THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Logical Statements and Proofs – Week 3 Tutorials

MATH1064: Discrete Mathematics for Computing

1. Write the following propositions in symbolic form:

(a) All hungry crocodiles are not amiable.

Solution: Suppose that the universal set is the set of all crocodiles x. Then suppose

$$\begin{array}{lll} H(x) & \text{means} & \text{“}x \text{ is hungry.”} \\ M(x) & \text{means} & \text{“}x \text{ is amiable.”} \end{array}$$

Note that there are different ways of writing the propositions in symbolic form. We will present one of them.

The proposition “all hungry crocodiles are not amiable” can be written in symbolic form as:

$$(\forall x)\big(H(x) \to \neg M(x)\big).$$

(b) Some crocodiles, if not hungry, are amiable.

Solution: The proposition “some crocodiles, when not hungry, are amiable” can be written in symbolic form as:

$$(\exists x)\big(\neg H(x) \to M(x)\big).$$

2. Write the following propositions in symbolic form:

(a) The square of every real number is non negative.

Solution: There are generally many ways of writing a given proposition in symbolic form. Three appropriate answers would be:

$$(\forall x)((x \in \mathbb{R}) \to (x^2 \notin (-\infty, 0)))$$

or

$$(\forall x)((x \in \mathbb{R}) \to (x^2 \geq 0)))$$

or

$$(\forall x)((x \in \mathbb{R}) \to \neg(x^2 \in (-\infty, 0))).$$

(b) There is an x in the set A which is not in the set B.

Solution: The answers would be:

$$(\exists x)((x \in A) \wedge (x \notin B))$$

or

$$(\exists x)((x \in A) \wedge \neg(x \in B)).$$

3. For each of the following arguments: If you think it is valid, give a formal derivation stating in each step the appropriate rule of inference. If you think it is invalid, justify your answer.

(a) A nice day is sufficient for children to play or adults to nap. Adults are napping. Therefore it is a nice day.

Solution: The solution to this question has two parts. In the first part we convince ourselves, that this argument is probably invalid by linking it to a logical fallacy that may have played a prominent role in writing down this argument.

We then proceed to give a formal proof of invalidity by finding truth variables to our propositions satisfying all of the premises but falsifying the conclusion.

Mathematically, we do not need the first part. This is simply a demonstration on how to analyse an argument.

First part: We define propositions as follows:

p = "It is a nice day."

q = "Children are playing."

r = "Adults are napping."

The argument is of the form:

1. $p \to (q \vee r)$
2. r

c. $\therefore\ p$

This can be expanded to show it is a fallacy of affirming the conclusion:

1. $p \to (q \vee r)$
2. r
3. $r \vee q$ (addition on 2.)
4. $q \vee r$ (commutative law on 3.)

c. $\therefore\ p$ (fallacy of affirming the conclusion on 1. and 4.)

Second part: We now have a conjecture that this argument is invalid. If we assign false to p, true to r, and true or false to q, then premises $p \to (q \vee r)$ and r are certainly true, but the conclusion is false. So $1. \wedge 2. \to c.$ is not a tautology and our argument is invalid.

(b) If interest rates fall, then the stock market will rise. The stock market is not rising. Therefore buyers are losing money or interest rates are not falling.

Solution: We define propositions as follows:

p = "Interest rates fall."

q = "The stock market rises."

r = "Buyers are losing money."

The argument is of the form:

1. $p \to q$
2. $\neg q$

c. $\therefore\ r \vee \neg p$

This can be expanded to a correct argument (albeit with spurious information added to shock value) as follows:

1. $p \to q$
2. $\neg q$
3. $\neg p$ (modus tollens on 1. and 2.)
4. $\neg p \vee r$ (addition on 3.)

c. $\therefore\ r \vee \neg p$ (commutative law on 4.)

4. Taking the universal set to be the set $\mathbb{R}$ of all real numbers, determine the truth and falsity of the following propositions.

(a) $(\forall x)((x > 2) \to (x^2 > 4))$

Solution: The sentence is clearly true.

(b) $(\forall x)((x^2 > 4) \to (x > 2))$

Solution: The sentence is false, by taking $x = -3$. Then $x^2 = 9 > 4$ is true, but $x > 2$ is false.

(c) $(\exists x)((x > 2) \to (x^2 > 4))$

Solution: The sentence is true, by taking $x = 3$.

(d) $(\exists x)((x^2 > 4) \to (x > 2))$

Solution: The sentence is true, by taking $x = 3$.

5. Prove the following statement: If x is an odd integer, then $x^2 + 3x + 5$ is odd.

Solution: If x is odd then there exists an integer c such that $x = 2c + 1$. Then

$$x^2 + 3x + 5 = (2c+1)^2 + 3(2c+1) + 5 = 4c^2 + 10c + 9 = 2(2c^2 + 10c + 4) + 1,$$

so $x^2 + 3x + 5$ is odd.

6. Prove the following statement: If n is an odd integer, then there is a unique integer k such that n is the sum of $k-2$ and $k+3$.

Solution: On a piece of scrap paper, we scribble $n = (k-2) + (k+3) = 2k+1$. From this we guess that we can just use the definition of an odd integer and rewrite it! So now on a nice piece of paper:

Let n be an arbitrary odd integer. Then there is an integer k such that $n = 2k+1$. We have $2k+1 = (k+k) + (-2+3) = (k-2) + (k+3)$ using the usual commutativity of addition. This implies $n = (k-2) + (k+3)$. So we have found an integer with the desired property.

Now assume that the integer m also has the property that n is the sum of $m-2$ and $m+3$. Then $(k-2) + (k+3) = n = (m-2) + (m+3)$ implies that $2k+1 = 2m+1$. But then $k = m$. Hence there is a unique integer with this property.

7. Prove that for all real numbers x and y, if $x + y \geq 100$, then $x \geq 50$ or $y \geq 50$.

Solution: We prove the contrapositive. Assume that it is not the case that $x \geq 50$ or $y \geq 50$. Then $x < 50$ and $y < 50$. This implies that $x + y < 50 + 50 = 100$.

8. Prove or disprove: The difference of the squares of any two consecutive integers is odd.

Solution: Suppose a and b are consecutive integers. Without loss of generality we may assume that $b = a + 1$. Now

$$\begin{aligned} a^2 - b^2 &= a^2 - (a+1)^2 \\ &= a^2 - (a^2 + 2a + 1) \\ &= -2a - 1 \\ &= -2a - 2 + 1 \\ &= 2(-a-1) + 1. \end{aligned}$$

Since $-a - 1$ is an integer, this shows that $a^2 - b^2$ is an odd integer.