MATH 1002 LECTURE 4-E

SYSTEMS OF LINEAR EQUATIONS

Definition: A linear equation in n variables $x_1, x_2, \ldots x_n$ has the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

where $a_1, \ldots, a_n, b \in \mathbb{R}$ are constants.

- We call the a_i's the coefficients and b the constant term.

Examples

- The general equation for a line in $\mathbb{R}^2$:

$$ax + by = c$$

two variables

- The general equation for a plane in $\mathbb{R}^3$:

$$ax + by + cz = d$$

three variables

Exercise: Which of the following equations are linear equations?

1) $3x + 2y = 7$
2) $x^2 + 2y = 8$
3) $\sin x + \cos y = 4$
4) $\sqrt{3}x + \pi y = \sin \frac{\pi}{4}$
5) $xy + 2z = 0$
6) $5x + y = 3z + 2$.

Solution 1), 4), 6) only

(for 6, rearrange: $5x + y - 3z = 2$)

Remark Here "linear" means adding or scaling.

We are not allowed to multiply variables together, take powers, exponentiate, apply trigonometric functions, etc.

Definition: Let $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ be a linear equation in the variables $x_1, \dots, x_n$.

A **solution** of this linear equation is a sequence of real numbers $(s_1, s_2, \dots, s_n)$ such that substituting $x_i = s_i$ into the LHS of the equation gives $a_1s_1 + \dots + a_ns_n = b$.

Remark/non-example: In the last lecture when we learned about linear independence, we looked for solutions to an equation

$$c_1\vec{x}_1 + \dots + c_k\vec{x}_k = \vec{0}$$

- Here the variables are the c_i
- But $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k, \vec{0}$ are vectors, not scalars (like our constants a_i, b).

⇒ this is a vector equation, not a "linear equation".

Solving a linear equation:

Example: Let's find all solutions to the equation

$$2x - 4y = 6.$$

Observation: This is the general equation of a line in $\mathbb{R}^2$.

So there will be infinitely many solutions.

What does it mean to "find them all"?

Answer: Give the line in parametric form.

Recall that to do this, we can isolate one of the variables : $x = 2y + 3$

and set the other variable equal to the parameter

$y = t$.

Combining, $\begin{cases} x = 2t + 3 \\ y = t \end{cases} \quad t \in \mathbb{R}$

All solutions to the equation are of this form.

Exercise: Solve the linear equation

$$2x + 4y - 6z = 6.$$

Solution: - isolate one variable: $x = 3 - 2y + 3z$

- let the others be parameters: $y = s$, $z = t$, $s, t \in \mathbb{R}$.

- obtain parametric equations

$$\begin{cases} x = 3 - 2s + 3t \\ y = s \\ z = t \end{cases} \qquad s, t \in \mathbb{R}$$

describing all solutions.

Remark: Again, we have infinitely many solutions, in this case forming a plane in $\mathbb{R}^3$.

General pattern:

- Solving one linear equation in 2 variables amounts to describing a line in $\mathbb{R}^2$ in parametric form.

  3 variables $\rightarrow$ plane in $\mathbb{R}^3$

  n variables $\rightarrow$ "hyperplane" in $\mathbb{R}^n$.

- In all of these cases, solving one equation at a time is not difficult.
  - The problem is when we need to solve several equations at once!

Definition: A system of linear equations is a finite set of linear equations, each with the same variables.

More precisely, a system of m linear equations in n variables looks like

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\qquad & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

where the a_{ij} and b_i (for $1 \le i \le m$ and $1 \le j \le n$) are constants in $\mathbb{R}$.

- a_{ij}'s are called coefficients
- b_i's are called constant terms

Note: The first subscript i counts the rows (equations)
The second subscript j counts the columns (variables)

Definition: The system is called homogeneous if all b_i's are 0.

Definition: A solution to the above system of linear equations is a sequence of real numbers $(s_1, s_2, \ldots, s_n)$ which is a solution to each of the equations simultaneously.

i.e. for each row i, we have

$$a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n = b_i.$$

Definition: A system is said to be consistent if it has at least one solution.

Otherwise, it is called inconsistent.

Solving systems of linear equations

Example: Solve the following system:

$$2x + y = 3 \quad (1)$$

$$3x - y = 7 \quad (2).$$

Solution: • Adding (1) & (2): $5x = 10$

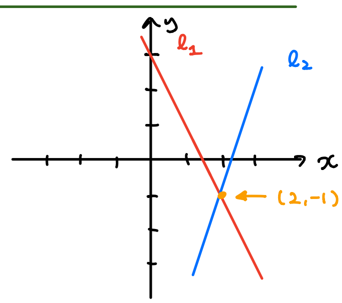
$\Rightarrow x = 2.$

• then (1) becomes $4 + y = 3 \Rightarrow y = -1$

So the system has a unique solution $x = 2$, $y = -1$.

$\Rightarrow$ it is consistent.

Geometric interpretation: (1) & (2) correspond to lines in $\mathbb{R}^2$



• A point lies in $\ell_1 \cap \ell_2$

$\Leftrightarrow$ its coordinates give a solution to both (1) & (2)

In this case $\ell_1 \cap \ell_2 = \{(2,-1)\}$

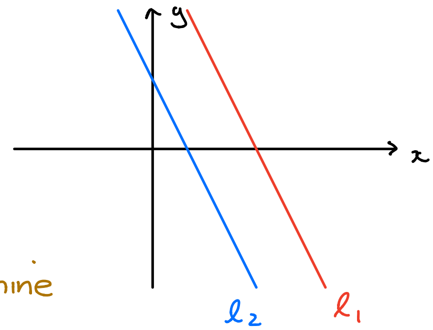
Example:

Solve the following system: $2x + y = 6 \quad (1)$ $2x + y = 2 \quad (2)$

Solution: Consider (1) − (2): $0 = 4$
this is not true!

So the system has no solution
→ it is inconsistent.

Geometric interpretation:



The equations determine two parallel lines.

They have no intersection!

Example:

> Solve the following system:
> $$6x + 3y = 6 \quad (1)$$
> $$2x + y = 2 \quad (2)$$

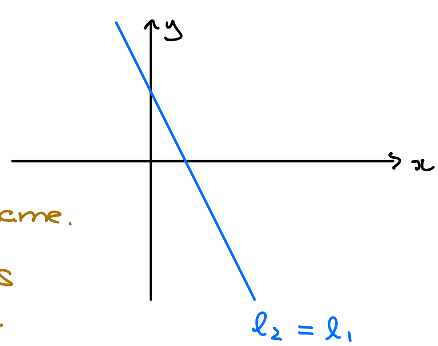
Solution: $3\cdot(2) - (1)$ gives $0 = 0$, which is always true.

Let $x = t$; then $y = 2 - 2x$
$= 2 - 2t$. where $t \in \mathbb{R}$.

So this system has infinitely many solutions;
they are of the form $x = t$, $y = 2 - 2t$ $(t \in \mathbb{R})$.

It is consistent.

Geometric interpretation:



The lines are the same.
So the intersection is the whole line.

<u>Theorem</u>: Every system of linear equations with coefficients in $\mathbb{R}$ has either

- a unique solution
- infinitely many solutions

  ] consistent

- no solution] inconsistent.

<u>Recall</u>: intersections of lines & planes in $\mathbb{R}^3$.

<u>Summary of the lecture</u>:

<u>New terminology</u>:

- (system of) linear equations
- solution to a (system of) linear equations
- coefficients, constant terms
- homogeneous
- consistent, inconsistent.

<u>You should be able to</u>:

- Use these terms
- Not get confused by all the i's and j's.
- Solve small systems by adding & subtracting equations from each other.
- In $\mathbb{R}^2$, $\mathbb{R}^3$: • understand that solutions to linear equations give lines and planes and solutions to systems of linear equations correspond to intersections of these.

<u>Next time</u>: towards an algorithm for solving systems of linear equations