THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Sets – Week 4 Tutorials

MATH1064: Discrete Mathematics for Computing

1. Let $A = \{a,b,c,\{a,d\}\}$. Determine each of the sets A_k and its size $|A_k|$.

(a) $A_1 = A \cup \{b,d,e\}$.

Solution: $A_1 = A \cup \{b,d,e\} = \{a,b,c,d,e,\{a,d\}\}$ and $|A_1| = 6$.

(b) $A_2 = A \cap \{b,d,e\}$.

Solution: $A_2 = A \cap \{b,d,e\} = \{b\}$ and $|A_2| = 1$.

(c) $A_3 = A \setminus \{a,b\}$.

Solution: $A_3 = A \setminus \{a,b\} = \{c,\{a,d\}\}$ and $|A_3| = 2$.

(d) $A_4 = A \setminus \{c,d\}$.

Solution: $A_4 = A \setminus \{c,d\} = \{a,b,\{a,d\}\}$ and $|A_4| = 3$.

(e) $A_5 = A \setminus \{\{a,d\}\}$.

Solution: $A_5 = A \setminus \{\{a,d\}\} = \{a,b,c\}$ and $|A_5| = 3$.

(f) $A_6 = A \setminus \{a,\{a,d\}\}$.

Solution: $A_6 = A \setminus \{a,\{a,d\}\} = \{b,c\}$ and $|A_6| = 2$.

2. List the elements in each of the six sets P, Q, $P \cup Q$, $P \cap Q$, $P \setminus Q$ and $Q \setminus P$, where

$$P = \{x \mid x \in \mathbb{Z} \text{ and } 4 \le x \le 10\},$$
$$Q = \{y \mid y \in \mathbb{Z} \text{ and } \frac{y}{2} \in \mathbb{Z} \text{ and } 0 \le y^2 \le 50\}.$$

Solution: The answers are:

$$P = \{4,5,6,7,8,9,10\}, \quad Q = \{-6,-4,-2,0,2,4,6\},$$
$$P \cup Q = \{-6,-4,-2,0,2,4,5,6,7,8,9,10\}, \quad P \cap Q = \{4,6\},$$
$$P \setminus Q = \{5,7,8,9,10\}, \quad Q \setminus P = \{-6,-4,-2,0,2\}.$$

3. Let $A = \{a, b, c, d\}$. Write down all the subsets of A. How many are there?

Solution: The subsets of A are

$$\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{d\},$$
$$\{a,b\},\ \{a,c\},\ \{a,d\},\ \{b,c\},\ \{b,d\},\ \{c,d\},$$
$$\{a,b,c\},\ \{a,b,d\},\ \{a,c,d\},\ \{b,c,d\},\ \{a,b,c,d\}.$$

There are 16 subsets of A.

4. Can you conclude that $A = B$ if A, B, and C are sets such that

(a) $A \cup C = B \cup C$? (b) $A \cap C = B \cap C$? (c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$?

Solution: No. To see this note that the above equation is satisfied by $A = C$ and $B = \emptyset$.

Solution: No. Set $C = \emptyset$ then the above equation is satisfied for all choices of A and B.

Solution: Yes. To prove this, first note that the statement is symmetric in A and B. It follows that it is enough to show that $A \subseteq B$.

Take an element $a \in A$. We must show that $a \in B$.

Because $A \cup C = B \cup C$ we have $a \in B \cup C$. Hence, a is either in B or in C (or both).

If $a \in B$ we are done.

If $a \in C$, then, by assumption, $a \in A \cap C$ and since $A \cap C = B \cap C$ we have that $a \in B \cap C$. In particular, $a \in B$.

Give a proof if you can conclude that $A = B$. Otherwise, give a counterexample.

5. Let A, B, and C be sets. Show that

(a) $(A \cup B) \subseteq (A \cup B \cup C)$. (b) $(A \cap B \cap C) \subseteq (A \cap B)$. (c) $(A \setminus C) \cap (C \setminus B) = \emptyset$.

Solution: Take $x \in A \cup B$. Hence, $x \in A$ or $x \in B$. In both cases we have that $x \in A \cup B \cup C$.

Solution: Take $x \in A \cap B \cap C$. Hence, $x \in A$ and $x \in B$ and $x \in C$. This means, in particular, that $x \in A \cap B$.

Solution: Suppose that $(A \setminus C) \cap (C \setminus B) \neq \emptyset$. Then there exists an element $x \in (A \setminus C) \cap (C \setminus B)$. Hence $x \in A \setminus C$ and so $x \notin C$. But also $x \in C \setminus B$ and hence $x \in C$. A contradiction.

It follows that $(A \setminus C) \cap (C \setminus B) = \emptyset$.

6. If A and B are subsets of a set X, prove that

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B).$$

Solution: Before we start this proof we need to know a couple of things.

- We need to know that if $x \notin A \cap B$, then either $x \notin A$ or $x \notin B$ (or both). This follows from our laws of logic, using the contrapositive: If both $x \in A$ and $x \in B$, then $x \in A \cap B$.
- We also need to know that if $x \notin A$ then $x \notin A \cap B$. Again, this follows from our laws of logic; since $x \in A \cap B$ means both $x \in A$ and $x \in B$, so if $x \notin A$ then certainly $x \notin A \cap B$.

Proof, Part 1. Suppose that $x \in X \setminus (A \cap B)$.
Then $x \in X$ and $x \notin A \cap B$. Since $x \notin A \cap B$, $x \notin A$ or $x \notin B$.
Suppose that $x \notin A$. Since we know $x \in X$, then $x \in X \setminus A$.
Similarly, if $x \notin B$ then $x \in X \setminus B$.

Since $x \notin A$ or $x \notin B$, at least one of $x \in X \setminus A$, $x \in X \setminus B$ is true.
Thus $x \in (X \setminus A) \cup (X \setminus B)$.

This first part of the proof shows that $X \setminus (A \cap B) \subseteq (X \setminus A) \cup (X \setminus B)$.

Proof, Part 2. Suppose that $x \in (X \setminus A) \cup (X \setminus B)$.
Then $x \in X \setminus A$ or $x \in X \setminus B$.
If $x \in X \setminus A$ then $x \in X$ and $x \notin A$.
Since $x \notin A$, certainly $x \notin A \cap B$, and so $x \in X \setminus (A \cap B)$.
Similarly if $x \in X \setminus B$ then $x \in X \setminus (A \cap B)$.
So in either case $x \in X \setminus (A \cap B)$.

This second part of the proof shows that $(X \setminus A) \cup (X \setminus B) \subseteq X \setminus (A \cap B)$.

The two parts of the proof together show that $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. ∎

This proof is written out with a lot of English words. In advanced work, proofs do tend to be written out with quite a lot of English words; symbols like $\therefore$ are not used. However, the more advanced the work, the bigger the jumps that the reader has to fill in!

7. How many elements does each of these sets have where a and b are distinct elements?

(a) $\mathscr{P}(\{a,b,\{a,b\}\})$ (b) $\mathscr{P}(\{\emptyset,a,\{a\},\{\{a\}\}\})$ (c) $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset)))$

Solution: The set $\{a,b,\{a,b\}\}$ contains three elements and hence $|\mathscr{P}(\{a,b,\{a,b\}\})| = 2^3 = 8$

Solution: The set $\{\emptyset,a,\{a\},\{\{a\}\}\}$ contains four elements (Here the empty set is an *element* of the set. Do not confuse with the empty set as a subset of every set.) and hence $|\mathscr{P}(\{\emptyset,a,\{a\},\{\{a\}\}\})| = 2^4 = 16$

Solution: We have $|\emptyset| = 0$ and $|\mathscr{P}(\emptyset)| = |\{\emptyset\}| = 2^0 = 1$. It follows that $|\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset)))| = 4$.

8. Prove that $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ if and only if $A \subseteq B$.

Solution: If $A \subseteq B$, then every subset of A is a subset of B. But, by the definition of the power set, this is equivalent to saying that every element of $\mathscr{P}(A)$ is an element of $\mathscr{P}(B)$.

To prove the converse we use the contrapositive: If $A \nsubseteq B$, then there exists an element $a \in A$ such that $a \notin B$. It follows that $\{a\} \in \mathscr{P}(A)$ but $\{a\} \notin \mathscr{P}(B)$. In particular $\mathscr{P}(A) \nsubseteq \mathscr{P}(B)$.

9. Let $A = \{a,b,c\}$, $B = \{x,y\}$, and $C = \{0,1\}$. Find

(a) $A \times B \times C$. (b) $C \times B \times A$. (c) $C \times A \times B$. d) $B \times B \times B$.

Solution:

$$\begin{aligned}
A \times B \times C &= \{(d,e,f) \,|\, d \in A, e \in B, f \in C\} \\
&= \{(a,x,0),(a,x,1),(a,y,0),(a,y,1),(b,x,0),(b,x,1), \\
&\qquad (b,y,0),(b,y,1),(c,x,0),(c,x,1),(c,y,0),(c,y,1)\}
\end{aligned}$$

Solution:

$$\begin{aligned} C \times B \times A &= \{(d,e,f) \mid d \in C, e \in B, f \in A\} \\ &= \{(0,x,a),(0,x,b),(0,x,c),(0,y,a),(0,y,b),(0,y,c), \\ &\quad (1,x,a),(1,x,b),(1,x,c),(1,y,a),(1,y,b),(1,y,c)\} \end{aligned}$$

Solution:

$$\begin{aligned} C \times A \times B &= \{(d,e,f) \mid d \in C, e \in A, f \in B\} \\ &= \{(0,a,x),(0,a,y),(0,b,x),(0,b,y),(0,c,x),(0,c,y), \\ &\quad (1,a,x),(1,a,y),(1,b,x),(1,b,y),(1,c,x),(1,c,y)\} \end{aligned}$$

Solution:

$$\begin{aligned} B \times B \times B &= \{(d,e,f) \mid d \in B, e \in B, f \in B\} \\ &= \{(x,x,x),(x,x,y),(x,y,x),(x,y,y), \\ &\quad (y,x,x),(y,x,y),(y,y,x),(y,y,y)\} \end{aligned}$$

10. Show that if $A \subseteq C$ and BD, then $A \times B \subseteq C \times D$.

Solution: We have $A \times B = \{(a,b) | a \in A, b \in B\}$ and $C \times D = \{(c,d) | c \in C, d \in D\}$. Let $(x,y) \in A \times B$. Then $x \in A$ and thus $x \in C$. Similarly, $y \in B$ and thus $y \in D$.

It follows that, by definition, $(x,y) \in C \times D$. Since $(x,y) \in A \times B$ was arbitrary this proves the statement.