Discrete Mathematics

MATH1064, Lecture 20

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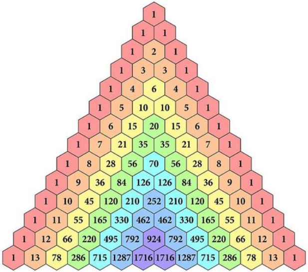


Extra exercises for Lecture 20

Section 6.3: Problems 14–18, 23–25

Recap: Choosing k items from a set of n items

Problem

Select k elements from a set containing n elements.

How many ways are there to do the selection?

1. Does order matter?
2. Is repetition allowed?

Combining (1) and (2) gives a total of four different cases!

Recap: Choosing k items from a set of n items

Select k elements from a set containing n elements.

- Case 1: If order matters and repetition is allowed: $\prod n_i$
  - How many possible NSW licence plates are there?

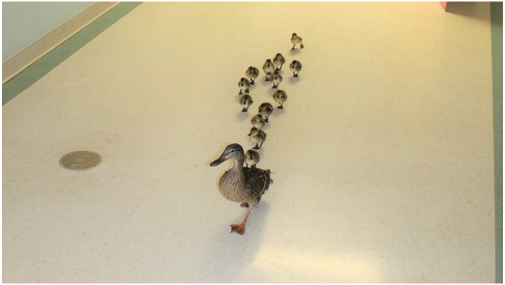
  - How many passport numbers are there?
  - How many safe passwords are there (limited rules)?

- Case 2: If order matters and repetition is not allowed:

$$P(n,k) = \frac{n!}{(n-k)!}$$

  - How many possible Top Ten charts are there?
  - Rank your five most favourite movies
  - Number of distinct weekly special boards

Case 3: Order does not matter, repetition not allowed



Example: Remember Mother duck? She now has seven tickets to go and see the pond – which she needs to distribute among her 13 little ducklings.

Other examples:

- How many poker hands are there? (Choose 5 from 52)
- Choosing numbers for Tatts Lotto (Choose 6 from 45)
- n people shake hands at a party. What is the total number of handshakes? (Choose 2 from n)

Example: Tatts Lotto (Choose 6 from 45)

If we count as before, assuming order matters, there are

$$45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \cdot 40 = 5,864,443,200$$

ways of choosing 6 from 45.

But order does not matter and we have no repetition, so we are interested in the numbers not as a 6-tuple, but as a set:

$$\{11, 2, 33, 4, 51, 6\} = \{6, 33, 4, 51, 2, 11\}.$$

How many times have we counted the same set?

That is, how many ordered 6-tuples come from the same set?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

So: the actual number of lottery choices is

$$\frac{5,864,443,200}{720} = 8,145,060.$$

Your probability of winning is $\frac{1}{8,145,060}$

$=$ Probability of dying in a plane crash

n choose k

Let's generalise! To choose k items from n:

- If order does matter, there are $P(n, k) = \dfrac{n!}{(n-k)!}$ possibilities.
- We have counted each unordered set $k!$ times.

  So the final answer is $\dfrac{n!}{k!(n-k)!}$.

We write this as $\dbinom{n}{k}$, pronounced "n choose k".

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{k!}$$

n choose k

Example: MATH1064 has 380 students. We use two rooms for the final exam: one room fits 220, and one room fits 160. How many ways can we distribute students across the rooms?

1. 220×160,
2. $\binom{380}{220}$,
3. $\binom{380}{160}$,
4. $\binom{380}{220} \times \binom{380}{160}$

An observation

Lemma

For all $n, k \in \mathbb{Z}$ with $0 \leq k \leq n$:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Why?

- The formula says so:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$$

- Because choosing which k objects to take is the same as choosing which $n-k$ objects to leave behind.

We have two proofs: an algebraic proof and a counting proof!

Case 4: Order does not matter, repetition allowed

Example 1: How many ways are there to put 2 balls (not distinguished) into 3 boxes (distinguished)?

- Why does order not matter?
- Why is repetition allowed?

Case 4: Order does not matter, repetition allowed (ctd)

Example 2: Seven people order a main from a list of four dishes. What are the possible orders given by the waiter to the kitchen?

- Why does order not matter?
- Why is repetition allowed?

This works in general

Choosing k items from a set of n items
such that order does not matter and repetition is allowed

is the same as

counting all possible arrangements of k crosses and $n-1$ bars, and this number is

$$\frac{(k+n-1)!}{k!(n-1)!} = \binom{n+k-1}{n-1}.$$

Summary

There are four ways of choosing k objects from a set of n:

Choose k from n	order matters	order does not matter
repetition allowed	n^k	$\binom{k+n-1}{n-1}$
repetition not allowed	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

In Eurovision:

- There are 16 countries in the first semifinal;
  10 of these will progress to the final.
- There are 17 countries in the second semifinal;
  10 of these will progress to the final.
- 7 additional countries have a guaranteed place in the final.

How many possible running orders are there for the final?

[1] $27!$ [2] $\binom{16}{10} \cdot \binom{17}{10} \cdot 7!$ [3] $\binom{16}{10} \cdot \binom{17}{10} \cdot 27!$

[4] $\binom{16}{10} \cdot \binom{17}{10} \cdot 7! \cdot 10! \cdot 10!$ [5] $\binom{27}{10} \cdot \binom{27}{10} \cdot \binom{27}{7}$ [6] Something else?

What decisions do we need to make?

- Choose which 10 countries progress from the first semifinal: $\binom{16}{10}$ options
- Choose which 10 countries progress from the second semifinal: $\binom{17}{10}$ options
- Now we know exactly which 27 countries are in the final, we must order them: 27! options

The answer: $\binom{16}{10} \cdot \binom{17}{10} \cdot 27! \simeq 1.7 \times 10^{36}$