Discrete Mathematics

MATH1064, Lecture 25

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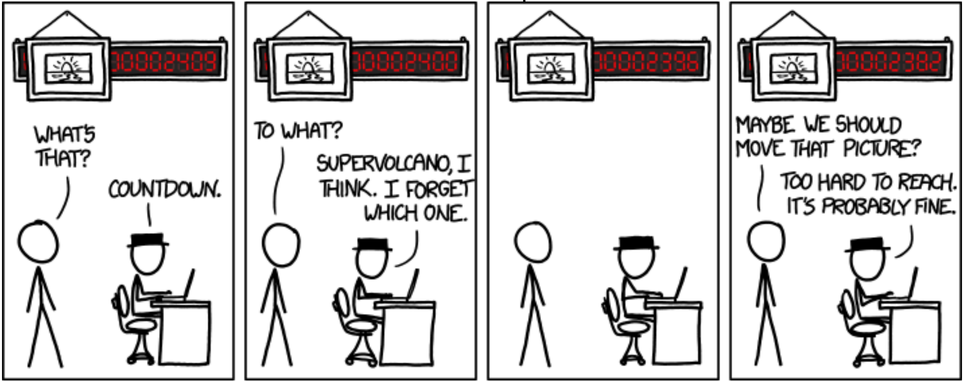
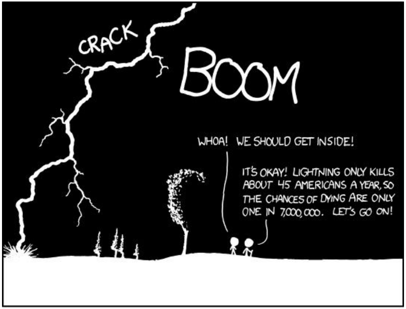


Extra exercises for Lecture 25

Section 7.1: Problems 1–15, 32, 33

Section 7.3: Problems 1–2



THE ANNUAL DEATH RATE AMONG PEOPLE WHO KNOW THAT STATISTIC IS ONE IN SIX.

Way back in Lecture 19

Sample space S: All possible outcomes of a random process or experiment

Coin 1 can show heads or tails, H or T
Coin 2 can show heads or tails, H or T
$S = \{\ (H,H),(H,T),(T,H),(T,T)\}$

Event E: Subset of the sample space: $E \subseteq S$

$E_1 =$ "2 heads" $= \{\ (H,H)\ \}$
$E_2 =$ "1 head and 1 tail" $= \{\ (H,T),(T,H)\ \}$
$E_3 =$ "2 tails" $= \{\ (T,T)\ \}$

Equally likely probability formula for a finite sample space:
If all outcomes in S are equally likely, then:

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}$$

So $P(E_1) = \frac{1}{4}, \quad P(E_2) = \frac{2}{4} = \frac{1}{2}, \quad P(E_3) = \frac{1}{4}.$

More interesting set-up

- Sample space S (also called probability space). Here: $|S| < \infty$.
- $x \in S$ is called an outcome, or (incorrectly) an elementary event
- $E \subseteq S$ is called an event
- Set of events $\mathcal{E}$ is a subset of the power set of S, so $\mathcal{E} \subseteq \mathcal{P}(S)$
- For $x \in S$, $\{x\}$ is called an elementary event

Probability distribution: Assign probabilities to outcomes

$$p : S \to [0,1] \text{ s.t. } \sum_{x \in S} p(x) = 1$$

Extend to events $E \subseteq S$ by adding up probabilities of outcomes of E

Example:

$S =$ "roll of two dice" $= \{(\text{⚀},\text{⚀}), \dots, (\text{⚅},\text{⚅})\}$

What is the probability of the event E that we roll doubles?

$E = \{(\text{⚀},\text{⚀}), (\text{⚁},\text{⚁}), (\text{⚂},\text{⚂}), (\text{⚃},\text{⚃}), (\text{⚄},\text{⚄}), (\text{⚅},\text{⚅})\}$

With a pair of fair dice: $p : S \to [0,1]; \quad x \mapsto \frac{1}{36}$

$p(E) = p((\text{⚀},\text{⚀})) + p((\text{⚁},\text{⚁})) + \dots + p((\text{⚅},\text{⚅})) = 6 \cdot \frac{1}{36} = \frac{1}{6}$

Random variable

We may want to assign values to events.

For example, $(\text{⚁}, \text{⚂}) \mapsto 2 + 3 = 5$ or $(\text{⚁}, \text{⚂}) \mapsto 2 \cdot 3 = 6$

Definition

A random variable is a function $X : S \to \mathbb{R}$ defined on the outcomes of a sample space

Example: S outcomes of the roll of two fair dice, so $|S| = 36$.

$X : S \to \mathbb{R}$, $X(s) =$ "sum of the values of the two dice in outcome $s \in S$"

Then $X(S) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

We have:

$$
\begin{aligned}
2 &= X(\text{⚀}, \text{⚀}) \\
3 &= X(\text{⚀}, \text{⚁}) = X(\text{⚁}, \text{⚀}) \\
4 &= X(\text{⚀}, \text{⚂}) = X(\text{⚁}, \text{⚁}) = X(\text{⚂}, \text{⚀}) \\
5 &= X(\text{⚀}, \text{⚃}) = X(\text{⚁}, \text{⚂}) = X(\text{⚂}, \text{⚁}) = X(\text{⚃}, \text{⚀}) \\
6 &= X(\text{⚀}, \text{⚄}) = X(\text{⚁}, \text{⚃}) = X(\text{⚂}, \text{⚂}) = X(\text{⚃}, \text{⚁}) = X(\text{⚄}, \text{⚀}) \\
7 &= X(\text{⚀}, \text{⚅}) = X(\text{⚁}, \text{⚄}) = X(\text{⚂}, \text{⚃}) = X(\text{⚃}, \text{⚄}) = X(\text{⚄}, \text{⚁}) = X(\text{⚅}, \text{⚀}) \\
8 &= X(\text{⚁}, \text{⚅}) = X(\text{⚂}, \text{⚄}) = X(\text{⚃}, \text{⚃}) = X(\text{⚄}, \text{⚂}) = X(\text{⚅}, \text{⚁}) \\
9 &= X(\text{⚂}, \text{⚅}) = X(\text{⚃}, \text{⚄}) = X(\text{⚄}, \text{⚃}) = X(\text{⚅}, \text{⚂}) \\
10 &= X(\text{⚃}, \text{⚅}) = X(\text{⚄}, \text{⚄}) = X(\text{⚅}, \text{⚃}) \\
11 &= X(\text{⚄}, \text{⚅}) = X(\text{⚅}, \text{⚄}) \\
12 &= X(\text{⚅}, \text{⚅})
\end{aligned}
$$

Random variable

We may want to assign values to events.

For example, (⚁, ⚂) $\mapsto 2 + 3 = 5$ or (⚁, ⚂) $\mapsto 2 \cdot 3 = 6$

Definition

A random variable is a function $X : S \to \mathbb{R}$ defined on the outcomes of a sample space

Example: S outcomes of the roll of two fair dice, so $|S| = 36$.
$X : S \to \mathbb{R}$, $X(s) =$ "sum of the values of the two dice in outcome $s \in S$"
Then $X(S) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.
We have:

$$\begin{array}{llll}
p(X = 2) & = 1/36 & & = p(X = 12) \\
p(X = 3) & = 2 \cdot 1/36 & = 1/18 & = p(X = 11) \\
p(X = 4) & = 3 \cdot 1/36 & = 1/12 & = p(X = 10) \\
p(X = 5) & = 4 \cdot 1/36 & = 1/9 & = p(X = 9) \\
p(X = 6) & = 5 \cdot 1/36 & = 5/36 & = p(X = 8) \\
p(X = 7) & = 6 \cdot 1/36 & = 1/6 &
\end{array}$$

Distribution of random variable

Distribution of random variable: Set of all pairs $(r, p(X = r))$

Can think of the image $X(S) = \{r \mid \exists s \in S(r = X(s))\}$ of the random variable as a new sample space with probability distribution $p(r) = p(X = r)$ because

$$\sum_{r \in X(S)} p(X = r) = 1$$

Example from above:

The sample space is $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

The new probability distribution is $p : S \to [0, 1]$ given by

$$p(2) = p(12) = \tfrac{1}{36}; \qquad p(3) = p(11) = \tfrac{1}{18};$$
$$p(4) = p(10) = \tfrac{1}{12}; \qquad p(5) = p(9) = \tfrac{1}{9};$$
$$p(6) = p(8) = \tfrac{5}{36}; \qquad p(7) = \tfrac{1}{6}$$

Conditional probability

Roll a die with your eyes closed.

Your friend tells you that you've rolled an even number.

Question: What is the probability that you have rolled a six?

Conditional probability

Conditional probability

Let E and F be events with $p(F) > 0$.
The conditional probability of E given F is

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

What is the difference between $p(E \mid F)$ and $p(E \cap F)$?

Consider a biased die with

$p(\epsdice{1}) = 0.1,\ p(\epsdice{2}) = 0.3,\ p(\epsdice{3}) = 0.2,$
$p(\epsdice{4}) = 0.1,\ p(\epsdice{5}) = 0.2,\ p(\epsdice{6}) = 0.1.$

Let F be the event "roll is even number"

This gives a new probability distribution:

$p(\epsdice{1} \mid F) = 0,\ p(\epsdice{2} \mid F) = ?,\ p(\epsdice{3} \mid F) = 0,$
$p(\epsdice{4} \mid F) = ?,\ p(\epsdice{5} \mid F) = 0,\ p(\epsdice{6} \mid F) = ?.$

Use the definition $p(E \mid F)$ and $p(E \cap F)$ to compute

$p(\epsdice{2} \mid F) =$

$p(\epsdice{4} \mid F) =$

$p(\epsdice{6} \mid F) =$

Independence

Assume that $p(E) > 0$ and $p(F) > 0$.

The conditional probability of E given F is $p(E \mid F) = \frac{p(E \cap F)}{p(F)}$.

The following are equivalent:

1. $p(E \mid F) = p(E)$ (or, equivalently, $p(F \mid E) = p(F)$)
2. $p(E \cap F) = p(E)p(F)$

If any of the above hold, E and F are called independent.

Example

Let S be the space of all bit strings of length 4, with equal probability distribution.

$$S = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111,$$
$$1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$$

Let E be the event that a bit string starts with 1.
Then $E = \{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$.

Let F be the event that a bit string contains an even number of 1s.
Then $F = \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}$.

So $p(E) = \frac{8}{16} = \frac{1}{2}$ and $p(F) = \frac{8}{16} = \frac{1}{2}$.

Q: Are E and F independent?
Equivalently, is $p(E \cap F) = p(E)p(F)$?

We have $E \cap F = \{1001, 1010, 1100, 1111\}$.

So $p(E \cap F) = \frac{4}{16} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = p(E) \cdot p(F)$.

A: Yes!

Question

In your local juggling club, $\frac{1}{3}$ of all members are computer scientists. Exactly $\frac{1}{4}$ of all members pass clubs, $\frac{2}{5}$ of these are computer scientists.

1. What is the percentage of computer scientists who pass clubs amongst all members?
2. What is the percentage of computer scientists who pass clubs amongst the computer scientists?
3. What relationships hold between all percentages involved?

$S =$ Juggling club
$E =$ computer scientists in S
$F =$ passes clubs in S

$\frac{1}{3}$ of all members are computer scientists, so $p(E) = \frac{1}{3}$
Exactly $\frac{1}{4}$ of all members pass clubs, so $p(F) = \frac{1}{4}$.
Now $\frac{2}{5}$ of these are computer scientists, so $p(E \mid F) = \frac{2}{5}$.

1. What is the percentage of computer science students who pass clubs amongst all members?
2. What is the percentage of computer science students who pass clubs amongst the computer science students?

1. $p(E \cap F) =$
2. $p(F \mid E) =$

Summary

Sample space S: Finite or at worst countably infinite.

Probability distribution is function $p : S \to [0,1]$ s.t. $\sum_{x \in S} p(x) = 1$

Random variable is function $X : S \to \mathbb{R}$ defined on the outcomes of S.

Distribution of random variable is set of all pairs $(r, p(X = r))$

Conditional probability of E given F is $p(E \mid F) = \frac{p(E \cap F)}{p(F)}$.

Independence of E and F if and only if $p(E \cap F) = p(E)p(F)$