Discrete Mathematics

MATH1064, Lecture 4

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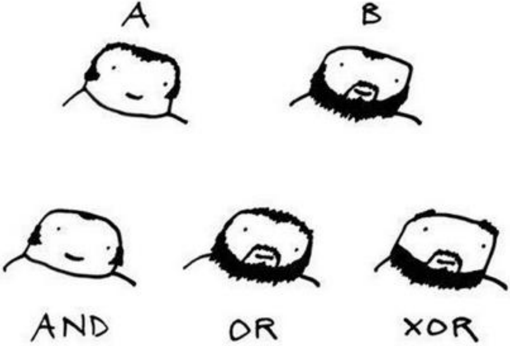
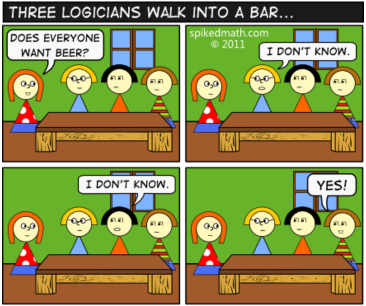


Extra exercises for Lecture 4

Section 1.3: Problems 11, 16–33 (no truth tables this time), 40, 41

Section 1.4: Problems 1–4



The universal quantifier

Five equivalent statements:

Every real number is non-negative or non-positive. (1)

For all $r \in \mathbb{R}$, $r \geq 0$ or $r \leq 0$. (2)

$\forall r \in \mathbb{R},\ r \geq 0 \ \vee \ r \leq 0.$ (3)

$\forall$ is the universal quantifier, pronounced "for all".
Note how we write the domain next to $\forall$: $\forall r \in \mathbb{R}$

Let $Q(r)$ be the predicate "$r \geq 0 \ \vee \ r \leq 0$".
$\forall r \in \mathbb{R},\ Q(r).$ (4)

Let $P(r)$ be the predicate $r \geq 0$, and $N(r)$ be the predicate $r \leq 0$.
$\forall r \in \mathbb{R},\ P(r) \ \vee \ N(r).$ (5)

The existential quantifier

Two equivalent statements:

There is a real number whose square equals 2. (1)

$\exists r \in \mathbb{R}$ such that $r^2 = 2$. (2)

$\exists$ is the existential quantifier, pronounced "there exists".

Even shorter:

$\exists r \in \mathbb{R} : r^2 = 2$. (3)

: is pronounced "such that".

If you want to express that something does NOT exist, you can use $\nexists$, pronounced "there does not exist".

Discussion time!

Which of the following propositions are true?

1. $\forall x \in \mathbb{R},\ x^2 \geq x.$

2. $\exists x \in \mathbb{R}$ such that $x^2 \geq x.$

3. $\forall x \in \mathbb{Z},\ x^2 \geq x$

Negation of quantified statements

Universal statement:

$$\forall x \in D,\ Q(x)$$

The negation of this statement is logically equivalent to:

$$\exists x \in D : \neg Q(x)$$

Existential statement:

$$\exists x \in D : R(x)$$

The negation of this statement is logically equivalent to:

$$\forall x \in D,\ \neg R(x)$$

Multiple quantifiers

Recall: $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

Which of the following sentences says "there is a smallest natural number"?

1. $\exists s \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, $s \leq n$.
2. $\forall n \in \mathbb{N}$, $\exists s \in \mathbb{N}$ such that $s \leq n$.

In words:

1. There is some number s that is less than or equal to every n. s must be a single fixed choice.
2. Every number n has some number s less than or equal to it. You may (if you wish) choose a different s for each n.

Negating multiple quantifiers

$\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R},\ xy = 0$

negates to ...

$\forall x \in \mathbb{R},\ \neg\,(\forall y \in \mathbb{R},\ xy = 0)$,

which is ...

$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}$ such that $\neg(xy = 0)$.

That is:

$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}$ such that $xy \neq 0$.

Remember! When negating, switch $\exists$ and $\forall$.

Some "light entertainment" about quantifiers and basic logic:
Search Trefor Bazett quantifiers on YouTube.

Valid and invalid arguments

Suppose we know that the following premises are true:

- If it is raining, then there must be clouds. $(p \rightarrow q)$
- It is raining. (p)

Can we conclude that there are clouds? $(\therefore q)$

Yes! This is a valid argument form, known as modus ponens (method of affirming).

Suppose we know that the following premises are true:

- If you are in Europe, you are invited to Eurovision. $(p \rightarrow q)$
- Australia has been invited to Eurovision. (q)

Can we conclude that Australia is in Europe? $(\therefore p)$

No! This is an invalid argument form, known as the converse error.

Valid and invalid arguments

An argument form is a sequence of compound propositions.

All but the last proposition form are called premises.
The last compound proposition is called the conclusion, and is sometimes written with a "therefore" sign: $\therefore$

1. $p \rightarrow q$ (premise)
2. p (premise)

c. $\therefore q$ (conclusion)

Definition

An argument form is valid if, whenever all of the premises are true, then the conclusion is true also.

Otherwise the argument form is invalid.

Discuss

Is the following argument valid?

1. $p \to q$ (premise)
2. $\neg p$ (premise)

c. $\therefore\ \neg q$ (conclusion)

1. If it rains then there are clouds. $(p \to q)$
2. It is not raining. $(\neg p)$

c. Therefore there are no clouds. $(\therefore\ \neg q)$

This argument is invalid. It is called the inverse error.
If we make p false but q true, then
all premises are true but the conclusion is false!

Checking validity

You can use a truth table to check if an argument form is valid. To check modus ponens:

1. $p \to q$ (premise)
2. p (premise)

c. $\therefore q$ (conclusion)

p	q	Premise: $p \to q$	Premise: p	Conclusion: q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

In every row where all premises are true, the conclusion is also true. Therefore this argument form is valid.

Checking validity

The second example (the converse error):

1. $p \to q$ (premise)
2. q (premise)

c. $\therefore p$ (conclusion)

p	q	Premise: $p \to q$	Premise: q	Conclusion: p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	F

There is a row where all premises are true but the conclusion is false. Therefore this argument form is invalid.

Remember ...

- To be valid, in every row where the premises are all true, the conclusion must be true.
- It does not matter what happens in rows where some of the premises are false.

Observation

The argument form with premises $p_1, \ldots, p_k$ and conclusion c is valid if and only if

$$p_1 \wedge \ldots \wedge p_k \to c$$

is a tautology!

Rules of inference

Table 1 on page 72 contains several important rules of inference:

- modus ponens and modus tollens (method of denying)
- hypothetical and disjunctive syllogism
- addition (generalisation) and simplification (specialisation)
- conjunction
- resolution

These are argument forms that you will use throughout computer science and mathematics.

You should know these!

More extensive lists of rules of inference and logical fallacies:

- https://en.wikipedia.org/wiki/List_of_rules_of_inference
- https://en.wikipedia.org/wiki/List_of_fallacies

Exercise: Negate the following three sentences

It wouldn't be inaccurate to assume that I couldn't exactly not say that it is or isn't almost partially incorrect.

In every job that has to be done, there is an element of fun.

If $2 \times 2 = -4$, then $2 = -2$.

Another Exercise

Which of the following statements is true?

1. $\forall x \in \mathbb{R},\ x \geq 2 \vee x^2 \geq 4$
2. $\forall x \in \mathbb{R},\ x \geq 2 \wedge x^2 \geq 4$
3. $\forall x \in \mathbb{R},\ x \geq 2 \rightarrow x^2 \geq 4$
4. $\forall x \in \mathbb{R},\ x \geq 2 \leftrightarrow x^2 \geq 4$