MATH 1002 LECTURE 4-B

PLANES IN $\mathbb{R}^3$: VECTOR FORM AND PARAMETRIC EQUATIONS

Recall from last week

- In order to specify a line ℓ we need to give one of the following sets of information

  - In $\mathbb{R}^2$: a point $P \in \ell$ + a normal vector $\vec{n}$.

  OR

  - In $\mathbb{R}^n$: a point $P \in \ell$ + a direction vector $\vec{d}$

  OR

  - In $\mathbb{R}^n$: two points $P, Q \in \ell$.

Recall from last time:

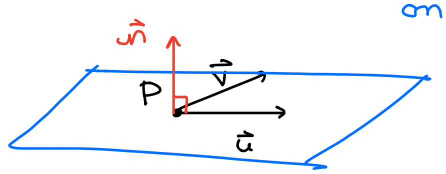
- In order to specify a plane $\mathcal{P}$ in $\mathbb{R}^3$, we can specify
  - a point $P \in \mathcal{P}$
  - a vector $\vec{n}$ normal to $\mathcal{P}$.

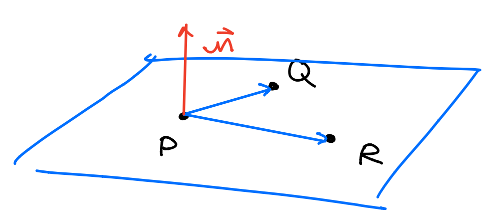
Exercise: What other pieces of information are enough to determine a plane?

Possible answers (A) a point $P \in \mathcal{P}$ and two direction vectors $\vec{u}$ & $\vec{v}$ (not parallel!)

OR

(B) three points $P, Q, R \in \mathcal{P}$ ← they must not all lie on the same line



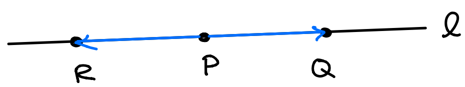


Definition: • Given three points $P, Q, R \in \mathbb{R}^n$, if there is a line ℓ which passes through all three of them, we say that P, Q, R are collinear

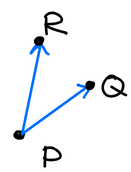
• If there is no such line, we say that P, Q, R are non-collinear.

> **Exercise** P, Q, R are collinear $\Leftrightarrow$ $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel.

proof



parallel, collinear



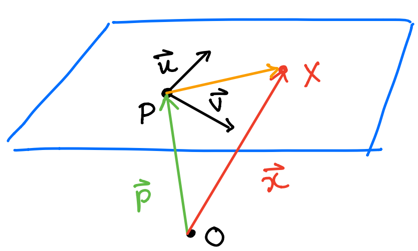
non-parallel, non-collinear.

(Try to come up with an algebraic proof on your own.)

So if we are in situation (B) where we have 3 non-collinear points P, Q, R, we can reduce to situation (A) by taking $P \in \mathcal{P}$ and letting $\vec{u} = \overrightarrow{PQ}$, $\vec{v} = \overrightarrow{PR}$.

Goal: Given the data of $P \in \mathcal{P}$ and $\vec{u}, \vec{v}$ non-parallel direction vectors parallel to the plane $\mathcal{P}$, write down an equation for $\mathcal{P}$.

Idea



Given any point $X \in \mathcal{P}$, we can always find $s, t \in \mathbb{R}$ such that

$$\overrightarrow{PX} = s\vec{u} + t\vec{v}.$$

- We can write $\overrightarrow{PX} = \overrightarrow{OX} - \overrightarrow{OP} = \vec{x} - \vec{p}$.
- So we get an equation for $\vec{x}$:

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}, \quad s,t \in \mathbb{R}$$

! Important!

Each choice of s,t gives a single point in $\mathbb{R}^3$.

By allowing s,t to vary, we get the whole plane $\mathcal{P}$

Definition: This equation is called a vector form for the plane $\mathcal{P}$ passing through the point P, with two direction vectors $\vec{u}, \vec{v}$.

- Recall that $\vec{u}, \vec{v}$ must be non-zero non-parallel vectors which are parallel to $\mathcal{P}$.

- We can expand the vector equation into three linear equations:

- Suppose that $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $P = (p_1, p_2, p_3)$.

Writing $X = (x, y, z)$ so $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the vector form becomes:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + s \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad s,t \in \mathbb{R}.$$

- Equating components, we get

$$\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases}, \quad s,t \in \mathbb{R}.$$

Definition: These equations are called parametric equations for the plane $\mathcal{P}$ passing through the point $P = (P_1, P_2, P_3)$ with two direction vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

Example: Find a vector form and parametric equations for the plane $\mathcal{P}$ passing through the points $P = (2,1,0)$, $Q = (-1,1,2)$, $R = (4,0,2)$.

Solution: We can take $P \in \mathcal{P}$, and as direction vectors $\vec{u} = \vec{PQ} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v} = \vec{PR} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

(Note that $\vec{u}$ & $\vec{v}$ are not parallel; P, Q, R are non-collinear)

> **Exercise:** Write down the corresponding vector form and parametric equations for $\mathcal{P}$.

Solution: Vector form:

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v} \qquad s,t \in \mathbb{R}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}. \qquad s,t \in \mathbb{R}$$

Parametric equations:

$$\begin{cases} x = 2 - 3s + 2t \\ y = 1 - t \\ z = 2s + 2t \end{cases}, s,t \in \mathbb{R}.$$

Exercise: Use these equations to write down two points in $\mathcal{P}$ (other than P, Q, R).

Solution: plug in values of s & t:

$s=1, t=0 \rightsquigarrow Q$

$s=0, t=1 \rightsquigarrow R$

Many other examples, such as

$s=1, t=1 \rightsquigarrow x=1, \; y=0, \; z=4$

$X = (1,0,4) \in \mathcal{P}.$

Exercise: Use these equations to determine whether the point $X = (-1,1,0) \in \mathcal{P}$.

Solution: We ask whether we can find $s,t \in \mathbb{R}$ so that

$$
\begin{aligned}
2 - 3s + 2t &= -1 \quad (1)\\
1 - t &= 1 \quad (2)\\
2s + 2t &= 0 \quad (3)
\end{aligned}
$$

$(3) \Rightarrow s = -t$

$(2) \Rightarrow t = 0$

then (1) becomes

$2 - 3(0) + 2(0) = -1$

$2 = -1$

which is not true!

$\therefore$ there is no solution, and $X \notin \mathcal{P}$. $\square$.

Remark: This is easier to check using the normal form.

Next time: Translating between normal/general & vector/parametric forms.

Summary of the lecture

- In order to specify a plane, we can give

  1) 3 non-collinear points $P, Q, R \in \mathcal{P}$

     OR

  2) 1 point $P \in \mathcal{P}$ & two non-parallel direction vectors $\vec{u}, \vec{v}$.

- Given the data of (2) $(P, \vec{u}, \vec{v})$, we can write down the

  vector equation : $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$, $s,t \in \mathbb{R}$

  or

  parametric equations
$$\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases} \quad s,t \in \mathbb{R}.$$

You should be able to:

- Given the data of (1) or (2), write down vector or parametric equations.

- Given vector or parametric equations, write down the data (1) or (2).