

# A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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ABSTRACT. We construct a quadratic double Dirichlet series  $Z(s, w)$  built from single variable quadratic Dirichlet  $L$ -functions  $L(s, \chi)$  over  $\mathbb{Q}$ . We prove that  $Z(s, w)$  admits meromorphic continuation to the  $(s, w)$ -plane and satisfies a group of functional equations.

## 1. PRELIMINARIES

We present an overview of quadratic Dirichlet  $L$ -functions over  $\mathbb{Q}$ . We begin with the Riemann zeta-function. The zeta function  $\zeta(s)$  is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m \geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for  $\operatorname{Re}(s) > 1$ . The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Now we recall characters on  $\mathbb{Z}$ . They are multiplicative functions  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ . They form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions  $\chi_d : \mathbb{Z} \rightarrow \mathbb{C}$  modulo  $d \geq 1$  (in that they are  $d$ -periodic) and such that  $\chi_d(m) = 0$  if  $(m, d) > 1$ .
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. If  $\chi$  is a Dirichlet character then its conjugate  $\bar{\chi}$  is also a Dirichlet character. Moreover,  $\bar{\chi}$  is the multiplicative inverse to  $\chi$  and the Dirichlet characters modulo  $m$  form a group under multiplication. This group is always finite and its order is  $\phi(d) = |(\mathbb{Z}/d\mathbb{Z})^*|$ . Dirichlet characters also satisfy orthogonality relations:

**Theorem 1.1** (Orthogonality relations).

(i) For any two Dirichlet characters  $\chi$  and  $\psi$  modulo  $d$ ,

$$\frac{1}{\phi(d)} \sum_{a \pmod{d}}' \chi(a) \bar{\psi}(a) = \delta_{\chi, \psi}.$$

(ii) For any  $a, b \in (\mathbb{Z}/d\mathbb{Z})^*$ ,

$$\frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \chi(a) \bar{\chi}(b) = \delta_{a, b}.$$

The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on  $\mathbb{Z}$ . First let us recall this symbol. For any odd prime  $p$  and any  $m \geq 1$ , we define the quadratic residue symbol  $\left(\frac{m}{p}\right)$  by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod{p} = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod{p} \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod{p} \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod{p}. \end{cases}$$

This symbol only depends upon  $m$  modulo  $p$  and is multiplicative in  $m$ . We can extend the quadratic residue symbol multiplicatively in the denominator. If  $d = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  is the prime factorization of  $d$ , then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \leq i \leq k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any odd  $d \geq 1$ . We can extend this symbol further and allow  $d \geq 1$  to be even. To this end, we define

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases}$$

and extend  $\left(\frac{m}{d}\right)$  multiplicatively in  $d$  when  $d$  is even. Now the quadratic residue symbol makes sense for any  $m, d \geq 1$ . Moreover, it is multiplicative in both  $m$  and  $d$  but no longer depends upon only  $m$  modulo  $d$  (it also depends upon  $m$  modulo 8). In particular,

$$\left(\frac{-1}{d}\right) = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases}$$

and if  $d \not\equiv 0 \pmod{2}$ , we can compactly write

$$\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}} = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = (-1)^{\frac{d^2-1}{8}} = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}. \end{cases}$$

The quadratic residue symbol also admits the following reciprocity law:

**Theorem 1.2** (Quadratic reciprocity). *If  $d, m \geq 1$ , then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2} \frac{m^{(2)}-1}{2}} \left(\frac{m}{d}\right),$$

where  $d^{(2)}$  and  $m^{(2)}$  are the parts of  $d$  and  $m$  relatively prime to 2 respectively.

We can now define the quadratic Dirichlet characters. For any odd square-free  $d \in \mathbb{Z}$ , define the quadratic Dirichlet character  $\chi_d$  by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod{4}, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod{4}. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension  $\mathbb{Q}(\sqrt{d})$ . We extend  $\chi_d$  multiplicatively in the denominator so that  $\chi_d$  makes sense for any odd  $d$ . In particular,  $\chi_d(m) = \pm 1$  provided  $d$  and  $m$  are relatively prime and  $\chi_d(m) = 0$  if  $(m, d) > 1$ . Quadratic reciprocity implies that  $\chi_d$  is a Dirichlet character modulo  $d$  if  $d \equiv 1 \pmod{4}$  and is a Dirichlet character modulo  $4d$  if  $d \equiv 2, 3 \pmod{4}$ . Indeed, if  $d \equiv 1 \pmod{4}$  then  $d^{(2)} = d$  and the sign is always 1. If  $d \equiv 3 \pmod{4}$ , then  $d^{(2)} = d$  and the sign is

$\left(\frac{-1}{m}\right)$  which is a character modulo 4. If  $d \equiv 2 \pmod{4}$ , then  $d^{(2)} \equiv 1, 3 \pmod{4}$  and we are reduced to one of the previous two cases. We will also require an associated character. For each  $\chi_d$ , we define  $\tilde{\chi}_d$  by

$$\tilde{\chi}_d(m) = (-1)^{\frac{d^{(2)}-1}{2} \frac{m^{(2)}-1}{2}} \chi_d(m).$$

Equivalently,  $\tilde{\chi}_d(m)$  can be expressed as

$$\tilde{\chi}_d(m) = \begin{cases} \chi_d(m) & \text{if } d \equiv 1, 2 \pmod{4}, \\ \chi_{-1}(m)\chi_d(m) & \text{if } d \equiv 3 \pmod{4}, \end{cases}$$

and this implies that  $\tilde{\chi}_d(m)$  is a quadratic Dirichlet character of the same modulus as  $\chi_d$ . We now discuss the Hilbert characters. We will only need four of them: the quadratic Dirichlet characters modulo 8. They are given as follows:

$$\begin{aligned} \chi_1(m) &= \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod{2}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} & \chi_{-1}(m) &= \begin{cases} 1 & \text{if } m \equiv 1 \pmod{4}, \\ -1 & \text{if } m \equiv 3 \pmod{4}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} \\ \chi_2(m) &= \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} & \chi_{-2}(m) &= \begin{cases} 1 & \text{if } m \equiv 1, 3 \pmod{8}, \\ -1 & \text{if } m \equiv 5, 7 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}. \end{cases} \end{aligned}$$

In general, we will denote a Hilbert character by  $\chi_a$  with  $a \in \{\pm 1, \pm 2\}$ . Note that

$$\chi_{-1}(m) = \left(\frac{-1}{m}\right) \quad \text{and} \quad \chi_2(m) = \left(\frac{m}{2}\right).$$

Moreover, we have the relations

$$\chi_{-2}(m) = \chi_{-1}(m)\chi_2(m), \quad \chi_1(m) = \chi_{-1}(m)\chi_{-1}(m), \quad \text{and} \quad \chi_{-1}(m) = \chi_2(m)\chi_{-2}(m).$$

Suppose  $d$  is square-free. If  $d \equiv 1, 2, 5 \pmod{8}$ , then  $d^{(2)} \equiv 1 \pmod{4}$  so that the sign in the statement of quadratic reciprocity is 1. If  $d \equiv 3, 6, 7 \pmod{8}$ , then  $d^{(2)} \equiv 3 \pmod{4}$  and the sign is  $(-1)^{\frac{m^{(2)}-1}{2}}$ . This fact together with the relations for the quadratic characters modulo 8 imply

$$\chi_d(m) = \begin{cases} \chi_m(d) & \text{if } d \equiv 1 \pmod{4}, \\ \chi_{-1}(m)\chi_m(d) & \text{if } d \equiv 3 \pmod{4}, \\ \chi_2(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 2 \pmod{8}, \\ \chi_{-2}(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 6 \pmod{8}. \end{cases}$$

With the Dirichlet and Hilbert characters introduced, we are ready to discuss the  $L$ -functions associated to quadratic Dirichlet characters. We define the  $L$ -function  $L(s, \chi_d)$  attached to  $\chi_d$  by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \geq 1} \frac{\chi_d(m)}{|m|^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi_d(p)}{|p|^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character,  $L(s, \chi_d) \ll \zeta(s)$  for  $\text{Re}(s) > 1$  so that  $L(s, \chi_d)$  is locally absolutely uniformly convergent in this region.  $L(s, \chi_d)$  also admits meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  if  $d$  is a perfect square. For square-free  $d$ , the completed  $L$ -function is defined as

$$L^*(s, \chi_d) = \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi_d) & \text{if } \chi_d \text{ is even,} \\ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi_d) & \text{if } \chi_d \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_d) = \begin{cases} \varepsilon_\chi q^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \chi_d \text{ is even,} \\ -\varepsilon_\chi q^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \chi_d \text{ is odd.} \end{cases}$$

Note that the gamma factors depend upon the parity of  $\chi_d$ . This is the root cause of an important technical issue later when deriving functional equations for the quadratic double Dirichlet series.

### THE QUADRATIC DOUBLE DIRICHLET SERIES

We will now define the quadratic double Dirichlet series  $Z(s, w)$ . For any integer  $d \geq 1$ , write  $d = d_0 d_1^2$  where  $d_0$  is square-free. Equivalently,  $d_0$  is the square-free part of  $d$  and  $\frac{d}{d_0}$  is a perfect square. The **quadratic double Dirichlet series**  $Z(s, w)$  is defined as

$$Z(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{d_0}) Q_{d_0 d_1^2}(s)}{d^w},$$

where the superscript (2) indicates that the local factor at 2 has been removed,  $Q_{d_0 d_1^2}(s)$  is the **correction polynomial** defined by

$$Q_{d_0 d_1^2}(s) = \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{d_0}(e_1) e_1^{-s} e_2^{1-s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{d_0}(e_1) e_1^{-s} e_2^{1-s},$$

and  $\mu$  is the usual Möbius function. For  $\text{Re}(s) > 1$ , there is the trivial estimate

$$Q_{d_0 d_1^2}(s) \ll \sum_{e_1 e_2 | d_1} 1 \ll \sigma_0(d_1)^2 \ll_\varepsilon |d_1|^\varepsilon \ll_\varepsilon |d|^\varepsilon,$$

for any  $\varepsilon > 0$ . As  $L(s, \chi_{d_0}) \ll 1$  for  $\text{Re}(s) > 1$ ,  $Z(s, w)$  is locally absolutely uniformly convergent in the region  $\Lambda = \{(s, w) \in \mathbb{C}^2 : \text{Re}(s) > 1, \text{Re}(w) > 1\}$ . It will also be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters  $\chi_{a_1}$  and  $\chi_{a_2}$ . The **quadratic double Dirichlet series**  $Z_{a_1, a_2}(s, w)$  twisted by  $\chi_{a_1}$  and  $\chi_{a_2}$  is defined as

$$Z_{a_1, a_2}(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{|d|^w},$$

where  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  is the **correction polynomial** twisted by  $\chi_{a_1}$  defined by

$$Q_{d_0 d_1^2}(s, \chi_{a_1}) = \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{-s} |e_2|^{1-2s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{-s} |e_2|^{1-2s},$$

and  $\mu$  is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound  $Q_{d_0 d_1^2}(s, \chi_{a_1}) \ll |d|^\varepsilon$  so that  $Z_{a_1, a_2}(s, w)$  converges locally absolutely uniformly in the same region as  $Z(s, w)$  does. In particular,  $Z(s, w) = Z_{1, 1}(s, w)$ .

### THE INTERCHANGE

As defined,  $Z_{a_1, a_2}(s, w)$  is a sum of  $L$ -functions, and hence Euler products, in  $s$ . We will prove an interchange formula for  $Z_{a_1, a_2}(s, w)$  which will show that it can be expressed as a sum of  $L$ -functions in  $w$ . That is, we want the variables  $s$  and  $w$  to change places. Precisely:

**Theorem 1.3** (Interchange). *Wherever  $Z_{a_1, a_2}(s, w)$  converges locally absolutely uniformly,*

$$Z_{a_1, a_2}(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{|d|^w} = \sum_{m \text{ odd}} \frac{L^{(2)}(w, \tilde{\chi}_{a_2 m_0}) \chi_{a_1}(m) Q_{m_0 m_1^2}(w, \chi_{a_2})}{|m|^s}.$$

*Proof.* Only the second equality needs to be proved. To do this, first expand the  $L$ -function  $L^{(2)}(s, \chi_{a_1 d_0})$  and polynomial  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  to get

$$\begin{aligned} Z(s, w) &= \sum_{d \text{ odd}} \frac{L(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{|d|^w} \\ &= \sum_{d \text{ odd}} \left( \sum_{m \text{ odd}} \chi_{a_1 d_0}(m) |m|^{-s} \right) \left( \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{-s} |e_2|^{1-2s} \right) \chi_{a_2}(d) |d|^{-w} \\ &= \sum_{m, d \text{ odd}} \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m e_1) |e_1|^{-s} |e_2|^{1-2s} |m|^{-s} |d|^{-w}. \end{aligned}$$

Now  $\chi_{a_1 d_0}(m e_1) = 0$  unless  $(d_0, m e_1) = 1$ . We make this restriction on the sum giving

$$\sum_{m, d \text{ odd}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, m e_1) = 1}} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m e_1) |e_1|^{-s} |e_2|^{1-2s} |m|^{-s} |d|^{-w}.$$

Making the change of variables  $m e_1 \rightarrow m$  yields

$$\sum_{d \text{ odd}} \sum_{\substack{m \text{ odd} \\ e_1 | m}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, m) = 1}} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m) |e_2|^{1-2s} |m|^{-s} |d|^{-w}.$$

For fixed  $d = d_0 d_1^2$  and  $e_2$ , the subsum over  $m$  and  $e_1$  is

$$\sum_{\substack{m \text{ odd} \\ e_1 | m}} \sum_{\substack{e_1 | \frac{d_1}{e_2} \\ (d_0, m) = 1}} \mu(e_1) \chi_{a_1 d_0}(m) |m|^{-s} = \sum_{\substack{m \text{ odd} \\ (d_0, m) = 1}} \chi_{a_1 d_0}(m) |m|^{-s} \left( \sum_{e_1 | \left( \frac{d_1}{e_2}, m \right)} \mu(e_1) \right).$$

The inner sum over  $e_1$  of the Möbius function vanishes unless  $\left( \frac{d_1}{e_2}, m \right) = 1$  in which case it is 1. Therefore the triple sum above becomes

$$\sum_{m, d \text{ odd}} \sum_{\substack{e_2 | d_1 \\ \left( \frac{d_0 d_1}{e_2}, m \right) = 1}} \chi_{a_2}(d) \chi_{a_1 d_0}(m) |e_2|^{1-2s} |m|^{-s} |d|^{-w}.$$

Making the change of variables  $d \rightarrow d e_2^2$ , the condition  $\left( \frac{d_0 d_1}{e_2}, m \right) = 1$  becomes  $(d_0 d_1, m) = 1$  which is equivalent to  $(d, m) = 1$ . Moreover,  $\chi_{a_2}(d e_2^2) = \chi_{a_2}(d)$ . Altogether, we obtain

$$\sum_{\substack{m, d \text{ odd} \\ (d, m) = 1}} \sum_{e_2} \chi_{a_2}(d) \chi_{a_1 d_0}(m) |e_2|^{1-2s-2w} |m|^{-s} |d|^{-w}.$$

Writing  $m = m_0 m_1^2$  analogously as for  $d$ , quadratic reciprocity implies  $\chi_{d_0}(m) = \tilde{\chi}_m(d_0) = \tilde{\chi}_{m_0}(d)$  where the last equality holds because  $(d, m) = 1$  and both  $d_0$  and  $m_0$  differ from  $d$  and  $m$  respectively by perfect squares. This implies  $\chi_{a_2}(d) \chi_{a_1 d_0}(m) = \chi_{a_1}(m) \tilde{\chi}_{a_2 m_0}(d)$  and so our expression becomes

$$\sum_{\substack{m, d \text{ monic} \\ (d, m) = 1}} \sum_{e_2} \chi_{a_1}(m) \tilde{\chi}_{a_2 m_0}(d) |e_2|^{1-2s-2w} |m|^{-s} |d|^{-w}.$$

But now we can reverse the argument with the roles of  $d$ ,  $m$ ,  $\chi_{a_1}$ , and  $\tilde{\chi}_{a_2}$  interchanged respectively to obtain

$$Z(s, w) = \sum_{m \text{ monic}} \frac{L(w, \tilde{\chi}_{a_2 m_0}) \chi_{a_1}(m) Q_{m_0 m_1^2}(w, \chi_{a_2})}{|m|^s}.$$

□

Note that the interchange is not completely symmetric because of the character  $\tilde{\chi}_{a_2 m_0}$  in the second expression for  $Z_{a_1, a_2}(s, w)$ . This is due to the fact that reciprocity is not perfect. In even more general settings the correction polynomials in  $w$  need not be equal to those in  $s$ .

#### WEIGHTING TERMS

#### FUNCTIONAL EQUATIONS

#### MEROMORPHIC CONTINUATION

#### POLES AND RESIDUES

#### REFERENCES

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