MATH1002 LECTURE 2-A

LINEAR COMBINATIONS OF VECTORS

Last week: • vectors are directed line segments, denoted $\vec{AB}, \vec{OC}$ etc or $\vec{u}, \vec{v}$ etc.

• represented by arrows

• a vector in $\mathbb{R}^n$ is of the form $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = [\underbrace{u_1, \dots, u_n}_{\text{real numbers}}]$

vector addition: $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}$

scalar multiplication: $c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$

Definitions: • the zero vector is $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

• two vectors $\vec{u}, \vec{v}$ are parallel if there is $c \in \mathbb{R}$ such that $\vec{u} = c\vec{v}$ or $v = c\vec{u}$.

> **Exercise:** the zero vector $\vec{0} \in \mathbb{R}^n$ is parallel to every other vector in $\mathbb{R}^n$

Solution: take $\vec{u} = 0$, $\vec{v} \in \mathbb{R}^n$ any vector, $c = 0$.
$\vec{u} = c\vec{v} \Rightarrow \vec{u}$ & $\vec{v}$ are parallel.

Definition: a vector $\vec{v}$ is a linear combination of k vectors $\vec{v}_1, \dots, \vec{v}_k$ if we can find scalars $c_1, c_2, \dots, c_k \in \mathbb{R}$ such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k.$$

• the scalars $c_1, \dots, c_k$ are called the coefficients of the linear combination.

Example # 1

$\vec{v} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$ is a linear combination of $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Indeed, we can write

$$\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Exercise: Can you write $\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$ as a linear combination of $\vec{e}_1$, $\vec{e}_2$ & $\vec{e}_3$?

Solution: Yes.

$$\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

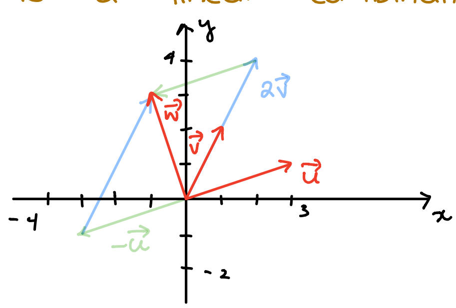
$$= -\vec{e}_1 + 3\vec{e}_2 + \underbrace{0}_{\text{don't need to write this.}}\vec{e}_3.$$

Example # 2:

Let $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Then $\vec{w} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2\vec{v} - \vec{u}$.

So $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$.



Example #3 Show that $\begin{bmatrix}2\\3\end{bmatrix}$ is not a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\end{bmatrix}$.

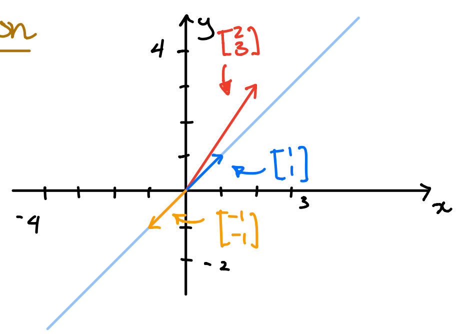
Solution: If it is a linear combination, we can find $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix}2\\3\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}c_1 - c_2\\c_1 - c_2\end{bmatrix}$$

$$\left.\begin{array}{l}\Rightarrow\ 2 = c_1 - c_2\\ \text{and } 3 = c_1 - c_2\end{array}\right\}$$ there are no numbers c_1, c_2 for which both equations hold!

Therefore, $\begin{bmatrix}2\\3\end{bmatrix}$ is not a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}$.

Geometric reason



We see that $\begin{bmatrix}1\\1\end{bmatrix}$ & $\begin{bmatrix}-1\\-1\end{bmatrix}$ are parallel.

$$\begin{bmatrix}-1\\-1\end{bmatrix} = (-1)\begin{bmatrix}1\\1\end{bmatrix}$$

So any linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\end{bmatrix}$ has the form

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\-1\end{bmatrix} = (c_1 - c_2)\begin{bmatrix}1\\1\end{bmatrix},$$

which means it's parallel to $\begin{bmatrix}1\\1\end{bmatrix}$.

Exercise: Is $\begin{bmatrix}1\\-2\\0\end{bmatrix}$ a linear combination of $\begin{bmatrix}4\\1\\3\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$?

Solution: We want to know if we can find $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix}1\\-2\\0\end{bmatrix} = c_1\begin{bmatrix}4\\1\\3\end{bmatrix} + c_2\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}4c_1 + c_2\\ c_1 + c_2\\ 3c_1 + c_2\end{bmatrix}$$

$$\Rightarrow \begin{cases} 1 = 4c_1 + c_2 & = 4c_1 - 3c_1 = c_1 \\ -2 = c_1 + c_2 & = c_1 - 3c_1 = -2c_1 \\ 0 = 3c_1 + c_2 & \leftarrow \Rightarrow c_2 = -3c_1 \end{cases}$$

$\therefore$ we can take $c_1 = 1$, $c_2 = -3$.

Double check: $\begin{bmatrix}4\\1\\3\end{bmatrix} - 3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}4-3\\1-3\\3-3\end{bmatrix} = \begin{bmatrix}1\\-2\\0\end{bmatrix}$ ☺.

Summary of the lecture

A vector $\vec{v}$ is a linear combination of a collection of vectors $\vec{v}_1, \ldots, \vec{v}_k$ if we can find numbers $c_1, c_2, \ldots, c_k \in \mathbb{R}$ (the coefficients of the linear combination) such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

You should be able to:

- Identify whether or not one vector is a linear combination of some other vectors
- If yes, find coefficients.
  Often requires setting up and solving a system of linear equations.
- If no, prove it.
  by showing the system has no solution.