Discrete Structures I
COMP1805

Introduction to Discrete Mathematics

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Knights and Knaves

imagine that you are visiting an island where
All the Inhabitants can be Classified as either
"Knights", who always tell the Truth,
or "Knaves", who always tell Lies

you meet a Small Group of inhabitants...

Knights and Knaves

the Man in Green says "the Man in Yellow is a Knave"

the Man in Yellow says "We are Both Knights"

Which are Knights and which are Knaves?

What does it mean to "Tell the Truth" or "Tell a Lie"?

Statements can typically be categorized as **Imperative**, **Declarative**, or **Interrogative**

Imperatives are **"Commands"**,
Interrogatives are **"Questions"**,

but **Declaratives** communicate **Statements**...

...and these statements are **Associated** with **Values**

Knights and Knaves

the Values that can be Associated with Declarative statements are "True" and "False"

"Knights" can only make Declarative statements that are Associated with a value of True

"Knaves" can only make Declarative statements that are Associated with a value of False

Knights and Knaves

the Man in Green says "the Man in Yellow is a Knave"

the Man in Yellow says "We are Both Knights"

How do you Solve this Question (or Others like it)?

What are the Different Possibilities?

Both men are Knights

Both men are Knaves

Green is a Knight and Yellow is a Knave

Green is a Knave and Yellow is a Knight

Can you Apply a Process of Elimination?

Could it be Possible that Both Men are Knaves?

If Both men are Knaves then the statement "the Man in Yellow is a Knave" would be True, but If they are Both Knaves then the man in Green is Only allowed to make False statements

this is a Contradiction

Could it be Possible that Both Men are Knights?

If they are Both Knights then the man in Green can Only make True statements, but he made the statement "the Man in Yellow is a Knave" and if the man in Yellow is a Knave then he Cannot be a Knight

this is a Contradiction

Could the Man in Green be the Only Knave?

If the man in Green is a Knave then the statement "the Man in Yellow is a Knave" must be False (i.e., man in Yellow is a Knight), but the man in Yellow said Both were Knights this Cannot be True

this is a Contradiction

Could the Man in Green be the Only Knight?

If the man in Green is a Knight then the statement "the Man in Yellow is a Knave" must be True, and If the man in Yellow stated they were Both Knights, this would be a False statement, which is Consistent

this Must Be the Correct Answer

Fork in the Road / Two Guard Variant

Two clowns stand at a Fork in the Road, and you Know that One is a Knight and the Other is a Knave, and that One of the Two Paths leads to certain Doom

By Asking Only a Single "Yes" or "No" Question Can you Determine Which of the Paths is Safe?

Fork in the Road / Two Guard Variant

asking a Single "Yes" or "No" Question is essentially Equivalent to Asking for the Truth Value that is Associated with a declarative statement

but it is Important to Realize that you Don't Know Whether the clown you ask is a Knight or a Knave

What is the Question to Ask?

Fork in the Road / Two Guard Variant

Would the Other Clown tell me that Your Path Leads to Safety (and not Certain Doom)?

if the clown you Ask says "Yes" then His Path is the Wrong path (i.e., his path leads to certain Doom)

How does this Question Work?

if the clown you Ask is the Knight, you are really Asking "would the Knave say that your path is Safe?"

the Knave would Lie, so do the Opposite

How does this Question Work?

if the clown you Ask is the Knave, you are really Asking "would the Knight say that your path is Safe?"

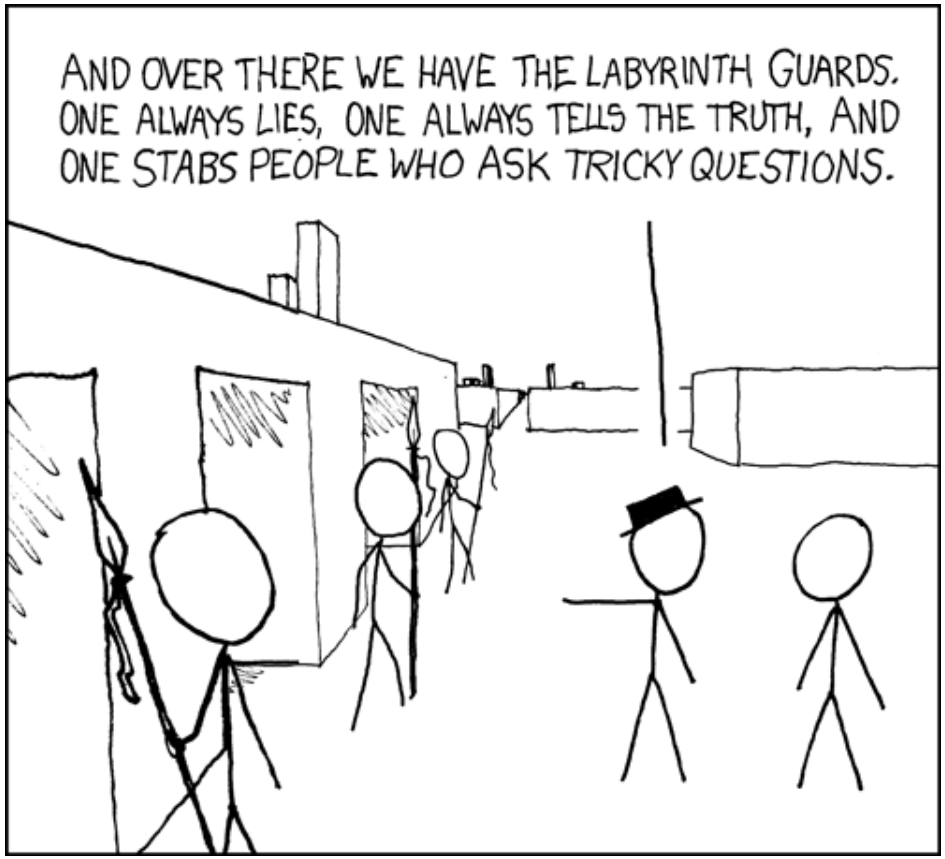
the Answer would be a Lie, so do the Opposite

Systematic Elimination

these classical Puzzles can be Solved through a Systematic (i.e., Methodical) Process of Elimination

only One of the Possible (candidate) Solutions can be Described without using Multiple Statements that Cannot all be Simultaneously True

My Favourite Variant: "Labyrinth Puzzle"



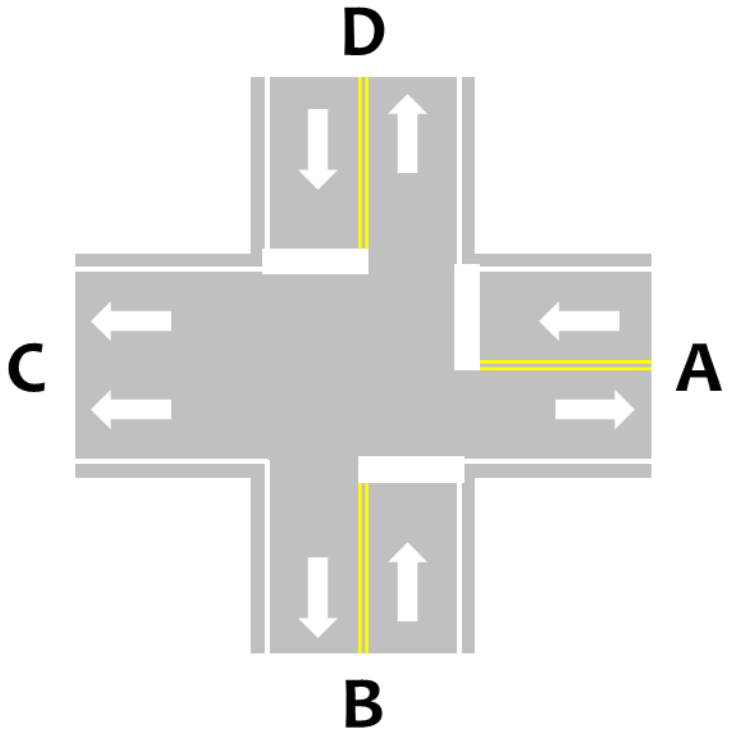
https://xkcd.com/246/

Traffic Light Scheduling

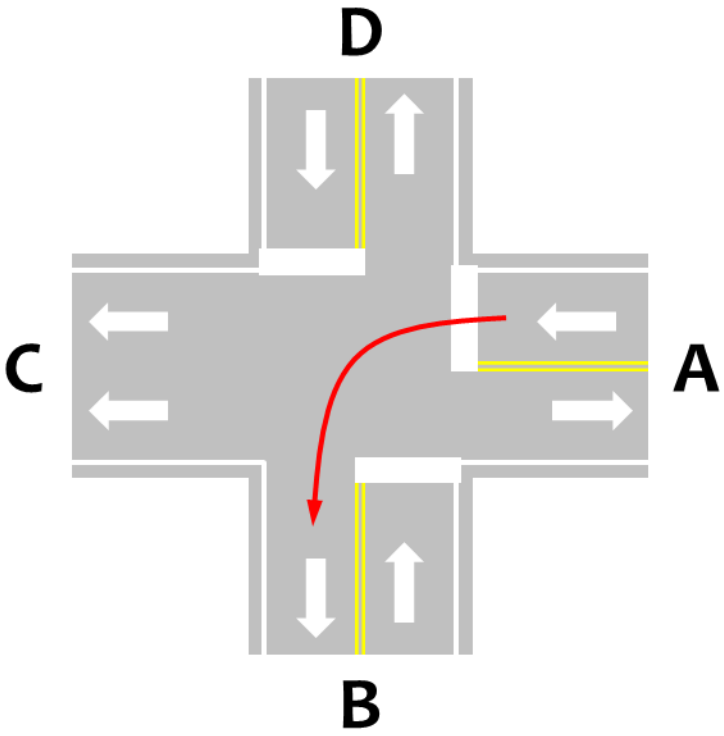
consider the following Intersection and decide on an Efficient Programming for the Traffic Lights...

i.e.,

write the Schedule for which Paths can have the Green Light at the Same Time

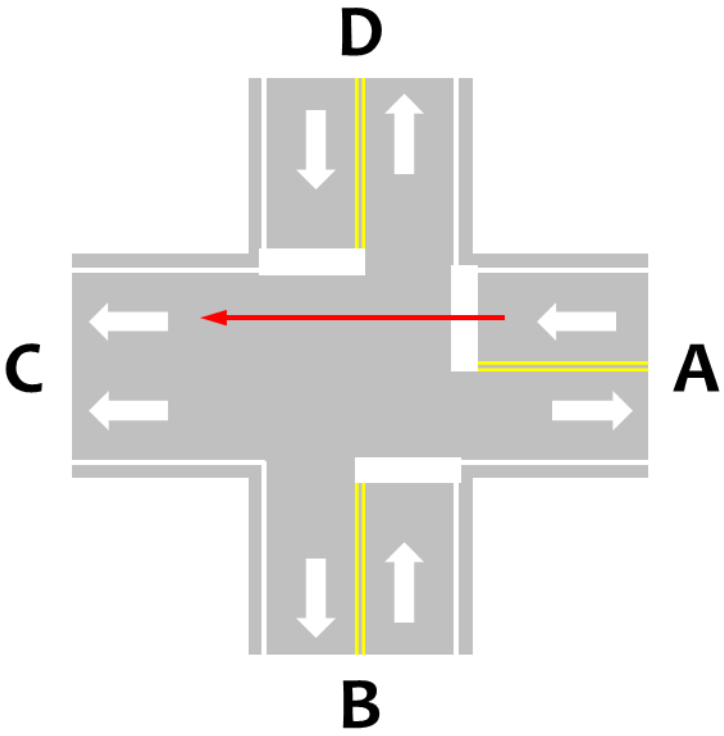


the Collection of Possible Paths includes:



the path from A to B, Denoted "AB"

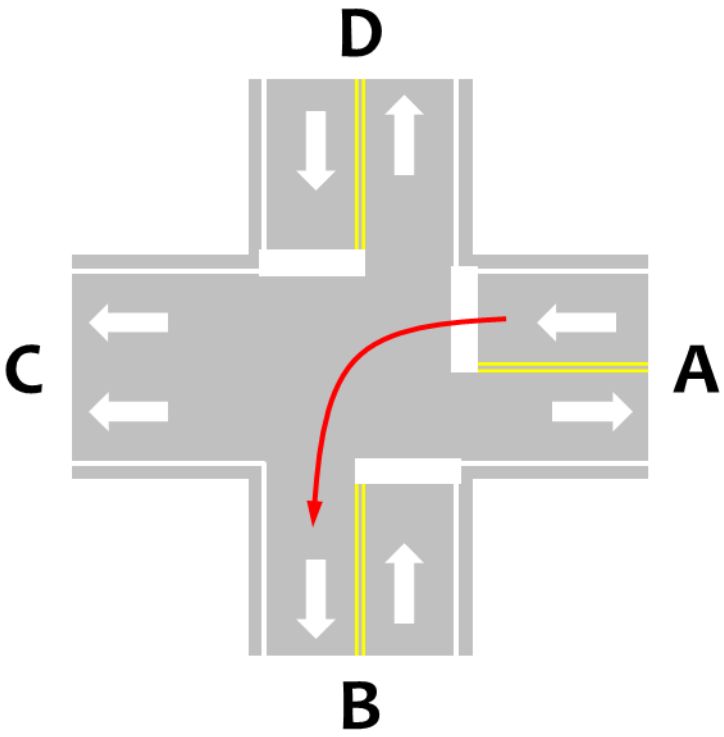
the Collection of Possible Paths includes:



the path from A to B, Denoted "AB"

the path from A to C, Denoted "AC"

the Collection of Possible Paths includes:

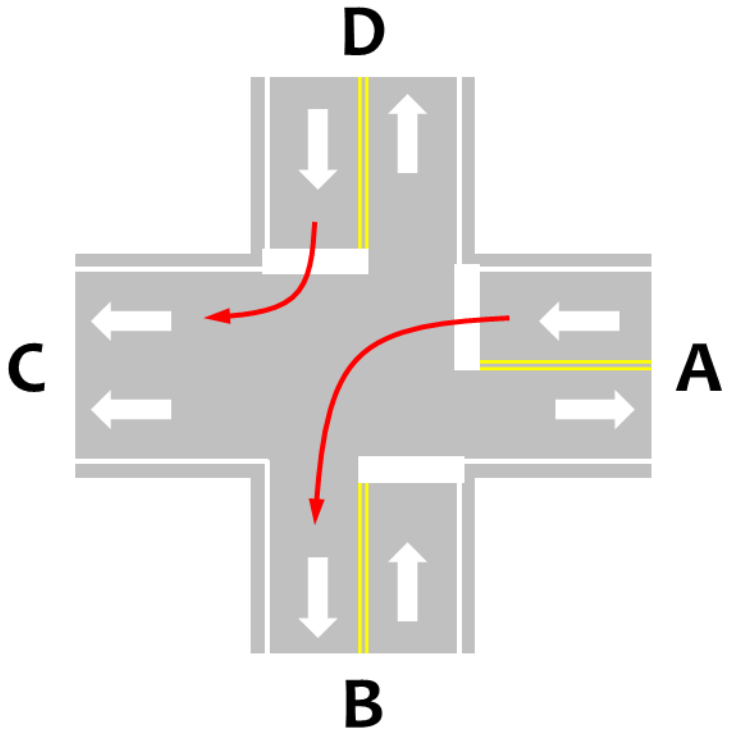


the path from A to B, Denoted "AB"
the path from A to C, Denoted "AC"
the path from A to D, Denoted "AD"
the path from B to A, Denoted "BA"
the path from B to C, Denoted "BC"
the path from B to D, Denoted "BD"
the path from D to A, Denoted "DA"
the path from D to B, Denoted "DB"
the path from D to C, Denoted "DC"

Paths that are Not in Direct Conflict can be traversed Simultaneously

e.g.,

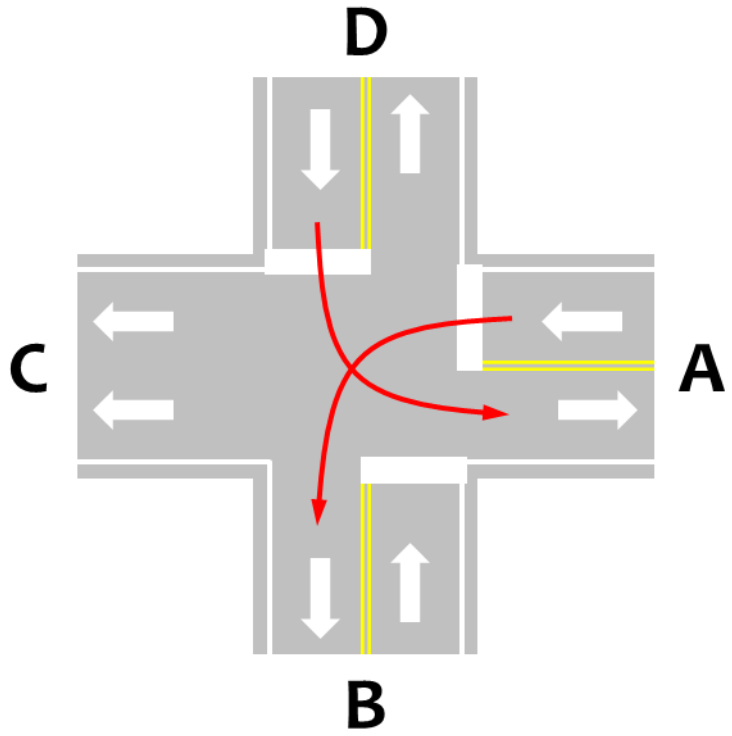
Paths AB and DC are Not in Conflict and can be Greenlit at the Same Time (Without Danger)



Paths that are in Direct Conflict Cannot be traversed Simultaneously

e.g.,

Paths AB and DA are in Conflict and Cannot be Greenlit at the Same Time (Without Danger)



How Many Possibilities Schedules Exist?

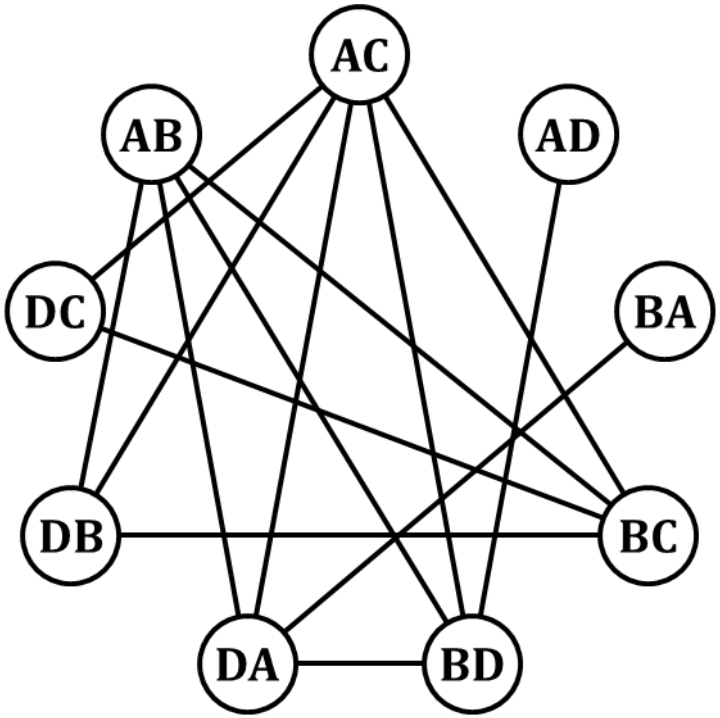
this is Not a Simple problem;

there are Nine Paths that could each be Greenlit (or Redlit) at Any Time in the schedule

Assuming that it is Never the Case that Every Light is Red, there are $2^9 - 1 = 511$ Options

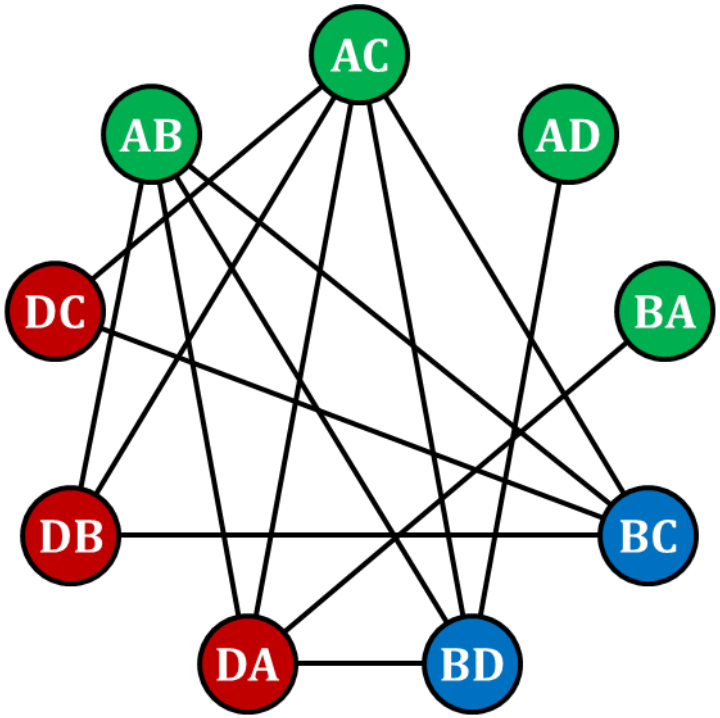
(n.b., this wouldn't even be a complete schedule yet!)

Abstract Representation



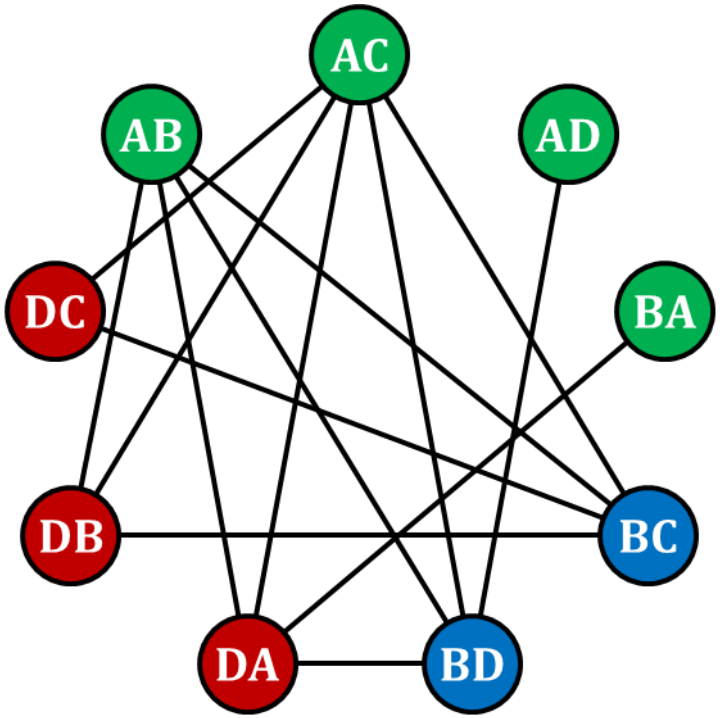
Consider a Diagram that has One Symbol for Each Path and Connects Conflicting Paths

Abstract Representation



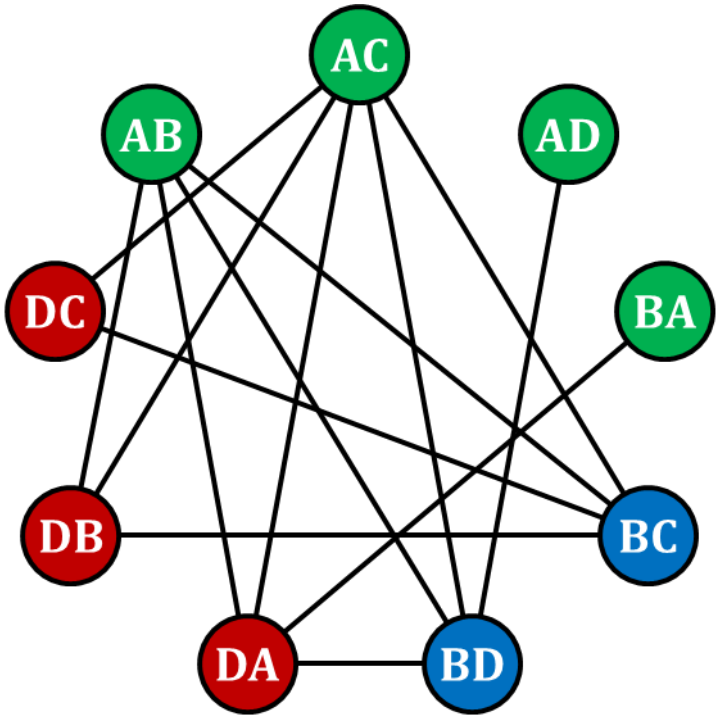
now **Assign Colours** to the symbols so that **No Pair** of **Connected** symbols has the **Same Colour**

Abstract Representation



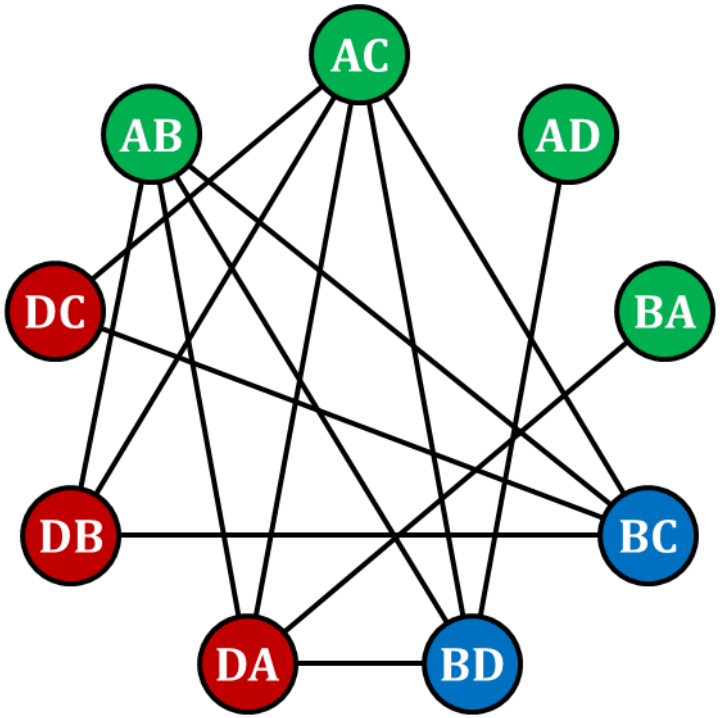
this Colouring actually Contains a Solution to the Traffic Light Scheduling Problem

Abstract Representation



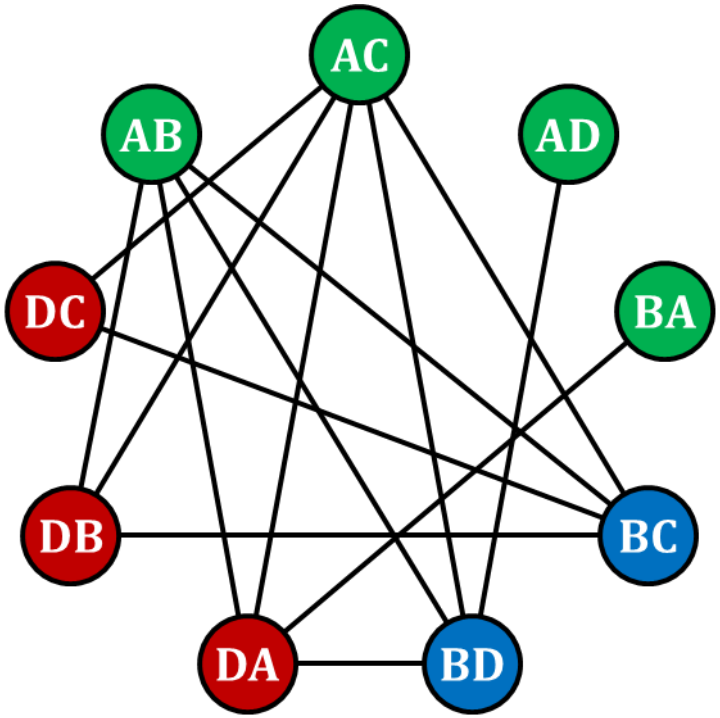
1st Assign the Green light to the Paths that were coloured with Green

Abstract Representation



2nd Assign the Green light to the Paths that were coloured with Blue

Abstract Representation



3rd Assign the Green light to the Paths that were coloured with Red

Abstract Representation

these Puzzles are Simple Instances
of Real-World Problems

the Solutions we have Discussed are
Applications of "Discrete Mathematics"

Resources

the Discrete Mathematics Study Center contains a Collection of Course Notes and a large Database of Practice Exercises

http://cglab.ca/~discmath/notes.html

http://cglab.ca/~discmath/exercises.html