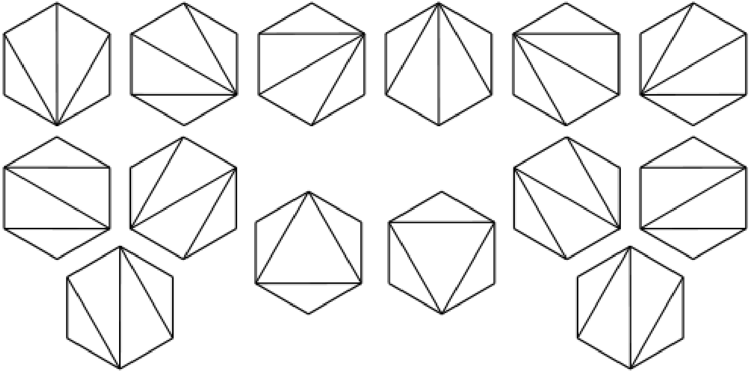
Discrete Mathematics

MATH1064, Lecture 23

Jonathan Spreer



Balanced strings

Let's count balanced strings of n left-brackets and n right-brackets.

- string means a sequence of brackets where order matters

  e.g.))((and ()() are different strings
- balanced means that in each initial substring, there are no more ")" than there are "("

  e.g.))((is not balanced, and ()() is balanced

Let b_n be the number of balanced strings on n left-brackets and n right-brackets.

Monotonic lattice paths

Let's count special paths on an $n \times n$ grid going from the bottom left corner to the top right corner. Namely, paths that only step right or up, and stay below the diagonal.

Let p_n be the number of such paths.

A bijection

Each monotonic lattice path is a sequence of n R's and n U's such that in each initial subsequence there are no more U's than R's (since otherwise the path would cross the diagonal).

We therefore have a well-defined map from monotonic lattice paths to balanced strings defined by

$$R \mapsto \ (\qquad \text{and} \qquad U \mapsto \ )$$

This map is bijective, with inverse defined by

$$(\ \mapsto R \qquad \text{and} \qquad) \ \mapsto U$$

e.g. $RURRUU \leftrightarrow ()(())$.

Hence $b_n = p_n$ for all $n \geq 0$.

Can we find a recurrence relation for this sequence? A closed form?

A recurrence relation

$$b_{n+1} = b_0 b_n + b_1 b_{n-1} + b_2 b_{n-2} + \ldots + b_n b_0 = \sum_{k=0}^{n} b_k b_{n-k}$$

Examples:

$$b_2 = b_0 b_1 + b_1 b_0 = 1 \cdot 1 + 1 \cdot 1 = 2$$
$$b_3 = b_0 b_2 + b_1 b_1 + b_2 b_0 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$$

How do you prove this?

Hint:

- Which bracket pairs up with the first "("?
- How can you use these two to organise the structure?

Rooted binary trees

Let's count rooted binary trees with exactly n vertices.

A closed formula

We have

$$b_n = p_n = t_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

You can find six proofs on:

`https://en.wikipedia.org/wiki/Catalan_number`

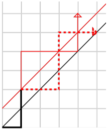
The second and third proofs are accessible with what you have learned so far!

A proof

We show that $p_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$ by André's reflection method

1. Count monotonic paths from $(0,0)$ to (n,n) on the $n \times n$ grid
2. n rightward, n upward steps $\rightarrow \binom{2n}{n}$ monotonic paths of this type
3. Good paths: paths staying below the diagonal
4. Bad paths: paths not staying below the diagonal
5. Idea: count the number of bad paths
6. Bad paths cross the main diagonal and touch the next higher diagonal
7. Good paths don't

A proof



1. Idea: For every bad path: flip accross the red diagonal everything after first "violation"
2. Portion before reflection goes upward one more than rightward
3. Flipped portion of path goes upward one more than rightward
4. $\rightarrow$ flipped path goes upward two more than rightward
5. $\rightarrow$ $n+1$ upward steps and $n-1$ rightward steps
6. Why? Flipped path goes from $(0,0)$ to $(n-1, n+1)$
7. Key observation: Monotonic paths in $(n-1) \times (n+1)$ grid are in bijection to bad paths in $n \times n$

A proof

1. Number of monotonic paths in $(n-1) \times (n+1)$ grid: $\binom{2n}{n-1}$
2. $\rightarrow$ number of bad paths in $n \times n$ grid: $\binom{2n}{n-1}$
3. $\rightarrow p_n = \binom{2n}{n} - \binom{2n}{n-1}$

$$\begin{aligned}
\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n! \cdot n!} - \frac{(2n)!}{(n+1)! \cdot (n-1)!} \\
&= \frac{(n+1) \cdot (2n)! - n \cdot (2n)!}{(n+1) \cdot n! \cdot n!} \\
&= \frac{(2n!)}{(n+1) \cdot n! \cdot n!} \\
&= \frac{1}{n+1} \binom{2n}{n}
\end{aligned}$$

One more thing the Catalan numbers count

Triangulations of the $(n+2)$-gon.

Summary and Exercise

We have

$$b_n = p_n = t_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

You can find five other proofs on:

https://en.wikipedia.org/wiki/Catalan_number

Exercise: Prove that $C_n < 4^n$ for $n \geq 1$.

The Catalan numbers on OEIS: http://oeis.org/A000108