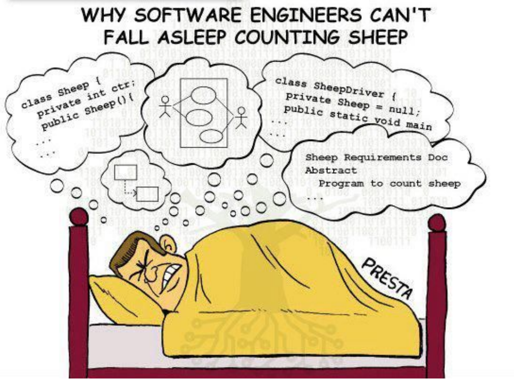
Discrete Mathematics

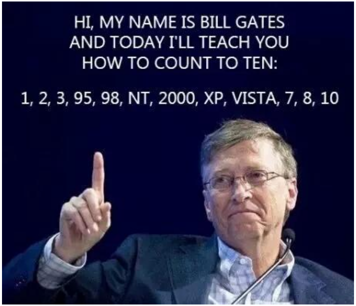
MATH1064, Lecture 19

Jonathan Spreer



Extra exercises for Lecture 20

Section 6.1: Problems 1–17



Counting

A warm-up question (we need to get this out of the way):

Suppose $m, n \in \mathbb{Z}$ and $m \leq n$.
How many integers are there from m to n inclusively?

1. $n - m - 1$
2. $n - m$
3. $n - m + 1$
4. none of the above

Counting

A simple question:
There will be 17 songs in the second Eurovision semifinal.
How many possible running orders could there be?

Counting

A harder question:

- There are 16 countries in the first semifinal;
  10 of these will progress to the final.
- There are 17 countries in the second semifinal;
  10 of these will progress to the final.
- There are 7 additional countries with a guaranteed place
  in the final.

Counting and probability

Part 1:

Imagine you toss two coins. You'll either get

1. 2 heads,
2. 1 head and 1 tail,
3. 2 tails.

Do all three events have the same likelihood of occurring?

Counting and probability

Part 2:

Everyone take two coins and toss them.

For instance here:

`https://www.random.org/coins/?num=2&cur=60-aud.1dollar`

1. 2 heads,
2. 1 head and 1 tail,
3. 2 tails.

Result:

Counting and probability

Sample space S: All possible outcomes of a random process or experiment

Coin 1 can show heads or tails, H_1 or T_1
Coin 2 can show heads or tails, H_2 or T_2
$S = \{ \{H_1, H_2\}, \{H_1, T_2\}, \{T_1, H_2\}, \{T_1, T_2\} \}$

Event E: Subset of the sample space

$E_1 =$ "2 heads" $= \{ \{H_1, H_2\} \}$
$E_2 =$ "1 head and 1 tail" $= \{ \{H_1, T_2\}, \{T_1, H_2\} \}$
$E_3 =$ "2 tails" $= \{ \{T_1, T_2\} \}$

Equally likely probability formula for a finite sample space:
If all outcomes in S are equally likely, then:

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}$$

Conclusion

If you have a finite sample space, and you can count,

then you can determine probabilities!

A common counting problem

Problem

Select k elements from a set containing n elements.

How many ways are there to do the selection?

Before this question can be answered, we need to know:

1. In the selection, does order matter?
2. Is repetition allowed?
   That is, can an element be chosen more than once?

Combining (1) and (2) gives a total of four different cases!

Case 1: Order matters, repetition allowed

Example: We have a new ATM card, and need to select a PIN.

We may choose four digits from the set

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Order matters, and we may choose the same digit multiple times.

Examples: 0000, 0505, 1243.

They really are elements of $S \times S \times S \times S = S^4$:
$(0, 0, 0, 0)$, $(5, 0, 5, 0)$, $(1, 2, 3, 4)$

How many possible PINs are there?

A more general example

Suppose you're in a restaurant.

If there are 3 entrees on the menu, 7 mains and 5 desserts,
how many different three-course meals are there to choose from?

The answer is $3 \cdot 7 \cdot 5 = 105$.

If E is the set of all entrees,

M the set of all mains and

D the set of all desserts,

then the set of all possible three-course meals is

$$E \times M \times D.$$

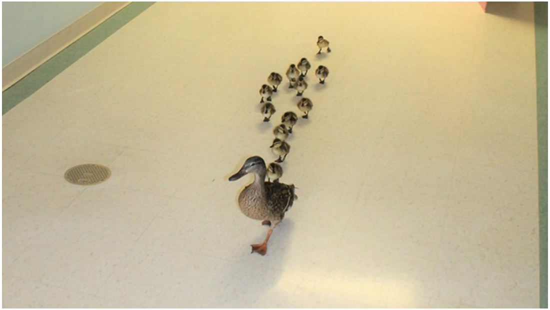
The multiplication rule

If a decision process can be broken down
into a finite number of independent steps,
and there are n_i possible outcomes for the ith step,
then the entire process can be carried out
in $\prod n_i$ possible ways.

In particular: for finite sets $S_1, S_2, \dots, S_k$,

$$|S_1 \times S_2 \times \ldots \times S_k| = \prod |S_i|.$$

Case 2: Order matters, repetition not allowed



Example: Mother duck has five bread crumbs of different sizes to give to her 13 little ducklings. Every duckling gets at most one bread crumb. In how many ways can she distribute the bread crumbs among her ducklings?

Why can't we have multiple selections?

Why does order matter?

Case 2: Order matters, repetition not allowed

Example: Mother duck has five distinct bread crumbs to distribute. She has 13 little ducklings to choose from.

The total number of possibilities is:

Another example

Example: A zoo has 78 distinct animals to look at. A family plans a visit to the zoo. They estimate they can look at around 12 animals during their visit. There is much debate about which animals to watch and in which order. How many distinct trips to the zoo can the family plan?

The total number of possibilities is:

Yet another example

Example: A very modern restaurant offers a special type of three-course meal. Every course must consist of two distinct dishes. No two courses can be identical and the order of the courses matter. There are five dishes to choose from. How many distinct three-course meals are there?

The total number of possibilities is:

Summary (Order matters, repetition not allowed)

If you choose k elements from a set S with n elements,
and order matters and repetition is not allowed,
then there are:

n possibilities for the first choice,
then $n-1$ for the second,
then $n-2$ for the third,
...
then $n-k+1$ for the kth.

So the total number of choices is:

$$n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

We call this number $P(n,k)$. The related sample space is

$$\{(a_1, \dots a_k) \in S^k \mid a_i \neq a_j \text{ if } i \neq j\}.$$