LECTURE 4-D

SPAN AND LINEAR INDEPENDENCE

Recall: A vector $\vec{v}$ is a linear combination of vectors $\vec{v}_1, \dots, \vec{v}_k$ in $\mathbb{R}^n$ if there are scalars $c_1, c_2, \dots, c_k \in \mathbb{R}$ so that

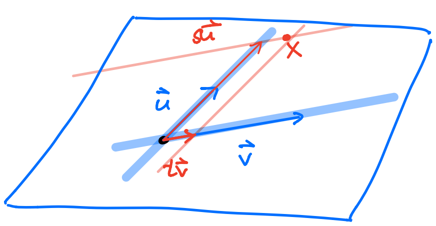
$$\vec{v} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k.$$

- The scalars $c_1, \dots, c_k$ are called the coefficients of the linear combination.

- Let $\vec{u}, \vec{v}$ be non-parallel vectors in $\mathbb{R}^3$.

  We know that the plane through the origin O and with direction vectors $\vec{u}, \vec{v}$ is given by

$$\mathcal{P} = \{\vec{x} = s\vec{u} + t\vec{v};\ s, t \in \mathbb{R}\}$$



$\vec{x} = s\vec{u} + t\vec{v}$

On the other hand, $\{\vec{x} = s\vec{u} + t\vec{v};\ s, t \in \mathbb{R}\}$ is just the set of all linear combinations of $\vec{u}, \vec{v}$.

Definition: If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the span of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ is denoted

$$\text{span}(S) \quad \text{or} \quad \text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$$

and is the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_k$.

So $\operatorname{span}(S) = \operatorname{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$

$$= \{ \vec{v} \in \mathbb{R}^n \mid \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k \text{ for some } c_1, \dots, c_k \in \mathbb{R} \}.$$

Example: We just saw that $\mathcal{P} = \operatorname{span}(\vec{u}, \vec{v})$.

Definition: If $\operatorname{span}(S) = \mathbb{R}^n$, we say that S is a spanning set for $\mathbb{R}^n$, or that $\vec{v}_1, \dots, \vec{v}_k$ span $\mathbb{R}^n$.

i.e. every vector in $\mathbb{R}^n$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_k$.

Example: The standard vectors $\vec{e}_1, \dots, \vec{e}_n$ span $\mathbb{R}^n$, because every $\vec{v} \in \mathbb{R}^n$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ can be written as

$$\vec{v} = v_1\vec{e}_1 + \dots + v_n\vec{e}_n.$$

Example Let $S = \{\vec{v}\}$, one non-zero vector.

$$\operatorname{span}(\vec{v}) = \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} = t\vec{v} \text{ for some } t \in \mathbb{R} \}.$$

This is the vector form of the line passing through the origin with direction vector $\vec{v}$.

Exercise: What if $\vec{v} = \vec{0}$? Then what is $\operatorname{span}(\vec{v})$?

Solution $\operatorname{span}(\vec{v}) = \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} = \underbrace{t\vec{0}}_{=\vec{0}} \text{ for some } t \in \mathbb{R} \}$

$$= \{\vec{0}\}.$$

Example Let $\vec{u}, \vec{v}$ be two non-zero vectors in $\mathbb{R}^n$.

Then $\text{span}(\vec{u}, \vec{v}) = \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} = s\vec{u} + t\vec{v},\ s,t \in \mathbb{R} \}$.

- We know that if $\vec{u}, \vec{v}$ are not parallel, then

  $\text{span}(\vec{u}, \vec{v})$ = plane P through O with direction vectors $\vec{u}, \vec{v}$.

Exercise: What if $\vec{u}$ and $\vec{v}$ are parallel?

Solution: we can write $\vec{v} = c\vec{u}$ for some $c \in \mathbb{R}$.

Then
$$\text{span}(\vec{u}, \vec{v}) = \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} = s\vec{u} + t\vec{v},\ s,t \in \mathbb{R} \}$$
$$= \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} = \underbrace{s\vec{u} + tc\vec{u}}_{=(s+tc)\vec{u}},\ s,t \in \mathbb{R} \}$$

So $\text{span}(\vec{u}, \vec{v}) = \text{span}(\vec{u}) = \{$all scalar multiples of $\vec{u}\}$

This is the line through O with direction vector $\vec{u}$.

Remark: It seems like often the span of n vectors gives an n-dimensional space

1 → line
2 → plane

but sometimes we get a space that is smaller than this.

Goal: What is the property of the set S that corresponds to this behaviour?

Definition: • A set of vectors $\vec{v}_1, \dots, \vec{v}_k$ is called linearly independent if the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$$

has exactly one solution, which is

$$c_1 = c_2 = \dots = c_k = 0.$$

i.e. there is no non-trivial linear combination of the $\vec{v}_i$ which gives $\vec{0}$.

- If there is more than one solution, we say the set is linearly dependent.

Example: If $\vec{u}, \vec{v}$ are parallel, then they are linearly dependent

proof: suppose $\vec{v} = c\vec{u}$, $c \in \mathbb{R}$

Then $c\vec{u} - \vec{v} = \vec{0}$.

So $\vec{0}$ is a non-trivial linear combination of $\vec{u}, \vec{v}$.

Equivalently, the equation

$$c_1\vec{u} + c_2\vec{v} = \vec{0}$$

has solution $(c_1 = c,\ c_2 = -1)$.

And so $\{\vec{u}, \vec{v}\}$ is linearly dependent.

Example: Show that if $\vec{u}, \vec{v}$ are not parallel, then they are linearly independent.

Hint: Suppose $c_1\vec{u} + c_2\vec{v} = \vec{0}$.

We want to show that $c_1 = c_2 = 0$.

Solution: Suppose that $c_1 \neq 0$, for example.

Then $\vec{u} = \frac{-c_2}{c_1}\vec{v}$, which would imply

$\vec{u}$ & $\vec{v}$ are parallel.

But we know that they are not parallel, so we must have $c_1 = 0$.

Similarly for c_2. $\square$

Example: The set $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is linearly independent in $\mathbb{R}^3$.

Solution: Suppose we have

$$c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 = \vec{0} \quad (*)$$

We need to show that $c_1 = c_2 = c_3 = 0$.

But $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$= \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

$(*)$: $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \iff c_1 = c_2 = c_3 = 0$, as desired.

Example: Let $\vec{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $\vec{v} = \begin{bmatrix}3\\0\\-2\end{bmatrix}$, $\vec{w} = \begin{bmatrix}5\\4\\0\end{bmatrix}$

The set S is linearly dependent because

$$\vec{w} = 2\vec{u} + \vec{v}$$

So $2\vec{u} + \vec{v} + (-1)\vec{w} = \vec{0}$.

i.e. $c_1 = 2$, $c_2 = 1$, $c_3 = -1$ give a non-trivial solution to $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$.

Some facts about linear independence

- A set $S = \{\vec{v}\}$ with just one vector is linearly independent

$\Leftrightarrow \vec{v} \neq \vec{0}$.

- in this case, it spans a line through the origin.

- A set $S = \{\vec{u}, \vec{v}\}$ of two vectors is linearly independent

  $\Leftrightarrow$ $\vec{u}$ & $\vec{v}$ are not parallel.

  - in this case, it spans a plane through the origin.

- Let $S = \{\vec{u}, \vec{v}, \vec{w}\}$, with $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$.

  - S spans $\mathbb{R}^3$ $\Leftrightarrow$ $\vec{u}, \vec{v}, \vec{w}$ are linearly independent.

More generally, if $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$, a set of n vectors in $\mathbb{R}^n$.

  - then S spans $\mathbb{R}^n \Leftrightarrow S$ is linearly independent.

Summary of the lecture:

- Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$.

  - The span of S is the set of all linear combinations of the vectors $\vec{v}_1, \ldots, \vec{v}_k$.

  - S is linearly independent if

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0} \Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$

  - otherwise, S is linearly dependent.

You should be able to:

- use and define these terms
- determine when a set of one or two vectors is linearly independent & write down its span.