Analytic L-functions

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Throughout, $s = \sigma + it$ and $u = \tau + ir$ will stand for complex variables with σ, τ, t, and r real. We also write

$$(u)_k = u(u+1)\cdots(u+(k-1)),$$

for the Pochhammer symbol.

1 Analytic Data

An *analytic L-function* $L(s)$ is a Dirichlet series

$$L(s) = \sum_{n\geq 1} \frac{a_L(n)}{n^s},$$

with coefficients $a_L(n) \in \mathbb{C}$ such that $a_L(1) = 1$ and satisfying the following properties:

Analyticity: There exists a nonnegative integer m_L such that the Dirichlet series of $L(s)$ is absolutely convergent for $\sigma > 1$ and admits meromorphic continuation to $\mathbb{C}$ with at most a pole at $s = 1$ of order m_L. Moreover, $(s-1)^{m_L}L(s)$ is order 1.

Functional Equation: There is a positive integer q_L, called the *conductor* of $L(s)$, a positive integer d_L called the *degree* of $L(s)$, complex numbers $(\mu_j)_{1\leq j\leq d_L}$ called the *gamma parameters* of $L(s)$ that are either real or occur in conjugate pairs and satisfy $\mathrm{Re}(\mu_j) \geq 0$, and a complex number ε_L called the *root number* of $L(s)$ with $|\varepsilon_L| = 1$, all of which determine functions

$$\Lambda(s, L) = q_L^{\frac{s}{2}}\gamma_L(s)L(s) \quad \text{and} \quad \gamma_L(s) = \pi^{-\frac{d_L s}{2}}\prod_j \Gamma\left(\frac{s+\mu_j}{2}\right),$$

called the *completion* of $L(s)$ and the *gamma factor* of $L(s)$ respectively, satisfying

$$\Lambda(s, L) = \varepsilon_L\overline{\Lambda(1-\overline{s}, L)}.$$

This is called the *functional equation* of $L(s)$. The tuple $(\varepsilon_L, q_L, \mu_1, \ldots, \mu_{d_L})$ is called the *functional equation data* of $L(s)$.

Euler product: For every prime p, there exist complex numbers $(\alpha_j(p))_{1\le j\le d_L}$ called the *Euler parameters* of $L(s)$ at p satisfying $|\alpha_j(p)| \le p^\theta$ for some $0 \le \theta < 1$ and are such that $L(s)$ admits the Euler product

$$L(s) = \prod_p (1-\alpha_1(p)p^{-s})^{-1} \cdots (1-\alpha_{d_L}(p)p^{-s})^{-1},$$

for $\sigma > 1$. The polynomial L_p defined by

$$L_p(T) = (1-\alpha_1(p)T)\cdots(1-\alpha_{d_L}(p)T),$$

is called the *Euler factor* of $L(s)$ at p. If $p \nmid q_L$ then L_p is of degree d_L while if $p \mid q_L$ then L_p is of degree less than d_L.

An analytic L-function is said to be *Selberg class* if it satisfies the following additional property:

Ramanujan bound: We may take $\theta = 0$ so that $|\alpha_j(p)| \le 1$.

Some comments on the definition of analytic L-functions are in order.

(i) As the Dirichlet series of $L(s)$ converges absolutely for $\sigma > 1$ its converges locally absolutely uniformly in this half-plane and therefore defines a holomorphic function there. Moreover, the Euler product necessarily converges locally absolutely uniformly in the same half-plane and hence defines a holomorphic function there as well which agrees with the Dirichlet series.

(ii) The bound $\mathrm{Re}(\mu_j) \ge 0$ ensures that the gamma factor is holomorphic in the half-plane $\sigma > 0$. As this factor is then guaranteed to be finite and nonzero at $s = 1$, the completion possesses a pole at $s = 1$ of order r. By the functional equation, the completion also has a pole at $s = 0$ of the same order.

(iii) If we merely assume $(s-1)^{m_L}L(s)$ is finite order then the functional equation forces the order to be 1. This can be deduced by considering the entire function

$$s^{m_L}(s-1)^{m_L}\Lambda(s,L),$$

and applying the Phragmén-Lindelöf convexity principle in vertical strips.

(iv) The condition that the root number satisfies $|\varepsilon_L| = 1$ isn't strictly necessary. For applying the functional equation twice shows $|\varepsilon_L|^2 = 1$.

(v) As the gamma function is conjugation-equivariant and the gamma parameters are real or occur in conjugate pairs, the gamma factor is also conjugation-equivariant. This means that we can write the functional equation in the form

$$q_L^{\frac{s}{2}}\gamma_L(s)L(s) = \varepsilon_L q_L^{\frac{1-s}{2}}\gamma_L(1-s)\overline{L(1-\overline{s})}.$$

(vi) The Euler product implies that the coefficients $a_L(n)$ are multiplicative and are determined on prime powers by

$$a_L(p^k) = \sum_{1\le j_1\le\cdots\le j_k\le d_L} \alpha_{j_1}(p)\cdots\alpha_{j_k}(p).$$

In other words, $a_L(p^k)$ is the complete symmetric polynomial of degree k in the Euler parameters at p. If the Ramanujan bound holds, then it follows that $a_L(n) \ll \sigma_{d_L}(n)$. This implies the slightly weaker, but more practical, estimate $a_L(n) \ll_\varepsilon n^\varepsilon$.

It it clear from the definition that analytic L-functions are closed under multiplication and that the degree is additive. Moreover, this closure respects Selberg class L-functions. This makes the set of analytic L-functions into a graded commutative semigroup where the grading is induced by degree. It follows that there exist irreducible elements with respect to the grading and we say that an analytic L-function is *primitive* if it such an irreducible. Clearly any analytic L-function of degree 1 is primitive. Moreover, every analytic L-function factors into a product of primitive L-functions. Such a factorization is only conjectured to be unique for Selberg class L-functions. A sufficient condition would be *Selberg's orthogonality conjecture.*

Conjecture (Selberg's orthogonality conjecture). *For any two primitive Selberg class L-functions $L_1(s)$ and $L_2(s)$, we have*

$$\sum_{p\le x}\frac{a_{L_1}(p)\overline{a_{L_2}(p)}}{p} = \delta_{L_1,L_2}\log\log(x) + O(1).$$

Indeed, we have the following result:

Proposition 1.1. *Assume Selberg's orthogonality conjecture. Then every Selberg class L-function factors uniquely into a product of primitive Selberg class L-functions.*

Proof. If the factorization into primitive unity L-functions were not unique, then we would have distinct factorizations satisfying

$$L_1(s)\cdots L_n(s) = M_1(s)\cdots M_m(s),$$

for some primitive Selberg class L-functions L_i and M_j. By uniqueness of coefficients of Dirichlet series, compare the p-th coefficient to see that

$$\sum_i a_{L_i}(p) = \sum_j a_{M_j}(p).$$

Now consider

$$\sum_{p<x}\frac{1}{p}\left(\sum_i a_{L_i}(p)\right)\left(\sum_j \overline{a_{M_j}(p)}\right) = \sum_{i,j}\sum_{p<x}\frac{a_{L_i}(p)\overline{a_{M_j}(p)}}{p}.$$

Selberg's orthogonality conjecture implies

$$O(1) = c\log\log(x) + O(1),$$

for some positive integer c since the factorizations are distinct. This is impossible. $\square$

To an analytic L-function $L(s)$, we associate its *analytic conductor* $\mathfrak{q}(s,L)$ defined by

$$\mathfrak{q}(s,L) = q_L \mathfrak{q}_\infty(s,L),$$

where

$$\mathfrak{q}_\infty(s,L) = \prod_j (|s+\mu_j|+3).$$

The choice of 3 in $|s+\mu_j|+3$ is a matter of convenience as could use any positive constant. In particular, it is useful when taking logarithms as $\log(|s+\mu_j|+3) \geq 1$. For legibility, we write also write

$$\mathfrak{q}(L) = \mathfrak{q}(0,L) \quad \text{and} \quad \mathfrak{q}_\infty(L) = \mathfrak{q}_\infty(0,L).$$

Estimates for the analytic conductor follow from those of the gamma function. Recall from Stirling's formula that

$$\Gamma(s) \ll_\varepsilon (|t|+3)^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|} \quad \text{and} \quad \frac{1}{\Gamma(s)} \ll (|t|+3)^{\frac{1}{2}-\sigma} e^{\frac{\pi}{2}|t|}, \tag{1}$$

for bounded σ provided that in the former estimate s is at least distance ε away from the poles of the gamma function. Hence

$$\frac{\Gamma(1-s)}{\Gamma(s)} \ll_\varepsilon (|t|+3)^{1-2\sigma},$$

for bounded σ provided s is at least distance ε away from the poles of $\Gamma(1-s)$. Then the estimates

$$\frac{\gamma_L(1-s)}{\gamma_L(s)} \ll_\varepsilon \mathfrak{q}_\infty(s,L)^{\frac{1}{2}-\sigma} \quad \text{and} \quad q_L^{\frac{1}{2}-s} \frac{\gamma_L(1-s)}{\gamma_L(s)} \ll_\varepsilon \mathfrak{q}(s,L)^{\frac{1}{2}-\sigma}, \tag{2}$$

hold for bounded σ provided s is at least distance ε away from the poles of $\gamma_L(1-s)$. An associated estimate can be obtain for the logarithmic derivative of the analytic conductor. Recall from the logarithm of Stirling's formula that

$$\frac{\Gamma'}{\Gamma}(s) \ll \log(|s|+3),$$

provided σ is bounded and s is at least distance ε away from the poles of the gamma function. Then the estimates

$$\frac{\gamma_L'}{\gamma_L}(s) \ll_\varepsilon \log \mathfrak{q}_\infty(s,L) \quad \text{and} \quad \log q_L + \frac{\gamma_L'}{\gamma_L}(s) \ll_\varepsilon \log \mathfrak{q}(s,L), \tag{3}$$

hold for bounded σ provided s is at least distance ε away from the poles of the gamma factor.

The *critical strip* of an analytic L-function is the vertical strip left invariant by the transformation $s \mapsto 1-s$. This region can also be described as

$$\left\{ s \in \mathbb{C} : \left| \sigma - \frac{1}{2} \right| \leq \frac{1}{2} \right\}.$$

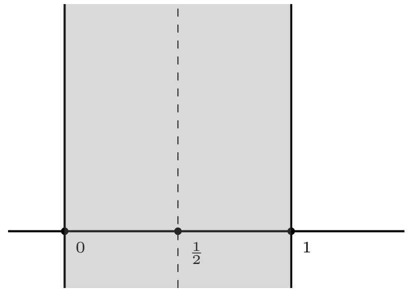

Figure 1: The critical strip.

The *critical line* is the vertical line left invariant by the transformation $s \mapsto 1-s$ which is also given by $\sigma = \frac{1}{2}$. The critical line bisects the critical strip vertically. The *central point* is the fixed point of the transformation $s \mapsto 1-s$, in other words, the point $s = \frac{1}{2}$. Clearly the central point is also the center of the critical line. The critical strip, critical line, and central point are all displayed in Figure 1.

In the half-plane $\sigma > 1$ we may study the analytic properties of $L(s)$ via its Dirichlet series. Using the functional equation, we may write

$$L(s) = \varepsilon_L q_L^{\frac{1}{2}-s} \frac{\gamma_L(1-s)}{\gamma_L(s)} \overline{L(1-\overline{s})}.$$

This permits the study of $L(s)$ in the half-plane $\sigma < 0$ by using the Dirichlet series of $\overline{L(1-\overline{s})}$ and the the functional equation data. The interior of the critical strip is exactly the region where we cannot use either of these methods to study the analytic properties of $L(s)$. Of course, anything may be possible on the boundary lines $\sigma = 0$ and $\sigma = 1$ (for example Landau's theorem).

2 The Approximate Functional Equation

Despite not being able to study an analytic L-function $L(s)$ in the critical strip by means of Dirichlet series, there is formula which acts as a compromise between the functional equation and the Dirichlet series. This formula is known as the approximate functional equation. The usefulness comes from the fact that the approximate functional equation is valid inside of the critical strip and therefore can be used to obtain analytic properties about $L(s)$ in that region. We will first derive a preliminary result showing $L(s)$ has at most polynomial growth.

Proposition 2.1. *Let $L(s)$ be an analytic L-function with σ bounded and σ at least distance ε away from the possible pole at $s = 1$. Then there is a positive constant A such that*

$$L(s) \ll_\varepsilon (|t|+3)^A.$$

Proof. Observe that $(s-1)^{m_L} L(s) \ll_\varepsilon (|t|+3)^{m_L}$ on the vertical line $\sigma = \max(1+\varepsilon, \sigma_2)$. Now suppose $\sigma = \min(-\varepsilon, \sigma_1)$. On this vertical line, the functional equation and Equation (2) together show

$$L(s) \ll_\varepsilon \mathfrak{q}(s,L)^{\frac{1}{2}-\sigma}\overline{L(1-\overline{s})}.$$

Hence there exists a positive constant A'' with

$$(s-1)^{m_L}L(s) \ll_\varepsilon (|t|+3)^{A''},$$

on the vertical line $\sigma = \min(-\varepsilon, \sigma_1)$. As $(s-1)^{m_L}L(s)$ is entire and of order 1, we can apply the Phragmén-Lindelöf convexity principle to $(s-1)^{m_L}L(s)$ the vertical strip $\min(-\varepsilon,\sigma_1) \le \sigma \le \max(1+\varepsilon,\sigma_2)$. Whence there is a positive constant A' such that

$$(s-1)^{m_L}L(s) \ll_\varepsilon (|t|+3)^{A'},$$

provided σ is bounded. Assuming s is at least distance ε away from the possible pole at $s=1$, diving by $(s-1)^{m_L}$ completes the proof. □

We are almost ready to prove the approximate function equation for an analytic L-function $L(s)$. The formula itself consists of two sums representing the Dirichlet series at s and a dualized Dirichlet series at $1-s$ as well as a potential residue term. The dual sum comes equip with a term containing the data of the functional equation. Both sums will also be dampened by a smooth cutoff function. In the statement of the approximate function equation, we will make use of a test function $\Phi(u)$. We require $\Phi(u)$ be an even holomorphic function bounded in the vertical strip $|\tau| < a+1$ for some $a > 1$ and such that $\Phi(0) = 1$. For s in the critical strip, we let $V_s(y)$ be the inverse Mellin transform

$$V_s(y) = \frac{1}{2\pi i}\int_{(a)} \frac{\gamma_L(s+u)}{\gamma_L(s)}\Phi(u)y^{-u}\frac{du}{u},$$

defined for $y > 0$. Stirling's formula implies

$$\frac{\Gamma(s+u)}{\Gamma(s)} \ll_\varepsilon \frac{(|t+r|+3)^{\sigma+\tau-\frac{1}{2}}}{(|t|+3)^{\sigma-\frac{1}{2}}}e^{-\frac{\pi}{2}(|t+r|-|t|)},$$

for s in the critical strip and bounded τ provided $s+u$ is at least distance ε away from the poles of the gamma function. Whence

$$\frac{\gamma_L(s+u)}{\gamma_L(s)} \ll_\varepsilon \frac{\mathfrak{q}_\infty(s+u)^{\frac{\sigma+\tau}{2}-\frac{1}{4}}}{\mathfrak{q}_\infty(s)^{\frac{\sigma}{2}-\frac{1}{4}}}e^{-d_L\frac{\pi}{2}(|t+r|-|t|)}, \tag{4}$$

for s in the critical strip and bounded τ provided $s+u$ is at least distance ε away from the poles of the gamma factor. Since $\Phi(u)$ is bounded in the vertical strip $|\tau| < a+1$, the integrand exhibits exponential decay. Therefore the integral is locally absolutely uniformly bounded and hence $V_s(y)$ is smooth. The function $V_s(y)$ is the smooth cutoff function mentioned previously. We will also let $\varepsilon_L(s)$ be given by

$$\varepsilon_L(s) = \varepsilon_L q_L^{\frac{1}{2}-s}\frac{\gamma_L(1-s)}{\gamma_L(s)}.$$

This term appears as a factor in the dual sum accounting for the functional equation data. We now prove the *approximate function equation*:

Theorem (Approximate functional equation). *Suppose $L(s)$ is an analytic L-function, $\Phi(u)$ is an even holomorphic function bounded in the vertical strip $|\tau| < a+1$ for some $a > 1$ and such that $\Phi(0) = 1$, and let $X > 0$. Then for s in the critical strip, we have*

$$L(s) = \sum_{n\geq 1} \frac{a_L(n)}{n^s} V_s\left(\frac{n}{\sqrt{q_L}X}\right) + \varepsilon_L(s) \sum_{n\geq 1} \frac{\overline{a_L(n)}}{n^{1-s}} V_{1-s}\left(\frac{nX}{\sqrt{q_L}}\right) + \frac{R(s,X,L)}{q_L^{\frac{s}{2}}\gamma_L(s)},$$

where $R(s,X,L)$ is given by

$$R(s,X,L) = \operatorname*{Res}_{u=1-s} \frac{\Lambda(s+u,L)\Phi(u)X^u}{u} + \operatorname*{Res}_{u=-s} \frac{\Lambda(s+u,L)\Phi(u)X^u}{u}.$$

Proof. Consider the integral

$$\frac{1}{2\pi i}\int_{(a)} \Lambda(s+u,L)\Phi(u)X^u \,\frac{du}{u}.$$

Stirling's formula shows that $\gamma_L(s+u)$ exhibits exponential decay while $L(s+u)$ has at most polynomial growth by Proposition 2.1. Since $\Phi(u)$ is bounded in the vertical strip $|\tau| < a+1$, it follows that the integrand exhibits exponential decay. Therefore the integral is locally absolutely uniformly bounded. We will evaluate the integral in two ways. On the one hand, we can expand $L(s+u)$ inside the integrand as a Dirichlet series and by absolute boundedness of the integral we may interchange the sum and integral. A short computation shows that this is

$$q_L^{\frac{s}{2}}\gamma_L(s) \sum_{n\geq 1} \frac{a_L(n)}{n^s} V_s\left(\frac{n}{\sqrt{q_L}X}\right).$$

On the other hand, we can shift the line of integration to $(-a)$. In doing so we pass by a simple pole at $u = 0$ and possible poles at $u = 1-s$ and $u = -s$ giving

$$\frac{1}{2\pi i}\int_{(-a)} \Lambda(s+u,L)\Phi(u)X^u \,\frac{du}{u} + \Lambda(s,L) + R(s,X,L).$$

Apply the functional equation and perform the change of variables $u \mapsto -u$ to rewrite this as

$$-\varepsilon_L \frac{1}{2\pi i}\int_{(a)} \overline{\Lambda(1-\overline{s+u},L)}\Phi(u)X^{-u} \,\frac{du}{u} + \Lambda(s,L) + R(s,X,L).$$

Analogous to the above, we can now expand $\overline{L(1-\overline{s+u})}$ inside the integrand as a Dirichlet series and by absolute boundedness of the integral we may interchange the sum and integral. A short computation shows that our previous expression becomes

$$-\varepsilon_L q_L^{\frac{1-s}{2}}\gamma_L(1-s) \sum_{n\geq 1} \frac{\overline{a_L(n)}}{n^{1-s}} V_s\left(\frac{nX}{\sqrt{q_L}}\right) + \Lambda(s,L) + R(s,X,L).$$

Equating these two evaluations and isolating the completion gives

$$\Lambda(s,L) = q_L^{\frac{s}{2}}\gamma_L(s)\sum_{n\geq 1}\frac{a_L(n)}{n^s}V_s\left(\frac{n}{\sqrt{q_L}X}\right)$$
$$+\,\varepsilon_L q_L^{\frac{1-s}{2}}\gamma_L(1-s)\sum_{n\geq 1}\frac{\overline{a_L(n)}}{n^{1-s}}V_{1-s}\left(\frac{nX}{\sqrt{q_L}}\right) + R(s,X,L).$$

Diving by $q_L^{\frac{s}{2}}\gamma_L(s)$ completes the proof. □

The approximate functional equation was first developed by Hardy and Littlewood in the series [**?**, **?**, **?**]. Let us now show how $V_s(y)$ has the effect of dampening the two dual sums appearing on the right-hand side of the approximate functional equation. In practice, it is common to choose $\Phi(u)$ such that it has exponential decay and we can make the vertical strip on which it is bounded arbitrarily wide. For example, let

$$\Phi(u) = \cos^{-4d_L M}\left(\frac{\pi u}{4M}\right),$$

for some positive integer M. Clearly $\Phi(u)$ is an even holomorphic function in the vertical strip $|\tau| < (2M-1)+1$ and satisfies $\Phi(0) = 1$. In view of the identity $\cos(u) = \frac{e^{iu}+e^{-iu}}{2}$, we find that

$$\cos^{-4d_L M}\left(\frac{\pi u}{4M}\right) \ll_\varepsilon e^{-d_L\pi|r|}, \tag{5}$$

for $|\tau| < (2M-1)+1$ provided u is at least distance ε away from the poles of $\Phi(u)$. Therefore $\Phi(u)$ admits exponential decay. For this choice of $\Phi(u)$, $V_s(y)$ and its derivatives will possess rapid decay.

Proposition 2.2. *Let $L(s)$ be an analytic L-function, set $\Phi(u) = \cos^{-4d_L M}\left(\frac{\pi u}{4M}\right)$ for some positive integer M, and let $V_s(y)$ be the inverse Mellin transform defined by*

$$V_s(y) = \frac{1}{2\pi i}\int_{(2M-1)}\frac{\gamma_L(s+u)}{\gamma_L(s)}\Phi(u)y^{-u}\frac{du}{u}.$$

Then for s in the critical strip, any nonnegative integer k, and positive integer N with $N < 2M$, $V_s(y)$ satisfies the estimates

$$(-y)^k V_s^{(k)}(y) = \begin{cases} \delta_{k,0} + O_\varepsilon\left(\left(\frac{y}{\sqrt{\mathfrak{q}_\infty(s,L)}}\right)^N\right) & \text{if } y \ll \sqrt{\mathfrak{q}_\infty(s,L)}, \\ O_\varepsilon\left(\left(\frac{y}{\sqrt{\mathfrak{q}_\infty(s,L)}}\right)^{-N}\right) & \text{if } y \gg \sqrt{\mathfrak{q}_\infty(s,L)}. \end{cases}$$

In particular,

$$(-y)^k V_s^{(k)}(y) \ll_\varepsilon \left(1 + \frac{y}{\sqrt{\mathfrak{q}_\infty(s,L)}}\right)^{-N}.$$

Proof. As we have already seen, the integrand defining $V_s(y)$ admits exponential decay. This permits us to differentiate under the integral sign and shift the line of integration. The former shows

$$(-y)^k V_s^{(k)}(y) = \frac{1}{2\pi i}\int_{(2M-1)} \frac{\gamma_L(s+u)}{\gamma_L(s)}\Phi(u)(u)_k y^{-u}\frac{du}{u}.$$

Shifting to $(-N)$, we pass by a simple pole at $u = 0$ of residue 1 if and only if $k = 0$. This gives

$$(-y)^k V_s^{(k)}(y) = \delta_{k,0} + \frac{1}{2\pi i}\int_{(-N)} \frac{\gamma_L(s+u)}{\gamma_L(s)}\Phi(u)(u)_k y^{-u}\frac{du}{u}.$$

This integrand also exhibits exponential decay since the Pochhammer symbol grows polynomially. Therefore the integral is dominated by the contribution when $u \ll 1$ and the Equations (4) and (5) together show

$$(-y)^k V_s^{(k)}(y) = \delta_{k,0} + O_\varepsilon\left(\left(\frac{y}{\sqrt{\mathfrak{q}_\infty(s,L)}}\right)^N\right).$$

If we instead shift to (N), we do not pass by any poles and obtain

$$(-y)^k V_s^{(k)}(y) = \frac{1}{2\pi i}\int_{(N)} \frac{\gamma_L(s+u)}{\gamma_L(s)}\Phi(u)(u)_k y^{-u}\frac{du}{u}.$$

An analogous argument to estimate the remaining integral shows

$$(-y)^k V_s^{(k)}(y) = O_\varepsilon\left(\left(\frac{y}{\sqrt{\mathfrak{q}_\infty(s,L)}}\right)^{-N}\right).$$

From these O-estimates we obtain nontrivial bounds in the ranges $y \ll \sqrt{\mathfrak{q}_\infty(s,L)}$ and $y \gg \sqrt{\mathfrak{q}_\infty(s,L)}$ respectively. Combining both of these estimates produces the bound

$$(-y)^k V_s^{(k)}(y) \ll_\varepsilon \left(1 + \frac{y}{\sqrt{\mathfrak{q}_\infty(s,L)}}\right)^{-N}.$$

□

With our choice of $\Phi(u)$, this result shows that $V_s(y)$ is essentially 1 up to some admissible error for $y \ll \sqrt{\mathfrak{q}_\infty(s,L)}$ and exhibits rapid decay thereafter as we can take M (and hence N) to be arbitrarily large.

In a similar spirit to the approximate functional equation, a useful summation formula can be derived from the functional equation. Let $\psi(y)$ be a smooth weight where $\Psi(s)$ is its Mellin transform. Then we will let $\psi(y,L)$ be the inverse Mellin transform

$$\psi(y,L) = \frac{1}{2\pi i}\int_{(a)} q_L^s \frac{\gamma_L(s)}{\gamma_L(1-s)} y^{-s}\Psi(1-s)\,ds,$$

defined for $y > 0$ where $a > 1$. By our choice of $\psi(y)$, its inverse Mellin transform $\Psi(s)$ has rapid decay. Since $L(s)$ has at most polynomial growth by Proposition 2.1, the integrand has rapid decay as well. Therefore the integral is locally absolutely uniformly bounded and hence $\psi(y, L)$ is smooth for $y > 0$. Our result is the following:

Theorem 2.3. *Let $L(s)$ be an analytic L-function and let $\psi(y)$ be a smooth weight where $\Psi(s)$ is its Mellin transform. Then*

$$\sum_{n\geq 1} a_L(n)\psi(n) = \frac{\varepsilon_L}{\sqrt{q_L}} \sum_{n\geq 1} \overline{a_L(n)}\psi(n, L) + R(L)\Psi(1),$$

where $R(L)$ is given by

$$R(L) = \operatorname*{Res}_{s=1} L(s).$$

Proof. Smoothed Perron's formula implies

$$\sum_{n\geq 1} a_L(n)\psi(n) = \frac{1}{2\pi i}\int_{(a)} L(s)\Psi(s)\,ds.$$

By our choice of $\psi(y)$, its inverse Mellin transform $\Psi(s)$ has rapid decay. Since $L(s)$ has at most polynomial growth by Proposition 2.1, the integrand has rapid decay as well. Therefore the integral is locally absolutely uniformly bounded which permits us to shift the line of integration. Shifting to $(1-a)$, we pass by a potential pole at $s = 1$ and obtain

$$\sum_{n\geq 1} a_L(n)\psi(n) = \frac{1}{2\pi i}\int_{(1-a)} L(s)\Psi(s)\,ds + R(L)\Psi(1).$$

Apply the functional equation to rewrite this equality in the form

$$\sum_{n\geq 1} a_L(n)\psi(n) = \frac{1}{2\pi i}\int_{(1-a)} \varepsilon_L q_L^{\frac{1}{2}-s}\frac{\gamma_L(1-s)}{\gamma_L(s)}\overline{L(1-\overline{s})}\Psi(s)\,ds + R(L)\Psi(1).$$

Performing the change of variables $s \mapsto 1-s$ in this latter integral gives

$$\sum_{n\geq 1} a_L(n)\psi(n) = \frac{1}{2\pi i}\int_{(a)} \varepsilon_L q_L^{s-\frac{1}{2}}\frac{\gamma_L(s)}{\gamma_L(1-s)}\overline{L(\overline{s})}\Psi(1-s)\,ds + R(L)\Psi(1).$$

The proof is complete upon expanding $\overline{L(\overline{s})}$ as a Dirichlet series, interchanging the sum and integral by absolute boundedness of the integral, and factoring out $\frac{\varepsilon_L}{\sqrt{q_L}}$. $\square$

3 The Riemann Hypothesis and Nontrivial Zeros

The zeros of an L-functions $L(s)$ has interesting behavior. From the Euler product we immediately see that $L(s)$ has no zeros in the half-plane $\sigma > 1$. We can use the functional equation to determine the zeros for $\sigma < 0$. Indeed, write the functional equation in the form

$$L(s) = \varepsilon_L q_L^{\frac{1}{2}-s} \frac{\gamma_L(1-s)}{\gamma_L(s)} \overline{L(1-\overline{s})}.$$

So for $\sigma < 0$, we see that $\overline{L(1-\overline{s})}$ is nonzero. Moreover, $\gamma_L(1-s)$ is as well. Together this means that for $\sigma < 0$ the poles of $\gamma_L(s)$ are zeros of $L(s)$. Such a zero is called a *trivial zero* of $L(s)$. From the definition of the gamma factor, they are all simple and of the form $s = -(\mu_j + 2n)$ for some gamma parameter μ_j and some nonnegative integer n.

Any other zero of $L(s)$ is called a *nontrivial zero* and it necessarily lies inside of the critical strip. Let ρ be a nontrivial zero of $L(s)$. Then the functional equation implies that $1-\overline{\rho}$ is also a nontrivial zero of $L(s)$. This means that nontrivial zeros occur in pairs

$$\rho \quad \text{and} \quad 1-\overline{\rho}.$$

It is possible to say more when $L(s)$ takes real values for $s > 1$. For in this case, the Schwarz reflection principle implies $L(\overline{s}) = \overline{L(s)}$ and that $L(s)$ takes real values on the entire real axis save for the possible pole at $s = 1$. It follows from the functional equation that $\overline{\rho}$ and $1-\rho$ are also nontrivial zeros. Therefore the nontrivial zeros of $L(s)$ come in sets of four

$$\rho, \quad \overline{\rho}, \quad 1-\rho, \quad \text{and} \quad 1-\overline{\rho}.$$

See Figure 2 for an example of this symmetry.

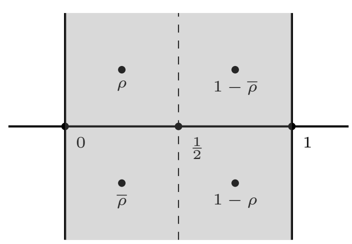

Figure 2: Symmetric nontrivial zeros.

The *Riemann hypothesis* for $L(s)$ says that this symmetry should be as simple as possible.

Conjecture (Riemann hypothesis, $L(s)$). *The nontrivial zeros of the analytic L-function $L(s)$ all lie on the vertical line $\sigma = \frac{1}{2}$.*

While stated as a conjecture, we not expect the Riemann hypothesis to hold for just any analytic L-function. We do however expect it to hold for Selberg class L-functions. In particular, the *Selberg class Riemann hypothesis* says that this symmetry should hold for any Selberg class L-function:

Conjecture (Selberg class Riemann hypothesis). *The nontrivial zeros of any Selberg class L-function $L(s)$ lie on the vertical line $\sigma = \frac{1}{2}$.*

So far, the Riemann hypothesis remains completely out of reach for any analytic L-function and thus the Selberg class Riemann hypothesis does as well.

4 The Lindelöf Hypothesis and Estimates on the Critical Line

Instead of asking about the zeros of an analytic L-function $L(s)$ on the critical line we can ask about its growth there. Let us begin this investigation by deriving an upper bound for $L(s)$ on the critical line using the Phragmén-Lindelöf convexity principle. This amounts to a more refined version of the argument used in Proposition 2.1 for the critical strip.

Theorem 4.1. *Let $L(s)$ be an analytic L-function. Then for $-\varepsilon \leq \sigma \leq 1+\varepsilon$, we have*

$$L(s) \ll_\varepsilon \mathfrak{q}(s,L)^{\frac{1-\sigma}{2}+\varepsilon},$$

provided s is at least distance ε away from the possible pole at $s=1$.

Proof. Since $(s-1)^{r_L}L(s)$ is entire and of order 1, we will apply the Phragmén-Lindelöf convexity principle just outside the edges of the critical strip. Just past the right edge along the vertical line $\sigma = 1+\varepsilon$, we have

$$(s-1)^{r_L}L(s) \ll_\varepsilon (s-1)^{r_L}.$$

Just past the left edge along the vertical line $\sigma = -\varepsilon$, the functional equation and Equation (2) together imply the bound

$$(s-1)^{r_L}L(s) \ll_\varepsilon (s-1)^{r_L}\mathfrak{q}(s,L)^{\frac{1}{2}+\varepsilon}.$$

By the Phragmén-Lindelöf convexity principle, we obtain

$$(s-1)^{r_L}L(s) \ll_\varepsilon (s-1)^{r_L}\mathfrak{q}(s,L)^{\frac{1-\sigma}{2}+\varepsilon},$$

provided $-\varepsilon \leq \sigma \leq 1+\varepsilon$. Assuming s is at least distance ε away from the possible pole at $s=1$, dividing by $(s-1)^{r_L}$ completes the proof. $\square$

At the critical line this theorem produces the bound

$$L\left(\frac{1}{2}+it\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it, L\right)^{\frac{1}{4}+\varepsilon}.$$

This is known as the *convexity bound* for $L(s)$. The *Lindelöf hypothesis* for $L(s)$ says that the exponent can be reduced to ε:

Conjecture (Lindelöf hypothesis, $L(s)$). *The analytic L-function $L(s)$ satisfies*

$$L\left(\frac{1}{2}+it\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it, L\right)^{\varepsilon}.$$

Unlike the Riemann hypothesis, the Lindelöf hypothesis is expected to hold for any analytic L-function. In any case, the *Selberg class Lindelöf hypothesis* says that the exponent can be reduced to ε for any Selberg class L-function:

Conjecture (Selberg class Lindelöf hypothesis). *Any Selberg class L-function $L(s)$ satisfies*

$$L\left(\frac{1}{2}+it\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it, L\right)^{\varepsilon}.$$

Like the Riemann hypothesis, we have been unable to prove the Lindelöf hypothesis for any analytic L-function. However, in practice the Lindelöf hypothesis is more tractable. Generally speaking, any improvement upon the exponent in the convexity bound in any aspect of the analytic conductor is called a *subconvexity estimate* (or a *convexity breaking bound*). In the uniform case, we would like to prove bounds of the form

$$L\left(\frac{1}{2}+it\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it, L\right)^{\delta+\varepsilon},$$

for some $0 \leq \delta \leq \frac{1}{4}$. The convexity bound says that we may take $\delta = \frac{1}{4}$ while the Lindelöf hypothesis for $L(s)$ implies that we may take $\delta = 0$. Any uniform subconvexity estimate would give a δ strictly less than $\frac{1}{4}$. Subconvexity estimates are viewed with great interest. This is primarily because of their connection to the Lindelöf hypothesis, but also because any improvement upon the convexity bound in any aspect of the analytic conductor often has drastic consequences in applications.

Remark 4.2. Some subconvexity bounds are deserving of names. The cases $\delta = \frac{3}{16}$ and $\delta = \frac{1}{6}$ are referred to as the *Burgess bound* and *Weyl bound* respectively.

With a little more work, we can obtain a a similar bound for the derivatives of analytic L-functions. First observe that Theorem 4.1 gives the estimate

$$L(s) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it, L\right)^{\frac{1}{4}+\varepsilon},$$

in the vertical strip $\left|\sigma - \frac{1}{2}\right| \leq \frac{\varepsilon}{2}$. This is just slightly stronger than the convexity bound. By Cauchy's integral formula, we may write

$$L^{(k)}\left(\frac{1}{2}+it\right) = \frac{k!}{2\pi i}\int_\eta \frac{L(s)}{(s-\frac{1}{2}-it)^{k+1}}\,ds,$$

where η is the circle about $\frac{1}{2}+it$ of radius $\frac{\varepsilon}{2}$. The parameterization $u \mapsto \frac{1}{2}+it+\frac{\varepsilon}{2}e^{i\theta}$ for η shows that

$$L^{(k)}\left(\frac{1}{2}+it\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it, L\right)^{\frac{1}{4}+\varepsilon}.$$

This is known as the *convexity bound* for $L^{(k)}(s)$.

We now turn to the question of how to obtain estimates on the critical line. There are many methods one can apply often depending upon the particular analytic L-function of interest. Here we will provide a method that works in vast generality and reduces the estimation of $L\left(\frac{1}{2}+it\right)$ to that of estimating smoothed finite sums of coefficients. To achieve this we need to establish the existence of certain bump functions. We say that a function $\omega(y)$ is a *smooth dyadic weight* if it is a positive bump function such that

$$\sum_{k\in\mathbb{Z}} \omega\left(\frac{y}{2^k}\right) = 1,$$

for all positive y. This last condition means $\left(\omega\left(\frac{y}{2^k}\right)\right)_{k\in\mathbb{Z}}$ is a partition of unity on $(0,\infty)$. To see that dyadic weights exist, let $\psi(y)$ be a smooth weight that is identically 1 on $[1,2]$ and supported in $\left[\frac{1}{2},4\right]$. Write

$$\sigma(y) = \sum_{k\in\mathbb{Z}} \psi\left(\frac{y}{2^k}\right).$$

Then $\sigma(y)$ is finite as finitely many of its summands are nonzero since any positive y satisfies $2^k \le y < 2^{k+1}$ for some unique integer k. This forces $\sigma(y)$ to be smooth and bounded. It is also bounded away from zero as $\sigma(y) \ge 1$ because $\psi(y)$ is identically 1 on $[1,2]$. Then we may take

$$\omega(y) = \frac{\psi(y)}{\sigma(y)}.$$

This shows that smooth dyadic weights exist. For any $t \in \mathbb{R}$ and $X > 0$, set

$$A_\omega(X,t,L) = \sum_{n\ge1} a_L(n) n^{-it} \omega\left(\frac{n}{X}\right).$$

Our result estimates $L(s)$ at $s = \frac{1}{2} + it$ in terms of $A_\omega(X,t,L)$.

Theorem 4.3. *Let $L(s)$ be an analytic L-function and let $\omega(y)$ be a smooth dyadic weight. Then*

$$L\left(\frac{1}{2}+it\right) \ll \mathfrak{q}\left(\frac{1}{2}+it, L\right)^{\varepsilon} \sup_{X \ll \sqrt{\mathfrak{q}\left(\frac{1}{2}+it,L\right)}} \left(\frac{|A_\omega(X,t,L)|}{\sqrt{X}}\right).$$

Proof. Take $s = \frac{1}{2} + it$, $X = 1$, and $\Phi(u) = \cos^{-4d_L M}\left(\frac{\pi u}{4M}\right)$ for some large positive integer M in the approximate functional equation. This permits us to write

$$\begin{aligned} L\left(\frac{1}{2}+it\right) &= \sum_{n\ge1} \frac{a_L(n)}{n^{\frac{1}{2}+it}} V_{\frac{1}{2}+it}\left(\frac{n}{\sqrt{q_L}}\right) \\ &\quad + \varepsilon_L\left(\frac{1}{2}+it\right) \sum_{n\ge1} \frac{\overline{a_L(n)}}{n^{\frac{1}{2}-it}} V_{\frac{1}{2}-it}\left(\frac{n}{\sqrt{q_L}}\right) + \frac{R\left(\frac{1}{2}+it,1,L\right)}{q_L^{\frac{1}{4}+i\frac{t}{2}} \gamma_L\left(\frac{1}{2}+it\right)}. \end{aligned}$$

Equation (2) implies $\varepsilon_L\left(\frac{1}{2}+it\right) \ll 1$ while Equations (1) and (5) together imply that the residue term exhibits exponential decay. Whence

$$L\left(\frac{1}{2}+it\right) \ll \left|\sum_{n\ge1} \frac{a_L(n)n^{-it}}{\sqrt{n}} V_{\frac{1}{2}+it}\left(\frac{n}{\sqrt{q_L}}\right)\right| + \left|\sum_{n\ge1} \frac{\overline{a_L(n)n^{it}}}{\sqrt{n}} V_{\frac{1}{2}-it}\left(\frac{n}{\sqrt{q_L}}\right)\right| + e^{-c|t|},$$

for some $c > 0$ say. Note that by our choice of $\Phi(u)$, the estimates in Proposition 2.2 for $V_{\frac{1}{2}+it}(y)$ and $V_{\frac{1}{2}-it}(y)$ are valid and they show that the two sums are absolutely convergent. We turn to estimating these sums which will be accomplished by applying the smooth dyadic weight. Indeed, write

$$V_s(y) = \sum_{k\in\mathbb{Z}} \omega\left(\frac{y}{2^k}\right) V_s(y).$$

Apply this identity to the first sum to be estimated and interchange sums by absolute convergence to obtain

$$\sum_{k\in\mathbb{Z}}\sum_{n\geq 1} \frac{a_L(n)n^{-it}}{\sqrt{n}}\omega\left(\frac{n}{2^k\sqrt{q_L}}\right) V_{\frac{1}{2}+it}\left(\frac{n}{\sqrt{q_L}}\right).$$

By definition of the partition of unity, for each k the inner sum over n is supported on the dyadic block $n \asymp X_k$ where $X_k = 2^k\sqrt{q_L}$. Taking absolute values and applying Proposition 2.2, our sum is O of

$$\sum_{k\in\mathbb{Z}} \left(1 + \frac{X_k}{\sqrt{\mathfrak{q}\left(\frac{1}{2}+it, L\right)}}\right)^{-(2M-1)} \frac{|A_\omega(X_k, t, L)|}{\sqrt{X_k}}.$$

In the range $X_k \gg \sqrt{\mathfrak{q}\left(\frac{1}{2}+it, L\right)}$, using trivial bound $A_\omega(X_k, t, L) \ll_\varepsilon X_k^{2+\varepsilon}$ (as $a_L(n) \ll_\varepsilon n^{1+\varepsilon}$ by absolute convergence for $\sigma > 1$) shows that the corresponding contribution has rapid decay. While in the range $X_k \ll \sqrt{\mathfrak{q}\left(\frac{1}{2}+it, L\right)}$ all but finitely many of the k are such that $A_\omega(X_k, t, L) = 0$. There are at most $O_\varepsilon\left(\mathfrak{q}\left(\frac{1}{2}+it, L\right)^\varepsilon\right)$ exceptions and each contributes a term O of

$$\sup_{X\ll\sqrt{\mathfrak{q}\left(\frac{1}{2}+it,L\right)}} \left(\frac{|A_\omega(X, t, L)|}{\sqrt{X}}\right).$$

All together this shows that the sum is O_ε of

$$\mathfrak{q}\left(\frac{1}{2}+it, L\right)^\varepsilon \sup_{X\ll\sqrt{\mathfrak{q}\left(\frac{1}{2}+it,L\right)}} \left(\frac{|A_\omega(X, t, L)|}{\sqrt{X}}\right).$$

The estimate for the second sum is handled analogously in view of the identity

$$\left|\sum_{n\geq 1} \frac{\overline{a_L(n)n^{it}}}{\sqrt{n}} V_{\frac{1}{2}-it}\left(\frac{n}{\sqrt{q_L}}\right)\right| = \left|\sum_{n\geq 1} \frac{a_L(n)n^{-it}}{\sqrt{n}} \overline{V_{\frac{1}{2}-it}\left(\frac{n}{\sqrt{q_L}}\right)}\right|.$$

Whence

$$L\left(\frac{1}{2}+it\right) \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2}+it, L\right)^\varepsilon \sup_{X\ll\sqrt{\mathfrak{q}\left(\frac{1}{2}+it,L\right)}} \left(\frac{|A_\omega(X, t, L)|}{\sqrt{X}}\right),$$

upon absorbing the exponential decay term. $\square$

As an interesting observation, we obtain the convexity bound as an application of Theorem 4.4. Since the Dirichlet series of $L(s)$ is absolutely convergent for $\sigma < 1$, we have a bound of shape

$$\sum_{n \le X} |a_L(n)| \ll_\varepsilon X^{1+\varepsilon}.$$

Estimating $A_\omega(X, t, L)$ trivially by this bound, we obtain

$$\sup_{X \ll \sqrt{\mathfrak{q}\left(\frac{1}{2}+it, L\right)}} \frac{|A_\omega(X, t, L)|}{\sqrt{X}} \ll_\varepsilon \mathfrak{q}\left(\frac{1}{2} + it, L\right)^{\frac{1}{4}+\varepsilon}, \tag{6}$$

which gives the convexity bound. As we believe $L(s)$ satisfies the Lindelöf hypothesis, we should expect lots of cancellation in the sum $A_\omega(X, t, f)$. In fact, the expected cancellation should be $O\left(\mathfrak{q}\left(\frac{1}{2} + it, f\right)^{\frac{1}{4}}\right)$ in light of Equation (6). As the coefficients $a_L(n)$ may very well be real, this expected cancellation should come from the terms $n^{-it} = e^{-it\log(n)}$ which acts on $a_L(n)$ by rotation.

Sometimes, however, we are only concerned with the size of $L(s)$ at the central point. The value of $L(s)$ at the central point is called the *central value* of $L(s)$. Many important properties about $L(s)$ can be connected to its central value. Any argument used to estimate the central value of an L-function is called a *central value estimate*. Taking $t = 0$ in Theorem 4.3 gives a way to obtain central value estimates.

Theorem 4.4. *Let $L(s)$ be an analytic L-function and let $\omega(y)$ be a smooth dyadic weight. Then*

$$L\left(\frac{1}{2}\right) \ll \max_{X \ll \sqrt{\mathfrak{q}\left(\frac{1}{2}, L\right)}} \left(\frac{|A_\omega(0, X, L)|}{\sqrt{X}}\right).$$

Proof. Take $t = 0$ in Theorem 4.3. □

5 Logarithmic Derivatives

Recall that $L(s)$ nonzero in the half-plane $\sigma > 1$. As this region is simply connected, there exists a unique holomorphic logarithm $\log L(s)$ there. By absolute convergence of the Euler product, we may write

$$\log L(s) = -\sum_p \sum_j \log(1 - \alpha_j(p)p^{-s}).$$

Moreover, the bound $|\alpha_j(p)| \le p^\theta$ ensures that $|\alpha_j(p)p^{-s}| < 1$ in this half-plane. Therefore the Taylor series of the logarithm is valid in this region and we may further write

$$\log L(s) = \sum_p \sum_j \sum_{k \ge 1} \frac{\alpha_j(p)^k}{kp^{ks}}.$$

While this triple sum converges for $\sigma > 1$, the bound $|\alpha_j(p)p^{-s}| < p^{\theta-\sigma}$ only guarantees absolutely convergence for $\sigma > 1+\theta$. It is in this half-plane that we may differentiate termwise. A short computation shows

$$\frac{L'}{L}(s) = -\sum_{n\geq 1}\frac{\Lambda_L(n)}{n^s},$$

where

$$\Lambda_L(n) = \begin{cases} \sum_j \alpha_j(p)^k \log(p) & \text{if } n = p^k, \\ 0 & \text{otherwise.} \end{cases}$$

We call $\Lambda_L(n)$ the *Von Mangoldt coefficients* of $L(s)$.

There is an incredibly useful formula for the logarithmic derivative of $L(s)$ which is often the starting point for deeper analytic investigations. To deduce it, we require a more complete understanding of the completion $\Lambda(s, L)$. First observe that the zeros ρ of $\Lambda(s, L)$ are contained within the critical strip. Indeed, by our earlier discussion of zeros, $\Lambda(s, L)$ is nonzero for $\sigma > 1$ and the functional equation implies that $\Lambda(s, L)$ is nonzero for $\sigma < 0$ too. This means that the zeros of $\Lambda(s, L)$ are exactly nontrivial zeros of $L(s)$. Now let us setup some notation. We define $\xi(s, L)$ by

$$\xi(s, L) = (s(1-s))^{r_L}\Lambda(s, L).$$

Then $\xi(s, L)$ is just $\Lambda(s, L)$ with the potential poles at $s = 0$ and $s = 1$ removed. This means $\xi(s, L)$ is entire. The functional equation also implies

$$\xi(s, L) = \varepsilon_L \overline{\xi(1-\overline{s}, L)}.$$

Our result will compute the Hadamard factorization of $\xi(s, L)$ and taking its logarithmic derivative will give a formula relating the logarithmic derivative of $L(s)$ to the zeros of $L(s)$.

Proposition 5.1. *For any analytic L-function $L(s)$, there exist constants A_L and B_L such that*

$$\xi(s, L) = e^{A_L + B_L s}\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}},$$

and hence the sum

$$\sum_{\rho\neq 0,1}\frac{1}{|\rho|^{1+\varepsilon}},$$

is convergent provided the product and sum are both counted with multiplicity and ordered with respect to the size of the ordinate. Moreover,

$$-\frac{L'}{L}(s) = \frac{r_L}{s} + \frac{r_L}{s-1} + \frac{1}{2}\log q_L + \frac{\gamma_L'}{\gamma_L}(s) - B_L - \sum_{\rho\neq 0,1}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

Proof. Recall that $\xi(s,L)$ is entire and its zeros are exactly the nontrivial zeros of $L(s)$. We claim $\xi(s,L)$ is of order 1. This follows from Stirling's formula and that $(s-1)^{r_L}L(s)$ is of order 1. Then the Hadamard factorization theorem implies

$$\xi(s,L) = e^{A_L+B_Ls}\prod_{\rho\neq 0,1}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}},$$

for some constants A_L and B_L and that the desired sum converges. This proves the first statement. We will compute the logarithmic derivative of $\xi(s,L)$ in two ways to prove the second statement. On the one hand, taking the logarithmic derivative of the definition of $\xi(s,L)$ yields

$$\frac{\xi'}{\xi}(s,L) = \frac{r_L}{s} + \frac{r_L}{s-1} + \frac{1}{2}\log q_L + \frac{\gamma_L'}{\gamma_L}(s) + \frac{L'}{L}(s).$$

On the other hand, taking the logarithmic derivative of the Hadamard factorization gives

$$\frac{\xi'}{\xi}(s,L) = B_L + \sum_{\rho\neq 0,1}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

Equating these expressions to obtain

$$B_L + \sum_{\rho\neq 0,1}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) = \frac{r_L}{s} + \frac{r_L}{s-1} + \frac{1}{2}\log q_L + \frac{\gamma_L'}{\gamma_L}(s) + \frac{L'}{L}(s).$$

This is equivalent to the second statement. $\square$

Explicit evaluation of the constants A_L and B_L can be challenging and heavily depend upon the particular L-function under investigation. However, useful estimates are not too difficult to obtain and this is often sufficient for applications.