A QUADRATIC DOUBLE DIRICHLET SERIES I: THE FUNCTION FIELD CASE

HENRY TWISS

Abstract. We construct a quadratic double Dirichlet series $Z(s, w)$ built from single variable quadratic Dirichlet L-functions $L(s, \chi)$ attached to the functional field $\mathbb{F}_q(t)$. We prove that $Z(s, w)$ admits meromorphic continuation to the (s, w)-plane and satisfies a group of functional equations. We also outline an application of the usefulness of $Z(s, w)$ and show that it is a rational function in $x = q^{-s}$ and $y = q^{-w}$. This is the simplest construction of a quadratic Weyl group multiple Dirichlet series over a global field.

1. Preliminaries

We will give an overview of the zeta function and quadratic Dirichlet L-functions attached to $\mathbb{F}_q(t)$. For proofs of these facts and a more detailed analysis see [2]. Let q be a power of an odd prime and let $\mathbb{F}_q[t]$ be the polynomial ring in t with coefficients in the finite field $\mathbb{F}_q$. This is a principal ideal domain. Moreover, the nonzero prime ideals in $\mathbb{F}_q[t]$ are generated by irreducible polynomials. Let $\mathbb{F}_q(t)$ denote the quotient field. Define the norm function $N(m)$ by

$$N(m) = |m| = q^{\deg(m)},$$

for any $m \in \mathbb{F}_q[t]$. The zeta function $\zeta(s)$ on $\mathbb{F}_q[t]$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m \text{ monic}} \frac{1}{|m|^s} = \prod_{P \text{ monic irr}} \left(1 - \frac{1}{|P|^s}\right)^{-1},$$

where the second equality holds since $\mathbb{F}_q[t]$ is a unique factorization domain. As for questions of convergence, there are q^n monic polynomials of degree n so, provided $\mathrm{Re}(s) > 1$, we can sum up the Dirichlet series according to degree and obtain an explicit expression:

$$\zeta(s) = \sum_{n \geq 0} \frac{\#\text{ of monic poly of deg } n}{q^{ns}} = \sum_{n \geq 1} \frac{1}{q^{n(1-s)}} = \frac{1}{1 - q^{1-s}}.$$

The latter expression is meromorphic on $\mathbb{C}$ with a simple pole at $s = 1$ of residue $\frac{1}{\log(q)}$. Therefore $\zeta(s)$ admits meromorphic continuation to $\mathbb{C}$. The zeta function also satisfies a functional equation. Define the completed zeta function $\zeta^*(s)$ (this is also the zeta function attached to $\mathbb{F}_q(t)$) by

$$\zeta^*(s) = \frac{1}{1 - q^{-s}} \zeta(s).$$

Then

$$\zeta^*(s) = q^{2s-1} \zeta^*(1 - s).$$

Recall that characters on $\mathbb{F}_q[t]$ are multiplicative functions $\chi : \mathbb{F}_q[t] \to \mathbb{C}$. They form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions $\chi_d : \mathbb{F}_q[t] \to \mathbb{C}$ modulo $d \in \mathbb{F}_q[t]$ (in that they are d-periodic) and such that $\chi_d(m) = 0$ if $(m, d) > 1$.
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

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The image of a Dirichlet character always lands in the roots of unity. If χ is a Dirichlet character then its conjugate $\overline{\chi}$ is also a Dirichlet character. Moreover, $\overline{\chi}$ is the multiplicative inverse to χ and the Dirichlet characters modulo d form a group under multiplication. This group is always finite and its order is $\phi(d) = |(\mathbb{F}_q[t]/d\mathbb{F}_q[t])^\times|$. Dirichlet characters also satisfy orthogonality relations:

Theorem 1.1 (Orthogonality relations)**.**

(i) For any two Dirichlet characters χ and ψ modulo d,

$$\frac{1}{\phi(d)} \sideset{}{'}\sum_{f \pmod d} \chi(f)\overline{\psi}(f) = \delta_{\chi,\psi}.$$

(ii) For any $f, g \in (\mathbb{F}_q[t]/d\mathbb{F}_q[t])^\times$,

$$\frac{1}{\phi(d)} \sum_{\chi \pmod d} \chi(f)\overline{\chi}(g) = \delta_{f,g}.$$

The Dirichlet characters that are of interest to us are those attached to quadratic extensions of $\mathbb{F}_q[t]$ which, in turn, arise from the quadratic residue symbol. First let us recall this symbol. For any irreducible $p \in \mathbb{F}_q[t]$ and any $m \in \mathbb{F}_q[t]$, we define the quadratic residue symbol $\left(\frac{m}{p}\right)$ by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{|p|-1}{2}} \pmod p = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod p \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod p \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod p). \end{cases}$$

This symbol is only dependent upon m modulo p and is multiplicative in m. Moreover, if $b \in \mathbb{F}_q^*$, we have

$$\left(\frac{b}{p}\right) = \operatorname{sgn}(b)^{\deg(p)}.$$

where $\operatorname{sgn}(b) = \pm 1$ depending on if $b \in (\mathbb{F}_q^\times)^2$ or not. Moreover, for $m \in \mathbb{F}_q[t]$ we have $\operatorname{sgn}(m) = \operatorname{sgn}(b_n)$ if $m(t) = b_n t^n + b_{n-1} t^{n+1} + \cdots + b_0$ (with $b_n \neq 0$). We can extend the quadratic residue symbol multiplicatively in the denominator. If $d = bp_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of d (with $b \in \mathbb{F}_q^*$), then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \le i \le k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any nonzero $d \in \mathbb{F}_q[t]$. Morevoer, it only depends upon m modulo d, the ideal generated by d, and is multiplicative in d. The quadratic residue symbol also admits the following reciprocity law:

Theorem 1.2 (Quadratic reciprocity)**.** *If $d, m \in \mathbb{F}_q[t]$ are nonzero, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{q-1}{2}\deg(d)\deg(m)} \operatorname{sgn}(d)^{\deg(m)} \operatorname{sgn}(m)^{-\deg(d)} \left(\frac{m}{d}\right).$$

Note that if $q \equiv 1 \pmod 4$ and d and m are monic, then the sign in the statement of quadratic reciprocity is always 1 so that reciprocity is perfect:

$$\left(\frac{d}{m}\right) = \left(\frac{m}{d}\right).$$

We can now define the quadratic Dirichlet characters. For any nonzero square-free monic $d \in \mathbb{F}_q[t]$, define the quadratic Dirichlet character χ_d by the following quadratic residue symbol:

$$\chi_d(m) = \left(\frac{d}{m}\right).$$

This quadratic Dirichlet character is attached to the quadratic extension $\mathbb{F}_q(t)(\sqrt{d})$. If $b \in \mathbb{F}_q^\times$, we define χ_b by

$$\chi_b(m) = \left(\frac{b}{m}\right) = \operatorname{sgn}(b)^{\deg(m)}.$$

We extend χ_d multiplicatively in the denominator so that χ_d makes sense for any nonzero $d \in \mathbb{F}_q[t]$. In particular, $\chi_d(m) = \pm 1$ provided d and m are relatively prime and $\chi_d(m) = 0$ if $(m, d) > 1$. Quadratic reciprocity implies that χ_d is a Dirichlet character modulo d. Indeed, for any m we can take d modulo m so that $\deg(d + m) = \deg(m)$ and $\operatorname{sgn}(d + m) = \operatorname{sgn}(m)$. Lastly, note that if $q \equiv 1 \pmod 4$ and d and m are monic,

$$\chi_d(m) = \chi_m(d),$$

is a reformulation of reciprocity being perfect. We now discuss the Hilbert characters. We will only need two of them: one nontrivial and one trivial. The nontrivial Hilbert character is χ_θ where $\theta \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$:

$$\chi_\theta(m) = (-1)^{\deg(m)}.$$

Note that $\overline{\chi_\theta} = \chi_\theta$. The other Hilbert character is the trivial Dirichlet character $\chi_\theta^2 = \chi_{\theta\theta} = \chi_1$. In general, we denote a Hilbert character by χ_a where $a \in \{1, \theta\}$. More generally, we would also require Hilbert characters to keep track of the $(-1)^{\frac{q-1}{2}\deg(d)\deg(m)}$ factor in the statement of quadratic reciprocity but as we will assume $q \equiv 1 \pmod 4$, we will not require this additional difficulty. With the Dirichlet and Hilbert characters introduced, we are ready to discuss the L-functions associated to quadratic Dirichlet characters. We define the Dirichlet L-function $L(s, \chi_d)$ attached to χ_d for square-free d, by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \text{ monic}} \frac{\chi_d(m)}{|m|^s} = \prod_{P \text{ monic irr}} \left(1 - \frac{\chi_d(P)}{|P|^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character, $L(s, \chi_d) \ll \zeta(s)$ for $\operatorname{Re}(s) > 1$ so that $L(s, \chi_d)$ is locally absolutely uniformly convergent in this region. $L(s, \chi_d)$ also admits analytic continuation to $\mathbb{C}$ (see [2] for a proof). Moreover, $L(s, \chi_d)$ is a polynomial in q^{-s} of degree at most $\deg(d) - 1$. The associated completed Dirichlet L-function $L^*(s, \chi_d)$ is defined as

$$L^*(s, \chi_d) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_d) & \text{if } \deg(d) \text{ is even,} \\ L(s, \chi_d) & \text{if } \deg(d) \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_d) = \begin{cases} q^{2s-1}|d|^{\frac{1}{2}-s} L^*(1 - s, \chi_d) & \text{if } \deg(d) \text{ is even,} \\ q^{2s-1}(q|d|)^{\frac{1}{2}-s} L^*(1 - s, \chi_d) & \text{if } \deg(d) \text{ is odd.} \end{cases}$$

Note that in the case $\deg(d)$ is even, the conductor is $|d|$ and in the case $\deg(d)$ is odd, the conductor is $q|d|$. Moreover, the gamma factors depend upon the degree of d. This will cause a small but important technical issue later when we want to derive functional equations for the quadratic double Dirichlet series.

Remark 1.1. *The definitions for $L(s, \chi_d)$ and $L^*(s, \chi_d)$ hold even when d is not square-free (however the functional equations do not hold). We will make use of slightly modified defintions in this case and so we purposely do not define these L-functions yet.*

2. The Quadratic Double Dirichlet Series

From now on we assume $q \equiv 1 \pmod 4$. This assumption is not strictly necessary to build the quadratic double Dirichlet series but it does allow for some technical simplifications as the statement of quadratic reciprocity is perfect. We are ready to define the quadratic double Dirichlet series $Z(s, w)$. For any monic

$d \in \mathbb{F}_q[t]$, write $d = d_0 d_1^2$ where d_0 is square-free. In other words, d_0 is the square-free part of d so that $\frac{d}{d_0}$ is a perfect square. The **quadratic double Dirichlet series** $Z(s,w)$ is defined as

$$Z(s,w) = \sum_{d \text{ monic}} \frac{L(s,\chi_{d_0}) Q_{d_0 d_1^2}(s)}{|d|^w},$$

where $Q_{d_0 d_1^2}(s)$ is the **correction polynomial** defined by

$$Q_{d_0 d_1^2}(s) = \sum_{e_1 e_2 | d_1} \mu(e_1)\chi_{d_0}(e_1)|e_1|^{-s}|e_2|^{1-2s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{d_0}(e_1)|e_1|^{-s}|e_2|^{1-2s},$$

and μ is the usual Möbius function on $\mathbb{F}_q[t]$. For $\mathrm{Re}(s) > 1$, we have the trivial bound

$$Q_{d_0 d_1^2}(s) \ll \sum_{e_1 e_2 | d_1} 1 \ll \sigma_0(d_1)^2 \ll_\varepsilon |d_1|^{2\varepsilon} \ll_\varepsilon |d|^\varepsilon,$$

for any $\varepsilon > 0$. Combining this estimate with the bound $L(s,\chi_{d_0}) \ll 1$ for $\mathrm{Re}(s) > 1$, we see that $Z(s,w)$ is locally absolutely uniformly convergent in the region $\Lambda = \{(s,w) \in \mathbb{C}^2 : \mathrm{Re}(s) > 1, \mathrm{Re}(w) > 1\}$. While $Z(s,w)$ is the double Dirchlet series we are after, it will be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters χ_{a_1} and χ_{a_2}. The **quadratic double Dirichlet series** $Z_{a_1,a_2}(s,w)$ twisted by χ_{a_1} and χ_{a_2} is defined as

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ monic}} \frac{L(s,\chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s,\chi_{a_1})}{|d|^w},$$

where $Q_{d_0 d_1^2}(s,\chi_{a_1})$ is the **correction polynomial** twisted by χ_{a_1} defined by

$$Q_{d_0 d_1^2}(s,\chi_{a_1}) = \sum_{e_1 e_2 | d_1} \mu(e_1)\chi_{a_1 d_0}(e_1)|e_1|^{-s}|e_2|^{1-2s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{a_1 d_0}(e_1)|e_1|^{-s}|e_2|^{1-2s},$$

and μ is the usual Möbius function on $\mathbb{F}_q[t]$. As the Hilbert characters are given by quadratic Dirichlet characters, we have the analogous bound $Q_{d_0 d_1^2}(s,\chi_{a_1}) \ll |d|_\varepsilon$ so that $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly in the same region as $Z(s,w)$ does. In this generalized setup, $Z(s,w) = Z_{1,1}(s,w)$.

3. The Interchange

Since L-functions attached to quadratic Dirichlet characters admit Euler products, $Z_{a_1,a_2}(s,w)$ is a sum of Euler products in s. We will now argue that $Z_{a_1,a_2}(s,w)$ can be written as a sum of Euler products in w. In effect, we want the variable s to appear in the denominator of $Z_{a_1,a_2}(s,w)$ and the L-functions in the numerator to be in the variable w. To be precise, we will prove the following:

Theorem 3.1 (Interchange)**.** *Wherever $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly,*

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ monic}} \frac{L(s,\chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s,\chi_{a_1})}{|d|^w} = \sum_{m \text{ monic}} \frac{L(w,\chi_{a_2 m_0})\chi_{a_1}(m) Q_{m_0 m_1^2}(w,\chi_{a_2})}{|m|^s},$$

Proof. Only the second equality needs to be justified since the first is the definition of $Z(s,w)$. Expanding the L-function $L(s,\chi_{a_1d_0})$ and polynomial $Q_{d_0d_1^2}(s,\chi_{a_1})$ gives

$$\begin{aligned} Z(s,w) &= \sum_{d \text{ monic}} \frac{L(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{|d|^w} \\ &= \sum_{d \text{ monic}} \left(\sum_{m \text{ monic}} \chi_{a_1d_0}(m)|m|^{-s}\right)\left(\sum_{e_1e_2|d_1} \mu(e_1)\chi_{a_1d_0}(e_1)|e_1|^{-s}|e_2|^{1-2s}\right)\chi_{a_2}(d)|d|^{-w} \\ &= \sum_{m,d \text{ monic}} \sum_{e_1e_2|d_1} \mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(me_1)|e_1|^{-s}|e_2|^{1-2s}|m|^{-s}|d|^{-w}. \end{aligned}$$

Now $\chi_{a_1d_0}(me_1) = 0$ unless $(d_0, me_1) = 1$. So we may make this restriction on the sum giving

$$\sum_{m,d \text{ monic}} \sum_{\substack{e_1e_2|d_1 \\ (d_0,me_1)=1}} \mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(me_1)|e_1|^{-s}|e_2|^{1-2s}|m|^{-s}|d|^{-w}.$$

Making the change of variables $me_1 \to m$ yields

$$\sum_{d \text{ monic}} \sum_{\substack{m \text{ monic} \\ e_1|m}} \sum_{\substack{e_1e_2|d_1 \\ (d_0,m)=1}} \mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(m)|e_2|^{1-2s}|m|^{-s}|d|^{-w}.$$

Now for fixed $d = d_0d_1^2$ and e_2, the resulting subsum over m and e_1 is

$$\sum_{\substack{m \text{ monic} \\ e_1|m}} \sum_{\substack{e_1|\frac{d_1}{e_2} \\ (d_0,m)=1}} \mu(e_1)\chi_{a_1d_0}(m)|m|^{-s} = \sum_{\substack{m \text{ monic} \\ (d_0,m)=1}} \chi_{a_1d_0}(m)|m|^{-s}\left(\sum_{e_1|\left(\frac{d_1}{e_2},m\right)} \mu(e_1)\right).$$

The inner sum over e_1 of the Möbius function vanishes unless $\left(\frac{d_1}{e_2}, m\right) = 1$ in which case it is 1. Therefore the triple sum above becomes

$$\sum_{m,d \text{ monic}} \sum_{\substack{e_2|d_1 \\ \left(\frac{d_0d_1}{e_2},m\right)=1}} \chi_{a_2}(d)\chi_{a_1d_0}(m)|e_2|^{1-2s}|m|^{-s}|d|^{-w}.$$

Making the change of variables $d \to de_2^2$, the condition $\left(\frac{d_0d_1}{e_2}, m\right) = 1$ becomes $(d_0d_1, m) = 1$ which is equivalent to $(d,m) = 1$. Moreover, $\chi_{a_2}(de_2^2) = \chi_{a_2}(d)$. Altogether, the previous double sum becomes

$$\sum_{\substack{m,d \text{ monic} \\ (d,m)=1}} \sum_{e_2} \chi_{a_2}(d)\chi_{a_1d_0}(m)|e_2|^{1-2s-2w}|m|^{-s}|d|^{-w}.$$

Writing $m = m_0m_1^2$ analogously as for d, quadratic reciprocity implies $\chi_{d_0}(m) = \chi_m(d_0) = \chi_{m_0}(d)$ where the last equality holds because $(d,m) = 1$ and both d_0 and m_0 differ from d and m respectively by perfect squares. Therefore $\chi_{a_2}(d)\chi_{a_1d_0}(m) = \chi_{a_1}(m)\chi_{a_2m_0}(d)$ and our expression becomes

$$\sum_{\substack{m,d \text{ monic} \\ (d,m)=1}} \sum_{e_2} \chi_{a_1}(m)\chi_{a_2m_0}(d)|e_2|^{1-2s-2w}|m|^{-s}|d|^{-w}.$$

But now we can reverse the argument with the roles of d, m, χ_{a_1}, and χ_{a_2} interchanged respectively to obtain

$$Z(s,w)=\sum_{m \text{ monic}} \frac{L(w,\chi_{a_2 m_0})\chi_{a_1}(m) Q_{m_0 m_1^2}(w,\chi_{a_2})}{|m|^s}.$$

□

The symmetry in the interchange is not typical of a larger reality. In more general setting the argument is more complicated than the proof presented. One does not usually arrive at such a nice symmetric expression because the correction polynomials in w need not be equal to those in s. In other words, when $Z(s,w)$ is represented as a sum of L-functions in s the corretion polynomials are $Q_{d_0 d_1^2}(s,\chi_{a_1})$, but when $Z(s,w)$ is represented as a sum of L-functions in w the correction polynomials take the form $P_{m_0 m_1^2}(w,\chi_{a_2})$. In our setting, it is a special case that these polnomials are the same when $d=m$ and $s=w$.

Remark 3.1. *When $a_1=a_2=1$, the interchange implies that $Z(s,w)$ is symmetric in s and w. That is,*

$$Z(s,w)=Z(w,s).$$

Moreover, the interchange implies the following relations for twisted quadratic double Dirichlet series:

$$Z_{1,\theta}(s,w)=Z_{\theta,1}(w,s) \quad \textit{and} \quad Z_{\theta,\theta}(s,w)=Z_{\theta,\theta}(w,s).$$

4. Weighting Terms

We will begin investigating the coefficients of $Z_{a_1,a_2}(s,w)$ further. Expanding $L(s,\chi_{a_1 d_0})Q_{d_0 d_1^2}(s,\chi_{a_1})$ in the numerator of $Z_{a_1,a_2}(s,w)$, we obtain

$$Z_{a_1,a_2}(s,w)=\sum_{d \text{ monic}} \frac{L(s,\chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s,\chi_{a_1})}{|d|^w}=\sum_{m,\,d \text{ monic}} \frac{\chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) a(m,d)}{|m|^s |d|^w},$$

where $\widehat{m}$ is the part of m relatively prime to d_0 and the **weighting coefficient** $a(m,d)$ is given by

$$a(m,d)=\sum_{\substack{e_1 e_2^2 e_3=m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3)=1}} \mu(e_1)|e_2|.$$

Indeed, the coefficient of $|m|^{-s}|d|^{-w}$ in the definition of $Z_{a_1,a_2}(s,w)$ is

$$\begin{aligned}
\chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3=m \\ e_1 e_2 | d_1}} \mu(e_1)\chi_{a_1 d_0}(e_1 e_3)|e_2| &= \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3=m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3)=1}} \mu(e_1)\chi_{a_1 d_0}(e_1 e_3)|e_2| \\
&= \chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3=m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3)=1}} \mu(e_1)|e_2| \\
&= \chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) a(m,d),
\end{aligned}$$

where the first equality holds because $\chi_{d_0}(e_1 e_3)=0$ unless $(d_0,e_1 e_3)=1$ and the second equality holds because when $(d_0,e_1 e_3)=1$ we know that $\widehat{m}$ differs from $e_1 e_3$ by a perfect square (they differ by divisors of (d_0,e_2)) and so $\chi_{d_0}(e_1 e_3)=\chi_{d_0}(\widehat{m})$.

Remark 4.1. *Note that $a(m,d)=0$ unless there is a decomposition $m=e_1 e_2^2 e_3$ of m such that $(d_0,e_1 e_3)=1$ and $e_1 e_2^2 \mid d_1$.*

We will define $L(s,\chi_{a_1 d})$ to be the Dirichlet series given by

$$L(s,\chi_{a_1 d}) = L(s,\chi_{a_1 d_0}) Q_{d_0 d_1^2}(s,\chi_{a_1}) = \sum_{m \text{ monic}} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m,d)}{|m|^s}.$$

With this definition, $L(s,\chi_d)$ makes sense even when d is not square-free and agrees with the previous definition when d is square-free. Moreover, we can write

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ monic}} \frac{\chi_{a_2}(d) L(s,\chi_{a_1 d})}{|d|^w}.$$

Performing the same procedure but with the interchange, we have

$$Z_{a_1,a_2}(s,w) = \sum_{m \text{ monic}} \frac{L(w,\chi_{a_2 m_0})\chi_{a_1}(m) Q_{m_0 m_1^2}(w,\chi_{a_2})}{|m|^s} = \sum_{m,d \text{ monic}} \frac{\chi_{a_2 m_0}(\widehat{d})\chi_{a_1}(m) a(d,m)}{|m|^s |d|^w},$$

where $\widehat{d}$ is the part of d relatively prime to m_0. Again, we let $L(w,\chi_{a_2 m})$ be the Dirichlet series given by

$$L(w,\chi_{a_2 m}) = L(w,\chi_{a_2 m_0}) Q_{m_0 m_1^2}(w,\chi_{a_2}) = \sum_{d \text{ monic}} \frac{\chi_{a_2 m_0}(\widehat{d}) a(d,m)}{|d|^w},$$

so that

$$Z_{a_1,a_2}(s,w) = \sum_{m \text{ monic}} \frac{\chi_{a_1}(m) L(w,\chi_{a_2 m})}{|m|^s}.$$

We will now turn our attention to studing the weighting coefficients $a(m,d)$. Their structure controls much of the information about both the double Dirichlet series and the correction polynomials. We first show that the weighting coefficients posess a multiplicativity property:

Proposition 4.1. *We have $a(m,1) = a(1,d) = 1$ and*

$$a(m,d) = \prod_{\substack{P^\alpha || m \\ P^\beta || d}} a(P^\alpha, P^\beta).$$

Proof. It is clear that $a(m,1) = a(1,d) = 1$ from the definition of the weighting coefficients. We first prove multiplicativity in m. So letting $m = m' P^\alpha$, we have to show

$$a(m,d) = a(m',d) a(P^\alpha, d).$$

Indeed, for $e_1 e_2^2 e_3 = m$, let $e_1 = c_1 d_1$, $e_2 = c_2 d_2$, and $e_3 = c_3 d_3$ with $c_1, c_2, c_3 \mid m'$ and $d_1, d_2, d_3 \mid P^\alpha$. Because $(m', P^\alpha) = 1$, as $e_1 e_2^2 e_3$ runs over decompositions of m, $c_1 c_2^2 c_3$ and $d_1 d_2^2 d_3$ run over decompositions of m' and P^α respectively. Moreover, as $e_1 e_2$ runs over the divisors of d_1 so does $c_1 d_1 c_2 d_2$. Then using multiplicativity of the Möbius function, we have

$$\begin{aligned}
a(m,d) &= \sum_{\substack{e_1e_2^2e_3=m \\ e_1e_2|d_1 \\ (d_0,e_1e_3)=1}} \mu(e_1)|e_2| \\
&= \sum_{\substack{c_1c_2^2c_3=m' \\ d_1d_2^2d_3=P^\beta \\ c_1d_1c_2d_2|d_1 \\ (d_0,c_1d_1c_3d_3)=1}} \mu(c_1)(d_1)|c_2||d_2| \\
&= \left(\sum_{\substack{c_1c_2^2c_3=m' \\ c_1c_2|d_1 \\ (d_0,c_1c_3)=1}} \mu(c_1)|c_2|\right)\left(\sum_{\substack{d_1d_2^2d_3=P^\alpha \\ d_1d_2|d_1 \\ (d_0,d_1d_3)=1}} \mu(d_1)|d_2|\right) \\
&= a(m',d)a(P^\alpha,d),
\end{aligned}$$

as desired. Now we prove multiplicativity in d. Since we have already shown multiplicativity in m, we may assume that $m = P^\alpha$. Letting $d = d'P^\beta$, we have to show

$$a(P^\alpha, d) = a(P^\alpha, P^\beta).$$

As $e_1e_2^2e_3 = P^\alpha$, the e_i are powers of P for $1 \le i \le 3$. It follows that $e_1e_2 \mid d_1$ is equivalent to $e_1e_2 \mid P^\beta$. Moreover, $(d_0, e_1e_2) = 1$ is equivalent to $(1, e_1e_2) = 1$ or $(P, e_1e_2) = 1$ depending on of β is even or odd. These facts together imply the desired identity. □

There is a connection between the correction polynomials $Q_{d_0d_1^2}(s,\chi_{a_1})$ and weighting coefficients $a(m,d)$. It will turn out that $Q_{d_0d_1^2}(s,\chi_{a_1})$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients. We first establish this relationship when d is an odd irreducible power:

Lemma 4.1. *For any irreducible P and $\alpha \ge 1$, we have*

$$Q_{P^{2\alpha+1}}(s) = \sum_{k\le 2\alpha} \frac{a(P^k, P^{2\alpha+1})}{|P|^{ks}}.$$

Moreover, the same holds for $Q_{P^{2\alpha+1}}(w)$.

Proof. Expanding the correction polynomial in $|P|^{-s}$ yields

$$Q_{P^{2\alpha+1}}(s) = \sum_{e_1e_2|P^\alpha} \mu(e_1)\chi_P(e_1)|e_1|^{-s}|e_2|^{1-2s} = \sum_{k\le 2\alpha} \frac{b(P^k, P^{2\alpha+1})}{|P|^{ks}}.$$

where

$$b(P^k, P^{2\alpha+1}) = \sum_{e_1e_2^2=P^k} \mu(e_1)\chi_P(e_1)|e_2|.$$

The proof will be complete if we can show $b(P^k, P^{2\alpha+1}) = a(P^k, P^{2\alpha+1})$. To see this, first observe $\mu(e_1)\chi_P(e_1) = 0$ unless $e_1 = 1$ in which case it is 1. So $b(P^k, P^{2\alpha+1}) = 0$ if k is odd and $|P|^{\frac{k}{2}}$ if k is even. Compactly stated,

$$b(P^k, P^{2\alpha+1}) = \begin{cases} |P|^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

On the other hand, $k \leq \alpha$ so that

$$a(P^k, P^{2\alpha+1}) = \sum_{\substack{e_1 e_2^2 e_3 = P^k \\ e_1 e_2 | P^\alpha \\ (P, e_1 e_3)=1}} \mu(e_1)|e_2| = \sum_{\substack{e_1 e_2^2 | P^k \\ (P, e_1 e_3)=1}} \mu(e_1)|e_2| = \sum_{e_2^2 = P^k} |e_2| = \begin{cases} |P|^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

This completes the proof. Clearly the same holds for $Q_{P^{2\alpha+1}}(w)$. □

We have a similar statement if d is an even irreducible power up to a square-free factor and relatively prime factor:

Lemma 4.2. *For any square-free monic d_0, $a_1 \in \{1, \theta\}$, irreducible P not dividing d_0, and $\beta \geq 1$, we have*

$$Q_{d_0 P^{2\beta}}(s, \chi_{a_1}) = (1 - \chi_{a_1 d_0}(P)|P|^{-s}) \sum_{k \leq 2\beta} \frac{\chi_{a_1 d_0}(P^k) a(P^k, P^{2\beta})}{|P|^{ks}}.$$

Moreover, the same holds for $Q_{m_0 P^{2\beta}}(w, \chi_{a_2})$.

Proof. Expand the correction polynomial in $|P|^{-s}$ to get

$$Q_{d_0 P^{2\beta}}(s, \chi_{a_1}) = \sum_{e_1 e_2 | P^\alpha} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{-s} |e_2|^{1-2s} = \sum_{k \leq 2\beta} \frac{b(P^k, P^{2\beta})}{|P|^{ks}}.$$

where

$$b(P^k, P^{2\beta}) = \sum_{e_1 e_2^2 = P^k} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_2|.$$

It suffices to show $b(P^k, P^{2\beta}) = \chi_{a_1 d_0}(P^k) \left(a(P^k, P^{2\beta}) - a(P^{k-1}, P^{2\beta}) \right)$. On the one hand, $\mu(e_1) = 0$ unless $e_1 = 1, P$ in which case $\mu(e_1) = \pm 1$ accordingly. So

$$b(P^k, P^{2\beta}) = \sum_{e_1 e_2^2 = P^k} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_2| = \begin{cases} \chi_{a_1 d_0}(P^k) |P|^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(P^k) |P|^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

where we have used the identity $\chi_{a_1 d_0}(e_1) = \chi_{a_1 d_0}(P^k)$ which holds because this quadratic Dirichlet character only depends upon the parity of k. On the other hand, as in the proof of Lemma 4.1

$$a(P^k, P^{2\beta}) = \begin{cases} |P|^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

But then

$$\chi_{a_1 d_0}(P^k) \left(a(P^k, P^{2\beta}) - a(P^{k-1}, P^{2\beta}) \right) = \begin{cases} \chi_{a_1 d_0}(P^k) |P|^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(P^k) |P|^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

which completes the proof. Clearly the same holds for $Q_{m_0 P^{2\beta}}(w, \chi_{a_2})$. □

From Lemmas 4.1 and 4.2 we see that $Q_{d_0 d_1^2}(s, \chi_{a_1})$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients $a(m, d)$ when d is an irreducible power. From the proofs of these two lemmas, we have evaulated $a(P^k, P^l)$ and we collect this as a corollary:

Corollary 4.1. *For any irreducible P,*

$$a(P^k, P^l) = \begin{cases} \min \left(|P|^{\frac{k}{2}}, |P|^{\frac{l}{2}} \right) & \textit{if } \min(k, l) \textit{ is even,} \\ 0 & \textit{otherwise.} \end{cases}$$

Using Proposition 4.1 and Corollary 4.1 together lets us evaulate $a(m, d)$ in general:

Corollary 4.2. *For any monics d and m,*

$$a(m,d) = \begin{cases} |(m,d)|^{\frac{1}{2}} & \textit{if } (m,d) \textit{ is a perfect square,} \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, Corollary 4.2 shows that $a(m,d)$ is symmetric in m and d. Since the weighting coefficients are multiplicative, it will follow that $Q_{d_0d_1^2}(s,\chi_{a_1})$ admits an Euler product. To describe this Euler product explicitely, let $d = d_0d_1^2d_2^2$ with d_0 square-free and monic, d_2 relatively prime to d_0d_1, and such that every irreducible divisor of d_1 divides d_0. In other words, d_0 is the square-free part of d, d_1 is the square part of d whose irreducible factors divide d to odd power, and d_2 is the square part of d whose irreducible factors divide d to even power. Then we have the following Euler product:

Theorem 4.1. *Let $d = d_0d_1^2d_2^2$ be the square decomposition of d stratified by even and odd powers. Then for any $a_1 \in \{1,\theta\}$,*

$$Q_{d_0d_1^2d_2^2}(s,\chi_{a_1}) = \prod_{P^\alpha || d_1} Q_{P^{2\alpha+1}}(s) \cdot \prod_{P^\beta || d_2} Q_{d_0P^{2\beta}}(s,\chi_{a_1}).$$

Moreover, the same holds for $Q_{m_0m_1^2m_2^2}(w,\chi_{a_2})$.

Proof. Recall that

$$L(s,\chi_{a_1d}) = L(s,\chi_{a_1d_0})Q_{d_0d_1^2d_2^2}(s,\chi_{a_1}) = \sum_{m \text{ monic}} \frac{\chi_{a_1d_0}(\widehat{m})a(m,d)}{|m|^s}.$$

We will now derive an alterate expression for $L(s,\chi_{a_1d})$. By Proposition 4.1, the coefficients of $L(s,\chi_{a_1d})$ are multiplicative. Therefore $L(s,\chi_{a_1d})$ admits the Euler product

$$L(s,\chi_{a_1d}) = \prod_{P \text{ monic irr}} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(\widehat{P^k})a(P^k,d)}{|P|^{ks}} \right).$$

Decomposing the product according to irreducibles dividing $d = d_0d_1^2d_2^2$, we see that

$$\begin{aligned}
&L(s,\chi_{a_1d}) \\
&= \prod_{P \text{ monic irr}} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(\widehat{P^k})a(P^k,d)}{|P|^{ks}} \right) \\
&= \prod_{P\nmid d} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(\widehat{P^k})a(P^k,1)}{|P|^{ks}} \right) \prod_{P^\alpha||d_1} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(\widehat{P^k})a(P^k,P^{2\alpha+1})}{|P|^{ks}} \right) \cdot \prod_{P^\beta||d_2} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(\widehat{P^k})a(P^k,P^\beta)}{|P|^{ks}} \right) \\
&= \prod_{P\nmid d} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(\widehat{P^k})}{|P|^{ks}} \right) \prod_{P^\alpha||d_1} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(\widehat{P^k})a(P^k,P^{2\alpha+1})}{|P|^{ks}} \right) \cdot \prod_{P^\beta||d_2} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(\widehat{P^k})a(P^k,P^\beta)}{|P|^{ks}} \right) \\
&= \prod_{P\nmid d} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(\widehat{P^k})}{|P|^{ks}} \right) \prod_{P^\alpha||d_1} \left(\sum_{k\geq 0} \frac{a(P^k,P^{2\alpha+1})}{|P|^{ks}} \right) \cdot \prod_{P^\beta||d_2} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(P^k)a(P^k,P^\beta)}{|P|^{ks}} \right).
\end{aligned}$$

Including the factors corresponding to irreducibles $P \mid d_2$ into the first product, we must multiply the last factor by the inverse of $\sum_{k\geq 0} \chi_{a_1d_0}(P)|P|^{-ks} = (1-\chi_{a_1d_0}(P)|P|^{-s})^{-1}$ obtaining

$$\prod_{P\nmid d_0} \left(\sum_{k\geq 0} \frac{\chi_{a_1d_0}(\widehat{P^k})}{|P|^{ks}} \right) \prod_{P^\alpha||d_1} \left(\sum_{k\geq 0} \frac{a(P^k,P^{2\alpha+1})}{|P|^{ks}} \right) \cdot \prod_{P^\beta||d_2} \left((1-\chi_{a_1d_0}(P)|P|^{-s}) \sum_{k\geq 0} \frac{\chi_{a_1d_0}(P^k)a(P^k,P^\beta)}{|P|^{ks}} \right),$$

since every irreducible divisor of d_1 divdes d_0. We recognize the first product as $L(s, \chi_{a_1 d_0})$. For the second and third products, Remark 4.1 implies that the sums run up to $k \leq 2\alpha$ and $k \leq 2\beta$ respectively. We then recognize the sums as $Q_{P^{2\alpha+1}}(s)$ and $Q_{d_0 P^{2\beta}}(s, \chi_{a_1})$ respectively. It follows that

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) \cdot \prod_{P^\alpha || d_1} Q_{P^{2\alpha+1}}(s) \cdot \prod_{P^\beta || d_2} Q_{d_0 P^{2\beta}}(s, \chi_{a_1}).$$

This is our alternate expression for $L(s, \chi_{a_1 d})$. Equating the two gives

$$L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = L(s, \chi_{a_1 d_0}) \cdot \prod_{P^\alpha || d_1} Q_{P^{2\alpha+1}}(s) \cdot \prod_{P^\beta || d_2} Q_{d_0 P^{2\beta}}(s, \chi_{a_1}),$$

which completes the proof as $L(s, \chi_{a_1 d_0}) \neq 0$ for $\mathrm{Re}(s) > 1$ (so that we may divide by $L(s, \chi_{a_1 d_0})$). Clearly the same holds for $Q_{m_0 m_1^2 m_2^2}(w, \chi_{a_2})$. □

It is worth noting that for $d = d_0 d_1^2 d_2^2$, the irreducible factors that divide $d_1 d_2$ are exactly those factors that divide d to power larger than 1. In particular, from Theorem 4.1 we see that the Euler product for $Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1})$ is supported on exactly the irreducibles dividing d to order larger than 1 but also depends upon the character $\chi_{a_1 d_0}$. Also, Theorem 4.1 is typical of a larger reality. In more general settigns when the correction polynomials $P_{m_0 m_1^2}(w, \chi_{a_2})$ on the other side of the interchange are not of the form $Q_{d_0 d_1^2}(s, \chi_{a_1})$, the correction polynomials $P_{m_0 m_1^2}(w, \chi_{a_2})$ still posess Euler products.

5. Functional Equations

We are now ready to derive functional equations for $Z_{a_1, a_2}(s, w)$. In order to do this we require a functional equation for $L(s, \chi_{a_1 d})$. This latter functional equation amounts to having a functional equation for the correction polynomials $Q_{d_0 d_1^2}(s, \chi_{a_1})$:

Theorem 5.1. *$Q_{d_0 d_1^2}(s, \chi_{a_1})$ admits the functional equation.*

$$Q_{d_0 d_1^2}(s, \chi_{a_1}) = |d_1|^{1-2s} Q_{d_0 d_1^2}(1 - s, \chi_{a_1}).$$

Moreover, the same holds for $Q_{m_0 m_1^2}(w, \chi_{a_2})$.

Proof. The idea is that e_2 and e_3 are interchanged in the sum. The computation is simple:

$$\begin{aligned} |d_1|^{1-2s} Q_{d_0 d_1^2}(1 - s) &= |d_1|^{1-2s} \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{s-1} |e_2|^{2s-1} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{s-1} \left| \frac{d_1}{e_2} \right|^{1-2s} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{s-1} |e_1 e_3|^{1-2s} \\ &= \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{-s} |e_3|^{1-2s} \\ &= Q_{d_0 d_1^2}(s, \chi_{a_1}). \end{aligned}$$

Clearly the same holds for $Q_{m_0 m_1^2}(w, \chi_{a_2})$. □

Define the completed L-function $L^*(s, \chi_{a_1 d})$ by

$$L^*(s, \chi_{a_1 d}) = L^*(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1}).$$

In particular, $L^*(s,\chi_d)$ makes sense even when d is not square-free and agrees with the previous definition when d is square-free. Combining Theorem 5.1, the functional equation for $L^*(s,\chi_{a_1d_0})$, and that $\deg(d)\equiv\deg(d_0) \pmod 2$, we obtain a functional equation for $L^*(s,\chi_{a_1d})$:

$$L^*(s,\chi_{a_1d}) = \begin{cases} q^{2s-1}|d|^{\frac{1}{2}-s}L^*(1-s,\chi_{a_1d}) & \text{if } \deg(d) \text{ is even,}\\ q^{s-\frac{1}{2}}|d|^{\frac{1}{2}-s}L^*(1-s,\chi_{a_1d}) & \text{if } \deg(d) \text{ is odd.}\end{cases}$$

We would like the functional equation for $L^*(s,\chi_{a_1d})$ to induce a functional equation for $Z_{a_1,a_2}(s,w)$ but there is one technical issue. The gamma factor for $L^*(s,\chi_{a_1d})$ depend upon the degree of d and in order to obtain a functional equation we need the gamma factor to be constant. We will be able to move past this obstruction by exploiting the orthogonality for Dirichlet characters, in particular, the Hilbert characters. Define $Z^+_{a_1,a_2}(s,w)$ and $Z^-_{a_1,a_2}(s,w)$ by

$$Z^+_{a_1,a_2}(s,w) = \frac{Z_{a_1,a_2}(s,w)+Z_{a_1,\theta a_2}(s,w)}{2} \quad \text{and} \quad Z^-_{a_1,a_2}(s,w) = \frac{Z_{a_1,a_2}(s,w)-Z_{a_1,\theta a_2}(s,w)}{2}.$$

In terms of the representation

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ monic}} \frac{\chi_{a_2}(d)L(s,\chi_{a_1d})}{|d|^w},$$

and the orthogonality relation (ii) for the Hilbert characters, we see that $Z^{\pm}_{a_1,a_2}(s,w)$ is the subseries containing only those d with $\chi_\theta(d)=\pm 1$. Equivalently, $Z^+_{a_1,a_2}(s,w)$ and $Z^-_{a_1,a_2}(s,w)$ are the subseries consisting of only thos d with even and odd degree respectively. But then $Z^+_{a_1,a_2}(s,w)$ and $Z^-_{a_1,a_2}(s,w)$ are sums of L-functions with a fixed gamma factor we can obtain functional equations for them. The fact that $Z_{a_1,a_2}(s,w)$ is a linear combination of these series will induce a function equation for $Z_{a_1,a_2}(s,w)$ in terms of other twisted quadratic double Dirichlet series:

Theorem 5.2. *$Z_{a_1,a_2}(s,w)$ admits the functional equations*

$$Z_{a_1,a_2}(s,w) = \frac{1}{2}\left(\frac{q^{2s-1}(1-q^{-s})}{1-q^{s-1}} + q^{s-\frac{1}{2}}\right)Z_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{q^{2s-1}(1-q^{-s})}{1-q^{s-1}} - q^{s-\frac{1}{2}}\right)Z_{a_1,\theta a_2}\left(1-s,s+w-\frac{1}{2}\right),$$

and

$$Z_{a_1,a_2}(s,w) = \frac{1}{2}\left(\frac{q^{2w-1}(1-q^{-w})}{1-q^{w-1}} + q^{w-\frac{1}{2}}\right)Z_{a_2,a_1}\left(s+w-\frac{1}{2},1-w\right) + \frac{1}{2}\left(\frac{q^{2w-1}(1-q^{-w})}{1-q^{w-1}} - q^{w-\frac{1}{2}}\right)Z_{a_2,\theta a_1}\left(s+w-\frac{1}{2},1-w\right).$$

Proof. As the interchange implies $Z_{a_1,a_2}(s,w)=Z_{a_2,a_1}(w,s)$, the second functional equation will follow from the first one. The functional equation for the L-function attached to a quadratic Dirichlet character induces the functional equations

$$Z^+_{a_1,a_2}(s,w) = \frac{q^{2s-1}(1-q^{-s})}{1-q^{s-1}}Z^+_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right),$$

and

$$Z^-_{a_1,a_2}(s,w) = q^{s-\frac{1}{2}}Z^-_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right).$$

Since $Z_{a_1,a_2}(s,w)=Z^+_{a_1,a_2}(s,w)+Z^-_{a_1,a_2}(s,w)$ the functional equations just stated give

$$Z_{a_1,a_2}(s,w) = \frac{q^{2s-1}(1-q^{-s})}{1-q^{s-1}}Z^+_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right) + q^{s-\frac{1}{2}}Z^-_{a_1,a_2}\left(1-s,s+w-\frac{1}{2}\right).$$

The desired functional equation for $Z_{a_1,a_2}(s,w)$ follows by expressing $Z^{\pm}_{a_1,a_2}(s,w)$ in terms of $Z_{a_1,a_2}(s,w)$ and $Z_{a_1,\theta a_2}(s,w)$. $\square$

We can more compactly state Theorem 5.2 in vector form. To do this, define

$$\mathbf{Z}(s,w) = \begin{pmatrix} Z_{1,1}(s,w) \\ Z_{1,\theta}(s,w) \\ Z_{\theta,1}(s,w) \\ Z_{\theta,\theta}(s,w) \end{pmatrix}.$$

Also let

$$\gamma^{+}(s) = \frac{1}{2}\left(\frac{q^{2s-1}(1-q^{-s})}{1-q^{s-1}} + q^{s-\frac{1}{2}}\right) \quad \text{and} \quad \gamma^{-}(s) = \frac{1}{2}\left(\frac{q^{2s-1}(1-q^{-s})}{1-q^{s-1}} - q^{s-\frac{1}{2}}\right),$$

be the gamma factors appearing in the functional equation. Then by Theorem 5.2, the 4×4 matrices

$$\Phi(s) = \begin{pmatrix} \gamma^{+}(s) & \gamma^{-}(s) & 0 & 0 \\ \gamma^{-}(s) & \gamma^{+}(s) & 0 & 0 \\ 0 & 0 & \gamma^{+}(s) & \gamma^{-}(s) \\ 0 & 0 & \gamma^{-}(s) & \gamma^{+}(s) \end{pmatrix} \quad \text{and} \quad \Psi(w) = \begin{pmatrix} \gamma^{+}(w) & 0 & \gamma^{-}(w) & 0 \\ \gamma^{-}(w) & 0 & \gamma^{+}(w) & 0 \\ 0 & \gamma^{+}(w) & 0 & \gamma^{-}(w) \\ 0 & \gamma^{-}(w) & 0 & \gamma^{+}(w) \end{pmatrix},$$

satisfy

$$\mathbf{Z}(s,w) = \Phi(s)\mathbf{Z}\left(1-s, s+w-\frac{1}{2}\right) \quad \text{and} \quad \mathbf{Z}(s,w) = \Psi(w)\mathbf{Z}\left(s+w-\frac{1}{2}, 1-w\right).$$

We can state this even more compactly by disregarding the matrix $\Psi(w)$. Indeed, using Remark 3.1 we can swap the middle two columns of $\Psi(w)$ at the expense of using $\mathbf{Z}\left(1-w, s+w-\frac{1}{2}\right)$ so that we have the identities

$$\mathbf{Z}(s,w) = \Phi(s)\mathbf{Z}\left(1-s, s+w-\frac{1}{2}\right) \quad \text{and} \quad \mathbf{Z}(s,w) = \Phi(w)\mathbf{Z}\left(1-w, s+w-\frac{1}{2}\right).$$

In more general settings, the functional equations for $Z_{a_1,a_2}(s,w)$ need not be so symmetric. Indeed, when the correction polynomials $P_{m_0 m_1^2}(w)$ are not of the form $Q_{d_0 d_1^2}(s)$, we require functional equations for both of these correction polynomials. We then have to use each functional equation for each correction polynomial. Moreover, if there are more Hilbert characters we need a more refined decomposition of $Z_{a_1,a_2}(s,w)$. In any case, from Theorem 5.2 we have two functional equations of shapes

$$\sigma_1 : (s,w) \to \left(1-s, s+w-\frac{1}{2}\right) \quad \text{and} \quad \sigma_2 : (s,w) \to \left(s+w-\frac{1}{2}, 1-w\right).$$

These transformations also naturally act on the (s,w)-plane. Moreover, we have the relation

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 : (s,w) \to (1-w, 1-s) \quad \text{or equivalently} \quad (\sigma_1\sigma_2)^3 = 1 : (s,w) \to (s,w).$$

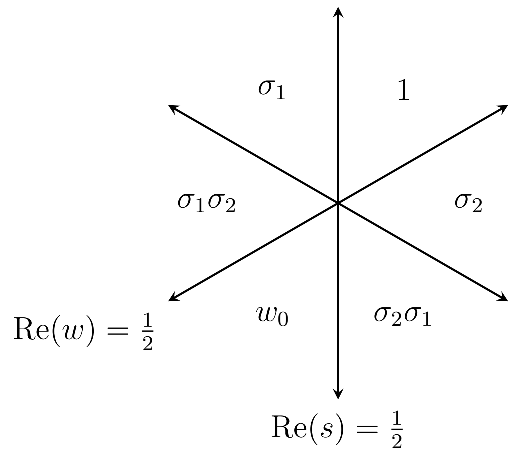
As $\sigma_1^2 = \sigma_2^2 = 1$, σ_1 and σ_2 generate the group

$$W = \langle \sigma_1, \sigma_2 : \sigma_1^2 = \sigma_2^2 = (\sigma_1\sigma_2)^3 = 1 \rangle \cong D_6 \cong S_3.$$

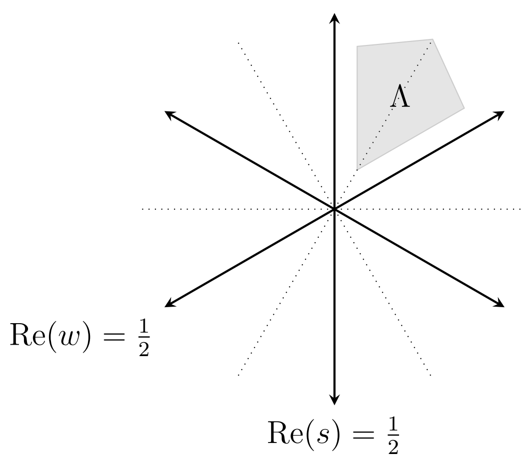
For convience we will set $w_0 = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. It follows that $Z_{a_1,a_2}(s,w)$ doesn't just posess two functional equations, it has a group of 6 functional equations. Actually, if we include the functional equation $(s,w) \to (w,s)$ coming from the interchange, there are 12 functional equations and the group is isomorphic to D_{12}. We can use these functional equations to meromorphically continue $Z(s,w)$ to the entire (s,w)-plane.

6. Meromorphic Continuation

Now it is time to establish the meromorphic continuation of $Z(s,w)$ to the entire (s,w)-plane. We could do this for each twisted quadratic double Dirichlet series $Z_{a_1,a_2}(s,w)$, but we will not need this level of generality here or further on. We will start by describing the action of W on the (s,w)-plane geometrically. First, it is easy to see from the definition of the actions σ_1 and σ_2 that there is a unique W-invariant point $\left(\frac{1}{2},\frac{1}{2}\right)$. Representing the point (s,w) by $(\mathrm{Re}(s),\mathrm{Re}(w))$ we have a geometric visualization of the action of W on the (s,w)-plane:



In the diagram, we we have shifted the (s,w)-plane so that the origin lies at $\left(\frac{1}{2},\frac{1}{2}\right)$ and we have tiled the (s,w)-axes so that they are no longer perpendicular. The effect of these adjustments is that σ_1 and σ_2 act by rigid motions sending the region enclosing 1 (corresponding to the identity) to either of the adjacent triangles. The other regions are obtained by acting by the corresponding element of W. The initial region Λ that $Z(s,w)$ is defined on is displayed in the figure below:



To meromorphically continue $Z(s,w)$ to all of the (s,w)-plane, we first need to analytically continue the quadratic double Dirichlet series $Z(s,w)$ and $Z_{1,\theta}(s,w)$ to a slightly larger region than Λ. This continuation will be achieved by the Phragmén-Lindelöf convexity principal. Fix some small $\varepsilon>0$. The functional equations for $L^*(s,\chi_d)$ and $L^*(w,\chi_m)$ provide the estimates

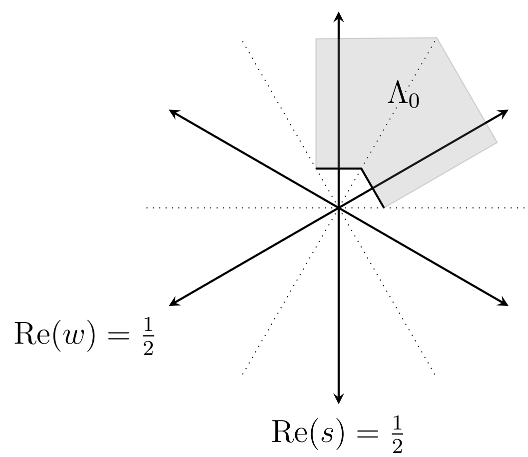
$$L(-\varepsilon,\chi_d)\ll|d|^{\frac{1}{2}+\varepsilon}\quad\text{and}\quad L(-\varepsilon,\chi_m)\ll|m|^{\frac{1}{2}+\varepsilon},$$

because $L(1+\varepsilon,\chi_d)\ll 1$ and $L(1+\varepsilon,\chi_m)\ll 1$. Since both of these L-functions at most have a simple pole at $s=1$ and $w=1$ respectively, the Phragmén-Lindelöf convexity principal implies the weak estimates

$$(s-1)L(s,\chi_d)\ll|d|^{\frac{1}{2}+\varepsilon}\quad\text{and}\quad(w-1)L(w,\chi_m)\ll|m|^{\frac{1}{2}+\varepsilon},$$

for $\text{Re}(s) > -\varepsilon$ and $\text{Re}(w) > -\varepsilon$. But then from the interchange we see that $(s-1)(w-1)Z(s,w)$ and $(s-1)(w-1)Z_{1,\theta}(s,w)$ are locally absolutely uniformly convergent on the region

$$\Lambda_0 = \Lambda \cup \left\{(s,w) \in \mathbb{C}^2 : \text{Re}(s) > 0, \text{Re}(w) > \frac{3}{2}\right\} \cup \left\{(s,w) \in \mathbb{C}^2 : \text{Re}(s) > \frac{3}{2}, \text{Re}(w) > 0\right\}.$$



In particular, $Z(s,w)$ and $Z_{1,\theta}(s,w)$ are meromorphic on this region with at most polar lines at $s = 1$ and $w = 1$. The advantage of the region Λ_0 over the intital region Λ is that Λ_0 intersects the hyperplains $s = \frac{1}{2}$ and $w = \frac{1}{2}$ so that the union of the reflections $w\Lambda_0$ for $w \in \Omega$ is connected. The idea now is to reflect Λ_0 via the functional equations and then apply a theorem of Bochner. To state this theorem we need a small definition. We say that a domain $\Omega \subset \mathbb{C}^n$ is a **tube domain** if there is an open set $\omega \subset \mathbb{R}^n$ such that

$$\Omega = \{(s_1, \ldots, s_n) \in \mathbb{C}^n : \text{Re}((s_1, \ldots, s_n)) \in \omega\}.$$

Tube domains are generalizations of vertical strips in the complex plane. Now we can state the theorem of Bochner (see [1] for a proof):

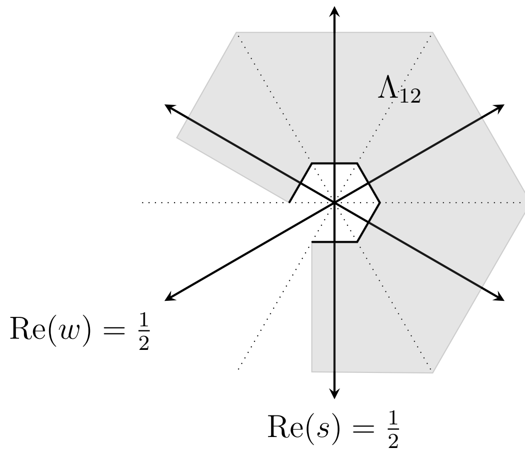
Theorem 6.1 (Bochner's continuation theorem)**.** *If Ω is a connected tube domain, then any holomorphic function on Ω can be extended to a holomorphic functon on the convex hull $\widehat{\Omega}$.*

By clearing polar divisors, Bochner's continuation theorem implies that any meromorphic function on a connected tube domain posessing a finite amount of hyperplane polar divisors can be extended to a meromorphic function on the convex hull. This is exactly the situation for $Z(s,w)$. Indeed, it is clear that a union of tube domains is a tube domain and so, in particular, Λ_0 is a tube domain. But on Λ_0 there are a most polar lines at $s = 1$ and $w = 1$. Reflecting these hyperplanes via W we obtain the finite set of possible polar divisors:

$$\left\{s = 1, w = 1, s = 0, w = 0, s + w = \frac{1}{2}, s + w = \frac{3}{2}\right\}.$$

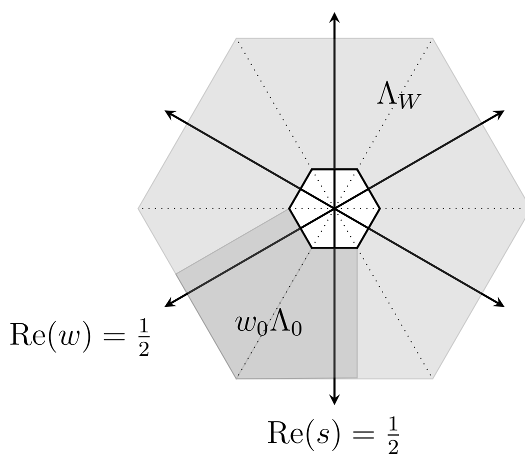
Therefore we are reduced to extending $Z(s,w)$ meromorphically. By applying the functional equations corresponding to σ_1, σ_2, and $\sigma_1\sigma_2$, $Z(s,w)$ admits meromorphic continuation to the region

$$\Lambda_{12} = \Lambda_0 \cup \sigma_1\Lambda_0 \cup \sigma_2\Lambda_0 \cup \sigma_1\sigma_2\Lambda_0.$$



Because the functional equation for $Z(s, w)$ involves both $Z(s, w)$ and $Z_{1,\theta}(s, w)$, it was necessarily to analytically continue both of these quadratic double Dirichlet series to Λ_0 before we applied the functional equation. Now Λ_{12} is a connected tube domain whose convex hull is $\mathbb{C}^2$. So by applying Bochner's continuation theorem (or rather our comment for meromorphic functions) we see that $Z(s, w)$ admits meromorphic continuation to the (s, w)-plane with at most a finite set of polar divisors. This method is better than repeatedly applying the functional equations corresponding to every $w \in W$. Indeed, if we did we would obtain meromorphic continuation to the region

$$\Lambda_W = \bigcup_{w \in W} w\Lambda_0.$$



There are two issues here. The first is that $Z(s, w)$ has two meromorphic continuations to the region $w_0\Lambda_0$ given by the functional equations corresponding to $w_0 = \sigma_1\sigma_2\sigma_1$ and $w_0 = \sigma_2\sigma_1\sigma_2$ and we would need to show that these agree. The second is that we have not obtained meromorphic continuation to $\mathbb{C}^2 - \Lambda_W$ which is a compact hexagon about the origin. By using Bochner's theorem after meromorphically continuning to Λ_{12}, we have avoided these issues and as a consquence shown that the meromorphic continuations given by $w_0 = \sigma_1\sigma_2\sigma_1$ and $w_0 = \sigma_2\sigma_1\sigma_2$ agree.

7. Poles and Residues

We will inespect the polar divisors of $Z(s, w)$ more carefully. It turns out that the set of polar divisors is smaller than

$$\left\{ s = 1, w = 1, s = 0, w = 0, s + w = \frac{1}{2}, s + w = \frac{3}{2} \right\}.$$

Indeed, there are no poles on the hyperplanes $s = 0$, $w = 0$, and $s + w = \frac{3}{2}$. To see this, first note that by our earlier application of the Phragmén-Lindelöf convexity principal we actually obtained continuation to an open set containing Λ_0 (because our estimates held for $\mathrm{Re}(s) > -\varepsilon$ and $\mathrm{Re}(w) > -\varepsilon$). We did not need this larger region for the meromorphic continuation but we do need it to study the poles. Now consider the possible polar divisor $s = 0$. We know $(s-1)(w-1)Z(s,w)$ and $(s-1)(w-1)Z_{1,\theta}(s,w)$ are holomorphic on an open set containing Λ_0 which contains half of the hyperplane defined by $s = 0$ outside of the hexagon $\mathbb{C}^2 - \Lambda_W$. As $(s-1)(w-1)$ is holomorphic on this region it follows that $Z(s,w)$ and $Z_{1,\theta}(s,w)$ do not have a polar divisor at $s = 0$ on an open set containing Λ_0. Now note that an open set containing $\sigma_1\sigma_2\Lambda_0$ contains the other half of the hyperplane defined by $s = 0$ outside of the hexagon $\mathbb{C}^2 - \Lambda_W$. Upon applying the functional equation corresponding to $\sigma_1\sigma_2$, Theorem 5.2 implies that the gamma factors in the corresponding functional equation have a simple pole when $s + w = \frac{3}{2}$ (the gamma factors in the functional equation for σ_1 have a simple pole at $s = 1$ and $s - 1 \to s + w - \frac{3}{2}$ under σ_2). Therefore $Z(s,w)$ and $Z_{1,\theta}(s,w)$ do not have polar divisors at $s = 0$ on an open set containing $\sigma_1\sigma_2\Lambda_0$ away from $s + w = \frac{3}{2}$. In particular, $Z(s,w)$ does not have a polar divisor at $s = 0$ on Λ_W and away from the other polar divisors. By Bochner's continuation theorem (after clearing all of the other possible polar divisors), we see that $Z(s,w)$ does not have a polar divisors at $s = 0$ on all of $\mathbb{C}^2$ and away from the other polar divisors. An identical argument holds for the case $w = 0$ with the regions Λ_0 and $\sigma_2\sigma_1\Lambda_0$. For the polar divisor $s + w = \frac{1}{2}$, we argue in the same way with the regions $\sigma_2\sigma_1\Lambda_0$, $\sigma_1\sigma_2\Lambda_0$, and $w_0\Lambda_0$. The only difference is that for these regions the gamma factors in the corresponding functional equations are different. For the first two regions $\sigma_2\sigma_1\Lambda_0$ and $\sigma_1\sigma_2\Lambda_0$ the gamma factors have a simple pole when $s + w = \frac{3}{2}$. For the third region $w_0\Lambda_0$ the gamma factors have simple poles at $s = 1$ and $w = 1$ which is seen by using both representations $w_0 = \sigma_1\sigma_2\sigma_1$ and $w_0 = \sigma_2\sigma_1\sigma_2$. So in conclusion, there are no poles on the hyperplanes $s = 0$, $w = 0$, and $s + w = \frac{1}{2}$ and away from the other polar divisors. As for the hyperplanes $s = 1$, $w = 1$, and $s + w = \frac{3}{2}$, there are clearly genuine poles for $s = 1$ and $w = 1$ coming from $L(s,\chi_{d_0})$ and $L(w,\chi_{m_0})$ when d and m are perfect squares (so that $d_0 = m_0 = 1$). For $s + w = \frac{3}{2}$, we have a pole coming from the gamma factors corresponding to the functional equations for $\sigma_2\sigma_1$ and $\sigma_1\sigma_2$. We collect all of our work as a theorem:

Theorem 7.1. *$Z(s,w)$ admits meromorphic continuation to $\mathbb{C}^2$ with polar divisors $s = 1$, $w = 1$, and $s + w = \frac{3}{2}$.*

We can now look at the residue of $Z(s,w)$ at these poles. Since all of the poles are obtained from each other by applying the functional equations of $Z(s,w)$, we begin by looking at the pole at $w = 1$. To compute the residue we use the representation

$$Z(s,w) = \sum_{m \text{ monic}} \frac{L(w,\chi_{m_0})Q_{m_0m_1^2}(w)}{|m|^s},$$

coming from the interchange. For a fixed m, the numerator $L(w,\chi_{m_0})Q_{m_0m_1^2}(w)$ in the summand corresponding to m has a pole at $w = 1$ if and only if m_0 is square-free, that is $m_0 = 1$, or equivalently $m = m_1^2$ itself is a perfect square. In this case, $L(w,\chi_{m_0}) = \zeta(w)$ so that

$$\operatorname*{Res}_{w=1} L(w,\chi_{m_0})Q_{m_0m_1^2}(w) = \frac{1}{\log(q)}Q_{m_1^2}(1).$$

But from Lemma 4.2 and Theorem 4.1 we see that $Q_{m_1^2}(1) = 1$, and so

$$\operatorname*{Res}_{w=1} Z(s,w) = \frac{1}{\log(q)} \sum_{m \text{ monic perfect square}} \frac{Q_{m_1^2}(1)}{|m|^s} = \frac{1}{\log(q)} \sum_{m \text{ monic}} \frac{1}{|m|^{2s}} = \frac{1}{\log(q)}\zeta(2s).$$

Notice that this expression has a simple pole at $s = \frac{1}{2}$ which is exactly when the polar lines $w = 1$ and $s + w = \frac{3}{2}$ intersect. The residue of $Z(s,w)$ at $s = 1$ is computed in the same way. Indeed, by applying the interchange, the exact same argument holds with the roles of s and w interchanged so that

$$\operatorname*{Res}_{s=1} Z(s,w) = \frac{1}{\log(q)}\zeta(2w).$$

Again, this expression has a simple pole at $w = \frac{1}{2}$ which is when the polar lines $s = 1$ and $s + w = \frac{3}{2}$ intersect. The other residues of the simple poles can be computed by apply the functional equations for $Z(s,w)$ and using the resiudes at $s = 1$ and $w = 1$. Now consider the point where the polar lines $w = 1$ and $s + w = \frac{3}{2}$ intersect. Setting $s = \frac{1}{2}$, we see that $Z\left(\frac{1}{2}, w\right)$ has a pole at $w = 1$ and we would like to understand this pole better. To acomplish this, the Mittag-Leffler theorem applied to $Z(s,w)$ (in w) implies that

$$Z(s,w) = \frac{R_1(s)}{w-1} + \frac{R_2(s)}{s+w-\frac{3}{2}} + V(s,w),$$

in some neighborhood of $\left(\frac{1}{2}, 1\right)$, where $V(s,w)$ is holomorphic, and we have set

$$R_1(s) = \operatorname*{Res}_{w=1} Z(s,w) \quad \text{and} \quad R_2(s) = \operatorname*{Res}_{w=\frac{3}{2}-s} Z(s,w).$$

From our residue computations, $R_1(s) = \frac{1}{\log(q)}\zeta(2s)$ which implies that it has a simple pole at $s = \frac{1}{2}$. The residue is given by $A = \frac{1}{2\log(q)}$. On the other hand, $Z\left(\frac{1}{2}, w\right)$ is holomorphic for $\operatorname{Re}(w) > 1$. These two facts together imply that $R_2(s)$ must have a simple pole at $s = \frac{1}{2}$ which cancels the simple pole coming from $R_1(s)$. So by Mittag-Leffler again, we may write

$$R_1(s) = \frac{A}{s-\frac{1}{2}} + R_3(s) \quad \text{and} \quad R_2(s) = -\frac{A}{s-\frac{1}{2}} + R_4(s),$$

in a neighborhood of $s = \frac{1}{2}$ and where $R_3(s)$ and $R_4(s)$ are holomorphic. It follows that

$$\begin{aligned}
Z(s,w) &= \frac{R_1(s)}{w-1} + \frac{R_2(s)}{s+w-\frac{3}{2}} + V(s,w) \\
&= \frac{A}{(w-1)\left(s-\frac{1}{2}\right)} + \frac{R_3(s)}{w-1} - \frac{A}{\left(s+w-\frac{3}{2}\right)\left(s-\frac{1}{2}\right)} + \frac{R_4(s)}{s+w-\frac{3}{2}} + V(s,w) \\
&= \frac{A}{(w-1)\left(s+w-\frac{3}{2}\right)} + \frac{R_3(s)}{w-1} + \frac{R_4(s)}{s+w-\frac{3}{2}} + V(s,w).
\end{aligned}$$

We can now set $s = \frac{1}{2}$ and let $B = R_3\left(\frac{1}{2}\right) + R_4\left(\frac{1}{2}\right)$ so that

$$Z\left(\frac{1}{2}, w\right) = \frac{A}{(w-1)^2} + \frac{B}{w-1} + O(1).$$

It follows that $Z\left(\frac{1}{2}, w\right)$ has a double pole at $w = 1$. This can be thought of as follows: the polar lines $w = 1$ and $s + w = \frac{3}{2}$ correspond to simple poles of $Z(s,w)$ except in the case when they intersect and the order of the poles combine to give $Z\left(\frac{1}{2}, w\right)$ a double pole at $w = 1$. Applying the interchange, the exact same argument holds to show that $Z\left(s, \frac{1}{2}\right)$ has a double pole at $s = 1$. We collect this work as a theorem:

Theorem 7.2. *$Z\left(\frac{1}{2}, w\right)$ and $Z\left(s, \frac{1}{2}\right)$ have double poles at $w = 1$ and $s = 1$ respectively. In particular, in neighborhoods of $w = 1$ and $s = 1$ respectively, we have*

$$Z\left(\frac{1}{2}, w\right) = \frac{A}{(w-1)^2} + \frac{B}{w-1} + O(1) \quad \text{and} \quad Z\left(s, \frac{1}{2}\right) = \frac{A}{(s-1)^2} + \frac{B}{s-1} + O(1),$$

for some constants A and B with $A = \frac{1}{2\log(q)}$.

An Application

As an application of the usefulness of quadratic double Dirichlet series, we show that the analytic properties of $Z(s,w)$ can be used to obtain a simultaneous non-vanishing result. In particular, we show that $L\left(\frac{1}{2},\chi_d\right)$ is nonzero for infinitely many d. Since the complex analysis involved comes from standard analytic number theory techniques, strictly speaking, we only sketch the result. To being, Perron's formula implies

$$\sum_{\deg(d)\leq X} L\left(\frac{1}{2},\chi_{d_0}\right) Q_{d_0 d_1^2}\left(\frac{1}{2}\right) = \frac{1}{2\pi i}\int_{(2)} Z\left(\frac{1}{2},w\right) X^w\, dw,$$

for some $X>1$. It can be shown that $Z(s,w)$ is bounded in vertical strips provided we are away from poles. It follows that Theorem 7.2 holds in vertical strips about the lines $\mathrm{Re}(w)=1$ and $\mathrm{Re}(s)=1$ respectively. Therefore

$$\frac{1}{2\pi i}\int_{(2)} Z\left(\frac{1}{2},w\right) X^w\, dw = \frac{1}{2\pi i}\int_{(2)} \left(\frac{A}{(w-1)^2}+\frac{B}{w-1}+O(1)\right) X^w\, dw.$$

Shifting the line of integration to the left (say to the line $\mathrm{Re}(w)=-2$) and using the residue theorem, we pick up a residue from the pole at $w=1$. The contribution from the first two terms to the residue are $AX\log(X)$ and BX respectively while the third term vanishes (recall that the integrand has a factor of X^w). The remaining vertical integral is $o(X)$. Therefore

$$\frac{1}{2\pi i}\int_{(2)} Z\left(\frac{1}{2},w\right) X^w\, dw = AX\log(X)+BX+o(X),$$

and it follows that

$$\sum_{\deg(d)\leq X} L\left(\frac{1}{2},\chi_{d_0}\right) = AX\log(X)+BX+o(X).$$

But as $A=\frac{1}{2\log(q)}\neq 0$ (see Theorem 7.2) this means that $L\left(\frac{1}{2},\chi_d\right)$ must be nonzero for infinitely many d (in particular for infinitely many square-free d). For otherwise, $\sum_{\deg(d)\leq X} L\left(\frac{1}{2},\chi_{d_0}\right)=O(1)$ which gives a contradiction.

$Z(s,w)$ as a Rational Function

Recall that $L(s,\chi_d)$ is a polynomial in q^{-s} of degree at most $\deg(d)-1$. A similar situation happens for the quadratic double Dirichlet series $Z(s,w)$, it will be a rational function in the variables $x=q^{-s}$ and $y=q^{-w}$. Since this property is a special case of Dirichlet series over function fields, we present the argument but supress the more detailed computations. Before we begin, we recall some properties of Hadamard products of power series. The details can be found in [3]. For any two power series

$$R_1(x,y)=\sum_{k,l\geq 0} r_1(k,l)x^k y^l \quad \text{and} \quad R_2(x,y)=\sum_{k,l\geq 0} r_2(k,l)x^k y^l,$$

or more generally generating series, their **Hadamard product** $(R_1 * R_2)(x,y)$ is defined by

$$(R_1 * R_2)(x,y)=\sum_{k,l\geq 0} r_1(k,l)r_2(k,l)x^k y^l.$$

If we assume $R_1(x,y)$ and $R_2(x,y)$ are regular around the origin $x=y=0$, then the Hadamard product can be expressed as two contour integrals around the origin:

$$(R_1 * R_2)(x,y)=\frac{1}{(2\pi i)^2}\int_{|z|=\rho}\int_{|w|=\rho} R_1(z,w)R_2\left(\frac{x}{z},\frac{y}{w}\right)\frac{dz}{z}\frac{dw}{w},$$

for sufficiently small $\rho > 0$. By the residue theorem,

$$(R_1 * R_2)(x, y) = \sum_{s=s(x,y)} \operatorname*{Res}_{s} \left(\frac{R_1(z, w) R_2\left(\frac{x}{z}, \frac{y}{w}\right)}{zw} \right),$$

where the sum is over all poles $s = s(x, y)$ of the integrand such that $\lim_{x,y\to 0} s(x, y) = 0$. This formula can be used to compute the Hadamard product of rational functions. In the following argument where computing a Hadamard product is necessary, this method can be used.

Now we show that $Z(s, w)$ is a rational function in $x = q^{-s}$ and $y = q^{-w}$. Throughout, we work in the region of local absolute uniform convergence for $Z(s, w)$. Consider the representation

$$Z(s, w) = \sum_{d \text{ monic}} \frac{L(s, \chi_d)}{|d|^w} = \sum_{m,\, d \text{ monic}} \frac{\chi_{d_0}(\widehat{m}) a(m, d)}{|m|^s |d|^w}.$$

Since $L(s, \chi_d)$ is a polynomial in q^{-s} of degree at most $\deg(d) - 1$ and correction polynomials $Q_{d_0 d_1^2}(s)$ are Dirichlet polynomials, one of the two following cases occur:

- If d is a perfect square, then $L(s, \chi_{d_0}) = \zeta(s)$ and
$$L(s, \chi_d) = \zeta(s) Q_{d_1^2}(s),$$
where $Q_{d_1^2}(s)$ is a polynomial in q^{-s} of degree $\deg(d)$.
- If d is not a perfect square, then
$$L(s, \chi_d) = L(s, \chi_{d_0}) Q_{d_0 d_1^2}(s),$$
is a polynomial in q^{-s} of degree at most $\deg(d) - 1$.

So if d is not a perfect square, then

$$L(s, \chi_d) = \sum_{\deg(m) \le \deg(d) - 1} \frac{\chi_{d_0}(\widehat{m}) a(m, d)}{|m|^s},$$

which is to say that

$$\sum_{\deg(m)=k} \chi_{d_0}(\widehat{m}) a(m, d) = 0,$$

for every $k \ge \deg(d)$ provided d is not a perfect square. To exploit this fact, we will decompose $Z(s, w)$ according to whether $\deg(d) \ge \deg(m)$ or $\deg(d) \le \deg(m)$. So we write

$$Z(s, w) = Z_0(s, w) + Z_0(w, s) - Z_1(s, w),$$

where

$$Z_0(s, w) = \sum_{0 \le l \le k} \frac{1}{q^{ls} q^{kw}} \sum_{\substack{m,\, d \text{ monic} \\ \deg(m)=k \\ \deg(d)=l}} \chi_{d_0}(\widehat{m}) a(m, d) \quad \text{and} \quad Z_1(s, w) = \sum_{k \ge 0} \frac{1}{q^{ks} q^{kw}} \sum_{\substack{m,\, d \text{ monic} \\ \deg(m)=k \\ \deg(d)=k}} \chi_{d_0}(\widehat{m}) a(m, d).$$

We will now show that $Z_0(s, w)$ and $Z_1(s, w)$ are rational functions in q^{-s} and q^{-w} via a convolution procedure. For $Z_0(s, w)$, our remarks about $L(s, \chi_d)$ imply the term in the inner sum of $Z_0(s, w)$ vanishes unless d is a perfect square and in this case $\chi_{d_0}(\widehat{m}) = 1$. Therefore

$$Z_0(s, w) = \sum_{0 \le l \le k} \frac{1}{q^{ls} q^{kw}} \sum_{\substack{m,\, d \text{ monic} \\ d \text{ a perfect square} \\ \deg(m)=l \\ \deg(d)=k}} a(m, d).$$

Now consider

$$Y(s,w) = \sum_{\substack{m,d \text{ monic} \\ d \text{ monic perfect square}}} \frac{a(m,d)}{|m|^s|d|^w} \quad \text{and} \quad K_0(s,w) = \sum_{0\le l\le k} \frac{1}{q^{ls}q^{kw}}.$$

We can express both of these as rational functions in q^{-s} and q^{-w}. Indeed, for $Y(s,w)$, Proposition 4.1 we that $Y(s,w)$ posesses a Euler product. Using Corollary 4.1, we compute

$$Y(s,w) = \frac{1-q^{1-s-2w}}{(1-q^{1-2w})(1-q^{1-s})(1-q^{2-2s-2w})},$$

which can also be seen by comparing coefficients of the power series expansion of the right-hand side. $K_0(s,w)$ is even easier because it is a geometric series:

$$K_0(s,w) = \sum_{0\le l\le k} \frac{1}{q^{ls}q^{kw}} = \frac{1}{(1-q^{-s})(1-q^{-s-w})}.$$

Then $Z_0(s,w)$ is expressed as a Hadamard product of power series in q^{-s} and q^{-w}:

$$Z_0(s,w) = Y(s,w) * K_0(s,w).$$

Then using the countour integral representation of the Hadamard product, we compute

$$Z_0(s,w) = \frac{1}{(1-q^{1-s})(1-q^{3-2s-2w})}.$$

For $Z_1(s,w)$, our remarks about $L(s,\chi_d)$ similarly imply

$$Z_1(s,w) = \sum_{k\ge 0} \frac{1}{q^{ks}q^{kw}} \sum_{\substack{m,d \text{ monic} \\ d \text{ a perfect square} \\ \deg(m)=k \\ \deg(d)=k}} a(m,d).$$

But then we may repeat the same argument as for $Z_0(s,w)$ with

$$Y(s,w) = \sum_{\substack{m,d \text{ monic} \\ d \text{ monic perfect square}}} \frac{a(m,d)}{|m|^s|d|^w} \quad \text{and} \quad K_1(s,w) = \sum_{k\ge 0} \frac{1}{q^{ks}q^{kw}},$$

and arrive at

$$Z_1(s,w) = \frac{1}{(1-q^{3-2s-2w})}.$$

Combining these representations for $Z_0(s,w)$ and $Z_1(s,w)$ with our decomposition of $Z(s,w)$ yields

$$Z(s,w) = \frac{1-q^{2-s-w}}{(1-q^{1-s})(1-q^{1-w})(1-q^{3-2s-2w})}.$$

Setting $x = q^{-s}$ and $y = q^{-w}$ gives

$$Z(x,y) = \frac{1-q^2xy}{(1-qx)(1-qy)(1-q^3x^2y^2)},$$

which is a rational function in x and y.

Remark 7.1. *Some authors, especially those using the Chinta-Gunnells construction of building multiple Dirichlet series, prefer the W-invariant point to be at the origin. For this, we make the change of variables* $\left(s+\frac{1}{2}, w+\frac{1}{2}\right) \to (s,w)$ *which gives*

$$Z(s,w) = \frac{1-q^{1-s-w}}{(1-q^{\frac{1}{2}-s})(1-q^{\frac{1}{2}-w})(2-q^{3-2s-2w})},$$

and upon setting $x = q^{-s}$ and $y = q^{-w}$, we have

$$Z(x,y) = \frac{1-qxy}{(1-q^{\frac{1}{2}}x)(1-q^{\frac{1}{2}}y)(1-q^2x^2y^2)}.$$

Generating Functions for the Weighting Coefficients

In this appendix we discuss properties of the generating function for the weighting coefficients $a(P^k,P^l)$ for a fixed irreducible P. This generating function has the same functional equations as $Z(s,w)$, and it is this property that motivated the Chinta-Gunnells construction. For a fixed irreducible P, define

$$H(s,w) = \sum_{k,l\geq 0} \frac{a(P^k,P^l)}{|P|^{ks}|P|^{lw}}.$$

From Corollary 4.1, $H(x,y)$ is locally absolutely uniformly convergent provided $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(w) > 1$. To obtain the generating function, we set $x = |P|^{-s}$ and $y = |P|^{-w}$ so that

$$H(x,y) = \sum_{k,l\geq 0} a(P^k,P^l)x^ky^l,$$

where this series is locally absolutely uniformly convergent for $|x| < |P|^{-1}$ and $|y| < |P|^{-1}$. Now let $x = |P|^{-s}$ in $Q_{P^{2\alpha+1}}(s)$ and $Q_{P^{2\beta}}(s)$ and denote the resulting functions as $Q_{P^{2\alpha+1}}(x)$ and $Q_{P^{2\beta}}(x)$. Then by Lemmas 4.1 and 4.2 and Corollary 4.1, we have

$$Q_{P^{2\alpha+1}}(x) = \sum_{k\geq 0} a(P^k,P^{2\alpha+1})x^k \quad \text{and} \quad \frac{1}{1-x}Q_{P^{2\beta}}(x) = \sum_{k\geq 0} a(P^k,P^{2\beta})x^k.$$

Note that these series are over every $k \geq 0$. In the first equality we have used the fact that $a(P^k,P^{2\alpha+1}) = 0$ if $k > 2\alpha$ and in the second equality we have used the geometric series representation $\frac{1}{1-x} = \sum_{k\geq 0} x^k$. Upon taking the subseries of $H(x,y)$ with l fixed, namely $\sum_{k\geq 0} a(P^k,P^l)x^k$, we obtain

$$\sum_{k\geq 0} a(P^k,P^l)x^k = \begin{cases} Q_{P^l}(x) & \text{if } l \text{ is odd,} \\ \frac{1}{1-x}Q_{P^l}(x) & \text{if } l \text{ is even.} \end{cases}$$

The functional equation Theorem 5.1 will induce functional equations for $Q_{P^l}(x)$ (l is even or odd) and this will further induce functional equations for $H(x,y)$. To see this, if $s \to 1-s$ then $x \to \frac{1}{|P|x}$ and so Theorem 5.1 implies the functional equation

$$Q_{P^{2l+i}}(x) = |P|^l x^{2l} Q_{P^{2l+i}}\left(\frac{1}{|P|x}\right),$$

for $i = 0,1$. Now let

$$H_y^+(x,y) = \frac{H(x,y)+H(x,-y)}{2} \quad \text{and} \quad H_y^-(x,y) = \frac{H(x,y)-H(x,-y)}{2},$$

be the even and odd parts of $H(x,y)$ with respect to y. That is, $H_y^+(x,y)$ is the subseries of $H(x,y)$ consisting of those terms with l even while $H_y^-(x,y)$ is the subseries of $H(x,y)$ consisting of those terms with l odd. Equivalently,

$$H_y^+(x,y) = \sum_{l\geq 0} Q_{P^{2l}}(x)y^{2l} \quad \text{and} \quad H_y^-(x,y) = \sum_{l\geq 0} Q_{P^{2l+1}}(x)y^{2l+1}.$$

Applying the functional equation for $Q_{P^{2l+i}}(x)$ to $H_y^+(x,y)$ and $H_y^-(x,y)$, we obtain functional equations

$$(1-x)H_y^+(x,y) = \left(1-\frac{1}{|P|x}\right)H_y^+\left(\frac{1}{|P|x}, \sqrt{|P|}xy\right) \quad \text{and} \quad H_y^-(x,y) = \frac{1}{\sqrt{|P|x}}H_y^-\left(\frac{1}{|P|x}, \sqrt{|P|}xy\right).$$

Similarly, if we look at the subseries of $H(x,y)$ with k fixed, and denote the analogous even and odd parts of $H(x,y)$ with respect to x by $H_x^+(x,y)$ and $H_x^-(x,y)$, then we have functional equations

$$(1-y)H_x^+(x,y) = \left(1-\frac{1}{|P|y}\right) H_x^+\left(\sqrt{|P|}xy, \frac{1}{|P|y}\right) \quad \text{and} \quad H_x^-(x,y) = \frac{1}{\sqrt{|P|y}} H_x^-\left(\sqrt{|P|}xy, \frac{1}{|P|y}\right).$$

We can state these four functional equations in a more compact form. For $f \in \mathbb{C}(x,y)$, let $f_y^{\pm}$ and $f_x^{\pm}$ be the even and odd parts of f with respect to y and x respectively. As $H(x,y) = H_y^+(x,y) + H_y^-(x,y)$ and $H(x,y) = H_x^+(x,y) + H_x^-(x,y)$, these pairs of functional equations are equivalent to $H(x,y)$ being invariant under the actions $|^{\mathrm{CG}}\sigma_1$ and $|^{\mathrm{CG}}\sigma_2$ on $\mathbb{C}(x,y)$ defined by

$$f(x,y)|^{\mathrm{CG}}\sigma_1 = -\frac{1-|P|x}{|P|x(1-x)} f_y^+\left(\frac{1}{|P|x}, \sqrt{|P|}xy\right) + \frac{1}{\sqrt{|P|x}} f_y^-\left(\frac{1}{|P|x}, \sqrt{|P|}xy\right),$$

and

$$f(x,y)|^{\mathrm{CG}}\sigma_2 = -\frac{1-|P|y}{|P|y(1-y)} f_x^+\left(\sqrt{|P|}xy, \frac{1}{|P|y}\right) + \frac{1}{\sqrt{|P|y}} f_x^-\left(\sqrt{|P|}xy, \frac{1}{|P|x}\right).$$

These two actions are precisely the Chinta-Gunnels actions corresponding to the simple reflections σ_1 and σ_2. It can be verified directly that these actions extend to an action of W on $\mathbb{C}(x,y)$. Therefore $H(x,y)$ has the same group of functional equations as $Z(s,w)$.

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