## 1. Reconstruction model

The $\ell$-th coil $k$-space data $g_\ell$ is modeled as follows:

$$g_\ell = \mathcal{P}\mathcal{F}\mathcal{S}_\ell u + \eta_\ell,\ \ell = 1, \cdots, p,$$

where $\eta_\ell$ is the additive noise, $\mathcal{S}_\ell$ the diagonal sensitivity matrix, $\mathcal{F}$ is the discrete Fourier transform matrix and $\mathcal{P}$, called sampling matrix, is a diagonal matrix with 0 and 1, $u$ is target slice image. In real application, the sensitivity $\mathcal{S}_\ell$ and target image $u$ are unknown variables. We adopt an alternative scheme to solve problem, and need to fix one and then solve another.

### *1.1. Update image $u$*

When the sensitivity $\hat{\mathcal{S}}_\ell$ is given, we want to solve the target image $u$. We combine all the coil data $g_\ell$ and have a compact formula:

$$g = Q_p M u + \eta,$$

where $g$ is the stacked $k$-space data, $M$ is the composition of $\mathcal{F}$ and $\mathcal{S}_\ell$, and $Q_p$ is the concatenation of $\mathcal{P}$:

$$g := \begin{bmatrix} g_1 \\ \vdots \\ g_p \end{bmatrix}, \mathcal{F}_p := \begin{bmatrix} \mathcal{F} & & \\ & \ddots & \\ & & \mathcal{F} \end{bmatrix}, M = \mathcal{F}_p \begin{bmatrix} \hat{\mathcal{S}}_1 \\ \vdots \\ \hat{\mathcal{S}}_p \end{bmatrix}, Q_p = \begin{bmatrix} \mathcal{P} & & \\ & \ddots & \\ & & \mathcal{P} \end{bmatrix}, \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_p \end{bmatrix}$$

The proposed optimization model by tight framelet regularization for SENSE-based MRI reconstruction is:

$$\hat{u} = \arg\min_u \left\{ \frac{1}{2}\|Q_p M u - g\|_2^2 + \|\Gamma W \mathcal{F}_p^{-1}(Q_o M u + g)\|_1 \right\} \tag{1}$$

Where $Q_o = I - Q_p$ is the sampling matrix at the missing positions, $\Gamma$ is a diagonal matrix with non-negative diagonal elements and the $W$ is a transformation matrix by 3-dimension and 2-dimension tight framelet system.

Let $x$ be a multi coil image of size $m \times n \times c$ in $\mathcal{C}^{m\times n\times c}$, therefore the tight frame transform using the filters for $u \in \mathcal{C}^{m\times n\times c}$ is given by

$$Wx = \begin{bmatrix} W_{3D,l} \circledast x & W_{3D,x} \circledast x & W_{3D,y} \circledast x & W_{2D} W_{3D,z} \circledast x & W_{3D,xy} \circledast x \end{bmatrix}^T$$

where the $W_{3D,l}$ denotes the lowpass filter of the 3-D transformation, $W_{2D}$ is the 2-D transformer matrix, while others represent the highpass filter at different directions of the 3-D transformation.

## 2. Algorithm

Let $K \doteq Q_p M$, $A \doteq W\mathcal{F}_p^{-1} Q_o M$ and $b \doteq W\mathcal{F}_p^{-1} g$, and identify the functions $f$, and $p$ as follows:

$$f(u) = \frac{1}{2}\|Ku - g\|^2, \quad p(s) = \|\Gamma(s + b)\|_1 \tag{2}$$

Our optimization model (1) is written into the

$$\min_u \left\{ \frac{1}{2}\|Ku - g\|_2^2 + p(Au) \right\}. \tag{3}$$

The PD3O algorithm is provided as:

$$\begin{aligned} s^{k+1} &= \operatorname{prox}_{\delta p^*}\left( (I - \gamma\delta A A^\top) s^k + \delta A(u^k - \gamma \nabla f(u^k)) \right) \\ &= \operatorname{prox}_{\delta p^*}\left( s^k - A(\delta\gamma A^T s^k - \delta(u^k - \gamma\nabla f(u^k)) \right) \\ u^{k+1} &= u^k - \gamma\nabla f(u^k) - \gamma A^\top s^{k+1} \end{aligned}$$

In the model (3), the operator $A \doteq W\mathcal{F}_p^{-1} Q_o M$, and its odjoint can be formulated as $A^T \doteq M^T Q_o^T \mathcal{F}_p W^T$, where $W^T$ denotes the adjoint matirx of the $W$. Assuming the $\omega$ obtained by the $Wx$, where $\omega$ is the tight frame coefficient of the multi coil image $x$ and satisfies the perfect reconstruction formula, i.e.,

$$x = W^T W x = W^T \omega$$

$$W^T\omega = W_{3D,l}^T \circledast \omega_l + W_{3D,x}^T \circledast \omega_x + W_{3D,y}^T \circledast \omega_y + W_{3D,z}^T W_{2D}^T \circledast \omega_z + W_{3D,xy}^T \circledast \omega_{xy}$$