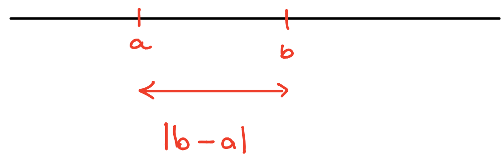
MATH1002 LECTURE 2-E

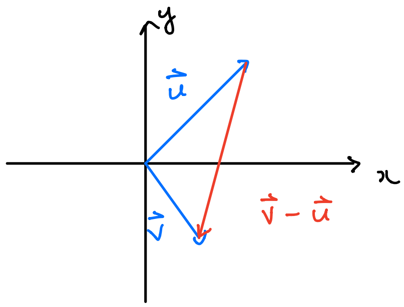
DISTANCE AND ANGLES BETWEEN VECTORS

Recall: The distance between two points $a, b \in \mathbb{R}$ is the absolute value of their difference.



$$|b-a| = \sqrt{(b-a)^2}$$

Similarly, we define the distance between two vectors $\vec{u}, \vec{v}$ to be the length/magnitude of their difference:



Definition: the distance between two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is

$$d(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\|$$

$$= \sqrt{(u_1-v_1)^2 + \cdots + (u_n-v_n)^2}$$

↳ a measure of how "different" $\vec{u}$ & $\vec{v}$ are.

Key fact: $d(\vec{u}, \vec{v}) = 0 \iff \vec{u} = \vec{v}$.

Exercise: prove this.

Solution: $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = 0$

$\iff \vec{u} - \vec{v} = \vec{0}$

$\iff \vec{u} = \vec{v}$. $\square$.

Another key fact $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$

Example

Exercise: Find the distance between $\vec{u} = \begin{bmatrix} 1 \\ \sqrt{2} \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$.

Solution: We have $\vec{u} - \vec{v} = \begin{bmatrix} 1-2 \\ \sqrt{2}-0 \\ -1-(-2) \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

$$\Rightarrow d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(-1)^2 + \sqrt{2}^2 + 1^2}$$

$$= \sqrt{1+2+1} = \sqrt{4}$$

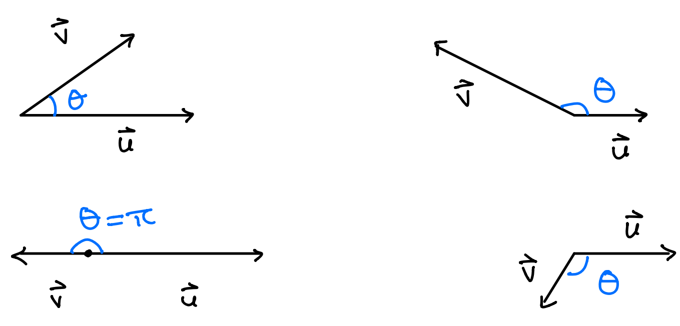
$$= 2.$$

The angle between two vectors:

- measures the difference between the directions of two vectors (length is not relevant here)
- gives a geometric interpretation of the dot product.

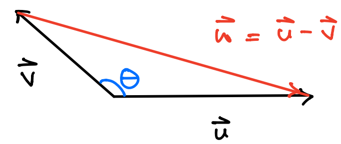
Start with $\mathbb{R}^2$, $\mathbb{R}^3$:

We can measure the angle between two vectors, to find a value $\theta \in [0, \pi]$



Recall from school: The cosine rule

Consider a triangle with sides $\vec{u}$, $\vec{v}$ & $\vec{w} = \vec{u} - \vec{v}$



Let $\theta \in [0, \pi]$ be the angle between $\vec{u}$ & $\vec{v}$.

Then $\quad \|\vec{w}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$

On the other hand, we now know:

$$\|\vec{w}\|^2 = \|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$$

$$= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v}.$$

Comparing the two formulas for $\|\vec{w}\|^2$, we conclude that

$$\|\vec{u}\| \|\vec{v}\| \cos\theta = \vec{u} \cdot \vec{v}.$$

Motivated by this, we make the following definition:

Definition: For two non-zero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, we define the angle between $\vec{u}, \vec{v}$ to be the unique value $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

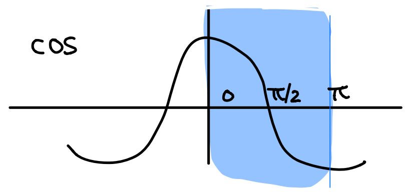
Remark: This only makes sense if we know $\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \in [-1, 1]$.

Cauchy-Schwarz: $\quad |\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$

$$\Rightarrow \frac{|\vec{u} \cdot \vec{v}|}{\|\vec{u}\| \|\vec{v}\|} \leq 1$$

$$\Rightarrow -1 \leq \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \leq 1 \qquad ☺$$

Remark:



If $\vec{u}\cdot\vec{v} > 0$, then $\cos\theta > 0 \Rightarrow \theta$ is acute.

If $\vec{u}\cdot\vec{v} < 0$ then $\cos\theta < 0 \Rightarrow \theta$ is obtuse.

If $\vec{u}\cdot\vec{v} = 0$ then $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$, a right angle.

Definition: We say that two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal if $\vec{u}\cdot\vec{v} = 0$.

- By the above remark, this means that the angle between $\vec{u}$ & $\vec{v}$ is $\pi/2$.

  i.e. $\vec{u}$ & $\vec{v}$ are perpendicular.

Example 1 Determine whether the angle between $\vec{u} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is acute, obtuse, or a right angle.

Solution: We have $\vec{u}\cdot\vec{v} = 1\times 1 + (-3)\times 1 + 0\times 1 = 1-3$
$= -2 < 0.$

So $\cos\theta < 0$; the angle θ is obtuse.

Exercise: Find $\cos\theta$ and hence (using a calculator) find θ.

Solution: $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

- $\vec{u}\cdot\vec{v} = -2$ above
- $\|\vec{u}\| = \sqrt{1^2 + (-3)^2 + 0^2} = \sqrt{10}$

- $\|\vec{v}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\Rightarrow \cos\theta = -\dfrac{2}{\sqrt{30}}$.

$\Rightarrow$ (using a calculator) $\theta \approx 111°$

(which is obtuse!)

Example # 2 Find $a \in \mathbb{R}$ such that $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\begin{bmatrix}a^2\\a\\-3\end{bmatrix}$ are orthogonal.

Solution: We need to find a such that

$$\begin{bmatrix}1\\-1\\2\end{bmatrix} \cdot \begin{bmatrix}a^2\\a\\-3\end{bmatrix} = 0$$

i.e. $1 \times a^2 + (-1) \times a + 2 \times (-3)$

$= a^2 - a - 6$

$= (a-3)(a+2) = 0$

So the two vectors are orthogonal if $a = 3$ or $a = -2$

Summary of the lecture

- We defined
  - distance
  - angle

  between $\vec{u}, \vec{v} \in \mathbb{R}^n$.
- $\vec{u}, \vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$

You should be able to:

- Calculate the distance & angle between two vectors
- Use $\vec{u} \cdot \vec{v}$ to determine whether θ is acute, right, obtuse
- Solve for variables to ensure two vectors are orthogonal.