MATH 1002 — LECTURE 3-C

GEOMETRIC MEANING OF THE CROSS PRODUCT: LENGTH

Recall: $\vec{u} \times \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$

determined by its
- direction ← use RH Rule
- length ← topic of this lecture.

What is the length of $\vec{u} \times \vec{v}$?

Theorem Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^3$, and let θ be the angle between them.
Then the length of $\vec{u} \times \vec{v}$ is

$$\boxed{\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta}$$

Remark: $\theta \in [0, \pi] \Rightarrow \sin\theta \in [0, 1]$.

proof: **Exercise:** show that $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - |\vec{u} \cdot \vec{v}|^2$

LHS: $\|\vec{u} \times \vec{v}\|^2 = (\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v})$

left-hand side

$$= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2$$

$$= u_2^2v_3^2 + u_3^2v_2^2 + u_3^2v_1^2 + u_1^2v_3^2 + u_1^2v_2^2 + u_2^2v_1^2 - 2u_2u_3v_2v_3 - 2u_1u_3v_1v_3 - 2u_1u_2v_1v_2.$$

RHS: $\|\vec{u}\|^2\|\vec{v}\|^2 - |\vec{u} \cdot \vec{v}|^2$

$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2$$

$$= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2$$

$$- \left(u_1^2 v_1^2 + 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_2u_3v_2v_3 + u_2^2v_2^2 + u_3^2v_3^2 \right)$$

$\Rightarrow$ LHS = RHS ☺

So $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - |\vec{u} \cdot \vec{v}|^2$

$$= \|\vec{u}\|^2 \|\vec{v}\|^2 - (\|\vec{u}\| \|\vec{v}\| \cos\theta)^2$$

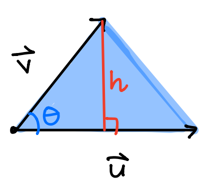
$$= \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2\theta)$$

$$= \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2\theta$$

since $\|\vec{u} \times \vec{v}\|$, $\|\vec{u}\| \|\vec{v}\| \sin\theta \geq 0$, we can take square roots:

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta \quad \text{as claimed} \quad \square.$$

Geometric content of this equality



$\vec{u}, \vec{v}$ determine a triangle T with area

$$A(T) = \tfrac{1}{2} bh$$

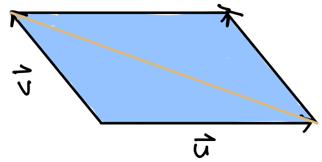
$$= \tfrac{1}{2} \|\vec{u}\| \|\vec{v}\| \sin\theta$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{h}{\|\vec{v}\|}$$

$$= \tfrac{1}{2} \|\vec{u} \times \vec{v}\|.$$

Conclusion: The area of the triangle spanned by $\vec{u}$ & $\vec{v}$ is $\tfrac{1}{2} \|\vec{u} \times \vec{v}\|$.

Equivalently: $\vec{u}, \vec{v}$ determine a parallelogram P



Area of parallelogram

$= A(P) = 2A(T)$

$= \|\vec{u} \times \vec{v}\|.$

Example: Suppose that $\vec{u}, \vec{v} \in \mathbb{R}^3$ such that

$\|\vec{u}\| = \sqrt{6}, \quad \|\vec{v}\| = \sqrt{35}, \quad \vec{u} \cdot \vec{v} = 14.$

What is $\|\vec{u} \times \vec{v}\|$?

Solution

- We know $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$ angle between $\vec{u}$ & $\vec{v}$.

- We also know $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

$$\Rightarrow \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{14}{\sqrt{6}\sqrt{35}} = \frac{14}{\sqrt{210}}$$

- Now $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \frac{14^2}{210} = \frac{210 - 196}{210} = \frac{14}{210}$$

$\Rightarrow$ since $\sin\theta \in [0,1)$, $\sin\theta = \dfrac{\sqrt{14}}{\sqrt{210}}$.

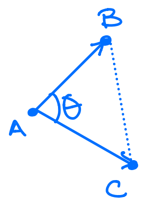
So $\|\vec{u} \times \vec{v}\| = \sqrt{6}\sqrt{35}\dfrac{\sqrt{14}}{\sqrt{210}} = \sqrt{14}.$

Example:

If $A = (1,1,2)$, $B = (3,1,1)$, $C = (0,4,2)$, find the area of the triangle ΔABC.

Solution

Area $= \frac{1}{2} \|\vec{AB} \times \vec{AC}\|$ (have three other options!)



$$\vec{AB} = \begin{bmatrix} 3-1 \\ 1-1 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} 0-1 \\ 4-1 \\ 2-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{AB} \times \vec{AC} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+3 \\ 1-0 \\ 6-0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$$

$$\Rightarrow \|\vec{AB} \times \vec{AC}\| = \sqrt{3^2+1^2+6^2}$$
$$= \sqrt{9+1+36}$$
$$= \sqrt{46}.$$

$\therefore$ Area $= \frac{1}{2}\sqrt{46}$.

Remark : Trick # 4 for remembering the cross product formula.

It's enough to remember
$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$$
$$\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$$
$$\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$$

use right hand rule and area of parallelogram

 $=1$

Then given $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$,

we write
$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$$
$$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$$

$$\Rightarrow \vec{u} \times \vec{v} = (u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3) \times (v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3)$$

Exercise : expand this and see that you recover the formula for $\vec{u} \times \vec{v}$.

Solution: $\overset{0}{u_1v_1\vec{e}_1\times\vec{e}_1} + \overset{\vec{e}_3}{u_1v_2\,\vec{e}_1\times\vec{e}_2} + \overset{-\vec{e}_2}{u_1v_3\,\vec{e}_1\times\vec{e}_3}$

$+ \overset{-\vec{e}_3}{u_2v_1\,\vec{e}_2\times\vec{e}_1} + \overset{0}{u_2v_2\,\vec{e}_2\times\vec{e}_2} + \overset{\vec{e}_1}{u_2v_3\,\vec{e}_2\times\vec{e}_3}$

$+ \overset{\vec{e}_2}{u_3v_1\,\vec{e}_3\times\vec{e}_1} + \overset{-\vec{e}_1}{u_3v_2\,\vec{e}_3\times\vec{e}_2} + \overset{0}{u_3v_3\,\vec{e}_3\times\vec{e}_3}$

$$= (u_2v_3 - u_3v_2)\,\vec{e}_1 + (u_3v_1 - u_1v_3)\,\vec{e}_2 + (u_1v_2 - u_2v_1)\,\vec{e}_3$$

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Summary of the lecture

The cross product $\vec{u}\times\vec{v}$ is determined by its length and direction

1) the direction is determined by the right-hand rule.

2) the length satisfies

$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$$

= area of parallelogram with sides $\vec{u}, \vec{v}$

= $\overset{2}{\cancel{\tfrac{1}{2}}}$ (area of triangle with sides $\vec{u}, \vec{v}$)

You should be able to

- use these properties to identify the direction or the length of a cross product.
- find area of triangles & parallelograms.