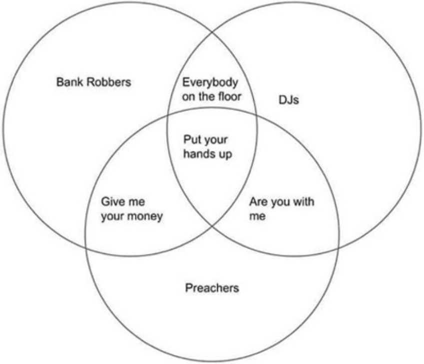
Discrete Mathematics

MATH1064, Lecture 8

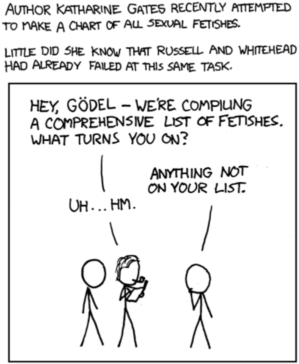
Jonathan Spreer



Extra exercises for Lecture 8

Section 2.1: Problems 1, 3–8, 13–16, 19, 20

Section 2.2: Problems 1, 3, 4, 11, 14, 18, 19, 20



Cardinality

If S is a finite set, then the cardinality of S is the number of distinct elements that S contains.

We write this as $|S|$.

Example: $S = \{1, 2, 3\}$ $|S| =$
Example: $S = \{2, 4, 2, 4\}$ $|S| =$
Example: $S = \{3, \{3\}, 3\}$ $|S| =$

If S is an infinite set, we often write $|S| = \infty$. However, cardinality is more interesting and more subtle: there are many different infinities, and two infinite sets might not have the same cardinality!

For example, $|\mathbb{R}| \neq |\mathbb{Z}|$, even though both are infinite!
We will see this again later in the semester.

Difference

For sets S and T, their difference is written $S \setminus T$ (or sometimes written $S - T$).

It contains all elements that belong to S but not T:

$$S \setminus T = \{x \mid x \in S \land x \notin T\}$$

Example:

$$\{1, 2, 3\} \setminus \{2, 5\} =$$

Example:

$$\{0, 1, 2, 3, \ldots\} \setminus \{0, -1, -2, -3, \ldots\} =$$

Example:

$$\mathbb{N} \setminus \mathbb{Z} =$$

Complement

Let U be some universal set in which we are working.

For example, the universal set might be:

- $\mathbb{R}$ if we are doing calculus;
- $\mathbb{Z}$ if we are doing number theory;
- the set of all points on the plane if we are doing geometry.

For any set $S \subseteq U$, the complement of S is written $\overline{S}$
(or sometimes written S^c).

It contains all elements of U that do not belong to S:

$$\overline{S} = \{x \in U \mid x \notin S\} =$$

Example, for the universe $U = \mathbb{Z}$:
$\overline{\{1, 2, 3\}} =$
$\overline{\{x \in \mathbb{Z} \mid x \text{ is even}\}} =$
$\overline{\mathbb{Z}} =$

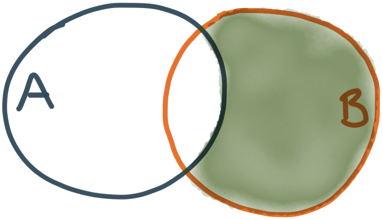
Summary: Relationships and operations

$$
\begin{array}{lcl}
S \subseteq T & \leftrightarrow & (x \in S \to x \in T) \\
S = T & \leftrightarrow & (x \in S \leftrightarrow x \in T) \\
S = T & \leftrightarrow & (S \subseteq T \wedge T \subseteq S)
\end{array}
$$

$$
\begin{array}{lcl}
S \cup T & = & \{x \mid x \in S \vee x \in T\} \\
S \cap T & = & \{x \mid x \in S \wedge x \in T\} \\
S \setminus T & = & \{x \mid x \in S \wedge x \notin T\} \\
\overline{S} & = & \{x \in U \mid x \notin S\}
\end{array}
$$

Venn diagrams

Venn diagrams are a convenient way of visualising the relationships between different sets.



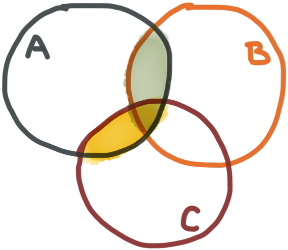
Difference: $B \setminus A$

Venn diagrams

Venn diagrams are not formal proofs,
but they can help visualise whether a statement is correct.

Is is always true that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$?

Is is always true that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$?



Building $(A \cap B) \cup (A \cap C)$

Is is always true that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$?

The Venn diagrams suggest: Yes!

Proposition

For all sets A, B, C: $\quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: To show $\mathrm{LHS} \subseteq \mathrm{RHS}$: Let $x \in A \cap (B \cup C)$.
Then $x \in A$.
Also, we have $x \in B \cup C$, which means *either* $x \in B$ or $x \in C$.

- If $x \in B$, then $x \in A \cap B$, and so $x \in (A \cap B) \cup (A \cap C)$.
- If $x \in C$, then $x \in A \cap C$, and so again $x \in (A \cap B) \cup (A \cap C)$.

Either way, $x \in (A \cap B) \cup (A \cap C)$.

(Continued over...)

Claim

For all sets A, B, C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (ctd.): To show $\mathrm{RHS} \subseteq \mathrm{LHS}$: Let $x \in (A \cap B) \cup (A \cap C)$.
Then *either* $x \in A \cap B$ or $x \in A \cap C$.

- Suppose $x \in A \cap B$.
  Then $x \in A$, and also $x \in B$. Since $x \in B$ we have $x \in B \cup C$, and so $x \in A \cap (B \cup C)$.
- Suppose $x \in A \cap C$.
  Then $x \in A$, and also $x \in C$. Since $x \in C$ we have $x \in B \cup C$, and again $x \in A \cap (B \cup C)$.

Either way, $x \in A \cap (B \cup C)$.

So: $\mathrm{LHS} \subseteq \mathrm{RHS}$ and $\mathrm{RHS} \subseteq \mathrm{LHS}$, which means $\mathrm{LHS} = \mathrm{RHS}$. $\square$

Claim

For all sets A, B, C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

A different proof:

$$\begin{aligned} x \in A \cap (B \cup C) &\leftrightarrow x \in A \wedge (x \in B \vee x \in C) \\ &\leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\ &\leftrightarrow x \in (A \cap B) \cup (A \cap C), \end{aligned}$$

using the distributive law for logical expressions! □

Set identities

The textbook has may more identities like this!
See Table 1 on page 130.

- $A \cup B = B \cup A$ and $A \cap B = B \cap A$
- $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup \emptyset = A$ and $A \cap U = A$, where $A \subseteq U$ for some universal set U
- ... and more

Do these look familiar?

These are just our logic identities in disguise!

Set identities and logic identities

Logic	**Sets**
$\wedge$ (and)	$\cap$ (intersection)
$\vee$ (or)	$\cup$ (union)
$\neg$ (not)	$\overline{A}$ (complement)
t (tautology)	U (universal set)
c (contradiction)	$\emptyset$ (empty set)

Proving that two statements are logically equivalent corresponds to proving that two sets are equal

Proving that $X \rightarrow Y$ for statements corresponds to proving that $X \subseteq Y$ for sets

Exercise

Let A and B be subsets of some universal set U.

True or false?

$$\overline{(A \cup B)} = \overline{A} \cup \overline{B}$$