Discrete Mathematics

MATH1064, Lecture 21

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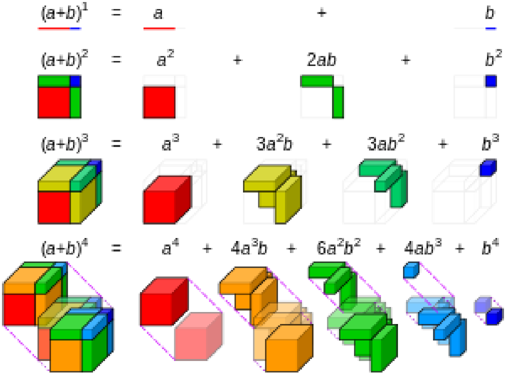
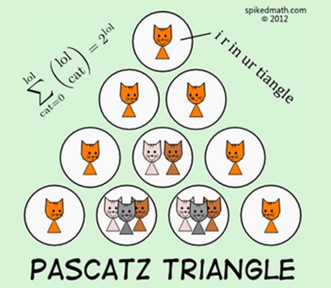


Extra exercises for Lecture 21

Section 6.4: Problems 1–9, 20



More counting problems

How many ways can you rearrange the letters of the word `CHILL`?

1. $\frac{5 \times 4 \times 3}{3!} = 10$
2. $5 + 4 + 3 + 2 + 1 = 15$
3. $5 \times 4 \times 3 = 60$
4. $5! = 120$
5. Something else

More counting problems

How many ways can you rearrange the letters of the word `KOKODA`?

1. 6
2. 120
3. 180
4. 450
5. 720

More counting problems

An alternate solution:

- If all six letters were different,
  there would be $6! = 720$ possibilities.
- Now make the two `K`s indistinguishable:
  Each solution has been counted twice.
  So there are now $720/2 = 360$ possibilities.
- Now make the two `O`s indistinguishable:
  Each solution has again been counted twice.
  So there are now $360/2 = 180$ possibilities.

Our two equivalent solutions:

$$\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1} = 180 = \frac{6!}{2! \cdot 2!}$$

More counting problems

Suppose you have n objects (e.g. balls, or letters), of which

n_1 are of type T_1 (e.g., blue balls)

n_2 are of type T_2 (e.g., red balls)

n_3 are of type T_3 (e.g., green balls)

etc. up to n_k of type T_k (e.g., fuchsia balls)

Assume that objects of the same type cannot be distinguished, but objects of different types can be distinguished.

Note: $n = n_1 + n_2 + \ldots + n_k$

Then the number of distinct permutations of the n objects is:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!n_2!\cdots n_k!}.$$

Even more counting problems

Ten politicians are being lined up for a photograph in Canberra.
How many ways can you arrange the politicians, if you must ensure that Julie is standing immediately to the left of Scott?

1. $9!$
2. $9! + 9!$
3. $10!$
4. $10 \times 8!$
5. $10! - 9!$
6. Something else

Even more counting problems

Ten politicians are being lined up for a photograph in Canberra. How many ways can you arrange the politicians, if you must ensure that Julie and Scott are adjacent?

Solution: Either Julie is immediately left of Scott, or Julie is immediately right of Scott.

Do what we did before: treat Julie and Scott as a single block. There are now two ways we can do this:

> If (Julie-Scott) is a single block $\rightarrow$ 9! possibilities
> If (Scott-Julie) is a single block $\rightarrow$ 9! possibilities

The total: $9! + 9! = 2 \cdot 9!$

We multiply if we must make decision A and then decision B.
We add if we must make either decision A or decision B.

Even more counting problems

Ten politicians are being lined up for a photograph in Canberra.
How many ways can you arrange the politicians, if you must ensure that Peter and Bill are *not* adjacent?

Solution: There are $2 \cdot 9!$ arrangements if Peter and Bill are adjacent.

There are $10!$ arrangements overall, with no constraints.

Therefore there are $(10! - 2 \cdot 9!)$ arrangements if Peter and Bill are *not* adjacent!

It's okay to overcount, as long as you subtract off the unwanted solutions later!

Formally

For finite sets:

- Multiplication: $|S \times T| = |S| \times |T|$
- Addition: If S and T are disjoint, then $|S \cup T| = |S| + |T|$
- Subtraction: If $T \subseteq S$, then $|S \setminus T| = |S| - |T|$

Binomial coefficients

If S is a set with $|S| = n$,
then the number of subsets of S with exactly k elements is

(1) n^k (2) $P(n,k)$ (3) $\binom{n}{k}$ (4) $\binom{n+k-1}{n-1}$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

This is called a binomial coefficient.

We saw that, for all $n, k \in \mathbb{N}$ satisfying $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

The reason: Choosing a subset $A \subseteq S$ (what to take) is equivalent to choosing a subset $S \setminus A$ (what to leave behind), and
$|A| = k$ if and only if $|S \setminus A| = n - k$.

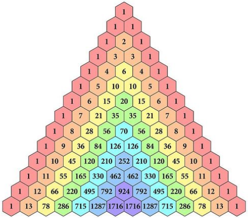
Another equation

Lemma

For all $n, k \in \mathbb{Z}$ with $1 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Again, we will see two proofs: an algebraic proof and a counting, or combinatorial, proof.



Algebraic proof of $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$
\begin{aligned}
& \binom{n-1}{k} + \binom{n-1}{k-1} \\
= \quad & \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
= \quad & \frac{(n-1)!}{(k-1)!(n-k-1)!} \cdot \left[\frac{1}{k} + \frac{1}{n-k}\right] \\
= \quad & \frac{(n-1)!}{(k-1)!(n-k-1)!} \cdot \left[\frac{n-k}{k(n-k)} + \frac{k}{k(n-k)}\right] \\
= \quad & \frac{(n-1)!}{(k-1)!(n-k-1)!} \cdot \frac{n}{k(n-k)} \\
= \quad & \frac{n!}{k!(n-k)!} \\
= \quad & \binom{n}{k}
\end{aligned}
$$

Combinatorial proof of $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Let $|S| = n$. Then $\binom{n}{k}$ is the number of ways of choosing a subset $A \subseteq S$ with $|A| = k$.

Let $S = \{s_1, s_2, \ldots, s_n\}$.
We take cases according to whether or not $s_1 \in A$:

- The number of subsets $A \subseteq S$ with $|A| = k$ and $s_1 \in A$ is $\binom{n-1}{k-1}$.
- The number of subsets $A \subseteq S$ with $|A| = k$ and $s_1 \notin A$ is $\binom{n-1}{k}$.

Therefore $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. $\square$

The Binomial Theorem

For all $a, b \in \mathbb{R}$ and all $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Example: Expand $(3+x)^4$:

$$\begin{aligned} & \binom{4}{0} 3^4 x^0 + \binom{4}{1} 3^3 x^1 + \binom{4}{2} 3^2 x^2 + \binom{4}{3} 3^1 x^3 + \binom{4}{4} 3^0 x^4 \\ = \quad & 81 + 4 \cdot 27x + 6 \cdot 9x^2 + 4 \cdot 3x^3 + x^4 \\ = \quad & 81 + 108x + 54x^2 + 12x^3 + x^4 \end{aligned}$$

We say that the coefficient of x^2 is 54,
the coefficient of x is 108, the coefficient of x^4 is 1, and so on.

The Binomial Theorem

For all $a, b \in \mathbb{R}$ and all $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof: Induction!

Proposition

For all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof #1: Expand $(1+1)^n$ using the binomial theorem:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

Proof #2: Count all subsets of $S = \{s_1, s_2, \dots, s_n\}$:

- For each s_i, there are two choices (use it, or don't).
  So the total number of subsets is $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$.
- There are $\binom{n}{k}$ subsets of size k, for $k = 0, 1, \dots, n$.
  So the total number of subsets is $\sum_{k=0}^{n} \binom{n}{k}$.

Proof #3: Use induction.

Another fact

Proposition

For any finite set S, the number of subsets of S with an even number of elements is equal to the number of subsets of S with an odd number of elements!

Proof: Let $|S| = n$.

- The number of subsets of even size is $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$.
  The number of subsets of odd size is $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$.
- By the binomial theorem,

$$(-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}.$$

So: $0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \binom{n}{5} + \cdots$.

Therefore: $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$. $\square$

Exercise

You are putting together a judging panel for a new reality TV show. The candidates for the panel include 12 celebrities and 4 experts. In how many ways can you make a panel of five, using at least one celebrity and at least one expert?

1. $12 \cdot 4 \cdot \binom{14}{3}$
2. $\binom{12}{5} + \binom{4}{5}$
3. $\binom{12}{3} \cdot \binom{4}{3}$
4. $\binom{16}{5}$
5. $\binom{16}{5} - \binom{12}{5}$

More identities

Look at

https://en.wikipedia.org/wiki/Binomial_coefficient

for many many more such identities involving binomial coefficients.

Similar identities form a whole field of mathematics: