MATH 1002 - LECTURE 1 - F

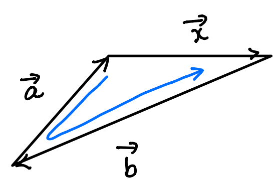
VECTOR ALGEBRA IN $\mathbb{R}^n$

Recall: So far we've seen vector addition and scalar multiplication in $\mathbb{R}^2$ and $\mathbb{R}^3$.

e.g. $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$, $\quad c\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}$

➽ There is no reason to stop at 2 or 3 (as long as we get comfortable with the fact that we can't draw accurate pictures.)

Warm-up:



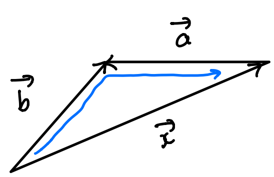
Exercise: Express $\vec{x}$ in terms of $\vec{a}$ & $\vec{b}$

Answer: $\vec{x} = -\vec{a} - \vec{b}$.

Remark: Going around the triangle $\vec{a} + \vec{x} + \vec{b}$, we end up where we started.

$$\Rightarrow \quad \vec{a} + \vec{x} + \vec{b} = \vec{0}$$

$$\text{So} \quad \vec{x} = -\vec{a} - \vec{b}$$



Express $\vec{x}$ in terms of $\vec{a}$ & $\vec{b}$.

Answer: $\vec{x} = \vec{b} - \vec{a}$

Remark: This time, going around the triangle gives
$$\vec{x} + \vec{a} - \vec{b} = \vec{0}$$

$$\Rightarrow \quad \vec{x} = \vec{b} - \vec{a}.$$

➽ We can use techniques from school algebra in vector algebra.

Definition: For n a positive natural number, a vector in $\mathbb{R}^n$ in component form is given by an n-tuple of real numbers:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

- the entries a_i are called the components of $\vec{a}$
  - → a_i is the i^{th} component
- two vectors $\vec{a}$ and $\vec{b}$ are equal if their components are equal: $a_i = b_i$ for $i = 1, 2, \dots, n$.

Vector addition and scalar multiplication are performed component-by-component:

- if $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, then $\vec{a} + \vec{b} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}$
- $c\vec{a} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$; $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

As for $\mathbb{R}^2$ and $\mathbb{R}^3$:

- the negative of a vector $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is

$$-\vec{a} := (-1)\vec{a} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix}.$$

↳ so $\vec{a} + (-\vec{a}) = \vec{0}$. $\boxed{\text{Exercise.}}$

- vector subtraction: $\vec{a} - \vec{b} := \vec{a} + (-\vec{b}) = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{bmatrix}$

- position vector: $A = (a_1, a_2, \dots, a_n)$

  $\Rightarrow \vec{OA} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

- displacement vector: $B = (b_1, b_2, \dots, b_n)$

  $\Rightarrow \vec{AB} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{bmatrix} = \vec{OB} - \vec{OA}$

  $= \vec{AO} + \vec{OB}$

Key properties of vector algebra:

Theorem: Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$, and take $c, d \in \mathbb{R}$.

Then (a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ [commutativity]

(b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ [associativity]

(c) $\vec{u} + \vec{0} = \vec{u}$

(d) $\vec{u} + (-\vec{u}) = \vec{0}$

(e) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

(f) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

$\Big\}$ [distributivity]

(g) $c(d\vec{u}) = (cd)\vec{u}$

(h) $1\vec{u} = \vec{u}$.

Why does this work?

Because addition and multiplication of real numbers satisfy the analogous properties.

Definition: A vector space is a set V equipped with

- addition: if $v, w \in V$, we can form $v+w \in V$.
- scalar multiplication: if $v \in V$ and $c \in \mathbb{R}$ we can form $cv \in V$.

satisfying the above properties.

$\Rightarrow$ So the theorem says that $\mathbb{R}^n$ is a vector space.

Examples: using these properties to simplify vector expressions.

Suppose $\vec{a}, \vec{b}, \vec{x}$ are vectors in $\mathbb{R}^n$.

- Goal: Simplify $3(2\vec{a} - \vec{b}) + (\vec{a} - 3\vec{b})$.

Exercise: label each step with the property used

$3(2\vec{a} - \vec{b}) + (\vec{a} - 3\vec{b})$

$= 3(2\vec{a} + (-1)\vec{b}) + (\vec{a} + (-1)(3\vec{b}))$ definition of subtraction

$= [3\cdot(2\vec{a}) + 3((-1)\vec{b})] + (\vec{a} + (-1)(3\vec{b}))$ distributivity

$= [6\vec{a} + (-3)\vec{b}] + (\vec{a} + (-3)\vec{b})$ property (g)

$= 6\vec{a} + (-3)\vec{b} + \vec{a} + (-3)\vec{b}$ associativity

$= 6\vec{a} + \vec{a} + (-3)\vec{b} + (-3)\vec{b}$ commutativity

$= 6\vec{a} + (1)\vec{a} + (-3)\vec{b} + (-3)\vec{b}$ property (h)

$= (6+1)\vec{a} + ((-3) + (-3))\vec{b}$ distributivity

$= 7\vec{a} + (-6)\vec{b}$ definition of addition in $\mathbb{R}$

$= 7\vec{a} - 6\vec{b}$ definition of subtraction.

! this is too long!

Usually we omit and combine a few steps:

$$3(2\vec{a} - \vec{b}) + (\vec{a} - 3\vec{b}) = 6\vec{a} - 3\vec{b} + \vec{a} - 3\vec{b}$$
$$= 7\vec{a} - 6\vec{b}.$$

- Goal #2 Suppose $\vec{a} + \vec{x} = -2\vec{b} + 2\vec{x}$. Solve for $\vec{x}$.

Using vector space properties

$$\vec{a} + \vec{x} = -2\vec{b} + 2\vec{x}$$
$$\Leftrightarrow \quad \vec{a} + \vec{x} + (-\vec{x}) = -2\vec{b} + 2\vec{x} + (-\vec{x})$$
$$\Leftrightarrow \quad \vec{a} + \vec{0} = -2\vec{b} + \vec{x}$$
$$\Leftrightarrow \quad \vec{a} = -2\vec{b} + \vec{x}$$
$$\Leftrightarrow \quad \vec{a} + 2\vec{b} = -2\vec{b} + \vec{x} + 2\vec{b}$$
$$\Leftrightarrow \quad \vec{a} + 2\vec{b} = \vec{x}$$

In practise, we skip a few steps:

$$\vec{a} + \vec{x} = -2\vec{b} + 2\vec{x}$$
$$\Leftrightarrow \quad \vec{a} = -2\vec{b} + 2\vec{x} - \vec{x} = -2\vec{b} + \vec{x}$$
$$\Leftrightarrow \quad \vec{a} + 2\vec{b} = \vec{x}$$

Summary of the lecture

- Vectors of $\mathbb{R}^n$ are described using columns of numbers $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

- Vector addition and scalar multiplication are performed component-wise.

  ⇒ they satisfy a list of properties making $\mathbb{R}^n$ into a vector space.

You should be able to:

- manipulate vectors in $\mathbb{R}^n$.

  symbolically (using our list of properties)

  and in actual examples.