A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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Abstract. We construct a quadratic double Dirichlet series $Z(s,w)$ built from single variable quadratic Dirichlet L-functions $L(s,\chi)$ over $\mathbb{Q}$. We prove that $Z(s,w)$ admits meromorphic continuation to the (s,w)-plane and satisfies a group of functional equations.

1. Preliminaries

We present an overview of quadratic Dirichlet L-functions over $\mathbb{Q}$. We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m\geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Now we recall characters on $\mathbb{Z}$. They are multiplicative functions $\chi : \mathbb{Z} \to \mathbb{C}$. They form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions $\chi_d : \mathbb{Z} \to \mathbb{C}$ modulo $d \geq 1$ (in that they are d-periodic) and such that $\chi_d(m) = 0$ if $(m,d) > 1$.
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. If χ is a Dirichlet character then its conjugate $\overline{\chi}$ is also a Dirichlet character. Moreover, $\overline{\chi}$ is the multiplicative inverse to χ and the Dirichlet characters modulo m form a group under multiplication. This group is always finite and its order is $\phi(d) = |(\mathbb{Z}/d\mathbb{Z})^*|$. Dirichlet characters also satisfy orthogonality relations:

Theorem 1.1 (Orthogonality relations)**.**

(i) For any two Dirichlet characters χ and ψ modulo d,

$$\frac{1}{\phi(d)} \sideset{}{'}\sum_{a \pmod d} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

(ii) For any $a, b \in (\mathbb{Z}/d\mathbb{Z})^*$,

$$\frac{1}{\phi(d)} \sum_{\chi \pmod d} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

Date: 2024.

The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{Z}$. First let us recall this symbol. For any odd prime p and any $m \geq 1$, we define the quadratic residue symbol $\left(\frac{m}{p}\right)$ by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod{p} = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod{p} \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod{p} \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod{p}. \end{cases}$$

This symbol only depends upon m modulo p and is multiplicative in m. We can extend the quadratic residue symbol multiplicatively in the denominator. If $d = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of d, then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \leq i \leq k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any odd $d \geq 1$. We can extend this symbol further and allow $d \geq 1$ to be even. To this end, we define

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases}$$

and extend $\left(\frac{m}{d}\right)$ multiplatively in d when d is even. Now the quadratic residue symbol makes sense for any $m, d \geq 1$. Moreover, it is multiplicative in both m and d but no longer depends upon only m modulo d (it also depends upon m modulo 8). In particular,

$$\left(\frac{-1}{d}\right) = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases}$$

and if $d \not\equiv 0 \pmod{2}$, we can compactly write

$$\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}} = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = (-1)^{\frac{d^2-1}{8}} = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}. \end{cases}$$

The quadratic residue symbol also admits the following reciprocity law:

Theorem 1.2 (Quadratic reciprocity)**.** *If $d, m \geq 1$ are relatively prime, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2}\frac{m^{(2)}-1}{2}} \left(\frac{m}{d}\right),$$

where $d^{(2)}$ and $m^{(2)}$ are the parts of d and m relatively prime to 2 respectively.

We can now define the quadratic Dirichlet characters. For any odd square-free $d \geq 1$, define the quadratic Dirichlet character χ_d by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod{4}, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod{4}. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension $\mathbb{Q}(\sqrt{d})$. We extend χ_d multiplicatively in the denominator so that χ_d makes sense for any odd $d \geq 1$. In particular, $\chi_d(m) = \pm 1$ provided d and m are relatively prime and $\chi_d(m) = 0$ if $(m, d) > 1$. Quadratic reciprocity implies that χ_d is a Dirichlet character modulo d if $d \equiv 1 \pmod{4}$ and is a Dirichlet character modulo $4d$ if $d \equiv 2, 3 \pmod{4}$. Indeed, if $d \equiv 1 \pmod{4}$ then $d^{(2)} = d$ and the sign is always 1. If $d \equiv 3 \pmod{4}$, then $d^{(2)} = d$ and the sign is $\left(\frac{-1}{m}\right)$ which is a character modulo 4. If $d \equiv 2 \pmod{4}$, then $d^{(2)} \equiv 1, 3 \pmod{4}$ and we are reduced to one of the previous two cases. We now discuss the Hilbert characters. We will only need four of them: the quadratic Dirichlet characters modulo 8. We define them as follows:

$$\psi_1(m) = \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod{2}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} \qquad \psi_{-1}(m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{4}, \\ -1 & \text{if } m \equiv 3 \pmod{4}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases}$$

$$\psi_2(m) = \begin{cases} 1 & \text{if } m \equiv 1,7 \pmod{8}, \\ -1 & \text{if } m \equiv 3,5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} \qquad \psi_{-2}(m) = \begin{cases} 1 & \text{if } m \equiv 1,3 \pmod{8}, \\ -1 & \text{if } m \equiv 5,7 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}. \end{cases}$$

In general, we will denote these Hilbert characters by ψ_a with $a \in \{\pm 1, \pm 2\}$. Now, we can write ψ_{-1} and ψ_2 in terms of Legendre symbols:

$$\psi_{-1}(m) = \left(\frac{-1}{m}\right) \quad \text{and} \quad \psi_2(m) = \left(\frac{m}{2}\right).$$

Moreover, these characters satisfy the relations

$$\psi_{-2}(m) = \psi_{-1}(m)\psi_2(m), \quad \psi_1(m) = \psi_{-1}(m)\psi_{-1}(m), \quad \text{and} \quad \psi_{-1}(m) = \psi_2(m)\psi_{-2}(m).$$

Suppose d is square-free. If $d \equiv 1,2,5 \pmod{8}$, then $d^{(2)} \equiv 1 \pmod{4}$ so that the sign in the statement of quadratic recipricty is 1. If $d \equiv 3,6,7 \pmod{8}$, then $d^{(2)} \equiv 3 \pmod{4}$ and the sign is $(-1)^{\frac{m^{(2)}-1}{2}}$. This fact together with the relations for the quadratic characters modulo 8 imply

$$\chi_d(m) = \begin{cases} \chi_m(d) & \text{if } d \equiv 1 \pmod{4}, \\ \chi_{-1}(m)\chi_m(d) & \text{if } d \equiv 3 \pmod{4}, \\ \chi_2(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 2 \pmod{8}, \\ \chi_{-2}(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 6 \pmod{8}. \end{cases}$$

With the Dirichlet and Hilbert characters introduced, we are ready to discuss the L-functions associated to quadratic Dirichlet characters. We define the L-function $L(s,\chi_d)$ attached to χ_d by a Dirichlet series or Euler product:

$$L(s,\chi_d) = \sum_{m \geq 1} \frac{\chi_d(m)}{|m|^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi_d(P)}{|P|^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character, $L(s,\chi_d) \ll \zeta(s)$ for $\mathrm{Re}(s) > 1$ so that $L(s,\chi_d)$ is locally absolutely uniformly convergent in this region. $L(s,\chi_d)$ also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s=1$ if d is a perfect square. For square-free d, the completed L-function is defined as

$$L^*(s,\chi_d) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) L(s,\chi_d) & \text{if } \chi_d \text{ is even}, \\ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right) L(s,\chi_d) & \text{if } \chi_d \text{ is odd}, \end{cases}$$

and satisfies the functional equation

$$L^*(s,\chi_d) = \begin{cases} \varepsilon_\chi q^{\frac{1}{2}-s} L^*(1-s,\chi_d) & \text{if } \chi_d \text{ is even}, \\ -\varepsilon_\chi q^{\frac{1}{2}-s} L^*(1-s,\chi_d) & \text{if } \chi_d \text{ is odd}. \end{cases}$$

Note that the gamma factors depend upon the partiy of χ_d. This the root cause of an important technical issue later when deriving functional equations for the quadratic double Dirichlet series.

The Quadratic Double Dirichlet Series

We will now define the quadratic double Dirichlet series $Z(s,w)$. For any integer $d \geq 1$, write $d = d_0 d_1^2$ where d_0 is square-free. Equivalently, d_0 is the square-free part of d and $\frac{d}{d_0}$ is a perfect square. The **quadratic double Dirichlet series** $Z(s,w)$ is defined as

$$Z(s,w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s,\chi_{d_0}) Q_{d_0 d_1^2}(s)}{d^w},$$

where the superscript (2) indicates that the local factor at 2 has been removed, $Q_{d_0 d_1^2}(s)$ is the **correction polynomial** defined by

$$Q_{d_0 d_1^2}(s) = \sum_{e_1 e_2 | d_1} \mu(e_1)\chi_{d_0}(e_1) e_1^{-s} e_2^{1-s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\chi_{d_0}(e_1) e_1^{-s} e_2^{1-s},$$

and μ is the usual Möbius function. For $\mathrm{Re}(s) > 1$, there is the trivial estimate

$$Q_{d_0 d_1^2}(s) \ll \sum_{e_1 e_2 | d_1} 1 \ll \sigma_0(d_1)^2 \ll_\varepsilon |d_1^2|^\varepsilon \ll_\varepsilon |d|^\varepsilon,$$

for any $\varepsilon > 0$. As $L(s,\chi_{d_0}) \ll 1$ for $\mathrm{Re}(s) > 1$, $Z(s,w)$ is locally absolutely uniformly convergent in the region $\Lambda = \{(s,w) \in \mathbb{C}^2 : \mathrm{Re}(s) > 1, \mathrm{Re}(w) > 1\}$. It will also be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters ψ_{a_1} and ψ_{a_2}. The **quadratic double Dirichlet series** $Z_{a_1,a_2}(s,w)$ twisted by ψ_{a_1} and ψ_{a_2} is defined as

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s,\psi_{a_1}\chi_{d_0})\psi_{a_2}(d) Q_{d_0 d_1^2}(s,\psi_{a_1})}{|d|^w},$$

where $Q_{d_0 d_1^2}(s,\psi_{a_1})$ is the **correction polynomial** twisted by ψ_{a_1} defined by

$$Q_{d_0 d_1^2}(s,\psi_{a_1}) = \sum_{e_1 e_2 | d_1} \mu(e_1)\psi_{a_1}(e_1)\chi_{d_0}(e_1)|e_1|^{-s}|e_2|^{1-2s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1)\psi_{a_1}(e_1)\chi_{d_0}(e_1)|e_1|^{-s}|e_2|^{1-2s},$$

and μ is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound $Q_{d_0 d_1^2}(s,\psi_{a_1}) \ll |d|_\varepsilon$ so that $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly in the same region as $Z(s,w)$ does. In particular, $Z(s,w) = Z_{1,1}(s,w)$.

The Interchange

Weighting Terms

Functional Equations

Meromorphic Continuation

Poles and Residues

References

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