The Poisson summation formula

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1 The Fourier Transform

Let $f(\mathbf{x})$ be absolutely integrable on $\mathbb{R}^n$. Its *Fourier transform* $(\mathcal{F}f)(\boldsymbol{\xi})$ is defined by

$$(\mathcal{F}f)(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{x})e^{-2\pi i\langle \boldsymbol{\xi},\mathbf{x}\rangle}\,d\mathbf{x},$$

for $\boldsymbol{\xi} \in \mathbb{R}^n$. It is absolutely bounded precisely because $f(\mathbf{x})$ is absolutely integrable on $\mathbb{R}^n$. Let us first demonstrate some properties of the Fourier transform.

Proposition 1.1. *Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be absolutely integrable on $\mathbb{R}^n$. Then the following properties hold:*

(i) For any $\alpha, \beta \in \mathbb{R}$, we have

$$(\mathcal{F}(\alpha f + \beta g))(\boldsymbol{\xi}) = \alpha(\mathcal{F}f)(\boldsymbol{\xi}) + \beta(\mathcal{F}g)(\boldsymbol{\xi}).$$

(ii) If $g(\mathbf{x}) = f(\mathbf{x} + \boldsymbol{\alpha})$ for any $\boldsymbol{\alpha} \in \mathbb{R}^n$ then

$$(\mathcal{F}g)(\boldsymbol{\xi}) = e^{2\pi i\langle \boldsymbol{\alpha},\boldsymbol{\xi}\rangle}(\mathcal{F}f)(\boldsymbol{\xi}).$$

(iii) If $g(\mathbf{x}) = f(A\mathbf{x})$ for any $A \in \mathrm{GL}_n(\mathbb{R})$ then

$$(\mathcal{F}g)(\boldsymbol{\xi}) = |\det(A)|^{-1}(\mathcal{F}f)((A^{-1})^t\boldsymbol{\xi}).$$

Proof. Property (i) is immediate from linearity of the integral while (ii) follows by applying the change of variables $\mathbf{x} \mapsto \mathbf{x} - \boldsymbol{\alpha}$. Property (iii) is proved by performing the change of variables $\mathbf{x} \mapsto A^{-1}\mathbf{x}$ which has Jacobian determinant is $|\det(A)|^{-1}$. □

2 Fourier Series

The Fourier transform is intimately related to periodic functions. Let Λ be a complete integral lattice in $\mathbb{R}^n$ with fundamental domain $\mathcal{M}$ and denote the dual lattice by $\Lambda^\vee$. Suppose $f(\mathbf{x})$ is Λ-periodic and integrable on $\mathcal{M}$. Then we define the $\lambda^\vee$-th *Fourier coefficient* $\hat{f}(\lambda^\vee)$ of $f(\mathbf{x})$ by

$$\hat{f}(\lambda^\vee) = \int_{\mathcal{M}} f(\mathbf{x})e^{-2\pi i\langle \lambda^\vee,\mathbf{x}\rangle}\,d\mathbf{x}.$$

The *Fourier series* $S_f(x)$ is defined by

$$S_f(\mathbf{x}) = \frac{1}{V_\Lambda} \sum_{\lambda^\vee \in \Lambda^\vee} \hat{f}(\lambda^\vee) e^{2\pi i \langle \lambda^\vee, \mathbf{x} \rangle}.$$

Notice that the Fourier coefficients are only defined for elements of the dual lattice and that the Fourier series is a sum over these elements. A crucial question is where the Fourier series of $f(\mathbf{x})$ converges, if at all, and if so does it even converge to $f(\mathbf{x})$ itself. Under quite reasonable conditions the Fourier series converges uniformly to the function itself, but we first make a reduction. Fix a basis $\lambda_1, \ldots, \lambda_n$ for Λ and let P be the associated generator matrix. This means P is the base change matrix from the standard basis to $\lambda_1, \ldots, \lambda_n$ so that P is the unique invertible linear transformation on $\mathbb{R}^n$ satisfying $\Lambda = P\mathbb{Z}^n$ and $\Lambda^\vee = (P^{-1})^t \mathbb{Z}^n$. Letting $f_P(\mathbf{x}) = f(P\mathbf{x})$, we see that $f_P(\mathbf{x})$ is 1-periodic in each component and integrable on $[0,1]^n$. As $\mathbb{Z}^n$ is self-dual and its covolume is 1, the $\mathbf{n}$-th Fourier coefficient $\hat{f}_P(\mathbf{n})$ of $f_P(\mathbf{x})$ is given by

$$\hat{f}_P(\mathbf{n}) = \int_{[0,1]^n} f(\mathbf{x}) e^{-2\pi i \langle \mathbf{n}, \mathbf{x} \rangle} \, d\mathbf{x}.$$

and the Fourier series is

$$S_{f_P}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^n} \hat{f}_P(\mathbf{n}) e^{2\pi i \langle \mathbf{n}, \mathbf{x} \rangle}.$$

As P is invertible, it suffices to study the convergence properties of this Fourier series. The reduction here this that we may assume our Fourier series are 1-periodic in each component and in this case we have the following well-known result:

Theorem 2.1. *Suppose $f(\mathbf{x})$ is a smooth function on $\mathbb{R}^n$ and is 1-periodic in each component. Then the Fourier series of $f(\mathbf{x})$ converges uniformly everywhere to $f(\mathbf{x})$.*

In the case of 1-periodic functions of a single variable, we can do better as we may merely assume $f(x)$ is of bounded variation. This is known as the *Dirichlet-Jordan test*:

Theorem (Dirichlet-Jodan test). *Suppose $f(x)$ is a function on $\mathbb{R}$ which is 1-periodic and of bounded variation. Then the Fourier series of $f(x)$ converges locally uniformly to $f(x)$ on every set where $f(x)$ is continuous. Moreover, at any jump discontinuity, the Fourier series of $f(x)$ converges to the average of the left-hand and right-hand limits of $f(x)$. In particular, this holds for all continuously differentiable functions with at most a finite number of jump discontinuities.*

3 Poisson Summation

Returning to the general setting, there are two ways of building a function from $f(\mathbf{x})$ that is λ-periodic. The first is to average $f(\mathbf{x})$ over all translates by elements of Λ while the second is to considering its Fourier series $f(\mathbf{x})$. This gives us the two series

$$\sum_{\lambda \in \Lambda} f(\mathbf{x} + \lambda) \quad \text{or} \quad \frac{1}{V_\Lambda} \sum_{\lambda^\vee \in \Lambda^\vee} (\mathcal{F}f)(\lambda^\vee) e^{2\pi i \langle \lambda^\vee, \mathbf{x} \rangle}.$$

The link between the Fourier transform and Fourier series is given by the *Poisson summation formula* which says that these two expressions are the same under some mild assumptions.

Theorem (Poisson summation formula). *Suppose Λ is a complete integral lattice in $\mathbb{R}^n$, $f(\mathbf{x})$ is absolutely integrable on $\mathbb{R}^n$, and the function*

$$F(\mathbf{x}) = \sum_{\lambda\in\Lambda} f(\mathbf{x}+\lambda),$$

is smooth. Then

$$\sum_{\lambda\in\Lambda} f(\mathbf{x}+\lambda) = \frac{1}{V_\Lambda} \sum_{\lambda^\vee\in\Lambda^\vee} (\mathcal{F}f)(\lambda^\vee) e^{2\pi i\langle\lambda^\vee,\mathbf{x}\rangle},$$

and

$$\sum_{\lambda\in\Lambda} f(\lambda) = \frac{1}{V_\Lambda} \sum_{\lambda^\vee\in\Lambda^\vee} (\mathcal{F}f)(\lambda^\vee).$$

Proof. Fix a basis $\lambda_1, \ldots, \lambda_n$ for Λ and let P be the associated generator matrix. Then $\Lambda = P\mathbb{Z}^n$ and $\Lambda^\vee = (P^{-1})^t\mathbb{Z}^n$. Letting $F_P(\mathbf{x}) = F(P\mathbf{x})$, we may write

$$F_P(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^n} f(P\mathbf{x}+P\mathbf{n}).$$

$F_P(\mathbf{x})$ is smooth and 1-periodic in each component because $F(\mathbf{x})$ is smooth and invariant under translation by Λ. Whence $F_P(\mathbf{x})$ admits a Fourier series converging uniformly everywhere to $F_P(\mathbf{x})$. Moreover, $F_P(\mathbf{x})$ is absolutely integrable on $[0,1]^n$. We will compute the $\mathbf{t}$-th Fourier coefficient of $F_P(\mathbf{x})$ given by

$$\hat{F}_P(\mathbf{t}) = \int_{[0,1]^n} F_P(\mathbf{x}) e^{-2\pi i\langle\mathbf{t},\mathbf{x}\rangle}\, d\mathbf{x}$$

The absolute integrability of $f(\mathbf{x})$ permits the interchange of sum and integral to obtain

$$\sum_{\mathbf{n}\in\mathbb{Z}^n} \int_{[0,1]^n} f(P\mathbf{x}+P\mathbf{n}) e^{-2\pi i\langle\mathbf{t},\mathbf{x}\rangle}\, d\mathbf{x}.$$

The change of variables $\mathbf{x} \mapsto P^{-1}\mathbf{x}$ gives

$$\frac{1}{V_\Lambda} \sum_{\mathbf{n}\in\mathbb{Z}^n} \int_{P[0,1]^n} f(\mathbf{x}+P\mathbf{n}) e^{-2\pi i\langle(P^{-1})^t\mathbf{t},\mathbf{x}\rangle},$$

because the Jacobian matrix is P^{-1} whose absolute determinant is $\frac{1}{V_\Lambda}$. This expression is simply

$$\frac{1}{V_\Lambda} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i\left\langle(P^{-1})^t\mathbf{t},\mathbf{x}\right\rangle}\, d\mathbf{x} = \frac{1}{V_\Lambda} (\mathcal{F}f)\left((P^{-1})^t\mathbf{t}\right).$$

Whence

$$\sum_{\lambda\in\Lambda} f(P\mathbf{x}+\lambda) = \frac{1}{V_\Lambda}\sum_{\mathbf{t}\in\mathbb{Z}^n}(\mathcal{F}f)\left((P^{-1})^t\mathbf{t}\right)e^{2\pi i\langle\mathbf{t},\mathbf{x}\rangle}.$$

Changing variables $\mathbf{x}\mapsto P^{-1}\mathbf{x}$ results in

$$\sum_{\lambda\in\Lambda} f(\mathbf{x}+\lambda) = \frac{1}{V_\Lambda}\sum_{\lambda^\vee\in\Lambda^\vee}(\mathcal{F}f)(\lambda^\vee)e^{2\pi i\langle\lambda^\vee,\mathbf{x}\rangle}.$$

This proves the first statement. Setting $\mathbf{x}=\mathbf{0}$ proves the second statement. □

For convenience, we state the Poisson summation formula in the simplified case for the complete integral lattice $\mathbb{Z}^n$ as a corollary since it is how the Poisson summation formula is usually applied.

Corollary 3.1. *Suppose $f(\mathbf{x})$ is absolutely integrable on $\mathbb{R}^n$, and the function*

$$F(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^n} f(\mathbf{x}+\mathbf{n}),$$

is smooth. Then

$$\sum_{\mathbf{n}\in\mathbb{Z}^n} f(\mathbf{x}+\mathbf{n}) = \sum_{\mathbf{t}\in\mathbb{Z}^n}(\mathcal{F}f)(\mathbf{t})e^{2\pi i\langle\mathbf{t},\mathbf{x}\rangle},$$

and

$$\sum_{\mathbf{n}\in\mathbb{Z}^n} f(\mathbf{n}) = \sum_{\mathbf{t}\in\mathbb{Z}^n}(\mathcal{F}f)(\mathbf{t}).$$

Proof. This is the Poisson summation formula for the complete integral lattice $\mathbb{Z}^n$ since $\mathbb{Z}^n$ is self-dual and $V_{\mathbb{Z}^n}=1$. □

Using the Dirichlet-Jordan test, we can prove a slightly stronger form of the Poisson summation formula in the single variable setting for the complete integral lattice $\mathbb{Z}$.

Theorem 3.2. *Suppose $f(x)$ is absolutely integrable on $\mathbb{R}$, and the function*

$$F(x) = {\sum_{n\in\mathbb{Z}}}^* f(x+n),$$

satisfies the Dirichlet-Jordan test, where the $*$ in the sum indicates that $f(x+n)$ is meant to represent the average of the left-hand and right-hand limits at jump discontinuities. Then

$${\sum_{n\in\mathbb{Z}}}^* f(x+n) = \sum_{t\in\mathbb{Z}}(\mathcal{F}f)(t)e^{2\pi itx},$$

and

$${\sum_{n\in\mathbb{Z}}}^* f(n) = \sum_{t\in\mathbb{Z}}(\mathcal{F}f)(t).$$

Proof. Observe that $F(x)$ is 1-periodic. As $F(x)$ satisfies the Dirichlet-Jordan test by assumption, it admits a Fourier series converging locally uniformly to $F(x)$ wherever $F(x)$ is continuous. In fact, by the construction of $F(x)$ and that the Fourier series of $F(x)$ converges to the average of the left-hand and right-hand limits at jump discontinuities, the Fourier series of $F(x)$ converges locally uniformly to $F(x)$ everywhere. Moreover, $F(x)$ is absolutely integrable on $[0,1]$. We compute the t-th Fourier coefficient of $F(x)$ given by

$$\hat{F}(t) = \int_0^1 F(x)e^{-2\pi itx}\,dx.$$

The absolute integrability of $f(x)$ permits the interchange of sum and integral to obtain

$$\int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx = (\mathcal{F}f)(t).$$

Whence

$$\sum_{n\in\mathbb{Z}}^{*} f(x+n) = \sum_{t\in\mathbb{Z}}(\mathcal{F}f)(t)e^{2\pi itx}.$$

This proves the first statement. Setting $x=0$ proves the second statement. $\square$

In practical settings, we need a class of functions $f(\mathbf{x})$ for which the assumptions of the Poisson summation formula hold. We say that $f(\mathbf{x})$ is of *Schwarz class* if $f \in C^\infty(\mathbb{R}^n)$ and $f(\mathbf{x})$ along with all of its partial derivatives have rapid decay. If $f(\mathbf{x})$ is of Schwarz class, the rapid decay implies that $f(\mathbf{x})$ and all of its derivatives are absolutely integrable over $\mathbb{R}^n$. Moreover, $F(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^n} f(\mathbf{x}+\mathbf{n})$ and all of its derivatives are locally absolutely uniformly convergent by the Weierstrass M-test. The uniform limit theorem then ensures $F(\mathbf{x})$ is smooth and thus the conditions of the Poisson summation formula are satisfied. So in most practical applications only only needed to ensure that $f(\mathbf{x})$ is Schwarz class.

We now introduce some explicit Schwarz class functions and compute their Fourier transforms. The classic example is $e^{-2\pi x^2}$. This function is particularly important because it is essentially its own Fourier transform.

Proposition 3.3. *Let $\alpha > 0$ and set $f(x) = e^{-2\pi\alpha x^2}$. Then*

$$(\mathcal{F}f)(\zeta) = \frac{e^{-\frac{\pi\zeta^2}{2\alpha}}}{\sqrt{2\alpha}}.$$

In particular, $e^{-\pi x^2}$ is its own Fourier transform.

Proof. Consider

$$(\mathcal{F}f)(\zeta) = \int_{-\infty}^{\infty} e^{-2\pi(\alpha x^2 + i\zeta x)}\,dx.$$

The change of variables $x \mapsto \frac{x}{\sqrt{\alpha}}$ transforms this integral into

$$\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-2\pi\left(x^2 + \frac{i\zeta x}{\sqrt{\alpha}}\right)}\,dx.$$

Complete the square in the exponent by observing

$$x^2 + \frac{i\zeta x}{\sqrt{\alpha}} = \left(x + \frac{i\zeta}{2\sqrt{\alpha}}\right)^2 + \frac{\zeta^2}{4\alpha},$$

so that the previous integral is equal to

$$\frac{e^{-\frac{\pi\zeta^2}{2\alpha}}}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-2\pi\left(x+\frac{i\zeta}{2\sqrt{\alpha}}\right)^2}\, dx.$$

By rapid decay of the integrand, the change of variables $x \mapsto \frac{x}{\sqrt{2}} - \frac{i\zeta}{\sqrt{\alpha}}$ is permitted giving

$$\frac{e^{-\frac{\pi\zeta^2}{2\alpha}}}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = \frac{e^{-\frac{\pi\zeta^2}{2\alpha}}}{\sqrt{2\alpha}},$$

because the remaining integral is the Gaussian integral. This proves the first statement. The second statement follows by taking $\alpha = \frac{1}{2}$. □

The analog of $e^{-2\pi x^2}$ on $\mathbb{R}^n$ is $e^{-2\pi\langle \mathbf{x},\mathbf{x}\rangle}$ which is Schwarz class because $e^{-2\pi x^2}$ is. We also obtain a generalization of the previous result this Schwarz class function as a corollary.

Corollary 3.4. *Let $\alpha > 0$ and set $f(\mathbf{x}) = e^{-2\pi\alpha\langle \mathbf{x},\mathbf{x}\rangle}$. Then*

$$(\mathcal{F}f)(\boldsymbol{\xi}) = \frac{e^{-\frac{\pi\langle \boldsymbol{\xi},\boldsymbol{\xi}\rangle}{2\alpha}}}{(2\alpha)^{\frac{n}{2}}}.$$

In particular, $e^{-\pi\langle \mathbf{x},\mathbf{x}\rangle}$ is its own Fourier transform.

Proof. Applying Proposition 3.3 to each variable separately proves the first statement. The second statement follows upon setting $\alpha = \frac{1}{2}$. □