

# A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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ABSTRACT. We construct a quadratic double Dirichlet series  $Z(s, w)$  built from single variable quadratic Dirichlet  $L$ -functions  $L(s, \chi)$  over  $\mathbb{Q}$ . We prove that  $Z(s, w)$  admits meromorphic continuation to the  $(s, w)$ -plane and satisfies a group of functional equations.

## 1. PRELIMINARIES

We present an overview of quadratic Dirichlet  $L$ -functions over  $\mathbb{Q}$ . We begin with the Riemann zeta-function. The zeta function  $\zeta(s)$  is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m \geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for  $\operatorname{Re}(s) > 1$ . The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Now we recall characters on  $\mathbb{Z}$ . They are multiplicative functions  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ . They form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions  $\chi_d : \mathbb{Z} \rightarrow \mathbb{C}$  modulo  $d \geq 1$  (in that they are  $d$ -periodic) and such that  $\chi_d(m) = 0$  if  $(m, d) > 1$ .
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. If  $\chi$  is a Dirichlet character then its conjugate  $\bar{\chi}$  is also a Dirichlet character. Moreover,  $\bar{\chi}$  is the multiplicative inverse to  $\chi$  and the Dirichlet characters modulo  $m$  form a group under multiplication. This group is always finite and its order is  $\phi(d) = |(\mathbb{Z}/d\mathbb{Z})^*|$ . Dirichlet characters also satisfy orthogonality relations:

**Theorem 1.1** (Orthogonality relations).

(i) For any two Dirichlet characters  $\chi$  and  $\psi$  modulo  $d$ ,

$$\frac{1}{\phi(d)} \sum_{a \pmod{d}}' \chi(a) \bar{\psi}(a) = \delta_{\chi, \psi}.$$

(ii) For any  $a, b \in (\mathbb{Z}/d\mathbb{Z})^*$ ,

$$\frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \chi(a) \bar{\chi}(b) = \delta_{a, b}.$$

The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on  $\mathbb{Z}$ . First let us recall this symbol. For any odd prime  $p$  and any  $m \geq 1$ , we define the quadratic residue symbol  $\left(\frac{m}{p}\right)$  by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod{p} = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod{p} \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod{p} \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod{p}. \end{cases}$$

This symbol only depends upon  $m$  modulo  $p$  and is multiplicative in  $m$ . We can extend the quadratic residue symbol multiplicatively in the denominator. If  $d = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$  is the prime factorization of  $d$ , then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \leq i \leq k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any odd  $d \geq 1$ . We can extend this symbol further and allow  $d \geq 1$  to be even. To this end, we define

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases}$$

and extend  $\left(\frac{m}{d}\right)$  multiplicatively in  $d$  when  $d$  is even. Now the quadratic residue symbol makes sense for any  $m, d \geq 1$ . Moreover, it is multiplicative in both  $m$  and  $d$  but no longer depends upon only  $m$  modulo  $d$  (it also depends upon  $m$  modulo 8). In particular,

$$\left(\frac{-1}{d}\right) = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases}$$

and if  $d \not\equiv 0 \pmod{2}$ , we can compactly write

$$\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}} = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = (-1)^{\frac{d^2-1}{8}} = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}. \end{cases}$$

The quadratic residue symbol also admits the following reciprocity law:

**Theorem 1.2** (Quadratic reciprocity). *If  $d, m \geq 1$ , then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2} \frac{m^{(2)}-1}{2}} \left(\frac{m}{d}\right),$$

where  $d^{(2)}$  and  $m^{(2)}$  are the parts of  $d$  and  $m$  relatively prime to 2 respectively.

We can now define the quadratic Dirichlet characters. For any odd square-free  $d \in \mathbb{Z}$ , define the quadratic Dirichlet character  $\chi_d$  by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod{4}, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod{4}. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension  $\mathbb{Q}(\sqrt{d})$ . We extend  $\chi_d$  multiplicatively in the denominator so that  $\chi_d$  makes sense for any odd  $d$ . In particular,  $\chi_d(m) = \pm 1$  provided  $d$  and  $m$  are relatively prime and  $\chi_d(m) = 0$  if  $(m, d) > 1$ . Quadratic reciprocity implies that  $\chi_d$  is a Dirichlet character modulo  $d$  if  $d \equiv 1 \pmod{4}$  and is a Dirichlet character modulo  $4d$  if  $d \equiv 2, 3 \pmod{4}$ . Indeed, if  $d \equiv 1 \pmod{4}$  then  $d^{(2)} = d$  and the sign is always 1. If  $d \equiv 3 \pmod{4}$ , then  $d^{(2)} = d$  and the sign is

$\left(\frac{-1}{m}\right)$  which is a character modulo 4. If  $d \equiv 2 \pmod{4}$ , then  $d^{(2)} \equiv 1, 3 \pmod{4}$  and we are reduced to one of the previous two cases. We will also require an associated character. For each  $\chi_d$ , we define  $\tilde{\chi}_d$  by

$$\tilde{\chi}_d(m) = (-1)^{\frac{d^{(2)}-1}{2} \frac{m^{(2)}-1}{2}} \chi_d(m).$$

Equivalently,  $\tilde{\chi}_d(m)$  can be expressed as

$$\tilde{\chi}_d(m) = \begin{cases} \chi_d(m) & \text{if } d \equiv 1, 2 \pmod{4}, \\ \chi_{-1}(m)\chi_d(m) & \text{if } d \equiv 3 \pmod{4}, \end{cases}$$

and it follows that  $\tilde{\chi}_d(m)$  is a quadratic Dirichlet character of the same modulus as  $\chi_d$ . We now discuss the Hilbert characters. We will only need four of them: the quadratic Dirichlet characters modulo 8. They are given as follows:

$$\begin{aligned} \chi_1(m) &= \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod{2}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} & \chi_{-1}(m) &= \begin{cases} 1 & \text{if } m \equiv 1 \pmod{4}, \\ -1 & \text{if } m \equiv 3 \pmod{4}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} \\ \chi_2(m) &= \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} & \chi_{-2}(m) &= \begin{cases} 1 & \text{if } m \equiv 1, 3 \pmod{8}, \\ -1 & \text{if } m \equiv 5, 7 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}. \end{cases} \end{aligned}$$

In general, we will denote a Hilbert character by  $\chi_a$  with  $a \in \{\pm 1, \pm 2\}$ . Note that

$$\chi_{-1}(m) = \left(\frac{-1}{m}\right) \quad \text{and} \quad \chi_2(m) = \left(\frac{m}{2}\right).$$

Moreover, we have the relations

$$\chi_{-2}(m) = \chi_{-1}(m)\chi_2(m), \quad \chi_1(m) = \chi_{-1}(m)\chi_{-1}(m), \quad \text{and} \quad \chi_{-1}(m) = \chi_2(m)\chi_{-2}(m).$$

Suppose  $d$  is square-free. If  $d \equiv 1, 2, 5 \pmod{8}$ , then  $d^{(2)} \equiv 1 \pmod{4}$  so that the sign in the statement of quadratic reciprocity is 1. If  $d \equiv 3, 6, 7 \pmod{8}$ , then  $d^{(2)} \equiv 3 \pmod{4}$  and the sign is  $(-1)^{\frac{m^{(2)}-1}{2}}$ . This fact together with the relations for the quadratic characters modulo 8 imply

$$\chi_d(m) = \begin{cases} \chi_m(d) & \text{if } d \equiv 1 \pmod{4}, \\ \chi_{-1}(m)\chi_m(d) & \text{if } d \equiv 3 \pmod{4}, \\ \chi_2(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 2 \pmod{8}, \\ \chi_{-2}(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 6 \pmod{8}. \end{cases}$$

With the Dirichlet and Hilbert characters introduced, we are ready to discuss the  $L$ -functions associated to quadratic Dirichlet characters. We define the  $L$ -function  $L(s, \chi_d)$  attached to  $\chi_d$  for square-free  $d$ , by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \geq 1} \frac{\chi_d(m)}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character,  $L(s, \chi_d) \ll \zeta(s)$  for  $\text{Re}(s) > 1$  so that  $L(s, \chi_d)$  is locally absolutely uniformly convergent in this region.  $L(s, \chi_d)$  also admits analytic continuation to  $\mathbb{C}$ . The completed  $L$ -function  $L^*(s, \chi_d)$  is defined as

$$L^*(s, \chi_d) = \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi_d) & \text{if } \chi_d \text{ is even,} \\ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi_d) & \text{if } \chi_d \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_d) = \begin{cases} \varepsilon_\chi q^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \chi_d \text{ is even,} \\ -\varepsilon_\chi q^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \chi_d \text{ is odd.} \end{cases}$$

Note that the gamma factors depend upon the parity of  $\chi_d$ . This is the root cause of an important technical issue later when deriving functional equations for the quadratic double Dirichlet series. Analogously, the Dirichlet  $L$ -function  $L(s, \tilde{\chi}_d)$  attached to  $\tilde{\chi}_d$  for square-free  $d$  is defined by a Dirichlet series or Euler product:

$$L(s, \tilde{\chi}_d) = \sum_{m \geq 1} \frac{\tilde{\chi}_d(m)}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{\tilde{\chi}_d(p)}{p^s}\right)^{-1}.$$

As for  $L(s, \chi_d)$ ,  $L(s, \tilde{\chi}_d) \ll \zeta(s)$  for  $\text{Re}(s) > 1$  so that  $L(s, \tilde{\chi}_d)$  is locally absolutely uniformly convergent in this region. Moreover,  $L(s, \tilde{\chi}_d)$  admits analytic continuation to  $\mathbb{C}$  and the completed  $L$ -function  $L^*(s, \tilde{\chi}_d)$  is defined as

$$L^*(s, \tilde{\chi}_d) = \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \tilde{\chi}_d) & \text{if } \tilde{\chi}_d \text{ is even,} \\ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \tilde{\chi}_d) & \text{if } \tilde{\chi}_d \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \tilde{\chi}_d) = \begin{cases} \varepsilon_{\tilde{\chi}} q^{\frac{1}{2}-s} L^*(1-s, \tilde{\chi}_d) & \text{if } \tilde{\chi}_d \text{ is even,} \\ -\varepsilon_{\tilde{\chi}} q^{\frac{1}{2}-s} L^*(1-s, \tilde{\chi}_d) & \text{if } \tilde{\chi}_d \text{ is odd.} \end{cases}$$

**Remark 1.1.** *The definitions for  $L(s, \chi_d)$  and  $L^*(s, \chi_d)$  work perfectly well even when  $d$  is not square-free (however the functional equations do not hold). We purposely do not define these  $L$ -functions, yet, for  $d$  not necessarily square-free.*

## THE QUADRATIC DOUBLE DIRICHLET SERIES

We will now define the quadratic double Dirichlet series  $Z(s, w)$ . For any integer  $d \geq 1$ , write  $d = d_0 d_1^2$  where  $d_0$  is square-free. Equivalently,  $d_0$  is the square-free part of  $d$  and  $\frac{d}{d_0}$  is a perfect square. The **quadratic double Dirichlet series**  $Z(s, w)$  is defined as

$$Z(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{d_0}) Q_{d_0 d_1^2}(s)}{d^w},$$

where the superscript (2) indicates that the local factor at 2 has been removed,  $Q_{d_0 d_1^2}(s)$  is the **correction polynomial** defined by

$$Q_{d_0 d_1^2}(s) = \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{d_0}(e_1) e_1^{-s} e_2^{1-s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{d_0}(e_1) e_1^{-s} e_2^{1-s},$$

and  $\mu$  is the usual Möbius function. For  $\text{Re}(s) > 1$ , there is the trivial estimate

$$Q_{d_0 d_1^2}(s) \ll \sum_{e_1 e_2 | d_1} 1 \ll \sigma_0(d_1)^2 \ll_\varepsilon d_1^{2\varepsilon} \ll_\varepsilon d^\varepsilon,$$

for any  $\varepsilon > 0$ . As  $L(s, \chi_{d_0}) \ll 1$  for  $\text{Re}(s) > 1$ ,  $Z(s, w)$  is locally absolutely uniformly convergent in the region  $\Lambda = \{(s, w) \in \mathbb{C}^2 : \text{Re}(s) > 1, \text{Re}(w) > 1\}$ . It will also be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters  $\chi_{a_1}$  and  $\chi_{a_2}$ . The **quadratic double Dirichlet series**  $Z_{a_1, a_2}(s, w)$  twisted by  $\chi_{a_1}$  and  $\chi_{a_2}$  is defined as

$$Z_{a_1, a_2}(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w},$$

where  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  is the **correction polynomial** twisted by  $\chi_{a_1}$  defined by

$$Q_{d_0 d_1^2}(s, \chi_{a_1}) = \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s},$$

and  $\mu$  is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound  $Q_{d_0 d_1^2}(s, \chi_{a_1}) \ll d_\varepsilon$  so that  $Z_{a_1, a_2}(s, w)$  converges locally absolutely uniformly in the same region as  $Z(s, w)$  does. In particular,  $Z(s, w) = Z_{1,1}(s, w)$ .

### THE INTERCHANGE

As defined,  $Z_{a_1, a_2}(s, w)$  is a sum of  $L$ -functions, and hence Euler products, in  $s$ . We will prove an interchange formula for  $Z_{a_1, a_2}(s, w)$  which will show that it can be expressed as a sum of  $L$ -functions in  $w$ . That is, we want the variables  $s$  and  $w$  to change places. Precisely:

**Theorem 1.3** (Interchange). *Wherever  $Z_{a_1, a_2}(s, w)$  converges locally absolutely uniformly,*

$$Z_{a_1, a_2}(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w} = \sum_{m \text{ odd}} \frac{L^{(2)}(w, \tilde{\chi}_{a_2 m_0}) \chi_{a_1}(m) Q_{m_0 m_1^2}(w, \chi_{a_2})}{m^s}.$$

*Proof.* Only the second equality needs to be proved. To do this, first expand the  $L$ -function  $L^{(2)}(s, \chi_{a_1 d_0})$  and polynomial  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  to get

$$\begin{aligned} Z(s, w) &= \sum_{d \text{ odd}} \frac{L(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w} \\ &= \sum_{d \text{ odd}} \left( \sum_{m \text{ odd}} \chi_{a_1 d_0}(m) m^{-s} \right) \left( \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} \right) \chi_{a_2}(d) d^{-w} \\ &= \sum_{m, d \text{ odd}} \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m e_1) e_1^{-s} e_2^{1-2s} m^{-s} d^{-w}. \end{aligned}$$

Now  $\chi_{a_1 d_0}(m e_1) = 0$  unless  $(d_0, m e_1) = 1$ . We make this restriction on the sum giving

$$\sum_{m, d \text{ odd}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, m e_1) = 1}} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m e_1) e_1^{-s} e_2^{1-2s} m^{-s} d^{-w}.$$

Making the change of variables  $m e_1 \rightarrow m$  yields

$$\sum_{d \text{ odd}} \sum_{\substack{m \text{ odd} \\ e_1 | m}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, m) = 1}} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m) e_2^{1-2s} m^{-s} d^{-w}.$$

For fixed  $d = d_0 d_1^2$  and  $e_2$ , the subsum over  $m$  and  $e_1$  is

$$\sum_{\substack{m \text{ odd} \\ e_1 | m}} \sum_{\substack{e_1 | \frac{d_1}{e_2} \\ (d_0, m) = 1}} \mu(e_1) \chi_{a_1 d_0}(m) m^{-s} = \sum_{\substack{m \text{ odd} \\ (d_0, m) = 1}} \chi_{a_1 d_0}(m) m^{-s} \left( \sum_{e_1 | \left( \frac{d_1}{e_2}, m \right)} \mu(e_1) \right).$$

The inner sum over  $e_1$  of the Möbius function vanishes unless  $\left( \frac{d_1}{e_2}, m \right) = 1$  in which case it is 1. Therefore the triple sum above becomes

$$\sum_{m, d \text{ odd}} \sum_{\substack{e_2 | d_1 \\ \left( \frac{d_0 d_1}{e_2}, m \right) = 1}} \chi_{a_2}(d) \chi_{a_1 d_0}(m) e_2^{1-2s} m^{-s} d^{-w}.$$

Making the change of variables  $d \rightarrow de_2^2$ , the condition  $\left(\frac{d_0 d_1}{e_2}, m\right) = 1$  becomes  $(d_0 d_1, m) = 1$  which is equivalent to  $(d, m) = 1$ . Moreover,  $\chi_{a_2}(de_2^2) = \chi_{a_2}(d)$ . Altogether, we obtain

$$\sum_{\substack{m, d \text{ odd} \\ (d, m)=1}} \sum_{e_2} \chi_{a_2}(d) \chi_{a_1 d_0}(m) e_2^{1-2s-2w} m^{-s} d^{-w}.$$

Writing  $m = m_0 m_1^2$  analogously as for  $d$ , quadratic reciprocity implies  $\chi_{d_0}(m) = \tilde{\chi}_m(d_0) = \tilde{\chi}_{m_0}(d)$  where the last equality holds because  $(d, m) = 1$  and both  $d_0$  and  $m_0$  differ from  $d$  and  $m$  respectively by perfect squares. This implies  $\chi_{a_2}(d) \chi_{a_1 d_0}(m) = \chi_{a_1}(m) \tilde{\chi}_{a_2 m_0}(d)$  and so our expression becomes

$$\sum_{\substack{d, m \geq 1 \\ (d, m)=1}} \sum_{e_2} \chi_{a_1}(m) \tilde{\chi}_{a_2 m_0}(d) e_2^{1-2s-2w} m^{-s} d^{-w}.$$

But now we can reverse the argument with the roles of  $d$ ,  $m$ ,  $\chi_{a_1}$ , and  $\tilde{\chi}_{a_2}$  interchanged respectively to obtain

$$Z(s, w) = \sum_{m \geq 1} \frac{L(w, \tilde{\chi}_{a_2 m_0}) \chi_{a_1}(m) Q_{m_0 m_1^2}(w, \chi_{a_2})}{m^s}.$$

□

Note that the interchange is not completely symmetric because of the character  $\tilde{\chi}_{a_2 m_0}$  in the second expression for  $Z_{a_1, a_2}(s, w)$ . This is due to the fact that reciprocity is not perfect. In even more general settings the correction polynomials in  $w$  need not be equal to those in  $s$ .

### WEIGHTING TERMS

We will now study the coefficients of  $Z_{a_1, a_2}(s, w)$  expanded in  $s$  and  $w$ . By expanding  $L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1})$  in the numerator of  $Z_{a_1, a_2}(s, w)$ , we can write

$$Z_{a_1, a_2}(s, w) = \sum_{d \geq 1} \frac{L(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w} = \sum_{d, m \geq 1} \frac{\chi_{a_1 d_0}(\hat{m}) \chi_{a_2}(d) a(m, d)}{m^s d^w},$$

where  $\hat{m}$  is the part of  $m$  relatively prime to  $d_0$  and the **weighting coefficient**  $a(m, d)$  is given by

$$a(m, d) = \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2.$$

To see this, the coefficient of  $m^{-s} d^{-w}$  in the definition of  $Z_{a_1, a_2}(s, w)$  is

$$\begin{aligned} \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1}} \mu(e_1) \chi_{a_1 d_0}(e_1 e_3) e_2 &= \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) \chi_{a_1 d_0}(e_1 e_3) e_2 \\ &= \chi_{a_1 d_0}(\hat{m}) \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2 \\ &= \chi_{a_1 d_0}(\hat{m}) \chi_{a_2}(d) a(m, d), \end{aligned}$$

where the first equality holds because  $\chi_{d_0}(e_1 e_3) = 0$  unless  $(d_0, e_1 e_3) = 1$  and the second equality holds because if  $(d_0, e_1 e_3) = 1$ ,  $\hat{m}$  differs from  $e_1 e_3$  by a perfect square (the divisors of which belong to  $(d_0, e_2)$ ) and so  $\chi_{d_0}(e_1 e_3) = \chi_{d_0}(\hat{m})$ .

**Remark 1.2.** Also,  $a(m, d) = 0$  unless  $m = e_1 e_2^2 e_3$  with  $(d_0, e_1 e_3) = 1$  and  $e_1 e_2^2 | d_1$ .

We will define  $L(s, \chi_{a_1 d})$  to be the Dirichlet series given by

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1}) = \sum_{m \geq 1} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m, d)}{m^s}.$$

In particular,  $L(s, \chi_d)$  now makes sense for  $d$  not necessarily square-free and this definition agrees with the former when  $d$  is square-free. Moreover, we have the representation

$$Z_{a_1, a_2}(s, w) = \sum_{d \geq 1} \frac{\chi_{a_2}(d) L(s, \chi_{a_1 d})}{d^w}.$$

If we perform the same procedure to the interchange, then

$$Z_{a_1, a_2}(s, w) = \sum_{m \geq 1} \frac{L(w, \widetilde{\chi}_{a_2 m_0}) \chi_{a_1}(m) Q_{m_0 m_1^2}(w, \chi_{a_2})}{m^s} = \sum_{d, m \geq 1} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d}) \chi_{a_1}(m) a(d, m)}{m^s d^w},$$

where  $\widehat{d}$  is the part of  $d$  relatively prime to  $m_0$ . Analogously, we define  $L(w, \widetilde{\chi}_{a_2 m})$  to be the Dirichlet series given by

$$L(w, \widetilde{\chi}_{a_2 m}) = L(w, \widetilde{\chi}_{a_2 m_0}) Q_{m_0 m_1^2}(w, \chi_{a_2}) = \sum_{d \geq 1} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d}) a(d, m)}{d^w},$$

so that

$$Z_{a_1, a_2}(s, w) = \sum_{m \geq 1} \frac{\chi_{a_1}(m) L(w, \widetilde{\chi}_{a_2 m})}{m^s}.$$

We now investigate the structure of the weighting coefficients  $a(m, d)$ . Their structure controls the majority of the information about both the double Dirichlet series and the correction polynomials. We first show that the weighting coefficients possess a multiplicativity property:

**Proposition 1.1.** *We have  $a(m, 1) = a(1, d) = 1$  and*

$$a(m, d) = \prod_{\substack{p^\alpha || m \\ p^\beta || d}} a(p^\alpha, p^\beta).$$

*Proof.* From the definition of the weighting coefficients,  $a(m, 1) = a(1, d) = 1$ . We will prove multiplicativity in  $m$  and then in  $d$ . Letting  $m = m' p^\alpha$ , we must show

$$a(m, d) = a(m', d) a(p^\alpha, d).$$

To accomplish this, for  $e_1 e_2^2 e_3 = m$ , let  $e_1 = c_1 d_1$ ,  $e_2 = c_2 d_2$ , and  $e_3 = c_3 d_3$  with  $c_1, c_2, c_3 \mid m'$  and  $d_1, d_2, d_3 \mid p^\alpha$ . Because  $(m', p^\alpha) = 1$ , as  $e_1 e_2^2 e_3$  runs over decompositions of  $m$ ,  $c_1 c_2^2 c_3$  and  $d_1 d_2^2 d_3$  run over decompositions of  $m'$  and  $p^\alpha$  respectively. Moreover, as  $e_1 e_2$  runs over the divisors of  $d_1$  so does  $c_1 d_1 c_2 d_2$ .

These facts combined with multiplicativity of the Möbius function gives

$$\begin{aligned}
a(m, d) &= \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2 \\
&= \sum_{\substack{c_1 c_2^2 c_3 = m' \\ d_1 d_2^2 d_3 = p^\beta \\ c_1 d_1 c_2 d_2 | d_1 \\ (d_0, c_1 d_1 c_3 d_3) = 1}} \mu(c_1) (d_1) |c_2| d_2 \\
&= \left( \sum_{\substack{c_1 c_2^2 c_3 = m' \\ c_1 c_2 | d_1 \\ (d_0, c_1 c_3) = 1}} \mu(c_1) |c_2| \right) \left( \sum_{\substack{d_1 d_2^2 d_3 = p^\alpha \\ d_1 d_2 | d_1 \\ (d_0, d_1 d_3) = 1}} \mu(d_1) d_2 \right) \\
&= a(m', d) a(p^\alpha, d),
\end{aligned}$$

as desired. Now we prove multiplicativity in  $d$ . Since we have already proven multiplicativity in  $m$ , we may assume  $m = p^\alpha$ . Letting  $d = d' p^\beta$ , we must show

$$a(p^\alpha, d) = a(p^\alpha, p^\beta).$$

As  $e_1 e_2^2 e_3 = p^\alpha$ , the  $e_i$  are powers of  $p$  for  $1 \leq i \leq 3$ . It follows that  $e_1 e_2 | d_1$  is equivalent to  $e_1 e_2 | p^\beta$ . Moreover,  $(d_0, e_1 e_2) = 1$  is equivalent to  $(1, e_1 e_2) = 1$  or  $(p, e_1 e_2) = 1$  depending on if  $\beta$  is even or odd. These facts imply the desired identity.  $\square$

The correction polynomials  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  are tightly connected to the weighting coefficients  $a(m, d)$ . In particular,  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients. We first prove this relationship when  $d$  is an odd prime power:

**Lemma 1.1.** *For any prime  $p$  and  $\alpha \geq 1$ , we have*

$$Q_{p^{2\alpha+1}}(s) = \sum_{k \leq 2\alpha} \frac{a(p^k, p^{2\alpha+1})}{p^{ks}}.$$

*Proof.* Expanding the correction polynomial in  $p^{-s}$  yields

$$Q_{p^{2\alpha+1}}(s) = \sum_{e_1 e_2 | p^\alpha} \mu(e_1) \chi_p(e_1) e_1^{-s} e_2^{1-2s} = \sum_{k \leq 2\alpha} \frac{b(p^k, p^{2\alpha+1})}{p^{ks}}.$$

where

$$b(p^k, p^{2\alpha+1}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1) \chi_p(e_1) e_2.$$

The proof will be finished if we can show  $b(p^k, p^{2\alpha+1}) = a(p^k, p^{2\alpha+1})$ . To see this, first observe  $\mu(e_1) \chi_p(e_1) = 0$  unless  $e_1 = 1$  in which case it is 1. So  $b(p^k, p^{2\alpha+1}) = 0$  if  $k$  is odd and  $p^{\frac{k}{2}}$  if  $k$  is even. Compactly stated,

$$b(p^k, p^{2\alpha+1}) = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$



On the other hand,  $k \leq \alpha$  so that

$$a(p^k, p^{2\alpha+1}) = \sum_{\substack{e_1 e_2^2 e_3 = p^k \\ e_1 e_2 | p^\alpha \\ (p, e_1 e_3) = 1}} \mu(e_1) e_2 = \sum_{\substack{e_1 e_2^2 | p^k \\ (p, e_1 e_3) = 1}} \mu(e_1) e_2 = \sum_{e_2^2 = p^k} e_2 = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

This finishes the proof.  $\square$

There is an analogous statement when  $d$  is an even prime power up to a square-free factor and relatively prime factor:

**Lemma 1.2.** *For any square-free integer  $d_0 \geq 1$ ,  $a_1 \in \{\pm 1, \pm 2\}$ , prime  $p$  not dividing  $d_0$ , and  $\beta \geq 1$ , we have*

$$Q_{d_0 p^{2\beta}}(s, \chi_{a_1}) = (1 - \chi_{a_1 d_0}(p) p^{-s}) \sum_{k \leq 2\beta} \frac{\chi_{a_1 d_0}(p^k) a(p^k, p^{2\beta})}{p^{ks}}.$$

*Proof.* Expand the correction polynomial in  $p^{-s}$  to get

$$Q_{d_0 p^{2\beta}}(s, \chi_{a_1}) = \sum_{e_1 e_2 | p^\alpha} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} = \sum_{k \leq 2\beta} \frac{b(p^k, p^{2\beta})}{p^{ks}}.$$

where

$$b(p^k, p^{2\beta}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1) \chi_{a_1 d_0}(e_1) e_2.$$

It suffices to show  $b(p^k, p^{2\beta}) = \chi_{a_1 d_0}(p^k) (a(p^k, p^{2\beta}) - a(p^{k-1}, p^{2\beta}))$ . On the one hand,  $\mu(e_1) = 0$  unless  $e_1 = 1, p$  in which case  $\mu(e_1) = \pm 1$  accordingly. So

$$b(p^k, p^{2\beta}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1) \chi_{a_1 d_0}(e_1) e_2 = \begin{cases} \chi_{a_1 d_0}(p^k) p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(p^k) p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

where we have used the identity  $\chi_{a_1 d_0}(e_1) = \chi_{a_1 d_0}(p^k)$  which holds because this quadratic Dirichlet character only depends upon the parity of  $k$ . On the other hand, as in the proof of Lemma 1.1

$$a(p^k, p^{2\beta}) = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

But then

$$\chi_{a_1 d_0}(p^k) (a(p^k, p^{2\beta}) - a(p^{k-1}, p^{2\beta})) = \begin{cases} \chi_{a_1 d_0}(p^k) p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(p^k) p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

which completes the proof.  $\square$

Lemmas 1.1 and 1.2 together show that  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients  $a(m, d)$  when  $d$  is an prime power. The proof of these lemmas also give the value of  $a(p^k, p^l)$  and we collect this as a corollary:

**Corollary 1.1.** *For any prime  $p$ ,*

$$a(p^k, p^l) = \begin{cases} \min\left(p^{\frac{k}{2}}, p^{\frac{l}{2}}\right) & \text{if } \min(k, l) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

If we combine Proposition 1.1 and Corollary 1.1 we can compute  $a(m, d)$  in general:

**Corollary 1.2.** *For any integers  $d, m \geq 1$ ,*

$$a(m, d) = \begin{cases} (m, d)^{\frac{1}{2}} & \text{if } (m, d) \text{ is a perfect square,} \\ 0 & \text{otherwise.} \end{cases}$$

As an immediate consequence of Corollary 1.2,  $a(m, d)$  is symmetric in  $m$  and  $d$ . As the weighting coefficients are multiplicative,  $Q_{d_0 d_1^2}(s, \chi_{a_1})$  will possess an Euler product. To state the Euler product explicitly, we write  $d = d_0 d_1^2 d_2^2$  with  $d_0$  square-free and,  $d_2$  relatively prime to  $d_0 d_1$ , and such that every prime divisor of  $d_1$  divides  $d_0$ . In other words,  $d_0$  is the square-free part of  $d$ ,  $d_1$  is the square part of  $d$  whose prime factors divide  $d$  to odd power, and  $d_2$  is the square part of  $d$  whose prime factors divide  $d$  to even power. We have the following Euler product:

**Theorem 1.4.** *Let  $d = d_0 d_1^2 d_2^2$  be the square decomposition of  $d$  stratified by even and odd powers. Then for any  $a_1 \in \{\pm 1, \pm 2\}$ ,*

$$Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = \prod_{p^\alpha || d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta || d_2} Q_{d_0 p^{2\beta}}(s, \chi_{a_1}).$$

*Proof.* Recall that

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = \sum_{m \geq 1} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m, d)}{m^s}.$$

We will now derive an alternate expression for  $L(s, \chi_{a_1 d})$ . By Proposition 1.1, the coefficients of  $L(s, \chi_{a_1 d})$  are multiplicative. Therefore  $L(s, \chi_{a_1 d})$  admits the Euler product

$$L(s, \chi_{a_1 d}) = \prod_{p \text{ prime}} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, d)}{p^{ks}} \right).$$

Decomposing the product according to primes dividing  $d = d_0 d_1^2 d_2^2$ , we get

$$\begin{aligned} & L(s, \chi_{a_1 d}) \\ &= \prod_{p \text{ prime}} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, d)}{p^{ks}} \right) \\ &= \prod_{p \nmid d} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, 1)}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^\beta)}{p^{ks}} \right) \\ &= \prod_{p \nmid d} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k}) a(p^k, p^\beta)}{p^{ks}} \right) \\ &= \prod_{p \nmid d} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left( \sum_{k \geq 0} \frac{a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(p^k) a(p^k, p^\beta)}{p^{ks}} \right). \end{aligned}$$

Including the factors corresponding to primes  $p \mid d_2$  into the first product, we must multiply the last factor by the inverse of  $\sum_{k \geq 0} \chi_{a_1 d_0}(p) p^{-ks} = (1 - \chi_{a_1 d_0}(p) p^{-s})^{-1}$  obtaining

$$\prod_{p \nmid d_0} \left( \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(\widehat{p^k})}{p^{ks}} \right) \prod_{p^\alpha || d_1} \left( \sum_{k \geq 0} \frac{a(p^k, p^{2\alpha+1})}{p^{ks}} \right) \cdot \prod_{p^\beta || d_2} \left( (1 - \chi_{a_1 d_0}(p) p^{-s}) \sum_{k \geq 0} \frac{\chi_{a_1 d_0}(p^k) a(p^k, p^\beta)}{p^{ks}} \right),$$

as every prime divisor of  $d_1$  divides  $d_0$ . The first product is  $L(s, \chi_{a_1 d_0})$ . For the second and third products, Remark 1.2 implies that the sums run up to  $k \leq 2\alpha$  and  $k \leq 2\beta$  respectively. Therefore they are  $Q_{p^{2\alpha+1}}(s)$

and  $Q_{d_0 p^{2\beta}}(s, \chi_{a_1})$  respectively. It follows that

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) \cdot \prod_{p^\alpha || d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta || d_2} Q_{d_0 p^{2\beta}}(s, \chi_{a_1}).$$

This is our alternate expression for  $L(s, \chi_{a_1 d})$  and equating the two results in

$$L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1}) = L(s, \chi_{a_1 d_0}) \cdot \prod_{p^\alpha || d_1} Q_{p^{2\alpha+1}}(s) \cdot \prod_{p^\beta || d_2} Q_{d_0 p^{2\beta}}(s, \chi_{a_1}),$$

from which the proof is complete since  $L(s, \chi_{a_1 d_0}) \neq 0$  for  $\text{Re}(s) > 1$  (so that we may divide by  $L(s, \chi_{a_1 d_0})$ ).  $\square$

Observe that for  $d = d_0 d_1^2 d_2^2$ , the prime factors that divide  $d_1 d_2$  are exactly those factors that divide  $d$  to power larger than 1. Thus, from Theorem 1.4 the Euler product for  $Q_{d_0 d_1^2 d_2^2}(s, \chi_{a_1})$  is supported on exactly the primes dividing  $d$  to order larger than 1 and also depends upon the character  $\chi_{a_1 d_0}$ .

## FUNCTIONAL EQUATIONS

## MEROMORPHIC CONTINUATION

## POLES AND RESIDUES

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