Pareto-Optimal Learning-Augmented Algorithms for Online Conversion Problems

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April 17, 2023

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Online Algorithm

- Design algorithms to make decisions in online settings where the entire input sequence is not available at the start but is revealed sequentially to the algorithm.

- **Objective**: Optimize the **competitive ratio** when the input sequence is the worst case

$$\alpha = \max_{\sigma} \frac{OPT(\sigma)}{ALG(\sigma)}$$

- **Pros**: Strong performance guarantee (competitive ratio)
- **Cons**: Worst cases are rare

Learning-Augmented Online Algorithm (LOA)

- Real world settings are not completely unknown or overly pessimistic. Can make use of **machine-learned predictions** to improve our design.

- Competitive ratio as a function of **prediction error** ϵ:

$$\alpha(\epsilon) = \max_{\sigma} \frac{OPT(\sigma)}{ALG(\sigma, \epsilon)}$$

- ***Consistency***: $\alpha(0)$ competitive ratio when $\epsilon = 0$

- ***Robustness***: $\max_{\epsilon} \alpha(\epsilon)$ worst competitive ratio over any ϵ

Paper Contributions:

- Designing **Pareto-optimal** algorithms for online conversion problems, which achieve optimal trade-offs between consistency and robustness.

- **Pareto Optimality**: for any γ, the LOA achieves the minimal consistency guarantee among all online algorithms that are γ-competitive.

- Proposing a novel method of deriving the **lower bound** on the **consistency-robustness trade-off**, which may be of use beyond online conversion problems

Paper Outline

1. Problem statement and a unified algorithm (Unified OTA)

2. Robustness and consistency:
   - Why it is challenging
   - Vectorized competitive ratio
   - Sufficient conditions for η-consistent and γ-robust OTA

3. Application of sufficient conditions for 1-max search and one-way trading

4. Numerical results for a real-world problem

Problem Statement: Online Conversion Problem

- How to convert one asset (e.g. dollars) to another (e.g. yens) over a trading period $[N] := \{1, 2, ..., N\}$:
  1. At each instance $n \in [N]$, observe exchange rate v_n
  2. Decides how much to trade ($x_n \in \chi_n$) to obtain return $v_n x_n$

- **Objective**: Maximize the cumulative return $\sum_{n \epsilon [N]} v_n x_n$

- The paper discusses 2 classic settings:
  1. **one-way trading**: fractional decision $\chi_n = [0, 1]$
  2. **1-max search**: integer decision $\chi_n = \{0, 1\}$

Online Threshold-Based Algorithms (OTA)

- A class of *reserve-and-greedy* algorithms where the idea is to use a threshold function to determine the amount of resources that need to be reserved based on resource utilization, and then greedily allocate resources respecting the reservation in each step.

- Known to be easy-to-use but hard-to-design.

Unified Approach

Algorithm 1: Online threshold-based algorithm with threshold function ϕ (OTA$_\phi$)

- Input: threshold function ϕ(w): [0, 1] $\rightarrow$ [L, U]
  initial utilization (how much has been traded): $w_0 \leftarrow 0$

- For n = 1, 2,..., N:
  Determine online decision $x_n^* = \underset{x_n \in \chi_n}{argmax} \; v_n x_n - \int_{w^{(n-1)}}^{w^{(n-1)}+x_n} \phi(u) du$

  Update resource utilization $w^{(n)} \leftarrow w^{(n-1)} + x_n^*$

Unified Approach

- **1-max search**: OTA sets ϕ = $\sqrt{LU}$. Select the first price that is at least $\sqrt{LU}$ OTA achieves the optimal competitive ratio $\sqrt{\theta}$ where θ = U/L.

- **One-way trading**: OTA sets ϕ = L + (α^*L − L) exp(α^*w), w ∈ [0, 1]. OTA achieves the optimal competitive ratio $\alpha^* = 1 + W((\theta - 1)/e)$ where W(.) is the Lambert - W function.

Competitive Algorithms for the Online Multiple Knapsack Problem with Application to Electric Vehicle Charging

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October 20, 2020

Learning-Augmented OTA

- If the maximum price V of an input sequence is known, an online algorithm trades all assets at price V to achieve maximum return.

- Therefore, it is appropriate to use a **machine-learned prediction** of the **maximum price P** to design online algorithms.

- **Objective:** Design learning-augmented OTA that achieves η-consistency and γ-robustness i.e.
  $\eta \geq \alpha(\epsilon = 0)$ and $\gamma \geq \max_{\epsilon} \alpha(\epsilon)$ where $\epsilon = |V - P|$ is the **prediction error**.

Design Challenges

- If we blindly use prediction as threshold price ϕ = P for 1-max search, while we achieve 1-consistent, robustness (when our prediction P is inaccurate) is the worst possible competitive ratio θ.

- Naive: Set ϕ as the linear combination between the optimal threshold for a pure online algorithm and P: $\phi = \lambda\sqrt{LU} + (1-\lambda)P$ where $\lambda \in [0, 1]$ is called the robustness parameter that changes depending on how much we trust P.
  . OTA is $(\lambda\sqrt{\theta} + (1-\lambda)\theta)$ - robust and $\sqrt{\theta}$ - consistent.

Not an improvement over $\sqrt{\theta}$

OTA with Robustness and Consistency

The paper's design of OTA:

1. Sufficient conditions: if the function consists of multiple segments and each segment satisfies a set of differential equations, OTA is with bounded consistency and robustness

2. Pareto-Optimality: Given a robustness (some γ), no algorithm can achieve a lower consistency than the paper's design.

Sufficient Conditions:

- Let ω be the set of all input sequences and $\mathscr{P} := \{\mathscr{P}_1, \ldots, \mathscr{P}_N\}$ be a partition of ω. Let $\omega_p \subseteq \omega$ be a subset of ω in which p is the maximum price of every instance.

Definition 4.2 (Generalized competitive ratio). $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_I)$ *is a generalized competitive ratio over* $\mathcal{P}$ *if* $\alpha_i = \max_{\mathcal{I} \in \mathcal{P}_i} \mathtt{OPT}(\mathcal{I})/\mathtt{ALG}(\mathcal{I})$ *is the worst-case ratio over* $\mathcal{P}_i$ *for all* $i \in [I]$.

Sufficient Conditions

Lemma 4.3. *Given a prediction $P \in [L, U]$, and parameters η and γ with $\eta \leq \gamma$,* OTA *for online conversion problems is η-consistent and γ-robust if there exists a partition $\mathcal{P} = \{\mathcal{P}_\eta, \mathcal{P}_\gamma\}$ with $\Omega = \mathcal{P}_\eta \cup \mathcal{P}_\gamma$ and $\Omega_P \subseteq \mathcal{P}_\eta$, and* OTA *is (η, γ)-competitive over $\mathcal{P}$.*

$$\gamma = max\{\eta, \gamma\} = \max_{I \in P_\eta \cup P_\gamma} \frac{OPT(I)}{ALG(I)} = \max_{\epsilon} \alpha(\epsilon)$$

robustness with arbitrary ϵ

$$\eta = \max_{I \in \mathcal{P}_\eta} \frac{OPT(I)}{ALG(I)} \geq \max_{I \in \omega_P} \frac{OPT(I)}{ALG(I)} = \alpha(0)$$

consistency $\epsilon = 0$

Sufficient Conditions

- Divide the range $[L, U]$ into l price segments $[M_0, M_1), \ldots, [M_{l-1}, M_l]$ with $L = M_0 < M_1 < \ldots < M_l = U$. We partition ω based on the price segments, i.e. $\omega = \{\omega_p\}_{p \in [M0,M1)} \cup \ldots \cup \{\omega_p\}_{p \in [Ml-1,Ml]}$.

- We consider a piece-wise threshold function φ created by concatenating a sequence of functions $\{\phi_i\}_{i \in [l]}$, where each piece ϕ_i is designed to guarantee α_i-competitiveness over P_i . In particular, divide the feasible region $[0, 1]$ into l resource segments $[\beta_0, \beta_1), \ldots, [\beta_{l-1}, \beta_l]$ with $0 = \beta_0 \le \beta_1 \le \cdots \le \beta_l = 1$, and $\phi_i(w) \in [M_{i-1}, M_i), w \in [\beta_{i-1}, \beta_i)$. We say ϕ_i is absorbed if $\beta_{i-1} = \beta_i$

Sufficient Conditions

Theorem 4.4. OTA *is* α*-competitive over* $\{\mathcal{P}_i\}_{i\in[I]}$ *for online conversion problems if* $\phi := \{\phi_i\}_{i\in[I]}$ *is a piece-wise and right-continuous function,* $\phi(1) \in \{M_i\}_{i\in[I]}$ *is one of the partition boundaries, and each threshold piece* $\phi_i(w) : [\beta_{i-1}, \beta_i) \to [M_{i-1}, M_i)$ *satisfies one of the following conditions:*

Case I*: if* $M_i \le \phi(0)$*, then* $M_i \le \alpha_i L$ *and* $\beta_i = 0$*;*

Case II*: if* $\phi(0) < M_i \le \phi(1)$*, then* ϕ_i *is in the form of*

$$\phi_i(w) = \begin{cases} M_{i-1} & w \in [\beta_{i-1}, \beta'_{i-1}) \\ \varphi_i(w) & w \in [\beta'_{i-1}, \beta_i) \end{cases}, \tag{2}$$

which consists of a flat segment in $[\beta_{i-1}, \beta'_{i-1})$ *and a strictly increasing segment* $\varphi_i(w)$ *that satisfies*

$$\begin{cases} \varphi_i(w) \le \alpha_i \left[\int_0^{\beta'_{i-1}} \phi(u)du + \int_{\beta'_{i-1}}^{w} \varphi_i(u)du + (1-w)L\right], \forall w \in [\beta'_{i-1}, \beta_i) \\ \varphi_i(\beta_i) = M_i \end{cases}; \tag{3}$$

Case III*: if* $M_i > \phi(1)$*, then* $M_i \le \alpha_i \int_0^1 \phi(u)du$ *and* $\beta_i = 1$*.*

Sufficient Conditions: Proof

- **Case I**: $\mathscr{P}_i$ with $M_i \leq \phi(0)$. Then $B_i = 0$.
  $B_{i-1} = B_i = 0$, threshold piece is absorbed
  For any $I \in \mathscr{P}_1$,
  $OPT(I) < M_i$.
  $ALG(I) = v_N \geq L$
  Then,
  $$\mathtt{OPT}(\mathcal{I})/\mathtt{ALG}(\mathcal{I}) < M_i/L \leq \alpha_i, \forall \mathcal{I} \in \mathcal{P}_i$$

Sufficient Conditions: Proof (cont.)

- **Case II**: $\mathscr{P}_i$ with $\phi(0) \leq M_i \leq \phi(1)$:

  WLOG, we only consider the instances whose price in the last step is not the unique maximum price. $w^{(N-1)}$ is the final utilization of OTA

$$\begin{aligned}
\mathtt{ALG}(\mathcal{I}) &= \sum_{n\in[N-1]} v_n \bar{x}_n + (1 - w^{(N-1)}) v_N \\
&\geq \sum_{n\in[N-1]} \int_{w^{(n-1)}}^{w^{(n)}} \phi(u)du + (1 - w^{(N-1)}) L \\
&= \int_0^{w^{(N-1)}} \phi(u)du + (1 - w^{(N-1)}) L,
\end{aligned}$$

Sufficient Conditions: Proof (cont.)

In this case:

$$\phi_i(w) = \begin{cases} M_{i-1} & w \in [\beta_{i-1}, \beta'_{i-1}) \\ \varphi_i(w) & w \in [\beta'_{i-1}, \beta_i) \end{cases},$$

Since $\phi_i(w)$ could be discontinuous at w = β'_{i-1}. We consider 2 subcases:

Case II(a): $M_{i-1} \leq p < \varphi_i(\beta'_{i-1})$

$$\frac{\mathtt{OPT}(\mathcal{I})}{\mathtt{ALG}(\mathcal{I})} \leq \frac{p}{\int_0^{w^{(N-1)}} \phi(u)du + (1 - w^{(N-1)})L} < \frac{\varphi_i(\beta'_{i-1})}{\int_0^{\beta'_{i-1}} \phi(u)du + (1 - \beta'_{i-1})L} \leq \alpha_i,$$

Sufficient Conditions: Proof (cont.)

Case II(b): $\varphi_i(\beta'_{i-1}) \leq p < \varphi_i(\beta_i)$

$$\frac{\mathtt{OPT}(\mathcal{I})}{\mathtt{ALG}(\mathcal{I})} \leq \frac{p}{\int_0^{w^{(N-1)}} \phi(u)du + (1 - w^{(N-1)})L} = \frac{\varphi_i(w^{(N-1)})}{\int_0^{w^{(N-1)}} \phi(u)du + (1 - w^{(N-1)})L} \leq \alpha_i,$$

Combining the two subcases give

$$\mathtt{OPT}(\mathcal{I})/\mathtt{ALG}(\mathcal{I}) \leq \alpha_i, \forall \mathcal{I} \in \Omega_p, \forall \Omega_p \subseteq \mathcal{P}_i$$

Sufficient Conditions: Proof (cont.)

- **Case III**: $\mathscr{P}_i$ with $M_i \geq \phi(1)$. Since $\phi(1)$ is one of the price segment boundaries, $M_{i-1} \geq \phi(1)$ and hence $\beta_{i-1} = 1$. Thus, ϕ_i is absorbed
  $w^{(N-1)} = 1$
  $\texttt{OPT}(\mathcal{I}) \leq M_i \quad \texttt{ALG}(\mathcal{I}) \geq \int_0^1 \phi(u)du$
  Therefore,
  $$\texttt{OPT}(\mathcal{I})/\texttt{ALG}(\mathcal{I}) \leq M_i/\int_0^1 \phi(u)du \leq \alpha_i, \forall \mathcal{I} \in \mathcal{P}_i$$

1-Max Search: Sufficient Conditions

- $\gamma(\lambda) = [\sqrt{(1-\lambda)^2 + 4\lambda\theta} - (1-\lambda)]/(2\lambda),$ and $\eta(\lambda) = \theta/\gamma(\lambda)$

  where $\gamma(\lambda)$ and $\eta(\lambda)$ are the solution of
  1. $\eta(\lambda) = \theta/\gamma(\lambda)$ and
  2. $\eta(\lambda) = \lambda\gamma(\lambda) + 1 - \lambda$.

- The first equation is the desired trade-off between robustness and consistency, which matches the Pareto lower bound, and thus represents a Pareto-optimal trade-off.

- The second equation sets η as a linear combination of 1 and γ, as λ increases from 0 to 1 , η increases from the best possible ratio 1 to the optimal competitive ratio $\sqrt{\theta}$, and γ decreases from the worst possible ratio θ to $\sqrt{\theta}$

1-Max Search: Sufficient Conditions

- Price threshold ϕ_P is set based on prediction P as follows:

$$
\begin{aligned}
&\text{when } P \in [L, L\eta),\ \Phi_P = L\eta;\\
&\text{when } P \in [L\eta, L\gamma),\ \Phi_P = \lambda L\gamma + (1-\lambda)P/\eta;\\
&\text{when } P \in [L\gamma, U],\ \Phi_P = L\gamma.
\end{aligned}
$$

1-Max Search Sufficient Conditions: Proof

Given **$P \in [L, L\eta)$**, we consider partition $[L, U]$ = **$[L, L\eta)$** $\cup$ **$[L\eta, U]$**

For ρ_1 = **$[L, L\eta)$**

$\Phi_P = \phi(0) = \phi(1) = L\eta$

Case I applies, OTA is η-competitive

For ρ_2 = **$[L\eta, U]$**

$M_2 = U > \phi(1)$

Case III applies, OTA is γ-competitive

So OTA is (η, γ)-competitive over $\rho = \{\rho_1, \rho_2\}$

1-Max Search Sufficient Conditions: Proof (cont.)

Given **$P \in [L\eta, L\gamma)$**, we consider partition $[L, U]$ = **$[L, \Phi_P)$** ∪ **$[\Phi_P, P]$** ∪ **$[P, U]$**

Note that $P = \eta P/\eta = \lambda\gamma P/\eta + (1-\lambda)P/\eta \geq \Phi_P$

For ρ_1 = **$[L, \Phi_P)$**

$\Phi_P = \phi(0) = \phi(1) = \lambda L\gamma + (1-\lambda)P/\eta$

Case I applies, OTA is γ-competitive since

$$\frac{M_1}{L} = \frac{\Phi_P}{L} = \lambda\gamma + \frac{1-\lambda}{L\eta}P \leq \lambda\gamma + (1-\lambda)\frac{\gamma}{\eta} \leq \gamma.$$

For ρ_2 = **$[\Phi_P, P]$**

$M_2 = P > \phi(1)$

Case III applies, OTA is η-competitive

$$\frac{M_2}{\int_0^1 \Phi_P du} = \frac{P}{\Phi_P} = \frac{P}{\lambda L\gamma + \frac{1-\lambda}{\eta}P} \leq \frac{1}{\lambda + \frac{1-\lambda}{\eta}} \leq \eta$$

1-Max Search Sufficient Conditions: Proof (cont.)

Given **$P \in [L\eta, L\gamma)$**, we consider partition $[L, U]$ = **$[L, \Phi_P)$** ∪ **$[\Phi_P, P]$** ∪ **$[P, U]$**

For ρ_3 = **$[P, U]$**

$M_3 = U > \phi(1)$

Case III applies, OTA is γ-competitive since

$$\frac{M_3}{\int_0^1 \Phi_P du} = \frac{U}{\Phi_P} = \frac{U}{\lambda L\gamma + \frac{1-\lambda}{\eta}P} \leq \frac{U}{\lambda L\gamma + (1-\lambda)L} = \frac{\theta}{\eta} = \gamma.$$

So OTA is (γ, η, γ)-competitive over $\rho = \{\rho_1, \rho_2, \rho_3\}$

1-Max Search Sufficient Conditions: Proof (cont.)

Given **$P \in [L\gamma, U)$**, we consider partition $[L, U]$ = **$[L, L\gamma)$** ∪ **$[L\gamma, U]$**

For ρ_1 = **$[L, L\gamma)$**

$M_1 = \Phi_P = \phi(0) = \phi(1) = L\gamma$

Case I applies, OTA is γ-competitive since $M_1/L = \gamma$

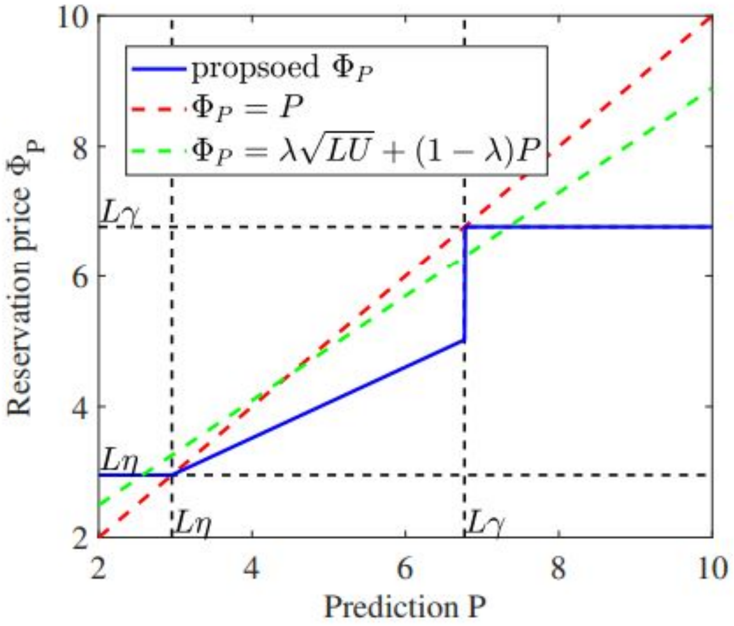
For ρ_2 = **$[L\gamma, U]$**

$M_2 = U > \phi(1)$

Case III applies, OTA is η-competitive since $\frac{M_3}{\int_0^1 \Phi_P du} = \frac{U}{\Phi_P} = \eta$

So OTA is (γ, η)-competitive over $\rho = \{\rho_1, \rho_2\}$

1-Max Search Sufficient Conditions: Visualization



- Figure 1 illustrates the form and compares it with two intuitive designs Φ_P = P and $\Phi_P = \lambda\sqrt{LU} + (1-\lambda)P$
- The proposed $\mathbf{\Phi_P}$ consists of three segments for predictions that are in boundary regions [L, Lη) and [Lγ, U] close to price lower and upper bounds, and in intermediate region [Lη, Lγ)

1-Max Search Sufficient Conditions: Intuitions

- Given any reservation price $\Phi \in [L, U]$, the robustness of OTA is $\max\{\Phi/L, U/\Phi\}$.

- The paper's choices of $\boldsymbol{\Phi_P}$ try to balance Φ/L and U/Φ by ensuring η-competitiveness.

- The intuitive design $\boldsymbol{\Phi_P} = P$ neglects this structure, so its robustness approaches the worst possible ratio θ.

1-Max Search Sufficient Conditions: Intuitions

- Given an accurate prediction P, the consistency of OTA is $\max\{\Phi/L, P/\Phi\}$. To guarantee a good consistency, we must avoid the case that $P < \Phi$, leading to the ratio Φ/L that cannot be properly bounded.

- The paper's choice of $\boldsymbol{\Phi_P}$ enforces $P \geq \Phi_P$.

- The intuitive design $\Phi P = \lambda\sqrt{LU} + (1 - \lambda)P$ fails to improve the consistency over $\sqrt{\theta}$ since it cannot always guarantee $P \geq \Phi_P$.

One-Way Trading: Sufficient Conditions

- Robustness - $\gamma(\lambda)$ and consistency - $\eta(\lambda)$:

$$\eta(\lambda) = \theta / \left[\frac{\theta}{\gamma(\lambda)} + (\theta - 1)\left(1 - \frac{1}{\gamma(\lambda)} \ln \frac{\theta - 1}{\gamma(\lambda) - 1}\right)\right], \text{ and } \gamma(\lambda) = \alpha^* + (1-\lambda)(\theta - \alpha^*), \quad (7)$$

- Price threshold ϕ_P is set based on prediction P as follows:

One-Way Trading: Sufficient Conditions

$$\text{when } P \in [L, M),\ \phi_P(w) = \begin{cases} L + (\eta L - L)\exp(\eta w) & w \in [0, \beta) \\ L + (U - L)\exp(\gamma(w-1)) & w \in [\beta, 1] \end{cases}, \tag{8a}$$

$$\text{when } P \in [M, U],\ \phi_P(w) = \begin{cases} L + (\gamma L - L)\exp(\gamma w) & w \in [0, \beta_1) \\ M_1 & w \in [\beta_1, \beta_1') \\ L + (M_1 - L)\exp(\eta(w - \beta_1')) & w \in [\beta_1', \beta_2] \\ L + (U - L)\exp(\gamma(w-1)) & w \in (\beta_2, 1] \end{cases}, \tag{8b}$$

where β and M are solutions of

$$\begin{cases} M = L + (\eta L - L)\exp(\eta\beta), \\ M\gamma/\eta = L + (U - L)\exp(\gamma(\beta - 1)); \end{cases} \tag{9}$$

and M_1, β_1, β_1', and β_2 are all functions of P and are determined by

$$\begin{cases} \beta_1 = \frac{1}{\gamma}\ln\frac{\max\{M_1/L, \gamma\} - 1}{\gamma - 1}, \\ \frac{M_1}{\eta} = \int_0^{\beta_1}\phi(u)du + (\beta_1' - \beta_1)M_1 + (1 - \beta_1')L, \\ P = L + (M_1 - L)\exp(\eta(\beta_2 - \beta_1')), \\ \beta_2 = 1 + \frac{1}{\gamma}\ln\frac{\min\{P\gamma/\eta, U\} - L}{U - L}. \end{cases} \tag{10}$$

Pareto Optimality

- The design of a learning-augmented OTA in this paper is Pareto-optimal, i.e. for a given robustness (γ), the design achieves the lower bound for consistency (η)

Theorem 5.1. *Any γ-robust deterministic* LOA *for 1-max-search must have consistency $\eta \geq \theta/\gamma$. Thus,* OTA *with the reservation price* (6) *is Pareto-optimal.*

Theorem 5.2. *If a deterministic* LOA *for one-way trading is γ-robust, its consistency is at least $\eta \geq \theta/[\frac{\theta}{\gamma} + (\theta - 1)(1 - \frac{1}{\gamma} \ln \frac{\theta-1}{\gamma-1})]$. Thus,* OTA *with the threshold function* (8) *is Pareto-optimal.*

Pareto Optimality: One-Way Trading

- To show a lower bound result, we first construct a special family of instances, and then show that for any γ-robust LOA (not necessarily being OTA), their consistency η is lower bounded under the special instances.

Definition 5.3 (p-instance). *Given $p \in [L, U]$ and a large N, an instance $\mathcal{I}_p := \{v_1, \ldots, v_N\}$ is called a p-instance if $v_n = L + (n-1)\delta, n \in [N-1]$ with $\delta = \frac{p-L}{N-2}$ and $v_N = L$.*

- When $N \to \infty$, the sequence of prices in I_p continuously increases from L to p, and drops to L in the last step

Pareto Optimality: One-Way Trading (cont.)

- Proof: Let $g(p) : [L, U] \rightarrow [0, 1]$ denote the total amount of converted dollar before the compulsory conversion.
- For a large N, executing the instance $I_{p+\delta}$ is equivalent to first executing I_p (excluding the last step) and then processing $p + \delta$ and L

  $\Rightarrow g(p+\delta) \geq g(p) \Rightarrow g(p)$ is non decreasing in [L, U]

- Also, $g(U) = 1$
- For any γ-robust online algorithm, given prediction $P \geq \gamma L$, no dollar needs to be converted under instances $\{I_p\}_{p \in [L, \gamma L)}$. This is because if a γ-robust online algorithm converts any dollar below the price γL, we can always design a new algorithm by letting it convert the dollar at the price γL instead.

Pareto Optimality: One-Way Trading (cont.)

- So for $P = U \geq \gamma L$

$$\mathtt{ALG}(\mathcal{I}_p) = g(\gamma L)\gamma L + \int_{\gamma L}^{p} u dg(u) + L(1 - g(p)) \geq \frac{p}{\gamma}, \quad \forall p \in [\gamma L, U]$$

- By integration by parts, the above is equivalent to

$$g(p) \geq \frac{p/\gamma - L}{p - L} + \frac{1}{p - L} \int_{\gamma L}^{p} g(u) du$$

- By Gronwall's Inequality,

$$g(p) \geq \frac{p/\gamma - L}{p - L} + \frac{1}{\gamma} \int_{\gamma L}^{p} \frac{u - \gamma L}{(u - L)^2} du = \frac{1}{\gamma} \ln \frac{p - L}{\gamma L - L}, \quad \forall p \in [\gamma L, U] \qquad \textbf{(1)}$$

THEOREM 1. *Let x and k be continuous and a and b Riemann integrable functions on $J = [\alpha, \beta]$ with b and k nonnegative on J.*

(i) *If*

$$x(t) \leq a(t) + b(t) \int_{\alpha}^{t} k(s)x(s)ds, \quad t \in J, \tag{4.1}$$

then

$$x(t) \leq a(t) + b(t) \int_{\alpha}^{t} a(s)k(s) \exp\left(\int_{s}^{t} b(r)k(r)dr \right) ds, \quad t \in J. \tag{4.1$'$}$$

Moreover, equality holds in (4.1)′ for a subinterval $J_1 = [\alpha, \beta_1]$ of J if equality holds in (4.1) for $t \in J_1$.

(ii) *The result remains valid if $\leq$ is replaced by $\geq$ in both (4.1) and (4.1)′.*

(iii) *Both* (i) *and* (ii) *remain valid if $\int_{\alpha}^{t}$ is replaced by $\int_{t}^{\beta}$ and $\int_{s}^{t}$ by $\int_{t}^{s}$ throuhgout.*

Pareto Optimality: One-Way Trading (cont.)

- To ensure η-consistency when the prediction is P = U, we must have

$$\mathtt{ALG}(\mathcal{I}_U) \geq \mathtt{OPT}(\mathcal{I}_U)/\eta$$

- Similarly, by integration by parts, the following must hold

$$\int_{\gamma L}^{U} g(u)du \leq (\eta - 1)U/\eta \qquad \textbf{(2)}$$

- Combining (1) and (2),

$$\eta \geq \theta/[\tfrac{\theta}{\gamma} + (\theta - 1)(1 - \tfrac{1}{\gamma} \ln \tfrac{\theta-1}{\gamma-1})]$$

Application: Bitcoin conversion

- BTC prices from 2015-2020 divided into 250 instances of length 1 week. 1 BTC is available for conversion in each instance.

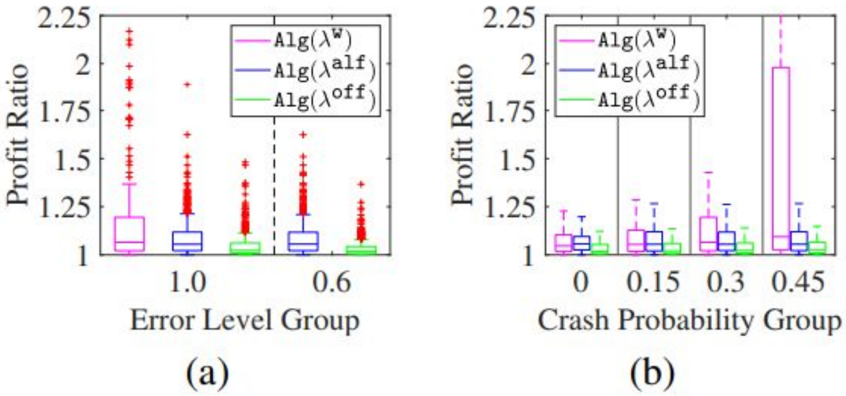
- For each instance, select an OTA indexed by $\lambda \in [0,1]$, sell one BTC fraction-by-fraction, and evaluate its empirical ratio.

- Answers two questions:
  (Q1) How does the learning-augmented OTA compare to pure online algorithms with different prediction qualities and drastic exchange rate crashes?
  (Q2) How should the OTA select the robustness parameter λ, and especially, would an online learning algorithm work in practice?

Numerical Results

- Compared 3 algorithms:
  1. ALG(λ^{w}): worst-case optimized online algorithm without prediction
  2. ALG(λ^{off}): an algorithm that selects the best λ offline, not practical since it is fed with the optimal parameter
  3. ALG(λ^{alf}): an algorithm that selects λ using online learning algorithms which selects the parameter using the adversarial Lipschitz algorithm in a full-information setting
  4. Alg(λ^{stc}), an online algorithm that uses the best static λ and serves as the baseline for Alg(λ alf)
- Prediction P is simply the maximum exchange rate of the previous week

Numerical Results: Q1

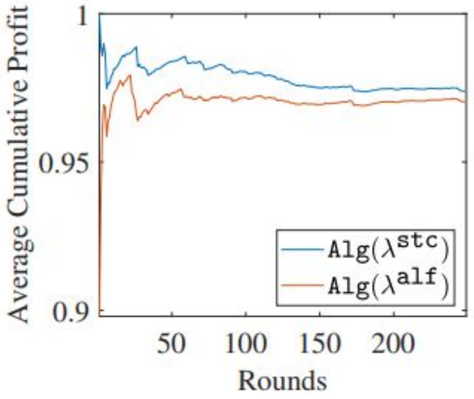


- To evaluate the impact of prediction quality, set error level between 0 and 1, where 0 indicates perfect predictions and 1 indicates unadjusted predictions

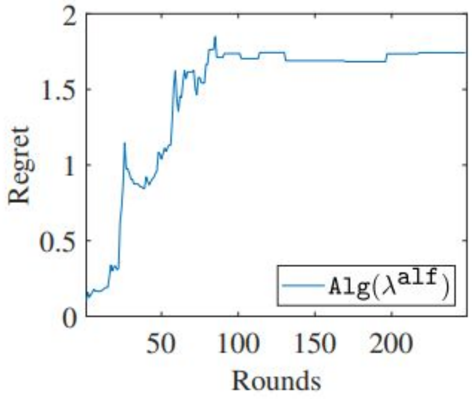
- The exchange rate of BTC at the last slot will crash to L with probability q

- The figures show that Alg(λ^{off}) and Alg(λ^{alf}) noticeably improve the performance of Alg(λ^{w}) and they are more stable at high crash probability

Numerical Results: Q2



(c)



(d)

- Figure (c) compares the average normalized profit of Alg(λ^{alf}) and Alg(λ^{stc}) and shows the reward of Alg(λ^{alf}) converges toward that of Alg(λ^{stc}) as the learning process moves forward.
- Figure (d) indicates that the regret of Alg(λ^{alf}) stabilizes