal contribution as

$$\begin{aligned} &\sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1) \bar{\chi}(\ell_2) \left(\frac{1}{2\pi i}\right)^3 \int_{(2)} \int_{(\sigma_{s_2})} \int_{(\sigma_{s_1})} \frac{1}{\zeta(2v)} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m) a(n) \sigma_{1-2v}(h)}{m^{s_1 + \frac{k-1}{2}} n^{s_2 + \frac{k-1}{2}}} \\ &\quad \cdot g(s_1) g(s_2) x^{s_1 + s_2} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} ds_1 ds_2 dv, \end{aligned}$$

with $\sigma_{s_1}, \sigma_{s_2} \gg 1$. Now make the following computation:

$$\begin{aligned} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m) a(n) \sigma_{1-2v}(h)}{m^{s_1 + \frac{k-1}{2}} n^{s_2 + \frac{k-1}{2}}} &= \ell_1^{s_1} \ell_2^{s_2} (\ell_1 \ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m) a(n) \sigma_{1-2v}(h)}{(\ell_1 m)^{s_1 + \frac{k-1}{2}} (\ell_2 n)^{s_2 + \frac{k-1}{2}}} \\ &= \ell_1^{s_1} \ell_2^{s_2} (\ell_1 \ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m) a(n) \sigma_{1-2v}(h)}{(\ell_2 n + h)^{s_1 + \frac{k-1}{2}} (\ell_2 n)^{s_2 + \frac{k-1}{2}}} \\ &= \ell_1^{s_1} \ell_2^{s_2} (\ell_1 \ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m) a(n) \sigma_{1-2v}(h)}{\left(1 + \frac{h}{\ell_2 n}\right)^{s_1 + \frac{k-1}{2}} (\ell_2 n)^{s_1 + s_2 + k-1}}. \end{aligned}$$

Recall the Mellin inversion formula

$$\frac{1}{(1+t)^\beta} = \frac{1}{2\pi i} \int_{(\sigma_u)} \frac{\Gamma(\beta-u) \Gamma(u)}{\Gamma(\beta)} t^{-u} du,$$

with $0 < \sigma_u < \text{Re}(\beta)$. Applying this formula with $t = \frac{h}{\ell_2 n}$ and $\beta = s_1 + \frac{k-1}{2}$, we have

$$\sum_{\ell_1 m = \ell_2 n + h} \frac{a(m) a(n) \sigma_{1-2v}(h)}{m^{s_1 + \frac{k-1}{2}} n^{s_2 + \frac{k-1}{2}}} = \frac{1}{2\pi i} \int_{(\sigma_u)} \ell_1^{s_1} \ell_2^{s_2} Z_f(s_1 + s_2 - u, v, u; \ell_1, \ell_2) \frac{\Gamma\left(s_1 - u + \frac{k-1}{2}\right) \Gamma(u)}{\Gamma\left(s_1 + \frac{k-1}{2}\right)} du.$$

Therefore the off-diagonal contribution can be expressed as

$$\sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1) \bar{\chi}(\ell_2) \left(\frac{1}{2\pi i} \right)^4 \int_{(2)} \int_{(\sigma_u)} \int_{(\sigma_{s_2})} \int_{(\sigma_{s_1})} \frac{1}{\zeta(2v)} \ell_1^{s_1} \ell_2^{s_2} Z_f(s_1 + s_2 - u, v, u; \ell_1, \ell_2) \\ \cdot \frac{\Gamma(s_1 - u + \frac{k-1}{2}) \Gamma(u)}{\Gamma(s_1 + \frac{k-1}{2})} g(s_1) g(s_2) x^{s_1+s_2} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} ds_1 ds_2 du dv.$$

Since $\sigma_{s_1}, \sigma_{s_2} \gg 1$ and $0 < \sigma_u < \sigma_{s_1} + \frac{k-1}{2}$, we may assume

$$\sigma_{s_1} + \sigma_{s_2} - \sigma_u > 1, \quad \sigma_u > \frac{k+1}{2}, \quad \text{and} \quad 2 \geq \frac{1}{2} - \delta(s, v, u).$$

This ensures that $Z_f(s_1 + s_2 - u, v, u; \ell_1, \ell_2)$ is absolutely convergent. Let us take

$$\sigma_{s_1} = 1 + \varepsilon_1, \quad \sigma_{s_2} = \frac{k+1}{2} + \varepsilon_2, \quad \text{and} \quad \sigma_u = \frac{k+1}{2} + \varepsilon_3,$$

with $\varepsilon_1 + \varepsilon_2 - \varepsilon_3 > 0$. The analytic continuation of $Z_f(s_1 + s_2 - u, v, u; \ell_1, \ell_2)$ exhibits no poles in the region

$$\sigma_{s_1} + \sigma_{s_2} - \sigma_u > \frac{1}{2} \quad \text{and} \quad \sigma_u > 0.$$

Therefore, we may shift the integral in u to $\sigma_u = \varepsilon_4$ without crossing over any poles. Then, we may shift the integrals in s_1 and s_2 to $\sigma_{s_1} + \sigma_{s_2} = \frac{1}{2} + \varepsilon_5$ provided $\varepsilon_5 > \varepsilon_4$ without crossing over any poles. Now we shift the integral in u so that

$$\sigma_{s_1} + \sigma_{s_2} - \sigma_u < \frac{1-k}{2} - \varepsilon_6 \quad \text{while} \quad \sigma_1 - \sigma_u + \frac{k-1}{2} > 0.$$

This is possible by moving the integral in u to $\sigma_u = \frac{k}{2} + \varepsilon_5 + \varepsilon_6$ and choosing $\sigma_1 = \frac{1}{2} + 2\varepsilon_5$ and $\sigma_2 = -\varepsilon_5$ with $\varepsilon_5 > \varepsilon_6$. In doing so, we pass over simple poles of $Z_f(s_1 + s_2 - u, v, u; \ell_1, \ell_2)$ occurring at $u = s_1 + s_2 - \frac{1}{2} + \ell - it_j$ for $0 \leq \ell \leq \frac{k}{2}$. We do not pass over any simple poles of gamma functions. since $\sigma_u > 0$ and $\sigma_1 - \sigma_u + \frac{k-1}{2} > 0$ throughout. Then the off-diagonal contribution can be expressed as

$$\sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1) \bar{\chi}(\ell_2) \left(\frac{1}{2\pi i} \right)^4 \int_{(2)} \int_{(\sigma_u)} \int_{(\sigma_{s_2})} \int_{(\sigma_{s_1})} \frac{1}{\zeta(2v)} \ell_1^{s_1} \ell_2^{s_2} (\ell_1 \ell_2)^{\frac{k-1}{2}} Z_f(s_1 + s_2 - u, v, u; \ell_1, \ell_2) \\ \cdot \frac{\Gamma(s_1 - u + \frac{k-1}{2}) \Gamma(u)}{\Gamma(s_1 + \frac{k-1}{2})} g(s_1) g(s_2) x^{s_1+s_2} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} ds_1 ds_2 du dv \\ + \sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1) \bar{\chi}(\ell_2) \left(\frac{1}{2\pi i} \right)^3 \int_{(2)} \int_{(\sigma_{s_2})} \int_{(\sigma_{s_1})} \sum_{t_j} \frac{1}{\zeta(2v)} \ell_1^{s_1} \ell_2^{s_2} (\ell_1 \ell_2)^{\frac{k-1}{2}} \text{Res}_{u=s_1+s_2-\frac{1}{2}+\ell-it_j} \\ \cdot \left[Z_f(s_1 + s_2 - u, v, u; \ell_1, \ell_2) \frac{\Gamma(s_1 - u + \frac{k-1}{2}) \Gamma(u)}{\Gamma(s_1 + \frac{k-1}{2})} \right] g(s_1) g(s_2) x^{s_1+s_2} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} ds_1 ds_2 dv.$$

Let us concern ourselves with the first term only. Here the first spectral expansion of $Z_f(s_1 + s_2 - u, v, u; \ell_1, \ell_2)$ is valid. The integral over v is

$$\frac{1}{2\pi i} \int_{(2)} \frac{L(s_1 + s_2 + 2v - \frac{3}{2})}{\zeta(2v) \zeta(2s_1 + 2s_2 + 2v - 2)} \frac{e^{\frac{\pi v^2}{Q^2}} Q^{2v}}{Q} dv \ll \sum_{\substack{q=q_1 q_2 q_3 \\ |q-Q| \ll Q^\varepsilon}} \mu(q_1 q_2) q_2^{1-2\varepsilon_5} q_3^{1-s_5} \lambda_j(q_3) \ll Q^{1+\theta+\varepsilon}.$$

Moreover $x \ll Q^{1+\varepsilon}$ so that

$$x^{s_1+s_2} \ll Q^{\frac{1}{2}+\varepsilon}$$

The contribution of L is

$$\ll \sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1) \overline{\chi}(\ell_2) \sum_{|t_j| \sim 1} \overline{\rho_j(-1) \langle V_{f,g}^{\ell_1, \ell_2}, \mu_j \rangle \ell_1^{s_1} \ell_2^{s_2} (\ell_1 \ell_2)^{\frac{k-1}{2}}} \ll L^{\frac{5}{2}}.$$

In total, the off-diagonal contribution is

$$\ll L^{\frac{5}{2}} Q^{\frac{3}{2} + \theta + \varepsilon}.$$

Balancing. We have the diagonal and off-diagonal estimates

$$\ll LQ^{2+\varepsilon} \quad \text{and} \quad \ll L^{\frac{5}{2}} Q^{\frac{3}{2} + \theta + \varepsilon}.$$

This implies

$$\left| L \left(\frac{1}{2}, f \otimes \chi \right) \right|^2 \ll \frac{1}{L^{2+\varepsilon} Q} \max_{x \ll Q^{1+\varepsilon}} \sum_{|q-Q| \ll Q^\varepsilon} S_\chi(x, q) \ll \frac{Q^{1+\varepsilon}}{L^{1+\varepsilon}} + L^{\frac{1}{2}-\varepsilon} Q^{\frac{1}{2} + \theta + \varepsilon}.$$

The terms are balanced when $L = Q^{\frac{1-2\theta}{3}}$. We then have

$$\left| L \left(\frac{1}{2}, f \otimes \chi \right) \right|^2 \ll Q^{\frac{2}{3} + \frac{2\theta}{3} + \varepsilon},$$

and it follows that

$$\left| L \left(\frac{1}{2}, f \otimes \chi \right) \right| \ll Q^{\frac{1}{3} + \frac{\theta}{3} + \varepsilon}.$$

4. HYBRID SUBCONVEXITY

Setup. Let $G(x)$ be a smooth function with compact support in the interval $[1, 2]$ and let $g(s)$ be the Mellin transform. For a Dirichlet character χ modulo Q and $T \leq |t| \leq 2T$, we define

$$B_\chi(x, t) = \sum_{m \geq 1} A(m) \chi(m) m^{-it} G\left(\frac{m}{x}\right) \quad \text{and} \quad B_{c_\chi}(x, t) = \sum_{m \geq 1} A(m) m^{-it} \overline{c_\chi}(m) G\left(\frac{m}{x}\right).$$

Using a smooth dyadic partition of unity and summation by parts, we have the bounds

$$L \left(\frac{1}{2} + it, f \otimes \chi \right) \ll (QT)^{-\frac{1}{2}} \max_{x \ll (QT)^{1+\varepsilon}} B_\chi(x, t) \quad \text{and} \quad L \left(\frac{1}{2} + it, f \otimes c_\chi \right) \ll (QT)^{-\frac{1}{2}} \max_{x \ll (QT)^{1+\varepsilon}} B_{c_\chi}(x, t).$$

Since $L(s, f \otimes \chi) \ll Q^{-\frac{1}{2}} L(s, f \otimes c_\chi)$, we have

$$\left| L \left(\frac{1}{2} + it, f \otimes \chi \right) \right|^2 \ll Q^{-2} T^{-1} \max_{x \ll (QT)^{1+\varepsilon}} |B_{c_\chi}(x, t)|^2.$$

So to obtain a subconvexity estimate for $L(s, f \otimes \chi)$ at $s = \frac{1}{2}$, it suffices to estimate $B_{c_\chi}(x, t)$ for $x \ll (QT)^{1+\varepsilon}$. Now let $q \geq 1$ and ψ be a Dirichlet character modulo q . We define

$$S_\chi(x, q, t) = \frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} |B_{c_\psi}(x, t)|^2 \left| \sum_{\ell \sim L} \chi(\ell) \overline{\psi}(\ell) \right|^2,$$

where $\ell \sim L$ means that $\ell \in [L, 2L]$ and is prime. As all of the terms in the sum are nonnegative, retaining only the term corresponding to $\psi = \chi$, the prime number theorem gives the lower bound

$$\frac{L^2}{Q \log^2(L)} |B_{c_\chi}(x, t)|^2 \ll S_\chi(x, q, t).$$

It follows that

$$\frac{L^2 Q}{\log^2(L)} \left| L \left(\frac{1}{2} + it, f \otimes \chi \right) \right|^2 \ll \frac{L^2}{QT \log^2(L)} \max_{x \ll (QT)^{1+\varepsilon}} |B_{c_\chi}(x, t)|^2 \ll \frac{1}{T} \max_{x \ll (QT)^{1+\varepsilon}} \sum_{|q-Q| \ll (QT)^\varepsilon} S_\chi(x, q, t).$$

Hence

$$\left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^2 \ll \frac{1}{L^{2+\varepsilon} QT} \max_{x \ll (QT)^{1+\varepsilon}} \sum_{|q-Q| \ll Q^\varepsilon} S_\chi(x, q, t).$$

Now recall the Mellin inverse

$$\frac{1}{2\pi i} \int_{(2)} \frac{e^{\frac{\pi v^2}{y^2}} Q^{2v}}{y} dv = e^{-\frac{y^2 \log^2(Q)}{\pi}} \ll \begin{cases} 1 & \text{if } |q-Q| \ll \frac{Q^{1+\varepsilon}}{y}, \\ Q^{-A} & \text{if } |q-Q| \gg \frac{Q^{1+\varepsilon}}{y}, \end{cases}$$

for any $A \gg 1$. From this integral transform, we conclude that

$$\sum_{|q-Q| \ll (QT)^\varepsilon} S_\chi(x, q, t) \ll \frac{1}{2\pi i} \int_{(2)} \sum_{q \geq 1} \frac{S_\chi(x, q, t)}{q^{2v}} \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} dv.$$

To estimate the right-hand side, we will rewrite the Dirichlet series over q . To do this, we first expand $S_\chi(x, q, t)$:

$$\begin{aligned} S_\chi(x, q, t) &= \frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} |B_{c_\psi}(x, t)|^2 \left| \sum_{\ell \sim L} \chi(\ell) \bar{\psi}(\ell) \right|^2 \\ &= \frac{1}{\varphi(q)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\psi \pmod{q}} \sum_{m, n \geq 1} A(m) A(n) m^{-it} n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_\psi(m) c_{\bar{\psi}}(n) \chi(\ell_1) \bar{\psi}(\ell_1) \bar{\chi}(\ell_2) \psi(\ell_2) \\ &= \frac{1}{\varphi(q)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\psi \pmod{q}} \sum_{m, n \geq 1} A(m) A(n) m^{-it} n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_\psi(\ell_1 m) c_{\bar{\psi}}(\ell_2 n) \chi(\ell_1) \bar{\chi}(\ell_2) \\ &= \sum_{\ell_1, \ell_2 \sim L} \sum_{m, n \geq 1} A(m) A(n) m^{-it} n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) c_q(\ell_1 m - \ell_2 n) \chi(\ell_1) \bar{\chi}(\ell_2), \end{aligned}$$

where in the last line we have used the identity

$$\frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} c_\psi(\ell_1 m) c_{\bar{\psi}}(\ell_2 n) = c_q(\ell_1 m - \ell_2 n).$$

Using the relation

$$\sum_{q \geq 1} \frac{c_q(\ell_1 m - \ell_2 n)}{q^{2v}} = \begin{cases} \frac{\zeta(2v-1)}{\zeta(2v)} & \text{if } \ell_1 m = \ell_2 n, \\ \frac{\sigma_{1-2v}(h)}{\zeta^{2v}} & \text{if } \ell_1 m = \ell_2 n + h, \end{cases}$$

we can express the Dirichlet series over q as a diagonal and off-diagonal term:

$$\begin{aligned} \sum_{q \geq 1} \frac{S_\chi(x, q, t)}{q^{2v}} &= \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n} \frac{\zeta(2v-1)}{\zeta(2v)} A(m) A(n) m^{-it} n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \\ &\quad + \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n + h} \frac{\sigma_{1-2v}(h)}{\zeta(2v)} A(m) A(n) m^{-it} n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \end{aligned}$$

Thus

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{(2)} \sum_{q \geq 1} \frac{S_\chi(x, q, t)}{q^{2v}} \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} dv \\
&= \frac{1}{2\pi i} \int_{(2)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n} \frac{\zeta(2v-1)}{\zeta(2v)} A(m) A(n) m^{-it} n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} dv \\
&+ \frac{1}{2\pi i} \int_{(2)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n + h} \frac{\sigma_{1-2v}(h)}{\zeta(2v)} A(m) A(n) m^{-it} n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} dv.
\end{aligned}$$

The Diagonal Contribution. We will estimate

$$\frac{1}{2\pi i} \int_{(2)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n} \frac{\zeta(2v-1)}{\zeta(2v)} A(m) A(n) m^{-it} n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} dv.$$

The integral over v is

$$\frac{1}{2\pi i} \int_{(2)} \frac{\zeta(2v-1)}{\zeta(2v)} \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} dv \ll \sum_{|q-Q| \ll (QT)^{\varepsilon}} \varphi(q) \ll Q^{1+\varepsilon} T^{\varepsilon},$$

Therefore the diagonal contribution is

$$\begin{aligned}
& \ll Q^{1+\varepsilon} \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n} A(m) A(n) m^{-it} n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \\
& \ll Q^{1+\varepsilon} \sum_{\ell_1, \ell_2 \sim L} \sum_{\substack{\ell_1 m = \ell_2 n \\ m, n \ll (QT)^{1+\varepsilon}}} A(m) A(n) m^{-it} n^{-it} \chi(\ell_1) \bar{\chi}(\ell_2) \\
& \ll Q^{1+\varepsilon} \sum_{\ell_1, \ell_2 \sim L} \sum_{\substack{d \geq 1 \\ d \ll \frac{(QT)^{1+\varepsilon}}{L}}} A(\ell_1 \ell_2 d) A(\ell_1 \ell_2 d) \chi(\ell_1) \bar{\chi}(\ell_2) \\
& \ll L Q^{2+\varepsilon} T^{1+\varepsilon}.
\end{aligned}$$

The Off-diagonal Contribution. We will estimate

$$\frac{1}{2\pi i} \int_{(2)} \sum_{\ell_1, \ell_2 \sim L} \sum_{\ell_1 m = \ell_2 n + h} \frac{\sigma_{1-2v}(h)}{\zeta(2v)} A(m) A(n) m^{-it} n^{-it} G\left(\frac{m}{x}\right) G\left(\frac{n}{x}\right) \chi(\ell_1) \bar{\chi}(\ell_2) \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} dv.$$

Applying the Mellin inversion formula to $G\left(\frac{m}{x}\right)$ and $G\left(\frac{n}{x}\right)$, we can express the off-diagonal contribution as

$$\begin{aligned}
& \sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1) \bar{\chi}(\ell_2) \left(\frac{1}{2\pi i}\right)^3 \int_{(2)} \int_{(\sigma_{s_2})} \int_{(\sigma_{s_1})} \frac{1}{\zeta(2v)} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m) a(n) \sigma_{1-2v}(h)}{m^{s_1 + \frac{k-1}{2} + it} n^{s_2 + \frac{k-1}{2} + it}} \\
& \cdot g(s_1) g(s_2) x^{s_1 + s_2} \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} ds_1 ds_2 dv,
\end{aligned}$$

with $\sigma_{s_1}, \sigma_{s_2} \gg 1$. Now make the following computation:

$$\begin{aligned} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m)a(n)\sigma_{1-2v}(h)}{m^{s_1 + \frac{k-1}{2} + it} n^{s_2 + \frac{k-1}{2} + it}} &= \ell_1^{s_1 + it} \ell_2^{s_2 + it} (\ell_1 \ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m)a(n)\sigma_{1-2v}(h)}{(\ell_1 m)^{s_1 + \frac{k-1}{2} + it} (\ell_2 n)^{s_2 + \frac{k-1}{2} + it}} \\ &= \ell_1^{s_1 + it} \ell_2^{s_2 + it} (\ell_1 \ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m)a(n)\sigma_{1-2v}(h)}{(\ell_2 n + h)^{s_1 + \frac{k-1}{2} + it} (\ell_2 n)^{s_2 + \frac{k-1}{2} + it}} \\ &= \ell_1^{s_1 + it} \ell_2^{s_2 + it} (\ell_1 \ell_2)^{\frac{k-1}{2}} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m)a(n)\sigma_{1-2v}(h)}{\left(1 + \frac{h}{\ell_2 n}\right)^{s_1 + \frac{k-1}{2} + it} (\ell_2 n)^{s_1 + s_2 + k - 1 + 2it}}. \end{aligned}$$

Recall the Mellin inversion formula

$$\frac{1}{(1+t)^\beta} = \frac{1}{2\pi i} \int_{(\sigma_u)} \frac{\Gamma(\beta - u)\Gamma(u)}{\Gamma(\beta)} t^{-u} du,$$

with $0 < \sigma_u < \operatorname{Re}(\beta)$. Applying this formula with $t = \frac{h}{\ell_2 n}$ and $\beta = s_1 + \frac{k-1}{2} + it$, we have

$$\begin{aligned} \sum_{\ell_1 m = \ell_2 n + h} \frac{a(m)a(n)\sigma_{1-2v}(h)}{m^{s_1 + \frac{k-1}{2} + it} n^{s_2 + \frac{k-1}{2} + it}} &= \frac{1}{2\pi i} \int_{(\sigma_u)} \ell_1^{s_1 + it} \ell_2^{s_2 + it} Z_f(s_1 + s_2 - u + 2it, v, u; \ell_1, \ell_2) \\ &\quad \cdot \frac{\Gamma(s_1 - u + \frac{k-1}{2} + 2it) \Gamma(u)}{\Gamma(s_1 + \frac{k-1}{2} + 2it)} du. \end{aligned}$$

Therefore the off-diagonal contribution can be expressed as

$$\begin{aligned} \sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1) \bar{\chi}(\ell_2) \left(\frac{1}{2\pi i}\right)^4 \int_{(2)} \int_{(\sigma_u)} \int_{(\sigma_{s_2})} \int_{(\sigma_{s_1})} \frac{1}{\zeta(2v)} \ell_1^{s_1 + it} \ell_2^{s_2 + it} Z_f(s_1 + s_2 - u + 2it, v, u; \ell_1, \ell_2) \\ \cdot \frac{\Gamma(s_1 - u + \frac{k-1}{2} + 2it) \Gamma(u)}{\Gamma(s_1 + \frac{k-1}{2} + 2it)} g(s_1) g(s_2) x^{s_1 + s_2} \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} ds_1 ds_2 du dv. \end{aligned}$$

Since $\sigma_{s_1}, \sigma_{s_2} \gg 1$ and $0 < \sigma_u < \sigma_{s_1} + \frac{k-1}{2}$, we may assume

$$\sigma_{s_1} + \sigma_{s_2} - \sigma_u > 1, \quad \sigma_u > \frac{k+1}{2}, \quad \text{and} \quad 2 \geq \frac{1}{2} - \delta(s, v, u).$$

This ensures that $Z_f(s_1 + s_2 - u + 2it, v, u; \ell_1, \ell_2)$ is absolutely convergent. Let us take

$$\sigma_{s_1} = 1 + \varepsilon_1, \quad \sigma_{s_2} = \frac{k+1}{2} + \varepsilon_2, \quad \text{and} \quad \sigma_u = \frac{k+1}{2} + \varepsilon_3,$$

with $\varepsilon_1 + \varepsilon_2 - \varepsilon_3 > 0$. The analytic continuation of $Z_f(s_1 + s_2 - u + 2it, v, u; \ell_1, \ell_2)$ exhibits no poles in the region

$$\sigma_{s_1} + \sigma_{s_2} - \sigma_u > \frac{1}{2} \quad \text{and} \quad \sigma_u > 0.$$

Therefore, we may shift the integral in u to $\sigma_u = \varepsilon_4$ without crossing over any poles. Then, we may shift the integrals in s_1 and s_2 to $\sigma_{s_1} + \sigma_{s_2} = \frac{1}{2} + \varepsilon_5$ provided $\varepsilon_5 > \varepsilon_4$ without crossing over any poles. Now we shift the integral in u so that

$$\sigma_{s_1} + \sigma_{s_2} - \sigma_u < \frac{1-k}{2} - \varepsilon_6 \quad \text{while} \quad \sigma_1 - \sigma_u + \frac{k-1}{2} > 0.$$

This is possible by moving the integral in u to $\sigma_u = \frac{k}{2} + \varepsilon_5 + \varepsilon_6$ and choosing $\sigma_1 = \frac{1}{2} + 2\varepsilon_5$ and $\sigma_2 = -\varepsilon_5$ with $\varepsilon_5 > \varepsilon_6$. In doing so, we pass over simple poles of $Z_f(s_1 + s_2 - u, v, u; \ell_1, \ell_2)$ occurring at $u =$

$s_1 + s_2 + 2it - \frac{1}{2} + \ell - it_j$ for $0 \leq \ell \leq \frac{k}{2}$. We do not pass over any simple poles of gamma functions. since $\sigma_u > 0$ and $\sigma_1 - \sigma_u + \frac{k-1}{2} > 0$ throughout. Then the off-diagonal contribution can be expressed as

$$\begin{aligned} & \sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1) \bar{\chi}(\ell_2) \left(\frac{1}{2\pi i} \right)^4 \int_{(2)} \int_{(\sigma_u)} \int_{(\sigma_{s_2})} \int_{(\sigma_{s_1})} \frac{1}{\zeta(2v)} \ell_1^{s_1+it} \ell_2^{s_2+it} (\ell_1 \ell_2)^{\frac{k-1}{2}} Z_f(s_1 + s_2 - u + 2it, v, u; \ell_1, \ell_2) \\ & \cdot \frac{\Gamma(s_1 - u + \frac{k-1}{2} + 2it) \Gamma(u)}{\Gamma(s_1 + \frac{k-1}{2} + 2it)} g(s_1) g(s_2) x^{s_1+s_2} \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} ds_1 ds_2 du dv \\ & + \sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1) \bar{\chi}(\ell_2) \left(\frac{1}{2\pi i} \right)^3 \int_{(2)} \int_{(\sigma_{s_2})} \int_{(\sigma_{s_1})} \sum_{t_j} \frac{1}{\zeta(2v)} \ell_1^{s_1+it} \ell_2^{s_2+it} (\ell_1 \ell_2)^{\frac{k-1}{2}} \text{Res}_{u=s_1+s_2+2it-\frac{1}{2}+\ell-it_j} \\ & \cdot \left[Z_f(s_1 + s_2 - u + 2it, v, u; \ell_1, \ell_2) \frac{\Gamma(s_1 - u + \frac{k-1}{2} + 2it) \Gamma(u)}{\Gamma(s_1 + \frac{k-1}{2} + 2it)} \right] g(s_1) g(s_2) x^{s_1+s_2} \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} ds_1 ds_2 dv. \end{aligned}$$

Let us concern ourselves with the first term only. Here the first spectral expansion of $Z_f(s_1 + s_2 - u + 2it, v, u; \ell_1, \ell_2)$ is valid. The integral over v is

$$\frac{1}{2\pi i} \int_{(2)} \frac{L(s_1 + s_2 + 2v - \frac{3}{2} + 2it)}{\zeta(2v) \zeta(2s_1 + 2s_2 + 2v - 2 + 4it)} \frac{e^{\frac{\pi v^2}{(QT^{-\varepsilon})^2}} Q^{2v}}{QT^{-\varepsilon}} dv \ll \sum_{\substack{q=q_1 q_2 q_3 \\ |q-Q| \ll (QT)^{\varepsilon}}} \mu(q_1 q_2) q_2^{1-2\varepsilon_5} q_3^{1-s_5} \lambda_j(q_3) \ll Q^{1+\theta+\varepsilon} T^{\varepsilon}.$$

Moreover $x \ll (QT)^{1+\varepsilon}$ so that

$$x^{s_1+s_2} \ll (QT)^{\frac{1}{2}+\varepsilon}$$

The contribution of L is

$$\ll \sum_{\ell_1, \ell_2 \sim L} \chi(\ell_1) \bar{\chi}(\ell_2) \sum_{|t_j| \sim 1} \overline{\rho_j(-1) \langle V_{f,g}^{\ell_1, \ell_2}, \mu_j \rangle} \ell_1^{s_1+it} \ell_2^{s_2+it} (\ell_1 \ell_2)^{\frac{k-1}{2}} \ll L^{\frac{5}{2}}.$$

In total, the off-diagonal contribution is

$$\ll L^{\frac{5}{2}} Q^{\frac{3}{2}+\theta+\varepsilon} T^{\frac{1}{2}+\varepsilon}.$$

Balancing. We have the diagonal and off-diagonal estimates

$$\ll L Q^{2+\varepsilon} T^{1+\varepsilon} \quad \text{and} \quad \ll L^{\frac{5}{2}} Q^{\frac{3}{2}+\theta+\varepsilon} T^{\frac{1}{2}+\varepsilon}.$$

This implies

$$\left| L \left(\frac{1}{2}, f \otimes \chi \right) \right|^2 \ll \frac{1}{L^{2+\varepsilon} QT} \max_{x \ll (QT)^{1+\varepsilon}} \sum_{|q-Q| \ll Q^{\varepsilon}} S_{\chi}(x, q) \ll \frac{Q^{1+\varepsilon} T^{\varepsilon}}{L^{1+\varepsilon}} + \frac{L^{\frac{1}{2}-\varepsilon} Q^{\frac{1}{2}+\theta+\varepsilon}}{T^{\frac{1}{2}-\varepsilon}}.$$

The terms are balanced when $L = Q^{\frac{1-2\theta}{3}}$. We then have

$$\left| L \left(\frac{1}{2}, f \otimes \chi \right) \right|^2 \ll Q^{\frac{2}{3}+\frac{2\theta}{3}+\varepsilon},$$

and it follows that

$$\left| L \left(\frac{1}{2}, f \otimes \chi \right) \right| \ll Q^{\frac{1}{3}+\frac{\theta}{3}+\varepsilon}.$$