Integral lattices

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1 Integral Lattices

Let F be a characteristic zero field and V be an n-dimensional F-vector space with nondegenerate symmetric inner product $\langle \cdot, \cdot \rangle$. We say that a subset Λ of V is a *lattice* if Λ is a free abelian group. In particular, any lattice Λ is of the form

$$\Lambda = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_m,$$

for some linearly independent vectors $v_1, \ldots, v_m$ of V with $m \leq n$. We say Λ is *complete* if $n = m$. This means $v_1, \ldots, v_n$ is necessarily a basis of V. If $e_1, \ldots, e_n$ is an orthonormal basis for V, write

$$v_i = \sum_i v_{ij} e_j,$$

with $v_{ij} \in F$, and define the associated *generator matrix* P by

$$P = \begin{pmatrix} v_{11} & \cdots & v_{n1} \\ \vdots & & \vdots \\ v_{1n} & \cdots & v_{nn} \end{pmatrix}.$$

Then P is the base change matrix from $e_1, \ldots, e_n$ to $v_1, \ldots, v_n$. The *covolume* V_Λ of a complete lattice Λ is defined to be

$$V_\Lambda = |\det(P)|.$$

As the base change matrix between any two orthonormal basses is an orthogonal matrix and hence has determinant ± 1, the covolume is independent of the choice of bases. In the case where V is a real or complex vector space, we will always take $e_1, \ldots, e_n$ to be the standard basis.

If Λ is a complete lattice in V then the *dual* $\Lambda^\vee$ of Λ is defined by

$$\Lambda^\vee = \{v \in V : \langle \lambda, v \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

In other words, the dual lattice consists of all of the vectors in V whose inner product with elements of the lattice are integers. Clearly the dual lattice is an abelian group. Less clear is that the dual lattice turns out to always be a complete lattice.

Proposition 1.1. *If $v_1, \ldots, v_n$ is a basis for a complete lattice Λ then the dual basis $v_1^\vee, \ldots, v_n^\vee$ is a basis for $\Lambda^\vee$. In particular, $\Lambda^\vee$ is a complete lattice.*

Proof. Let $v \in V$ and write

$$v = \sum_j a_i v_j^\vee,$$

with $v_j^\vee \in \mathbb{R}$. Since the dual basis is determined by $\langle v_i, v_j^\vee \rangle = \delta_{i,j}$, it follows that $v \in \Lambda^\vee$ if and only if $v_j^\vee \in \mathbb{Z}$. This means $v_1^\vee, \ldots, v_n^\vee$ is a basis for $\Lambda^\vee$ as a free abelian group and thus is a complete lattice. □

Since the dual of the dual basis is the original basis, $(\Lambda^\vee)^\vee = \Lambda$. We say that Λ is *self-dual* if $\Lambda^\vee = \Lambda$. For example, the lattice $\mathbb{Z}^n$ is self-dual because the standard basis is self-dual. It also turns out that the covolume of the dual lattice is the inverse of the volume of the original lattice:

Proposition 1.2. *Let Λ be a complete lattice in V. Then*

$$V_{\Lambda^\vee} = \frac{1}{V_\Lambda}.$$

Proof. Let $e_1, \ldots, e_n$ be an orthonormal basis for V, let $v_1, \ldots, v_n$ be a basis for Λ, and let P be the associated generator matrix. By Proposition 1.1, $v_1^\vee, \ldots, v_n^\vee$ is a basis for $\Lambda^\vee$ and $(P^{-1})^t$ is the base change matrix from $e_1, \ldots, e_n$ to $v_1^\vee, \ldots, v_n^\vee$. As $\det((P^{-1})^t) = \det(P)^{-1}$ the claim follows. □

We now turn to the case when V is an n-dimensional real inner product space. Let $d\lambda$ be the Lebesgue measure induced by the inner product with associated volume Vol given by

$$\mathrm{Vol}(M) = \int_M d\lambda,$$

for any measurable set M. Note that Λ acts on V by automorphisms given by translation. In other words, we have a group action

$$\Lambda \times V \to V \qquad (\lambda, v) \mapsto \lambda + v.$$

Moreover, Λ acts properly discontinuously on V. To see this, let $v \in V$ and let δ_v be such that $0 < \delta_v < \min_i(v - v_i)$. Taking U_v to be the ball of radius δ_v about v, the intersection $\lambda + U_v \cap U_v$ is empty unless $\lambda = 0$. As Λ is also discrete, it follows that V/Λ is also connected Hausdorff. Whence V/Λ admits a fundamental domain

$$\mathcal{M} = \{t_1 v_1 + \cdots + t_n v_n \in V : 0 \le t_i \le 1 \text{ for all } i\}.$$

Moreover, any translation of $\mathcal{M}$ by an element of Λ is also a fundamental domain. As we might expect, the covolume of Λ is equal to the volume of $\mathcal{M}$:

Proposition 1.3. *Let Λ be a complete lattice in a finite dimensional real inner product space V and let $\mathcal{M}$ be a fundamental domain for Λ. Then*

$$V_\Lambda = \mathrm{Vol}(\mathcal{M}).$$

Proof. Let $e_1, \ldots, e_n$ be an orthonormal basis of V, and $v_1, \ldots, v_n$ be a basis for Λ. Also let P be the associated generator matrix. The change of variables

$$x_1 e_1 + \cdots + x_n e_n \mapsto x_1 v_1 + \cdots + x_n v_n,$$

has Jacobian determinant V_Λ. Whence

$$\mathrm{Vol}(\mathcal{M}) = \int_{\mathcal{M}} d\lambda = V_\Lambda \int_{[0,1]^n} d\mathbf{x} = V_\Lambda.$$

□

This result shows that the covolume of a complete lattice in a real inner product space is a measure of the density of the lattice. The smaller the covolume the smaller the fundamental domain and the more dense the lattice is.

It turns out that for a real inner product space, being a lattice is equivalent to being a discrete subgroup:

Proposition 1.4. *Let Λ be a subset of a finite dimensional real inner product space V. Then Λ is a lattice if and only if it is a discrete subgroup.*

Proof. It is clear that if Λ is a lattice then it is a discrete subgroup. So suppose Λ is a discrete subgroup. Then Λ is closed. Let V' be the subspace spanned by Λ and let its dimension be m. Choosing a basis $v_1', \ldots, v_m'$ of V', set

$$\Lambda' = \mathbb{Z}v_1' + \cdots + \mathbb{Z}v_m'.$$

Then Λ' is a complete lattice in V' and $\Lambda' \subseteq \Lambda \subset V$. We claim Λ/Λ' is finite. To see this, let λ be a representative of a coset in Λ/Λ' and let $\mathcal{M}'$ be a fundamental domain for Λ' in V'. As $\mathcal{M}'$ is a fundamental domain, there exists unique $w' \in \mathcal{M}'$ and $\lambda' \in \Lambda'$ such that $\lambda = w' + \lambda'$. But then $w' = \lambda - \lambda' \in \mathcal{M}' \cap \Lambda$ and since $\mathcal{M}' \cap \Lambda$ is closed, discrete, and compact, it must be finite. Hence there are finitely many w' and thus finitely many cosets in Λ/Λ'. Letting $|\Lambda/\Lambda'| = q$, we have $q\Lambda \subseteq \Lambda'$ and therefore

$$\Lambda \subseteq \Lambda' = \mathbb{Z}\frac{1}{q}v_1' + \cdots + \mathbb{Z}\frac{1}{q}v_m'.$$

In particular, Λ is a subgroup of a free abelian group and therefore is free abelian. This means Λ is a lattice. □

As for complete lattices in V, they are equivalent to the existence of a bounded set whose translates cover V:

Proposition 1.5. *Let Λ be a lattice in a finite dimensional real inner product space V. Then Λ complete if and only if there exists a bounded subset M of V whose translates by Λ cover V.*

Proof. First suppose Λ is complete. Then we may take $M = \mathcal{M}$ to be a fundamental domain of Λ which is bounded and whose translates by Λ cover V. Now suppose Λ is a lattice and there exists a bounded subset M of V whose translates by Λ cover

V. Let W be the subspace of V spanned by Λ. Then W is closed. Moreover, Λ is complete if and only if $V = W$ and this is what we will show. So let $v \in V$. Since the translates of M by Λ cover V, for every positive integer n we may write

$$nv = w_n + \lambda_n,$$

with $v_n \in M$ and $\lambda_n \in \Lambda$. As M is bounded, $\lim_{n\to\infty} \frac{1}{n}v_n = 0$. Moreover, $\frac{1}{n}\lambda_n \in W$ and since W is closed we must have that $\lim_{n\to\infty} \frac{1}{n}\lambda_n$ exists and belongs to W. These two limits together imply

$$v = \lim_{n\to\infty} \frac{1}{n}\lambda_n,$$

and is an element of W. Thus $V = W$ which means Λ is complete. $\square$

The most important result we will require about lattices is *Minkowski's lattice point theorem* which states that, under some mild conditions, a set of sufficiently large volume in V contains a nonzero point of a complete lattice:

Theorem (Minkowski's lattice point theorem). *Suppose Λ is a lattice in an n-dimensional real inner product space V. Let $X \subset V$ is a compact convex symmetric set. If*

$$\mathrm{Vol}(X) \geq 2^n V_\Lambda,$$

then there exists a nonzero $\lambda \in \Lambda \cap X$.

Proof. We will prove the claim depending on if the inequality is strict or not. First suppose $\mathrm{Vol}(X) > 2^n V_\Lambda$. Consider the linear map

$$\phi : \frac{1}{2}X \to V/\Lambda \qquad \frac{1}{2}x \mapsto \frac{1}{2}x + \Lambda.$$

If ϕ were injective then we would have

$$\mathrm{Vol}\left(\frac{1}{2}X\right) = \frac{1}{2^n}\mathrm{Vol}(X) \leq V_\Lambda,$$

so that $\mathrm{Vol}(X) \leq 2^n V_\Lambda$. This is a contradiction, so ϕ cannot be injective. Hence there exists distinct $x_1, x_2 \in \frac{1}{2}X$ such that $\phi(x_1) = \phi(x_2)$. Thus $2x_1, 2x_2 \in X$. In particular, since X is symmetric we must have $-2x_2 \in X$. Convexity of X implies

$$\left(1 - \frac{1}{2}\right)2x_1 + \frac{1}{2}(-2x_2) = x_1 - x_2,$$

is an element of X. But $x_1 - x_2 \in \Lambda$ because $\phi(x_1) = \phi(x_2)$ and ϕ is linear. Then $\lambda = x_1 - x_2$ is nonzero with $\lambda \in \Lambda \cap X$. Now suppose $\mathrm{Vol}(X) = 2^n V_\Lambda$. Then

$$\mathrm{Vol}((1+\varepsilon)X) = (1+\varepsilon)^n 2^n V_\Lambda > 2^n V_\Lambda.$$

What we have just proved shows that there exists a nonzero $\lambda_\varepsilon \in \Lambda \cap (1+\varepsilon)X$. In particular, if $\varepsilon \leq 1$ then $\lambda_\varepsilon \in 2X \cap \Lambda$. The set $2X \cap \Lambda$ is compact and discrete and therefore finite. Taking $\varepsilon = \frac{1}{n}$, the sequence $(\lambda_{\frac{1}{n}})_{n\geq 1}$ belongs to the finite set $2X \cap \Lambda$ and so must converge to a point λ. Since Λ is discrete and the $\lambda_{\frac{1}{n}}$ are nonzero so too is λ. As λ is an element of

$$\bigcap_{n\geq 1} \left(1 + \frac{1}{n}\right) X,$$

and X is closed, $\lambda \in X$ as well. Thus we have found a nonzero $\lambda \in \Lambda \cap X$. $\square$