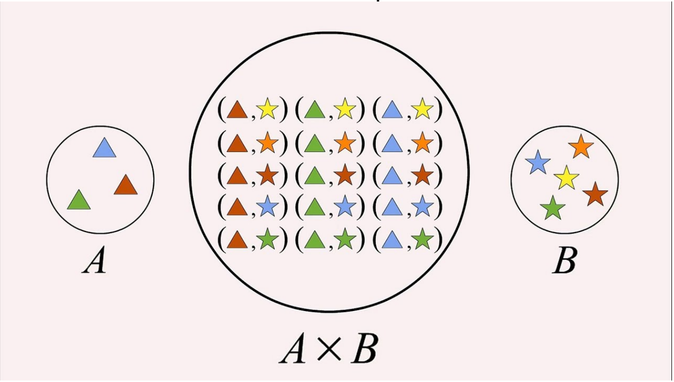
Discrete Mathematics

MATH1064, Lecture 9

Jonathan Spreer



Interval notation

For endpoints $a, b \in \mathbb{R}$ with $a \leq b$:

$$
\begin{aligned}
[a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\
(a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \\
[a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\
[a, \infty) &= \{x \in \mathbb{R} \mid a \leq x\} \\
(a, \infty) &= \{x \in \mathbb{R} \mid a < x\} \\
(-\infty, b] &= \{x \in \mathbb{R} \mid x \leq b\} \\
(-\infty, b) &= \{x \in \mathbb{R} \mid x < b\}
\end{aligned}
$$

All you need to remember is:

- A round bracket means exclude the endpoint
- A square bracket means include the endpoint
- You are never allowed to include ∞ or $-\infty$!

Power Sets

For any set S, the power set of S is the set of all subsets of S.
We write this as $\mathcal{P}(S)$:

$$\mathcal{P}(S) = \{X \mid X \subseteq S\}$$

If $S = \{3, 5\}$:

$$\mathcal{P}(S) = \{\emptyset,\ \{3\},\ \{5\},\ \{3,5\}\}$$

If $S = \{a, b, c\}$:

$$\mathcal{P}(S) = \{\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{b,c\},\ \{a,b,c\}\}$$

We **always** have $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$.

The cardinality of the power set

Suppose S is a finite set with n elements, i.e., $|S| = n$.

What is $|\mathcal{P}(S)|$? That is, how many subsets does S have?

1. n
2. $2n$
3. n^2
4. 2^n
5. $n!$

Cartesian product

For sets, order does not matter: $\{a, b\} = \{b, a\}$

We write (a, b) for an ordered pair.

Here order and repetition do matter:
$(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Example: $(3, 5) \neq (5, 3)$
Example: $(3, 3) \neq 3$

The Cartesian product $A \times B$ of sets A and B is:

$$A \times B = \{\ (a, b) \mid a \in A,\ \ b \in B\}.$$

Example: If $A = \{a, b\}$ and $B = \{2, 3\}$, then

$$A \times B = \{\ (a, 2),\ (a, 3),\ (b, 2),\ (b, 3)\ \}.$$

Question

What is the cardinality of $\{a, b, c\} \times \{d, e\}$?

Hint: Write down the elements!

The cardinality of the Cartesian product

Suppose X and Y are finite sets with $|X| = n$ and $|Y| = m$.

What is $|X \times Y|$?

How do we build an ordered pair $(x, y) \in X \times Y$?

We have n choices for x, and m choices for y.

Therefore we have $n \cdot m$ choices overall!

So $|X \times Y| = n \cdot m$.

If $A = \emptyset$ or $B = \emptyset$, then $A \times B =$

$A \times B = B \times A$ if and only if

We write $A^2 = A \times A$.

One set you know from geometry: the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Bigger Cartesian products

Can generalise from pairs to

– ordered triples (a_1, a_2, a_3)

– ordered quadruples (a_1, a_2, a_3, a_4)

– ordered n–tuples $(a_1, a_2, \ldots, a_n)$

where $a_k \in A_k$ for sets A_1, A_2, A_3, $\ldots$.

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$
$$\text{if and only if}$$
$$a_k = b_k \text{ for each } k \in \{1, \ldots n\}.$$

The set of all ordered n–tuples is denoted $A_1 \times A_2 \times \ldots A_n$.

What is a function?

Let X and Y be sets.

Then f is called a function from X to Y, written $f\colon X \to Y$, if it assigns to each $x \in X$ a unique element $y \in Y$.

In mathematics, "unique" means "one and only one".

We write $y = f(x)$, and sometimes abbreviate this to $x \mapsto y$.

Examples:

$f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$
$g\colon \mathbb{N} \to \mathbb{N}$ defined by $n \mapsto 2^n$
$\sin\colon \mathbb{R} \to \mathbb{R}$

A function is sometimes called a map, or a mapping.

Are these functions?

1. $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$
2. $f\colon \mathbb{N} \to \mathbb{N}$ where $f(n) = n!$
3. $f\colon \mathbb{N} \to \mathbb{N}$ where $f(n) = \frac{(n+1)(n+3)}{3}$
   No, since $f(n) \notin \mathbb{N}$ for some n.
4. $f\colon \mathbb{N} \to \mathbb{N}$ where $f(n) = \frac{(n+1)(n+2)}{2}$
5. $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = 42$
6. $f\colon \mathbb{Q} \to \mathbb{Z}$ where for all $m, n \in \mathbb{Z}$ with $n \neq 0$, $f(m/n) = m$
   No, since each rational has many different expressions m/n.
7. $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $f((x, y)) = |x - y|$
8. $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x)$ is the solution $z \in \mathbb{R}$ to $3x + 1 = z$
9. $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x)$ is the solution $z \in \mathbb{R}$ to $z^2 = x$
   No, since $z^2 = x$ might have two solutions, or none.

A formal definition

We can think of a function as a subset of a Cartesian product:

> A function $f\colon X \to Y$ is a subset $\Gamma \subseteq X \times Y$ such that, for each $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in \Gamma$. We write $f(x)$ to denote this unique y.

Example: For $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$,

$$\Gamma = \{\ (1,1),\ (2,4),\ (3,9),\ (4,16),\ \ldots\}$$

The subset

$$\Gamma = \{\ (x, y) \mid x \in X \ \wedge\ f(x) = y\} \subseteq X \times Y$$

is the graph of the function f. We have

$$\Gamma = \{\ (x, f(x)) \mid x \in X\} \subseteq X \times Y$$

If $f\colon X \to Y$ is a function, then:

1. X is called the domain of f.
2. Y is called the co-domain of f.
3. If $x \in X$, then $f(x)$ is called the image of x.
4. If $A \subseteq X$, then

$$f(A) = \{f(x) \mid x \in A\} = \{y \mid \exists x \in A : f(x) = y\} \subseteq Y$$

   is called the image of A.
   The entire set $f(X)$ is called the range of f.
5. If $y \in Y$, then $f^{-1}(y) = \{x \in X \mid f(x) = y\} \subseteq X$
   is called the preimage of y.
6. If $B \subseteq Y$, then $f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X$
   is called the preimage of B.

Don't confuse f^{-1} with $\frac{1}{f}$!

More functions

1. The assignment $f(x) = x$ for each $x \in X$ defines the identity function $\iota_X \colon X \to X$.

   Note: ι is the Greek letter iota.

2. If $f \colon X \to Y$ is a function, then a function $\mathcal{P}(X) \to \mathcal{P}(Y)$ is defined by $A \mapsto f(A)$.

3. If $f \colon X \to Y$ is a function, then a function $\mathcal{P}(Y) \to \mathcal{P}(X)$ is defined by $B \mapsto f^{-1}(B)$.

Exercise

Which of the following are functions?

- $f\colon \mathbb{N} \to \{a, b, c, \ldots, z\}$ = English alphabet defined by $f(n)$ = first letter in English spelling of n.
- $f\colon \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = \sin(n)$.
- $f\colon \mathbb{N} \to \mathbb{R}$ defined by $f(n) = \sqrt{n}$.