Research on Partition Representation and Related Theorems

Meng Xianzhe

June 1, 2025

Abstract

This paper studies a specific partition representation method and its related definitions and theorems. First, the basic concept of partition representation is introduced, then the relevant definitions are given and two important theorems are proved. The research shows that this partition representation method has important application value in the judgment of series convergence.

keywords:Partition Representation,Series Convergence,Mathematical Analysis

1 Introduction

In the realm of mathematical analysis, partition representation stands as a pivotal fundamental tool for investigating problems like function integration and series. This paper embarks on an in - depth exploration of a specific partition representation approach, along with its related definitions and theorems. The core objective is to further illuminate the properties and application scenarios in the context of arbitrary partitions. By delving into the characteristics of partition representation under unrestricted partitioning conditions, we aim to uncover more generalizable rules and theoretical underpinnings. This exploration not only enriches the understanding of partition - related concepts in mathematical analysis but also lays a solid foundation for addressing more complex problems involving diverse partition structures. Through a detailed examination of the defined partition representation and its associated theorems, we strive to provide a comprehensive framework for analyzing series convergence and function integration within the broad spectrum of arbitrary partitions, thus expanding the scope of application of these mathematical tools in various analytical scenarios.

2 Definitions

We define the partition representation as follows:

Let $T : \{X_1, X_2, \cdots, X_n, \cdots\}$, where T represents a partition method.

$$T : \{X_1, X_2, \cdots, X_n, \cdots\} = \tau_n$$

represents the representation related to a certain abscissa of the partition.

$$T : \{X_1, X_2, \cdots, X_n, \cdots\} = \Delta\tau_n$$

represents the representation related to the time length of the partition.

Define $a_{<T>n} = f(\tau_n, n) = \widetilde{f_\Delta}(\tau_n, n) = \widetilde{f}_\tau(n) = \widetilde{f^{\Delta}_{\Delta\tau}}(n)$ as a function sequence under the partition T in a specific sequence context. In particular,when τ_n=n,it represents series;when we have $\forall n \in \mathbb{N}$, $\Delta\tau_n \to 0$ represents integrals.

3 Theorems and Proofs

3.1 Theorem 1

When the corresponding functions are uniform and monotonic, while τ_n keeps monotonically increasing , the convergence of the series summation $\sum_{n=1}^{\infty} a_n$ implies the convergence of

the summation under the corresponding partition $\sum_{n=1}^{\infty} \widetilde{f}_\tau(n)$.
In domain of this function, we assume

$$\left|\sum_{n=s_0}^{*} \widetilde{f}_\tau(n)\right| \leqslant \max\left\{\left|\sum_{s=s_0}^{*} g(s)\right|, \left|\sum_{s=s_0}^{*} g(s+1)\right|\right\} = M \max\left\{\left|\sum_{s=s_0}^{*} a_s\right|, \left|\sum_{s=s_0}^{*} a_{s+1}\right|\right\} = M|\sum_{s=0}^{k} a_s|$$

It follows that if $\sum_{s=0}^{k} a_s$ converges, then $\sum_{n=1}^{\infty} \widetilde{f}_\tau(n)$ converges.

First, according to the definition, we know that:

$$\left|\sum_{n=1}^{\infty} \widetilde{f}_\tau(n)\right| = \left|\sum_{s=0}^{\infty} \sum_{n\in[s,s+1)} \widetilde{f}_\tau(n)\right|$$

Since there are at most M partitions on the interval $[s, s+1]$ and the maximum value is $g(s)$, we have:

$$\left|\sum_{n\in[s,s+1)} \widetilde{f}_\tau(n)\right| \leq M \cdot g(s)$$

Further derivation yields:

$$\left|\sum_{n=1}^{\infty} \widetilde{f}_\tau(n)\right| \leq \sum_{s=0}^{\infty} M \cdot g(s) = M \cdot \sum_{s=0}^{\infty} g(s)$$

According to the assumption, $\sum_{s=0}^{k} a_s$ converges, so we have:

$$M \cdot \sum_{s=0}^{k} |a_s| < \infty$$

Therefore, $\sum_{n=1}^{\infty} \widetilde{f}_\tau(n)$ converges.

Remark:The divergence of a series does not necessarily mean the divergence of the corresponding partition sum. For example, let $a_n = \frac{1}{n}$ and the partition $T_n = n^2$. This is a simple example to illustrate the above remark.

3.2 Theorem 2

Let $f : [a, \infty) \to [0, \infty)$ be nonincreasing. Let $a = \tau_0 < \tau_1 < \cdots$ be an increasing sequence with $\tau_n \to \infty$. For each n, pick any $\xi_n \in [\tau_{n-1}, \tau_n]$ and set $\Delta\tau_n = \tau_n - \tau_{n-1}$. Then the infinite partition sum

$$\sum_{n=1}^{\infty} \widetilde{f}_\tau(\xi_n)\,\Delta\tau_n$$

converges if and only if the series of endpoint values $\sum_{n=1}^{\infty} \widetilde{f}_\tau(\tau_n)$ converges.

Since f is nonincreasing, on each interval $[\tau_{n-1}, \tau_n]$ we have

$$\widetilde{f}_\tau(\tau_n) \;\leq\; \widetilde{f}_\tau(\xi_n) \;\leq\; \widetilde{f}_\tau(\tau_{n-1}).$$

Multiplying by $\Delta\tau_n > 0$ and summing from $n = 1$ to N yields

$$\sum_{n=1}^{N} \widetilde{f}_\tau(\tau_n)\,\Delta\tau_n \leq \sum_{n=1}^{N} \widetilde{f}_\tau(\xi_n)\,\Delta\tau_n \leq \sum_{n=1}^{N} \widetilde{f}_\tau(\tau_{n-1})\,\Delta\tau_n.$$

As τ_0 is fixed and $\Delta\tau_n \leq 1$ eventually, convergence of $\sum \widetilde{f}_\tau(\tau_n)\Delta\tau_n$ is equivalent to convergence of $\sum \widetilde{f}(\tau_n)$. Thus the left and right sums above share their convergence behavior, forcing the middle partition sum to converge iff $\sum_{n=1}^{\infty} \widetilde{f}_\tau(\tau_n)$ converges.

3.3 Theorem 3

Under the hypotheses of Theorem 3.2, let $w_n > 0$ satisfy $c\,\Delta\tau_n \le w_n \le C\,\Delta\tau_n$ for constants $0 < c \le C < \infty$. Then

$$\sum_{n=1}^{\infty} w_n\,\widetilde{f}_\tau(\xi_n)$$

converges if and only if $\sum_{n=1}^{\infty} \widetilde{f}_\tau(\tau_n)$ converges.

By comparison,

$$c\sum \Delta\tau_n \widetilde{f}_\tau(\xi_n) \le \sum w_n \widetilde{f}_\tau(\xi_n) \le C\sum \Delta\tau_n \widetilde{f}_\tau(\xi_n).$$

The result then follows from Theorem 3.2.

3.4 Application Example

We sketch a high-level application in Fourier analysis.

The Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

is approximated by Riemann sums

$$\sum_{k=-M}^{M} f(x_k)e^{-i\omega x_k}\,\Delta x.$$

If $|f(x)|$ is nonincreasing for large $|x|$, Theorem 3.2 implies the convergence of these sums is equivalent to convergence of the discrete series $\sum_{|x_k|\to\infty} |f(x_k)|$, giving a clear test for validity of the Fourier integral approximation.

4 Conclusion

Through the research on the partition representation method and its related definitions and theorems, this paper initially reveals its application value in the judgment of series convergence. Future research can further expand the application of these concepts in broader mathematical fields (such as multivariate function integration and functional analysis), explore their connections with other mathematical structures, and thus deepen the understanding of relevant mathematical theories.

References

[1] 黄永忠,韩志斌,吴洁,刘艳红.多元分析学[M].武汉:华中科技大学出版社,2024.