Calibration Observation Model for Parallel Magnetic Resonance Imaging

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1 Introduction

Magnetic Resonance Imaging (MRI) is a non-invasive medical imaging technology, which not only does'nt use ionizing radiation, but also can obtain high-resolution and high-quality pathological images. However, MRI is different from Computed Tomography(CT) and other imaging methods in that they have the characteristic of fast imaging time. The biggest obstacle of MRI is that its data acquisition is too slow, which result in patients feel uncomfortable.Thus, the main research purpose of MRI technology is to shorten the imaging time and improve the imaging quality.

Parallel MRI (pMRI) technology is used to shorten the imaging time by the hardware, it uses a series of coils to acqucire the undersampling MR signal at the meantime, then uses the reconstruction algorithm to predict high-resolution MR image by partial k-space data. The most commonly used reconstruction algorithms can divided into two catagories, the image domain method(e.g. SENSE) and the k-space domain method(e.g. GRAPPA).

SENSE and GRAPPA are widely used in clinical application, but these two models have some shortcomings. For the SENSE method, the coil sensitivity can be estimated, but it is still not accuracy. For the another method, the interpolation kernel can be estimated by the auto calibraion signal(ACS) while it's not accuracy when the number of the ACS line is far from enough. Therefore, we propose the calibration observation model incoporating the advantage of the SENSE and GRAPPA. We exploit the SENSE model to remove artifacts in image domain and use GRAPPA kernel to regularize the SENSE model in K-space domain.

1.1 SENSE-based pMRI reconstruction model

The coil data from the jth coil can modeled as follows:

$$g_l = PFS_l u + \eta \tag{1}$$

where u denotes the desired reconstruced image, η is gaussian white noisy, F is dicrete Fourier tranform matrix, and P denotes the sampling matrix.

Combining equations from p coils, we have

$$g = \mathcal{P}Mu + \eta \tag{2}$$

where,

$$g = \begin{bmatrix} g_1 \\ \vdots \\ g_p \end{bmatrix}, \mathcal{P} = \begin{bmatrix} P & & \\ & \ddots & \\ & & P \end{bmatrix}, M = \begin{bmatrix} FS_1 \\ \vdots \\ FS_p \end{bmatrix}, \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_p \end{bmatrix}$$

2 GRAPPA model

GRAPPA utilizes the interpolation kernel to estimate the missing point, assuming we use G to represent the interpolation kernel, then we have

$$\mathcal{P}GMu \approx g \tag{3}$$

3 Regularization with two level tight framelet systems

3.1 DHF

The tight framelets in the first level is the directional Haar framelet (DHF) system, the filters associated with the DHF are

$$\tau_0 = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tau_1 = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tau_2 = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tau_3 = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$
$$\tau_4 = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \tau_5 = \frac{1}{4}\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \tau_6 = \frac{1}{4}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

Let M_k denotes the associated matrix representation of the filters τ_k, therefore

$$B_{1l} = M_0, B_{1h} = [M_1^T, \cdots, M_6^T]^T$$

Let $\Phi_{1\Lambda} : R^{6n} \to R$ can be defined through a function $\varphi_1 : R^6 \to R$ and a non-negative parameter vector $\Lambda = [\lambda_1, \lambda_2, \cdots, \lambda_n]$ as follows

$$\Phi_{1\Lambda}(v) = \sum_{i=1}^{n} \lambda_i \varphi_1(v_i, v_{i+n}, \cdots, v_{i+5n})$$

In this model, $\varphi_1(x_1, x_2, x_3, x_4, x_5, x_6) = \sqrt{|x_1|^2 + |x_2|^2} + \sqrt{|x_3|^2 + |x_4|^2}$

3.2 DCT-based tight framelet

The tight framelet in the second level is generated from the standard 3×3 DCT-II orthogonal matrix , the filters are

$$\tau_0 = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tau_1 = \frac{\sqrt{6}}{18}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}, \tau_2 = \frac{\sqrt{2}}{18}\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix},$$
$$\tau_3 = \frac{\sqrt{6}}{18}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}, \tau_4 = \frac{1}{6}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \tau_5 = \frac{\sqrt{3}}{18}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{bmatrix},$$
$$\tau_6 = \frac{\sqrt{2}}{18}\begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}, \tau_7 = \frac{\sqrt{3}}{18}\begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix}, \tau_7 = \frac{1}{18}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix},$$

Let P_k denotes the associated matrix representation of the filters τ_k, therefore

$$B_{1l} = P_0, B_{1h} = [P_1^T, \cdots, P_8^T]^T$$

Let $\Phi_{2\Theta}: R^{8n} \to R$ be defined through a nonnegative parameter sequence $\Theta = \{\theta_i = (\theta_{i1}, \theta_{i2}, \cdots, \theta_{i8}) \in R^{8n} : 1 \leqslant i \leqslant n\}$ non-negative elements as follows

$$\Phi_{2\Theta}(v) = \sum_{i=1}^{n} \|[\theta_{i1} v_i, \theta_{i2} v_{i+n}, \cdots, \theta_{i8} v_{i+7n}]\|_1$$

4 Calibration Observation Model

4.1 Model

According to the model (2) and (3), we propose the following optimization model as

$$\min_u \left\{ \frac{1}{2}\|\mathcal{P}Mu - g\|_2^2 + \frac{\lambda}{2}\|\mathcal{P}GMu - g\|_2^2 + \Phi_{1\Lambda}(B_{1h}u) + \Phi_{2\Theta}(B_{2h}B_{1l}u) \right\} \tag{4}$$

where λ is equal to 1 by default. Define

$$f(u) = \frac{1}{2}\|\mathcal{P}Mu - g\|_2^2 + \frac{\lambda}{2}\|\mathcal{P}GMu - g\|_2^2, \quad h(s) = \Phi_{1\Lambda}(B_{1h}s_1) + \Phi_{2\Theta}(B_{2h}B_{1l}s_2), \quad A = \begin{bmatrix} B_{1h} \\ B_{2h}B_{1l} \end{bmatrix}$$

where $s = (s_1, s_2)$. Therefore, the model(4) can be rewriten as

$$\min_u \{f(u) + h(Au)\} \tag{5}$$

$f(u) = \frac{1}{2}\|\mathcal{P}Mu - g\|_2^2 + \frac{\lambda}{2}\|\mathcal{P}GMu - g\|_2^2$ can be rewritten as:

$$f(u) = \frac{1}{2}\left\| \begin{bmatrix} \mathcal{P}M \\ \sqrt{\lambda}\mathcal{P}GM \end{bmatrix} u - \begin{bmatrix} g \\ \sqrt{\lambda}g \end{bmatrix} \right\|_2^2 \tag{6}$$

Equation (6) can be written in a more concise form :

$$f(u) = \frac{1}{2}\|Ku - y\|_2^2 \tag{7}$$

where

$$K = \begin{bmatrix} \mathcal{P}M \\ \sqrt{\lambda}\mathcal{P}GM \end{bmatrix}, y = \begin{bmatrix} g \\ \sqrt{\lambda}g \end{bmatrix}$$

Accoding to the function f, we have that $\nabla f(u) = K^T(Ku - y) = M^T\mathcal{P}^T(\mathcal{P}Mu - g) + \lambda M^T G^T \mathcal{P}^T(\mathcal{P}GMu - g)$, and we can find that the gradient of f is $\|K\|^2$-Lipschitz continuous.

Choosing γ and δ such that $\gamma < 2/\|K\|^2$ and $\gamma\delta < 1$. The PD3O has the following iteration:

$$\begin{aligned} u^k &= real(v^k) \\ s^{k+1} &= prox_{\delta h^*}((I - \gamma\delta AA^T)s^k + \delta A(2u^k - v^k - \gamma\nabla f(u^k))) \\ v^{k+1} &= u^k - \gamma\nabla f(u^k) - \gamma A^T s^{k+1} \end{aligned}$$

where $\nabla f(u) = M^T\mathcal{P}^T(\mathcal{P}Mu - g) + \lambda M^T G^T \mathcal{P}^T(\mathcal{P}GMu - g)$.

According to the Moreau decomposition, we can get

$$s^{k+1} = x^k - \delta prox_{\delta^{-1}h}(\delta^{-1}x^k) \tag{8}$$

where $x^k = (I - \gamma\delta AA^T)s^k + \delta A(2u^k - v^k - \gamma\nabla f(u^k))$

Algorithm 1 Double Domain Reconstruction Algorithm

Require: g is acquired k-space data; A is tight frame operator; $k := 0$; $u^0 := sos(F^{-1}g)$; $v^0 = u^0$; $s^0 = Au^0$; $\epsilon := 1e-5$

Ensure: $\gamma < 2/\|K\|^2$ and $\gamma\delta < 1$

1: **while** $k \leq k_{max}$ **and** $\|u^{k+1} - u^k\| \geqslant \epsilon$ **do**
2: $u^k = real(v^k)$
3: $x^k = (I - \gamma\delta AA^T)s^k + \delta A(2u^k - v^k - \gamma\nabla f(u^k))$
4: $s^{k+1} = x^k - \delta prox_{\delta^{-1}h}(\delta^{-1}x^k)$
5: $v^{k+1} = u^k - \gamma\nabla f(u^k) - \gamma A^T s^{k+1}$
6: **end while**

4.2 The calculation of the Lipschitz constant of the gradient of f

To solve the model (4), we need to estimate the Lipschitz constant of the f. Note that $\nabla f(u) = K^T(Ku - y)$, then the gradient of f is $\|K\|_2^2$-Lipschitz continuous,

$$
\begin{aligned}
\|K\|_2^2 = \|K^T K\|_2 &= \left\| \begin{bmatrix} \mathcal{P}M \\ \mathcal{P}GM \end{bmatrix}_2^T \begin{bmatrix} \mathcal{P}M \\ \mathcal{P}GM \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} M^T\mathcal{P}^T & M^T G^T \mathcal{P}^T \end{bmatrix} \begin{bmatrix} \mathcal{P}M \\ \mathcal{P}GM \end{bmatrix} \right\|_2 \\
&= \left\| \begin{bmatrix} M^T\mathcal{P}^T\mathcal{P}M + M^T G^T\mathcal{P}^T\mathcal{P}GM \end{bmatrix} \right\|_2 \\
&\leqslant \left\| M^T\mathcal{P}^T\mathcal{P}M \right\|_2 + \left\| M^T G^T\mathcal{P}^T\mathcal{P}GM \right\|_2 \\
&= \left\| M^T\mathcal{P}^T\mathcal{P}M \right\|_2 + \left\| S^T F^T G^T\mathcal{P}^T\mathcal{P}GFS \right\|_2 \\
&= \left\| S^T F^T\mathcal{P}FS \right\|_2 + \left\| S^T F^T G^T\mathcal{P}GFS \right\|_2
\end{aligned}
$$

Based on the previous work, we have known a constant k related to the sensitivity S_i which is defined as

$$
k = \max_j \sum_i^p |s_j^i|^2 \tag{9}
$$

where s_j^i the kth diagonal element of the sensitivity matrix S_i, is the sensitivity coefficient of the ith coil at the kth pixel. And we have

$$
\begin{aligned}
\|K\|_2^2 &\leqslant \left\| S^T F^T\mathcal{P}FS \right\|_2 + \left\| S^T F^T G^T\mathcal{P}GFS \right\|_2 \\
&\leqslant k(\left\| F^T\mathcal{P}F \right\|_2 + \left\| F^T G^T\mathcal{P}GF \right\|_2) \\
&\leqslant k(1 + C)
\end{aligned} \tag{10}
$$