THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

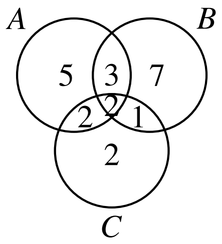
Solutions to Counting II – Week 9 Tutorials

MATH1064: Discrete Mathematics for Computing

1. If possible, compute $|A \cup B \cup C|$ from the given information. If it is not possible, explain why.

(a) $|A| = 12$, $|B| = 13$, $|C| = 7$, $|A \cap B| = 5$, $|A \cap C| = 4$, $|B \cap C| = 3$ and $|A \cap B \cap C| = 2$.

Solution: A complete Venn diagram is as shown:



In this case, we see that it is possible to compute $|A \cup B \cup C|$ and it is given by

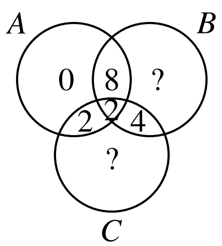
$$\begin{aligned}&|A \cup B \cup C|\\ =&|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|\\ =&12 + 13 + 7 - 5 - 4 - 3 + 2 = 22.\end{aligned}$$

(b) $|A| = 12$, $|B| = 13$, $|C| = 7$, $|A \cap B| = 10$, $|A \cap C| = 7$, $|B \cap C| = 13$ and $|A \cap B \cap C| = 2$.

Solution: Since $|C| = 7$, $|B \cap C|$ is at most equal to 7. But $|B \cap C|$ is given to be 13. This contradiction shows that sets A, B, C as described in the question do not exist. Thus it does not make sense to try to compute $|A \cup B \cup C|$.

(c) $|A| = 12$, $|B| = 13$, $|C| = 7$, $|A \cap B| = 10$, $|A \cap C| = 4$, $|B \cap C| = 6$ and $|A \cap B \cap C| = 2$.

Solution: If you start to draw a Venn diagram you will eventually run into an impossibibility. One approach gives the result shown.



The two question marks show impossibilities emerging. From the diagram we can see that $|B| \geq 14$, but $|B|$ is given to be 13, which is a contradiction. (Similarly we can see that $|C| \geq 8$, but $|C|$ is given to be 7.) Thus sets A, B, C as described in the question do not exist: it does not make sense to try to compute $|A \cup B \cup C|$.

2. In a group of 400 students, 300 are doing mathematics, 250 are doing physics and 200 are doing chemistry. Furthermore, 210 are doing mathematics and physics, 120 are doing mathematics and chemistry and 80 are doing physics and chemistry. Only 40 are doing all three subjects. How many students are not doing any of these subjects?

Solution: Let M be the set of students doing mathematics, P the set of students doing physics and C the set of students doing chemistry. Then from the given information, we see that $|M| = 300$, $|P| = 250$, $|C| = 200$, $|M \cap P| = 210$, $|M \cap C| = 120$, $|P \cap C| = 80$ and $|M \cap P \cap C| = 40$. Thus the number of students doing at least one of mathematics, physics and chemistry is $|M \cup P \cup C|$ which is equal to

$$\begin{aligned}|M| + |P| + |C| - |M \cap P| - |M \cap C| - |P \cap C| + |M \cap P \cap C|\\ = 300 + 250 + 200 - 210 - 120 - 80 + 40 = 380.\end{aligned}$$

Hence the number of students not doing any of these subjects is $400 - 380 = 20$.

3. In a group of 50 participants at a recent international meeting, 30 speak English, 18 speak German, 26 speak French, 9 speak both English and German, 16 speak both English and French, 8 speak both French and German, and 47 speak at least one of English, French or German.

(a) How many people in the group cannot speak English, French or German?

Solution: Since there are 50 participants and 47 can speak at least one of English, French or German, we see that the number of people who cannot speak English, French or German is $50 - 47 = 3$.

(b) How many people in the group can speak all three languages?

Solution: Let E be the set of the participants who speak English, G the set of those who speak German and F the set of those who speak French. Then from the information, we see that $|E| = 30$, $|G| = 18$, $|F| = 26$, $|E \cap G| = 9$, $|E \cap F| = 16$, $|F \cap G| = 8$ and $|E \cup G \cup F| = 47$. We want to find $|E \cap G \cap F|$. Now

$$|E \cup G \cup F| = |E| + |G| + |F| - |E \cap G| - |E \cap F| - |G \cap F| + |E \cap G \cap F|,$$

so that

$$47 = 30 + 18 + 26 - 9 - 16 - 8 + |E \cap G \cap F|.$$

Thus $|E \cap G \cap F| = 6$. Hence the number of people who can speak all three languages is 6.

4. How many numbers between 1 and 100 (inclusive) are divisible by at least one of the numbers 3, 5 or 7?

Solution: Let A be the set of all numbers between 1 and 100 that are divisible by 3, B be the set of all numbers between 1 and 100 that are divisible by 5, and C be the set of all numbers between 1 and 100 that are divisible by 7. Then

$$|A| = \left\lfloor \frac{100}{3} \right\rfloor = 33, \quad |B| = \left\lfloor \frac{100}{5} \right\rfloor = 20 \quad \text{and} \quad |C| = \left\lfloor \frac{100}{7} \right\rfloor = 14$$

(where $\lfloor x \rfloor$ is the greatest integer not exceeding x).

Now $|A\cap B|$ is the set of all numbers between 1 and 100 that are divisible by 3 and 5, i.e. by 15, and so

$$|A\cap B| = \left\lfloor \frac{100}{15} \right\rfloor = 6.$$

Similarly, $|A\cap C|$ is the set of all numbers between 1 and 100 that are divisible by 3 and 7, i.e. by 21, and so

$$|A\cap C| = \left\lfloor \frac{100}{21} \right\rfloor = 4.$$

Also, $|B\cap C|$ is the set of all numbers between 1 and 100 that are divisible by 5 and 7, i.e. by 35, and so

$$|B\cap C| = \left\lfloor \frac{100}{35} \right\rfloor = 2.$$

Moreover, $|A\cap B\cap C|$ is the set of all numbers between 1 and 100 that are divisible by 3, 5 and 7, i.e. by 105, and so

$$|A\cap B\cap C| = \left\lfloor \frac{100}{105} \right\rfloor = 0.$$

Hence the total number of numbers between 1 and 100 that are divisible by at least one of the numbers 3, 5 or 7 is $|A\cup B\cup C|$ which is equal to

$$|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C| = 55.$$

5. Given $A=\{1,2,3,4,5\}$. How many permutations (i.e. bijections) $f:A\to A$ have the property that $f(i)=i$ for at least one value of i?

Solution: For $i=1,2,3,4,5$, let A_i be the set of permutations $f:A\to A$ such that $f(i)=i$. Then $|A_i|=4!$, $(i=1,2,3,4,5)$ $|A_i\cap A_j|=3!$ for $i<j$, $|A_i\cap A_j\cap A_k|=2$ for $i<j<k$, $|A_i\cap A_j\cap A_k\cap A_l|=1$ for $i<j<k<l$, and $|A_1\cap\cdots\cap A_5|=1$. Hence the number of permutations $f:A\to A$ having the property that $f(i)=i$ for at least one value of i is $|A_1\cup A_2\cup A_3\cup A_4|$ which is equal to

$$\sum_{i=1}^{5}|A_i| - \sum_{i<j}|A_i\cap A_j| + \sum_{i<j<k}|A_i\cap A_j\cap A_k| - \sum_{i<j<k<l}|A_i\cap A_j\cap A_k\cap A_l| + |A_1\cap\cdots\cap A_5|$$
$$= 5\times 4! - 10\times 3! + 10\times 2 - 5 + 1 = 76.$$

6. In a soccer match between Teams A and B, the final score is $n-$all. At no time during the match was Team B in the lead. In how many different ways can the A-B scores be built up, starting at 0-0 and ending at n-n? Try the cases $n=1,2,3$ and write down the scoring patterns. Can you guess what the answer might be when $n=4$? Find a connection between the scoring patterns and Problem 1.1.

Solution: The possible scoring patterns for $n=1,2,3$ are as follows: (Left hand numbers represent A's score, right hand numbers represent B's score.)

$n=1$: 0-0,1-0,1-1.

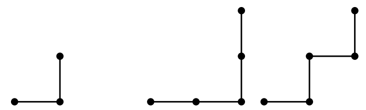
$n=2$: 0-0,1-0,1-1,2-1,2-2 and 0-0,1-0,2-0,2-1,2-2.

$n = 3$: 0-0,1-0,1-1,2-1,3-1,3-2,3-3; 0-0,1-0,1-1,2-1,2-2,3-2,3-3 ;
0-0,1-0,2-0,2-1,2-2,3-2,3-3; 0-0,1-0,2-0,2-1,3-1,3-2,3-3
and 0-0,1-0,2-0,3-0,3-1,3-2,3-3.

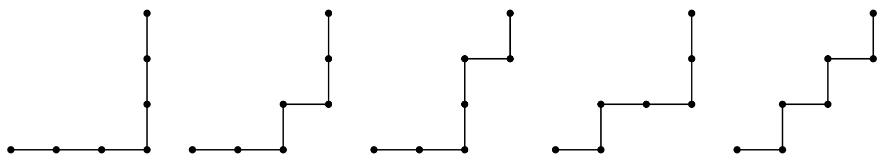
Hence there is 1 scoring pattern when $n = 1$, 2 when $n = 2$, and 5 when $n = 3$.

When $n = 4$ there are 14 scoring patterns, as the number of scoring patterns, for any n, is the Catalan number c_n.

A good way of illustrating the scoring patterns is by plotting a path between points with integer coefficients in the xy plane. For example, for $n = 1, 2, 3$, the scoring patterns are represented as follows:



$n = 1$ $n = 2$



$n = 3$

The connection between this problem and Problem 1.1 is as follows. Associate a left bracket "(" with every unit move to the right, and a right bracket ")" with every unit move upwards.

7. (a) Show that $x_n = (3^{n+1} - 1)/2$ is a solution to the recurrence relation

$$x_{n+1} - x_n = 3^{n+1}.$$

Solution: If $x_n = (3^{n+1} - 1)/2$, then $x_{n+1} = (3^{n+2} - 1)/2$ so that

$$x_{n+1} - x_n = \frac{3^{n+2} - 1}{2} - \frac{3^{n+1} - 1}{2} = \frac{3^{n+2} - 3^{n+1}}{2} = 3^{n+1}.$$

Hence $x_n = (3^{n+1} - 1)/2$ is a solution of the recurrence relation.

(b) Show that $x_n = n!$ is a solution to the recurrence relation

$$x_n - n(n-1)x_{n-2} = 0.$$

Solution: If $x_n = n!$ then $x_{n-2} = (n-2)!$ and so

$$x_n - n(n-1)x_{n-2} = n! - n(n-1)(n-2)! = n! - n! = 0.$$

Hence $x_n = n!$ is a solution to the recurrence relation.

8. Solve the following recurrence relations:

(a) $x_n - 5x_{n-1} + 6x_{n-2} = 0,\ n \geq 2,\ x_0 = 3,\ x_1 = 7$

Solution: The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$. That is, $(\lambda-2)(\lambda-3) = 0$ and so $\lambda = 2$ or $\lambda = 3$. Thus the general solution is

$$x_n = A2^n + B3^n,$$

for some constants A and B. Using the initial conditions $x_0 = 3$ and $x_1 = 7$, we obtain $3 = A + B$ and $7 = 2A + 3B$. Solving yields $A = 2$ and $B = 1$. Hence the solution is

$$x_n = 2 \cdot 2^n + 3^n.$$

(b) $x_n - 8x_{n-1} + 16x_{n-2} = 0,\ n \geq 2,\ x_0 = 3,\ x_1 = 20$.

Solution: The characteristic equation is $\lambda^2 - 8\lambda + 16 = 0$. That is, $(\lambda - 4)^2 = 0$ and so $\lambda = 4$ is a repeated root. Thus the general solution is

$$x_n = A4^n + Bn4^n,$$

for some constants A and B. Using the initial conditions $x_0 = 3$ and $x_1 = 20$, we obtain $3 = A$ and $20 = 4A + 4B$. Then $A = 3$ and $B = 2$. Hence the solution is

$$x_n = 3 \cdot 4^n + 2n4^n = (3 + 2n)4^n.$$

(c) $x_{n+3} - 6x_{n+2} + 11x_{n+1} - 6x_n = 0$, for $n \geq 0$, where $x_0 = 1$, $x_1 = 0$ and $x_2 = -1$.

Solution: The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

Then $(\lambda-1)(\lambda-2)(\lambda-3) = 0$, and so $\lambda = 1$, $\lambda = 2$ or $\lambda = 3$. Thus the general solution to the recurrence relation is

$$x_n = A + B2^n + C3^n,$$

for some constants where A, B and C. Now, using the initial values $x_0 = 1, x_1 = 0, x_2 = -1$, we obtain

$$\begin{aligned} A + \ \ B + \ \ C &= 1 \\ A + 2B + 3C &= 0 \\ A + 4B + 9C &= -1. \end{aligned}$$

Solving this system of linear equations yields $A = \frac{5}{2}$, $B = -2$, and $C = \frac{1}{2}$. Hence the solution is

$$x_n = \frac{5}{2} - 2 \cdot 2^n + \frac{1}{2} \cdot 3^n.$$