MATH1002 — LECTURE 3-F

LINES IN $\mathbb{R}^2$ & $\mathbb{R}^3$: OVERVIEW & COMPARISON.

Summary of the last two lectures:

	Normal form	General form	Vector form	Parametric form
Lines in $\mathbb{R}^2$	$\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{p}$ $\vec{n}$: normal vector	$ax + by = c$ $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$; $c = \vec{n}\cdot\vec{p}$	$\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$ $\vec{d}$: direction vector	$x = p_1 + td_1$ $y = p_2 + td_2$ $t \in \mathbb{R}$ $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$
Lines in $\mathbb{R}^3$	—	—	$\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$ $\vec{d}$: direction vector.	$x = p_1 + td_1$ $y = p_2 + td_2$ $z = p_3 + td_3$ $t \in \mathbb{R}$, $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Normal / general form: useful when I give you a point and ask, "Is it in ℓ?"

Vector / parametric form: useful when I ask, "what are some points in ℓ?"

In $\mathbb{R}^2$: Both options exist. How do we switch between them?

Example: Recall from last time the line $\ell \subset \mathbb{R}^2$ passing through $P = (1, -3)$ and $Q = (2, -1)$.

We saw it has vector equation

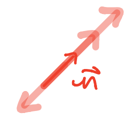
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Goal: Find its normal form and general equation.

Solution: We need to find the components of a normal vector $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ to ℓ.

→ It will satisfy $\vec{n} \cdot \vec{d} = 0$.

Taking $\vec{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we obtain $n_1 + 2n_2 = 0$



There will be lots of solutions, so we can take

$n_2 = -1$, and then $n_1 = 2$.

So $\vec{n} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a normal vector to ℓ,

and we can write down the normal form

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}_{\vec{p}}$$

$$= 2 + 3 = 5.$$

i.e. $\boxed{\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \vec{x} = 5}$ ← normal form

$\boxed{2x - y = 5}$ ← general equation

- We just saw: vector equation → normal form → general form.

We can also directly go:

- parametric equations → general form
- general form → parametric equations.

Example: from parametric equations to general form:

- Start from $\begin{cases} x = 1 + t \\ y = -3 + 2t \end{cases}$ $t \in \mathbb{R}$.

- Isolate t: $t = x - 1$

$$t = \frac{1}{2}(y+3)$$

- Obtain an equation: $x - 1 = \frac{1}{2}(y+3)$

$$\Leftrightarrow \quad 2(x-1) = y+3$$

$$\Leftrightarrow \quad 2x - 2 = y+3$$

$$\Leftrightarrow \quad 2x - y = 5$$ ← general equation!

Example: From general equation to parametric equations:

- Start from $2x - y = 5$
- Isolate one variable : e.g. $y = 2x - 5$.
- Let the other variable be a parameter: $x = s$, $s \in \mathbb{R}$.

$$\rightsquigarrow \begin{cases} x = s \\ y = 2s - 5 \end{cases} \quad s \in \mathbb{R}. \qquad (*)$$

Claim: these parametric equations give the same line

as $\begin{cases} x = 1 + t \\ y = -3 + 2t \end{cases} \quad t \in \mathbb{R}.$

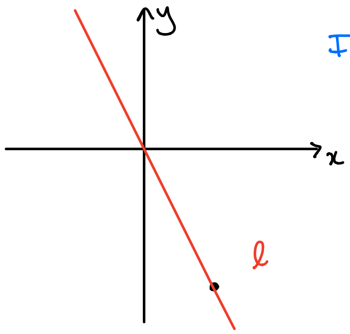
- indeed if we set $s = 1 + t$, $(*)$ becomes

$$\begin{cases} x = 1 + t, \\ y = 2(1+t) - 5 \\ \quad = 2t - 3 \end{cases} \quad t \in \mathbb{R}.$$

Example: Consider the line ℓ in $\mathbb{R}^2$ with parametric equations

$$\begin{cases} x = 2 - t \\ y = -4 + 2t \end{cases}, \quad t \in \mathbb{R}.$$

Exercise: Find a general equation for ℓ.



Solution: Solve for t: $t = 2 - x \implies 2t = 4 - 2x$

$2t = y + 4$

Identify two sides: $4 - 2x = y + 4$

$\implies 2x + y = 0.$

Exercise: Read off a direction vector & a normal vector for ℓ.

- direction vector from vector equation: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
- normal vector from general form $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

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Lines in $\mathbb{R}^3$			$\vec{x} = \vec{p} + t\vec{d}$, $t\in\mathbb{R}$ $\vec{d}$: direction vector.	$x = p_1 + td_1$ $y = p_2 + td_2$ $t\in\mathbb{R}$, $z = p_3 + td_3$ $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

- can read off normal vector   can read off direction vector.

You should be able to:

- identify different forms, switch between them, and use them to extract information about the line.