THE UNIVERSITY OF SYDNEY

MATH1002 LINEAR ALGEBRA

Semester 1 **Preparatory Exercises for Week 2** 2021

There are thirteen teaching weeks. There will be a set of preparatory exercises and a set of tutorial exercises each week except for Week 1. Each set of exercises covers the material from the lectures in the previous week. This is the first set of preparatory exercises; it is based on the lectures from Week 1. Preparatory exercises should be attempted before attending the tutorial. Answers are provided below.

Important Ideas and Useful Facts:

(i) A *vector* $\mathbf{v}$ is a directed line segment that corresponds to a displacement from a point A to a point B. The vector from A to B is denoted $\overrightarrow{AB}$, with A the *initial point* or *tail* and B the *terminal point* or *head*.

(ii) The set of all points in n-dimensional space corresponds to the set of all vectors with tails at the origin O. To each point A, there corresponds the vector $\overrightarrow{OA}$, called the *position vector* of A.

(iii) If A is the point in the plane with coordinates (a_1, a_2), then $\overrightarrow{OA} = [a_1, a_2]$. The entries a_1 and a_2 are the *components* of the vector $\overrightarrow{OA}$. We often use the *column vector* $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ rather than the *row vector* $[a_1, a_2]$. Similarly if A is the point in n-dimensional space with coordinates $(a_1, a_2, \ldots, a_n)$, then the vector $\overrightarrow{OA}$ is $[a_1, a_2, \ldots, a_n]$ or $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, and this vector has ith component a_i.

(iv) Two vectors are equal if and only if their corresponding components are equal. Geometrically, two vectors are equal if and only if they have the same length and the same direction, equivalently, we could translate one on to the other. A vector with its tail at the origin is in *standard position*.

(v) The set of all vectors with n components is denoted $\mathbb{R}^n$. This is the same as the set of all ordered n-tuples, written as row or column vectors.

(vi) If $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$ then their *sum* $\mathbf{u} + \mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$. Similarly in $\mathbb{R}^n$, if $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, then

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n].$$

(vii) Head-to-Tail Rule for Vector Addition: Translate $\mathbf{v}$ so its tail is at the head of $\mathbf{u}$. Then $\mathbf{u} + \mathbf{v}$ is the vector from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$.

(viii) Parallelogram Rule: If $\mathbf{u}$ and $\mathbf{v}$ are in standard position, then $\mathbf{u} + \mathbf{v}$ is the vector in standard position along the diagonal of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

(ix) The *zero vector* in $\mathbb{R}^n$ is $\mathbf{0} = [0, 0, \ldots, 0] = \overrightarrow{OO}$. This vector has length 0 and its direction is not defined. For all vectors $\mathbf{u}$ in $\mathbb{R}^n$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(x) A *scalar* is a real number. Given a scalar c and a vector $\mathbf{v}$, the *scalar multiple* $c\mathbf{v}$ is the vector obtained by multiplying each component of $\mathbf{v}$ by c. So if $\mathbf{v} = [v_1, v_2]$ then $c\mathbf{v} = [cv_1, cv_2]$. Similarly in $\mathbb{R}^n$, if $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ then $c\mathbf{v} = [cv_1, cv_2, \ldots, cv_n]$. For all vectors $\mathbf{v}$, $0\mathbf{v} = \mathbf{0}$ and $1\mathbf{v} = \mathbf{v}$, and for all scalars c and d, $c(d\mathbf{v}) = (cd)\mathbf{v}$. Geometrically, $c\mathbf{v}$ points in the same direction as $\mathbf{v}$ if $c > 0$ and the opposite direction if $c < 0$, and $c\mathbf{v}$ has length $|c|$ times the length of $\mathbf{v}$. Two vectors are scalar multiples of each other if and only if they are parallel.

(xi) The *negative* of a vector $\mathbf{v}$ is the vector $(-1)\mathbf{v} = -\mathbf{v}$. This has the same length as $\mathbf{v}$ but points in the opposite direction. For all vectors $\mathbf{v}$, $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. If A and B are points then $\overrightarrow{AB} = -\overrightarrow{BA}$.

(xii) We define *vector subtraction* by $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

(xiii) If P and Q are points in $\mathbb{R}^n$ then $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$.

(xiv) Commutative Law of Addition: For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

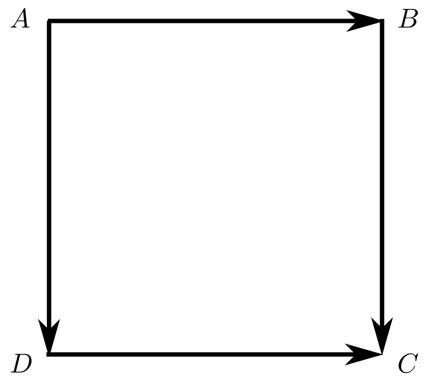
(xv) Associative Law of Addition: For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

(xvi) Distributive Laws: For all scalars c and d and all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ and $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.

Preparatory Exercises:

1. Let $\mathbf{u}=[3,1]$ and $\mathbf{v}=[-1,1]$ be vectors in $\mathbb{R}^2$. On the same diagram, draw and label the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ and $-2\mathbf{v}$ in standard position.

2. The edges of the square $ABCD$ are marked by vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$ and $\overrightarrow{DC}$, as shown.



True or false:

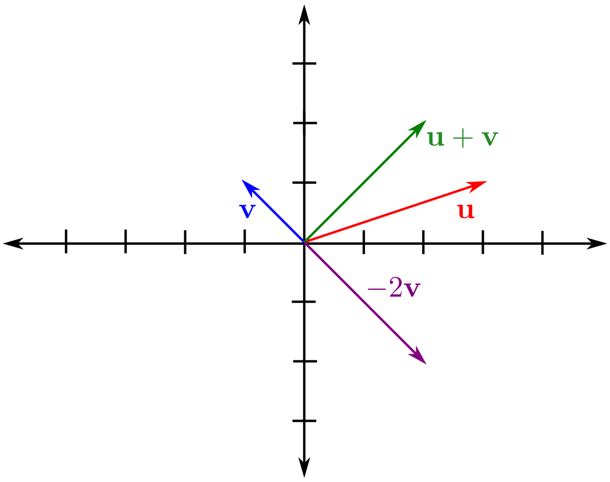
1. $\overrightarrow{AB}=\overrightarrow{BC}$
2. $\overrightarrow{AB}=\overrightarrow{CD}$
3. $\overrightarrow{AD}=\overrightarrow{BC}$
4. $\overrightarrow{AC}=\overrightarrow{BC}+\overrightarrow{DC}$

3. Simplify the following vector expressions, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{w}$, $\mathbf{z}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$.

1. $3\mathbf{a}+2\mathbf{b}-4(\mathbf{b}+\frac{1}{2}\mathbf{a})$
2. $-(\mathbf{w}-6\mathbf{z})-2\mathbf{w}+\mathbf{v}-2\mathbf{z}$

Answers to Preparatory Exercises

1. The diagram is:



2. (i) false (ii) false (iii) true (iv) true

3. (i) $\mathbf{a} - 2\mathbf{b}$ (ii) $\mathbf{v} - 3\mathbf{w} + 4\mathbf{z}$