SENSE-based pMRI Reconstruction by Double Optimization Models Using 3D Tight Frame Regularization

Abstract

The sensitivity encoding (SENSE) method is one of the multi-coil parallel magnetic resonance imaging (pMRI) reconstruction methods and is widely utilized in commercial application on clinical imaging for disease detection. Each coil image in pMRI system is an imaging slice modulated by corresponding coil sensitivity and similar with each other, which can be combined together as 3-dimension image data and be regularized by 3-dimension framelet. We propose SENSE-based double optimization models by 3-dimension framelet regularization together to reconstruct high quality images from sampled k-space data with high acceleration rate and reduce aliasing artifacts due to inaccurate estimation of each coil sensitivity. One optimization model for an imaging slice is regularized by the sparsity of its 3-dimension framelet coefficients of multi coil images and can be solved by ADMM method; and the other model for sensitivity estimation is regularized by the 3-dimension features of multi-coil sensitivities with a constrain of the normalization of all sensitivities, which is efficiently solved by the FISTA algorithm. Experiments on real phantoms and in-vivo data show that our double optimization models can be efficient to reconstruct high quality images and reduce aliasing artifacts.

Keywords: pMRI, Directional Haar tight framelets, 3-Dimension framelet regularization, SENSE, GRAPPA, , ADMM.
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1. Introduction

We emphasize two points: coil images are correlated and can be considered as 3-D image data and regularized together; Sensitivity is not easily to be estimated accurately, when the reconstruction image is more and more close to the imaging slice, then it can update the sensitivity by an optimization model; and updated sensitivity will reconstruct a better target image.

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The outline of the paper is as follows.

2. SENSE-based pMRI Model

2-Dimensional Haar Framelet [**?**] [**?**]

$$\hat{u} = \arg\min_{u} \left\{ \frac{1}{2} \|Q_p M u - g\|_2^2 + \|\Gamma W_{2D} u\|_1 \right\} \tag{1}$$

3. 3-Dimension Tight Frame

4. Double Optimization Models for SENSE Reconstruction by 3-Dimension regularization

5. pMRI Algorithm for Double Models

The ℓ-th coil k-space data g_ℓ is modeled as follows:

$$g_\ell = \mathcal{P}\mathcal{F}\mathcal{S}_\ell u + \eta_\ell,\ \ell = 1, \cdots, p,$$

where η_ℓ is the additive noise, $\mathcal{S}_\ell$ the diagonal sensitivity matrix, $\mathcal{F}$ is the discrete Fourier transform matrix and $\mathcal{P}$, called sampling matrix, is a diagonal matrix with 0 and 1, u is target slice image. In real application, the sensitivity $\mathcal{S}_\ell$ and target image u are unknown variables. We adopt an alternative scheme to solve problem, and need to fix one and then solve another.

5.1. Update image u

When the sensitivity $\hat{\mathcal{S}}_\ell$ is given, we want to solve the target image u. We combine all the coil data g_ℓ and have a compact formula:

$$g = Q_p M u + \eta,$$

where g is the stacked k-space data, M is the composition of $\mathcal{F}$ and $\mathcal{S}_\ell$, and Q_p is the concatenation of $\mathcal{P}$:

$$g := \begin{bmatrix} g_1 \\ \vdots \\ g_p \end{bmatrix}, \mathcal{F}_p := \begin{bmatrix} \mathcal{F} & & \\ & \ddots & \\ & & \mathcal{F} \end{bmatrix}, M = \mathcal{F}_p \begin{bmatrix} \hat{\mathcal{S}}_1 \\ \vdots \\ \hat{\mathcal{S}}_p \end{bmatrix}, Q_p = \begin{bmatrix} \mathcal{P} & & \\ & \ddots & \\ & & \mathcal{P} \end{bmatrix}, \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_p \end{bmatrix}$$

The proposed optimization model by 3-Dimension tight framelet regularization for SENSE-based MRI reconstruction is:

$$\hat{u} = \arg\min_u \left\{ \frac{1}{2} \|Q_p M u - g\|_2^2 + \|\Gamma W_{3D} \mathcal{F}_p^{-1} (Q_o M u + g)\|_1 \right\} \tag{2}$$

Where $Q_o = I - Q_p$ is the sampling matrix at the missing positions, Γ is a diagonal matrix with non-negative diagonal elements and the W_{3D} is a transformer matrix by 3-dimension tight framelet system.

5.2. Primal-Dual Three-Operator splitting (PD3O)-based Algorithm

Let $K \doteq Q_p M$, $A \doteq W_{3D} \mathcal{F}_p^{-1} Q_o M$ and $b \doteq W_{3D} \mathcal{F}_p^{-1} g$, and identify the functions f, and p as follows:

$$f(u) = \frac{1}{2} \|Ku - g\|^2, \quad p(s) = \|\Gamma(s + b)\|_1 \tag{3}$$

Our optimization model (2) is written into the

$$\min_u \left\{ \frac{1}{2} \|Ku - g\|_2^2 + p(Au) \right\}.$$

The PD3O algorithm is provided as:

$$\begin{aligned} s^{k+1} &= \mathrm{prox}_{\delta p^*} \left((I - \gamma \delta A A^\top) s^k + \delta A(u^k - \gamma \nabla f(u^k)) \right) \\ u^{k+1} &= u^k - \gamma \nabla f(u^k) - \gamma A^\top s^{k+1} \end{aligned}$$

To compute p^* and its proximity operator, due to the separability of the ℓ_1 norm, let us consider the one-dimensional case of the function $p(s) = \|\Gamma(s + b)\|_1$, that is, $p(s) = \lambda |s + b|$, where $\lambda \geqslant 0$.

Lemma 1. *Let $p(s) = \lambda |s + b|$, where $\lambda \geqslant 0$, $s \in \mathbb{R}$. Then*

$$p^*(t) = \begin{cases} \iota_{\{0\}}(t), & \text{if } \lambda = 0; \\ -\langle b, t \rangle + \iota_{[-\lambda, \lambda]}(t), & \text{if } \lambda > 0. \end{cases}$$

Proof. First, consider $\lambda = 0$. Then $p(s) = 0$ for all $s \in \mathbb{R}$. Therefore,

$$p^*(t) = \sup_{s \in \mathbb{R}} \langle s, t \rangle - p(s) = \sup_{s \in \mathbb{R}} \langle s, t \rangle = \iota_{\{0\}}(t).$$

Next, consider $\lambda > 0$.

$$\begin{aligned} p^*(t) &= \sup_{s \in \mathbb{R}} \left(\langle s, t \rangle - p(s) \right) \\ &= \sup_{s \in \mathbb{R}} \left(\langle s, t \rangle - \lambda |s + b| \right) \\ &= \sup_{\widetilde{s} \in \mathbb{R}} \left(\langle \widetilde{s} - b, t \rangle - \lambda |\widetilde{s}| \right) \\ &= \langle -b, t \rangle + \sup_{\widetilde{s} \in \mathbb{R}} \left(\langle \widetilde{s}, t \rangle - \lambda |\widetilde{s}| \right) \\ &= \langle -b, t \rangle + \iota_{[-\lambda, \lambda]}(t) \end{aligned}$$

□

Actually, we can write

$$p^*(t) = \begin{cases} +\infty, & \text{if } t \notin [-\lambda, \lambda]; \\ -\langle b, t\rangle, & \text{if } t \in [-\lambda, \lambda]. \end{cases}$$

Next, let compute the proximity of p^* in one-dimensional case.

Lemma 2. *Let $p(s) = \lambda|s + b|$, where $\lambda \geqslant 0$, $s \in \mathbb{R}$. Then*

$$\mathrm{prox}_{\delta p^*}(s) = \begin{cases} \lambda, & \text{if } s + \delta b \geqslant \lambda; \\ s + \delta b, & \text{if } |s + \delta b| \leqslant \lambda; \\ -\lambda, & \text{if } s + \delta b \leqslant -\lambda \end{cases}$$

Proof.

$$\begin{aligned} \mathrm{prox}_{\delta p^*}(s) &= \operatorname*{argmin}_{t\in\mathbb{R}} \frac{1}{2}(t-s)^2 + \delta p^*(t) \\ &= \operatorname*{argmin}_{t\in[-\lambda,\lambda]} \frac{1}{2}(t-s)^2 - \delta\langle b, t\rangle \\ &= \operatorname*{argmin}_{t\in[-\lambda,\lambda]} \frac{1}{2}(t-(s+\delta b))^2 \\ &= \begin{cases} \lambda, & \text{if } s + \delta b \geqslant \lambda; \\ s + \delta b, & \text{if } |s + \delta b| \leqslant \lambda; \\ -\lambda, & \text{if } s + \delta b \leqslant -\lambda \end{cases} \end{aligned}$$

□

We can write $\mathrm{prox}_{\delta p^*}$ in the following form

$$\mathrm{prox}_{\delta p^*}(s) = (s + \delta b) - \mathrm{soft}(s + \delta b, \lambda)$$

5.3. Update sensitivity $\mathcal{S}_\ell$

When the target image $\hat{u}$ is given, we want to update the sensitivity $\mathcal{S}_\ell$. The k-space of each coil can be approximated as $g_\ell + (I - \mathcal{P})\mathcal{F}\hat{\hat{\mathcal{S}}}_\ell\hat{u}$. Meanwhile the sensitivity of each coil is very smooth, then the Fourier coefficients of $\mathcal{S}_\ell$ tend to be zero when it is far away from the k-space center. Thus, we only collect data of square window around the center of k-space to estimate the sensitivity as

$$g_\ell^c = \mathcal{P}_c(g_\ell + (I - \mathcal{P})\mathcal{F}\hat{\mathcal{S}}_\ell\hat{u}), \tag{5}$$

where $\mathcal{P}_c$ is the diagonal matrix to collect data of square window around the center of k-space. Let $\mathcal{U} := diag(\hat{u})$ be the diagonal matrix form of the target image $\hat{u}$, and s_ℓ be the vector form of the ℓ-th coil diagonal sensitivity matrix $\mathcal{S}_\ell$, we have

$$\mathcal{S}_\ell\hat{u} = Us_\ell.$$

We combine all the coil data g_ℓ^c and have a compact formula:

$$g_c = Q_c\mathcal{F}_p\mathcal{U}_p s + \eta,$$

$$g_c := \begin{bmatrix} g_1^c \\ \vdots \\ g_p^c \end{bmatrix}, s := \begin{bmatrix} s_1 \\ \vdots \\ s_p \end{bmatrix}, \mathcal{U}_p := \begin{bmatrix} \mathcal{U} & & \\ & \ddots & \\ & & \mathcal{U} \end{bmatrix}, Q_c = \begin{bmatrix} \mathcal{P}_c & & \\ & \ddots & \\ & & \mathcal{P}_c \end{bmatrix}$$

We define a concave set

$$C := \left\{ s \in \mathcal{C}^{p\cdot\sharp u} : \sum_{\ell=1}^{p} \overline{s_\ell[k]} s_\ell[k] = 1,\ k = 1, \cdots, \sharp u \right\}$$

and its indicator function is defined as

$$\iota_C(s) := \begin{cases} 0, & \text{if } s \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\mathrm{env}_{a|\cdot|}(x) = \begin{cases} a|x| - \frac{1}{2}a^2, & \text{if } |x| \geqslant a; \\ \frac{1}{2}|x|^2, & \text{otherwise.} \end{cases} \tag{6}$$

The optimization model for sensitivity by 3-Dimension tight framelet regularization is proposed as

$$\hat{s} = \arg\min_{s} \left\{ \frac{1}{2}\|Q_c \mathcal{F}_p \mathcal{U}_p s - g_c\|_2^2 + \mathrm{env}_{\alpha\|\cdot\|_1 \circ \Gamma}(W_{3D}s), S.T. \sum_{\ell=1}^{p} \overline{s_\ell[k]} s_\ell[k] = 1,\ k = 1, \cdots, \sharp u, \right\} \tag{7}$$

Set

$$\begin{aligned} h(s) &:= \frac{1}{2}\|Q_c \mathcal{F}_p \mathcal{U}_p s - g_c\|_2^2 + \mathrm{env}_{\|\cdot\|_1 \circ \Gamma}(W_{3D}s) \\ \mathcal{C} &:= \{s : \sum_{\ell=1}^{p} \overline{s_\ell[k]} s_\ell[k] = 1,\ k = 1, \cdots, \sharp u\} \end{aligned}$$

Then, model (7) can be rewritten as

$$\hat{s} = \arg\min_{s} \{h(s) + \iota_{\mathcal{C}}(s)\} \tag{8}$$

If h has a gradient with constant L, then the forward and backward proximal by Attouch and Bolte is

$$s^{k+1} = \mathrm{prox}_{\iota_{\mathcal{C}}}(s^k - \frac{1}{L}\nabla h(s^k)).$$

The alternative scheme to solve problem is following as $\hat{u}_0$, $\hat{s}_0$, $\hat{u}_1$, $\hat{s}_1$, $\cdots$.

$$\begin{aligned} \hat{u}_k &= \arg\min_{u} \left\{ \frac{1}{2}\|Q_p M u - g\|_2^2 + \|\Gamma W_{3D} \mathcal{F}_p^{-1}(Q_q M u + g)\|_1 \right\} \\ \hat{s}_k &= \arg\min_{s} \left\{ \frac{1}{2}\|Q_c \mathcal{F}_p \mathcal{U}_p s - g_c\|_2^2 + \mathrm{env}_{\|\cdot\|_1 \circ \Gamma}(W_{3D}s) + \iota_{\mathcal{C}}(s) \right\} \end{aligned}$$

References