

A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

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ABSTRACT. We construct a quadratic double Dirichlet series $Z(s, w)$ built from single variable quadratic Dirichlet L -functions $L(s, \chi)$ over \mathbb{Q} . We prove that $Z(s, w)$ admits meromorphic continuation to the (s, w) -plane and satisfies a group of functional equations.

1. PRELIMINARIES

We present an overview of quadratic Dirichlet L -functions over \mathbb{Q} . We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m \geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\operatorname{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to \mathbb{C} with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Now we recall characters on \mathbb{Z} . They are multiplicative functions $\chi : \mathbb{Z} \rightarrow \mathbb{C}$. They form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions $\chi_d : \mathbb{Z} \rightarrow \mathbb{C}$ modulo $d \geq 1$ (in that they are d -periodic) and such that $\chi_d(m) = 0$ if $(m, d) > 1$.
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. If χ is a Dirichlet character then its conjugate $\bar{\chi}$ is also a Dirichlet character. Moreover, $\bar{\chi}$ is the multiplicative inverse to χ and the Dirichlet characters modulo m form a group under multiplication. This group is always finite and its order is $\phi(d) = |(\mathbb{Z}/d\mathbb{Z})^*|$. Dirichlet characters also satisfy orthogonality relations:

Theorem 1.1 (Orthogonality relations).

(i) For any two Dirichlet characters χ and ψ modulo d ,

$$\frac{1}{\phi(d)} \sum_{a \pmod{d}}' \chi(a) \bar{\psi}(a) = \delta_{\chi, \psi}.$$

(ii) For any $a, b \in (\mathbb{Z}/d\mathbb{Z})^*$,

$$\frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \chi(a) \bar{\chi}(b) = \delta_{a, b}.$$

The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on \mathbb{Z} . First let us recall this symbol. For any odd prime p and any $m \geq 1$, we define the quadratic residue symbol $\left(\frac{m}{p}\right)$ by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod{p} = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod{p} \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod{p} \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod{p}. \end{cases}$$

This symbol only depends upon m modulo p and is multiplicative in m . We can extend the quadratic residue symbol multiplicatively in the denominator. If $d = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of d , then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \leq i \leq k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any odd $d \geq 1$. We can extend this symbol further and allow $d \geq 1$ to be even. To this end, we define

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases}$$

and extend $\left(\frac{m}{d}\right)$ multiplicatively in d when d is even. Now the quadratic residue symbol makes sense for any $m, d \geq 1$. Moreover, it is multiplicative in both m and d but no longer depends upon only m modulo d (it also depends upon m modulo 8). In particular,

$$\left(\frac{-1}{d}\right) = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}, \\ 0 & d \equiv 0 \pmod{2}, \end{cases}$$

and if $d \not\equiv 0 \pmod{2}$, we can compactly write

$$\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}} = \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ -1 & d \equiv 3 \pmod{4}, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = (-1)^{\frac{d^2-1}{8}} = \begin{cases} 1 & d \equiv 1, 7 \pmod{8}, \\ -1 & d \equiv 3, 5 \pmod{8}. \end{cases}$$

The quadratic residue symbol also admits the following reciprocity law:

Theorem 1.2 (Quadratic reciprocity). *If $d, m \geq 1$, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2} \frac{m^{(2)}-1}{2}} \left(\frac{m}{d}\right),$$

where $d^{(2)}$ and $m^{(2)}$ are the parts of d and m relatively prime to 2 respectively.

We can now define the quadratic Dirichlet characters. For any odd square-free $d \in \mathbb{Z}$, define the quadratic Dirichlet character χ_d by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod{4}, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod{4}. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension $\mathbb{Q}(\sqrt{d})$. We extend χ_d multiplicatively in the denominator so that χ_d makes sense for any odd d . In particular, $\chi_d(m) = \pm 1$ provided d and m are relatively prime and $\chi_d(m) = 0$ if $(m, d) > 1$. Quadratic reciprocity implies that χ_d is a Dirichlet character modulo d if $d \equiv 1 \pmod{4}$ and is a Dirichlet character modulo $4d$ if $d \equiv 2, 3 \pmod{4}$. Indeed, if $d \equiv 1 \pmod{4}$ then $d^{(2)} = d$ and the sign is always 1. If $d \equiv 3 \pmod{4}$, then $d^{(2)} = d$ and the sign is

$\left(\frac{-1}{m}\right)$ which is a character modulo 4. If $d \equiv 2 \pmod{4}$, then $d^{(2)} \equiv 1, 3 \pmod{4}$ and we are reduced to one of the previous two cases. We will also require an associated character. For each χ_d , we define $\tilde{\chi}_d$ by

$$\tilde{\chi}_d(m) = (-1)^{\frac{d^{(2)}-1}{2} \frac{m^{(2)}-1}{2}} \chi_d(m).$$

Equivalently, $\tilde{\chi}_d(m)$ can be expressed as

$$\tilde{\chi}_d(m) = \begin{cases} \chi_d(m) & \text{if } d \equiv 1, 2 \pmod{4}, \\ \chi_{-1}(m)\chi_d(m) & \text{if } d \equiv 3 \pmod{4}, \end{cases}$$

and it follows that $\tilde{\chi}_d(m)$ is a quadratic Dirichlet character of the same modulus as χ_d . We now discuss the Hilbert characters. We will only need four of them: the quadratic Dirichlet characters modulo 8. They are given as follows:

$$\begin{aligned} \chi_1(m) &= \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod{2}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} & \chi_{-1}(m) &= \begin{cases} 1 & \text{if } m \equiv 1 \pmod{4}, \\ -1 & \text{if } m \equiv 3 \pmod{4}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} \\ \chi_2(m) &= \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } m \equiv 3, 5 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}, \end{cases} & \chi_{-2}(m) &= \begin{cases} 1 & \text{if } m \equiv 1, 3 \pmod{8}, \\ -1 & \text{if } m \equiv 5, 7 \pmod{8}, \\ 0 & \text{if } m \equiv 0 \pmod{2}. \end{cases} \end{aligned}$$

In general, we will denote a Hilbert character by χ_a with $a \in \{\pm 1, \pm 2\}$. Note that

$$\chi_{-1}(m) = \left(\frac{-1}{m}\right) \quad \text{and} \quad \chi_2(m) = \left(\frac{m}{2}\right).$$

Moreover, we have the relations

$$\chi_{-2}(m) = \chi_{-1}(m)\chi_2(m), \quad \chi_1(m) = \chi_{-1}(m)\chi_{-1}(m), \quad \text{and} \quad \chi_{-1}(m) = \chi_2(m)\chi_{-2}(m).$$

Suppose d is square-free. If $d \equiv 1, 2, 5 \pmod{8}$, then $d^{(2)} \equiv 1 \pmod{4}$ so that the sign in the statement of quadratic reciprocity is 1. If $d \equiv 3, 6, 7 \pmod{8}$, then $d^{(2)} \equiv 3 \pmod{4}$ and the sign is $(-1)^{\frac{m^{(2)}-1}{2}}$. This fact together with the relations for the quadratic characters modulo 8 imply

$$\chi_d(m) = \begin{cases} \chi_m(d) & \text{if } d \equiv 1 \pmod{4}, \\ \chi_{-1}(m)\chi_m(d) & \text{if } d \equiv 3 \pmod{4}, \\ \chi_2(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 2 \pmod{8}, \\ \chi_{-2}(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 6 \pmod{8}. \end{cases}$$

With the Dirichlet and Hilbert characters introduced, we are ready to discuss the L -functions associated to quadratic Dirichlet characters. We define the L -function $L(s, \chi_d)$ attached to χ_d for square-free d , by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \geq 1} \frac{\chi_d(m)}{|m|^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi_d(p)}{|p|^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character, $L(s, \chi_d) \ll \zeta(s)$ for $\text{Re}(s) > 1$ so that $L(s, \chi_d)$ is locally absolutely uniformly convergent in this region. $L(s, \chi_d)$ also admits analytic continuation to \mathbb{C} . The completed L -function $L^*(s, \chi_d)$ is defined as

$$L^*(s, \chi_d) = \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi_d) & \text{if } \chi_d \text{ is even,} \\ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi_d) & \text{if } \chi_d \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_d) = \begin{cases} \varepsilon_\chi q^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \chi_d \text{ is even,} \\ -\varepsilon_\chi q^{\frac{1}{2}-s} L^*(1-s, \chi_d) & \text{if } \chi_d \text{ is odd.} \end{cases}$$

Note that the gamma factors depend upon the parity of χ_d . This is the root cause of an important technical issue later when deriving functional equations for the quadratic double Dirichlet series. Analogously, the Dirichlet L -function $L(s, \tilde{\chi}_d)$ attached to $\tilde{\chi}_d$ for square-free d is defined by a Dirichlet series or Euler product:

$$L(s, \tilde{\chi}_d) = \sum_{m \geq 1} \frac{\tilde{\chi}_d(m)}{|m|^s} = \prod_{p \text{ prime}} \left(1 - \frac{\tilde{\chi}_d(p)}{|p|^s} \right)^{-1}.$$

As for $L(s, \chi_d)$, $L(s, \tilde{\chi}_d) \ll \zeta(s)$ for $\text{Re}(s) > 1$ so that $L(s, \tilde{\chi}_d)$ is locally absolutely uniformly convergent in this region. Moreover, $L(s, \tilde{\chi}_d)$ admits analytic continuation to \mathbb{C} and the completed L -function $L^*(s, \tilde{\chi}_d)$ is defined as

$$L^*(s, \tilde{\chi}_d) = \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \tilde{\chi}_d) & \text{if } \tilde{\chi}_d \text{ is even,} \\ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \tilde{\chi}_d) & \text{if } \tilde{\chi}_d \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \tilde{\chi}_d) = \begin{cases} \varepsilon_{\tilde{\chi}} q^{\frac{1}{2}-s} L^*(1-s, \tilde{\chi}_d) & \text{if } \tilde{\chi}_d \text{ is even,} \\ -\varepsilon_{\tilde{\chi}} q^{\frac{1}{2}-s} L^*(1-s, \tilde{\chi}_d) & \text{if } \tilde{\chi}_d \text{ is odd.} \end{cases}$$

Remark 1.1. *The definitions for $L(s, \chi_d)$ and $L^*(s, \chi_d)$ work perfectly well even when d is not square-free (however the functional equations do not hold). We purposely do not define these L -functions, yet, for d not necessarily square-free.*

THE QUADRATIC DOUBLE DIRICHLET SERIES

We will now define the quadratic double Dirichlet series $Z(s, w)$. For any integer $d \geq 1$, write $d = d_0 d_1^2$ where d_0 is square-free. Equivalently, d_0 is the square-free part of d and $\frac{d}{d_0}$ is a perfect square. The **quadratic double Dirichlet series** $Z(s, w)$ is defined as

$$Z(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{d_0}) Q_{d_0 d_1^2}(s)}{d^w},$$

where the superscript (2) indicates that the local factor at 2 has been removed, $Q_{d_0 d_1^2}(s)$ is the **correction polynomial** defined by

$$Q_{d_0 d_1^2}(s) = \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{d_0}(e_1) e_1^{-s} e_2^{1-s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{d_0}(e_1) e_1^{-s} e_2^{1-s},$$

and μ is the usual Möbius function. For $\text{Re}(s) > 1$, there is the trivial estimate

$$Q_{d_0 d_1^2}(s) \ll \sum_{e_1 e_2 | d_1} 1 \ll \sigma_0(d_1)^2 \ll_\varepsilon |d_1|^{2\varepsilon} \ll_\varepsilon |d|^\varepsilon,$$

for any $\varepsilon > 0$. As $L(s, \chi_{d_0}) \ll 1$ for $\text{Re}(s) > 1$, $Z(s, w)$ is locally absolutely uniformly convergent in the region $\Lambda = \{(s, w) \in \mathbb{C}^2 : \text{Re}(s) > 1, \text{Re}(w) > 1\}$. It will also be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters χ_{a_1} and χ_{a_2} . The **quadratic double Dirichlet series** $Z_{a_1, a_2}(s, w)$ twisted by χ_{a_1} and χ_{a_2} is defined as

$$Z_{a_1, a_2}(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{|d|^w},$$

where $Q_{d_0 d_1^2}(s, \chi_{a_1})$ is the **correction polynomial** twisted by χ_{a_1} defined by

$$Q_{d_0 d_1^2}(s, \chi_{a_1}) = \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{-s} |e_2|^{1-2s} = \sum_{e_1 e_2 e_3 = d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{-s} |e_2|^{1-2s},$$

and μ is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound $Q_{d_0 d_1^2}(s, \chi_{a_1}) \ll |d|_\varepsilon$ so that $Z_{a_1, a_2}(s, w)$ converges locally absolutely uniformly in the same region as $Z(s, w)$ does. In particular, $Z(s, w) = Z_{1,1}(s, w)$.

THE INTERCHANGE

As defined, $Z_{a_1, a_2}(s, w)$ is a sum of L -functions, and hence Euler products, in s . We will prove an interchange formula for $Z_{a_1, a_2}(s, w)$ which will show that it can be expressed as a sum of L -functions in w . That is, we want the variables s and w to change places. Precisely:

Theorem 1.3 (Interchange). *Wherever $Z_{a_1, a_2}(s, w)$ converges locally absolutely uniformly,*

$$Z_{a_1, a_2}(s, w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{|d|^w} = \sum_{m \text{ odd}} \frac{L^{(2)}(w, \tilde{\chi}_{a_2 m_0}) \chi_{a_1}(m) Q_{m_0 m_1^2}(w, \chi_{a_2})}{|m|^s}.$$

Proof. Only the second equality needs to be proved. To do this, first expand the L -function $L^{(2)}(s, \chi_{a_1 d_0})$ and polynomial $Q_{d_0 d_1^2}(s, \chi_{a_1})$ to get

$$\begin{aligned} Z(s, w) &= \sum_{d \text{ odd}} \frac{L(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{|d|^w} \\ &= \sum_{d \text{ odd}} \left(\sum_{m \text{ odd}} \chi_{a_1 d_0}(m) |m|^{-s} \right) \left(\sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_1 d_0}(e_1) |e_1|^{-s} |e_2|^{1-2s} \right) \chi_{a_2}(d) |d|^{-w} \\ &= \sum_{m, d \text{ odd}} \sum_{e_1 e_2 | d_1} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m e_1) |e_1|^{-s} |e_2|^{1-2s} |m|^{-s} |d|^{-w}. \end{aligned}$$

Now $\chi_{a_1 d_0}(m e_1) = 0$ unless $(d_0, m e_1) = 1$. We make this restriction on the sum giving

$$\sum_{m, d \text{ odd}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, m e_1) = 1}} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m e_1) |e_1|^{-s} |e_2|^{1-2s} |m|^{-s} |d|^{-w}.$$

Making the change of variables $m e_1 \rightarrow m$ yields

$$\sum_{d \text{ odd}} \sum_{\substack{m \text{ odd} \\ e_1 | m}} \sum_{\substack{e_1 e_2 | d_1 \\ (d_0, m) = 1}} \mu(e_1) \chi_{a_2}(d) \chi_{a_1 d_0}(m) |e_2|^{1-2s} |m|^{-s} |d|^{-w}.$$

For fixed $d = d_0 d_1^2$ and e_2 , the subsum over m and e_1 is

$$\sum_{\substack{m \text{ odd} \\ e_1 | m}} \sum_{\substack{e_1 | \frac{d_1}{e_2} \\ (d_0, m) = 1}} \mu(e_1) \chi_{a_1 d_0}(m) |m|^{-s} = \sum_{\substack{m \text{ odd} \\ (d_0, m) = 1}} \chi_{a_1 d_0}(m) |m|^{-s} \left(\sum_{e_1 | \left(\frac{d_1}{e_2}, m \right)} \mu(e_1) \right).$$

The inner sum over e_1 of the Möbius function vanishes unless $\left(\frac{d_1}{e_2}, m \right) = 1$ in which case it is 1. Therefore the triple sum above becomes

$$\sum_{m, d \text{ odd}} \sum_{\substack{e_2 | d_1 \\ \left(\frac{d_0 d_1}{e_2}, m \right) = 1}} \chi_{a_2}(d) \chi_{a_1 d_0}(m) |e_2|^{1-2s} |m|^{-s} |d|^{-w}.$$

Making the change of variables $d \rightarrow de_2^2$, the condition $\left(\frac{d_0 d_1}{e_2}, m\right) = 1$ becomes $(d_0 d_1, m) = 1$ which is equivalent to $(d, m) = 1$. Moreover, $\chi_{a_2}(de_2^2) = \chi_{a_2}(d)$. Altogether, we obtain

$$\sum_{\substack{m, d \text{ odd} \\ (d, m)=1}} \sum_{e_2} \chi_{a_2}(d) \chi_{a_1 d_0}(m) |e_2|^{1-2s-2w} |m|^{-s} |d|^{-w}.$$

Writing $m = m_0 m_1^2$ analogously as for d , quadratic reciprocity implies $\chi_{d_0}(m) = \tilde{\chi}_m(d_0) = \tilde{\chi}_{m_0}(d)$ where the last equality holds because $(d, m) = 1$ and both d_0 and m_0 differ from d and m respectively by perfect squares. This implies $\chi_{a_2}(d) \chi_{a_1 d_0}(m) = \chi_{a_1}(m) \tilde{\chi}_{a_2 m_0}(d)$ and so our expression becomes

$$\sum_{\substack{m, d \text{ monic} \\ (d, m)=1}} \sum_{e_2} \chi_{a_1}(m) \tilde{\chi}_{a_2 m_0}(d) |e_2|^{1-2s-2w} |m|^{-s} |d|^{-w}.$$

But now we can reverse the argument with the roles of d , m , χ_{a_1} , and $\tilde{\chi}_{a_2}$ interchanged respectively to obtain

$$Z(s, w) = \sum_{m \text{ monic}} \frac{L(w, \tilde{\chi}_{a_2 m_0}) \chi_{a_1}(m) Q_{m_0 m_1^2}(w, \chi_{a_2})}{|m|^s}.$$

□

Note that the interchange is not completely symmetric because of the character $\tilde{\chi}_{a_2 m_0}$ in the second expression for $Z_{a_1, a_2}(s, w)$. This is due to the fact that reciprocity is not perfect. In even more general settings the correction polynomials in w need not be equal to those in s .

WEIGHTING TERMS

We will now study the coefficients of $Z_{a_1, a_2}(s, w)$ expanded in s and w . By expanding $L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1})$ in the numerator of $Z_{a_1, a_2}(s, w)$, we can write

$$Z_{a_1, a_2}(s, w) = \sum_{d \text{ monic}} \frac{L(s, \chi_{a_1 d_0}) \chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{|d|^w} = \sum_{m, d \text{ monic}} \frac{\chi_{a_1 d_0}(\hat{m}) \chi_{a_2}(d) a(m, d)}{|m|^s |d|^w},$$

where \hat{m} is the part of m relatively prime to d_0 and the **weighting coefficient** $a(m, d)$ is given by

$$a(m, d) = \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) |e_2|.$$

To see this, the coefficient of $|m|^{-s} |d|^{-w}$ in the definition of $Z_{a_1, a_2}(s, w)$ is

$$\begin{aligned} \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1}} \mu(e_1) \chi_{a_1 d_0}(e_1 e_3) |e_2| &= \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) \chi_{a_1 d_0}(e_1 e_3) |e_2| \\ &= \chi_{a_1 d_0}(\hat{m}) \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) |e_2| \\ &= \chi_{a_1 d_0}(\hat{m}) \chi_{a_2}(d) a(m, d), \end{aligned}$$

where the first equality holds because $\chi_{d_0}(e_1 e_3) = 0$ unless $(d_0, e_1 e_3) = 1$ and the second equality holds because if $(d_0, e_1 e_3) = 1$, \hat{m} differs from $e_1 e_3$ by a perfect square (the divisors of which belong to (d_0, e_2)) and so $\chi_{d_0}(e_1 e_3) = \chi_{d_0}(\hat{m})$.

Remark 1.2. Also, $a(m, d) = 0$ unless $m = e_1 e_2^2 e_3$ with $(d_0, e_1 e_3) = 1$ and $e_1 e_2^2 | d_1$.

We will define $L(s, \chi_{a_1 d})$ to be the Dirichlet series given by

$$L(s, \chi_{a_1 d}) = L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1}) = \sum_{m \text{ monic}} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m, d)}{|m|^s}.$$

In particular, $L(s, \chi_d)$ now makes sense for d not necessarily square-free and this definition agrees with the former when d is square-free. Moreover, we have the representation

$$Z_{a_1, a_2}(s, w) = \sum_{d \text{ monic}} \frac{\chi_{a_2}(d) L(s, \chi_{a_1 d})}{|d|^w}.$$

If we perform the same procedure to the interchange, then

$$Z_{a_1, a_2}(s, w) = \sum_{m \text{ monic}} \frac{L(w, \tilde{\chi}_{a_2 m_0}) \chi_{a_1}(m) Q_{m_0 m_1^2}(w, \chi_{a_2})}{|m|^s} = \sum_{m, d \text{ monic}} \frac{\tilde{\chi}_{a_2 m_0}(\widehat{d}) \chi_{a_1}(m) a(d, m)}{|m|^s |d|^w},$$

where \widehat{d} is the part of d relatively prime to m_0 , and analogously we define $L(w, \tilde{\chi}_{a_2 m})$ by

$$L(w, \tilde{\chi}_{a_2 m}) = L(w, \tilde{\chi}_{a_2 m_0}) Q_{m_0 m_1^2}(w, \chi_{a_2}) = \sum_{d \text{ monic}} \frac{\tilde{\chi}_{a_2 m_0}(\widehat{d}) a(d, m)}{|d|^w},$$

so that

$$Z_{a_1, a_2}(s, w) = \sum_{m \text{ monic}} \frac{\chi_{a_1}(m) L(w, \tilde{\chi}_{a_2 m})}{|m|^s}.$$

We now investigate the structure of the weighting coefficients $a(m, d)$. They possess a multiplicativity property:

Proposition 1.1. *We have $a(m, 1) = a(1, d) = 1$ and*

$$a(m, d) = \prod_{\substack{P^\alpha || m \\ P^\beta || d}} a(P^\alpha, P^\beta).$$

Proof. **Todo:** [xxx]

□

FUNCTIONAL EQUATIONS

MEROMORPHIC CONTINUATION

POLES AND RESIDUES

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