MATH 1002 LECTURE 4-F

REPRESENTING A SYSTEM OF LINEAR EQUATIONS AS AN AUGMENTED MATRIX

Recall: Theorem Every system of linear equations in $\mathbb{R}$ has either

- 0 solutions ← inconsistent
- exactly one solution
- infinitely many solutions

} consistent.

Example: Consider the following system of 3 linear equations with 3 variables ("unknowns")

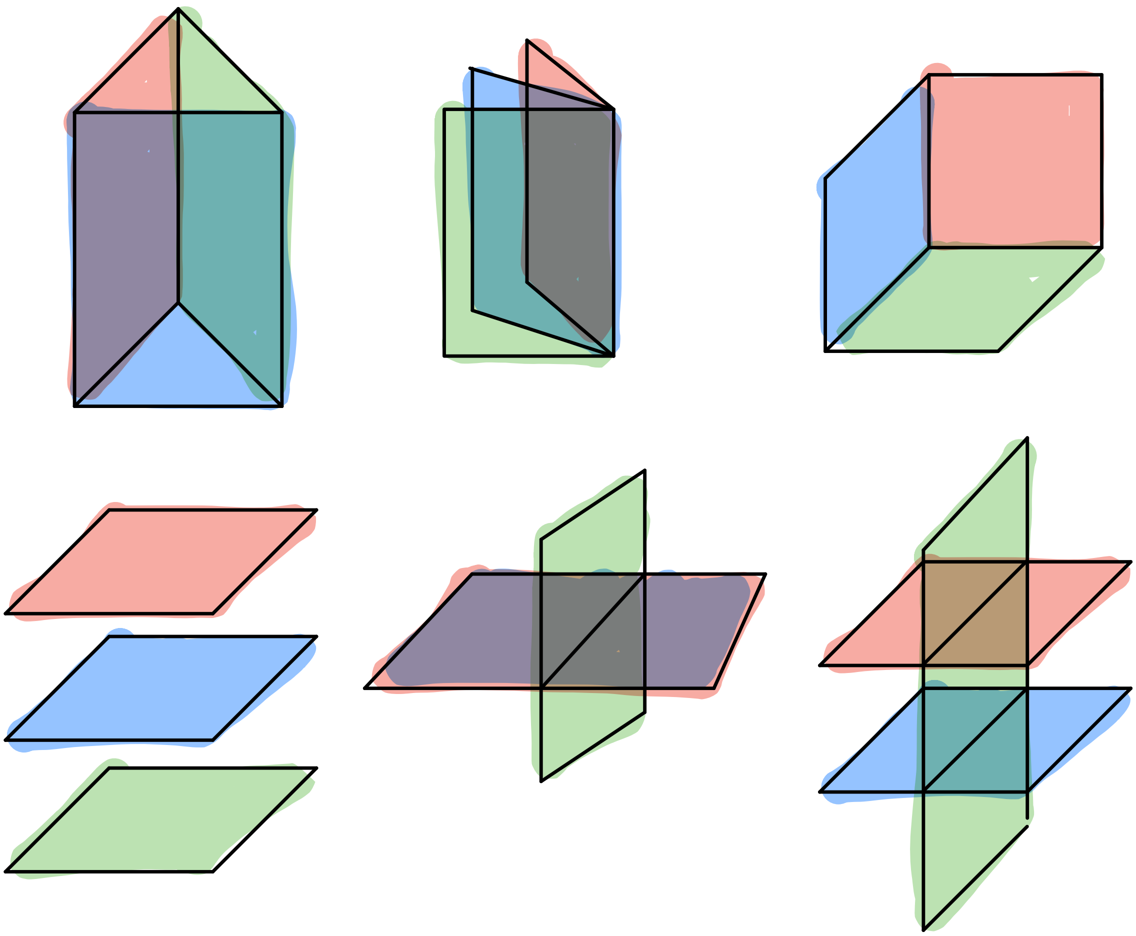
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Each of these equations gives a plane in $\mathbb{R}^3$.

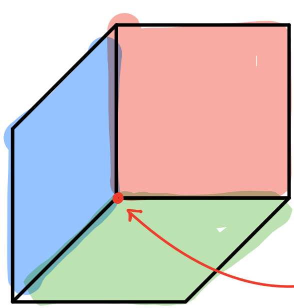
Solutions to the system correspond to points which lie in the intersection of all three planes.

→ We get different numbers of solutions depending on how the planes are arranged with respect to one another.

Exercise: consider the configurations of three planes shown below.
In each case, how many solutions are there?



Solution case 1 : the system has 1 solution



3 planes meet at a single point.

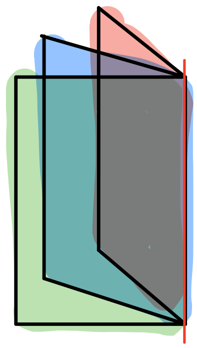
solution is this one point

Case 2 : the system has infinitely many solutions

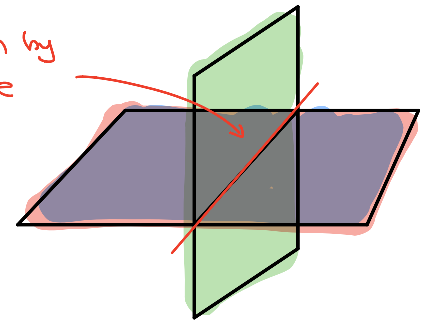
- 3 planes are all the same

  OR

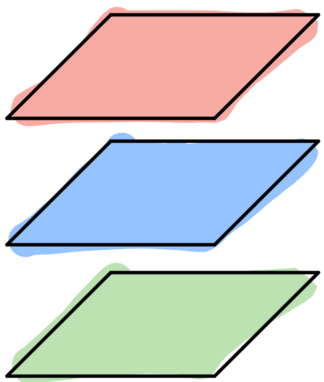
- 3 planes all meet at a common line



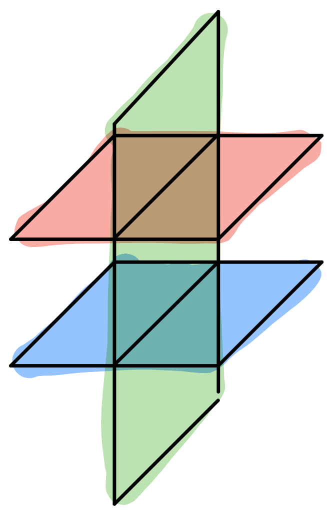
solutions are given by all points on the line



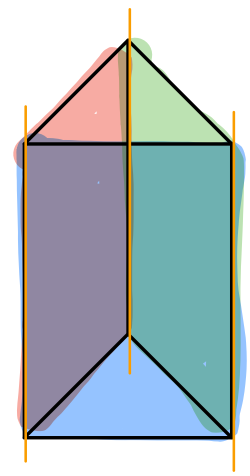
Case 3 : The system has no solutions:



3 parallel planes



2 parallel planes



lines ℓ_1, ℓ_2, ℓ_3 are all parallel

Recall : A system of m linear equations in n variables has the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad\; & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

- It is convenient to not draw all the x's and + and =, and just record the coefficients (a_{ij}) and constant terms

(b_i) in a box, called an augmented matrix

rows

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

columns

- Entry a_{ij} lies in row i and column j
  - First subscript i is for row
  - second subscript j is for column.

- We draw a vertical line separating the right hand side of the system (constant terms) from the system from the coefficients on the left.

- $[\, a_{i1} \quad a_{i2} \quad \cdots \quad a_{in} \mid b_i \,] = R_i$ "i^{th} row of the matrix"
  - corresponds to the i^{th} equation

Example: The following system

$$\begin{aligned} x - y + 3z &= 1 \\ y + 2z &= 4 \\ 5z &= 5 \end{aligned}$$

can be represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 5 \end{array}\right]$$

Exercise: What is a_{13}? What is a_{32}?

Solution: a_{ij} = entry in i^{th} row and j^{th} column.

$a_{13} = 3$ $\qquad a_{32} = 0$.

Example: Suppose that a system of linear equations has the following augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 3 & 6 & 7 & -2 \\ -2 & 0 & 4 & -1 & 3 \\ 7 & 0 & 0 & -5 & -10 \end{array}\right]$$

Exercise: How many variables does this system have? How many equations?

Solution: variables $\longleftrightarrow$ columns before the vertical line

$\hookrightarrow$ four variables x_1, x_2, x_3, x_4

equations $\longleftrightarrow$ rows

$\hookrightarrow$ three equations

Exercise: Write down this system of linear equations in the variables $x_1, x_2, \ldots$.

Solution:

$$\begin{aligned} x_1 + 3x_2 + 6x_3 + 7x_4 &= -2 \\ -2x_1 + 4x_3 - x_4 &= 3 \\ 7x_1 - 5x_4 &= -10. \end{aligned}$$

Next week: We will see how we can use these augmented matrices to solve the system of linear equations.

Here is a preview:

Linear system	Augmented matrix
(1) $x - y = 2$ (2) $2x + 3y = 5$	$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 2 & 3 & 5 \end{array}\right]$
Replace Eq. 2 by (2) − 2×(1)	Replace R_2 by $R_2 - 2R_1$
(1) $x - y = 2$ (2)' $5y = 1$	$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 5 & 1 \end{array}\right]$

solve:

$y = 1/5$

$x = 2 + y = 2 + 1/5 = 11/5$.

Summary of the lecture:

- We saw how systems of three equations in three variables correspond to arrangements of three planes in $\mathbb{R}^3$
  - solutions $\longleftrightarrow$ points of the intersection.

- We saw how systems of linear equations are encoded in augmented matrices
  - rows $\longleftrightarrow$ equations in the system
  - columns before the vertical line $\longleftrightarrow$ variables.

You should be able to:

- Translate between systems of linear equations & augmented matrices
  - identify entry a_{ij}, row R_i.