

# A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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ABSTRACT. We construct a quadratic double Dirichlet series  $Z(s, w)$  built from single variable quadratic Dirichlet  $L$ -functions  $L(s, \chi)$  over  $\mathbb{Q}$ . We prove that  $Z(s, w)$  admits meromorphic continuation to the  $(s, w)$ -plane and satisfies a group of functional equations.

## 1. PRELIMINARIES

We present an overview of quadratic Dirichlet  $L$ -functions over  $\mathbb{Q}$ . We begin with the Riemann zeta-function. The zeta function  $\zeta(s)$  is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m \geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for  $\operatorname{Re}(s) > 1$ . The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Now we recall characters on  $\mathbb{Z}$ . They are multiplicative functions  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$  and form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions  $\chi_d : \mathbb{Z} \rightarrow \mathbb{C}$  modulo  $d \geq 1$  (in that they are  $d$ -periodic) and such that  $\chi_d(m) = 0$  if  $(m, d) > 1$ .
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. Moreover,  $\bar{\chi}$  is the multiplicative inverse to  $\chi$  and the Dirichlet characters modulo  $d$  form a subgroup under multiplication. This group is always finite and its order is  $\phi(d) = |(\mathbb{F}_q[t]/d\mathbb{F}_q[t])^\times|$ . The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on  $\mathbb{Z}$ . First let us recall this symbol. For any odd prime  $p$  and any  $d \in \mathbb{Z}$ , we define the quadratic residue symbol  $\left(\frac{d}{p}\right)$  by

$$\left(\frac{d}{p}\right) \equiv d^{\frac{p-1}{2}} \pmod{p} = \begin{cases} 1 & \text{if } x^2 \equiv d \pmod{p} \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv d \pmod{p} \text{ is not solvable,} \\ 0 & \text{if } d \equiv 0 \pmod{p}. \end{cases}$$

This symbol only depends upon  $d$  modulo  $p$  and is multiplicative in  $d$ . We can extend the quadratic residue symbol multiplicatively in the denominator. First we define

$$\left(\frac{d}{-1}\right) = \begin{cases} 1 & \text{if } d \geq 0, \\ -1 & \text{if } d < 0, \end{cases} \quad \text{and} \quad \left(\frac{d}{2}\right) = \begin{cases} 1 & \text{if } d \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } d \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } d \equiv 0 \pmod{2}. \end{cases}$$

If  $m = up_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$  is the prime factorization of  $m$  (with  $u = \pm 1$ ), then we define

$$\left(\frac{d}{m}\right) = \left(\frac{d}{u}\right) \prod_{1 \leq i \leq k} \left(\frac{d}{p_i}\right)^{e_i}.$$

The quadratic residue symbol now makes sense for any  $m \in \mathbb{Z}$  and is multiplicative in both  $d$  and  $m$ . The quadratic residue symbol also admits the following reciprocity law:

**Theorem 1.1** (Quadratic reciprocity). *If  $d, m \in \mathbb{Z}$ , then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2} \frac{m^{(2)}-1}{2}} \left(\frac{m}{|d|}\right),$$

where  $d^{(2)}$  and  $m^{(2)}$  are the parts of  $d$  and  $m$  relatively prime to 2 respectively.

Moreover, we have the additional relations

$$\left(\frac{-1}{m}\right) = (-1)^{\frac{m^{(2)}-1}{2}} \quad \text{and} \quad \left(\frac{2}{m}\right) = (-1)^{\frac{m^2-1}{8}},$$

and if  $m \not\equiv 0 \pmod{2}$ , we can write

$$\left(\frac{-1}{m}\right) = (-1)^{\frac{m-1}{2}} = \begin{cases} 1 & m \equiv 1 \pmod{4}, \\ -1 & m \equiv 3 \pmod{4}, \end{cases} \quad \text{and} \quad \left(\frac{2}{m}\right) = (-1)^{\frac{m^2-1}{8}} = \begin{cases} 1 & m \equiv 1, 7 \pmod{8}, \\ -1 & m \equiv 3, 5 \pmod{8}. \end{cases}$$

We can now define the quadratic Dirichlet characters. For any square-free  $d \in \mathbb{Z}$ , define the quadratic Dirichlet character  $\chi_d$  by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod{4}, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod{4}. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension  $\mathbb{Q}(\sqrt{d})$ . We extend  $\chi_d$  multiplicatively in the denominator so that  $\chi_d$  makes sense for any odd  $d$ . In particular,  $\chi_d(m) = \pm 1$  provided  $d$  and  $m$  are relatively prime and  $\chi_d(m) = 0$  if  $(m, d) > 1$ . Quadratic reciprocity implies that  $\chi_d$  is a Dirichlet character modulo  $|d|$  if  $d \equiv 1 \pmod{4}$  and is a Dirichlet character modulo  $|4d|$  if  $d \equiv 2, 3 \pmod{4}$ . Indeed, if  $d \equiv 1 \pmod{4}$  then  $d^{(2)} = d$  and the sign is always 1. If  $d \equiv 3 \pmod{4}$ , then  $d^{(2)} = d$  and the sign is  $\left(\frac{-1}{m}\right)$  which is a character modulo 4. If  $d \equiv 2 \pmod{4}$ , then  $d^{(2)} \equiv 1, 3 \pmod{4}$  and we are reduced to one of the previous two cases. We will also set

$$q(d) = \begin{cases} |d| & \text{if } d \equiv 1 \pmod{4}, \\ |4d| & \text{if } d \equiv 2, 3 \pmod{4}, \end{cases} \quad \text{and} \quad \varepsilon_{\chi_d} = \frac{\tau(\chi_d)}{\sqrt{q(d)}} = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ 1 + i & \text{if } d \equiv 2, 3 \pmod{4}, \end{cases}$$

where  $\tau(\chi_d)$  is the Gauss sum attached to  $\chi_d$ . We will also require an associated character. For each  $\chi_m$  (here we are purposely interchanging the roles of  $d$  and  $m$  to keep consistency with the notation when discussing the quadratic double Dirichlet series later), we define  $\tilde{\chi}_m$  by

$$\tilde{\chi}_m(d) = (-1)^{\frac{m^{(2)}-1}{2} \frac{d^{(2)}-1}{2}} \chi_m(|d|).$$

By quadratic reciprocity,  $\tilde{\chi}_m$  is a quadratic Dirichlet character of the same modulus as  $\chi_m$  and is multiplicative in  $m$ . Moreover, we have the identity  $\tilde{\tilde{\chi}}_m(d) = \chi_m(|d|)$ . Analogously, we set

$$q(m) = \begin{cases} |m| & \text{if } m \equiv 1 \pmod{4}, \\ |4m| & \text{if } m \equiv 2, 3 \pmod{4}, \end{cases} \quad \text{and} \quad \varepsilon_{\tilde{\chi}_m} = \frac{\tau(\tilde{\chi}_m)}{\sqrt{q(m)}} = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{4}, \\ 1 + i & \text{if } m \equiv 2, 3 \pmod{4}, \end{cases}$$

where  $\tau(\tilde{\chi}_m)$  is the Gauss sum attached to  $\tilde{\chi}_m$ . We now discuss the Hilbert characters. We will only need four of them: the quadratic Dirichlet characters modulo 8. They are given as follows:

$$\begin{aligned} \chi_1(m) &= \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod{2}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} & \chi_{-1}(m) &= \begin{cases} 1 & \text{if } m \equiv 1 \pmod{4}, \\ -1 & \text{if } m \equiv 3 \pmod{4}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} \\ \chi_2(m) &= \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} & \chi_{-2}(m) &= \begin{cases} 1 & \text{if } m \equiv 1, 3 \pmod{8}, \\ -1 & \text{if } m \equiv 5, 7 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}. \end{cases} \end{aligned}$$

In general, we will denote a Hilbert character by  $\chi_a$  with  $a \in \{\pm 1, \pm 2\}$ . The Hilbert characters also satisfy an important orthogonality property:

**Theorem 1.2** (Orthogonality of Hilbert characters). *If  $d, m \in \mathbb{Z}$  are odd, then*

$$\frac{1}{4} \sum_{a \in \{\pm 1, \pm 2\}} \chi_a(dm) = \begin{cases} 1 & \text{if } d \equiv m \pmod{8}, \\ 0 & \text{if } d \not\equiv m \pmod{8}. \end{cases}$$

Also, we have the identities

$$\tilde{\chi}_a(m) = \chi_a(|m|), \quad \chi_{-1}(m) = \left(\frac{-1}{m}\right), \quad \text{and} \quad \chi_2(m) = \left(\frac{2}{m}\right),$$

and the relations

$$\chi_{-2}(m) = \chi_{-1}(m)\chi_2(m), \quad \chi_1(m) = \chi_{-1}(m)\chi_{-1}(m), \quad \text{and} \quad \chi_{-1}(m) = \chi_2(m)\chi_{-2}(m).$$

With the Dirichlet and Hilbert characters introduced, we are ready to discuss the  $L$ -functions associated to quadratic Dirichlet characters. We define the  $L$ -function  $L(s, \chi_d)$  attached to  $\chi_d$  for square-free  $d$ , by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \geq 1} \frac{\chi_d(m)}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character,  $L(s, \chi_d) \ll \zeta(s)$  for  $\text{Re}(s) > 1$  so that  $L(s, \chi_d)$  is locally absolutely uniformly convergent in this region.  $L(s, \chi_d)$  also admits analytic continuation to  $\mathbb{C}$ . The completed  $L$ -function  $L^*(s, \chi_d)$  is defined as

$$L^*(s, \chi_d) = \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi_d) & \text{if } d > 0, \\ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi_d) & \text{if } d < 0. \end{cases}$$

We have the functional equation

$$L^*(s, \chi_d) = \varepsilon_{\chi_d} q(d)^{\frac{1}{2}-s} L^*(1-s, \chi_d),$$

which can be equivalently expressed as

$$L^*(s, \chi_d) = \begin{cases} |d|^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } d \equiv 1, 5 \pmod{8}, \\ (1+i)|4d|^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } d \equiv 2, 3, 6, 7 \pmod{8}. \end{cases}$$

Note that the gamma factor depends upon  $d$  modulo 8. This is the root cause of an important technical issue later when deriving functional equations for the quadratic double Dirichlet series. Analogously, the Dirichlet  $L$ -function  $L(w, \tilde{\chi}_m)$  attached to  $\tilde{\chi}_m$  for square-free  $m$  is defined by a Dirichlet series or Euler product:

$$L(w, \tilde{\chi}_m) = \sum_{d \geq 1} \frac{\tilde{\chi}_m(d)}{d^w} = \prod_{p \text{ prime}} \left(1 - \frac{\tilde{\chi}_m(p)}{p^w}\right)^{-1}.$$

As for  $L(s, \chi_d)$ ,  $L(w, \tilde{\chi}_m) \ll \zeta(w)$  for  $\operatorname{Re}(w) > 1$  so that  $L(w, \tilde{\chi}_m)$  is locally absolutely uniformly convergent in this region. Moreover,  $L(w, \tilde{\chi}_m)$  admits analytic continuation to  $\mathbb{C}$  and the completed  $L$ -function  $L^*(w, \tilde{\chi}_m)$  is defined as

$$L^*(w, \tilde{\chi}_m) = \begin{cases} \pi^{-\frac{w}{2}} \Gamma\left(\frac{w}{2}\right) L(w, \tilde{\chi}_m) & \text{if } m \equiv 1, 2, 5 \pmod{8}, \\ \pi^{-\frac{w}{2}} \Gamma\left(\frac{w+1}{2}\right) L(w, \tilde{\chi}_m) & \text{if } m \equiv 3, 6, 7 \pmod{8}. \end{cases}$$

We have the functional equation

$$L^*(w, \tilde{\chi}_m) = \varepsilon_{\tilde{\chi}_m} q(m)^{\frac{1}{2}-w} L^*(1-w, \tilde{\chi}_m),$$

which can be equivalently expressed as

$$L^*(w, \tilde{\chi}_m) = \begin{cases} |m|^{\frac{1}{2}-w} L^*(1-w, \tilde{\chi}_m) & \text{if } m \equiv 1, 5 \pmod{8}, \\ (1+i)|4m|^{\frac{1}{2}-w} L^*(1-w, \tilde{\chi}_m) & \text{if } m \equiv 2, 3, 6, 7 \pmod{8}. \end{cases}$$

Analogously, note that the gamma factor depends upon  $m$  modulo 8.

**Remark 1.1.** *The definitions for  $L(s, \chi_d)$ ,  $L^*(s, \chi_d)$ ,  $L(w, \tilde{\chi}_m)$ , and  $L^*(w, \tilde{\chi}_m)$  work perfectly well even when  $d$  and  $m$  are not square-free (however the functional equations do not hold). We purposely do not define these  $L$ -functions, yet, for  $d$  and  $m$  not necessarily square-free.*

## 2. THE QUADRATIC DOUBLE DIRICHLET SERIES

We will now define the quadratic double Dirichlet series  $Z(s, w)$ . For any integer  $d \geq 1$ , write  $d = d_0 d_1^2$  where  $d_0$  is square-free. Equivalently,  $d_0$  is the square-free part of  $d$  and  $\frac{d}{d_0}$  is a perfect square. The **quadratic double Dirichlet series**  $Z(s, w)$  is defined as

$$Z(s, w) = \sum_{\substack{d \geq 1 \\ (d, 2) = 1}} \frac{L^{(2)}(s, \chi_{d_0}) Q_{d_0 d_1^2}(s)}{d^w},$$

where  $Q_{d_0 d_1^2}(s)$  is the **correction polynomial** defined by

$$Q_{d_0 d_1^2}(s) = \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{d_0}(e_1) e_1^{-s} e_2^{1-s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{d_0}(e_1) e_1^{-s} e_2^{1-s},$$

and  $\mu$  is the usual Möbius function. For  $\operatorname{Re}(s) > 1$ , there is the trivial estimate

$$Q_{d_0 d_1^2}(s) \ll \sum_{e_1 e_2 | d_1} 1 \ll \sigma_0(d_1)^2 \ll_\varepsilon d_1^{2\varepsilon} \ll_\varepsilon d^\varepsilon,$$

for any  $\varepsilon > 0$ . As  $L(s, \chi_{d_0}) \ll 1$  for  $\operatorname{Re}(s) > 1$ ,  $Z(s, w)$  is locally absolutely uniformly convergent in the region  $\Lambda = \{(s, w) \in \mathbb{C}^2 : \operatorname{Re}(s) > 1, \operatorname{Re}(w) > 1\}$ . It will also be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters  $\chi_{a_1}$  and  $\chi_{a_2}$ . The **quadratic double Dirichlet series**  $Z_{a_1, a_2}(s, w)$  twisted by  $\chi_{a_1}$  and  $\chi_{a_2}$  is defined as

$$Z_{a_1, a_2}(s, w) = \sum_{\substack{d \geq 1 \\ (d, 2) = 1}} \frac{L^{(2)}(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w},$$

where  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  is the **correction polynomial** twisted by  $\chi_{a_1}$  defined by

$$Q_{d_0 d_1^2}(s, \chi_{a_1}) = \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s},$$

and  $\mu$  is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound  $Q_{d_0 d_1^2}(s, \chi_{a_1}) \ll d_\varepsilon$  so that  $Z_{a_1, a_2}(s, w)$  converges locally absolutely uniformly in the same region as

$Z(s, w)$  does. In particular,  $Z(s, w) = Z_{1,1}(s, w)$ . We will also require quadratic double Dirichlet series corresponding to the characters  $\tilde{\chi}_m$ . Analogously writing  $m = m_0 m_1^2$ , the **quadratic double Dirichlet series**  $\tilde{Z}(s, w)$  is defined as

$$\tilde{Z}(w, s) = \sum_{\substack{m \geq 1 \\ (m, 2)=1}} \frac{L^{(2)}(w, \tilde{\chi}_{m_0}) Q_{m_0 m_1^2}(w)}{m^s},$$

where  $Q_{d_0 d_1^2}(w)$  is the **correction polynomial** defined by

$$Q_{m_0 m_1^2}(w) = \sum_{e_1 e_2 | m_1} \mu(e_1) \chi_{m_0}(e_1) e_1^{-w} e_2^{1-w} = \sum_{e_1 e_2 e_3 = m_1} \mu(e_1) \chi_{m_0}(e_1) e_1^{-w} e_2^{1-w},$$

and  $\mu$  is the usual Möbius function. We have the analogous estimate  $Q_{m_0 m_1^2}(w) \ll_\varepsilon m^\varepsilon$  and as  $L(w, \tilde{\chi}_{m_0}) \ll 1$  for  $\text{Re}(w) > 1$ ,  $\tilde{Z}(w, s)$  is locally absolutely uniformly convergent in the same region as  $Z(s, w)$ . We also need to consider twists by a pair of Hilbert characters  $\tilde{\chi}_{a_2}$  and  $\tilde{\chi}_{a_1}$ . The **quadratic double Dirichlet series**  $\tilde{Z}_{a_2, a_1}(w, s)$  twisted by  $\tilde{\chi}_{a_2}$  and  $\tilde{\chi}_{a_1}$  is defined as

$$\tilde{Z}_{a_2, a_1}(w, s) = \sum_{\substack{m \geq 1 \\ (m, 2)=1}} \frac{L^{(2)}(w, \tilde{\chi}_{a_2 m_0}) \tilde{\chi}_{a_1}(m) Q_{m_0 m_1^2}(w, \tilde{\chi}_{a_2})}{m^s},$$

where  $Q_{m_0 m_1^2}(w, \tilde{\chi}_{a_2})$  is the **correction polynomial** twisted by  $\tilde{\chi}_{a_2}$  defined by

$$Q_{m_0 m_1^2}(w, \tilde{\chi}_{a_2}) = \sum_{e_1 e_2 | m_1} \mu(e_1) \tilde{\chi}_{a_2 m_0}(e_1) e_1^{-w} e_2^{1-2w} = \sum_{e_1 e_2 e_3 = m_1} \mu(e_1) \tilde{\chi}_{a_2 m_0}(e_1) e_1^{-w} e_2^{1-2w}.$$

and  $\mu$  is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound  $Q_{m_0 m_1^2}(w, \tilde{\chi}_{a_2}) \ll_\varepsilon m^\varepsilon$  so that  $\tilde{Z}_{a_2, a_1}(w, s)$  converges locally absolutely uniformly in the same region as  $\tilde{Z}(w, s)$  does. In particular,  $\tilde{Z}(w, s) = \tilde{Z}_{1,1}(w, s)$ .

### 3. THE INTERCHANGE

As defined,  $Z_{a_1, a_2}(s, w)$  is a sum of  $L$ -functions, and hence Euler products, in  $s$ . We will prove an interchange formula for  $Z_{a_1, a_2}(s, w)$  which will show that it can be expressed as a sum of  $L$ -functions in  $w$ . That is, we want the variables  $s$  and  $w$  to change places. Precisely:

**Theorem 3.1** (Interchange). *Wherever  $Z_{a_1, a_2}(s, w)$  converges locally absolutely uniformly,*

$$Z_{a_1, a_2}(s, w) = \sum_{\substack{d \geq 1 \\ (d, 2)=1}} \frac{L^{(2)}(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w} = \sum_{\substack{m \geq 1 \\ (m, 2)=1}} \frac{L^{(2)}(w, \tilde{\chi}_{a_2 m_0}) \tilde{\chi}_{a_1}(m) Q_{m_0 m_1^2}(w, \tilde{\chi}_{a_2})}{m^s}.$$

Moreover, the same holds for  $\tilde{Z}_{a_2, a_1}(w, s)$ .

*Proof.* Only the second equality needs to be proved. To do this, first expand the  $L$ -function  $L^{(2)}(s, \chi_{a_1 d_0})$  and polynomial  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  to get

$$\begin{aligned} Z(s, w) &= \sum_{\substack{d \geq 1 \\ (d, 2)=1}} \frac{L^{(2)}(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w} \\ &= \sum_{\substack{d \geq 1 \\ (d, 2)=1}} \left( \sum_{\substack{m \geq 1 \\ (m, 2)=1}} \chi_{a_1 d_0}(m) m^{-s} \right) \left( \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} \right) \chi_{a_2}(d) d^{-w} \\ &= \sum_{\substack{d, m \geq 1 \\ (dm, 2)=1}} \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m e_1) e_1^{-s} e_2^{1-2s} m^{-s} d^{-w}. \end{aligned}$$

Now  $\chi_{a_1 d_0}(m e_1) = 0$  unless  $(d_0, m e_1) = 1$ . We make this restriction on the sum giving

$$\sum_{\substack{d, m \geq 1 \\ (dm, 2)=1}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, m e_1)=1}} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m e_1) e_1^{-s} e_2^{1-2s} m^{-s} d^{-w}.$$

Making the change of variables  $m e_1 \rightarrow m$  yields

$$\sum_{\substack{d \geq 1 \\ (d, 2)=1}} \sum_{\substack{m \geq 1 \\ (m, 2)=1}} \sum_{\substack{e_1 e_2 | d_1 \\ e_1 | m \\ (d_0, m)=1}} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m) e_2^{1-2s} m^{-s} d^{-w}.$$

For fixed  $d = d_0 d_1^2$  and  $e_2$ , the subsum over  $m$  and  $e_1$  is

$$\sum_{\substack{m \geq 1 \\ (m, 2)=1 \\ e_1 | m}} \sum_{\substack{e_1 | \frac{d_1}{e_2} \\ (d_0, m)=1}} \mu(e_1) \chi_{a_1 d_0}(m) m^{-s} = \sum_{\substack{m \geq 1 \\ (m, 2)=1 \\ (d_0, m)=1}} \chi_{a_1 d_0}(m) m^{-s} \left( \sum_{e_1 | \left(\frac{d_1}{e_2}, m\right)} \mu(e_1) \right).$$

The inner sum over  $e_1$  of the Möbius function vanishes unless  $\left(\frac{d_1}{e_2}, m\right) = 1$  in which case it is 1. Therefore the triple sum above becomes

$$\sum_{\substack{d, m \geq 1 \\ (dm, 2)=1}} \sum_{\substack{e_2 | d_1 \\ \left(\frac{d_0 d_1}{e_2}, m\right)=1}} \chi_{a_2}(d) \chi_{a_1 d_0}(m) e_2^{1-2s} m^{-s} d^{-w}.$$

Making the change of variables  $d \rightarrow d e_2^2$ , the condition  $\left(\frac{d_0 d_1}{e_2}, m\right) = 1$  becomes  $(d_0 d_1, m) = 1$  which is equivalent to  $(d, m) = 1$ . Moreover,  $\chi_{a_2}(d e_2^2) = \chi_{a_2}(d)$ . Altogether, we obtain

$$\sum_{\substack{d, m \geq 1 \\ (dm, 2)=1 \\ (d, m)=1}} \sum_{e_2} \chi_{a_2}(d) \chi_{a_1 d_0}(m) e_2^{1-2s-2w} m^{-s} d^{-w}.$$

Writing  $m = m_0 m_1^2$  analogously as for  $d$ , quadratic reciprocity and positivity of  $m$  and  $d$  together imply  $\chi_{d_0}(m) = \tilde{\chi}_m(d_0) = \tilde{\chi}_{m_0}(d)$  where the last equality holds because  $(d, m) = 1$  and both  $d_0$  and  $m_0$  differ from  $d$  and  $m$  respectively by perfect squares. As  $\chi_{a_1}(m) = \tilde{\chi}_{a_1}(m)$  and  $\chi_{a_2}(d) = \tilde{\chi}_{a_2}(d)$  (again we use the

positivity of  $m$  and  $d$ ), the previous fact implies  $\chi_{a_2}(d)\chi_{a_1d_0}(m) = \tilde{\chi}_{a_1}(m)\tilde{\chi}_{a_2m_0}(d)$  and so our expression becomes

$$\sum_{\substack{d,m \geq 1 \\ (dm,2)=1 \\ (d,m)=1}} \sum_{e_2} \tilde{\chi}_{a_1}(m)\tilde{\chi}_{a_2m_0}(d)e_2^{1-2s-2w}m^{-s}d^{-w}.$$

But now we can reverse the argument with the roles of  $d$ ,  $m$ ,  $\chi_{a_1}$ , and  $\chi_{a_2}$  interchanged respectively, but with  $\tilde{\chi}_{a_1}$  and  $\tilde{\chi}_{a_2}$ , to obtain

$$Z(s, w) = \sum_{\substack{m \geq 1 \\ (m,2)=1}} \frac{L^{(2)}(w, \tilde{\chi}_{a_2m_0})\tilde{\chi}_{a_1}(m)Q_{m_0m_1^2}(w, \tilde{\chi}_{a_2})}{m^s}.$$

Clearly the same holds for  $\tilde{Z}_{a_2,a_1}(w, s)$ . □

Note that the interchange is not completely symmetric because of the characters  $\tilde{\chi}_{a_2m_0}$ ,  $\tilde{\chi}_{a_1}$ , and  $\tilde{\chi}_{a_2}$  in the second expression for  $Z_{a_1,a_2}(s, w)$ . This is due to the fact that reciprocity is not perfect. In even more general settings the correction polynomials in  $w$  need not be equal to those in  $s$ .

**Remark 3.1.** When  $a_1 = a_2 = 1$ , the interchange implies

$$Z(s, w) = \tilde{Z}(w, s).$$

More generally, the interchange implies the following relations for twisted quadratic double Dirichlet series:

$$Z_{a_1,a_2}(s, w) = \tilde{Z}_{a_2,a_1}(w, s),$$

for  $a_1, a_2 \in \{\pm 1, \pm 2\}$ .

#### 4. WEIGHTING TERMS

We will now study the coefficients of  $Z_{a_1,a_2}(s, w)$  expanded in  $s$  and  $w$ . Expanding  $L^{(2)}(s, \chi_{a_1d_0})Q_{d_0d_1^2}(s, \chi_{a_1})$  in the numerator of  $Z_{a_1,a_2}(s, w)$ , we can write

$$Z_{a_1,a_2}(s, w) = \sum_{\substack{d \geq 1 \\ (d,2)=1}} \frac{L^{(2)}(s, \chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s, \chi_{a_1})}{d^w} = \sum_{\substack{d,m \geq 1 \\ (dm,2)=1}} \frac{\chi_{a_1d_0}(\hat{m})\chi_{a_2}(d)a(m, d)}{m^s d^w},$$

where  $\hat{m}$  is the part of  $m$  relatively prime to  $d_0$  and the **weighting coefficient**  $a(m, d)$  is given by

$$a(m, d) = \sum_{\substack{e_1e_2^2e_3=m \\ e_1e_2|d_1 \\ (d_0,e_1e_3)=1}} \mu(e_1)e_2.$$

To see this, the coefficient of  $m^{-s}d^{-w}$  in the definition of  $Z_{a_1,a_2}(s, w)$  is

$$\begin{aligned} \chi_{a_2}(d) \sum_{\substack{e_1e_2^2e_3=m \\ e_1e_2|d_1}} \mu(e_1)\chi_{a_1d_0}(e_1e_3)e_2 &= \chi_{a_2}(d) \sum_{\substack{e_1e_2^2e_3=m \\ e_1e_2|d_1 \\ (d_0,e_1e_3)=1}} \mu(e_1)\chi_{a_1d_0}(e_1e_3)e_2 \\ &= \chi_{a_1d_0}(\hat{m})\chi_{a_2}(d) \sum_{\substack{e_1e_2^2e_3=m \\ e_1e_2|d_1 \\ (d_0,e_1e_3)=1}} \mu(e_1)e_2 \\ &= \chi_{a_1d_0}(\hat{m})\chi_{a_2}(d)a(m, d), \end{aligned}$$

where the first equality holds because  $\chi_{d_0}(e_1e_3) = 0$  unless  $(d_0, e_1e_3) = 1$  and the second equality holds because if  $(d_0, e_1e_3) = 1$ ,  $\hat{m}$  differs from  $e_1e_3$  by a perfect square (the divisors of which belong to  $(d_0, e_2)$ )

and so  $\chi_{d_0}(e_1 e_3) = \chi_{d_0}(\widehat{m})$ . For completeness, we extend the definition of  $a(m, d)$  to all  $d, m \geq 1$ . In particular,  $a(m, d)$  makes sense when  $m$  or  $d$  may be even.

**Remark 4.1.** Also,  $a(m, d) = 0$  unless  $m = e_1 e_2^2 e_3$  with  $(d_0, e_1 e_3) = 1$  and  $e_1 e_2^2 \mid d_1$ .

We will define  $L(s, \chi_{a_1 d})$  to be the Dirichlet series given by

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1}) = \sum_{m \geq 1} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m, d)}{m^s}.$$

Clearly  $L(s, \chi_{a_1 d})$  is locally absolutely uniformly convergent for  $\text{Re}(s) > 1$ . In particular,  $L(s, \chi_d)$  now makes sense for  $d$  not necessarily square-free and this definition agrees with the former when  $d$  is square-free. Moreover, we have the representation

$$Z_{a_1, a_2}(s, w) = \sum_{\substack{d \geq 1 \\ (d, 2) = 1}} \frac{\chi_{a_2}(d) L^{(2)}(s, \chi_{a_1 d})}{d^w}.$$

If we perform the same procedure but with the interchange, we get

$$\widetilde{Z}_{a_2, a_1}(w, s) = \sum_{\substack{m \geq 1 \\ (m, 2) = 1}} \frac{L^{(2)}(w, \widetilde{\chi}_{a_2 m_0}) \widetilde{\chi}_{a_1}(m) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2})}{m^s} = \sum_{\substack{d, m \geq 1 \\ (dm, 2) = 1}} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d}) \widetilde{\chi}_{a_1}(m) a(d, m)}{m^s d^w},$$

where  $\widehat{d}$  is the part of  $d$  relatively prime to  $m_0$ . Analogously, we define  $L(w, \widetilde{\chi}_{a_2 m})$  to be the Dirichlet series given by

$$L(w, \widetilde{\chi}_{a_2 m}) = L(w, \widetilde{\chi}_{a_2 m_0}) Q_{m_0 m_1^2}(w, \widetilde{\chi}_{a_2}) = \sum_{d \geq 1} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d}) a(d, m)}{d^w}.$$

Again,  $L(w, \widetilde{\chi}_{a_2 m})$  is locally absolutely uniformly convergent for  $\text{Re}(w) > 1$ . We also have

$$\widetilde{Z}_{a_2, a_1}(w, s) = \sum_{\substack{m \geq 1 \\ (m, 2) = 1}} \frac{\widetilde{\chi}_{a_1}(m) L^{(2)}(w, \widetilde{\chi}_{a_2 m})}{m^s}.$$

We now investigate the structure of the weighting coefficients  $a(m, d)$ . Their structure controls the majority of the information about both the quadratic double Dirichlet series and the correction polynomials. We first show that the weighting coefficients possess a multiplicativity property:

**Proposition 4.1.** We have  $a(m, 1) = a(1, d) = 1$  and

$$a(m, d) = \prod_{\substack{p^\alpha \parallel m \\ p^\beta \parallel d}} a(p^\alpha, p^\beta).$$

*Proof.* From the definition of the weighting coefficients,  $a(m, 1) = a(1, d) = 1$ . We will prove multiplicativity in  $m$  and then in  $d$ . Letting  $m = m' p^\alpha$ , we must show

$$a(m, d) = a(m', d) a(p^\alpha, d).$$

To accomplish this, for  $e_1 e_2^2 e_3 = m$ , let  $e_1 = c_1 d_1$ ,  $e_2 = c_2 d_2$ , and  $e_3 = c_3 d_3$  with  $c_1, c_2, c_3 \mid m'$  and  $d_1, d_2, d_3 \mid p^\alpha$ . Because  $(m', p^\alpha) = 1$ , as  $e_1 e_2^2 e_3$  runs over decompositions of  $m$ ,  $c_1 c_2^2 c_3$  and  $d_1 d_2^2 d_3$  run over decompositions of  $m'$  and  $p^\alpha$  respectively. Moreover, as  $e_1 e_2$  runs over the divisors of  $d_1$  so does  $c_1 d_1 c_2 d_2$ .



These facts combined with multiplicativity of the Möbius function gives

$$\begin{aligned}
a(m, d) &= \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2 \\
&= \sum_{\substack{c_1 c_2^2 c_3 = m' \\ d_1 d_2^2 d_3 = p^\beta \\ c_1 d_1 c_2 d_2 | d_1 \\ (d_0, c_1 d_1 c_3 d_3) = 1}} \mu(c_1) (d_1) |c_2| d_2 \\
&= \left( \sum_{\substack{c_1 c_2^2 c_3 = m' \\ c_1 c_2 | d_1 \\ (d_0, c_1 c_3) = 1}} \mu(c_1) |c_2| \right) \left( \sum_{\substack{d_1 d_2^2 d_3 = p^\alpha \\ d_1 d_2 | d_1 \\ (d_0, d_1 d_3) = 1}} \mu(d_1) d_2 \right) \\
&= a(m', d) a(p^\alpha, d),
\end{aligned}$$

as desired. Now we prove multiplicativity in  $d$ . Since we have already proven multiplicativity in  $m$ , we may assume  $m = p^\alpha$ . Letting  $d = d' p^\beta$ , we must show

$$a(p^\alpha, d) = a(p^\alpha, p^\beta).$$

As  $e_1 e_2^2 e_3 = p^\alpha$ , the  $e_i$  are powers of  $p$  for  $1 \leq i \leq 3$ . It follows that  $e_1 e_2 | d_1$  is equivalent to  $e_1 e_2 | p^\beta$ . Moreover,  $(d_0, e_1 e_2) = 1$  is equivalent to  $(1, e_1 e_2) = 1$  or  $(p, e_1 e_2) = 1$  depending on if  $\beta$  is even or odd. These facts imply the desired identity.  $\square$

The correction polynomials  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  are tightly connected to the weighting coefficients  $a(m, d)$ . In particular,  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients. We first prove this relationship when  $d$  is an odd prime power:

**Lemma 4.1.** *For any prime  $p$  and  $\alpha \geq 1$ , we have*

$$Q_{p^{2\alpha+1}}(s) = \sum_{k \leq 2\alpha} \frac{a(p^k, p^{2\alpha+1})}{p^{ks}}.$$

Moreover, the same holds for  $Q_{p^{2\alpha+1}}(w)$ .

*Proof.* Expanding the correction polynomial in  $p^{-s}$  yields

$$Q_{p^{2\alpha+1}}(s) = \sum_{e_1 e_2 | p^\alpha} \mu(e_1) \chi_p(e_1) e_1^{-s} e_2^{1-2s} = \sum_{k \leq 2\alpha} \frac{b(p^k, p^{2\alpha+1})}{p^{ks}}.$$

where

$$b(p^k, p^{2\alpha+1}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1) \chi_p(e_1) e_2.$$

The proof will be finished if we can show  $b(p^k, p^{2\alpha+1}) = a(p^k, p^{2\alpha+1})$ . To see this, first observe  $\mu(e_1) \chi_p(e_1) = 0$  unless  $e_1 = 1$  in which case it is 1. So  $b(p^k, p^{2\alpha+1}) = 0$  if  $k$  is odd and  $p^{\frac{k}{2}}$  if  $k$  is even. Compactly stated,

$$b(p^k, p^{2\alpha+1}) = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

On the other hand,  $k \leq \alpha$  so that

$$a(p^k, p^{2\alpha+1}) = \sum_{\substack{e_1 e_2^2 e_3 = p^k \\ e_1 e_2 | p^\alpha \\ (p, e_1 e_3) = 1}} \mu(e_1) e_2 = \sum_{\substack{e_1 e_2^2 | p^k \\ (p, e_1 e_3) = 1}} \mu(e_1) e_2 = \sum_{e_2^2 = p^k} e_2 = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

This finishes the proof. Clearly the same holds for  $Q_{p^{2\alpha+1}}(w)$ .  $\square$

There is an analogous statement when  $d$  is an even prime power up to a square-free factor and relatively prime factor:

**Lemma 4.2.** *For any square-free integer  $d_0 \geq 1$ ,  $a_1 \in \{\pm 1, \pm 2\}$ , prime  $p$  not dividing  $d_0$ , and  $\beta \geq 1$ , we have*

$$Q_{d_0 p^{2\beta}}(s, \chi_{a_1}) = (1 - \chi_{a_1 d_0}(p) p^{-s}) \sum_{k \leq 2\beta} \frac{\chi_{a_1 d_0}(p^k) a(p^k, p^{2\beta})}{p^{ks}}.$$

Moreover, the same holds for  $Q_{m_0 p^{2\beta}}(w, \tilde{\chi}_{a_2})$ .

*Proof.* Expand the correction polynomial in  $p^{-s}$  to get

$$Q_{d_0 p^{2\beta}}(s, \chi_{a_1}) = \sum_{e_1 e_2 | p^\alpha} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} = \sum_{k \leq 2\beta} \frac{b(p^k, p^{2\beta})}{p^{ks}},$$

where

$$b(p^k, p^{2\beta}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1) \chi_{a_1 d_0}(e_1) e_2.$$

It suffices to show  $b(p^k, p^{2\beta}) = \chi_{a_1 d_0}(p^k) (a(p^k, p^{2\beta}) - a(p^{k-1}, p^{2\beta}))$ . On the one hand,  $\mu(e_1) = 0$  unless  $e_1 = 1, p$  in which case  $\mu(e_1) = \pm 1$  accordingly. So

$$b(p^k, p^{2\beta}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1) \chi_{a_1 d_0}(e_1) e_2 = \begin{cases} \chi_{a_1 d_0}(p^k) p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(p^k) p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

where we have used the identity  $\chi_{a_1 d_0}(e_1) = \chi_{a_1 d_0}(p^k)$  which holds because this quadratic Dirichlet character only depends upon the parity of  $k$ . On the other hand, as in the proof of Lemma 4.1

$$a(p^k, p^{2\beta}) = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

But then

$$\chi_{a_1 d_0}(p^k) (a(p^k, p^{2\beta}) - a(p^{k-1}, p^{2\beta})) = \begin{cases} \chi_{a_1 d_0}(p^k) p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(p^k) p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

which completes the proof. Clearly the same holds for  $Q_{m_0 p^{2\beta}}(w, \tilde{\chi}_{a_2})$ .  $\square$

Lemmas 4.1 and 4.2 together show that  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients  $a(m, d)$  when  $d$  is an prime power. The proof of these lemmas also give the value of  $a(p^k, p^l)$  and we collect this as a corollary:

**Corollary 4.1.** *For any prime  $p$ ,*

$$a(p^k, p^l) = \begin{cases} \min\left(p^{\frac{k}{2}}, p^{\frac{l}{2}}\right) & \text{if } \min(k, l) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

If we combine Proposition 4.1 and Corollary 4.1 we can compute  $a(m, d)$  in general:

**Corollary 4.2.** *For any integers  $d, m \geq 1$ ,*

$$a(m, d) = \begin{cases} (m, d)^{\frac{1}{2}} & \text{if } (m, d) \text{ is a perfect square,} \\ 0 & \text{otherwise.} \end{cases}$$

As an immediate consequence of Corollary 4.2,  $a(m, d)$  is symmetric in  $m$  and  $d$ . As the weighting coefficients are multiplicative,  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  will possess an Euler product. To state the Euler product explicitly, we write  $d = d_0 d_1^2 d_2^2$  with  $d_0$  square-free and,  $d_2$  relatively prime to  $d_0 d_1$ , and such that every prime divisor of  $d_1$  divides  $d_0$ . In other words,  $d_0$  is the square-free part of  $d$ ,  $d_1$  is the square part of  $d$  whose prime factors divide  $d$  to odd power, and  $d_2$  is the square part of  $d$  whose prime factors divide  $d$  to even power. We have the following Euler product:

**Theorem 4.1.** *Let  $d = d_0 d_1^2 d_2^2$  be the square decomposition of  $d$  stratified by even and odd powers. Then for any  $a_1 \in \{\pm 1, \pm 2\}$ ,*

$$Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = \prod_{p^\alpha || d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta || d_2} Q_{d_0 p^{2\beta}}(s, \chi_{a_1}).$$

Moreover, the same holds for  $Q_{m_0 m_1^2 m_2^2}(w, \tilde{\chi}_{a_2})$ .

*Proof.* Recall that

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = \sum_{m \geq 1} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m, d)}{m^s}.$$

We will now derive an alternate expression for  $L(s, \chi_{a_1 d})$ . By Proposition 4.1, the coefficients of  $L(s, \chi_{a_1 d})$  are multiplicative. Therefore  $L(s, \chi_{a_1 d})$  admits the Euler product

$$L(s, \chi_{a_1 d}) = \prod_{p \text{ prime}} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, d)}{p^{ks}} \right).$$

Decomposing the product according to primes dividing  $d = d_0 d_1^2 d_2^2$ , we get

$$\begin{aligned} L(s, \chi_{a_1 d}) &= \prod_{p \text{ prime}} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, d)}{p^{ks}} \right) \\ &= \prod_{p \nmid d} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, 1)}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^\beta)}{p^{ks}} \right) \\ &= \prod_{p \nmid d} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^\beta)}{p^{ks}} \right) \\ &= \prod_{p \nmid d} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left( \sum_{k \geq 0} \frac{a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(p^k) a(p^k, p^\beta)}{p^{ks}} \right). \end{aligned}$$

Including the factors corresponding to primes  $p \mid d_2$  into the first product, we must multiply the last factor by the inverse of  $\sum_{k \geq 0} \chi_{a_1 d_0}(p^k) p^{-ks} = (1 - \chi_{a_1 d_0}(p) p^{-s})^{-1}$  obtaining

$$\prod_{p \nmid d_0} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left( \sum_{k \geq 0} \frac{a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left( (1 - \chi_{a_1 d_0}(p) p^{-s}) \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(p^k) a(p^k, p^\beta)}{p^{ks}} \right),$$

as every prime divisor of  $d_1$  divides  $d_0$ . The first product is  $L(s, \chi_{a_1 d_0})$ . For the second and third products, Remark 4.1 implies that the sums run up to  $k \leq 2\alpha$  and  $k \leq 2\beta$  respectively. Therefore they are  $Q_{p^{2\alpha+1}}(s)$  and  $Q_{d_0 p^{2\beta}}(s, \chi_{a_1})$  respectively. It follows that

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) \cdot \prod_{p^\alpha \parallel d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta \parallel d_2} Q_{d_0 p^{2\beta}}(s, \chi_{a_1}).$$

This is our alternate expression for  $L(s, \chi_{a_1 d})$  and equating the two results in

$$L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = L(s, \chi_{a_1 d_0}) \cdot \prod_{p^\alpha \parallel d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta \parallel d_2} Q_{d_0 p^{2\beta}}(s, \chi_{a_1}),$$

from which the proof is complete since  $L(s, \chi_{a_1 d_0}) \neq 0$  for  $\text{Re}(s) > 1$  (so that we may divide by  $L(s, \chi_{a_1 d_0})$ ). Clearly the same holds for  $Q_{m_0 m_1^2 m_2^2}(w, \tilde{\chi}_{a_2})$ .  $\square$

Observe that for  $d = d_0 d_1^2 d_2^2$ , the prime factors that divide  $d_1 d_2$  are exactly those factors that divide  $d$  to power larger than 1. Thus, from Theorem 4.1 the Euler product for  $Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1})$  is supported on exactly the primes dividing  $d$  to order larger than 1 and also depends upon the character  $\chi_{a_1 d_0}$ .

## 5. FUNCTIONAL EQUATIONS

We can now derive functional equations for  $Z_{a_1, a_2}(s, w)$ . These functional equations will be induced from the functional equations for  $L(s, \chi_{a_1 d})$  and  $L(s, \tilde{\chi}_{a_2 m})$ . To prove these latter functional equations, we require a functional equation for the correction polynomials:

**Theorem 5.1.**  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  admits the functional equation.

$$Q_{d_0 d_1^2}(s, \chi_{a_1}) = d_1^{1-2s} Q_{d_0 d_1^2}(1-s, \chi_{a_1}).$$

Moreover, the same holds for  $Q_{m_0 m_1^2}(w, \tilde{\chi}_{a_2})$ .

*Proof.* The strategy is to interchange  $e_2$  and  $e_3$  in the sum defining  $Q_{d_0 d_1^2}(s, \chi_{a_1})$ :

$$\begin{aligned} d_1^{1-2s} Q_{d_0 d_1^2}(1-s) &= d_1^{1-2s} \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{s-1} e_2^{2s-1} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{s-1} \left( \frac{d_1}{e_2} \right)^{1-2s} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{s-1} (e_1 e_3)^{1-2s} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_3^{1-2s} \\ &= Q_{d_0 d_1^2}(s, \chi_{a_1}). \end{aligned}$$

Clearly the same holds for  $Q_{m_0 m_1^2}(w, \tilde{\chi}_{a_2})$ .  $\square$

We will define the completed  $L$ -function  $L^*(s, \chi_{a_1 d})$  by

$$L^*(s, \chi_{a_1 d}) = L^*(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1}).$$

In particular,  $L^*(s, \chi_d)$  makes sense even when  $d$  is not square-free and agrees with the previous definition when  $d$  is square-free. Combining Theorem 5.1, the functional equation for  $L^*(s, \chi_{a_1 d_0})$ , and that  $d \equiv d_0 \pmod{4}$ , we obtain a functional equation for  $L^*(s, \chi_{a_1 d})$ :

$$L^*(s, \chi_{a_1 d}) = \begin{cases} |d|^{\frac{1}{2}-s} L^*(1-s, \chi_{a_1 d}) & \text{if } a_1 d \equiv 1, 5 \pmod{8}, \\ (1+i)|8d|^{\frac{1}{2}-s} L^*(1-s, \chi_{a_1 d}) & \text{if } a_1 d \equiv 2, 3, 6, 7 \pmod{8}. \end{cases}$$

Analogously, define the completed  $L$ -function  $L^*(w, \tilde{\chi}_{a_2m})$  by

$$L^*(w, \tilde{\chi}_{a_2m}) = L^*(w, \tilde{\chi}_{a_2m_0})Q_{m_0m_1^2}(w, \tilde{\chi}_{a_2}).$$

Then, as before, we have a functional equation for  $L^*(w, \tilde{\chi}_{a_2m})$ :

$$L^*(w, \tilde{\chi}_{a_2m}) = \begin{cases} |m|^{\frac{1}{2}-w} L^*(1-w, \tilde{\chi}_{a_2m}) & \text{if } a_2m \equiv 1, 5 \pmod{8}, \\ (1+i)|8m|^{\frac{1}{2}-w} L^*(1-w, \tilde{\chi}_{a_2m}) & \text{if } a_2m \equiv 2, 3, 6, 7 \pmod{8}. \end{cases}$$

The functional equations for  $L^*(s, \chi_{a_1d})$  and  $L^*(w, \tilde{\chi}_{a_2m})$  will induce functional equations for  $Z_{a_1,a_2}(s, w)$  and  $\tilde{Z}_{a_2,a_1}(w, s)$ . However, there is an obstruction caused by the gamma factors. Indeed, the gamma factors for  $L^*(s, \chi_{a_1d})$  and  $L^*(w, \tilde{\chi}_{a_2m})$  depend  $a_1d$  and  $a_2m$  modulo 8 respectively. To induce functional equations we need the gamma factors to be constant. Orthogonality of the Hilbert characters will allow us to get past this issue. For  $b \in \{1, 3, 5, 7\}$ , define  $Z_{a_1,a_2}^b(s, w)$  and  $\tilde{Z}_{a_2,a_1}^b(w, s)$  by

$$Z_{a_1,a_2}^b(s, w) = \frac{1}{4} \sum_{a \in \{\pm 1, \pm 2\}} \chi_a(b) Z_{a_1,aa_2}(s, w) \quad \text{and} \quad \tilde{Z}_{a_2,a_1}^b(w, s) = \frac{1}{4} \sum_{a \in \{\pm 1, \pm 2\}} \tilde{\chi}_a(b) \tilde{Z}_{a_2,aa_1}(w, s).$$

In terms of the representations

$$Z_{a_1,a_2}(s, w) = \sum_{\substack{d \geq 1 \\ (d,2)=1}} \frac{\chi_{a_2}(d) L^{(2)}(s, \chi_{a_1d})}{d^w} \quad \text{and} \quad \tilde{Z}_{a_2,a_1}(w, s) = \sum_{\substack{m \geq 1 \\ (m,2)=1}} \frac{\tilde{\chi}_{a_1}(m) L^{(2)}(w, \tilde{\chi}_{a_2m})}{m^s},$$

and orthogonality of the Hilbert characters,  $Z_{a_1,a_2}^b(s, w)$  and  $\tilde{Z}_{a_2,a_1}^b(w, s)$  are the subseries containing only those  $d$  and  $m$  equivalent to  $b$  modulo 8 respectively. Then  $Z_{a_1,a_2}^b(s, w)$  and  $\tilde{Z}_{a_2,a_1}^b(w, s)$  are sums of  $L$ -functions with a fixed gamma factor and so we can obtain functional equations. The fact that  $Z_{a_1,a_2}(s, w)$  and  $\tilde{Z}_{a_2,a_1}(w, s)$  are linear combinations of these series will induce function equations. Precisely, we have the following statement:

**Theorem 5.2.**  $Z_{a_1,a_2}(s, w)$  admits the functional equations

$$\begin{aligned} Z_{a_1,a_2}(s, w) &= \frac{1}{4} \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \sum_{\substack{a_1b > 0 \\ a_1b \equiv 1,5 \pmod{8} \\ a \in \{\pm 1, \pm 2\}}} \chi_a(b) \frac{(1 - \chi_{a_1b}(2)2^{s-1})^{-1}}{(1 - \chi_{a_1b}(2)2^{-s})^{-1}} Z_{a_1,aa_2} \left(1 - s, s + w - \frac{1}{2}\right) \\ &+ \frac{1}{4} \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{-s}{2})}{\Gamma(\frac{s+1}{2})} \sum_{\substack{a_1b < 0 \\ a_1b \equiv 1,5 \pmod{8} \\ a \in \{\pm 1, \pm 2\}}} \chi_a(b) \frac{(1 - \chi_{a_1b}(2)2^{s-1})^{-1}}{(1 - \chi_{a_1b}(2)2^{-s})^{-1}} Z_{a_1,aa_2} \left(1 - s, s + w - \frac{1}{2}\right) \\ &+ \frac{1+i}{8^s} \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \sum_{\substack{a_1b > 0 \\ a_1b \equiv 2,3,6,7 \pmod{8} \\ a \in \{\pm 1, \pm 2\}}} \chi_a(b) \frac{(1 - \chi_{a_1b}(2)2^{s-1})^{-1}}{(1 - \chi_{a_1b}(2)2^{-s})^{-1}} Z_{a_1,aa_2} \left(1 - s, s + w - \frac{1}{2}\right) \\ &+ \frac{1+i}{8^s} \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{-s}{2})}{\Gamma(\frac{s+1}{2})} \sum_{\substack{a_1b < 0 \\ a_1b \equiv 2,3,6,7 \pmod{8} \\ a \in \{\pm 1, \pm 2\}}} \chi_a(b) \frac{(1 - \chi_{a_1b}(2)2^{s-1})^{-1}}{(1 - \chi_{a_1b}(2)2^{-s})^{-1}} Z_{a_1,aa_2} \left(1 - s, s + w - \frac{1}{2}\right). \end{aligned}$$

and

$$\begin{aligned}
Z_{a_1, a_2}(s, w) &= \frac{1}{4} \pi^{w-\frac{1}{2}} \frac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} \sum_{\substack{a_1 b \equiv 1, 5 \pmod{8} \\ a \in \{\pm 1, \pm 2\}}} \chi_a(b) \frac{(1 - \chi_{a_2 b}(2) 2^{w-1})^{-1}}{(1 - \chi_{a_2 b}(2) 2^{-w})^{-1}} Z_{aa_1, a_2} \left(s + w - \frac{1}{2}, 1 - w\right) \\
&+ \frac{1+i}{8^w} \pi^{w-\frac{1}{2}} \frac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} \sum_{\substack{a_1 b \equiv 2 \pmod{8} \\ a \in \{\pm 1, \pm 2\}}} \chi_a(b) \frac{(1 - \chi_{a_2 b}(2) 2^{w-1})^{-1}}{(1 - \chi_{a_2 b}(2) 2^{-w})^{-1}} Z_{aa_1, a_2} \left(s + w - \frac{1}{2}, 1 - w\right) \\
&+ \frac{1+i}{8^w} \pi^{w-\frac{1}{2}} \frac{\Gamma\left(\frac{-w}{2}\right)}{\Gamma\left(\frac{w+1}{2}\right)} \sum_{\substack{a_1 b \equiv 3, 6, 7 \pmod{8} \\ a \in \{\pm 1, \pm 2\}}} \chi_a(b) \frac{(1 - \chi_{a_2 b}(2) 2^{w-1})^{-1}}{(1 - \chi_{a_2 b}(2) 2^{-w})^{-1}} Z_{aa_1, a_2} \left(1 - w, s + w - \frac{1}{2}\right).
\end{aligned}$$

*Proof.* Set

$$L_{a_1 b}(s) = \frac{(1 - \chi_{a_1 b}(2) 2^{s-1})^{-1}}{(1 - \chi_{a_1 b}(2) 2^{-s})^{-1}}.$$

For the first functional equation, the functional equation for  $L$ -functions attached to quadratic Dirichlet characters implies the functional equation

$$Z_{a_1, a_2}^b(s, w) = \begin{cases} \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} L_{a_1 b}(s) Z_{a_1, a_2}^b \left(1 - s, s + w - \frac{1}{2}\right) & \text{if } a_1 b > 0, a_1 b \equiv 1, 5 \pmod{8}, \\ \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} L_{a_1 b}(s) Z_{a_1, a_2}^b \left(1 - s, s + w - \frac{1}{2}\right) & \text{if } a_1 b < 0, a_1 b \equiv 1, 5 \pmod{8}, \\ \frac{1+i}{8^{s-\frac{1}{2}}} \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} L_{a_1 b}(s) Z_{a_1, a_2}^b \left(1 - s, s + w - \frac{1}{2}\right) & \text{if } a_1 b > 0, a_1 b \equiv 2, 3, 6, 7 \pmod{8}, \\ \frac{1+i}{8^{s-\frac{1}{2}}} \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} L_{a_1 b}(s) Z_{a_1, a_2}^b \left(1 - s, s + w - \frac{1}{2}\right) & \text{if } a_1 b < 0, a_1 b \equiv 2, 3, 6, 7 \pmod{8}. \end{cases}$$

But as

$$Z_{a_1, a_2}(s, w) = \sum_{b \in \{1, 3, 5, 7\}} Z_{a_1, a_2}^b(s, w),$$

the functional equation above gives

$$\begin{aligned}
Z_{a_1, a_2}(s, w) &= \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \sum_{\substack{a_1 b > 0 \\ a_1 b \equiv 1, 5 \pmod{8}}} L_{a_1 b}(s) Z_{a_1, a_2}^b \left(1 - s, s + w - \frac{1}{2}\right) \\
&+ \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \sum_{\substack{a_1 b < 0 \\ a_1 b \equiv 1, 5 \pmod{8}}} L_{a_1 b}(s) Z_{a_1, a_2}^b \left(1 - s, s + w - \frac{1}{2}\right) \\
&+ \frac{1+i}{8^{s-\frac{1}{2}}} \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \sum_{\substack{a_1 b > 0 \\ a_1 b \equiv 2, 3, 6, 7 \pmod{8}}} L_{a_1 b}(s) Z_{a_1, a_2}^b \left(1 - s, s + w - \frac{1}{2}\right) \\
&+ \frac{1+i}{8^{s-\frac{1}{2}}} \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \sum_{\substack{a_1 b < 0 \\ a_1 b \equiv 2, 3, 6, 7 \pmod{8}}} L_{a_1 b}(s) Z_{a_1, a_2}^b \left(1 - s, s + w - \frac{1}{2}\right).
\end{aligned}$$

The first functional equation for  $Z_{a_1,a_2}(s, w)$  follows by writing  $Z_{a_1,a_2}^b(s, w)$  in terms of  $Z_{a_1,aa_2}(s, w)$  for  $a \in \{\pm 1, \pm 2\}$ . For the second functional equation, first set

$$L_{a_2b}(w) = \frac{(1 - \tilde{\chi}_{a_2b}(2)2^{w-1})^{-1}}{(1 - \tilde{\chi}_{a_2b}(2)2^{-w})^{-1}}.$$

Now argue as before but for  $\tilde{Z}_{a_2,a_1}(w, s)$  using the functional equation

$$\tilde{Z}_{a_2,a_1}^b(w, s) = \begin{cases} \pi^{w-\frac{1}{2}} \frac{\Gamma(\frac{1-w}{2})}{\Gamma(\frac{w}{2})} L_{a_2b}(w) \tilde{Z}_{a_2,a_1}^b \left(1-w, s+w-\frac{1}{2}\right) & \text{if } a_2b \equiv 1, 5 \pmod{8}, \\ \frac{1+i}{8^{w-\frac{1}{2}}} \pi^{w-\frac{1}{2}} \frac{\Gamma(\frac{1-w}{2})}{\Gamma(\frac{w}{2})} L_{a_2b}(w) L_{a_2b}(w) \tilde{Z}_{a_2,a_1}^b \left(1-w, s+w-\frac{1}{2}\right) & \text{if } a_2b \equiv 2 \pmod{8}, \\ \frac{1+i}{8^{w-\frac{1}{2}}} \pi^{w-\frac{1}{2}} \frac{\Gamma(\frac{-w}{2})}{\Gamma(\frac{w+1}{2})} L_{a_2b}(w) \tilde{Z}_{a_2,a_1}^b \left(1-w, s+w-\frac{1}{2}\right) & \text{if } a_2b \equiv 3, 6, 7 \pmod{8}, \end{cases}$$

induced from the corresponding  $L$ -function. We then apply the interchange in the form  $\tilde{Z}_{a_2,a_1}(w, s) = Z_{a_1,a_2}(s, w)$  and the fact that

$$L_{a_2b}(w) = \frac{(1 - \chi_{a_2b}(2)2^{w-1})^{-1}}{(1 - \chi_{a_2b}(2)2^{-w})^{-1}}.$$

to obtain the second functional equation for  $Z_{a_1,a_2}(s, w)$ . Clearly analogous functional equations hold for  $\tilde{Z}_{a_2,a_1}(w, s)$ .  $\square$

These functional equations are quite unruly and it is often far more simple to compactify them in terms of vectors. For simplicity we do this only for  $Z_{a_1,a_2}(s, w)$ . Define  $\mathbf{Z}(s, w)$  by

$$\mathbf{Z}(s, w) = (Z_{a_1,a_2}(s, w))_{a_1,a_2 \in \{\pm 1, \pm 2\}},$$

with the lexicographical ordering determined by  $1 > -1 > 2 > -2$ . Also, for  $i \in \{1, 2, 3\}$ , set

$$\gamma_{a_1,a,b}^{\pm,i}(s) = \begin{cases} \frac{1}{4} \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \chi_a(b) \frac{(1 - \chi_{a_1b}(2)2^{s-1})^{-1}}{(1 - \chi_{a_1b}(2)2^{-s})^{-1}} & \text{if } + \text{ and } i = 1, \\ \frac{1}{4} \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{-s}{2})}{\Gamma(\frac{s+1}{2})} \chi_a(b) \frac{(1 - \chi_{a_1b}(2)2^{s-1})^{-1}}{(1 - \chi_{a_1b}(2)2^{-s})^{-1}} & \text{if } - \text{ and } i = 1, \\ \frac{1+i}{8^s} \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \chi_a(b) \frac{(1 - \chi_{a_1b}(2)2^{s-1})^{-1}}{(1 - \chi_{a_1b}(2)2^{-s})^{-1}} & \text{if } + \text{ and } i = 2, 3, \\ \frac{1+i}{8^s} \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{-s}{2})}{\Gamma(\frac{s+1}{2})} \chi_a(b) \frac{(1 - \chi_{a_1b}(2)2^{s-1})^{-1}}{(1 - \chi_{a_1b}(2)2^{-s})^{-1}} & \text{if } - \text{ and } i = 2, 3, \end{cases}$$

and

$$\gamma_{a_2,a,b}^i(w) = \begin{cases} \frac{1}{4} \pi^{w-\frac{1}{2}} \frac{\Gamma(\frac{1-w}{2})}{\Gamma(\frac{w}{2})} \chi_a(b) \frac{(1 - \chi_{a_2b}(2)2^{w-1})^{-1}}{(1 - \chi_{a_2b}(2)2^{-w})^{-1}} & \text{if } i = 1, \\ \frac{1+i}{8^w} \pi^{w-\frac{1}{2}} \frac{\Gamma(\frac{1-w}{2})}{\Gamma(\frac{w}{2})} \chi_a(b) \frac{(1 - \chi_{a_2b}(2)2^{w-1})^{-1}}{(1 - \chi_{a_2b}(2)2^{-w})^{-1}} & \text{if } i = 2, \\ \frac{1+i}{8^w} \pi^{w-\frac{1}{2}} \frac{\Gamma(\frac{-w}{2})}{\Gamma(\frac{w+1}{2})} \chi_a(b) \frac{(1 - \chi_{a_2b}(2)2^{w-1})^{-1}}{(1 - \chi_{a_2b}(2)2^{-w})^{-1}} & \text{if } i = 3. \end{cases}$$

These are the gamma factors appearing in the functional equations. By Theorem 5.2, there exist  $16 \times 16$  matrices

$$\Phi(s) \quad \text{and} \quad \Psi(w),$$

whose coefficients are functions of  $\gamma_{a_1,a,b}^{\pm,i}(s)$  and  $\gamma_{a_2,a,b}^i(w)$  respectively, and satisfy functional equations

$$\mathbf{Z}(s, w) = \Phi(s)\mathbf{Z}\left(1-s, s+w-\frac{1}{2}\right) \quad \text{and} \quad \mathbf{Z}(s, w) = \Psi(s)\mathbf{Z}\left(s+w-\frac{1}{2}, 1-w\right),$$

which are equivalent to those for  $Z_{a_1,a_2}(s, w)$  given in Theorem 5.2. So we have two functional equations of shapes

$$\sigma_1 : (s, w) \rightarrow \left(1-s, s+w-\frac{1}{2}\right) \quad \text{and} \quad \sigma_2 : (s, w) \rightarrow \left(s+w-\frac{1}{2}, 1-w\right).$$

These transformations also act on the  $(s, w)$ -plane and satisfy the relations

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 : (s, w) \rightarrow (1-w, 1-s) \quad \text{or equivalently} \quad (\sigma_1\sigma_2)^3 = 1 : (s, w) \rightarrow (s, w).$$

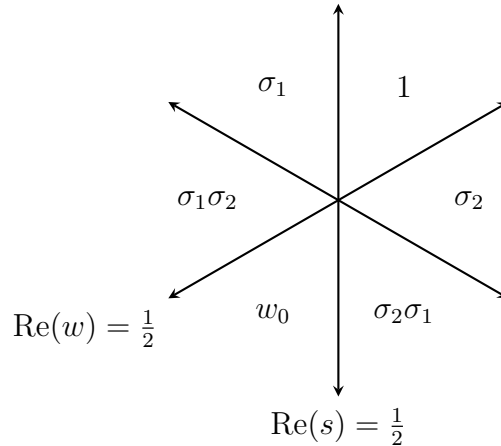
As  $\sigma_1^2 = \sigma_2^2 = 1$ ,  $\sigma_1$  and  $\sigma_2$  generate the group

$$W = \langle \sigma_1, \sigma_2 : \sigma_1^2 = \sigma_2^2 = (\sigma_1\sigma_2)^3 = 1 \rangle \cong D_6 \cong S_3.$$

For convenience we set  $w_0 = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ . It follows that  $Z_{a_1,a_2}(s, w)$  possess a group of 6 functional equations. These functional equations can be used to meromorphically continue  $Z_{a_1,a_2}(s, w)$  to the entire  $(s, w)$ -plane. Of course, all the same can be achieved for  $\tilde{Z}_{a_2,a_1}(w, s)$  as well.

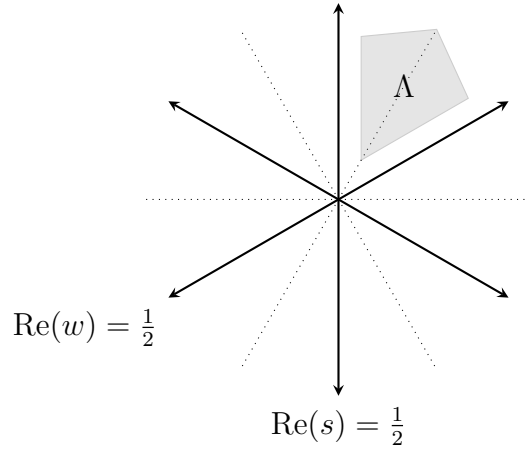
## 6. MEROMORPHIC CONTINUATION

We will now show how to meromorphically continue  $Z(s, w)$  to the entire  $(s, w)$ -plane. We could do this for each twisted quadratic double Dirichlet series  $Z_{a_1,a_2}(s, w)$  and  $\tilde{Z}_{a_2,a_1}(w, s)$ , but we will not be concerned with this level of generality here or further on. We will start by describing the action of  $W$  on the  $(s, w)$ -plane. It is clear from the definition of the actions  $\sigma_1$  and  $\sigma_2$  that there is a unique  $W$ -invariant point  $(\frac{1}{2}, \frac{1}{2})$ . Representing the point  $(s, w)$  by  $(\text{Re}(s), \text{Re}(w))$  we represent the action of  $W$  on the  $(s, w)$ -plane as follows:



In this diagram we have transformed the  $(s, w)$ -plane so that the origin lies at  $(\frac{1}{2}, \frac{1}{2})$  and the  $(s, w)$ -axes intersect at  $\frac{\pi}{3}$  angles. We have done this so that  $\sigma_1$  and  $\sigma_2$  act by rigid motions sending the region enclosing 1 (corresponding to the identity) to either of the adjacent triangles. The other regions are obtained by acting by the corresponding element of  $W$ . The initial region  $\Lambda$  that  $Z(s, w)$  is defined on is displayed in the figure below:





To meromorphically continue  $Z(s, w)$  to all of the  $(s, w)$ -plane, we first need to show that the quadratic double Dirichlet series  $Z_{a_1, a_2}(s, w)$  are locally absolutely uniformly convergent on a slightly larger region than  $\Lambda$ . This will be achieved by the Phragmén-Lindelöf convexity principal. Fix some small  $\varepsilon > 0$ . The functional equations for  $L^*(s, \chi_{a_1 d})$  and  $L^*(w, \tilde{\chi}_{a_2 m})$  and Stirling's formula together imply the estimates

$$L(-\varepsilon, \chi_{a_1 d}) \ll (a_1 d)^{\frac{1}{2}+\varepsilon} \quad \text{and} \quad L(-\varepsilon, \tilde{\chi}_{a_2 m}) \ll (a_2 m)^{\frac{1}{2}+\varepsilon},$$

because  $L(1 + \varepsilon, \chi_{a_1 d}) \ll 1$  and  $L(1 + \varepsilon, \tilde{\chi}_{a_2 m}) \ll 1$ . As both of these  $L$ -functions have at most a simple pole at  $s = 1$  and  $w = 1$  respectively, the Phragmén-Lindelöf convexity principal gives the weak estimates

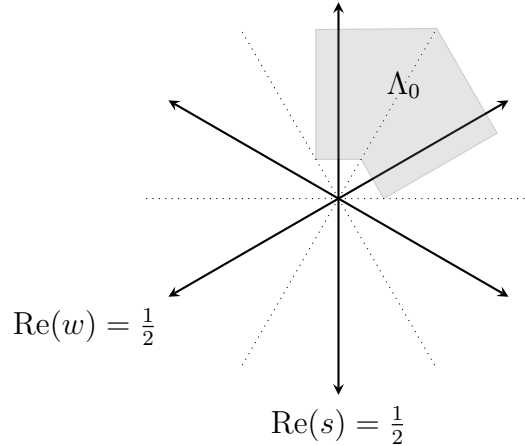
$$(s - 1)L(s, \chi_{a_1 d}) \ll (a_1 d)^{\frac{1}{2}+\varepsilon} \quad \text{and} \quad (w - 1)L(w, \tilde{\chi}_{a_2 m}) \ll (a_2 m)^{\frac{1}{2}+\varepsilon},$$

and hence

$$(s - 1)L^{(2)}(s, \chi_{a_1 d}) \ll (a_1 d)^{\frac{1}{2}+\varepsilon} \quad \text{and} \quad (w - 1)L^{(2)}(w, \tilde{\chi}_{a_2 m}) \ll (a_2 m)^{\frac{1}{2}+\varepsilon},$$

for  $\text{Re}(s) > -\varepsilon$  and  $\text{Re}(w) > -\varepsilon$ . It follows from the interchange that  $(s - 1)(w - 1)Z_{a_1, a_2}(s, w)$  is locally absolutely uniformly convergent on the region

$$\Lambda_0 = \Lambda \cup \left\{ (s, w) \in \mathbb{C}^2 : \text{Re}(s) > 0, \text{Re}(w) > \frac{3}{2} \right\} \cup \left\{ (s, w) \in \mathbb{C}^2 : \text{Re}(s) > \frac{3}{2}, \text{Re}(w) > 0 \right\}.$$



Therefore  $Z_{a_1, a_2}(s, w)$  is meromorphic on this region with at most polar lines at  $s = 1$  and  $w = 1$ . The key difference between  $\Lambda$  and  $\Lambda_0$  is that  $\Lambda_0$  intersects the hyperplanes  $s = \frac{1}{2}$  and  $w = \frac{1}{2}$  so that the union of the reflections  $w\Lambda_0$  for  $w \in W$  is connected. This guarantees that the functional equations imply meromorphic continuation since adjacent reflections of  $\Lambda_0$  overlap on open sets. We now reflect  $\Lambda_0$  via the functional equations and then apply a theorem of Bochner which we now state. First, we say that a domain  $\Omega \subset \mathbb{C}^n$  is a **tube domain** if there is an open set  $\omega \subset \mathbb{R}^n$  such that

$$\Omega = \{(s_1, \dots, s_n) \in \mathbb{C}^n : \text{Re}((s_1, \dots, s_n)) \in \omega\}.$$

Tube domains are generalizations of vertical strips in the complex plane. Now we can state the theorem of Bochner (see [1] for a proof):

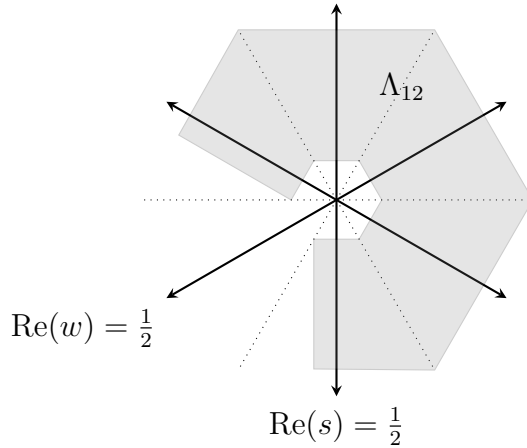
**Theorem 6.1** (Bochner's continuation theorem). *If  $\Omega$  is a connected tube domain, then any holomorphic function on  $\Omega$  can be extended to a holomorphic function on the convex hull  $\widehat{\Omega}$ .*

Clearing polar divisors if necessary, Bochner's continuation theorem implies that any meromorphic function on a connected tube domain possessing a finite amount of hyperplane polar divisors can be extended to a meromorphic function on the convex hull. This is exactly the situation for  $Z(s, w)$ . Clearly  $\Lambda_0$  is a tube domain and on  $\Lambda_0$  there are at most polar lines at  $s = 1$  and  $w = 1$ . Reflecting these hyperplanes via  $W$  we obtain the finite set of possible polar divisors:

$$\left\{ s = 1, w = 1, s = 0, w = 0, s + w = \frac{1}{2}, s + w = \frac{3}{2} \right\}.$$

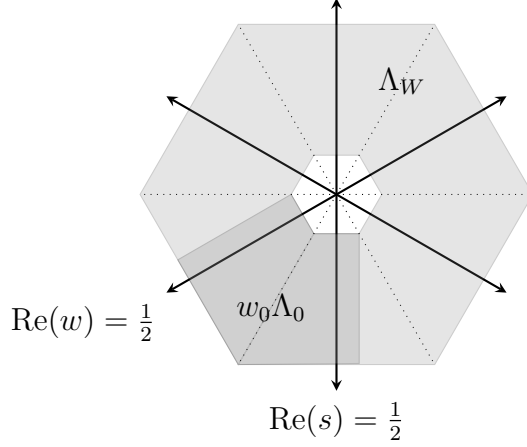
So by the previous argument, we are reduced to extending  $Z(s, w)$  meromorphically. By applying the functional equations corresponding to  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_1\sigma_2$ ,  $Z(s, w)$  admits meromorphic continuation to the region

$$\Lambda_{12} = \Lambda_0 \cup \sigma_1\Lambda_0 \cup \sigma_2\Lambda_0 \cup \sigma_1\sigma_2\Lambda_0.$$



Now  $\Lambda_{12}$  is a connected tube domain whose convex hull is  $\mathbb{C}^2$ . So by applying Bochner's continuation theorem (or rather our comment for meromorphic functions) we see that  $Z(s, w)$  admits meromorphic continuation to the  $(s, w)$ -plane with at most a finite set of polar divisors. This argument is more elegant than repeatedly applying the functional equations corresponding to every  $w \in W$ . Indeed, if we did we would obtain meromorphic continuation to the region

$$\Lambda_W = \bigcup_{w \in W} w\Lambda_0.$$



There are two issues here. The first is that  $Z(s, w)$  has two meromorphic continuations to the region  $w_0\Lambda_0$  given by the functional equations corresponding to  $w_0 = \sigma_1\sigma_2\sigma_1$  and  $w_0 = \sigma_2\sigma_1\sigma_2$  and we would need to show that these agree. The second is that we have not obtained meromorphic continuation to  $\mathbb{C}^2 - \Lambda_W$  which is a compact hexagon about the origin. By using Bochner's theorem after meromorphically continuing to  $\Lambda_{12}$ , we have avoided these issues and as a consequence shown that the meromorphic continuations given by  $w_0 = \sigma_1\sigma_2\sigma_1$  and  $w_0 = \sigma_2\sigma_1\sigma_2$  agree.

## 7. POLES AND RESIDUES

We can now determine the poles and residues of  $Z(s, w)$ . Recall that the set of possible polar divisors is

$$\left\{ s = 1, w = 1, s = 0, w = 0, s + w = \frac{1}{2}, s + w = \frac{3}{2} \right\}.$$

The poles of  $Z(s, w)$  is actually smaller than this set as there are no poles on the hyperplanes  $s = 0$ ,  $w = 0$ , and  $s + w = \frac{3}{2}$ . To see this, first observe that by our earlier application of the Phragmén-Lindelöf convexity principal we actually obtained continuation to an open set containing  $\Lambda_0$  (because our estimates held for  $\text{Re}(s) > -\varepsilon$  and  $\text{Re}(w) > -\varepsilon$ ). We did not need this slightly larger region for the meromorphic continuation but we do require it to study the poles. Now consider the possible polar divisor  $s = 0$ . Therefore  $(s - 1)(w - 1)Z_{a_1, a_2}(s, w)$  is holomorphic on an open set containing  $\Lambda_0$  which contains half of the hyperplane defined by  $s = 0$  outside of the hexagon  $\mathbb{C}^2 - \Lambda_W$ . Since  $(s - 1)(w - 1)$  is holomorphic on this region,  $Z_{a_1, a_2}(s, w)$  can not have a polar divisor at  $s = 0$  on an open set containing  $\Lambda_0$ . Moreover, an open set containing  $\sigma_1\sigma_2\Lambda_0$  contains the other half of the hyperplane defined by  $s = 0$  outside of the hexagon  $\mathbb{C}^2 - \Lambda_W$ . Upon applying the functional equation corresponding to  $\sigma_1\sigma_2$ , Theorem 5.2 implies that the gamma factors in the corresponding functional equation have a simple pole when  $s + w = \frac{3}{2}$  (the gamma factors in the functional equation for  $\sigma_1$  have a simple pole at  $s = 1$  and  $s - 1 \rightarrow s + w - \frac{3}{2}$  under  $\sigma_2$ ). Therefore  $Z_{a_1, a_2}(s, w)$  does not have polar divisors at  $s = 0$  on an open set containing  $\sigma_1\sigma_2\Lambda_0$  away from  $s + w = \frac{3}{2}$ . In particular,  $Z(s, w)$  does not have a polar divisor at  $s = 0$  on  $\Lambda_W$  and away from the other polar divisors. By Bochner's continuation theorem (after clearing all of the other possible polar divisors),  $Z(s, w)$  does not have a polar divisors at  $s = 0$  on all of  $\mathbb{C}^2$  and away from the other polar divisors. An identical argument holds for the case  $w = 0$  using the regions  $\Lambda_0$  and  $\sigma_2\sigma_1\Lambda_0$ . For the polar divisor  $s + w = \frac{1}{2}$ , we argue in the same way with the regions  $\sigma_2\sigma_1\Lambda_0$ ,  $\sigma_1\sigma_2\Lambda_0$ , and  $w_0\Lambda_0$ , but there is one difference. For these regions, the gamma factors in the corresponding functional equations are different. For the first two regions  $\sigma_2\sigma_1\Lambda_0$  and  $\sigma_1\sigma_2\Lambda_0$ , the gamma factors have a simple pole when  $s + w = \frac{3}{2}$ . For the third region  $w_0\Lambda_0$ , the gamma factors have simple poles at  $s = 1$  and  $w = 1$  which is seen by using both representations  $w_0 = \sigma_1\sigma_2\sigma_1$  and  $w_0 = \sigma_2\sigma_1\sigma_2$ . Nevertheless, there are no poles on the hyperplanes  $s = 0$ ,  $w = 0$ , and  $s + w = \frac{1}{2}$  and away from the other polar divisors. As for the hyperplanes  $s = 1$ ,  $w = 1$ , and  $s + w = \frac{3}{2}$ , there are clearly genuine poles for  $s = 1$  and  $w = 1$  coming from  $L(s, \chi_{d_0})$  and  $L(w, \chi_{m_0})$

when  $d$  and  $m$  are perfect squares (so that  $d_0 = m_0 = 1$ ). For  $s + w = \frac{3}{2}$ , we have a pole coming from the gamma factors corresponding to the functional equations for  $\sigma_2\sigma_1$  and  $\sigma_1\sigma_2$ . We collect all of our work as a theorem:

**Theorem 7.1.**  $Z(s, w)$  admits meromorphic continuation to  $\mathbb{C}^2$  with polar divisors  $s = 1$ ,  $w = 1$ , and  $s + w = \frac{3}{2}$ .

We can also study the residues of  $Z(s, w)$  at these poles. Since all of the poles are obtained from each other by applying the functional equations of  $Z(s, w)$ , we begin by looking at the pole when  $w = 1$ . To compute the residue we use the representation

$$Z(s, w) = \sum_{\substack{m \geq 1 \\ (m, 2)=1}} \frac{L^{(2)}(w, \tilde{\chi}_{m_0}) Q_{m_0 m_1^2}(w)}{m^s},$$

coming from the interchange. The numerator  $L(w, \tilde{\chi}_{m_0}) Q_{m_0 m_1^2}(w)$  in the summand corresponding to  $m$  has a pole at  $w = 1$  if and only if  $m_0$  is a perfect square, that is  $m_0 = 1$ , or equivalently  $m = m_1^2$  itself is a perfect square. In this case,  $L(w, \tilde{\chi}_{m_0}) = \zeta(w)$  and

$$\text{Res}_{w=1} L^{(2)}(w, \chi_{m_0}) Q_{m_0 m_1^2}(w) = \frac{1}{2} Q_{m_1^2}(1).$$

But from Lemma 4.2 and Theorem 4.1 we see that  $Q_{m_1^2}(1) = 1$ , and hence

$$\text{Res}_{w=1} Z(s, w) = \frac{1}{2} \sum_{\substack{m \text{ perfect square} \\ (m, 2)=1}} \frac{Q_{m_1^2}(1)}{m^s} = \frac{1}{2} \sum_{\substack{m \geq 1 \\ (m, 2)=1}} \frac{1}{m^{2s}} = \frac{1}{2} \zeta^{(2)}(2s).$$

Notice that this expression has a simple pole at  $s = \frac{1}{2}$  which is exactly when the polar lines  $w = 1$  and  $s + w = \frac{3}{2}$  intersect. The residue of  $Z(s, w)$  at  $s = 1$  is computed in an analogous way. Indeed, by applying the interchange, the exact same argument holds with the roles of  $s$  and  $w$  interchanged so that

$$\text{Res}_{s=1} Z(s, w) = \frac{1}{2} \zeta^{(2)}(2w).$$

Again, this expression has a simple pole at  $w = \frac{1}{2}$  which is when the polar lines  $s = 1$  and  $s + w = \frac{3}{2}$  intersect. The other residues at the simple poles can be computed by applying the functional equations for  $Z(s, w)$  and using the residues at  $s = 1$  and  $w = 1$ . Now consider the point where the polar lines  $w = 1$  and  $s + w = \frac{3}{2}$  intersect. Setting  $s = \frac{1}{2}$ , we see that  $Z(\frac{1}{2}, w)$  has a pole at  $w = 1$  and we would like to study this pole more. To achieve this, the Mittag-Leffler theorem applied to  $Z(s, w)$  (in  $w$ ) implies

$$Z(s, w) = \frac{R_1(s)}{w - 1} + \frac{R_2(s)}{s + w - \frac{3}{2}} + V(s, w),$$

in some neighborhood of  $(\frac{1}{2}, 1)$ , where  $V(s, w)$  is holomorphic, and we have set

$$R_1(s) = \text{Res}_{w=1} Z(s, w) \quad \text{and} \quad R_2(s) = \text{Res}_{w=\frac{3}{2}-s} Z(s, w).$$

From our residue computations above,  $R_1(s) = \frac{1}{2} \zeta^{(2)}(2s)$  which implies that it has a simple pole at  $s = \frac{1}{2}$ . The residue is given by  $A = \frac{1}{8}$ . On the other hand,  $Z(\frac{1}{2}, w)$  is holomorphic for  $\text{Re}(w) > 1$ . These two facts together imply that  $R_2(s)$  must have a simple pole at  $s = \frac{1}{2}$  which cancels the simple pole coming from  $R_1(s)$ . So by Mittag-Leffler again, we may write

$$R_1(s) = \frac{A}{s - \frac{1}{2}} + R_3(s) \quad \text{and} \quad R_2(s) = -\frac{A}{s - \frac{1}{2}} + R_4(s),$$

in a neighborhood of  $s = \frac{1}{2}$  and where  $R_3(s)$  and  $R_4(s)$  are holomorphic. Then

$$\begin{aligned} Z(s, w) &= \frac{R_1(s)}{w-1} + \frac{R_2(s)}{s+w-\frac{3}{2}} + V(s, w) \\ &= \frac{A}{(w-1)(s-\frac{1}{2})} + \frac{R_3(s)}{w-1} - \frac{A}{(s+w-\frac{3}{2})(s-\frac{1}{2})} + \frac{R_4(s)}{s+w-\frac{3}{2}} + V(s, w) \\ &= \frac{A}{(w-1)(s+w-\frac{3}{2})} + \frac{R_3(s)}{w-1} + \frac{R_4(s)}{s+w-\frac{3}{2}} + V(s, w). \end{aligned}$$

We can now set  $s = \frac{1}{2}$  and let  $B = R_3(\frac{1}{2}) + R_4(\frac{1}{2})$  so that

$$Z\left(\frac{1}{2}, w\right) = \frac{A}{(w-1)^2} + \frac{B}{w-1} + O(1).$$

It follows that  $Z(\frac{1}{2}, w)$  has a double pole at  $w = 1$ . This can be thought of as follows: the polar lines  $w = 1$  and  $s + w = \frac{3}{2}$  correspond to simple poles of  $Z(s, w)$  except in the case when they intersect and the order of the poles combine to give  $Z(\frac{1}{2}, w)$  a double pole at  $w = 1$ . Applying the interchange, the exact same argument holds to show that  $Z(s, \frac{1}{2})$  has a double pole at  $s = 1$ . We collect this work as a theorem:

**Theorem 7.2.**  *$Z(\frac{1}{2}, w)$  and  $Z(s, \frac{1}{2})$  have double poles at  $w = 1$  and  $s = 1$  respectively. In particular, in neighborhoods of  $w = 1$  and  $s = 1$  respectively, we have*

$$Z\left(\frac{1}{2}, w\right) = \frac{A}{(w-1)^2} + \frac{B}{w-1} + O(1) \quad \text{and} \quad Z\left(s, \frac{1}{2}\right) = \frac{A}{(s-1)^2} + \frac{B}{s-1} + O(1),$$

for some constants  $A$  and  $B$  with  $A = \frac{1}{8}$ .

## REFERENCES

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