Discrete Mathematics

MATH1064, Lecture 18

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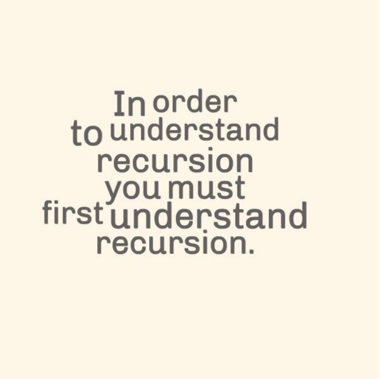
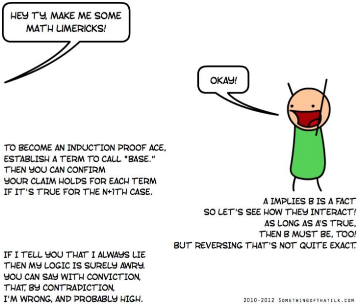


Extra exercises for Lecture 18

Section 5.1: Problems 3, 4, 10, 11, 20, 23, 24, 31–34

Section 5.3: Problems 1–4, 7, 23–25



Another way to think of induction

Remember modus ponens?

1. p
2. $p \to q$

c. $\therefore q$

Essentially, the principle of mathematical induction states that the following argument is valid:

1. $P(0)$
2. $\forall n \in \mathbb{N},\ P(n) \to P(n+1)$

c. $\therefore \forall n \in \mathbb{N},\ P(n)$

Yet another way to think of induction

The principle of mathematical induction (version 2)

Let X be a set of natural numbers with the following properties:

1. The number 0 is in X.
2. For all $n \in \mathbb{N}$, if n is in X then $n + 1$ is also in X.

Then X is the set of all natural numbers, i.e., $X = \mathbb{N}$.

Just think of X as the set of all n for which $P(n)$ is true.

Mathematical induction is not a theorem—it is one of the properties that defines the natural numbers.
This is another story for another course!

Another induction example

Question

For which $n \in \mathbb{N}$ is $2^n < n!$?

1. **Exploration**

2. **Conjecture:** $2^n < n!$ for all $n \geq 4$

3. **Proof:** We will use induction (surprise!).

Proposition

For all integers $n \geq 4$, $2^n < n!$.

3. Proof: Let $P(n)$ be the predicate: $2^n < n!$

Basis step: $2^4 = 16$ and $4! = 24$. Since $16 < 24$, $P(4)$ is true.

Inductive step: Suppose $P(n)$ is true. That is, we assume $2^n < n!$.
We must prove $P(n+1)$. That is, we must prove $2^{n+1} < (n+1)!$. We have:

$$2^{n+1} = 2 \cdot 2^n < 2 \cdot n!,$$

by the inductive hypothesis. So:

$$2^{n+1} < 2 \cdot n! < (n+1) \cdot n! = (n+1)!,$$

using the fact that $n \geq 4$. Therefore $P(n+1)$ is true.

Since $P(4)$ is true and $P(n) \rightarrow P(n+1)$ for $n \geq 4$, it follows by mathematical induction that $P(n)$ is true for all $n \geq 4$. $\square$

Yet another example

Bernoulli's inequality

For all real $x > 0$ and all integers $n \geq 2$, $(1+x)^n > 1 + nx$.

Proof. Let x be some particular (but arbitrary) positive real.

Basis step: Start with $n = 2$. We need to show $(1+x)^2 > 1 + 2x$.
This follows from $(1+x)^2 = 1 + 2x + x^2 > 1 + 2x$, since $x > 0$.

Inductive step: Assume that $(1+x)^n > 1 + nx$ for some $n \geq 2$.
We need to show that $(1+x)^{n+1} > 1 + (n+1)x$.

We can rewrite the left hand side as
$(1+x)^{n+1} = (1+x)^n(1+x) > (1+nx)(1+x)$,
using the inductive hypothesis and the fact that $x > 0$. Then:
$(1+x)^{n+1} > (1+nx)(1+x) = 1 + (n+1)x + nx^2 > 1 + (n+1)x$.

So: We showed that $(1+x)^2 > 1 + 2x$, and also showed that
$(1+x)^n > 1 + nx$ implies $(1+x)^{n+1} > 1 + (n+1)x$.
By mathematical induction, $(1+x)^n > 1 + nx$ for all $n \geq 2$. □

Loading the base case

The principle of strong mathematical induction

Let $P(n)$ be a predicate that is defined for all integers $n \geq a$, and let $b \geq a$. Suppose:

1. Basis step: $P(a), P(a+1), \ldots, P(b-1)$ and $P(b)$ are all true.
2. Inductive step: For all integers $n \geq b$, if $P(a), P(a+1), \ldots, P(n-1)$ and $P(n)$ are all true then $P(n+1)$ is also true.

Then $P(n)$ is true for all integers $n \geq a$.

It can be shown that this is equivalent to the ordinary principle of mathematical induction.

Try it!

Recursion

Recursion

Defining a mathematical expression using itself.

Easiest case: define $f : \mathbb{N} \to \mathbb{R}$ by

- $f(0) = a$ (initial condition)
- $f(n+1) = g(n, f(n))$ where g is known (recurrence relation)

Examples:

Factorial:

Fibonacci sequence:

Squares:

Another example

Remember the Tower of Hanoi?

The sequence $(T_n)_{n \geq 0}$ is defined by:

$T_0 = 0$ (initial conditions)

$T_n = 2T_{n-1} + 1$ for $n \geq 1$ (recurrence relation)

How to convert this recursive definition into an explicit formula?

This is difficult in general, and you cannot always do it.
But, if you can guess the formula, it is easy to prove!

Just check that it satisfies the recursive definition, by substituting in your formula and checking that equality holds.

Tower of Hanoi

The recursive definition:

$T_0 = 0, \quad T_n = 2T_{n-1} + 1$ for $n \geq 1$

Compute a few terms:

$T_0 = 0 \quad T_1 = 1 \quad T_2 = 3 \quad T_3 = 7 \quad T_4 = 15 \quad T_5 = 31$

Try to guess a formula: $T_n = 2^n - 1$

To prove this, substite your formula into the recursive definition and check that it works:

- Initial conditions:
  $T_0 = 2^0 - 1 = 1 - 1 = 0$, which satisfies the initial conditions
- Recurrence relation:
  $T_n = 2^n - 1 = 2 \cdot 2^{n-1} - 1 = 2 \cdot (2^{n-1} - 1) + 1 = 2T_{n-1} + 1$, which satisfies the recurrence relation

Therefore our explicit formula is correct!

How can you guess the formula?

Method 1: Expand how each term is constructed

Consider $(a_n)_{n\geq 0}$, with $a_0 = 7$ and $a_n = a_{n-1} + 3$ for $n \geq 1$.

$$a_0 = 7$$
$$a_1 = 7 + 3$$
$$a_2 = 7 + 3 + 3$$
$$a_3 = 7 + 3 + 3 + 3$$

Conjecture: $a_n = 7 + 3n$

Proof:

- $a_0 = 7 + 3 \cdot 0 = 7$, which satisfies the initial conditions
- $a_n = 7 + 3n = 7 + 3(n-1) + 3 = a_{n-1} + 3$, which satisfies the recurrence relation

Method 2: Compute several terms and look for a pattern

Consider $(a_n)_{n\geq 0}$, with:

$$a_0 = 0, \quad a_1 = 1, \quad a_n = \frac{1}{2}(a_{n-1} + a_{n-2} + 1) + 3(n-1) \text{ for } n \geq 2$$

Some initial terms:

$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 9, \quad a_4 = 16, \quad a_5 = 25$

Conjecture: $a_n = n^2$

Proof:

- $a_0 = 0^2 = 0$ and $a_1 = 1^2 = 1$, satisfying the initial conditions
- For the recurrence relation, we start on the right hand side:
  $\frac{1}{2}(a_{n-1} + a_{n-2} + 1) + 3(n-1) =$
  $\frac{1}{2}[(n-1)^2 + (n-2)^2 + 1] + 3(n-1) =$
  $\frac{1}{2}(2n^2 - 6n + 5 + 1) + 3n - 3 = n^2 - 3n + 3 + 3n - 3 = n^2$
  $= a_n$, which satisfies the recurrence relation

Other methods of guesswork

Method 3: The Online Encyclopaedia of Integer Sequences (oeis.org)

Method 4: Generating functions

And more: Transform the sequence, examine differences, ...

Guessing can be dangerous...

Fermat considered the sequence $(F_n)_{n \geq 0}$, where:

$$F_n = 2^{(2^n)} + 1$$

Some initial terms:

$F_0 = 3 \quad F_1 = 5 \quad F_2 = 17 \quad F_3 = 257 \quad F_4 = 65537$

These are all prime!

Fermat conjectured that F_n is prime for all n.

$F_5 = 4294967297 = 641 \times 6700417$

In fact, it is not known if any of these numbers are prime for $n > 4$!

Read about the history of the Fermat primes to learn more.