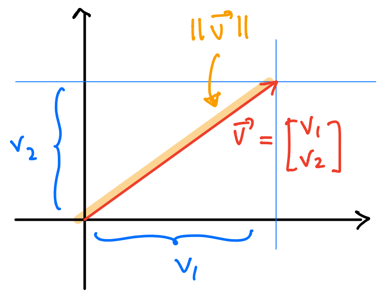
MATH1002 LECTURE 2-C

DOT PRODUCT & LENGTH

From last time: Given $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$, their dot product

is $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \in \mathbb{R}$.

Length of a vector in $\mathbb{R}^2$:



In $\mathbb{R}^2$, Pythagoras' theorem tells us that the length of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

is $\|\vec{v}\| := \sqrt{v_1^2 + v_2^2}$

Remark: $= \sqrt{\vec{v} \cdot \vec{v}}$

In $\mathbb{R}^n$:

Definition The length (or norm) of a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is

defined to be

$$\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}}$$

$$= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Remark: we already checked that $\vec{v} \cdot \vec{v} \geq 0$, so this makes sense.

Example

Exercise: Find the length of $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

Solution: $\|\vec{v}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$.

Properties of length

Theorem: For any vector $\vec{v}$ and scalar c, we have

(a) $\|\vec{v}\| \geq 0$

(b) $\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$ $\leftarrow$ because $\vec{v}\cdot\vec{v} = 0 \iff \vec{v} = \vec{0}$

(c) $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

Exercise: Use properties of the dot product to prove (c).

Solution: By definition,

$$\|c\vec{v}\| = \sqrt{(c\vec{v})\cdot(c\vec{v})} = \sqrt{c^2\,\vec{v}\cdot\vec{v}} = \sqrt{c^2}\,\sqrt{\vec{v}\cdot\vec{v}} = |c|\cdot\|\vec{v}\|. \quad \square$$

Rmk: This is what we expect from our definition of scalar multiplication.

Theorem (THE CAUCHY-SCHWARTZ INEQUALITY)

For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|.$$

proof Note that the inequality is clearly true if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

So assume $\vec{v} \neq \vec{0}$.

Exercise: For $\lambda \in \mathbb{R}$, show that

$$\|\vec{u} - \lambda\vec{v}\|^2 = \|\vec{u}\|^2 - 2\lambda\,\vec{u}\cdot\vec{v} + \lambda^2\|\vec{v}\|^2$$

Solution:

$$\begin{aligned}\|\vec{u} - \lambda\vec{v}\|^2 &= (\vec{u} - \lambda\vec{v})\cdot(\vec{u} - \lambda\vec{v}) \\ &= \vec{u}\cdot(\vec{u} - \lambda\vec{v}) - \lambda\,\vec{v}\cdot(\vec{u} - \lambda\vec{v}) \\ &= \vec{u}\cdot\vec{u} - \lambda\,\vec{u}\cdot\vec{v} - \lambda\,\vec{v}\cdot\vec{u} + \lambda^2\,\vec{v}\cdot\vec{v} \\ &= \|\vec{u}\|^2 - 2\lambda\,\vec{u}\cdot\vec{v} + \lambda^2\|\vec{v}\|^2 \quad \square\end{aligned}$$

Now taking $\lambda = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}$ ← $\vec{v} \neq \vec{0}$

we have

$$0 \le \left\| \vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2} \right) \vec{v} \right\|^2$$

$$= \|\vec{u}\|^2 - 2\frac{(\vec{u}\cdot\vec{v})(\vec{u}\cdot\vec{v})}{\|\vec{v}\|^2} + \frac{(\vec{u}\cdot\vec{v})^2}{\|\vec{v}\|^4}\|\vec{v}\|^2$$

$$= \|\vec{u}\|^2 - \frac{(\vec{u}\cdot\vec{v})^2}{\|\vec{v}\|^2}.$$

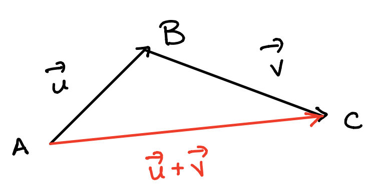
$$\Rightarrow \frac{(\vec{u}\cdot\vec{v})^2}{\|\vec{v}\|^2} \le \|\vec{u}\|^2$$

$$\Rightarrow (\vec{u}\cdot\vec{v})^2 \le \|\vec{u}\|^2\,\|\vec{v}\|^2$$

$$\Rightarrow |\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\cdot\|\vec{v}\|. \qquad \square$$

Theorem (THE TRIANGLE INEQUALITY)

For $\vec{u}, \vec{v} \in \mathbb{R}^n$, $\|\vec{u} + \vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$.



Geometric meaning: The length of any side of a triangle is less than the sum of the lengths of the other two sides.

proof: $\|\vec{u} + \vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$

$$= \vec{u}\cdot(\vec{u}+\vec{v}) + \vec{v}\cdot(\vec{u}+\vec{v}) \quad \text{(distributive law)}$$

$$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

$$\leq \|\vec{u}\|^2 + 2|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2 \quad (\text{since } \vec{u}\cdot\vec{v} \leq |\vec{u}\cdot\vec{v}|)$$

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$$

$$= (\|\vec{u}\| + \|\vec{v}\|)^2$$

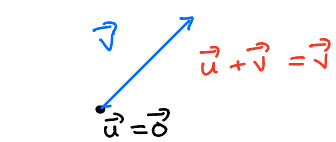
So $\|\vec{u}+\vec{v}\|^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2$.

Since $\|\vec{u}+\vec{v}\|,\ \|\vec{u}\| + \|\vec{v}\| \geq 0$, we can take square roots:

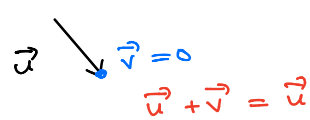
$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|. \qquad \square$$

Remark: We have equality $\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ in three situations

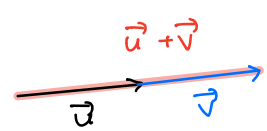
- $\vec{u} = \vec{0}$



- $\vec{v} = \vec{0}$



- $\vec{u}$ & $\vec{v}$ point in the same direction:



Summary of the lecture

The length (magnitude) of a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

It has nice properties.

- $\|\vec{v}\| \geq 0$ with $\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$
- $\|c\vec{v}\| = |c|\,\|\vec{v}\|$.
- Cauchy-Schwartz inequality:
$$\|\vec{u}\|\,\|\vec{v}\| \geq |\vec{u} \cdot \vec{v}|.$$
- Triangle inequality
$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

You should be able to:

- calculate the length of a vector
- use the above properties.

You do not need to memorize the proofs of the CS & triangle inequalities, but you should feel comfortable with manipulating formulas involving dot product, scalar multiplication, & vector addition.