Discrete Mathematics

MATH1064, Lecture 6

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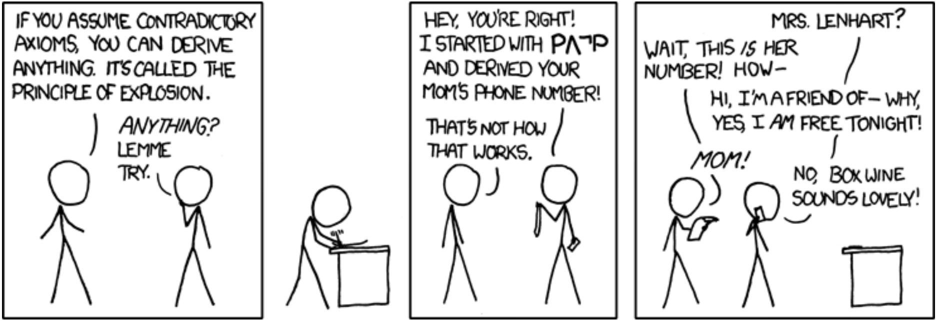
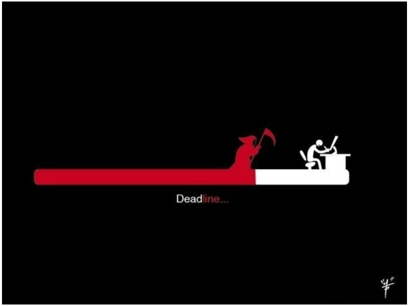


Extra exercises for Lecture 6

Section 1.7: Problems 5–8, 17, 18

Section 1.8: Problems 5, 6, 10, 11, 17



Is this argument valid or invalid?

> *Ringo does not play the drums, or Paul plays the bass. If John does not play the ukulele, then Paul does not play the bass. Ringo plays the drums, and John does not play the ukulele. Then George wears Lederhosen.*

R = "Ringo plays the drums"
P = "Paul plays the bass"
J = "John plays the ukulele"
G = "George wears Lederhosen"

What are the premises? What is the conclusion? Is this valid?

Our premises: $(\neg R \vee P)$, $(\neg J \to \neg P)$, $(R \wedge \neg J)$
Desired conclusion: G

1 $\neg R \vee P$

2 $\neg J \to \neg P$

3 $R \wedge \neg J$

4 R (Specialisation from (3))
5 $R \vee G$ (Generalisation from (4))
6 $\neg J$ (Specialisation from (3))
7 $\neg P$ (Modus ponens from (6) and (2))
8 $\neg R$ (Elimination from (7) and (1))
9 G (Elimination from (8) and (5))

Can show that this is a valid argument!

(Lines 1-3 (some people say lines 1-8) are the premises
and line 9 is the conclusion.)

Our premises: $(\neg R \vee P)$, $(\neg J \to \neg P)$, $(R \wedge \neg J)$
We were able to get the desired conclusion: G. But:

1 $\neg R \vee P$

2 $\neg J \to \neg P$

3 $R \wedge \neg J$

4	R	(Specialisation from (3))
5	$R \vee \neg G$	(Generalisation from (4))
6	$\neg J$	(Specialisation from (3))
7	$\neg P$	(Modus ponens from (6) and (2))
8	$\neg R$	(Elimination from (7) and (1))
9	$\neg G$	(Elimination from (8) and (5))

This is also a valid argument!

(Lines 1-3 (some people say lines 1-8) are the premises
and line 9 is the conclusion.)

Problem: The conjunction of the premises is a contradition!

$$(\neg R \vee P) \wedge (\neg J \to \neg P) \wedge (R \wedge \neg J)$$

In other words: the premises are inconsistent!

We have seen that one can derive anything from this contradiction:

$$\Big((\neg R \vee P) \wedge (\neg J \to \neg P) \wedge (R \wedge \neg J) \wedge p_4 \wedge \ldots \wedge p_8\Big) \to X,$$

where $X =$ your favourite statement.

Even and odd

Even numbers

The integer n is even if and only if n is twice some integer.

That is: n is even if and only if $\exists k \in \mathbb{Z}$ such that $n = 2k$.

14 is even:	$k = 7$ gives	$14 = 2 \cdot 7$
0 is even:	$k = 0$ gives	$0 = 2 \cdot 0$

Odd numbers

The integer n is odd if and only if n is twice some integer plus one.

That is: n is odd if and only if $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$.

9 is odd:	$k = 4$ gives	$9 = 2 \cdot 4 + 1$
-5 is odd:	$k = -3$ gives	$-5 = 2 \cdot (-3) + 1$

Direct proof

Key idea: To prove $\forall x, P(x) \rightarrow Q(x)$:
Choose some arbitrary x for which $P(x)$ is true,
and argue by logical inference that $Q(x)$ must be true also.

Lemma

For all $n \in \mathbb{Z}$, if n is odd then n^2 is odd.

Proof. Choose any odd $n \in \mathbb{Z}$. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$.

This means that $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Thus $\exists \ell \in \mathbb{Z}$ such that $n^2 = 2\ell + 1$, and so n^2 is odd. $\square$

Techniques used:

- Our definitions are precise – use them!
- To prove $\exists \ell$ such that $R(\ell)$, just find one example.

Proof by contradiction

Key idea: To prove P: Assume that P is false, and use logical inference to derive a contradiction.

Lemma

For all $n \in \mathbb{Z}$, n is not simultaneously both odd and even.

Proof. Suppose the lemma is false.
Then there exists some $n \in \mathbb{Z}$ that is both odd and even.
Thus $n = 2k$ for some $k \in \mathbb{Z}$, and $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$.

Therefore $2k = 2\ell + 1$, and so $1 = 2(k - \ell)$. This means that $k - \ell = \frac{1}{2}$, which is impossible since both k and ℓ are integers.

Therefore n cannot be simultaneously both odd and even. $\square$

Question: Why did we use two different symbols k and ℓ?

A more complex proof by contradiction

Lemma

For all $n \in \mathbb{N}$, n is either odd or even.

Proof: Assume the lemma is false. Choose the smallest $n \in \mathbb{N}$ that is neither odd nor even.

If $n > 0$, then it follows that $n - 1$ is either odd or even.

- If $n - 1$ is odd, then $n - 1 = 2k + 1$ for some $k \in \mathbb{Z}$.
  Thus $n = 2k + 2 = 2(k + 1)$, and so n is even.
- If $n - 1$ is even, then $n - 1 = 2k$ for some $k \in \mathbb{Z}$.
  Thus $n = 2k + 1$, and so n is odd.

If $n = 0$ then $n = 2 \cdot 0$, and so n is even.

In all cases, n is either odd or even, contradicting our choice of n.
Therefore every $n \in \mathbb{N}$ is either odd or even. $\square$

Questions

Lemma

For all $n \in \mathbb{N}$, n is either odd or even.

Why did we have to treat $n = 0$ separately?

Does the same proof work if we allow $n \in \mathbb{Z}$, as opposed to just $n \in \mathbb{N}$?

Proof by contraposition

Key idea: To prove $\forall x, P(x) \rightarrow Q(x)$:
Choose some arbitrary x for which $Q(x)$ is false,
and argue by logical inference that $P(x)$ must be false also.

Lemma

For all $n \in \mathbb{N}$, if n^2 is odd then n is odd.

Proof. Choose any $n \in \mathbb{N}$ that is not odd.
Then, by the previous result, n is even.

Therefore $n = 2k$ for some $k \in \mathbb{Z}$, and so $n^2 = (2k)^2 = 2 \cdot (2k^2)$.
Therefore n^2 is even also. By another of our previous results, it follows that n^2 is not odd.

So: if n is not odd, then n^2 is not odd. By the contrapositive, this means that if n^2 is odd then n is odd. $\square$

Proof by contraposition:
This technique is based on $(p \to q) \equiv (\neg q \to \neg p)$.

Incorrect proof techniques:
The following methods do not prove $\forall x, P(x) \to Q(x)$:

- Proof by example:
  Giving several examples for which $P(x)$ and $Q(x)$ are true
- Begging the question / Circular reasoning:
  Assuming $Q(x)$ within your proof, before you have proven it

See the text (pages 75, 89, 90) for others!

Disproof by counterexample

Key idea: To disprove a statement $\forall x, P(x)$ – that is, to show that the statement is false – we simply need to show one example of an x for which $P(x)$ is false. This x is called a counterexample.

Example

Disprove the following statement:
$\forall a, b \in \mathbb{R}$, if $a^2 = b^2$, then $a = b$.

Counterexample. Let $a = 1$ and $b = -1$.

Then $a^2 = 1$ and $b^2 = (-1)^2 = 1$, whence $a^2 = b^2$.

But $a = 1 \neq -1 = b$, which gives $a \neq b$.

Without loss of generality (WLOG)

Key idea: Use symmetry in the statement to reduce the number of cases to consider.

Example

$\forall a, b \in \mathbb{Z}$, if ab and $a+b$ are even, then both a and b are even.

By contraposition suppose not both a and b are even.
Without loss of generality we may assume a is odd.
So $a = 2k+1$ for some $k \in \mathbb{Z}$.

- Case 1: b is even. Then $b = 2\ell$ for some $\ell \in \mathbb{Z}$. This gives $a+b = (2k+1)+2\ell = 2(k+\ell)+1$. Hence $a+b$ is odd.
- Case 2: b is odd. Then $b = 2\ell+1$ for some $\ell \in \mathbb{Z}$. This gives $ab = (2k+1)(2\ell+1) = 2(2k\ell+k+\ell)+1$. Hence ab is odd.

In each case, not both $a+b$ and ab are even.
This completes the proof by contraposition.

Without loss of generality (WLOG)

WLOG means that no generality is lost by making a simplifying assumption: If the simple case is true then trivially all cases must be true.

More examples:

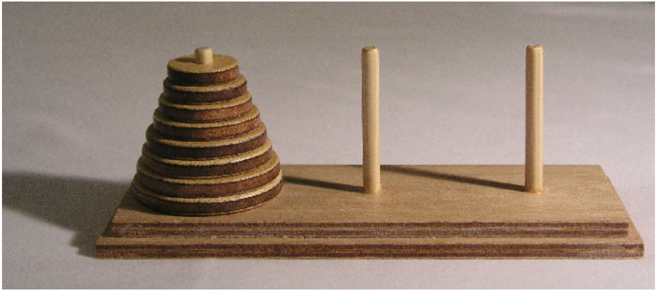
- **Statement**: If three objects are each painted either red or blue, then there must be at least two objects of the same color.
- **Proof**: Assume, WLOG, that the first object is red. If either of the other two objects is red, then we are finished; if not, then the other two objects must both be blue and we are still finished.

- **Statement**: $\forall x, y \in \mathbb{R}, |x+y| \leq |x| + |y|$.
- **Proof**: Assume, without loss of generality, that $x > 0$. ....

Using WLOG is often just saying

- "Assume ...", then
- "... and the other cases follow similarly".

Let's have some fun!



The Tower of Hanoi

Given: A tower of 8 discs in decreasing size on one of three pegs.

Problem: Transfer the entire tower to one of the other pegs.

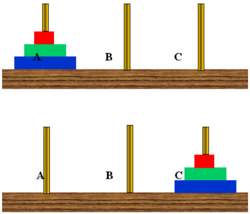
Rule 1: Move only one disc at a time.

Rule 2: Never move a larger disc onto a smaller disc.

1. Is there a solution?
2. What's the minimal number of moves necessary and sufficient for the task?

Key idea: Generalise! What if there are n discs?

Let T_n =minimal number of moves.



$T_0 = 0$ $T_1 = 1$ $T_2 = 3$ $T_3 = ?$

(1) $T_3 = 5$ (2) $T_3 = 7$ (3) $T_3 = 9$ (4) $T_3 = 10$ (5) $T_3 = 11$

The winning strategy is:

1. Move the $n-1$ smallest discs from A to B.
2. Move the big disc from A to C.
3. Move the $n-1$ smallest discs from B to C.

So it is possible for all n to move the tower (induction!) and:

$$T_n \leq 2T_{n-1} + 1.$$

Give an argument for the equality $T_n = 2T_{n-1} + 1$ at home!

This together with $T_0 = 0$ allows us to compute T_n for any n.
In particular T_8!

But a solution of T_n as a function of the number of discs n would be much prettier and more useful!

$$T_0 = 0$$
$$T_1 = 2 \cdot 0 + 1 = 1$$
$$T_2 = 2 \cdot 1 + 1 = 3$$
$$T_3 = 2 \cdot 3 + 1 = 7$$
$$T_4 = 2 \cdot 7 + 1 = 15$$
$$T_5 = 2 \cdot 15 + 1 = 31$$
$$T_6 = 2 \cdot 31 + 1 = 63$$

$$T_0 + 1 = 0 + 1 = 1$$
$$T_1 + 1 = 1 + 1 = 2$$
$$T_2 + 1 = 3 + 1 = 4$$
$$T_3 + 1 = 7 + 1 = 8$$
$$T_4 + 1 = 15 + 1 = 16$$
$$T_5 + 1 = 31 + 1 = 32$$
$$T_6 + 1 = 63 + 1 = 64$$

$T_n = 2^n - 1$ for $0 \leq n \leq 6$.

You'll prove that this holds for all n in a little while!

Conclusion

1. To find a solution to a mathematical problem, one needs to poke around and investigate. There is intuition and luck involved! Looking at small cases can help a lot.
2. Once one has found a mathematical expression or statement that seems right, one has to give sound arguments to establish its truth.
3. It sometimes pays off to be ambitious and shoot for more!

In this course, you're mostly given a mathematical formula or statement to prove or disprove. Even here, it is a good idea to start with small or special cases. This gives you some insight into the problem and help you find a proof or counterexample.

Assignment 1

Material for Assignment 1

This completes the material relevant for Assignment 1.

Suggestions for additional preparation:

Reading: Revise Chapter 1, Sections 1.1 to 1.8

Exercises: Attempt additional exercises from the book!