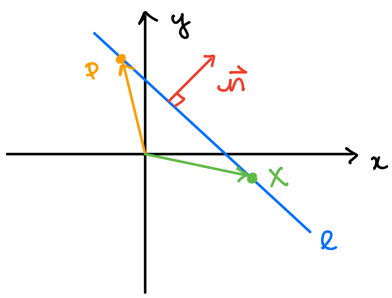
MATH 1002 LECTURE 8-D

LINES IN $\mathbb{R}^2$: NORMAL FORM & GENERAL FORM

<u>Definition</u>: Given a line ℓ in $\mathbb{R}^2$, a normal vector to (or for) ℓ is a vector $\vec{n} \neq \vec{0}$ which is orthogonal to ℓ.



<u>Idea</u>: The information of

- the normal vector $\vec{n}$

  and

- one point $P \in \ell$

completely determines ℓ.

<u>Remark</u>: Unlike for vectors, the <u>position</u> of a line matters, which is why we need P.

Let's fix $P \in \ell$, and let X be any point.

→ Can we find properties of X in terms of P & $\vec{n}$ that tell us whether X is on the line ℓ?

Write $X = (x, y)$, $\vec{OX} = \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{OP} = \vec{p}$.

Now $X \in \ell \iff \vec{PX}$ is parallel to ℓ

$\iff \vec{PX}$ is orthogonal to $\vec{n}$

$\iff \vec{PX} \cdot \vec{n} = 0$.

Now note that $\vec{PX} = \vec{OX} - \vec{OP} = \vec{x} - \vec{p}$

So $X \in \ell \iff \boxed{(\vec{x} - \vec{p}) \cdot \vec{n} = 0}$

$$\Leftrightarrow \quad \boxed{\vec{x} \cdot \vec{n} = \vec{p} \cdot \vec{n}.}$$

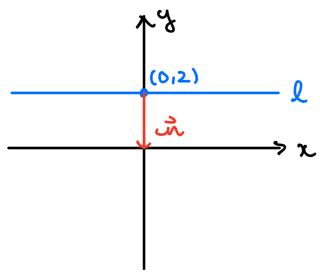
Definition: The normal form of the line ℓ in $\mathbb{R}^2$ is given by the equation $(\vec{x} - \vec{p}) \cdot \vec{n} = 0$

or equivalently $\vec{x} \cdot \vec{n} = \vec{p} \cdot \vec{n}$.

What does it mean? $\ell = \{(x,y) \mid \begin{bmatrix} x \\ y \end{bmatrix} \cdot \vec{n} = \vec{p} \cdot n\}$

"ℓ is the set of all points satisfying the equation"

Example



We can tell by looking that

$\ell = \{(x,2)\}$.

But let's check using the normal form.

We can take $P = (0,2)$ and $\vec{n} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$. (for example).

So $\ell = \{(x,y) \mid \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \end{bmatrix}\}$

$= \{(x,y) \mid -2y = -4\}$

$= \{(x,y) \mid y = 2\}$ as we saw before.

Recall: In school you probably saw equations for lines that looked like "$y = mx + b$"

or "$ax + by = c$".

We can rearrange the equation from the normal form to look like this.

- Suppose $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $P = (p_1, p_2)$, so $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$.

Then $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

becomes $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$

$\Leftrightarrow \quad ax + by = ap_1 + bp_2$

i.e. $ax + by = c$ where $c = ap_1 + bp_2$.
$= \vec{n} \cdot \vec{p}$

Definition: This equation is called the general equation for the line ℓ in $\mathbb{R}^2$.

$\Rightarrow$ it tells us: $\ell = \{ (x, y) \mid ax + by = c \}$.

Remark: can read off $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$.

- The two equations look different, but they are just different ways of specifying the same line

Exercise: if ℓ has general form given by the equation $3x - y = 9$, what is a normal vector to ℓ?

Solution: we can take $\vec{n} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

or any scalar multiple $\lambda \vec{n} = \begin{bmatrix} 3\lambda \\ -\lambda \end{bmatrix}$.

Example: Let's find the normal and general forms of an equation of the line ℓ passing through the point $(3,1)$ and perpendicular to the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Solution: We take $\vec{n} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $P = (3,1)$, $\vec{p} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

So $\vec{n} \cdot \vec{p} = 2 \times 3 + (-1) \times 1 = 5$

So the normal form of ℓ is $\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 5$

and the general form is $2x - y = 5$.

Remark These equations aren't unique.

For example, we could replace P by the point

$Q = (0, -5)$ $\left(\text{since } \begin{bmatrix} 0 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 5 \right)$

and we could replace $\vec{n}$ by $\vec{m} = \begin{bmatrix} 2\pi \\ -\pi \end{bmatrix}$.

Exercise Now what is the normal form for ℓ?

Solution $\vec{m} \cdot \vec{q} = \begin{bmatrix} 2\pi \\ -\pi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -5 \end{bmatrix} = 5\pi$

So we have $\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 2\pi \\ -\pi \end{bmatrix} = 5\pi$

$\hookrightarrow$ and in general form: $2\pi x - \pi y = 5\pi$.

Remark: We can recover our first equation by dividing by π:

$2x - y = 5$.

Summary of the lecture

- To write down the equation of a line in $\mathbb{R}^2$, it's enough to know

  1) the coordinates of a point P on ℓ

  2) a normal vector $\vec{n}$.

This allows us to write down the (an) equation in

normal form $\begin{bmatrix} x \\ y \end{bmatrix} \cdot \vec{n} = \vec{OP} \cdot \vec{n}$

or in

general form $\underbrace{a}_{}x + \underbrace{b}_{}y = \underbrace{c}_{\vec{OP}\cdot\vec{n}.}$

components of $\vec{n}$

You should be able to:

- Given P and $\vec{n}$, write down the normal form and general form.
- Given the equation in general or normal form
  - read off a normal vector $\vec{n}$
  - find some points of ℓ.