THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Functions, Sequences and Divisibility – Week 5 Tutorials

MATH1064: Discrete Mathematics for Computing

1. Define $\lfloor x \rfloor$ to be the largest integer that is $\leq x$. For example, $\lfloor \pi \rfloor = 3$.

Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = \left\lfloor \dfrac{1-6n}{3} \right\rfloor$.

(a) Is f injective?

Solution: f is injective. To show this, we assume that $f(n) = f(m)$, and then try to show that $n = m$.
Now $f(n) = \left\lfloor \dfrac{1-6n}{3} \right\rfloor = \left\lfloor -2n + \dfrac{1}{3} \right\rfloor = -2n$ since n and $2n$ are integers. Similarly, $f(m) = -2m$.
So if $f(n) = f(m)$, we have $-2n = -2m$, and so $n = m$; therefore f is injective.

(b) Is f surjective?

Solution: Since $f(n) = -2n$, we see that we never have f mapping to any odd integer. So f is not surjective $\mathbb{Z}$; for example, there is no integer w with $f(w) = 1$.

(c) Prove, or disprove with a counter-example, that $f(mn) = f(m)f(n)$.

Solution: It is false that $f(mn) = f(m)f(n)$. For example, consider $m = 1$ and $n = -2$.
Then $f(1) = -2$ and $f(-2) = 4$, so that $f(1) \times f(-2) = -8$; but $f(1 \times (-2)) = f(-2) = 4$.

2. Define $f : \mathbb{Q} \to \mathbb{Z}$ by the rule $f(x) = 2\lfloor x+1 \rfloor$, for all $x \in \mathbb{Q}$, where as above $\lfloor x \rfloor$ is the largest integer that is $\leq x$.

(a) Is the function f injective? Prove this, or give a counter-example.

Solution: f is *not* injective; for example

$$\begin{aligned} f(0) &= 2\lfloor 0+1 \rfloor = 2\lfloor 1 \rfloor = 2 \times 1 = 2, \quad \text{and} \\ f(\tfrac{1}{4}) &= 2\lfloor \frac{1}{4} + 1 \rfloor = 2\lfloor \frac{5}{4} \rfloor = 2 \times 1 = 2, \end{aligned}$$

but $0 \neq \frac{1}{4}$.

(b) Is the function f surjective? Prove this, or give a counter-example.

Solution: f is *not* surjective $\mathbb{Z}$, because $\lfloor x+1 \rfloor$ is always an integer, so $f(x) = 2\lfloor x+1 \rfloor$ is always an *even* integer. So for instance there is no $x \in \mathbb{Q}$ such that $f(x) = 1$ (because 1 is odd).

(c) What is the image (or range) of f? (Give your answer as a set.)

Solution: The image (or range) of f is $\{2n \mid n \in \mathbb{Z}\}$, the set of all even integers.

(d) The function $g : \mathbb{Z} \to \mathbb{Z}$ is given by the rule $g(x) = 2x - 1$, for all $x \in \mathbb{Z}$.
Determine the composition function $g \circ f$, and calculate $(g \circ f)(-\frac{3}{4})$.

Solution: Now $g : \mathbb{Z} \to \mathbb{Z}$, where $g(x) = 2x - 1$, for all $x \in \mathbb{Z}$. We have:

$$\begin{aligned}(g \circ f)(x) = g(f(x)) &= g(2\lfloor x+1 \rfloor) \\ &= 2 \times 2\lfloor x+1 \rfloor \ - \ 1 \\ &= 4\lfloor x+1 \rfloor - 1.\end{aligned}$$

So $(g \circ f)(-\frac{3}{4}) = 4\lfloor -\frac{3}{4} + 1 \rfloor - 1 = 4\lfloor \frac{1}{4} \rfloor - 1 \ = \ (4 \times 0) - 1 \ = \ -1$.

(e) What is the image (or range) of $g \circ f$? (Give your answer as a set.)

Solution: The image (or range) of $g \circ f$ is the set $(g \circ f)(\mathbb{Q}) \subseteq \mathbb{Z}$
Note that $(g \circ f)(x) = 4\lfloor x+1 \rfloor - 1$, which is 4 times an integer, minus 1.
So the image of $g \circ f$ is contained in the set $\{4m - 1 \mid m \in \mathbb{Z}\}$.
We claim that $(g \circ f)(\mathbb{Q}) = \{4m - 1 \mid m \in \mathbb{Z}\}$. We already have shown that LHS$\subseteq$RHS, so we now need to show RHS$\subseteq$LHS.
Let $y \in \{4m + 3 \mid m \in \mathbb{Z}\}$. Then $y = 4m - 1$ for some $m \in \mathbb{Z}$. Using the fact that the floor of an integer is that integer, we obtain:

$$y = 4m - 1 = 4\lfloor m \rfloor - 1 = (g \circ f)(m).$$

This shows that every element in $\{4m - 1 \mid m \in \mathbb{Z}\}$ is contained in the image.

3. Determine all real numbers x for which

$$x^2 - \lfloor x \rfloor = \frac{1}{2}.$$

Hint: You can deduce a lot of information about x after a small manipulation of this simple equation!

Solution: Answers: $1/\sqrt{2}$ and $\sqrt{3}/\sqrt{2}$
Note that the equation is equivalent to $2x^2 = 2\lfloor x \rfloor + 1$. So the real number $2x^2$ is a positive odd integer. Hence $x = \frac{a}{\sqrt{2}}$, where a^2 is a positive odd integer. Now

$$a^2 = 2\lfloor \frac{a}{\sqrt{2}} \rfloor + 1 \leq \sqrt{2}a + 1.$$

It follows that $a > 0$. Also, $0 \leq -a^2 + \sqrt{2}a + 1$. This implies that a is either 1 or $\sqrt{3}$. Hence the claimed answers.

4. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms

(a) $3, 6, 11, 18, 27, 38, 51, 66, 83, 102, \ldots$

(b) $7, 11, 15, 19, 23, 27, 31, 35, 39, 43, \ldots$

(c) $1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, \ldots$

(d) $1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, \ldots$

(e) $0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, \ldots$

(f) $1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, \dots$

(g) $1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, \dots$

(h) $2, 4, 16, 256, 65536, 4294967296, \dots$

Solution: For each of the sequences above go to `OEIS.org`, type in the numbers, and look at the results.

5. What is the term a_8 of the sequence $(a_n)_{n \geq 0}$ if a_n equals

(a) 2^{n-1}? (b) 7? (c) $-(-2)^n$? (d) $1+(-1)^n$? (e) $3^n - 2^n$? (f) $\lfloor -\sqrt{n} \rfloor$?

Solution: $a_8 = 2^7 = 128$.

Solution: $a_8 = 7$.

Solution: $a_8 = -(-2)^8 = -256$.

Solution: $a_8 = 1 + (-1)^8 = 1 + 1 = 2$.

Solution: $a_8 = 3^8 - 2^8 = 6561 - 256 = 6305$.

Solution: $a_8 = \lfloor -\sqrt{8} \rfloor = \lfloor -2.828... \rfloor = -3$.

6. Let $n, d \in \mathbb{Z}$. Recall that $d \mid n$ means d divides n. Does

(a) $4 \mid 72$? (b) $0 \mid 5$? (c) $-3 \mid 9$? (d) $2 \mid \sum_{k=1}^{n} k$ for each $n \in \mathbb{N}$?

Solution: Yes — $4 \cdot 18 = 72$

Solution: No, because 0 only divides 0.

Solution: Yes — $(-3) \cdot (-3) = 9$

Solution: No, for instance, it does not work for $n = 1$. For which $n \in \mathbb{N}$ does 2 divide the sum?

7. If a and b are integers such that $a = 4x + 1$ and $b = 4y + 1$ for some $x, y \in \mathbb{Z}$, then prove that the product ab is of the the form $4m + 1$, for some integer m. Can you generalise this fact?

Solution: We have $ab = (4x+1)(4y+1) = 4(4xy + x + y) + 1$. So $m = 4xy + x + y \in \mathbb{Z}$ satisfies $ab = 4m + 1$. There are several different generalisations one can come up with. One is by assuming $a = kx + 1$ and $b = ky + 1$, for some $k, x, y \in \mathbb{Z}$.

8. Determine whether each of the following statements is true or false.

(a) $20 \equiv 75 \pmod{5}$ (b) $112 \equiv 12 \pmod{9}$ (c) $11^2 \equiv 8^2 \pmod{3}$

Solution: Both 20 and 75 are divisible by 5, so the statement is true.

Solution: We have $112 - 12 = 100$. Since 100 is not divisible by 9, the statement is false. Another way to see this is that division by 9 leaves 112 with remainder 4 and 12 with remainder 3.

Solution: We have $11 \equiv 8 \pmod{3}$ since $11 - 8 = 3$. So the statement is true, since this implies $11^2 \equiv 8^2 \pmod{3}$.