

## 0.1 The Kuznetsov Trace Formula

The Kuznetsov trace formula is an analog of the Petersson trace formula for weight zero Maass forms. From ??,  $\mathcal{L}(N, \chi)$  admits an orthonormal basis of Maass forms for the point spectrum (these forms are generally not Hecke-Maass eigenforms because they need not be Hecke normalized or even cuspidal in the case of the discrete spectrum). However, by ?? and ?? we make take this orthonormal basis to consist of Hecke-Maass eigenforms and the constant function. Denote this basis by  $\{u_j\}_{j \geq 0}$  with  $u_0(z) = 1$  and let  $u_j$  be of type  $\nu_j$  for  $j \geq 1$ . In particular,  $\{u_j\}_{j \geq 1}$  is an orthonormal basis of Hecke-Maass eigenforms and each such form admits a Fourier series at the  $\mathfrak{a}$  cusp given by

$$(u_j|\sigma_{\mathfrak{a}})(z) = \sum_{n \neq 0} a_{j,\mathfrak{a}}(n) \sqrt{y} K_{\nu_j}(2\pi ny) e^{2\pi i n x}.$$

The Kuznetsov trace formula is an equation relating the Fourier coefficients  $a_{j,\mathfrak{a}}(n)$  and  $a_{j,\mathfrak{b}}(n)$  of the basis  $\{u_j\}_{j \geq 1}$  for two cusps  $\mathfrak{a}$  and  $\mathfrak{b}$  of  $\Gamma_0(N) \backslash \mathbb{H}$  to a sum of integral transforms involving test functions and Salié sums. Similar to the Petersson trace formula, we will compute the inner product of two Poincaré series  $P_{n,\chi,\mathfrak{a}}(z, \psi)(z)$  and  $P_{m,\chi,\mathfrak{b}}(z, \varphi)(z)$  in two different ways. The first will be geometric in nature while the second will be spectral. We first need to compute the Fourier series of such a Poincaré series. Although we will not need it explicitly, we will work over any congruence subgroup:

**Proposition 0.1.1.** *Let  $m \geq 1$ ,  $\chi$  be Dirichlet character with conductor dividing the level,  $\mathfrak{a}$  and  $\mathfrak{b}$  be cusps of  $\Gamma \backslash \mathbb{H}$ , and  $\psi(y)$  be a smooth function such that  $\psi(y) \ll_{\varepsilon} y^{1+\varepsilon}$  as  $y \rightarrow 0$ . The Fourier series of  $P_{m,\chi,\mathfrak{a}}(z, \psi)$  on  $\Gamma \backslash \mathbb{H}$  at the  $\mathfrak{b}$  cusp is given by*

$$(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z, \psi) = \sum_{t \in \mathbb{Z}} \left( \delta_{\mathfrak{a},\mathfrak{b}} \delta_{m,t} \psi(\operatorname{Im}(z)) + \sum_{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}} \psi(y, m, t, c) S_{\chi,\mathfrak{a},\mathfrak{b}}(m, t, c) \right) e^{2\pi i t z},$$

where  $\psi(y, m, t, c)$  is the integral transform given by

$$\psi(y, m, t, c) = \int_{\operatorname{Im}(z)=y} \psi \left( \frac{y}{|cz|^2} \right) e^{-\frac{2\pi i m}{c^2 z} - 2\pi i t z} dz.$$

*Proof.* From the cocycle condition and ??, we have

$$(P_{m,\chi,\mathfrak{a}}|\sigma_{\mathfrak{b}})(z, \psi) = \delta_{\mathfrak{a},\mathfrak{b}} \psi(\operatorname{Im}(z)) e^{2\pi i m z} + \sum_{\substack{c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}, d \in \mathbb{Z} \\ d \pmod{c} \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)}} \bar{\chi}(d) \psi \left( \frac{\operatorname{Im}(z)}{|cz+d|^2} \right) e^{2\pi i m \left( \frac{a}{c} - \frac{1}{c^2 z + cd} \right)},$$

where  $a$  and  $b$  are chosen such that  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$  and we have used the fact that

$$\frac{a}{c} - \frac{1}{c^2 z + cd} = \frac{az+b}{cz+d}.$$

Summing over all pairs  $(c, d)$  with  $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ ,  $d \in \mathbb{Z}$ , and  $d \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$  is the same as summing over all triples  $(c, \ell, r)$  with  $c \in \mathcal{C}_{\mathfrak{a},\mathfrak{b}}$ ,  $\ell \in \mathbb{Z}$ , and  $r$  taken modulo  $c$  with  $r \in \mathcal{D}_{\mathfrak{a},\mathfrak{b}}(c)$ . Indeed, this is seen by writing  $d = c\ell + r$ . Moreover, since  $ad - bc = 1$  we have  $a(c\ell + r) - bc = 1$  which further implies that

$ar \equiv 1 \pmod{c}$ . So we may take  $a$  to be the inverse for  $r$  modulo  $c$ . Then

$$\begin{aligned} \sum_{\substack{c \in \mathcal{C}_{a,b}, d \in \mathbb{Z} \\ d \pmod{c} \in \mathcal{D}_{a,b}(c)}} \bar{\chi}(d) \psi \left( \frac{\operatorname{Im}(z)}{|cz + d|^2} \right) e^{2\pi i m \left( \frac{a}{c} - \frac{1}{c^2 z + cd} \right)} &= \sum_{(c,\ell,r)} \bar{\chi}(c\ell + r) \psi \left( \frac{\operatorname{Im}(z)}{|cz + c\ell + r|^2} \right) e^{2\pi i m \left( \frac{a}{c} - \frac{1}{c^2 z + c^2 \ell + cr} \right)} \\ &= \sum_{(c,\ell,r)} \bar{\chi}(r) \psi \left( \frac{\operatorname{Im}(z)}{|cz + c\ell + r|^2} \right) e^{2\pi i m \left( \frac{a}{c} - \frac{1}{c^2 z + c^2 \ell + cr} \right)} \\ &= \sum_{\substack{c \in \mathcal{C}_{a,b} \\ r \in \mathcal{D}_{a,b}(c)}} \bar{\chi}(r) \sum_{\ell \in \mathbb{Z}} \psi \left( \frac{\operatorname{Im}(z)}{|cz + c\ell + r|^2} \right) e^{2\pi i m \left( \frac{a}{c} - \frac{1}{c^2 z + c^2 \ell + cr} \right)}, \end{aligned}$$

where on the right-hand side it is understood that we are summing over all triples  $(c, \ell, r)$  with the prescribed properties and the second line holds since  $\chi$  has conductor dividing the level and  $d \in \mathcal{D}_{a,b}(c)$  is determined modulo  $c$ . Now let

$$I_{c,r}(z, \psi) = \sum_{\ell \in \mathbb{Z}} \psi \left( \frac{\operatorname{Im}(z)}{|cz + c\ell + r|^2} \right) e^{2\pi i m \left( \frac{a}{c} - \frac{1}{c^2 z + c^2 \ell + cr} \right)}.$$

We apply the Poisson summation formula to  $I_{c,r}(z, \psi)$ . This is allowed since the summands are absolutely integrable by ??, as they exhibit polynomial decay of order  $\sigma > 1$  because  $\psi(y) \ll_\varepsilon y^{1+\varepsilon}$  as  $y \rightarrow 0$ , and  $I_{c,r}(z, \psi)$  is holomorphic because  $(P_{m,\chi,a}|\sigma_b)(z, \psi)$  is. By the identity theorem it suffices to apply the Poisson summation formula for  $z = iy$  with  $y > 0$ . So let  $f(x)$  be given by

$$f(x) = \psi \left( \frac{y}{|cx + r + icy|^2} \right) e^{2\pi i m \left( \frac{a}{c} - \frac{1}{c^2 x + cr + ic^2 y} \right)}.$$

As we have just noted,  $f(x)$  is absolutely integrable on  $\mathbb{R}$ . We compute the Fourier transform:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi itx} dx = \int_{-\infty}^{\infty} \psi \left( \frac{y}{|cx + r + icy|^2} \right) e^{2\pi i m \left( \frac{a}{c} - \frac{1}{c^2 x + cr + ic^2 y} \right)} e^{-2\pi itx} dx.$$

Complexify the integral to get

$$\int_{\operatorname{Im}(z)=0} \psi \left( \frac{y}{|cz + r + icy|^2} \right) e^{2\pi i m \left( \frac{a}{c} - \frac{1}{c^2 z + cr + ic^2 y} \right)} e^{-2\pi itz} dz.$$

Now make the change of variables  $z \rightarrow z - \frac{r}{c} - iy$  to obtain

$$e^{2\pi i m \frac{a}{c} + 2\pi i t \frac{r}{c} - 2\pi ty} \int_{\operatorname{Im}(z)=y} \psi \left( \frac{y}{|cz|^2} \right) e^{-\frac{2\pi i m}{c^2 z} - 2\pi itz} dz.$$

As the remaining integral is  $\psi(y, m, t, c)$ , it follows that

$$\hat{f}(t) = \psi(y, m, t, c) e^{2\pi i m \frac{a}{c} + 2\pi i t \frac{r}{c} - 2\pi ty}.$$

By the Poisson summation formula and the identity theorem, we have

$$I_{c,r}(z, \psi) = \sum_{t \in \mathbb{Z}} (\psi(y, m, t, c) e^{2\pi i m \frac{a}{c} + 2\pi i t \frac{r}{c}}) e^{2\pi i tz},$$

for all  $z \in \mathbb{H}$ . Substituting this back into the Eisenstein series gives a form of the Fourier series:

$$\begin{aligned}
(P_{m,\chi,a}|\sigma_b)(z, \psi) &= \delta_{a,b}\psi(\operatorname{Im}(z))e^{2\pi imz} + \sum_{\substack{c \in \mathcal{C}_{a,b} \\ r \in \mathcal{D}_{a,b}}} \bar{\chi}(r) \sum_{t \in \mathbb{Z}} \psi(y, m, t, c)e^{2\pi im\frac{a}{c} + 2\pi it\frac{r}{c}} e^{2\pi itz} \\
&= \sum_{t \in \mathbb{Z}} \left( \delta_{a,b}\delta_{m,t}\psi(\operatorname{Im}(z)) + \sum_{\substack{c \in \mathcal{C}_{a,b} \\ r \in \mathcal{D}_{a,b}}} \bar{\chi}(r)\psi(y, m, t, c)e^{2\pi im\frac{a}{c} + 2\pi it\frac{r}{c}} \right) e^{2\pi itz} \\
&= \sum_{t \in \mathbb{Z}} \left( \delta_{a,b}\delta_{m,t}\psi(\operatorname{Im}(z)) + \sum_{c \in \mathcal{C}_{a,b}} \psi(y, m, t, c) \sum_{r \in \mathcal{D}_{a,b}} \bar{\chi}(r)e^{2\pi im\frac{a}{c} + 2\pi it\frac{r}{c}} \right) e^{2\pi itz}.
\end{aligned}$$

We will simplify the innermost sum. Since  $a$  is the inverse for  $r$  modulo  $c$ , the innermost sum above becomes

$$\sum_{r \in \mathcal{D}_{a,b}} \bar{\chi}(r)e^{2\pi im\frac{a}{c} + 2\pi it\frac{r}{c}} = \sum_{r \in \mathcal{D}_{a,b}} \bar{\chi}(\bar{a})e^{2\pi im\frac{a}{c} + 2\pi it\frac{\bar{a}}{c}} = \sum_{r \in \mathcal{D}_{a,b}} \chi(a)e^{\frac{2\pi i(am+\bar{a}t)}{c}} = S_{\chi,a,b}(m, t, c).$$

So at last, we obtain our desired Fourier series:

$$(P_{m,\chi,a}|\sigma_b)(z) = \sum_{t \in \mathbb{Z}} \left( \delta_{a,b}\delta_{m,t}\psi(\operatorname{Im}(z)) + \sum_{c \in \mathcal{C}_{a,b}} \psi(y, m, t, c)S_{\chi,a,b}(m, t, c) \right) e^{2\pi itz}. \quad \square$$

We can now derive the first half of the Kuznetsov trace formula by computing the inner product between  $P_{n,\chi,a}(z, \psi)$  and  $P_{m,\chi,b}(z, \varphi)$ :

$$\begin{aligned}
\langle P_{n,\chi,a}(\cdot, \psi), P_{m,\chi,b}(\cdot, \varphi) \rangle &= \frac{1}{V_{\Gamma_0(N)}} \int_{\mathcal{F}_{\Gamma_0(N)}} P_{n,\chi,a}(z, \psi) \overline{P_{m,\chi,b}(z, \varphi)} d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}} \int_{\mathcal{F}_{\Gamma_0(N)}} \sum_{\gamma \in \Gamma_b \setminus \Gamma_0(N)} \chi(\gamma) P_{n,\chi,a}(z, \psi) \overline{\varphi(\operatorname{Im}(\sigma_b^{-1}\gamma z))} e^{-2\pi im\sigma_b^{-1}\gamma z} d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}} \int_{\mathcal{F}_{\Gamma_0(N)}} \sum_{\gamma \in \Gamma_b \setminus \Gamma_0(N)} P_{n,\chi,a}(\gamma z, \psi) \overline{\varphi(\operatorname{Im}(\sigma_b^{-1}\gamma z))} e^{-2\pi im\sigma_b^{-1}\gamma z} d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}} \int_{\mathcal{F}_{\sigma_b^{-1}\Gamma_0(N)\sigma_b}} \sum_{\gamma \in \Gamma_b \setminus \Gamma_0(N)} P_{n,\chi,a}(\gamma\sigma_b z, \psi) \overline{\varphi(\operatorname{Im}(\sigma_b^{-1}\gamma\sigma_b z))} e^{-2\pi im\sigma_b^{-1}\gamma\sigma_b z} d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}} \int_{\mathcal{F}_{\sigma_b^{-1}\Gamma_0(N)\sigma_b}} \sum_{\gamma \in \Gamma_\infty \setminus \sigma_b^{-1}\Gamma_0(N)\sigma_b^{-1}} P_{n,\chi,a}(\sigma_b\gamma z, \psi) \overline{\varphi(\operatorname{Im}(\gamma z))} e^{-2\pi im\gamma z} d\mu \\
&= \frac{1}{V_{\Gamma_0(N)}} \int_{\Gamma_\infty \setminus \mathbb{H}} (P_{n,\chi,a}|\sigma_b)(z, \psi) \overline{\varphi(\operatorname{Im}(z))} e^{-2\pi im\bar{z}} d\mu,
\end{aligned}$$

where in the third line we have used the automorphy of  $P_{n,\chi,a}(z, \psi)$ , in the forth and fifth lines we have made the change of variables  $z \rightarrow \sigma_b z$  and  $\gamma \rightarrow \sigma_b \gamma \sigma_b^{-1}$  respectively, and in the sixth line we have unfolded. Now substitute in the Fourier series of  $P_{n,\chi,a}(z, \psi)$  at the  $b$  cusp to obtain

$$\frac{1}{V_{\Gamma_0(N)}} \int_{\Gamma_\infty \setminus \mathbb{H}} \sum_{t \in \mathbb{Z}} \left( \delta_{a,b}\delta_{n,t}\psi(\operatorname{Im}(z)) + \sum_{c \in \mathcal{C}_{a,b}} \psi(y, n, t, c)S_{\chi,a,b}(n, t, c) \right) \overline{\varphi(\operatorname{Im}(z))} e^{2\pi itz - 2\pi im\bar{z}} d\mu,$$

which is equivalent to

$$\frac{1}{V_{\Gamma_0(N)}} \int_0^\infty \int_0^1 \sum_{t \geq 1} \left( \delta_{\mathfrak{a}, \mathfrak{b}} \delta_{n,t} \psi(y) + \sum_{c \in \mathcal{C}_{\mathfrak{a}, \mathfrak{b}}} \psi(y, n, t, c) S_{\chi, \mathfrak{a}, \mathfrak{b}}(n, t, c) \right) \overline{\varphi(y)} e^{2\pi i(t-m)x} e^{-2\pi(t+m)y} \frac{dx dy}{y^2}.$$

By the dominated convergence theorem, we can interchange the sum and the two integrals. Then ?? implies that the inner integral cuts off all of the terms except the diagonal  $t = m$ . This leaves

$$\frac{1}{V_{\Gamma_0(N)}} \int_0^\infty \left( \delta_{\mathfrak{a}, \mathfrak{b}} \delta_{n,m} \psi(y) + \sum_{c \in \mathcal{C}_{\mathfrak{a}, \mathfrak{b}}} \psi(y, n, m, c) S_{\chi, \mathfrak{a}, \mathfrak{b}}(n, m, c) \right) \overline{\varphi(y)} e^{-4\pi my} \frac{dy}{y^2}.$$

Interchanging the integral and the remaining sum by the dominated convergence theorem again, we arrive at

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), P_{m,\chi,\mathfrak{b}}(\cdot, \varphi) \rangle = \delta_{\mathfrak{a}, \mathfrak{b}} \delta_{n,m} (\psi, \varphi)_{n,m} + \sum_{c \in \mathcal{C}_{\mathfrak{a}, \mathfrak{b}}} S_{\chi, \mathfrak{a}, \mathfrak{b}}(n, m, c) V(n, m, c, \psi, \varphi),$$

where we have set

$$(\psi, \varphi)_{n,m} = \frac{1}{V_{\Gamma_0(N)}} \int_0^\infty \psi(y) \overline{\varphi(y)} e^{-2\pi(n+m)y} \frac{dy}{y^2},$$

and

$$V(n, m, c; \psi, \varphi) = \frac{1}{V_{\Gamma_0(N)}} \int_0^\infty \int_{\text{Im}(z)=y} \psi\left(\frac{y}{|cz|^2}\right) \overline{\varphi(y)} e^{-\frac{2\pi im}{c^2 z} - 2\pi inz - 4\pi my} \frac{dz dy}{y^2}.$$

This is the first half of the Kuznetsov trace formula. For the second half, ?? gives

$$P_{n,\chi,\mathfrak{a}}(\cdot, \psi) = \sum_{j \geq 0} \langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), u_j \rangle u_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^\infty \left\langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + ir\right) \right\rangle E_{\mathfrak{a}}\left(z, \frac{1}{2} + ir\right) dr,$$

and

$$P_{m,\chi,\mathfrak{a}}(\cdot, \varphi) = \sum_{j \geq 0} \langle P_{m,\chi,\mathfrak{a}}(\cdot, \varphi), u_j \rangle u_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^\infty \left\langle P_{m,\chi,\mathfrak{a}}(\cdot, \varphi), E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + ir\right) \right\rangle E_{\mathfrak{a}}\left(z, \frac{1}{2} + ir\right) dr.$$

By orthonormality, it follows that

$$\begin{aligned} \langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), P_{m,\chi,\mathfrak{a}}(\cdot, \varphi) \rangle &= \sum_j \langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), u_j \rangle \overline{\langle P_{m,\chi,\mathfrak{a}}(\cdot, \varphi), u_j \rangle} \\ &\quad + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^\infty \left\langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + ir\right) \right\rangle \overline{\left\langle P_{m,\chi,\mathfrak{a}}(\cdot, \varphi), E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + ir\right) \right\rangle} dr. \end{aligned}$$

Now we must simplify the remaining inner products. Let  $f \in \mathcal{L}(N, \chi)$  with Fourier series

$$f(z) = a^+(0) y^{\frac{1}{2}+\nu} + a^-(0) y^{\frac{1}{2}-\nu} + \sum_{n \neq 0} a(n) \sqrt{y} K_\nu(2\pi|n|y) e^{2\pi inx}.$$

By unfolding the integral in the Petersson inner product and cutting off everything except the diagonal using ?? exactly as in the case for  $\langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), P_{m,\chi,\mathfrak{a}}(\cdot, \varphi) \rangle$ , we see that

$$\langle P_{n,\chi,\mathfrak{a}}(\cdot, \psi), f \rangle = \frac{1}{V_\Gamma} \int_0^\infty \overline{a(n) \sqrt{y} K_\nu(2\pi ny)} \psi(y) e^{-4\pi my} \frac{dy}{y^2}.$$

Now set

$$\omega_\nu(n, \psi) = \frac{1}{V_\Gamma} \int_0^\infty \sqrt{y} K_\nu(2\pi|n|y) \overline{\psi(y)} e^{-4\pi my} \frac{dy}{y^2}.$$

Then it follows from the Fourier series of cusp forms and Eisenstein series that

$$\langle P_{n,\chi,a}(\cdot, \psi), u_j \rangle = \overline{a_j(n)\omega_{\nu_j}(n, \psi)},$$

for  $j \geq 1$  and

$$\left\langle P_{n,\chi,a}(\cdot, \psi), E_a\left(\cdot, \frac{1}{2} + ir\right) \right\rangle = \overline{\tau_a\left(n, \frac{1}{2} + ir\right) \omega_{ir}(n, \psi)}.$$

In particular,  $\langle P_{n,\chi,a}(\cdot, \psi), u_0 \rangle = 0$ . So we obtain

$$\begin{aligned} \langle P_{n,\chi,a}(\cdot, \psi), P_{m,\chi,a}(\cdot, \varphi) \rangle &= \sum_{j \geq 1} \overline{a_j(n)} a_j(m) \overline{\omega(n, \psi)} \omega(m, \varphi) \\ &\quad + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^\infty \overline{\tau_a\left(n, \frac{1}{2} + ir\right)} \tau_a\left(m, \frac{1}{2} + ir\right) \overline{\omega(n, \psi)} \omega(m, \varphi) dr. \end{aligned}$$

This is the second half of the Kuznetsov trace formula. Equating the first and second halves we get the **Kuznetsov trace formula**:

$$\begin{aligned} \delta_{n,m}(\psi, \varphi) + \sum_{\substack{c \geq 1 \\ c \equiv 0 \pmod{N}}} \frac{1}{c} S_\chi(n, m, c) V(n, m, c, \psi, \varphi) &= \sum_{j \geq 1} \overline{a_j(n)} a_j(m) \overline{\omega(n, \psi)} \omega(m, \varphi) \\ &\quad + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^\infty \overline{\tau_a\left(n, \frac{1}{2} + ir\right)} \tau_a\left(m, \frac{1}{2} + ir\right) \overline{\omega(n, \psi)} \omega(m, \varphi) dr. \end{aligned}$$

The left-hand side is called the **geometric side** and the right-hand side is called the **spectral side**. We collect our work as a theorem:

**Theorem 0.1.1 (Kuznetsov trace formula).** *Let  $\{u_j\}_{j \geq 1}$  be an orthonormal basis of Hecke-Maass eigenforms for  $\mathcal{L}(N, \chi)$  of types  $\nu_j$  with Fourier coefficients  $a_j(n)$ . Then for any positive integers  $n, m \geq 1$ , we have*

$$\begin{aligned} \delta_{n,m}(\psi, \varphi) + \sum_{\substack{c \geq 1 \\ c \equiv 0 \pmod{N}}} \frac{1}{c} S_\chi(n, m, c) V(n, m, c, \psi, \varphi) &= \sum_{j \geq 1} \overline{a_j(n)} a_j(m) \overline{\omega(n, \psi)} \omega(m, \varphi) \\ &\quad + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^\infty \overline{\tau_a\left(n, \frac{1}{2} + ir\right)} \tau_a\left(m, \frac{1}{2} + ir\right) \overline{\omega(n, \psi)} \omega(m, \varphi) dr. \end{aligned}$$

## 0.2 The Ideal Norm

Let us now prove some properties about the ideal norm. We first show that it respects localization:

**Proposition 0.2.1.** *Let  $\mathcal{O}/\mathcal{o}$  be a Dedekind extension of separable extension  $L/K$  and let  $D \subseteq \mathcal{o} - \{0\}$  be a multiplicative subset. Then for any fractional ideal  $\mathfrak{F}$  of  $\mathcal{O}$ , we have*

$$N_{\mathcal{O}D^{-1}/\mathcal{o}D^{-1}}(\mathfrak{F}D^{-1}) = N_{\mathcal{O}/\mathcal{o}}(\mathfrak{F})D^{-1}.$$

*Proof.* Since the ideal norm is multiplicative, it suffices to prove the claim in the case of a prime  $\mathfrak{P}$  of  $\mathcal{O}$ . Then we must show

$$N_{\mathcal{O}D^{-1}/\mathcal{O}D^{-1}}(\mathfrak{P}D^{-1}) = N_{\mathcal{O}/\mathcal{O}}(\mathfrak{P})D^{-1}.$$

This is immediate from ?? and the definition of the ideal norm.  $\square$

The ideal norm is also compatible with the field trace:

**Proposition 0.2.2.** *Let  $\mathcal{O}/\mathcal{O}$  be a Dedekind extension of degree  $n$  separable extension  $L/K$ . Then for any  $\lambda \in \mathcal{O}$ , we have*

$$N_{\mathcal{O}/\mathcal{O}}(\lambda\mathcal{O}) = N_{L/K}(\lambda)\mathcal{O}.$$

*Proof.* In light of Proposition 0.2.1, it suffices to assume  $\mathcal{O}/\mathcal{O}$  is a local Dedekind extension. Therefore  $\mathcal{O}$  is a discrete valuation ring,  $\mathcal{O}$  is a principal ideal domain, and  $\mathcal{O}/\mathcal{O}$  admits an integral basis  $\alpha_1, \dots, \alpha_n$  making  $\mathcal{O}$  a free  $\mathcal{O}$ -module of rank  $n$ . Let  $\mathfrak{p}$  be the unique prime of  $\mathcal{O}$  and  $\pi$  be a uniformizer so that  $\mathfrak{p} = \pi\mathcal{O}$ . Since the ideal norm and the field norm are both multiplicative and  $\mathcal{O}$  and  $\mathcal{O}$  are both unique factorization domains, we may assume that  $\lambda$  is prime. Then  $\lambda\mathcal{O} = \mathfrak{P}$  for some prime  $\mathfrak{P}$  of  $\mathcal{O}$ . So on the one hand,

$$N_{\mathcal{O}/\mathcal{O}}(\lambda\mathcal{O}) = \mathfrak{p}^{f_{\mathfrak{p}}(\mathfrak{P})}.$$

As  $\mathcal{O}$  is a discrete valuation ring, we have the prime factorization  $N_{L/K}(\lambda) = \mu\pi^f$ . So on the other hand,

$$N_{L/K}(\lambda)\mathcal{O} = \mathfrak{p}^f.$$

It now suffices to show that  $f = f_{\mathfrak{p}}(\mathfrak{P})$ . **Todo:** [xxx]  $\square$

The different and discriminant and related to each other via the ideal norm. In particular, the ideal norm of the different is the discriminant:

**Proposition 0.2.3.** *Let  $\mathcal{O}/\mathcal{O}$  be a Dedekind extension of a degree  $n$  separable extension  $L/K$ . Then*

$$\mathfrak{d}_{\mathcal{O}/\mathcal{O}} = N_{\mathcal{O}/\mathcal{O}}(\mathfrak{D}_{\mathcal{O}/\mathcal{O}}).$$

*Proof.* In view of ??, we may assume  $\mathcal{O}/\mathcal{O}$  is a local Dedekind extension. Therefore  $\mathcal{O}$  is a discrete valuation ring,  $\mathcal{O}$  is a principal ideal domain, and  $\mathcal{O}/\mathcal{O}$  admits an integral basis  $\alpha_1, \dots, \alpha_n$  making  $\mathcal{O}$  a free  $\mathcal{O}$ -module of rank  $n$ . Then  $\mathfrak{d}_{\mathcal{O}/\mathcal{O}}$  is a principal integral ideal where

$$\mathfrak{d}_{\mathcal{O}/\mathcal{O}} = d_{\mathcal{O}}(\mathcal{O})\mathcal{O}.$$

As  $\mathcal{O}$  is a principal ideal domain, every fractional ideal is also principal. So on the one hand,  $\mathfrak{C}_{\mathcal{O}/\mathcal{O}} = \lambda\mathcal{O}$  for some nonzero  $\lambda \in L$  and  $\lambda\alpha_1, \dots, \lambda\alpha_n$  is a basis of  $L/K$  contained in  $\mathfrak{C}_{\mathcal{O}/\mathcal{O}}$ . Moreover,

$$d_{L/K}(\lambda\alpha_1, \dots, \lambda\alpha_n) = N_{L/K}(\lambda)^2 d_{L/K}(\alpha_1, \dots, \alpha_n),$$

by ?? and that base change matrix from  $\alpha_1, \dots, \alpha_n$  to  $\lambda\alpha_1, \dots, \lambda\alpha_n$  is the multiplication by  $\lambda$  map. **Todo:** [xxx]  $\square$

## 0.3 Moments

Our first result will only require the Euler-Maclaurin summation formula. However, we first need the following lemma:

**Lemma 0.3.1.** *For any  $X > 0$  and fixed  $\delta > 0$ , we have*

$$\zeta(s) = \sum_{n \leq X} \frac{1}{n^s} - \frac{X^{1-s}}{1-s} + O\left(\frac{1}{X^\sigma}\right),$$

provided  $\delta < \sigma < 1$  and  $|t| \leq X$ .

*Proof.* Let  $N$  be an integer such that  $N > X$ , take  $\sigma > 1$ , and consider

$$\zeta(s) - \sum_{n \leq N} \frac{1}{n^s} = \sum_{n > N} \frac{1}{n^s}.$$

As  $A(M) \leq M$  and  $\sigma > 1$ , we have  $A(M)M^{-s} \rightarrow 0$  as  $M \rightarrow \infty$ . So ?? implies that we can rewrite the sum on the right-hand side as

$$\sum_{n > N} \frac{1}{n^s} = \frac{N^{-s}}{2} + \int_N^\infty u^{-s} + \left(u - \lfloor u \rfloor - \frac{1}{2}\right) - su^{-(s+1)} du.$$

Since

$$\int_N^\infty u^{-s} du = -\frac{N^{1-s}}{1-s},$$

we further have

$$\sum_{n > N} \frac{1}{n^s} = \frac{N^{-s}}{2} - \frac{N^{1-s}}{1-s} - s \int_N^\infty \left(u - \lfloor u \rfloor - \frac{1}{2}\right) u^{-(s+1)} du.$$

Therefore we arrive at

$$\zeta(s) = \sum_{n \leq N} \frac{1}{n^s} + \frac{N^{-s}}{2} - \frac{N^{1-s}}{1-s} - s \int_N^\infty \left(u - \lfloor u \rfloor - \frac{1}{2}\right) u^{-(s+1)} du, \quad (1)$$

for  $\sigma > 1$ . In fact, the right-hand side of Equation (1) is meromorphic for  $\sigma > 0$ . Indeed, as the first three terms are with the third term having a simple pole at  $s = 1$ , so it suffices to show that the integral is as well. To do so we will show that the integral is locally absolutely uniformly convergent in this region. Indeed, let  $K$  is a compact subset in this region and set  $\alpha = \min_{s \in K}(\sigma)$  and  $\beta = \max_{s \in K}(|s|)$ . Then we have to show that the integral is absolutely uniformly convergent on  $K$ . As  $\lfloor u \rfloor - u - \frac{1}{2} \ll 1$ , we have

$$s \int_N^\infty \left(u - \lfloor u \rfloor - \frac{1}{2}\right) u^{-(s+1)} du \ll \beta \int_x^\infty \frac{1}{u^{\alpha+1}} du = \frac{\beta}{\alpha x^\alpha} \ll_{\alpha, \beta} 1,$$

as desired. Therefore Equation (1) for  $\sigma > 0$ . Now let  $\delta < \sigma < 1$  and  $|t| \leq X$ . Then  $|s|$  is absolutely bounded and  $\sigma$  is bounded away from zero. From the analogous estimate

$$s \int_N^\infty \left(u - \lfloor u \rfloor - \frac{1}{2}\right) u^{-(s+1)} du = O\left(|s| \int_N^\infty \frac{1}{u^{\sigma+1}} du\right) = O\left(\frac{|s|}{N^\sigma}\right),$$

and Equation (1), we get

$$\zeta(s) = \sum_{n \leq N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + O\left(\frac{|s|+1}{N^\sigma}\right). \quad (2)$$

We will refine Equation (2) by truncating the remaining sum. Write

$$\sum_{n \leq N} \frac{1}{n^s} = \sum_{n \leq X} \frac{1}{n^s} + \sum_{X < n \leq N} \frac{1}{n^s}.$$

Applying the Euler-Maclaurin summation formula to the last sum gives

$$\begin{aligned} \sum_{X < n \leq N} \frac{1}{n^s} &= \left(X - \lfloor X \rfloor - \frac{1}{2}\right) X^{-s} + \frac{1}{2} N^{-s} \\ &\quad + \int_X^N u^{-s} du - s \int_X^N \left(u - \lfloor u \rfloor - \frac{1}{2}\right) u^{-(s+1)} du. \end{aligned} \quad (3)$$

Now observe that

$$\int_X^N u^{-s} du = \frac{N^{1-s}}{1-s} - \frac{X^{1-s}}{1-s}, \quad (4)$$

and

$$s \int_N^X \left(u - \lfloor u \rfloor - \frac{1}{2}\right) u^{-(s+1)} du = O\left(|s| \int_N^X \frac{1}{u^{\sigma+1}} du\right) = O\left(\frac{|s|}{X^\sigma}\right) + O\left(\frac{|s|}{N^\sigma}\right), \quad (5)$$

Inserting the estimates in Equations (4) and (5) into Equation (3) yields

$$\sum_{X < n \leq N} \frac{1}{n^s} = \frac{N^{1-s}}{1-s} - \frac{X^{1-s}}{1-s} + O\left(\frac{|s|+1}{N^\sigma}\right) + O\left(\frac{|s|+1}{X^\sigma}\right), \quad (6)$$

where we have also used the fact that  $(X - \lfloor X \rfloor - \frac{1}{2}) X^{-s} O = (\frac{1}{X^\sigma})$ . Upon combining Equations (3) and (6), we have

$$\zeta(s) = \sum_{n \leq X} \frac{1}{n^s} - \frac{X^{1-s}}{1-s} + O\left(\frac{|s|+1}{N^\sigma}\right) + O\left(\frac{|s|+1}{X^\sigma}\right).$$

Taking the limit as  $N \rightarrow \infty$  completes the proof. **Todo: [this proof needs correction]** □

Lemma 0.3.1 can be thought of as a weak approximate functional equation for the Riemann zeta function. Our first result bounds the growth rate of  $M_2(T, \zeta)$  only using Lemma 0.3.1:

**Theorem 0.3.1.** *For  $T > 2$ ,*

$$M_2(T, \zeta) = O(T \log(T)).$$

*Proof.* Taking  $s = \frac{1}{2} + it$  and  $X = t$  for  $t \geq 2$  in Lemma 0.3.1 gives

$$\zeta\left(\frac{1}{2} + it\right) = \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} - \frac{t^{\frac{1}{2}-it}}{\frac{1}{2}-it} + O\left(\frac{1}{\sqrt{t}}\right).$$

This implies the weaker estimate

$$\zeta\left(\frac{1}{2} + it\right) = \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} + O\left(\frac{1}{\sqrt{t}}\right). \quad (7)$$

Applying Equation (7) to the definition of  $M_2(T, \zeta)$  and recalling that  $\zeta\left(\frac{1}{2} + it\right)$  is absolutely bounded for  $0 \leq t \leq 2$ , yields

$$\begin{aligned} M_2(T, \zeta) &= \int_2^T \left| \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} \right|^2 dt \\ &\quad + O\left( \int_2^T \left| \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} \right| \frac{1}{\sqrt{t}} dt \right) + O\left( \int_2^T \frac{1}{t} dt \right). \end{aligned} \tag{8}$$

We will now simplify Equation (8). For the second term, the Cauchy-Schwarz inequality gives

$$\begin{aligned} \int_2^T \left| \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} \right| \frac{1}{\sqrt{t}} dt &= O\left( \left( \int_2^T \left| \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} \right|^2 dt \int_2^T \frac{1}{t} dt \right)^{\frac{1}{2}} \right) \\ &= O\left( \left( \int_2^T \left| \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} \right|^2 dt \right)^{\frac{1}{2}} \sqrt{\log(T)} \right), \end{aligned}$$

where in the second line we have made use of the estimate

$$\int_2^T \frac{1}{t} dt = O(\log(T)).$$

Applying this latter estimate to third term as well, Equation (8) becomes

$$\begin{aligned} M_2(T, \zeta) &= \int_2^T \left| \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} \right|^2 dt \\ &\quad + O\left( \left( \int_2^T \left| \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} \right|^2 dt \right)^{\frac{1}{2}} \sqrt{\log(T)} \right) + O(\log(T)). \end{aligned} \tag{9}$$

Therefore, it suffices to show that the remaining integral is  $O(T \log(T))$ . We first expand the sum and interchange it with the integral by the Fubini-Tonelli theorem:

$$\begin{aligned} \int_2^T \left| \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} \right|^2 dt &= \int_2^T \sum_{n,m \leq t} \frac{1}{n^{\frac{1}{2}+it} m^{\frac{1}{2}-it}} dt \\ &= \sum_{n,m \leq T} \int_{\max(n,m,2)}^T \frac{1}{n^{\frac{1}{2}+it} m^{\frac{1}{2}-it}} dt \\ &= \sum_{n \leq T} \frac{T-n}{n} + \sum_{\substack{n,m \leq T \\ n \neq m}} \frac{1}{\sqrt{nm}} \int_{\max(n,m,2)}^T \left(\frac{m}{n}\right)^{it} dt \\ &= T \sum_{n \leq T} \frac{1}{n} + O\left( \sum_{n \leq T} 1 \right) + \left( \sum_{\substack{n,m \leq T \\ n \neq m}} \frac{1}{\sqrt{nm} \log\left(\frac{m}{n}\right)} \right), \end{aligned} \tag{10}$$

where in the second to last line we have separated the terms for which  $n = m$  or not. We now estimate all of the remaining terms in Equation (10). For the first term, by ?? we have

$$T \sum_{n \leq T} \frac{1}{n} = T \log(T) + O(T) = O(T \log(T)). \quad (11)$$

The second term is easier since

$$\sum_{n \leq T} 1 = O(T) = O(T \log(T)). \quad (12)$$

For the last term, separate the sum into the terms for which  $n < \frac{m}{2}$  or not, so that

$$\sum_{\substack{n,m \leq T \\ n \neq m}} \frac{1}{\sqrt{nm} \log\left(\frac{m}{n}\right)} = \sum_{\substack{n,m \leq T \\ n < \frac{m}{2} \\ n \neq m}} \frac{1}{\sqrt{nm} \log\left(\frac{m}{n}\right)} + \sum_{\substack{n,m \leq T \\ n \geq \frac{m}{2} \\ n \neq m}} \frac{1}{\sqrt{nm} \log\left(\frac{m}{n}\right)}. \quad (13)$$

In the first sum on the right-hand side of Equation (13) we have  $\log\left(\frac{m}{n}\right) \geq \log(2)$  so that  $\log\left(\frac{m}{n}\right)$  is bounded from below. Hence

$$\sum_{\substack{n,m \leq T \\ n < \frac{m}{2}}} \frac{1}{\sqrt{nm} \log\left(\frac{m}{n}\right)} = O\left(\sum_{\substack{n,m \leq T \\ n < \frac{m}{2}}} \frac{1}{\sqrt{nm}}\right) = O\left(\left(\sum_{n \leq T} \frac{1}{\sqrt{n}}\right)^2\right).$$

But as

$$\sum_{n \leq T} \frac{1}{\sqrt{n}} = O\left(\int_1^T \frac{1}{\sqrt{x}} dx\right) = O(\sqrt{T}),$$

we find that

$$\sum_{\substack{n,m \leq T \\ n < \frac{m}{2}}} \frac{1}{\sqrt{nm} \log\left(\frac{m}{n}\right)} = O(T) = O(T \log(T)). \quad (14)$$

For the second sum, write  $n = m - r$  where  $1 \leq r \leq \frac{m}{2}$  so that  $\log\left(\frac{m}{n}\right) = \log\left(\frac{m}{m-r}\right) = -\log(1 - \frac{r}{m}) \geq \frac{r}{m}$  where the inequality follows from the Taylor series of the logarithm. Whence

$$\sum_{\substack{n,m \leq T \\ n \geq \frac{m}{2} \\ n \neq m}} \frac{1}{\sqrt{nm} \log\left(\frac{m}{n}\right)} = O\left(\sum_{m \leq T} \sum_{r \leq \frac{m}{2}} \frac{m}{r \sqrt{m(m-r)}}\right) = O\left(\sum_{m \leq T} \sum_{r \leq \frac{m}{2}} \frac{1}{r}\right).$$

To estimate the double sum, using ?? again, we have

$$\sum_{r \leq \frac{m}{2}} \frac{1}{r} = \log(m) + O(1).$$

This estimate together with

$$\sum_{m \leq T} \log(m) + O(1) = T \log(T) + O(T) = O(T \log(T)),$$

gives

$$\sum_{\substack{n,m \leq T \\ n \geq \frac{m}{2} \\ n \neq m}} \frac{1}{\sqrt{nm} \log\left(\frac{m}{n}\right)} = O(T \log(T)). \quad (15)$$

Substituting Equations (14) and (15) into Equation (13) yields

$$\sum_{\substack{n,m \leq T \\ n \neq m}} \frac{1}{\sqrt{nm} \log\left(\frac{m}{n}\right)} = O(T \log(T)). \quad (16)$$

Then Equations (11), (12) and (16) together with Equation (10) yields

$$\int_2^T \left| \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} \right|^2 = O(T \log(T)). \quad (17)$$

Applying Equation (17) to Equation (9), we at last obtain

$$M_2(T, \zeta) = O(T \log(T)) + O(\sqrt{T} \log(T)) + O(T) = O(T \log(T)),$$

as desired.  $\square$

## 0.4 Misc.

(ii) The gamma factor  $\gamma(s, f \otimes g)$  takes the form

$$\gamma(s, f \otimes g) = \pi^{-\frac{d_f \otimes g s}{2}} \prod_{\substack{1 \leq j \leq d_f \\ 1 \leq \ell \leq d_g}} \Gamma\left(\frac{s + \mu_{j,\ell}}{2}\right),$$

with the local roots at infinity satisfying the additional bounds  $\operatorname{Re}(\mu_{j,\ell}) \leq \operatorname{Re}(\kappa_j) + \operatorname{Re}(\nu_\ell)$  and  $|\mu_{j,\ell}| \leq |\kappa_j| + |\nu_\ell|$ .

- (iii) The root number  $q(f \otimes g)$  satisfies  $q(f \otimes g) \mid q(f)^{d_f} q(g)^{d_g}$ . If  $q(f \otimes g)$  is a proper divisor of  $q(f)^{d_f} q(g)^{d_g}$ , we say that  $L(s, f \otimes g)$  exhibits **conductor dropping**.
- (v)  $L(s, f \otimes g)$  has a pole of order  $r_{f \otimes g} \geq 1$  at  $s = 1$  if  $f = g$ .