Discrete Mathematics

MATH1064, Lecture 3

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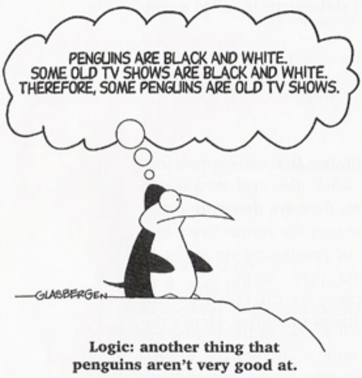

Extra exercises for Lecture 3

Section 1.1: Problems 1, 3, 5, 11, 12, 40, 41, 48, 49

Section 1.2: Problems 1, 5, 6, 15, 34, 37

Section 1.3: Problems 1, 5, 7, 10, 16–28

Logical equivalences

Know these (Rosen, pages 27-28):

- Commutative laws:
  $p \wedge q \equiv q \wedge p$
  $p \vee q \equiv q \vee p$
- Associative laws:
  $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
  $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- Distributive laws:
  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Logical equivalences (ctd.)

Know these also (Rosen, pages 27-28):

- Identity laws:
  $p \wedge (\text{tautology}) \equiv p$
  $p \vee (\text{contradiction}) \equiv p$
- Universal bound laws:
  $p \vee (\text{tautology}) \equiv (\text{tautology})$
  $p \wedge (\text{contradiction}) \equiv (\text{contradiction})$
- Negation laws:
  $p \ \vee \neg p \equiv (\text{tautology})$
  $p \ \wedge \neg p \equiv (\text{contradiction})$
- Double negative law:
  $\neg(\neg p) \equiv p$

Logical equivalences (ctd.)

And these (Rosen, pages 27-28):

- Idempotent laws:
  $p \wedge p \equiv p$
  $p \vee p \equiv p$
- De Morgan's laws:
  $\neg(p \wedge q) \equiv \neg p \vee \neg q$
  $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- Absorption laws:
  $p \vee (p \wedge q) \equiv p$
  $p \wedge (p \vee q) \equiv p$
- Negations:
  $\neg(\text{tautology})$ is a contradiction
  $\neg(\text{contradiction})$ is a tautology

Proofs using logical equivalences

From the first lecture:

$$((p \wedge \neg q) \vee (p \wedge q)) \wedge q \equiv p \wedge q$$

We have already seen how to prove this using truth tables.

An alternative proof using logical equivalences:

$$\begin{aligned}
&((p \wedge \neg q) \vee (p \wedge q)) \wedge q \\
&\quad \equiv (p \wedge (\neg q \vee q)) \wedge q && \textit{(distributive law)} \\
&\quad \equiv (p \wedge (\text{tautology})) \wedge q && \textit{(negation law)} \\
&\quad \equiv p \wedge q && \textit{(identity law)}
\end{aligned}$$

This method is *much* better when you have many variables!

Equivalences:

Which equivalence is correct?

1. $(p \vee q) \rightarrow r \equiv \ (p \rightarrow r) \wedge (q \rightarrow r)$
2. $(p \vee q) \rightarrow r \equiv \ (p \rightarrow r) \vee (q \rightarrow r)$

An example:

(If I cycle, I get tired) and (If I run, I get tired)
$(p \rightarrow r) \wedge (q \rightarrow r)$

If I cycle or run, I get tired.
$(p \vee q) \rightarrow r$

Recall that: $(p \rightarrow q) \equiv (\neg p \vee q)$

We have:

$$
\begin{aligned}
(p \vee q) \rightarrow r &\equiv \neg(p \vee q) \vee r && \textit{(express} \rightarrow \textit{using} \vee, \neg\textit{)} \\
&\equiv (\neg p \wedge \neg q) \vee r && \textit{(De Morgan's law)} \\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) && \textit{(distributive law)} \\
&\equiv (p \rightarrow r) \wedge (q \rightarrow r) && \textit{(express} \rightarrow \textit{using} \vee, \neg\textit{)}
\end{aligned}
$$

Similarly:

$$(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r)$$

More constructions

Contraposite

The contrapositive of $p \to q$ is $\neg q \to \neg p$.

These are equivalent!

$$
\begin{aligned}
p \to q &\equiv \neg p \vee q && \textit{(express } \to \textit{ using } \vee, \neg\textit{)} \\
&\equiv q \vee \neg p && \textit{(commutative law)} \\
&\equiv \neg(\neg q) \vee \neg p && \textit{(double negative law)} \\
&\equiv \neg q \to \neg p && \textit{(express } \to \textit{ using } \vee, \neg\textit{)}
\end{aligned}
$$

More constructions

Converse

The converse of $p \to q$ is $q \to p$.

Inverse

The inverse of $p \to q$ is $\neg p \to \neg q$.

Are either of these equivalent to $p \to q$?
Which of the following is true?

1. The converse is equivalent to $p \to q$.
2. The inverse is equivalent to $p \to q$.
3. Both are equivalent to $p \to q$.
4. Neither is equivalent to $p \to q$.

The biconditional

Let p and q be proposition variables; the biconditional from p to q, $p \leftrightarrow q$, is defined by the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

We say:

1. p if and only if q
2. p is a neccessary and sufficient condition for q
3. q is a neccessary and sufficient condition for p

The biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

In terms of operations that we already know:

$$p \leftrightarrow q \;\equiv\; (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \;\equiv\; (\neg p \vee q) \wedge (\neg q \vee p)$$

Order of operations:

- $\neg$ comes first
- $\wedge$ and $\vee$ are equal seconds
- $\to$ and $\leftrightarrow$ are equal thirds

Satisfiability

Satisfiable

A compound proposition is satisfiable if there is an assignment of truth values to its variables that makes it true. Otherwise it is unsatisfiable.

An unsatisfiable compound proposition is a contradiction!

Determine whether the following compound proposition is satisfiable:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

Predicates

Definition (Predicate)

A predicate is a sentence that contains finitely many variables, and which becomes a proposition (aka statement) if the variables are given specific values.

1. $x^2 \geq x$.
2. p is prime.
3. Is q composite?
4. $x = x + 1$.
5. $a^2 + b^2 = c^2$.
6. Factorise q, then divide by two.

Predicates

Definition (Domain)

The domain of a variable in a predicate is the set of all possible values that may be assigned to it.

Example:
Let $P(x)$ be the predicate "$x^2 = 4$".
We might define the domain of x to be $\mathbb{Z}$, the set of all integers.

Definition (Truth set)

The truth set of a predicate $P(x)$ is the set of all values in the domain that, when assigned to x, make $P(x)$ a true statement.

For the predicate $P(x)$ above, the truth set is $\{-2, 2\}$.

Common domains

The natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
Warning! In mathematics you often see $\mathbb{N} = \{1, 2, 3, \ldots\}$.

The integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The rationals: $\mathbb{Q} =$ all fractions $= \{\frac{a}{b} \mid a, b \in \mathbb{Z} \ \wedge \ b \neq 0\}$

The real numbers: $\mathbb{R} =$ the entire number line
You will see this defined "properly" in a different course.

Domains and truth sets

Whoever writes the predicate gets to choose the domain.

The truth set is something you must then deduce.

Consider the predicate "$2x = 1$".

- If the domain is $\mathbb{R}$ (the reals) or $\mathbb{Q}$ (the rationals), then the truth set is $\{1/2\}$.
- If the domain is $\mathbb{Z}$ (the integers) or $\mathbb{N}$ (the natural numbers), then the truth set is empty; that is, $\{\ \}$.

Summary

A predicate is a sentence that contains finitely many variables, and which becomes a proposition if the variables are given specific values.

The domain of a variable in a predicate is the set of all possible values that may be assigned to it.

The truth set of a predicate $P(x)$ is the set of all values in the domain that, when assigned to x, make $P(x)$ a true proposition.

Example:

Let $P(x)$ be the predicate "$x^2 = 4$".
Let $\mathbb{Z}$, the set of all integers, be the domain of x.
Then the the truth set of $P(x)$ is $\{-2, 2\}$.