Discrete Mathematics

MATH1064, Lecture 29

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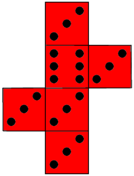
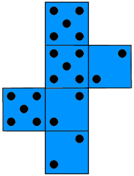


RED



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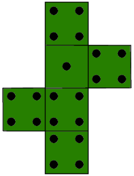
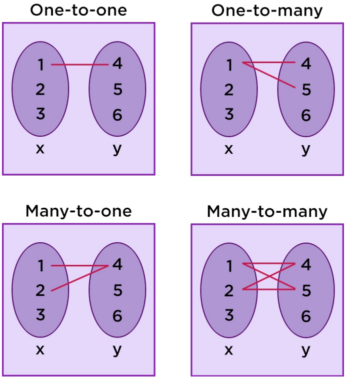


OLIVE

Extra exercises for Lecture 29

Section 9.1: Problems 42, 43

Section 9.5: Problems 5–9



From Lecture 28

Definition (Equivalence class)

If R is an equivalence relation on X and $x \in X$, then the set

$$[x] = \{y \in X \mid (x, y) \in R\}$$

is called the equivalence class of x.

We looked at the equivalence relation R on $\mathbb{Z}$ defined by

$$(m, n) \in R \text{ if and only if } 3 \mid (m - n).$$

There are three equivalence classes:

$$[0] =$$
$$[1] =$$
$$[2] =$$

General facts about equivalence classes

If R is an equivalence relation on the non-empty set X, then:

$$[x] \neq \emptyset \quad \text{for all} \quad x \in X$$

$$X = \bigcup_{x \in X} [x]$$

$$[x] \cap [y] = \begin{cases} \emptyset & \text{if} \quad (x,y) \notin R \\ [x] = [y] & \text{if} \quad (x,y) \in R \end{cases}$$

Conclusion: The equivalence classes partition the set X.

Partitions

If $A \cap B = \emptyset$, then we say that A and B are disjoint.

Definition

A set $\{S_1, S_2, \ldots\}$ is a partition of S if:

1. $S_i \neq \emptyset$ for all i
2. $S = S_1 \cup S_2 \cup \ldots$
3. $S_i \cap S_j = \emptyset$ whenever $i \neq j$

(3) in words:
The sets $S_1, S_2, \ldots$ are pairwise disjoint, or mutually disjoint.

Examples:

Examples of partitions

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Let $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ be the set of all even integers, and $O = \{\ldots, -3, -1, 1, 3, \ldots\}$ be the set of all odd integers.
Then $\{E, O\}$ is a partition of $\mathbb{Z}$.

Why?

Examples of partitions

Definition

A set $\{S_1, S_2, \dots\}$ is a partition of S if:

1. $S_i \neq \emptyset$ for all i
2. $S = S_1 \cup S_2 \cup \dots$
3. $S_i \cap S_j = \emptyset$ whenever $i \neq j$

For each $i = 0, 1, 2, 3, 4$, let $S_i = \{n \in \mathbb{Z} \mid n \equiv i \pmod 5\}$:

$$\begin{aligned}
S_0 &= \{\dots, -10, -5, 0, 5, 10, \dots\} \\
S_1 &= \{\dots, -9, -4, 1, 6, 11, \dots\} \\
S_2 &= \{\dots, -8, -3, 2, 7, 12, \dots\} \\
S_3 &= \{\dots, -7, -2, 3, 8, 13, \dots\} \\
S_4 &= \{\dots, -6, -1, 4, 9, 14, \dots\}
\end{aligned}$$

Then $\{S_0, S_1, S_2, S_3, S_4\}$ is a partition of $\mathbb{Z}$.

Direct proof:

1. To show that $S_i \neq \emptyset$ for each i:
   $0 \in S_0$, $1 \in S_1$, $2 \in S_2$, $3 \in S_3$ and $4 \in S_4$.
2. To show that $\mathbb{Z} = S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$:
   It is clear that $S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \subseteq \mathbb{Z}$, since each $S_i \subseteq \mathbb{Z}$.

   We must now show that $\mathbb{Z} \subseteq S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$.
   Consider any $n \in \mathbb{Z}$. By the quotient-remainder theorem, $n = 5q + r$ for some $r \in \{0, 1, 2, 3, 4\}$.
   So $n \equiv r \pmod 5$, and we have $n \in S_r$ for $r \in \{0, 1, 2, 3, 4\}$.
   Thus $n \in S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$.
3. To show that S_0, S_1, S_2, S_3 and S_4 are mutually disjoint:
   Suppose $n \in S_i \cap S_j$ for some $i \neq j$.
   Then $n \equiv i \equiv j \pmod 5$, and so $5 \mid i - j$.
   But since $-4 \leq i - j \leq 4$, this means that $i - j = 0$.
   Therefore $i = j$, a contradiction.

Indirect proof:

The sets S_0, S_1, S_2, S_3 and S_4 are the equivalence classes of the equivalence relation on $\mathbb{Z}$ defined by

$$n \equiv m \pmod 5$$

or

$$5 \mid (n - m).$$

Hence $\{S_0, S_1, S_2, S_3, S_4\}$ is a partition of $\mathbb{Z}$.

Another example

Consider:

$$\{\ [n, n+1) \mid n \in \mathbb{Z}\} = \{\ \ldots,\ [-2,-1),\ [-1,0),\ [0,1),\ [1,2),\ \ldots\ \}$$

Is this a partition of $\mathbb{R}$?

We saw that: An equivalence relation on X gives a partition of X.
Conversely: A partition of X gives an equivalence relation on X.

Suppose $\{X_1, X_2, \ldots\}$ is a partition of X.
Then the relation R defined by

$$x \; R \; y \iff \exists i \text{ such that } \; x \in X_i \text{ and } y \in X_i$$

is an equivalence relation.

In words:

x is related to y

if and only if

x and y lie in the same set X_i of the partition.

What can we do with equivalence classes?

Consider the relation R on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, defined by

$$(a, b) \; R \; (c, d) \text{ if and only if } ad = bc$$

Is this an equivalence relation? Yes!

What are its equivalence classes?

$$\begin{aligned}
&\{(1,2),\ (2,4),\ (3,6),\ (4,8),\ \ldots\} \\
&\{(1,3),\ (2,6),\ (3,9),\ (4,12),\ \ldots\} \\
&\{(-5,4),\ (-10,8),\ (-15,12),\ldots\}
\end{aligned}$$

We call the set of all equivalence classes $\mathbb{Q}$.
This is how we construct the rational numbers!

Anti-symmetric

Let R be a relation on the set X.

The relation R is symmetric if and only if:

For all $a, b \in X$, $(a, b) \in R$ implies $(b, a) \in R$.

The relation R is anti-symmetric if and only if:

For all $a, b \in X$, $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$.

Which statement is true?

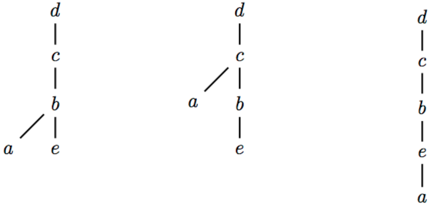
1. "anti-symmetric" $=$ "not symmetric"
2. "anti-symmetric" $\neq$ "not symmetric"

Partial and total orders

A relation R on a set X which is reflexive, transitive, and anti-symmetric is called a partial order on X.

If in addition, for all $a, b \in X$, aRb or bRa, then R is called a total order on X.

Partial vs. total order:



No No Yes

Examples

Example 1: The relation $\leq$ on $\mathbb{R}$

Example 2: Let X be a non-empty set. The relation R on $\mathcal{P}(X)$ defined by

$$(A, B) \in R \text{ if and only if } A \subseteq B$$

is reflexive, transitive and anti-symmetric.

Let's do some counting!

Let $X = \{1, 2, \ldots, n\}$.

1. How many relations can you define on X?
2. How many reflexive relations can you define on X?
3. How many symmetric relations can you define on X?
4. How many anti-symmetric relations can you define on X?

Hint: $R \subseteq X \times X$. For each element of $X \times X$, you need to decide whether it is in R or not!