A QUADRATIC DOUBLE DIRICHLET SERIES OVER NUMBER FIELDS

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ABSTRACT. We construct a quadratic double Dirichlet series $Z(s,w)$ built from single variable quadratic Dirichlet L-functions $L(s,\chi)$ over $\mathbb{Q}$. We prove that $Z(s,w)$ admits meromorphic continuation to the (s,w)-plane and satisfies a group of functional equations.

1. Preliminaries

We present an overview of quadratic Dirichlet L-functions over $\mathbb{Q}$. We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Now we recall the characters on $\mathbb{Z}$. They are multiplicative functions $\chi : \mathbb{Z} \to \mathbb{C}$. Their image always lands in the roots of unity. The two flavors of characters of interest to use are:

- Dirichlet characters: multiplicative functions $\chi_m : \mathbb{Z} \to \mathbb{C}$ modulo $m \geq 1$ (in that they are m-periodic) and such that $\chi_m(n) = 0$ if $(m,n) > 1$.
- Hilbert symbols: Dirichlet characters modulo 1.

If χ is a character then its conjugate $\overline{\chi}$ is also a character. Moreover, $\overline{\chi}$ is the multiplicative inverse to χ and the characters modulo m form a group under multiplication. This group is always finite and its order is $\phi(m)$. Characters also satisfy orthogonality properties:

Theorem 1.1 (Orthogonality relations)**.** *Let χ and ψ be any two Dirichlets character modulo m and let a and b be any two integers modulo m. Then*

(i)

$$\frac{1}{\phi(m)} \sum_{a \pmod m} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

(ii)

$$\frac{1}{\phi(m)} \sum_{\chi \pmod m} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

Todo: [continue here] The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{F}_q[t]$. If $f \in \mathbb{F}_q[t]$ is a monic non-constant irreducible, define the quadratic residue symbol χ_f by

$$\chi_f(g) = \left(\frac{f}{g}\right) = g^{\frac{|f|-1}{2}} \pmod f,$$

for any $g \in \mathbb{F}_q[t]$. Then $\chi_f(g) \in \{\pm 1\}$ provided f and g are relatively prime and $\chi_f(g) = 0$ if $(f, g) > 1$. If $b \in \mathbb{F}^\times$, then we define the quadratic residue symbol χ_b by

$$\chi_b(g) = \left(\frac{b}{m}\right) = \operatorname{sgn}(b)^{\deg(f)},$$

where $\operatorname{sgn}(b) = \pm 1$ depending on if $b \in (\mathbb{F}^\times)^2$ or not. Moreover, if $d \in \mathbb{F}_q[t]$ then we set $\operatorname{sgn}(d) = \operatorname{sgn}(b_n)$ if $d(t) = b_n t^n + b_{n-1} t^{n+1} + \cdots + b_0$ (with $b_n \neq 0$). Extending χ_f multiplicativity in f, χ_f is defined for any f not necessarily monic. The quadratic residue symbol also has the following reciprocity property:

Theorem 1.2 (Quadratic reciprocity)**.** *If $f, g \in \mathbb{F}_q[t]$ are monic, square-free, and relatively prime, then*

$$\left(\frac{f}{g}\right) = (-1)^{\frac{q-1}{2}\deg(f)\deg(g)} \left(\frac{g}{f}\right).$$

Note that if $q \equiv 1 \pmod 4$, the sign in the statement of quadratic reciprocity is always 1 so that the reciprocity is perfect. We now describe the Hilbert symbols on $\mathbb{F}_q[t]$. In fact, there are only two Hilbert symbols, one non-trivial, and one trivial. The non-trivial Hilbert symbol is χ_θ where $\theta \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$:

$$\chi_\theta(f) = (-1)^{\deg(f)}.$$

Note that $\overline{\chi_\theta} = \chi_\theta$. The other Hilbert symbol is the trivial character $\chi_\theta^2 = \chi_{\theta\theta} = \chi_1$. In general, we denote a Hilbert symbol by χ_a where $a \in \{1, \theta\}$.

We can now define the L-functions attached to the symbol χ_f for not necessarily monic f. We define the L-series $L(s, \chi_f)$ attached to χ_f by a Dirichlet series or Euler product:

$$L(s, \chi_f) = \sum_{g \text{ monic}} \frac{\chi_f(g)}{|g|^s} = \prod_{P \text{ monic irr}} \left(1 - \frac{\chi_f(P)}{|P|^s}\right)^{-1}.$$

By definition of the quadratic residue symbol, $L(s, \chi_f) \ll \zeta(s)$ for $\operatorname{Re}(s) > 1$ so that $L(s, \chi_f)$ is absolutely uniformly convergent on compacta in this region. $L(s, \chi_f)$ also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ if f is square-free and is analytic otherwise (see [1] for a proof). Moreover, $L(s, \chi_f)$ is a polynomial in q^{-s} of degree at most $\deg(f) - 1$. The completed L-function is defined as follows:

$$L^*(s, \chi_f) = \begin{cases} \frac{1}{1-q^{-s}} L(s, \chi_f) & \text{if } \deg(f) \text{ is even,} \\ L(s, \chi_f) & \text{if } \deg(f) \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s, \chi_f) = \begin{cases} q^{2s-1} |f|^{\frac{1}{2}-s} L^*(1-s, \chi_f) & \text{if } \deg(f) \text{ is even,} \\ q^{2s-1} (q|f|)^{\frac{1}{2}-s} L^*(1-s, \chi_f) & \text{if } \deg(f) \text{ is odd.} \end{cases}$$

Note that in the case $\deg(f)$ is even, the conductor is $|f|$ and in the case $\deg(f)$ is odd, the conductor is $q|f|$. In other words, the gamma factors depend upon the degree of f. This will cause a small but important technical issue later when we want to derive functional equations for the quadratic double Dirichlet series.

The Quadratic Double Dirichlet Series

The Interchange

Weighting Terms

Funcational Equations

Meromorphic Continuation

Poles and Residues

References

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