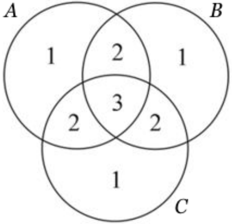
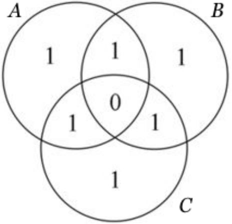
Discrete Mathematics

MATH1064, Lecture 22

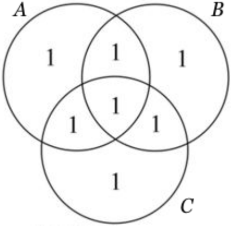
Jonathan Spreer



$|A| + |B| + |C|$



$|A| + |B| + |C|$
$- |A \cap B| - |A \cap C| - |B \cap C|$



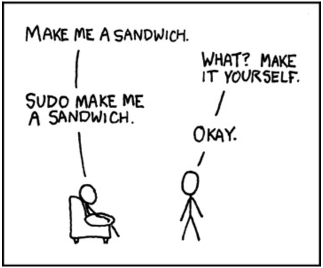
$|A| + |B| + |C|$
$- |A \cap B| - |A \cap C| - |B \cap C|$
$+ |A \cap B \cap C|$

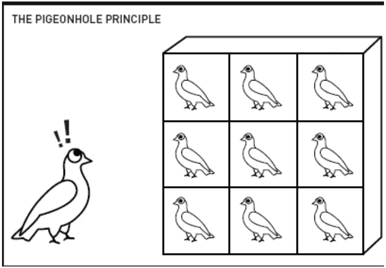
Extra exercises for Lecture 22

Section 6.1: Problems 62, 63

Section 6.2: Problems 1–5, 15–17

Section 8.5: Problems 1–5





A problem

MATH1984 (Introductory Navel Gazing) is compulsory for Science students and also compulsory for Arts students.

Nobody in their right mind would study MATH1984 unless they were forced to.

There are 300 students studying science and 400 studying arts.
How many people are studying MATH1984?

Choose:

1. 100
2. 700
3. 120 000
4. Something else

Suppose I tell you there are 300 Science students and 400 Arts students, and that 100 of these are dual Science-Arts students. How many study MATH1984?

Choose:

1. 100
2. 200
3. 300
4. 400
5. 500
6. 600
7. 700

Addition, revisited

Remember: For disjoint sets A and B: $|A \cup B| = |A| + |B|$.

Lemma

For any sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

What about three sets?

Suppose MATH1999 is a prerequisite for MATH2000, MATH2001 and MATH2002.

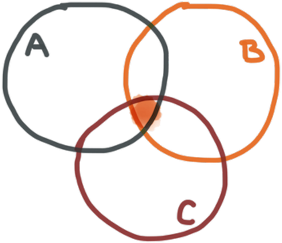
The class sizes are:

- MATH2000: 50 students
- MATH2001: 60 students
- MATH2002: 70 students
- 30 students take both MATH2000 and MATH2001
- 20 students take both MATH2000 and MATH2002
- 25 students take both MATH2001 and MATH2002
- 5 students take all three subjects

How many students take MATH1999?

In terms of sets

We need a formula for the size of a set union in terms of the sizes of set intersections:



$$
\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| \\
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\
&\quad + |A \cap B \cap C|
\end{aligned}
$$

Suppose MATH1999 is a prerequisite for MATH2000, MATH2001 and MATH2002.

The class sizes are:

- MATH2000: 50 students
- MATH2001: 60 students
- MATH2002: 70 students
- 30 students take both MATH2000 and MATH2001
- 20 students take both MATH2000 and MATH2002
- 25 students take both MATH2001 and MATH2002
- 5 students take all three subjects

How many students take MATH1999?

The inclusion-exclusion principle

For sets $A_1, A_2, \ldots, A_n$:

$$\begin{aligned} |A_1 \cup \ldots \cup A_n| &= \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| \\ &\quad + \sum_{i<j<k} |A_i \cap A_j \cap A_k| \\ &\quad - \sum_{i<j<k<\ell} |A_i \cap A_j \cap A_k \cap A_\ell| \\ &\quad + \ldots \pm |A_1 \cap A_2 \cap \ldots \cap A_n| \end{aligned}$$

To measure the size of the set union, we:

measure each set individually;

subtract all two-way intersections;

add all three-way intersections;

subtract all four-way intersections;

and so on!

Exercise

I have ten students, and I wish to give back their marked quizzes.

If I give the quizzes to students at random (without asking for student numbers, but ensuring that everybody receives a quiz), what is the probability that nobody receives their own quiz?

Let A_i be the set of all permutations of $1, \ldots, 10$ that keep i fixed. The probability that nobody receives their own exam is:

$$\frac{10! - |A_1 \cup \ldots \cup A_{10}|}{10!}$$

So... we need to measure $|A_1 \cup \ldots \cup A_{10}|$.

Measuring $|A_1 \cup \ldots \cup A_{10}|$

- The sum of all individual set sizes:
  For any i, $|A_i| = 9!$
  There are 10 sets $A_1, \ldots, A_{10}$, so $\sum |A_i| = 10 \cdot 9!$
- The sum of all two-way intersections:
  For any $i < j$, $|A_i \cap A_j| = 8!$
  There are $\binom{10}{2}$ pairwise intersections $A_i \cap A_j$,
  so $\sum |A_i \cap A_j| = \binom{10}{2} \cdot 8!$
- The sum of all three-way intersections:
  For any $i < j < k$, $|A_i \cap A_j \cap A_k| = 7!$
  There are $\binom{10}{3}$ triple intersections $A_i \cap A_j \cap A_k$,
  so $\sum |A_i \cap A_j \cap A_k| = \binom{10}{3} \cdot 7!$
- And so on...

By the inclusion-exclusion principle,
$|A_1 \cup \ldots \cup A_{10}| = 10 \cdot 9! - \binom{10}{2} \cdot 8! + \binom{10}{3} \cdot 7! - \ldots + \binom{10}{9} \cdot 1! - \binom{10}{10} \cdot 0!$

$$|A_1 \cup \ldots \cup A_{10}| = 10 \cdot 9! - \binom{10}{2} \cdot 8! + \binom{10}{3} \cdot 7! - \ldots + \binom{10}{9} \cdot 1! - \binom{10}{10} \cdot 0!$$

The final probability:

$$\frac{10! - 10 \cdot 9! + \binom{10}{2} \cdot 8! - \ldots - \binom{10}{9} \cdot 1! + \binom{10}{10} \cdot 0!}{10!} = \frac{1\,334\,961}{3\,628\,800}$$

This is $\simeq 0.3678795 \simeq \dfrac{1}{e}$.

Why $\frac{1}{e}$?

$$\frac{\binom{10}{k} \cdot (10-k)!}{10!} = \frac{10!}{k!(10-k)!} \cdot \frac{(10-k)!}{10!} = \frac{1}{k!}$$

So the probability is actually:

$$\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{1}{10!} \simeq \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} = e^{-1}$$

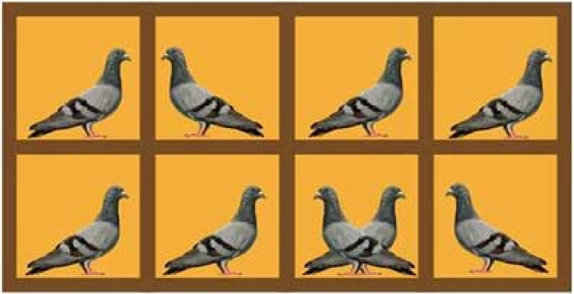
Another exercise

Everybody in the class has at least one mobile phone.
40 students have iPhones, 50 students have Android phones, and 30 students have Windows phones.
10 students have an iPhone *and* an Android phone, 15 students have an Android phone *and* a Windows phone, and 5 students have an iPhone *and* a Windows phone.
3 high-tech students have all three phones.

How many students in the class?

1. 87
2. 90
3. 93
4. 100
5. 120

Remember this from week 1?



The pigeonhole principle

If you have n pigeons sitting in k pigeonholes, and if $n > k$, then at least one of the pigeonholes contains at least two pigeons.

Let's generalise!

The generalised pigeonhole principle

If you have n pigeons sitting in k pigeonholes, and if $n > k \cdot m$, then at least one of the pigeonholes contains at least $m + 1$ pigeons.

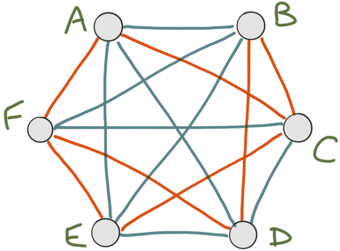
The "original" pigeonhole principle corresponds to the case $m = 1$.

Example: Suppose we have $n = 101$ pigeons to fit into $k = 20$ holes. Then, using the generalised pigeonhole principle with $m = 5$, we can conclude that some hole contains ≥ 6 pigeons.

Puzzle C, from practice sheet 2

Show that in any group of six people, either three people all know each other, or three people all don't know each other. We assume that whenever A knows B then B also knows A.

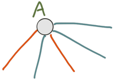
Represent people using dots, and relationships using lines.
Orange line: x knows y.
Blue line: x does not know y.



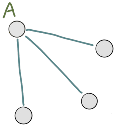
Show that there is always an orange triangle or a blue triangle!

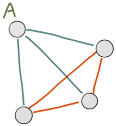
Proof

Dot A has five lines coming out of it.



By the pigeonhole principle, at least three of these lines must be the same colour! Without loss of generality, suppose this is blue.





Now look at the three "adjacent" dots at the other end of these blue lines. Are any of these dots joined by another blue line?

If so, this extra blue line forms a blue triangle with A. If not, the three adjacent dots form an orange triangle amongst themselves!

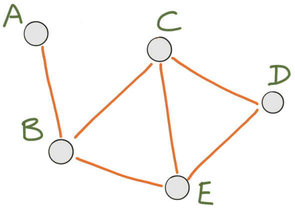
Either way, we have a blue triangle or an orange triangle. $\square$

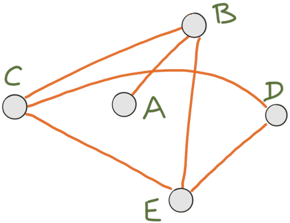
Graphs

We have a language for these types of arguments: graph theory.
We will study this in Week 11. Here is a sneak preview.

A graph is a collection of vertices (dots), some of which may be joined by edges (lines).

The vertices represent objects, and the edges represent relationships between these objects.





The drawing is irrelevant: both graphs above are identical.
What matters is which vertices are joined to which others.

Ramsey theory

We have shown: Every graph on six vertices has at least a triangle or has an independent set of size three.

The Ramsey number R(3,3) is six.

http://mathworld.wolfram.com/RamseyNumber.html