Discrete Mathematics

MATH1064, Lecture 28

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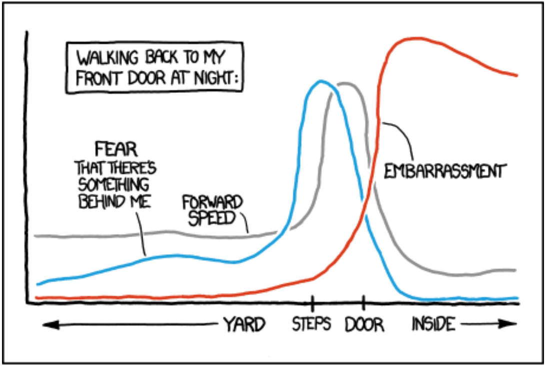
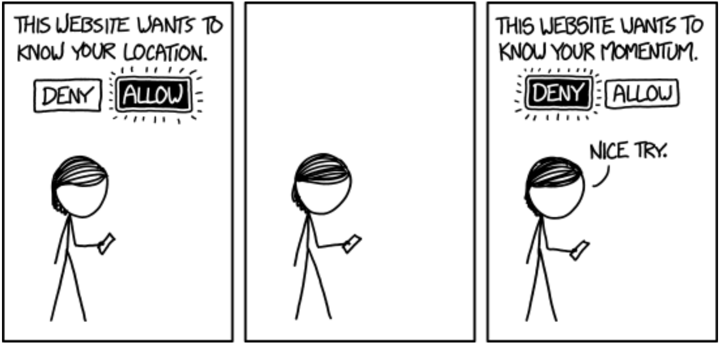


Extra exercises for Lectures 28

Section 9.1: Problems 1–9

Section 9.5: Problems 1–3



Relations

Henri Poincaré (1854–1912)

Mathematicians do not study objects,
but relations between objects.

Colloquial: There is a relation between two things if there is a connection between them.

Relations between people:

1. x has the same hair colour as y
2. x is friends on Facebook with y
3. x is taller than y

A long time ago...

A function $f\colon X \to Y$ is a subset $\Gamma \subseteq X \times Y$ such that,
for each $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in \Gamma$.

Example: For $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$,

$$\Gamma = \{\ (1,1),\ (2,4),\ (3,9),\ (4,16),\ \ldots\}$$

Definition (Relation)

Let X and Y be sets.
A relation R from X to Y is a subset of $X \times Y$.

This is a generalisation of a function.

We write $(x, y) \in R$ also as $x\ R\ y$, or $x \sim y$, and say that x is related to y.

The complementary relation to R is $\overline{R} = (X \times Y) \setminus R$.

If $X = Y$ we say that R is a relation on X.

Usually, a relation is defined implicitly:
Example: Define the relation R on $\mathbb{R}$ by: $x\ R\ y$ if and only if $\lfloor x \rfloor = y$.

True or False?

1. $\pi\ R\ 3$ is
2. $1\ R\ 1$ is
3. $3\ R\ \pi$ is

Three properties

Let R be a relation on the non-empty set X. Then:

1. R is reflexive provided that $(x, x) \in R$ for all $x \in X$.
2. R is symmetric provided that if $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive provided that if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Example: Let X be a non-empty set. Define a relation R on $\mathcal{P}(X)$ by

$$(A, B) \in R \text{ if and only if } A \subseteq B.$$

True or False?

1. R is reflexive is
2. R is symmetric is
3. R is transitive is

Three properties

Let R be a relation on the non-empty set X. Then:

1. R is reflexive provided that $(x, x) \in R$ for all $x \in X$.
2. R is symmetric provided that if $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive provided that if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Example: Consider the relation R on $\mathbb{R}$ defined by

$$(x, y) \in R \text{ if and only if } x < y.$$

True or False?

1. R is reflexive is
2. R is symmetric is
3. R is transitive is

Three properties

Let R be a relation on the non-empty set X. Then:

1. R is reflexive provided that $(x, x) \in R$ for all $x \in X$.
2. R is symmetric provided that if $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive provided that if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Example: Consider the relation R on $\mathbb{Z}$ defined by

$$(m, n) \in R \text{ if and only if } m \mid n.$$

True or False?

1. R is reflexive is
2. R is symmetric is
3. R is transitive is

Three properties

Let R be a relation on the non-empty set X. Then:

1. R is reflexive provided that $(x, x) \in R$ for all $x \in X$.
2. R is symmetric provided that if $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive provided that if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Example: Consider the relation R on $\mathbb{R}$ defined by

$$(m, n) \in R \text{ if and only if } \lfloor m \rfloor = \lceil n \rceil$$

True or False?

1. R is reflexive is
2. R is symmetric is
3. R is transitive is

Three properties

Let R be a relation on the non-empty set X. Then:

1. R is reflexive provided that $(x, x) \in R$ for all $x \in X$.
2. R is symmetric provided that if $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive provided that if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Example: Consider the relation R on $\mathbb{Z}$ defined by

$$(m, n) \in R \text{ if and only if } 3 \mid (m - n).$$

True or False?

1. R is reflexive is
2. R is symmetric is
3. R is transitive is

Combining relations

The relations R_1 and R_2 from X to Y are a subsets of $X \times Y$.

We can use operations on subsets to create new relations:

$R_1 \cup R_2, \;\; R_1 \cap R_2, \;\; R_1 \setminus R_2$

We can compose relations R from X to Y and S from Y to Z to get a new relation $S \circ R$ from X to Z by defining:

$$S \circ R = \{(a, c) \mid \exists b \in Y : aRb \;\wedge bSc\} \subseteq X \times Z.$$

Example: Let X be the set of all people (dead or alive) and the relation R be defined by xRy if x is a parent of y.

What is $R^2 = R \circ R$? A: x is a grandparent of y.

What is $R^3 = R^2 \circ R$? A: x is a great-grandparent of y.

What is $R^4 = R^3 \circ R$? A: x is a great-great-grandparent of y.

Definition (Equivalence relation)

If the relation R on the non-empty set X is reflexive, symmetric and transitive, then R is an equivalence relation on X.

Definition (Equivalence class)

If R is an equivalence relation on X and $x \in X$, then the set

$$[x] = \{y \in X \mid (x, y) \in R\}$$

is the equivalence class of x.

Back to the equivalence relation R on $\mathbb{Z}$ defined by

$$(m, n) \in R \text{ if and only if } 3 \mid (m - n).$$

What are the equivalence classes?

$$(m, n) \in R \text{ if and only if } 3 \mid (m - n)$$

There are three equivalence classes:

$$\{\ldots, -6, -3, 0, 3, 6, \ldots\}$$
$$\{\ldots, -5, -2, 1, 4, 7, \ldots\}$$
$$\{\ldots, -4, -1, 2, 5, 8, \ldots\}$$

We can (if we like) call these [0], [1] and [2].

There is one equivalence class for each remainder modulo 3!

Summary

Definition (Relation)

Let X and Y be sets.
A relation R from X to Y is a subset of $X \times Y$.

- We write $(x, y) \in R$ also as $x\ R\ y$, or $x \sim y$, and say that x is related to y
- The complementary relation to R is $\overline{R} = (X \times Y) \setminus R$
- If $X = Y$ we say that R is a relation on X
- The composition of relations R from X to Y and S from Y to Z is defined by

$$S \circ R = \{(a, c) \mid \exists b \in Y : aRb \ \wedge bSc\} \subseteq X \times Z$$

Exercise

An equivalence relation $R \subseteq X \times X$ on X must be:

1. reflexive: $\forall x \in X$, $(x, x) \in R$
2. symmetric: $(x, y) \in R \to (y, x) \in R$
3. transitive: $(x, y) \in R \wedge (y, z) \in R \to (x, z) \in R$

Consider the following relations R, S on $\mathcal{P}(\mathbb{Z})$.
For any sets $A, B \subseteq \mathbb{Z}$:

- $x \, R \, y$ if and only if there is an injection from A to B
- $x \, S \, y$ if and only if there is a bijection from A to B

Which of these are equivalence relations?