A QUADRATIC DOUBLE DIRICHLET SERIES II: THE NUMBER FIELD CASE

HENRY TWISS

Abstract. We construct a quadratic double Dirichlet series $Z(s, w)$ built from single variable quadratic Dirichlet L-functions $L(s, \chi)$ over $\mathbb{Q}$. We prove that $Z(s, w)$ admits meromorphic continuation to the (s, w)-plane and satisfies a group of functional equations.

1. Preliminaries

We present an overview of quadratic Dirichlet L-functions over $\mathbb{Q}$. We begin with the Riemann zeta-function. The zeta function $\zeta(s)$ is defined as the Dirichlet series or Euler product

$$\zeta(s) = \sum_{m \geq 1} \frac{1}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$. The second equality is an analytic reformulation of the fundamental theorem of arithmetic. The Riemann zeta function also admits meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1. The functional equation is

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Now we recall characters on $\mathbb{Z}$. They are multiplicative functions $\chi : \mathbb{Z} \to \mathbb{C}$. They form a group under multiplication. The two flavors we will care about are:

- Dirichlet characters: multiplicative functions $\chi_d : \mathbb{Z} \to \mathbb{C}$ modulo $d \geq 1$ (in that they are d-periodic) and such that $\chi_d(m) = 0$ if $(m, d) > 1$.
- Hilbert characters: The group of characters generated by those that appear in the sign change of reciprocity statements.

The image of a Dirichlet character always lands in the roots of unity. If χ is a Dirichlet character then its conjugate $\overline{\chi}$ is also a Dirichlet character. Moreover, $\overline{\chi}$ is the multiplicative inverse to χ and the Dirichlet characters modulo m form a group under multiplication. This group is always finite and its order is $\phi(d) = |(\mathbb{Z}/d\mathbb{Z})^*|$. Dirichlet characters also satisfy orthogonality relations:

Theorem 1.1 (Orthogonality relations)**.**

(i) For any two Dirichlet characters χ and ψ modulo d,

$$\frac{1}{\phi(d)} \sideset{}{'}\sum_{a \pmod{d}} \chi(a)\overline{\psi}(a) = \delta_{\chi,\psi}.$$

(ii) For any $a, b \in (\mathbb{Z}/d\mathbb{Z})^*$,

$$\frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \chi(a)\overline{\chi}(b) = \delta_{a,b}.$$

Date: 2024.

The Dirichlet characters that are of interest to us are those given by the quadratic residue symbol on $\mathbb{Z}$. First let us recall this symbol. For any odd prime p and any $m \geq 1$, we define the quadratic residue symbol $\left(\frac{m}{p}\right)$ by

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod p = \begin{cases} 1 & \text{if } x^2 \equiv m \pmod p \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv m \pmod p \text{ is not solvable,} \\ 0 & \text{if } m \equiv 0 \pmod p. \end{cases}$$

This symbol only depends upon m modulo p and is multiplicative in m. We can extend the quadratic residue symbol multiplicatively in the denominator. If $d = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of d, then we define

$$\left(\frac{m}{d}\right) = \prod_{1 \leq i \leq k} \left(\frac{m}{p_i}\right)^{e_i}.$$

So the quadratic residue symbol now makes sense for any odd $d \geq 1$. We can extend this symbol further and allow $d \geq 1$ to be even. To this end, we define

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod 8, \\ -1 & \text{if } m \equiv 3, 5 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

and extend $\left(\frac{m}{d}\right)$ multiplatively in d when d is even. Now the quadratic residue symbol makes sense for any $m, d \geq 1$. Moreover, it is multiplicative in both m and d but no longer depends upon only m modulo d (it also depends upon m modulo 8). In particular,

$$\left(\frac{-1}{d}\right) = \begin{cases} 1 & d \equiv 1 \pmod 4, \\ -1 & d \equiv 3 \pmod 4, \\ 0 & d \equiv 0 \pmod 2, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = \begin{cases} 1 & d \equiv 1, 7 \pmod 8, \\ -1 & d \equiv 3, 5 \pmod 8, \\ 0 & d \equiv 0 \pmod 2, \end{cases}$$

and if $d \not\equiv 0 \pmod 2$, we can compactly write

$$\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}} = \begin{cases} 1 & d \equiv 1 \pmod 4, \\ -1 & d \equiv 3 \pmod 4, \end{cases} \quad \text{and} \quad \left(\frac{2}{d}\right) = (-1)^{\frac{d^2-1}{8}} = \begin{cases} 1 & d \equiv 1, 7 \pmod 8, \\ -1 & d \equiv 3, 5 \pmod 8. \end{cases}$$

The quadratic residue symbol also admits the following reciprocity law:

Theorem 1.2 (Quadratic reciprocity)**.** *If $d, m \geq 1$, then*

$$\left(\frac{d}{m}\right) = (-1)^{\frac{d^{(2)}-1}{2}\frac{m^{(2)}-1}{2}} \left(\frac{m}{d}\right),$$

where $d^{(2)}$ and $m^{(2)}$ are the parts of d and m relatively prime to 2 respectively.

We can now define the quadratic Dirichlet characters. For any odd square-free $d \in \mathbb{Z}$, define the quadratic Dirichlet character χ_d by the following quadratic residue symbol:

$$\chi_d(m) = \begin{cases} \left(\frac{d}{m}\right) & \text{if } d \equiv 1 \pmod 4, \\ \left(\frac{4d}{m}\right) & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

This quadratic Dirichlet character is attached to the quadratic extension $\mathbb{Q}(\sqrt{d})$. We extend χ_d multiplicatively in the denominator so that χ_d makes sense for any odd d. In particular, $\chi_d(m) = \pm 1$ provided d and m are relatively prime and $\chi_d(m) = 0$ if $(m, d) > 1$. Quadratic reciprocity implies that χ_d is a Dirichlet character modulo d if $d \equiv 1 \pmod 4$ and is a Dirichlet character modulo $4d$ if $d \equiv 2, 3 \pmod 4$. Indeed, if $d \equiv 1 \pmod 4$ then $d^{(2)} = d$ and the sign is always 1. If $d \equiv 3 \pmod 4$, then $d^{(2)} = d$ and the sign is

$\left(\frac{-1}{m}\right)$ which is a character modulo 4. If $d \equiv 2 \pmod 4$, then $d^{(2)} \equiv 1, 3 \pmod 4$ and we are reduced to one of the previous two cases. We will also require an associated character. For each χ_d, we define $\widetilde{\chi}_d$ by

$$\widetilde{\chi}_d(m) = (-1)^{\frac{d^{(2)}-1}{2}\frac{m^{(2)}-1}{2}} \chi_d(m).$$

Equivalently, $\widetilde{\chi}_d(m)$ can be expressed as

$$\widetilde{\chi}_d(m) = \begin{cases} \chi_d(m) & \text{if } d \equiv 1, 2 \pmod 4, \\ \chi_{-1}(m)\chi_d(m) & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

and it follows that $\widetilde{\chi}_d(m)$ is a quadratic Dirichlet character of the same modulus as χ_d. We now discuss the Hilbert characters. We will only need four of them: the quadratic Dirichlet characters modulo 8. They are given as follows:

$$\chi_1(m) = \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod 2, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases} \qquad \chi_{-1}(m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4, \\ -1 & \text{if } m \equiv 3 \pmod 4, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

$$\chi_2(m) = \begin{cases} 1 & \text{if } m \equiv 1, 7 \pmod 8, \\ -1 & \text{if } m \equiv 3, 5 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2, \end{cases} \qquad \chi_{-2}(m) = \begin{cases} 1 & \text{if } m \equiv 1, 3 \pmod 8, \\ -1 & \text{if } m \equiv 5, 7 \pmod 8, \\ 0 & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

In general, we will denote a Hilbert character by χ_a with $a \in \{\pm 1, \pm 2\}$. Note that

$$\chi_{-1}(m) = \left(\frac{-1}{m}\right) \quad \text{and} \quad \chi_2(m) = \left(\frac{m}{2}\right).$$

Moreover, we have the relations

$$\chi_{-2}(m) = \chi_{-1}(m)\chi_2(m), \quad \chi_1(m) = \chi_{-1}(m)\chi_{-1}(m), \quad \text{and} \quad \chi_{-1}(m) = \chi_2(m)\chi_{-2}(m).$$

Suppose d is square-free. If $d \equiv 1, 2, 5 \pmod 8$, then $d^{(2)} \equiv 1 \pmod 4$ so that the sign in the statement of quadratic recipricty is 1. If $d \equiv 3, 6, 7 \pmod 8$, then $d^{(2)} \equiv 3 \pmod 4$ and the sign is $(-1)^{\frac{m^{(2)}-1}{2}}$. This fact together with the relations for the quadratic characters modulo 8 imply

$$\chi_d(m) = \begin{cases} \chi_m(d) & \text{if } d \equiv 1 \pmod 4, \\ \chi_{-1}(m)\chi_m(d) & \text{if } d \equiv 3 \pmod 4, \\ \chi_2(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 2 \pmod 8, \\ \chi_{-2}(m)\chi_m\left(\frac{d}{2}\right) & \text{if } d \equiv 6 \pmod 8. \end{cases}$$

With the Dirichlet and Hilbert characters introduced, we are ready to discuss the L-functions associated to quadratic Dirichlet characters. We define the L-function $L(s, \chi_d)$ attached to χ_d for square-free d, by a Dirichlet series or Euler product:

$$L(s, \chi_d) = \sum_{m \geq 1} \frac{\chi_d(m)}{m^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1}.$$

By definition of the quadratic Dirichlet character, $L(s, \chi_d) \ll \zeta(s)$ for $\mathrm{Re}(s) > 1$ so that $L(s, \chi_d)$ is locally absolutely uniformly convergent in this region. $L(s, \chi_d)$ also admits analytic continuation to $\mathbb{C}$. The completed L-function $L^*(s, \chi_d)$ is defined as

$$L^*(s, \chi_d) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) L(s, \chi_d) & \text{if } \chi_d \text{ is even}, \\ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right) L(s, \chi_d) & \text{if } \chi_d \text{ is odd}, \end{cases}$$

and satisfies the functional equation

$$L^*(s,\chi_d) = \begin{cases} \varepsilon_\chi q^{\frac{1}{2}-s}L^*(1-s,\chi_d) & \text{if } \chi_d \text{ is even,} \\ -\varepsilon_\chi q^{\frac{1}{2}-s}L^*(1-s,\chi_d) & \text{if } \chi_d \text{ is odd.} \end{cases}$$

Note that the gamma factors depend upon the partiy of χ_d. This the root cause of an important technical issue later when deriving functional equations for the quadratic double Dirichlet series. Analogously, the Dirichlet L-function $L(s,\widetilde{\chi}_d)$ attached to $\widetilde{\chi}_d$ for square-free d is defined by a Dirichlet series or Euler product:

$$L(s,\widetilde{\chi}_d) = \sum_{m\geq 1}\frac{\widetilde{\chi}_d(m)}{m^s} = \prod_{p \text{ prime}}\left(1-\frac{\widetilde{\chi}_d(p)}{p^s}\right)^{-1}.$$

As for $L(s,\chi_d)$, $L(s,\widetilde{\chi}_d) \ll \zeta(s)$ for $\mathrm{Re}(s)>1$ so that $L(s,\widetilde{\chi}_d)$ is locally absolutely uniformly convergent in this region. Moreover, $L(s,\widetilde{\chi}_d)$ admits analytic continuation to $\mathbb{C}$ and the completed L-function $L^*(s,\widetilde{\chi}_d)$ is defined as

$$L^*(s,\widetilde{\chi}_d) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\widetilde{\chi}_d) & \text{if } \widetilde{\chi}_d \text{ is even,} \\ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\widetilde{\chi}_d) & \text{if } \widetilde{\chi}_d \text{ is odd,} \end{cases}$$

and satisfies the functional equation

$$L^*(s,\widetilde{\chi}_d) = \begin{cases} \varepsilon_{\widetilde{\chi}} q^{\frac{1}{2}-s}L^*(1-s,\widetilde{\chi}_d) & \text{if } \widetilde{\chi}_d \text{ is even,} \\ -\varepsilon_{\widetilde{\chi}} q^{\frac{1}{2}-s}L^*(1-s,\widetilde{\chi}_d) & \text{if } \widetilde{\chi}_d \text{ is odd.} \end{cases}$$

Remark 1.1. *The definitions for $L(s,\chi_d)$ and $L^*(s,\chi_d)$ work perfectly well even when d is not square-free (however the functional equations do not hold). We purposely do not define these L-functions, yet, for d not necessarily square-free.*

The Quadratic Double Dirichlet Series

We will now define the quadratic double Dirichlet series $Z(s,w)$. For any integer $d\geq 1$, write $d=d_0d_1^2$ where d_0 is square-free. Equivalently, d_0 is the square-free part of d and $\frac{d}{d_0}$ is a perfect square. The **quadratic double Dirichlet series** $Z(s,w)$ is defined as

$$Z(s,w) = \sum_{d \text{ odd}}\frac{L^{(2)}(s,\chi_{d_0})Q_{d_0d_1^2}(s)}{d^w},$$

where the superscript (2) indicates that the local factor at 2 has been removed, $Q_{d_0d_1^2}(s)$ is the **correction polynomial** defined by

$$Q_{d_0d_1^2}(s) = \sum_{e_1e_2\mid d_1}\mu(e_1)\chi_{d_0}(e_1)e_1^{-s}e_2^{1-s} = \sum_{e_1e_2e_3=d_1}\mu(e_1)\chi_{d_0}(e_1)e_1^{-s}e_2^{1-s},$$

and μ is the usual Möbius function. For $\mathrm{Re}(s)>1$, there is the trivial estimate

$$Q_{d_0d_1^2}(s) \ll \sum_{e_1e_2\mid d_1} 1 \ll \sigma_0(d_1)^2 \ll_\varepsilon d_1^{2\varepsilon} \ll_\varepsilon d^\varepsilon,$$

for any $\varepsilon>0$. As $L(s,\chi_{d_0}) \ll 1$ for $\mathrm{Re}(s)>1$, $Z(s,w)$ is locally absolutely uniformly convergent in the region $\Lambda = \{(s,w)\in\mathbb{C}^2 : \mathrm{Re}(s)>1, \mathrm{Re}(w)>1\}$. It will also be necessary to consider quadratic double Dirichlet series twisted by a pair of Hilbert characters χ_{a_1} and χ_{a_2}. The **quadratic double Dirichlet series** $Z_{a_1,a_2}(s,w)$ twisted by χ_{a_1} and χ_{a_2} is defined as

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ odd}}\frac{L^{(2)}(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{d^w},$$

where $Q_{d_0d_1^2}(s,\chi_{a_1})$ is the **correction polynomial** twisted by χ_{a_1} defined by

$$Q_{d_0d_1^2}(s,\chi_{a_1}) = \sum_{e_1e_2|d_1} \mu(e_1)\chi_{a_1d_0}(e_1)e_1^{-s}e_2^{1-2s} = \sum_{e_1e_2e_3=d_1} \mu(e_1)\chi_{a_1d_0}(e_1)e_1^{-s}e_2^{1-2s},$$

and μ is the usual Möbius function. By definition of the Hilbert characters, we have the analogous bound $Q_{d_0d_1^2}(s,\chi_{a_1}) \ll d_\varepsilon$ so that $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly in the same region as $Z(s,w)$ does. In particular, $Z(s,w) = Z_{1,1}(s,w)$.

The Interchange

As defined, $Z_{a_1,a_2}(s,w)$ is a sum of L-functions, and hence Euler products, in s. We will prove an interchange formula for $Z_{a_1,a_2}(s,w)$ which will show that it can be expressed as a sum of L-functions in w. That is, we want the variables s and w to change places. Precisely:

Theorem 1.3 (Interchange)**.** *Wherever $Z_{a_1,a_2}(s,w)$ converges locally absolutely uniformly,*

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ odd}} \frac{L^{(2)}(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{d^w} = \sum_{m \text{ odd}} \frac{L^{(2)}(w,\widetilde{\chi}_{a_2m_0})\chi_{a_1}(m)Q_{m_0m_1^2}(w,\chi_{a_2})}{m^s}.$$

Proof. Only the second equality needs to be proved. To do this, first expand the L-function $L^{(2)}(s,\chi_{a_1d_0})$ and polynomial $Q_{d_0d_1^2}(s,\chi_{a_1})$ to get

$$\begin{aligned} Z(s,w) &= \sum_{d \text{ odd}} \frac{L(s,\chi_{a_1d_0})\chi_{a_2}(d)Q_{d_0d_1^2}(s,\chi_{a_1})}{d^w} \\ &= \sum_{d \text{ odd}} \left(\sum_{m \text{ odd}} \chi_{a_1d_0}(m)m^{-s}\right)\left(\sum_{e_1e_2|d_1} \mu(e_1)\chi_{a_1d_0}(e_1)e_1^{-s}e_2^{1-2s}\right)\chi_{a_2}(d)d^{-w} \\ &= \sum_{m,d \text{ odd}} \sum_{e_1e_2|d_1} \mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(me_1)e_1^{-s}e_2^{1-2s}m^{-s}d^{-w}. \end{aligned}$$

Now $\chi_{a_1d_0}(me_1) = 0$ unless $(d_0, me_1) = 1$. We make this restriction on the sum giving

$$\sum_{m,d \text{ odd}} \sum_{\substack{e_1e_2|d_1 \\ (d_0,me_1)=1}} \mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(me_1)e_1^{-s}e_2^{1-2s}m^{-s}d^{-w}.$$

Making the change of variables $me_1 \to m$ yields

$$\sum_{d \text{ odd}} \sum_{\substack{m \text{ odd} \\ e_1|m}} \sum_{\substack{e_1e_2|d_1 \\ (d_0,m)=1}} \mu(e_1)\chi_{a_2}(d)\chi_{a_1d_0}(m)e_2^{1-2s}m^{-s}d^{-w}.$$

For fixed $d = d_0d_1^2$ and e_2, the subsum over m and e_1 is

$$\sum_{\substack{m \text{ odd} \\ e_1|m}} \sum_{\substack{e_1|\frac{d_1}{e_2} \\ (d_0,m)=1}} \mu(e_1)\chi_{a_1d_0}(m)m^{-s} = \sum_{\substack{m \text{ odd} \\ (d_0,m)=1}} \chi_{a_1d_0}(m)m^{-s}\left(\sum_{e_1|\left(\frac{d_1}{e_2},m\right)} \mu(e_1)\right).$$

The inner sum over e_1 of the Möbius function vanishes unless $\left(\frac{d_1}{e_2},m\right) = 1$ in which case it is 1. Therefore the triple sum above becomes

$$\sum_{m,d \text{ odd}} \sum_{\substack{e_2|d_1 \\ \left(\frac{d_0d_1}{e_2},m\right)=1}} \chi_{a_2}(d)\chi_{a_1d_0}(m)e_2^{1-2s}m^{-s}d^{-w}.$$

Making the change of variables $d \to de_2^2$, the condition $\left(\frac{d_0 d_1}{e_2}, m\right) = 1$ becomes $(d_0 d_1, m) = 1$ which is equivalent to $(d, m) = 1$. Moreover, $\chi_{a_2}(de_2^2) = \chi_{a_2}(d)$. Altogether, we obtain

$$\sum_{\substack{m,\, d \text{ odd} \\ (d,m)=1}} \sum_{e_2} \chi_{a_2}(d)\chi_{a_1 d_0}(m) e_2^{1-2s-2w} m^{-s} d^{-w}.$$

Writing $m = m_0 m_1^2$ analogously as for d, quadratic reciprocity implies $\chi_{d_0}(m) = \widetilde{\chi}_m(d_0) = \widetilde{\chi}_{m_0}(d)$ where the last equality holds because $(d, m) = 1$ and both d_0 and m_0 differ from d and m respectively by perfect squares. This implies $\chi_{a_2}(d)\chi_{a_1 d_0}(m) = \chi_{a_1}(m)\widetilde{\chi}_{a_2 m_0}(d)$ and so our expression becomes

$$\sum_{\substack{m,\, d \text{ monic} \\ (d,m)=1}} \sum_{e_2} \chi_{a_1}(m)\widetilde{\chi}_{a_2 m_0}(d) e_2^{1-2s-2w} m^{-s} d^{-w}.$$

But now we can reverse the argument with the roles of d, m, χ_{a_1}, and $\widetilde{\chi}_{a_2}$ interchanged respectively to obtain

$$Z(s, w) = \sum_{m \text{ monic}} \frac{L(w, \widetilde{\chi}_{a_2 m_0})\chi_{a_1}(m) Q_{m_0 m_1^2}(w, \chi_{a_2})}{m^s}.$$

□

Note that the interchange is not completely symmetric because of the character $\widetilde{\chi}_{a_2 m_0}$ in the second expression for $Z_{a_1,a_2}(s, w)$. This is due to the fact that reciprocty is not perfect. In even more general settings the correction polynomials in w need not be equal to those in s.

Weighting Terms

We will now study the coefficients of $Z_{a_1,a_2}(s, w)$ expanded in s and w. By expanding $L(s, \chi_{a_1 d_0}) Q_{d_0 d_1^2}(s, \chi_{a_1})$ in the numerator of $Z_{a_1,a_2}(s, w)$, we can write

$$Z_{a_1,a_2}(s, w) = \sum_{d \text{ monic}} \frac{L(s, \chi_{a_1 d_0})\chi_{a_2}(d) Q_{d_0 d_1^2}(s, \chi_{a_1})}{d^w} = \sum_{m,\, d \text{ monic}} \frac{\chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) a(m, d)}{m^s d^w},$$

where $\widehat{m}$ is the part of m relatively prime to d_0 and the **weighting coefficient** $a(m, d)$ is given by

$$a(m, d) = \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2.$$

To see this, the coefficient of $m^{-s} d^{-w}$ in the definition of $Z_{a_1,a_2}(s, w)$ is

$$\begin{aligned}
\chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1}} \mu(e_1)\chi_{a_1 d_0}(e_1 e_3) e_2 &= \chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1)\chi_{a_1 d_0}(e_1 e_3) e_2 \\
&= \chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) \sum_{\substack{e_1 e_2^2 e_3 = m \\ e_1 e_2 | d_1 \\ (d_0, e_1 e_3) = 1}} \mu(e_1) e_2 \\
&= \chi_{a_1 d_0}(\widehat{m})\chi_{a_2}(d) a(m, d),
\end{aligned}$$

where the first equality holds because $\chi_{d_0}(e_1 e_3) = 0$ unless $(d_0, e_1 e_3) = 1$ and the second equality holds because if $(d_0, e_1 e_3) = 1$, $\widehat{m}$ differs from $e_1 e_3$ by a perfect square (the divisors of which belong to (d_0, e_2)) and so $\chi_{d_0}(e_1 e_3) = \chi_{d_0}(\widehat{m})$.

Remark 1.2. *Also,* $a(m, d) = 0$ *unless* $m = e_1 e_2^2 e_3$ *with* $(d_0, e_1 e_3) = 1$ *and* $e_1 e_2^2 \mid d_1$.

We will define $L(s,\chi_{a_1 d})$ to be the Dirichlet series given by

$$L(s,\chi_{a_1 d}) = L(s,\chi_{a_1 d_0}) Q_{d_0 d_1^2}(s,\chi_{a_1}) = \sum_{m \text{ monic}} \frac{\chi_{a_1 d_0}(\widehat{m}) a(m,d)}{m^s}.$$

In particular, $L(s,\chi_d)$ now makes sense for d not necessarily square-free and this defintion agrees with the former when d is square-free. Moreover, we have the representation

$$Z_{a_1,a_2}(s,w) = \sum_{d \text{ monic}} \frac{\chi_{a_2}(d) L(s,\chi_{a_1 d})}{d^w}.$$

If we perform the same procedure to the interchange, then

$$Z_{a_1,a_2}(s,w) = \sum_{m \text{ monic}} \frac{L(w,\widetilde{\chi}_{a_2 m_0})\chi_{a_1}(m) Q_{m_0 m_1^2}(w,\chi_{a_2})}{m^s} = \sum_{m,d \text{ monic}} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d})\chi_{a_1}(m) a(d,m)}{m^s d^w},$$

where $\widehat{d}$ is the part of d relatively prime to m_0. Analogously, we define $L(w,\widetilde{\chi}_{a_2 m})$ to be the Dirichlet series given by

$$L(w,\widetilde{\chi}_{a_2 m}) = L(w,\widetilde{\chi}_{a_2 m_0}) Q_{m_0 m_1^2}(w,\chi_{a_2}) = \sum_{d \text{ monic}} \frac{\widetilde{\chi}_{a_2 m_0}(\widehat{d}) a(d,m)}{d^w},$$

so that

$$Z_{a_1,a_2}(s,w) = \sum_{m \text{ monic}} \frac{\chi_{a_1}(m) L(w,\widetilde{\chi}_{a_2 m})}{m^s}.$$

We now investigate the structure of the weighting coefficients $a(m,d)$. Their structure controls the majority of the information about both the double Dirichlet series and the correction polynomials. We first show that the weighting coefficients posess a multiplicativity property:

Proposition 1.1. *We have $a(m,1) = a(1,d) = 1$ and*

$$a(m,d) = \prod_{\substack{p^\alpha || m \\ p^\beta || d}} a(p^\alpha, p^\beta).$$

Proof. From the defintion of the weighting coefficients, $a(m,1) = a(1,d) = 1$. We will prove multiplicativity in m and then in d. Letting $m = m' p^\alpha$, we must show

$$a(m,d) = a(m',d) a(p^\alpha, d).$$

To acomplish this, for $e_1 e_2^2 e_3 = m$, let $e_1 = c_1 d_1$, $e_2 = c_2 d_2$, and $e_3 = c_3 d_3$ with $c_1, c_2, c_3 \mid m'$ and $d_1, d_2, d_3 \mid p^\alpha$. Because $(m', p^\alpha) = 1$, as $e_1 e_2^2 e_3$ runs over decompositions of m, $c_1 c_2^2 c_3$ and $d_1 d_2^2 d_3$ run over decompositions of m' and p^α respectively. Moreover, as $e_1 e_2$ runs over the divisors of d_1 so does $c_1 d_1 c_2 d_2$.

These facts combined with multiplicativity of the Möbius function gives

$$\begin{aligned} a(m,d) &= \sum_{\substack{e_1e_2^2e_3=m \\ e_1e_2|d_1 \\ (d_0,e_1e_3)=1}} \mu(e_1)e_2 \\ &= \sum_{\substack{c_1c_2^2c_3=m' \\ d_1d_2^2d_3=p^\beta \\ c_1d_1c_2d_2|d_1 \\ (d_0,c_1d_1c_3d_3)=1}} \mu(c_1)(d_1)|c_2|d_2 \\ &= \left(\sum_{\substack{c_1c_2^2c_3=m' \\ c_1c_2|d_1 \\ (d_0,c_1c_3)=1}} \mu(c_1)|c_2| \right) \left(\sum_{\substack{d_1d_2^2d_3=p^\alpha \\ d_1d_2|d_1 \\ (d_0,d_1d_3)=1}} \mu(d_1)d_2 \right) \\ &= a(m',d)a(p^\alpha,d), \end{aligned}$$

as desired. Now we prove multiplicativity in d. Since we have already proven multiplicativity in m, we may assume $m=p^\alpha$. Letting $d=d'p^\beta$, we must show

$$a(p^\alpha,d)=a(p^\alpha,p^\beta).$$

As $e_1e_2^2e_3=p^\alpha$, the e_i are powers of p for $1\le i\le 3$. It follows that $e_1e_2 \mid d_1$ is equivalent to $e_1e_2 \mid p^\beta$. Moreover, $(d_0,e_1e_2)=1$ is equivalent to $(1,e_1e_2)=1$ or $(p,e_1e_2)=1$ depending on of β is even or odd. These facts imply the desired identity. □

The correction polynomials $Q_{d_0d_1^2}(s,\chi_{a_1})$ are tightly connected to the weighting coefficients $a(m,d)$. In particular, $Q_{d_0d_1^2}(s,\chi_{a_1})$ is a Dirichlet polynomial whose coefficients are essentially given by the weighting coefficients. We first prove this relationship when d is an odd irreducible power:

Lemma 1.1. *For any irreducible p and $\alpha\ge 1$, we have*

$$Q_{p^{2\alpha+1}}(s)=\sum_{k\le 2\alpha}\frac{a(p^k,p^{2\alpha+1})}{p^{ks}}.$$

Proof. Expanding the correction polynomial in p^{-s} yields

$$Q_{p^{2\alpha+1}}(s)=\sum_{e_1e_2|p^\alpha}\mu(e_1)\chi_p(e_1)e_1^{-s}e_2^{1-2s}=\sum_{k\le 2\alpha}\frac{b(p^k,p^{2\alpha+1})}{p^{ks}}.$$

where

$$b(p^k,p^{2\alpha+1})=\sum_{e_1e_2^2=p^k}\mu(e_1)\chi_p(e_1)e_2.$$

The proof will be finished if we can show $b(p^k,p^{2\alpha+1})=a(p^k,p^{2\alpha+1})$. To see this, first observe $\mu(e_1)\chi_p(e_1)=0$ unless $e_1=1$ in which case it is 1. So $b(p^k,p^{2\alpha+1})=0$ if k is odd and $p^{\frac{k}{2}}$ if k is even. Compactly stated,

$$b(p^k,p^{2\alpha+1})=\begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.}\end{cases}$$

On the other hand, $k \leq \alpha$ so that

$$a(p^k, p^{2\alpha+1}) = \sum_{\substack{e_1 e_2^2 e_3 = p^k \\ e_1 e_2 | p^\alpha \\ (p, e_1 e_3) = 1}} \mu(e_1) e_2 = \sum_{\substack{e_1 e_2^2 | p^k \\ (p, e_1 e_3) = 1}} \mu(e_1) e_2 = \sum_{e_2^2 = p^k} e_2 = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

This finishes the proof. □

There is an analogous statement when d is an even irreducible power up to a square-free factor and relatively prime factor:

Lemma 1.2. *For any square-free monic d_0, $a_1 \in \{\pm 1, \pm 2\}$, irreducible p not dividing d_0, and $\beta \geq 1$, we have*

$$Q_{d_0 p^{2\beta}}(s, \chi_{a_1}) = (1 - \chi_{a_1 d_0}(p) p^{-s}) \sum_{k \leq 2\beta} \frac{\chi_{a_1 d_0}(p^k) a(p^k, p^{2\beta})}{p^{ks}}.$$

Proof. Expand the correction polynomial in p^{-s} to get

$$Q_{d_0 p^{2\beta}}(s, \chi_{a_1}) = \sum_{e_1 e_2 | p^\alpha} \mu(e_1) \chi_{a_1 d_0}(e_1) e_1^{-s} e_2^{1-2s} = \sum_{k \leq 2\beta} \frac{b(p^k, p^{2\beta})}{p^{ks}}.$$

where

$$b(p^k, p^{2\beta}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1) \chi_{a_1 d_0}(e_1) e_2.$$

It suffices to show $b(p^k, p^{2\beta}) = \chi_{a_1 d_0}(p^k) \left(a(p^k, p^{2\beta}) - a(p^{k-1}, p^{2\beta}) \right)$. On the one hand, $\mu(e_1) = 0$ unless $e_1 = 1, p$ in which case $\mu(e_1) = \pm 1$ accordingly. So

$$b(p^k, p^{2\beta}) = \sum_{e_1 e_2^2 = p^k} \mu(e_1) \chi_{a_1 d_0}(e_1) e_2 = \begin{cases} \chi_{a_1 d_0}(p^k) p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(p^k) p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

where we have used the identity $\chi_{a_1 d_0}(e_1) = \chi_{a_1 d_0}(p^k)$ which holds because this quadratic Dirichlet character only depends upon the parity of k. On the other hand, as in the proof of Lemma 1.1

$$a(p^k, p^{2\beta}) = \begin{cases} p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

But then

$$\chi_{a_1 d_0}(p^k) \left(a(p^k, p^{2\beta}) - a(p^{k-1}, p^{2\beta}) \right) = \begin{cases} \chi_{a_1 d_0}(p^k) p^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ -\chi_{a_1 d_0}(p^k) p^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

which completes the proof. □

Functional Equations

Meromorphic Continuation

Poles and Residues

References

[1] Rosen, M. (2002). Number theory in function fields (Vol. 210). Springer Science & Business Media.
[2] Hormander, L. (1973). An introduction to complex analysis in several variables. Elsevier.
[3] Chinta, G., & Gunnells, P. E. (2007). Weyl group multiple Dirichlet series constructed from quadratic characters. Inventiones mathematicae, 167, 327-353.
[4] Stanley, R. (2023). Enumerative Combinatorics: Volume 2. Cambridge University Press.