THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Set Theory – Week 4 Practice Class

MATH1064: Discrete Mathematics for Computing

Here is a list of **problems** for the practice class. Try to solve them before you go to class! There are more problems here than can be solved in the hour, so you should get started on them!

1. Let $A = \{x \in \mathbb{Z} \mid \exists p \in \mathbb{Z} : x = 4p-1\}$ and $B = \{y \in \mathbb{Z} \mid \exists q \in \mathbb{Z} : y = 4q-5\}$. Prove that $A = B$, by showing that if $x \in A$ then $x \in B$, and if $y \in B$ then $y \in A$.

Solution: We have to show $A \subseteq B$ and $B \subseteq A$.

To show that $A \subseteq B$, let $x \in A$, that is, $x = 4p-1$ for some $p \in \mathbb{Z}$. But then

$$x = 4(p+1)-5,$$

and since $q = (p+1) \in \mathbb{Z}$ we have that $x \in B$. Since $x \in A$ was chosen arbitrarily, we have that $A \subseteq B$.

To show that $B \subseteq A$, let $y \in B$, that is, $y = 4q-5$ for some $q \in \mathbb{Z}$. But then

$$y = 4(q-1)-1,$$

and since $p = (q-1) \in \mathbb{Z}$ we have that $y \in A$. Since $y \in B$ was chosen arbitrarily, we have that $B \subseteq A$.

2. Let U be some universal set, and let $A \subseteq U$. Show that:

1. $\overline{A} = U \setminus A$;
2. $A \cap \overline{A} = \emptyset$;
3. $A \cup \overline{A} = U$;
4. $\overline{U} = \emptyset$;
5. $\overline{\overline{A}} = A$;

Try proving these using the "element method", and also using logical equivalences.

Solution: Let $P(x) =$ "$x \in A$", $Q(x) =$ "$x \in U$", and $R(x) =$ "$x \in \emptyset$". Note that the second predicate is a tautology and the third predicate is a contradiction.

1. Element method: Let $x \in U$. If $x \in \overline{A}$, then by definition $x \notin A$ and hence $x \in U \setminus A$. If $x \in U \setminus A$, then $x \notin A$ and thus, by definition, $x \in \overline{A}$.

   Logical equivalence:

$$\begin{aligned} x \in \overline{A} &\equiv \neg P(x) \\ &\equiv \text{ (tautology) } \wedge \neg P(x) \\ &\equiv Q(x) \wedge \neg P(x) \\ &\equiv (x \in U) \wedge (x \notin A) \\ &\equiv x \in U \setminus A \end{aligned}$$

   See above for defintions of $P(x)$ and $Q(x)$.

2. Element method: Let $x \in U$. If $x \in A$ then $x \notin A \cap \overline{A}$. If $x \in A$ then $x \notin \overline{A}$ and hence $x \notin A \cap \overline{A}$ and hence $A \cap \overline{A} \subseteq \emptyset$. On the other hand, $\emptyset \subseteq A \cap \overline{A}$ is always true.

   Logical equivalence:

$$\begin{aligned} P(x) \wedge \neg P(x) &\equiv \text{ (contradiction)} \\ &\equiv R(x) \end{aligned}$$

   See above for defintions of $P(x)$ and $R(x)$.

3. Element method: Let $x \in U$. Then $x \in A$ or $x \in \overline{A}$ and hence $U \subseteq A \cup \overline{A}$. Let $x \in A \cup \overline{A}$. If $x \in A$ then $x \in U$. If $x \in \overline{A}$ then $x \in U$. Altogether $x \in U$ and $A \cup \overline{A} \subseteq U$.

   Logical equivalence:

$$\begin{aligned} P(x) \wedge \neg P(x) &\equiv \text{ (tautology)} \\ &\equiv Q(x) \end{aligned}$$

   See above for defintions of $P(x)$ and $Q(x)$.

4. Element method: Let $x \in U$. If $x \in \overline{U}$ then $x \notin U$ a contradiction. Hence such an x cannot exist and $\overline{U} \subseteq \emptyset$. Again, $\emptyset \subseteq \overline{U}$ is always true.

   Logical equivalence:

$$\begin{aligned} \neg Q(x) &\equiv \neg \text{ (tautology)} \\ &\equiv \text{ (contradiction)} \\ &\equiv R(x) \end{aligned}$$

   See above for definitions of $Q(x)$ and $R(x)$.

5. Element method: Let $x \in U$. If $x \in A$ then $x \notin \overline{A} = B$. But if $x \notin B$, then $x \in \overline{B} = \overline{\overline{A}}$ and $A \subseteq \overline{\overline{A}}$. If $x \in \overline{\overline{A}} = \overline{B}$ then, by definition, $x \notin B = \overline{A}$. But if $x \notin \overline{A}$ then, by definition, $x \in A$ and $\overline{\overline{A}} \subseteq A$.

   Logical equivalence:

$$\neg(\neg P(x)) \equiv P(x)$$

   Double negative. See above for definition of $P(x)$

3. Let the universal set be $U = \{1,2,\ldots,10\}$, and consider the following subsets of U:

$$A = \{1,2,3,4\}, \quad B = \{2,4,6,8,10\}, \quad C = \{1,3,5,7,9\}.$$

Determine $A \cap B$, $B \cup C$, $B \setminus A$, $A \setminus C$, $\overline{B}$, $\overline{A}$, and $\overline{A} \cup B$.

Solution:

- $A \cap B = \{2,4\}$

- $B \cup C = \{1,2,3,4,5,6,7,8,9,10\} = U$
- $B \setminus A = \{6,8,10\}$
- $A \setminus C = \{2,4\}$
- $\overline{B} = \{1,3,5,7,9\}$
- $\overline{A} = \{5,6,7,8,9,10\}$
- $\overline{A} \cup B = \{2,4,5,6,7,8,9,10\}$

4. What are the following sets? Give proofs with your answers.

1. $\displaystyle\bigcup_{i=1}^{\infty} [i, i+1]$?
2. $\displaystyle\bigcap_{i=1}^{\infty} [i, i+1]$?
3. $\displaystyle\bigcup_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$?
4. $\displaystyle\bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$?

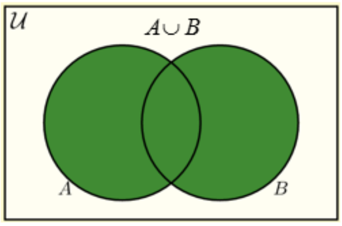
Solution:

1. $\displaystyle\bigcup_{i=1}^{\infty} [i, i+1] = \{x \in \mathbb{R} | x \geq 1\}$.
For every $i \in \mathbb{Z}$, $i \geq 1$, we have that $[i, i+1] \subseteq \{x \in \mathbb{R} | x \geq 1\}$. On the other hand, for every $y \in \{x \in \mathbb{R} | x \geq 1\}$ we have $\lfloor y \rfloor \in \mathbb{Z}$, $\lfloor y \rfloor \geq 1$, and $y \in [\lfloor y \rfloor, \lfloor y \rfloor + 1]$. Hence, $\{x \in \mathbb{R} | x \geq 1\} \subseteq \bigcup_{i=1}^{\infty} [i, i+1]$.
2. $\displaystyle\bigcap_{i=1}^{\infty} [i, i+1] = \emptyset$.
First note that $[1,2] \cap [3,4] = \emptyset$. To see this observe that all $x \in [1,2] \cap [3,4]$ must satisfy $x \leq 2$ and $x \geq 3$, a contradiction. The equality follows since $\emptyset \cap A = \emptyset$ for all sets A.
3. $\displaystyle\bigcup_{i=1}^{\infty} \left[0, \frac{1}{i}\right] = [0, 1]$
First note that, for $i \in \mathbb{Z}$, $i \geq 1$, we have $0 < 1/i \leq 1$. Hence, $[0, 1/i] \subseteq [0, 1]$. The equality now follows from the fact that, for $A \subseteq B$ we must always have $A \cup B = B$.
4. $\displaystyle\bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right] = \{0\}$
Let $x \in \{0\}$, that is, $x = 0$. Since $0 \in [0, 1/i]$ for all $i \in \mathbb{Z}$, $i \geq 1$, we have that $x \in \bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$ and $\{0\} \subseteq \bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$.
Let $x \in \bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$. Since, as before, we can assume that $[0, 1/i] \subseteq [0, 1]$ for all $i \in \mathbb{Z}$, $i \geq 1$, we have that $\bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right] \subseteq [0, 1]$. It follows that $0 \leq x \leq 1$.
If $x > 0$, then $x > \frac{1}{\lceil 1/x \rceil}$. It follows that $x \notin [1, \frac{1}{\lceil 1/x \rceil}]$, and since $\lceil 1/x \rceil \in \mathbb{Z}$, $\lceil 1/x \rceil \geq 1$, we have that $x \notin \bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$. It follows that $x \in \bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$ at most if $x = 0$ and hence $\bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right] \subseteq \{0\}$.

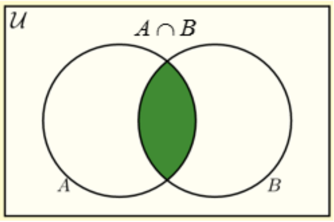
5. For each statement below, draw a Venn diagram and decide whether or not the statement is true for all sets A and B. For each statement that is true, give a proof. For each statement that is false, give a counterexample.

1. $A \subseteq A \cup B$;
2. $A \subseteq A \cap B$;
3. $A \setminus B \subseteq A$;
4. $A \setminus B \subseteq B$;
5. $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

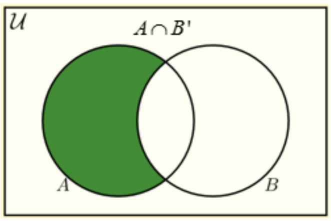
Solution:

1. 

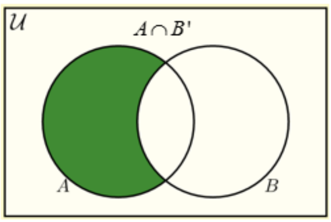
   The green area in the Venn diagram is $A \cup B$ which always contains A. Hence the statement $A \subseteq A \cup B$ is always true.

2. 

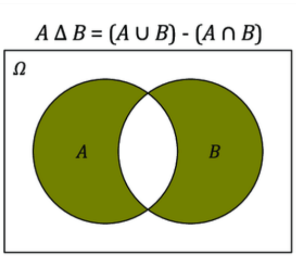
   The green area in the Venn diagram is $A \cap B$. For the two sets A and B in the picture A is not a subset of $A \cap B$. Hence, we have found a counter example and the statement $A \subseteq A \cap B$ is false in general.

3. 

   The green area in the Venn diagram is $A \setminus B$. The statement $A \setminus B \subseteq A$ is immediate: $x \in A \setminus B$ directly implies that $x \in A$.

4. 

   The green area in the Venn diagram is $A \setminus B$. The statement $A \setminus B \subseteq B$ is not true in the picture and, thus, false in general.

5. 

Let $x \in (A \setminus B) \cup (B \setminus A)$. If $x \in (A \setminus B)$ then $x \in A$ but $x \notin B$. Hence $x \in A \cup B$ but $x \notin A \cap B$ and hence $x \in (A \cup B) \setminus (A \cap B)$. If $x \in (B \setminus A)$ then $x \in B$ but $x \notin A$. Hence $x \in A \cup B$ but $x \notin A \cap B$ and hence $x \in (A \cup B) \setminus (A \cap B)$.

Let $x \in (A \cup B) \setminus (A \cap B)$. If $x \in A$ then $x \notin B$ (because otherwise $x \in A \cap B$) and thus $x \in A \setminus B$. If $x \in B$ then $x \notin A$ and thus $x \in B \setminus A$. Hence $x \in (A \setminus B) \cup (B \setminus A)$.

6. Show that:

1. $A \setminus (A \cap B) = A \setminus B$;
2. $(A \setminus B) \setminus C = A \setminus (B \cup C)$;
3. $\bigcap_{i=1}^{n} (A_i \setminus B) = \left(\bigcap_{i=1}^{n} A_i \right) \setminus B$.

Again, try proving these using the "element method", and also using logical equivalences.

Solution:

1. Let $x \in A \setminus (A \cap B)$. Then $x \in A$ and $x \notin A \cap B$. But since $x \in A$ necessarily $x \notin B$. It follows that $x \in A \setminus B$.

   Let $x \in A \setminus B$. Hence $x \in A$ and $x \notin B$. In particular, $x \notin A \cap B$. Hence $x \in A \setminus (A \cap B)$.

   Logival equivalence: Consider predicates $P(x) =$ "$x \in A$", $Q(x) =$ "$x \in B$".

$$\begin{aligned} P(x) \land \neg(P(x) \land Q(x)) &\equiv P(x) \land (\neg P(x) \lor \neg Q(x)) \\ &\equiv (P(x) \land \neg P(x)) \lor (P(x) \land \neg Q(x)) \\ &\equiv \text{ (contradiction) } \lor (P(x) \land \neg Q(x)) \\ &\equiv (P(x) \land \neg Q(x)) \end{aligned}$$

2. Let $x \in (A \setminus B) \setminus C$. So, on the one hand we have $x \in A$, and on the other hand we have $x \notin B$ and $x \notin C$, hence $x \notin B \cup C$. Altogether this means that $x \in A \setminus (B \cup C)$.

   Let $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Hence $x \notin B$ and $x \notin C$. Altogether $x \in (A \setminus B) \setminus C$.

   Logival equivalence: Consider predicates $P(x) =$ "$x \in A$", $Q(x) =$ "$x \in B$", and $R(x) =$ "$x \in C$".

$$P(x) \land \neg Q(x) \land \neg R(x) \equiv P(x) \land \neg(Q(x) \lor R(x))$$

3. Let $x \in \bigcap_{i=1}^{n}(A_i \setminus B)$. Then $x \in A_i$, for all $1 \le i \le n$, and $x \notin B$. Hence $x \in (\bigcap_{i=1}^{n} A_i) \setminus B$.

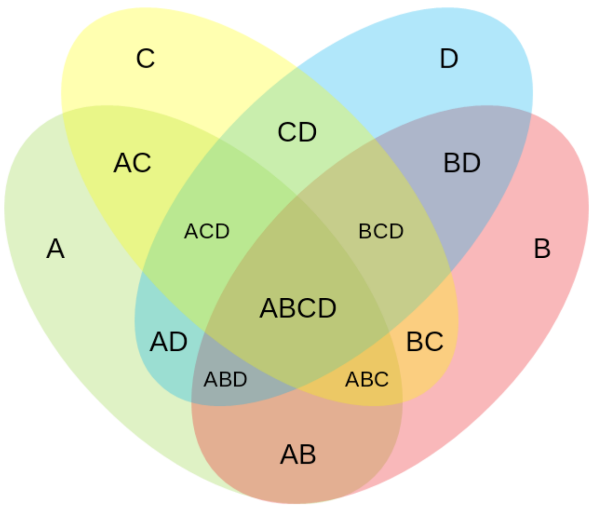
   Let $x \in (\bigcap_{i=1}^{n} A_i) \setminus B$. Then $x \in A_i$, for all $1 \le i \le n$, and $x \notin B$. Hence $x \in \bigcap_{i=1}^{n}(A_i \setminus B)$.

   Logival equivalence: Consider predicates $P_i(x) =$ "$x \in A_i$", $Q(x) =$ "$x \in B$".

$$\bigwedge_{i=1}^{n}(P_i(x) \wedge \neg Q(x)) \equiv \left(\bigwedge_{i=1}^{n} P_i(x)\right) \wedge \neg Q(x)$$

7. How would you draw a Venn diagram for *four* sets A, B, C, D?

Solution:



8. Write down all the elements of the following sets:

1. $\mathscr{P}(\{1,2\}) \times \mathscr{P}(\{a,b\})$
2. $\mathscr{P}(\{1,2\} \times \{a,b\})$

Solution:

1. $|\mathscr{P}(\{1,2\})| = 4 = |\mathscr{P}(\{a,b\})|$. So $|\mathscr{P}(\{1,2\}) \times \mathscr{P}(\{a,b\})| = 16$.

$(\emptyset,\emptyset)$	$(\emptyset,\{a\})$	$(\emptyset,\{b\})$	$(\emptyset,\{a,b\})$
$(\{1\},\emptyset)$	$(\{1\},\{a\})$	$(\{1\},\{b\})$	$(\{1\},\{a,b\})$
$(\{2\},\emptyset)$	$(\{2\},\{a\})$	$(\{2\},\{b\})$	$(\{2\},\{a,b\})$
$(\{1,2\},\emptyset)$	$(\{1,2\},\{a\})$	$(\{1,2\},\{b\})$	$(\{1,2\},\{a,b\})$

2. $|\{1,2\} \times \{a,b\}| = 4$. So $|\mathscr{P}(\{1,2\} \times \{a,b\})| = 16$.

$\emptyset$	$\{(1,a)\}$	$\{(1,b)\}$	$\{(2,a)\}$
$\{(2,b)\}$	$\{(1,a),(1,b)\}$	$\{(1,a),(2,a)\}$	$\{(1,a),(2,b)\}$
$\{(1,b),(2,a)\}$	$\{(1,b),(2,b)\}$	$\{(1,a),(1,b),(2,b)\}$	$\{(1,a),(1,b),(2,a)\}$
$\{(2,a),(2,b)\}$	$\{(1,a),(2,a),(2,b)\}$	$\{(1,b),(2,a),(2,b)\}$	$\{(1,a),(1,b),(2,a),(2,b)\}$

9. Are the following statements true or false for all sets A, B and C? If true, give a proof. If false, give a counterexample.

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \cup (B \times C) = (A \cup B) \times (A \cup C)$

Solution:

1. Let $(x,y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. If $y \in B$ then $(x,y) \in A \times B$. If $y \in C$ then $(x,y) \in A \times C$. Altogether, $(x,y) \in (A \times B) \cup (A \times C)$.

   Let $(x,y) \in (A \times B) \cup (A \times C)$. If $(x,y) \in A \times B$ then $x \in A$ and $y \in B$. If $(x,y) \in A \times C$ then $x \in A$ and $y \in C$. In both cases we have $(x,y) \in A \times (B \cup C)$.
2. $A = \{a\}$, $B = \{b\}$, $C = \{c\}$. Then

$$A \cup (B \times C) = \{a, (b,c)\},$$

but

$$(A \cup B) \times (A \cup C) = \{(a,a), (a,c), (b,a), (b,c)\}.$$

Puzzles on next page!

Here are two **puzzles** that you can think about during week 3. Feel free to ask your tutors or lecturer for more hints!

E After doing some mathematics, Anne and Jonathan went for a brisk walk. When they returned, Jonathan wondered how far they had gone, and Anne promptly replied "A bit more than 2.9km—I counted more than 4096 steps, and each step is about 70cm." She can't have been counting her steps in her head as they were continuously discussing mathematics during their walk. In hindsight, Jonathan realised that Anne's fingers are never still when she walks, especially the ones on her right hand. How did she count her steps whilst maintaining a conversation?

F A hat contains 42 slips of paper, numbered $1, 2, \ldots, 42$ respectively. Two slips are drawn at random from the hat, and the difference of their numbers is written on a new slip which is put in the hat while the two old slips are destroyed. This procedure is repeated until the hat contains only one slip. Show that this last slip bears an odd number.

Puzzle hints:

Stuck on the puzzles from week 3? Here are some hints!

C The pigeonhole principle (again!). Pick one person x. How many other people does x know? How many other people does x not know? What can you deduce from here?

D Try replacing 2015 with smaller odd numbers (5, 7, 9, ...). Look for patterns!