THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Counting and Recurrence Relations – Week 9 Practice Class

MATH1064: Discrete Mathematics for Computing

Here is a list of **problems** for the practice class. Try to solve them before you go to class! There are more problems here than can be solved in the hour, so you should get started on them!

1. Give a recursive definition for the set of all strings of 0s and 1s for which all the 0s precede all the 1s (for example, 00111, 0001, 00, or 111).

Solution: First we stipulate that the empty string is valid. We define all other valid strings by the following recurrence relations. Let S be an arbitrary string.

1. S is valid $\Rightarrow$ $0S$ is valid.
2. S is valid $\Rightarrow$ $S1$ is valid.

2.

1. Out of 100 people surveyed, 40 people like apples, 50 like bananas and 40 like cherries. 15 like apples and bananas, 16 like bananas and cherries, 17 like cherries and apples and 10 people like all three. How many people do not like any of the three fruits?
2. A total of 1232 students have taken a course in Statistics, 879 have taken a course in Calculus and 114 have taken a course in Logic. Furthermore, 103 have taken courses in both Statistics and Calculus, 23 have taken courses in both Statistics and Logic, and 14 have taken courses in Calculus and Logic. If 2092 students have taken at least one course in Statistics, Calculus or Logic, how many students have taken courses in all three subjects?

Solution:

1. We apply the inclusion-exclusion principle to find the number of people who like at least one fruit. Denote this number by F. The inclusion-exclusion principle tells us that

$$\begin{aligned} F &= 40+50+40-15-16-17+10 \\ &= 92 \end{aligned}$$

Therefore the number of people who do not like any fruit is 8.

2. As in the previous part we apply the inclusion-exclusion principle. Let x be the number of students who have taken courses in Statistics, Calculus and Logic.

$$2092 = 1232+879+114-103-23-14+x$$

Therefore $x = 7$.

3. How many integers between 1 and 600 inclusive are coprime with 30? That is, how many integers $n \in \mathbb{N}$ satisfy $1 \leq n \leq 600$ and $\gcd(n, 30) = 1$?

Solution: We use the fact that $30 = 2 \times 3 \times 5$ is the unique prime decomposition of 30. Therefore we are looking for all of the numbers between 1 and 600 which do not have $2, 3$ or 5 as a prime factor. In order to do so we will find all of the numbers which are not coprime with 30. Define the following set for an arbitrary $z \in \mathbb{Z}$.

$$s_z = \{x \in \mathbb{Z} \,|1 \leq x \leq 600 \text{ and } z|x\}$$

Now denote $S_z = |s_z|$ that is, S_z is the cardinality of s_z. We know that every second integer is even so $S_2 = 300$. Similarly, $S_3 = 200$ and $S_5 = 120$.

Note that for $x, y, z \in \mathbb{Z}$ where x and y are coprime numbers we have $xy|z$ if and only if $x|z$ and $y|z$. Therefore we have

$$\begin{aligned} s_6 &= s_2 \cap s_3 \\ s_{10} &= s_2 \cap s_5 \\ s_{15} &= s_3 \cap s_5 \\ s_{30} &= s_2 \cap s_3 \cap s_5 \end{aligned}$$

As above we use the fact that every sixth number is divisible by 6 and the fact that 600 is divisible by 6 to determine that $S_6 = \frac{600}{6} = 100$. Similarly $S_{10} = 60$, $S_{15} = 40$ and $S_{30} = 20$.

Now we may apply the inclusion-exclusion principle to the set of numbers between 1 and 600 which are not coprime with 30. Let x be the cardinality of the set of all such numbers.

$$\begin{aligned} x &= 300 + 200 + 120 - 100 - 60 - 40 + 20 \\ &= 440 \end{aligned}$$

Therefore the number of numbers between 1 and 600 which are coprime with 30 is 160.

4. Five people are to be chosen from a group of ten people, to form a committee.

1. How many different committees of five can be chosen?
2. Amongst the ten people, we have Alice, Bob, Eve and Oscar. Alice and Bob are extremely talkative during meetings, and so at most one of these two can be chosen. Eve and Oscar are inseparable, and so a committee must either contain both of them or neither of them. With these constraints, how many different five-person committees can be formed?

Solution:

1. This is an un-ordered choice without repetition so the answer is $\binom{10}{5} = 252$.
2. We use our answer from part $a)$ and exclude the committees which are not allowed due to the restraints in this part. First we exclude all of the committees which include both Alice and Bob. There are exactly $\binom{8}{3}$ such choices. Then we must exclude all committees which include exactly one of Eve and Oscar. Therefore $2\binom{8}{4}$ such choices. However we have now overcounted the committees which include both Alice and Bob exactly one of Eve and Oscar. There are $2\binom{6}{2}$ such choices. Applying the inclusion-exclusion principle. The total number committees which we must exclude in this part is

$$\binom{8}{3} + 2\binom{8}{4} - 2\binom{6}{2} = 56 + 140 - 30 = 166$$

Therefore the answer is $252 - 166 = 86$.

5. Show that $x_n = 6 \cdot 2^n - 4$ is a solution to the recurrence relation

$$x_n = 3x_{n-1} - 2x_{n-2}.$$

Solution: If $x_n = 6 \cdot 2^n - 4$ then $x_{n-1} = 6 \cdot 2^{n-1} - 4$ and $x_{n-2} = 6 \cdot 2^{n-2} - 4$ so that

$$\begin{aligned} 3x_{n-1} - 2x_{n-2} &= 3(6 \cdot 2^{n-1} - 4) - 2(6 \cdot 2^{n-2} - 4) \\ &= 3 \cdot 6 \cdot 2^{n-1} - 12 - 2 \cdot 6 \cdot 2^{n-2} + 8 \\ &= 9 \cdot 2^n - 3 \cdot 2^n - 4 \\ &= 6 \cdot 2^n - 4 = x_n. \end{aligned}$$

Hence $x_n = 6 \cdot 2^n - 4$ is a solution of the recurrence relation.

6. Solve the following recurrence relation: $x_n = 7x_{n-1}$, for $n \geq 1$, where $x_0 = 2$.

Solution: We have

$$\begin{aligned} x_1 &= 7x_0 \\ x_2 &= 7x_1 = 7^2 x_0 \\ x_3 &= 7x_2 = 7^3 x_0 \\ &\vdots \\ x_n &= 7x_{n-1} = 7^2 x_{n-2} = \cdots = 7^n x_0. \end{aligned}$$

Since $x_0 = 2$, the solution is $x_n = 2 \cdot 7^n$.

7. Solve the following recurrence relation: $x_n = 4x_{n-1} - 3x_{n-2}$, where $x_0 = 1$ and $x_1 = 2$.

Solution: The recurrence relation can be written as

$$x_n - 4x_{n-1} + 3x_{n-2} = 0.$$

Then the characteristic equation is

$$\lambda^2 - 4\lambda + 3 = 0.$$

That is, $(\lambda - 1)(\lambda - 3) = 0$ so that $\lambda = 1$ or $\lambda = 3$. Thus the general solution for the given recurrence relation is

$$x_n = A1^n + B3^n,$$

for some constants A and B. Now, using the initial values $x_0 = 1$ and $x_1 = 2$, we obtain $1 = A + B$ and $2 = A + 3B$ which implies that $A = B = \frac{1}{2}$. Hence the solution is

$$x_n = \frac{1}{2} + \frac{1}{2} \cdot 3^n.$$

8. Solve the following recurrence relation: $x_n = 10x_{n-1} - 25x_{n-2}$, for $n \geq 2$, where $x_0 = -1$ and $x_1 = 5$.

Solution: The recurrence relation can be written as

$$x_n - 10x_{n-1} + 25x_{n-2} = 0.$$

Then the characteristic equation is

$$\lambda^2 - 10\lambda + 25 = 0.$$

That is $(\lambda - 5)^2 = 0$ so that $\lambda = 5$ is a repeated root with multiplicity 2. Thus the general solution to the recurrence relation is

$$x_n = A5^n + Bn5^n = (A + Bn)5^n,$$

where A and B are some constants. Now, using the initial values $x_0 = -1$ and $x_1 = 5$, we have $-1 = A$ and $5 = (A + B)5$. Solving yields $A = -1$ and $B = 2$. Hence the solution is

$$x_n = (-1 + 2n)5^n.$$

9. Solve the following recurrence relation: $x_n = 3x_{n-1} - 3x_{n-2} + x_{n-3}$, where $x_0 = 0$, $x_1 = 1$ and $x_2 = 3$.

Solution: The recurrence relation can be written as

$$x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} = 0.$$

Then the characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0.$$

That is, $(\lambda - 1)^3 = 0$ so that $\lambda = 1$ is a repeated root with multiplicity 3. Thus the general solution for the recurrence relation is

$$x_n = A1^n + Bn1^n + Cn^2 1^n = A + Bn + Cn^2,$$

where A, B and C are some constants. Now, using the initial values $x_0 = 0$, $x_1 = 1$ and $x_2 = 3$, we have

$$\begin{aligned} 0 &= A \\ 1 &= A + B + C \\ 3 &= A + 2B + 4C. \end{aligned}$$

Solving the system yields $A = 0$, $B = C = \frac{1}{2}$. Hence the solution is

$$x_n = \frac{1}{2}(n + n^2) = \frac{1}{2}n(n+1).$$

10. Find a recurrence relation for the following sum: $S_n = \sum_{i=1}^{n} i^2$.

Solution: We have $S_n = \sum_{i=1}^{n} i^2$, i.e.

$$S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

Then

$$S_{n+1} = 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2$$

and so a recurrence relation for the sum is

$$S_{n+1} - S_n = (n+1)^2.$$

Puzzles on next page!

Here are two **puzzles** that you can think about during the week. Feel free to ask your tutors or lecturer for more hints!

M The (finite) sequence $a_0, a_1, \ldots, a_{2015}$ has the following properties:

- $0 \leq a_n \leq 1$ for each $n \geq 0$;
- $a_n \geq (a_{n-1} + a_{n+1})/2$ for each $n \geq 1$.

Prove that $a_{2015} - a_{2014} \leq 1/2015$ for any such sequence. Can you an example of such a sequence with $a_{2015} - a_{2014} = 1/2015$?

N The list

$$\begin{array}{c} 10234567 \\ 10234576 \\ 10234657 \\ \vdots \\ 76543210 \end{array}$$

contains all the 8-digit numbers which use every digit 0, 1, 2, 3, 4, 5, 6 and 7 exactly one each, listed in order from smallest to largest. What is the 20000th number on the list?

Puzzle hints:

Stuck on the puzzles from the last sheet? Here are some hints!

K If it were true that $12 \mid n$, then what other integers d in the range $1, \ldots, 25$ would also have to satisfy $d \mid n$? If it were true that $4 \mid n$ *and* $5 \mid n$, then what other integers d in the range $1, \ldots, 25$ would also have to satisfy $d \mid n$? Look for more relationships of this type.

L Consider the nth cut. What is the largest number of *other* cuts that it can intersect? What is the largest number of *regions* (pieces of pizza) that it can pass through?