MATH1002

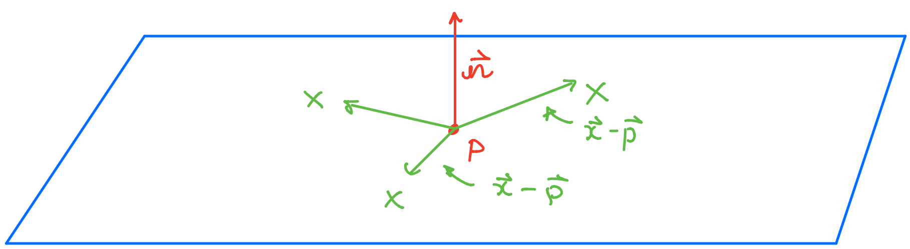
LECTURE 4-A

NORMAL PLANES IN $\mathbb{R}^3$

Recall In $\mathbb{R}^2$, specifying a normal vector $\vec{n}$ to ℓ and a point $P \in \ell$ completely determines ℓ.

We saw that this doesn't work in $\mathbb{R}^3$.

Lots of lines can be orthogonal to a given vector $\vec{n}$:



In fact: the equation $\vec{n}\cdot(\vec{x}-\vec{p}) = 0$ describes a plane in $\mathbb{R}^3$

↳ there is exactly one plane in $\mathbb{R}^3$ passing through the point P and orthogonal to $\vec{n}$.

Definition: a normal vector to a plane is a non-zero vector $\vec{n}$ which is orthogonal (perpendicular) to the plane.

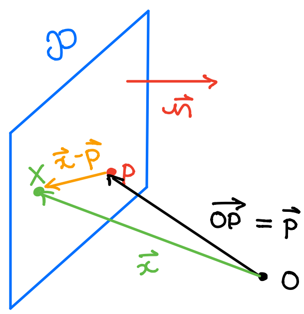
↳ if A, B are any two points in the plane, then $\vec{n}\cdot(\overrightarrow{AB})$ must be 0.

So from the above discussion, the following definition makes sense:

Definition: The normal form of the equation of the plane $\mathcal{P}$ in $\mathbb{R}^3$ passing through the point P & with normal vector $\vec{n}$ is

$$\boxed{\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}}$$

- Here $\vec{p}$ is the position vector of P
  $\vec{x}$ is the position vector of an arbitrary point of $\mathcal{P}$.



$\vec{PX}$ is orthogonal to $\vec{n}$
$\parallel$
$(\vec{x}-\vec{p})$

$\Leftrightarrow \vec{n} \cdot (\vec{x}-\vec{p}) = 0$

General form: If we know $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $P = (p_1, p_2, p_3)$, then $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$.

We can write $X = (x, y, z)$, so $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and then the equation $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ becomes

$$\boxed{ax + by + cz = d,} \qquad \text{where } d = ap_1 + bp_2 + cp_3.$$

"general form" for the equation of the plane $\mathcal{P}$.

This tells us that $\mathcal{P} = \{(x, y, z) \mid ax + by + cz = d\}$

"such that"

Example Find a normal form and a general form for the equation of a plane passing through $P = (1, 0, -2)$ and having a normal vector $\vec{n} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

Solution:
- Normal form: $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

  i.e. $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

- To find the general form, expand and simplify.

$$2x - y + 3z = -4$$

Exercise: Is the point $(0, 7, -1)$ in this plane?

Solution: We just need to check if $x = 0$, $y = 7$, $-1 = z$ satisfies the general equation:

LHS: $2 \times 0 - 7 + 3(-1) = -7 - 3 = -10$

RHS: 4

They are not equal, so no, the point does not lie on the plane. $\square$.

Now consider the line $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$, $t \in \mathbb{R}$

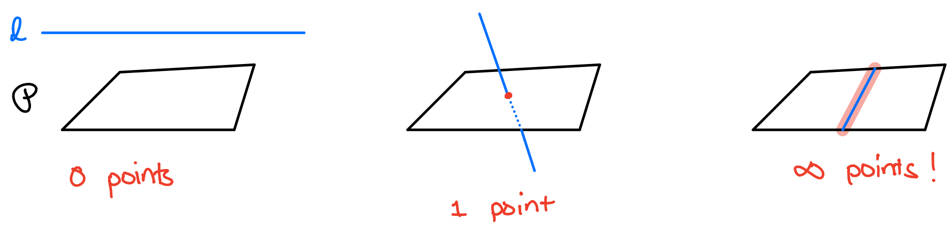
Question: Does it intersect the plane $2x - y + 3z = -4$, and if so, what is the intersection?

Warm-up question:

Let $\mathcal{P}$ be any plane in $\mathbb{R}^3$ and ℓ any line.

How many points can be in the intersection $\mathcal{P} \cap \ell$?

Solution: $\mathcal{P} \cap \ell$ can consist of 0, 1 or infinitely many points.



Back to the question:

How do we know if a point $X = (x, y, z)$ is in both

the line $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$, $t \in \mathbb{R}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-t \\ 2t \\ -1+4t \end{bmatrix}$ for some t.

and the plane $2x - y + 3z = -4$?

▶ want $2(1-t) - (2t) + 3(-1+4t) = -4$

Exercise: Solve for t!

(By previous exercise, you expect 0, 1 or ∞-many solutions.)

Solution

$$2 - 3 + 8t = -4$$

$$\Rightarrow 8t = -3$$

$$\Rightarrow t = -3/8.$$

Remarks: • There is exactly one solution

$\Rightarrow$ the intersection consists of exactly one point $X = (1 + 3/8,\ 2(3/8),\ -1 + 4(-3/8))$

$= (11/8,\ -6/8,\ -5/2)$

- you can check that $X \in \ell$ and $X \in \mathcal{P}$.

Summary of the lecture:

- To define a plane in $\mathbb{R}^3$ we can specify one point P in the plane and one vector $\vec{n}$ normal to the plane

  ▸ normal form: $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

  general form: $ax + by + cz = d$

  where $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $d = \vec{n} \cdot \vec{p}$.

- Given a line ℓ and a plane $\mathcal{P}$ in $\mathbb{R}^3$, the intersection will consist of
  - nothing
  - exactly one point

    OR
  - infinitely many points (the whole line ℓ)

You should be able to:

- write down equations for planes in $\mathbb{R}^3$ in normal form & general form
- given such an equation, identify points in the plane & calculate the intersection with a line.