THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

# Solutions to Probaility – Week 10 Tutorials

MATH1064: Discrete Mathematics for Computing

**1.** A coin is biased so that the probability of heads is 0.7 and the probability of tails is 0.3. Suppose that the coin is tossed ten times and that the results are mutually independent.

(a) What is the probabilty of obtaining exactly seven heads?

***Solution:*** Let $E_7$ be the event that exactly seven heads are obtained. Seven heads can be obtained in 120 different ways in ten tosses (why?). Hence

$$p(E_7) = 120 \cdot (0.7)^7 \cdot (0.3)^3 \cong 0.267 = 26.7\%.$$

(b) What is the probability of obtaining exactly ten heads?

***Solution:*** Let $E_{10}$ be the event that exactly ten heads are obtained. Ten heads can be obtained in only one way in ten tosses. Hence

$$p(E_{10}) = (0.7)^{10} \cong 0.028 = 2.8\%.$$

In contrast, with a fair coin the probability of this event is less than $0.01\%$.

(c) What is the probability of obtaining no heads?

***Solution:*** Let $E_0$ be the event that no heads are obtained. This is the same as the event that only tails are obtained. Hence

$$p(E_0) = (0.3)^{10} \cong 0.000006.$$

(d) What is the probability of obtaining at least one head?

***Solution:*** The event of obtaining at least one head is the complement of the previous event. Hence its probability is $1 - p(E_0) \cong 99.999994$.

**2.** A student taking a multiple-choice exam does not know the answer to two questions. All have five choices for the answer. For one of the two questions, the student can eliminate two answer choices as incorrect but has no idea about the other answer choices. For the other question, the student has no clue about the correct answer at all. Assume that whether the student chooses the correct answer on one of the questions does not affect whether the student chooses the correct answer on the other question.

***Solution:*** Let $E$ be the event that the student answers the first question correctly and $F$ be the event that the student answers the second question correctly. Because two choices can be eliminated on the first question, $p(E) = \frac{1}{3}$ and we have $p(B) = \frac{1}{5}$. So $p(\overline{E}) = \frac{2}{3}$ and $p(\overline{F}) = \frac{4}{5}$.

(a) What is the probability that the student answers both questions correctly?

***Solution:*** $p(E \cap F) = p(E)p(F) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$

(b) What is the probability that the student answers exactly one question correctly?

***Solution:*** We are interested in the probability of the event $(E \cap \overline{F}) \cup (\overline{E} \cap F)$. We have:

$$p((E \cap \overline{F}) \cup (\overline{E} \cap F)) = p(E \cap \overline{F}) + p(\overline{E} \cap F)$$
$$= p(E)p(\overline{F}) + p(\overline{E})p(\cap F) = \frac{1}{3} \cdot \frac{4}{5} + \frac{2}{3} \cdot \frac{1}{5} = \frac{2}{5}.$$

(c) What is the probability that the student answers neither question correctly?

***Solution:*** This is the complement of the previous parts, so
$p(\overline{E} \cap \overline{F}) = 1 - (\frac{1}{15} + \frac{2}{5}) = \frac{8}{15}$
Alternatively, this can be computed directly as in the previous parts.

**3.** A pair of fair dice, one red and the other one green, are rolled. Let $E$ be the event that the number face up on the red die is 2, and let $F$ be the event that the number face up on the green die is 4 or 5. Show that the events $E$ and $F$ are independent.

***Solution:*** We record the outcomes of rolling the dice as $kn$, where the red die shows $k$ and the green die shows $n$. The sample space $S$ is the set of all 36 outcomes. Let $E$ be the event that the red die shows 2 and let $F$ be the event that the green die shows 4 or 5. Then

$$E = \{21, 22, 23, 24, 25, 26\},$$
$$F = \{14, 15, 24, 25, 34, 35, 44, 45, 54, 55, 64, 65\}$$

It follows that $E \cap F = \{24, 25\}$. Since the dice are fair, we have $p(E) = \frac{6}{36} = \frac{1}{6}$, $p(F) = \frac{12}{36} = \frac{1}{3}$ and $p(E \cap F) = \frac{2}{36} = \frac{1}{18}$. Now

$$p(E)p(F) = \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18} = p(E \cap F)$$

and so the events are independent.

**4.** Consider a medical test that screens for a disease found in 5 people in 1,000. Suppose that the false positive rate is 3% and the false negative rate is 1%. Then 99% of the time a person who has the condition tests positive for it, and 97% of the time person who does not have the condition tests negative for it.

***Solution:*** Let $A$ be the event that a person tests positive for the disease, $E$ be the event that a person has the disease; $F$ be the event that a person does not have the disease. Then $p(A \mid E) = 0.99$, $p(\overline{A} \mid E) = 0.01$, $p(A \mid F) = 0.03$, $p(\overline{A} \mid F) = 0.97$. Also, $p(E) = \frac{5}{1000} = 0.005$ and so $p(F) = 0.995$.

(a) What is the probability that a randomly chosen person who tests positive for the disease actually has the disease?

***Solution:*** So by Bayes' theorem $p(E \mid A) = \frac{p(A|E)p(E)}{p(A|E)p(E)+p(A|F)p(F)} = 0.1422...$ so about 14.4%.

(b) What is the probability that a randomly chosen person who tests negative for the disease does not indeed have the disease?

***Solution:*** By Bayes' theorem $p(F \mid \overline{A}) = \frac{p(\overline{A}|F)p(F)}{p(\overline{A}|E)p(E)+p(\overline{A}|F)p(F)} = 0.999948...$ so about 99.995%.

**5.** Suppose $E$ and $F$ are events in a sample space $S$ and that $p(E)$, $p(F)$ and $p(E \mid F)$ are known. Derive a formula for $p(E \mid \overline{F})$ in terms of these.

***Solution:*** Hint: use $E = (E \cap F) \cup (E \cap \overline{F})$.

The answer is $p(E \mid \overline{F}) = \frac{p(E) - p(E|F)p(F)}{1 - p(F)}$.

**6.** One urn contains 12 blue balls and 7 white balls, and a second urn contains 8 blue balls and 19 white balls. An urn is selected at random, and a ball is chosen from the urn.

(a) What is the probability that the chosen ball is blue?

***Solution:*** Let $U_1$ be the event that the first urn is chosen, $U_2$ be the event that the second urn is chosen, and $B$ be the event that the chosen ball is blue. Then $p(B \mid U_1) = \frac{12}{19}$ and $p(B \mid U_2) = \frac{8}{27}$. We have

$$p(B \cap U_1) = p(B \mid U_1)p(U_1) = \frac{12}{19} \cdot \frac{1}{2} = \frac{6}{19}$$

$$p(B \cap U_2) = p(B \mid U_2)p(U_2) = \frac{8}{27} \cdot \frac{1}{2} = \frac{4}{27}$$

Now $B = (B \cap U_1) \cup (B \cap U_2)$ and $(B \cap U_1) \cap (B \cap U_2) = \emptyset$, so

$$p(B) = p(B \cap U_1) + p(B \cap U_2) = \frac{6}{19} + \frac{4}{27} \cong 46.4\%.$$

(b) If the chosen ball is blue, what is the probability that it came from the first urn?

***Solution:*** Given that the ball is blue, the probability that it came from the first urn is $p(U_1 \mid B)$. Applying Bayes' theorem gives:

$$p(U_1 \mid B) = \frac{p(B \mid U_1)p(U_1)}{p(B \mid U_1)p(U_1) + p(B \mid U_2)p(U_2)} = \frac{(6/19)}{6/19 + 4/27} \cong 68.1\%.$$