MATH 1002 LECTURE 8-B

GEOMETRIC MEANING OF THE CROSS PRODUCT: DIRECTION

Recall from last time:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

➽ We know that this vector $\vec{u} \times \vec{v}$ contains two kinds of information

(1) direction

(2) length.

We will show: Both of these can be understood geometrically based on the geometric properties of $\vec{u}$ & $\vec{v}$.

This lecture: What is the direction of $\vec{u} \times \vec{v}$?

Key insights from last time

- $\vec{u} \times \vec{u} = \vec{0}$ $\Rightarrow$ if $\vec{v} = c\vec{u}$ (parallel) then $\vec{u} \times \vec{v} = c(\vec{u} \times \vec{u}) = \vec{0}$.

- $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ & to $\vec{v}$:

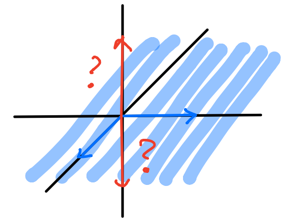
$$(\vec{u} \times \vec{v}) \cdot \vec{u} = 0 \,; \quad (\vec{u} \times \vec{v}) \cdot \vec{v} = 0.$$

We need to consider the case where $\vec{u}, \vec{v}$ are <u>not</u> parallel.

Then the set of all linear combinations of $\vec{u}$ & $\vec{v}$ is a plane in $\mathbb{R}^3$.

Example: The set of linear combinations of $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the set $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$

i.e. the x-y plane:



Any vector perpendicular to both $\vec{u}$ & $\vec{v}$ must point directly out of this plane, so there are only two possible directions.

THE RIGHT-HAND RULE tells us which of these two options is the direction of $\vec{u} \times \vec{v}$.

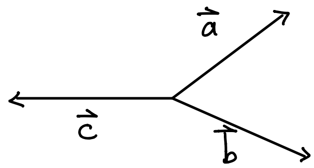
1) Point your right hand with your fingers held straight pointing in the direction of $\vec{u}$.

2) Curl your fingers to point in the direction of $\vec{v}$ (flip your hand over if necessary!)

3) What direction is your thumb pointing?

→ this is the direction of $\vec{u} \times \vec{v}$.

Example: Consider the vectors $\vec{a}, \vec{b}, \vec{c}$ lying flat on the iPad/computer screen.



Then • $\vec{a} \times \vec{b}$ points into the iPad screen.

• $\vec{b} \times \vec{a}$ points out of the screen.

Exercise: What about $\vec{a} \times \vec{c}$?   Out.

Remark: The Right-Hand Rule confirms the fact we showed algebraically: $\vec{a} \times \vec{b}$ & $\vec{b} \times \vec{a}$ point in the opposite direction.

Summary of the lecture

- If $\vec{u}, \vec{v}$ are parallel, $\vec{u} \times \vec{v} = \vec{0}$.
- If not, $\vec{u}, \vec{v}$ span a plane and $\vec{u} \times \vec{v}$ is perpendicular to the plane.

  Its direction is determined by the Right-Hand Rule.

You should be able to:

- use the right-hand rule.

  (look up other examples/drawings on the internet!)

Next lecture: geometric content of $\|\vec{u} \times \vec{v}\|$.