THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Counting – Week 8 Practice Class

MATH1064: Discrete Mathematics for Computing

Here is a list of **problems** for the practice class. Try to solve them before you go to class! There are more problems here than can be solved in the hour, so you should get started on them!

1. The Sydney Uni drama club is auditioning for its annual production of *Romeo and Juliet*. Six people are auditioning for the role of Romeo, and eight other people are auditioning for the role of Juliet. In how many ways can the director cast the lead couple?

Solution: Assume we cast the role of Romeo first. There are six choices for the role of Romeo. Second we cast the role of Juliet. It is given that the eight people auditioning for Juliet are different from those auditioning for Romeo so there are no restrictions on how the director can cast the role of Juliet. Therefore, for each choice of Romeo there are eight choices for the role of Juliet, each giving rise to a distinct lead couple. Therefore the total number of choices is:

$$6 \times 8 = 48$$

2. There are eleven questions on the final exam, but you decide based on your successful grade for your quizzes that you will only do five of them. How many combinations of questions are there which you could complete?

Hint: Don't try this in November!

Solution: First assume that we pick which questions to do in order. For the first question we choose to attempt we have eleven choices, for the second we have ten and so on. So if we were considering ordered choices of questions there would be

$$11 \times 10 \times 9 \times 8 \times 7 = \frac{11!}{6!}$$

possibilities. However, in this question it doesn't matter what order we complete the questions so we need to divide out by the number of ways in which we could have completed the same questions in a different order. That is, the number of ways in which you can re-order five objects, i.e. 5!. Therefore the final answer is

$$\frac{11!}{6! \times 5!} = \binom{11}{5}$$

3. For a university soccer tournament, the Schools of IT and of Maths and Stats need to nominate five computer scientists and six mathematicians. If there are 28 computer scientists and 40 mathematicians who would like to play, how many teams can be chosen if you don't take into account the positions they play? How does the number change if you do take into account the positions they play?

Solution: For the first part of the question we are making a choice of people in which the order that we choose people does not matter and we do not allow the same person to be chosen twice. Therefore the answer to the first part is

$$\binom{28}{5}\binom{40}{6}$$

In the second part of the question we care about what position each person plays. One way to think about this is that we now care about what order we choose the people from each school and we also need to take into account what positions are filled by which school. First, the number of ways to select 5 students out of 28 from the school of IT if we care about order is:

$$28 \times 27 \times 26 \times 25 \times 24 = \frac{28!}{23!}$$

Similarly the number of ways to select 6 students out of 40 from the school of maths and stats if we care about order is:

$$40 \times 39 \times 38 \times 37 \times 36 \times 35 = \frac{40!}{34!}$$

The number of different ways to allocate positions between the two schools is exactly the number of ways to choose five positions out of eleven. Or, equivalently to choose six positions out of eleven. In either case we don't care about order in this selection.

$$\binom{11}{5} = \binom{11}{6}$$

Therefore the final answer is:

$$\frac{28!}{23!} \times \frac{40!}{34!} \times \binom{11}{5}$$

4. Without a calculator, compute the values of $\binom{9}{3}$ and $\binom{200}{198}$.

Solution:

$$\binom{9}{3} = \frac{9!}{3!6!} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 3 \times 4 \times 7 = 84$$

$$\binom{200}{198} = \frac{200!}{198!2!} = \frac{200 \times 199}{2 \times 1} = 19900$$

5. A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind as you enter the shop, in how many ways can you select a dozen donuts?

Solution: We are making a choice of 12 doughnuts from 20 options. We are allowed to choose the same doughnut twice but the order in which we choose them doesn't matter. Therefore the answer is

$$\binom{12+20-1}{12} = \binom{31}{12}$$

6. Suppose you have six bottles which you wish to put in five fridges. How many possible outcomes are there if the bottles are not distinguished from each other, but the fridges are?

Solution: Imagine lining all six bottles in a row and dividing them with four lines. All bottles to the left of the left-most line will go in the first fridge. All bottles between the first and second lines from the left will go in the second fridge and so on. We need to calculate the number of ways in which we can order six identical bottles and four identical lines. The answer is

$$\frac{10!}{6! \times 4!} = \binom{10}{6}.$$

Alternatively, we have six choices (one for each bottle) with five options (the fridge the bottle has to go to). This is another istance of a counting problem of type "order does not matter, repetition allowed" and hence the solution is

$$\binom{6+5-1}{6} = \binom{10}{6}.$$

7. In how many ways can you distribute seven bananas and six oranges between four children so that each child receives at least one banana?

Solution: We will assume that the the bananas are all identical so first we just give a banana to each child. Now the question is reduced to 'in how many ways can you distribute three bananas and six oranges between four children?' It is not specified in the question whether or not the children are distinguished, we will assume that they are. We consider the bananas and oranges separately but the method of proof is the same for both, and the same as in Question 6. Imagine we line up three bananas and draw three lines between them so that the bananas to the left of the left-most line are given to the first child and so on.

$$\frac{6!}{3!3!} = \binom{6}{3}$$

Similarly the number of ways in which we can distibute the oranges is

$$\frac{9!}{6!3!} = \binom{9}{3}$$

Therefore the final answer is

$$\binom{6}{3} \times \binom{9}{3}$$

8. Prove that

$$n \cdot \binom{2n}{n} = (n+1) \cdot \binom{2n}{n-1}.$$

Give an algebraic proof, and also a combinatorial proof.

Solution: First the algebraic proof

$$n \cdot \binom{2n}{n} = n \cdot \frac{(2n)!}{n! \times n!} = \frac{(2n)!}{n!(n-1)!} = (n+1) \cdot \frac{(2n)!}{(n+1)!(n-1)!}$$
$$= (n+1) \cdot \binom{2n}{n-1}$$

A combinatorial proof would go as follows. Imagine we are trying to colour $2n$ objects so that n of them are red, 1 is blue and $(n-1)$ of them are green. Suppose we first choose which objects should be red. There are

$$\binom{2n}{n}$$

ways to do this. Now choose which of the remaining objects should be blue. There are n ways to do this. So if we count by first painting the n objects then painting the blue objects and painting all remaining objects green, we have

$$n \cdot \binom{2n}{n}$$

ways to perform the painting. However if we first painted the green objects then the blue object and painted all remaining objects red, we have

$$(n+1) \cdot \binom{2n}{n-1}$$

ways to perform the colouring. Therefore we have shown

$$n \cdot \binom{2n}{n} = (n+1) \cdot \binom{2n}{n-1}$$

9. Give a combinatorial proof of the fact that if $n = 2k$, where $k > 0$, then $\frac{n!}{2^k}$ is an integer.

Hint: $\frac{n!}{2^k}$ is counting something. But what? Think about the n symbols $a_1, a_1, a_2, a_2, \dots, a_k, a_k$.

Solution: For a combinatorial proof ask what is the number of ways we can order the set of n symbols $a_1, a_1, a_2, a_2, \dots, a_k, a_k$. The total number of symbols is n and the number of ways we can order n symbols is $n!$. However there are k pairs of identical symbols so we need to divide our answer by 2 for each pair. The final answer is

$$\frac{n!}{2^k}$$

Therefore $\frac{n!}{2^k}$ must be an integer.

10. 1. Explain why, in a group of 50 people, at least five people were born in the same month.

2. How many people do you need to ensure that at least seven people were born in the same month?
3. Show that, if we select six distinct integers from the set $\{1,2,\ldots,10\}$, two of these must add to give 11.

Solution:

1. Apply the pigeonhole principle. There are 12 holes, each representing the month in which a person was born. There are 50 people. Of the first 48 people it is possible that there are no five share the same month of birth. However this is only possible if four of the first 48 people are born in each of the twelve months. This means that the 49^{th} person must share a birthday with four other people.
2. We need $6 \times 12 + 1 = 73$.
3. Suppose there was such a choice of six distinct integers. Note first that for each integer $x \in \{1,2,\ldots 10\}$ there is a unique integer $y = 11 - x$. Therefore if x is included in our choice of six distinct integers then we can't include y.

   If we first choose the number x_1 as part of our set, we have 8 choices for the next number in the set because we can't choose x_1 and we can't choose $11 - x_1$. Similarly we have 6 choices for the third number, 4 for the fourth, 2 for the fifth and 0 for the sixth. Hence it is not possible to make such a selection.

11. Given a positive integer n, in how many ways can you choose four integers a,b,c,d with the property $0 \leq a \leq b \leq c \leq d \leq n$?

For example, if $n = 1$ then the answer is 5: (a,b,c,d) = $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,1,1)$, or $(1,1,1,1)$.

Solution: Imagine we are dividing up eleven dots using four lines so that the number of dots to the left of the left-most line is a, the number of dots to the left of the line second from the left is b and so on. The number of ways in which we can do this is

$$\frac{(11+4)!}{4! \times 11!} = \binom{15!}{11!}$$

Puzzles on next page!

Here are two **puzzles** that you can think about during the week. Feel free to ask your tutors or lecturer for more hints!

K Josh and Scott are having trouble balancing their budgets, and so they decide to return to school. At the end of the school year, the teacher wants to test the prime division skills of the entire class of 25 students. They are all lined up in a queue, and Josh and Scott stand next to each other.

The teacher writes a natural number on the board. The first student in the queue says "That number is divisible by 1." Then the second student in the queue says "That number is divisible by 2," and so on, the kth student says "That number is divisible by k," until the final student in the queue says "That number is divisible by 25."

"Well done," the teacher exclaims, "except for Josh and Scott everyone gave a correct answer." Where did Josh and Scott stand in the queue?

L Suppose you have a pizza and a knife. Into how many pieces can you cut the pizza with n straight cuts? If you cut the pizza in the usual way, using diameters, you get $2n$ pieces. If all of your cuts are parallel, you get $n+1$ pieces.

What is the maximal number of pieces into which you can cut your pizza using n cuts?

This problem is similar to the Tower of Hanoi problem: you can first find a recursion for the maximal number, and then a closed form.

Puzzle hints:

Stuck on the puzzles from the last sheet? Here are some hints!

I Look at the example $\frac{3}{8} = \frac{1}{3} + \frac{1}{24}$. You can rewrite this as $\frac{3}{8} = \frac{8}{24} + \frac{1}{24}$; notice that 8 and 1 are both distinct common divisors of the common denominator 24.

J It is useful to know *how many* digits are in each number, even if you cannot see what those digits are.