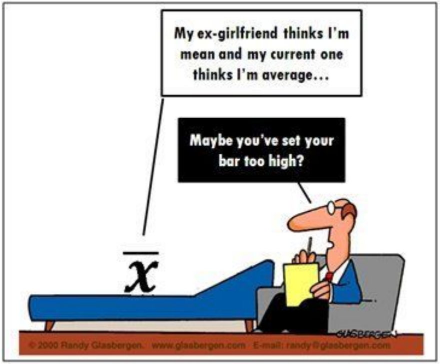
Discrete Mathematics

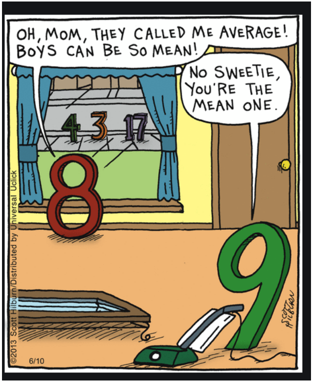
MATH1064, Lecture 27

Jonathan Spreer



Extra exercises for Lecture 27

Section 7.4: Problems 1–8, 27, 28



Question

There are 25 students in your tutorial group. What is the probability, that there exist two students with their birthdays on the same day?

More on random variables

Two very natural questions are:

1. What is the typical value of a random variable?
2. How much are the values of a random variable spread out?

Examples:

- You roll two dice. What is the sum of their values on average?
- How many tails are expected when you flip a coin 100 times?
- 10 students come to my office and want to collect their quiz.
  I give the papers back at random.
  How many students do you expect to receive their quiz?

These properties are studied using the

1. Expected value of a random variable
2. Variance of a random variable

Expected value

Expected value

The expected value, also called the expectation or mean, of a random variable X on the sample space S is equal to

$$E(X) = \sum_{s \in S} p(s) X(s)$$

Example 1: Flipping two coins
$S = \{(H,H),(H,T),(T,H),(T,T)\}$, $p : S \to [0,1]$, $p(s) = \frac{1}{|S|} = \frac{1}{4}$,
$X : S \to \mathbb{R}$, $X(s) =$ "number of heads"

$$\begin{aligned} E(X) &= \sum_{s \in S} p(s) X(s) \\ &= p((H,H)) \cdot 2 + p((T,H)) \cdot 1 + p((H,T)) \cdot 1 + p((T,T)) \cdot 0 \\ &= \frac{1}{4}(2+1+1+0) = 1 \end{aligned}$$

Expected value

Example 2: Throwing two dice.

$S = \{(\text{⚀}, \text{⚀}), \ldots, (\text{⚅}, \text{⚅})\}$,
$p : S \to [0,1],\ p(s) = \frac{1}{|S|} = \frac{1}{36}$,
$X : S \to \mathbb{R},\ X(s) =$ "sum of the values of the two dice in outcome $s \in S$"

$$\begin{aligned}
E(X) &= \sum_{s \in S} p(s) X(s) \\
&= p((\text{⚀}, \text{⚀})) \cdot 2 + p((\text{⚀}, \text{⚁})) \cdot 3 + p((\text{⚁}, \text{⚀})) \cdot 3 + \ldots \\
&= \frac{1}{36} \cdot 2 + \frac{1}{36} \cdot 3 + \frac{1}{36} \cdot 3 + \frac{1}{36} \cdot 4 \ldots \\
&= \frac{1}{36}(2 + 3 + 3 + 4 + 4 + 4 + 5 + 5 + 5 + 5 + \ldots) \\
&= 7
\end{aligned}$$

Expected value

Example 2: Throwing two dice.
$X(S) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,

$$p(X = 1) = \frac{1}{36} = p(X = 12), \qquad p(X = 3) = \frac{2}{36} = p(X = 11)$$
$$p(X = 4) = \frac{3}{36} = p(X = 10), \qquad p(X = 5) = \frac{4}{36} = p(X = 9)$$
$$p(X = 6) = \frac{5}{36} = p(X = 8), \qquad p(X = 7) = \frac{6}{36}$$

$$\begin{aligned} E(X) &= \sum_{r \in X(S)} p(X = r) r \\ &= \frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \cdots = 7 \end{aligned}$$

Variance

Variance

The variance of the random variable X on the sample space S is equal to

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 \ p(s).$$

Example: Flipping two coins
$S = \{(H,H),(H,T),(T,H),(T,T)\}$, $p : S \to [0,1]$, $p(s) = \frac{1}{|S|} = \frac{1}{4}$,
$X : S \to \mathbb{R}$, $X(s) =$ "number of heads", $E(X) = 1$

$$\begin{aligned} V(X) =& \sum_{s \in S} (X(s) - E(X))^2 \ p(s) \\ =& p((H,H)) \cdot (2-1)^2 + p((T,H)) \cdot (1-1)^2 \\ &+ p((H,T)) \cdot (1-1)^2 + p((T,T)) \cdot (0-1)^2 \\ =& \frac{1}{4}(1+0+0+1) = \frac{1}{2} \end{aligned}$$

Useful identities:

Linearity of expectation: X, Y random variables, $a, b \in \mathbb{R}$, then

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n),$$

$$E(aX + b) = aE(X) + b.$$

Variance:

$$V(X) = E(X^2) - E(X)^2 = E((X - E(X))^2).$$

If X and Y are independent, then

$$E(XY) = E(X) \cdot E(Y) \quad \text{and} \quad V(X + Y) = V(X) + V(Y).$$

A complicated example made simple

10 students come to my office and want to collect their quiz.

I give the papers back at random.

How many students do you expect to receive their quiz?

Suppose there are n students and n quizzes.
Let S be the sample space of all $n!$ permutations of $\{1, \ldots, n\}$.
We are interested in the random variable $X : S \to \mathbb{N}$ defined by:

$$X(s) = \text{number of fixed points of } s$$

i.e. the number of students receiving their own quiz.

Define the new random variable $X_k : S \to \{0, 1\}$ by
$X_k(s) = 1$ if s fixes k and $X_k(s) = 0$ if s does not fix k.

Then $X(s) = X_1(s) + X_2(s) + \ldots + X_n(s)$
Hence $E(X) = E(X_1) + \ldots + E(X_n)$.
Now $E(X_k) = p(X_k = 1) = \frac{(n-1)!}{n!} = \frac{1}{n}$.
Hence $E(X) = E(X_1) + \ldots + E(X_n) = \frac{1}{n} + \ldots + \frac{1}{n} = \frac{n}{n} = 1$.

So on average, a permutation has one fixed point.
So on average, one student receives their quiz!

What is the variance?
$V(X) = E(X^2) - E(X)^2$, so we need to work out $E(X^2)$.

$$
\begin{aligned}
E(X^2) &= E\Big(\big(\textstyle\sum_{k=1}^n X_k\big)^2\Big) \\
&= E\Big(\textstyle\sum_{k=1}^n \sum_{j=1}^n X_k X_j\Big) \\
&= \textstyle\sum_{k=1}^n \sum_{j=1}^n E(X_k X_j) \\
&= \textstyle\sum_{k=1}^n E(X_k^2) + 2\sum_{1 \le k < j \le n} E(X_k X_j)
\end{aligned}
$$

Now $X_k^2 = X_k$ hence $E(X_k^2) = E(X_k) = \frac{1}{n}$.

If $k < j$ then $E(X_k X_j) = p(\text{ } s \text{ fixes } j \text{ and } k \text{ }) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$.

So for $n \geq 2$ we have

$$E(X^2) = \frac{n}{n} + \binom{n}{2}\frac{2}{n(n-1)} = 2$$

So the variance is $V(X) = E(X^2) - E(X)^2 = 2 - 1 = 1$.
A random permutation of $n \geq 2$ elements has 1 ± 1 fixed points!

Summary

Sample space S: Finite or at worst countably infinite.

Random variable is function $X : S \to \mathbb{R}$ defined on the events of S.

Distribution of random variable is set of all pairs $(r, p(X = r))$

Expected value is

$$E(X) = \sum_{s \in S} p(s)X(s) = \sum_{r \in X(S)} p(X = r)\ r$$

Deviation at s is $X(s) - E(X)$.

Linearity I: $E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$.

Linearity II: $E(aX + b) = aE(X) + b$.

Exercise

Your student society sells lottery tickets to raise funds.

A ticket has probability 0.93 of losing and 0.07 of winning.

You buy a lottery ticket every day for 30 days.

Q: What is the probability that you win a prize at least once?