Here is a series of exercises proving various summation formulas. For any sequence $(a_n)_{n\geq 1}$ of complex numbers and $X > 0$, set $X \geq 0$ let

$$A(X) = \sum_{n\leq X} a_n.$$

1. Summation by parts: Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of complex numbers. Then for any positive integers N and M with $1 \leq M \leq N$, prove that

$$\sum_{M\leq n\leq N} a_n(b_{n+1} - b_n) = (a_{N+1}b_{N+1} - a_M b_M) - \sum_{M\leq n\leq N} b_{n+1}(a_{n+1} - a_n).$$

2. Summation by parts: Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$ be two sequences of complex numbers. Then for any integers M and N with $0 \leq M \leq N$, prove that

$$\sum_{M+1\leq n\leq N} a_n b_n = A(N)b_N - A(M)b_M - \sum_{M\leq n\leq N-1} A(n)(b_{n+1} - b_n).$$

3. Abel's summation formula: Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. For any X and Y with $0 \leq X < Y$ and continuously differentiable function $\phi : [X, Y] \to \mathbb{C}$, prove that

$$\sum_{X<n\leq Y} a_n\phi(n) = A(Y)\phi(Y) - A(X)\phi(X) - \int_X^Y A(u)\phi'(u)\,du.$$

   Hint: First prove in the case $X = M$ and $Y = N$. Then try to generalize by letting $M = \lfloor X\rfloor$ and $N = \lfloor Y\rfloor$. For the first case, start by using summation by parts.

4. Euler-Maclaurin summation formula: For any X and Y with $0 \leq X < Y$ and continuously differentiable function $\phi : [X, Y] \to \mathbb{C}$, we have

$$\sum_{X<n\leq Y} \phi(n) = \left(X - \lfloor X\rfloor - \frac{1}{2}\right)\phi(X) - \left(Y - \lfloor Y\rfloor - \frac{1}{2}\right)\phi(Y) + \int_X^Y \phi(u) + \left(u - \lfloor u\rfloor - \frac{1}{2}\right)\phi'(u)\,du.$$

   Hint: Start as you would in proving Abel's summation formula but then try applying integration by parts.

5. Use the Euler-Maclaurin summation formula to obtain an integral representation for $\zeta(s)$ that is meromorphic for $\sigma > 0$. Could you continue further? How?