THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Predicates – Week 3 Practice Class

MATH1064: Discrete Mathematics for Computing

Exercises: Do a selection of problems from each Section 1.1 to Section 1.8. There is no need to do every single exercise, but you should feel confident that you can do each type of question!

Here is a list of **problems** for the contact class. Try to solve them before you go to class!

1. For each of the following arguments: If you think it is valid, give a formal derivation stating in each step the appropriate rule of inference. If you think it is invalid, justify your answer.

1. If Parramatta elects a new mayor, then it will raise taxes. If Parramatta does not raise taxes, then it will not build a new stadium. Parramatta does not elect a new mayor. Therefore it will not build a new stadium.
2. Every baby that eats will make a mess and drool. Every baby that drools will smile. There is a baby who eats and screams. Therefore there is a baby who smiles.

Solution:

1. The solution to this question has two parts. In the first part we convince ourselves, that this argument is probably invalid by linking it to a logical fallacy that may have played a prominent role in writing down this argument.

   We then proceed to give a formal proof of invalidity by finding truth variables to our propositions satisfying all of the premises but falsifying the conclusion.

   Mathematically, we do not need the first part. This is simply a demonstration on how to analyse an argument.

   First part: We have three statements: $m =$ "Parramatta elects a new major." $t =$ "Parramatta raises taxes." $s =$ "Parramatta builds a new stadium."

   Our example of a formal derivation is then:

   (a) $m \to t$ (Premise)
   (b) $\neg t \to \neg s$ (Premise)
   (c) $\neg m$ (Premise)
   (d) $\neg t \to \neg m$ (Contrapositive of 1)
   (e) $\neg m \to \neg t$ (Fallacy: Affirming the consequent of 4)
   (f) $\neg t$ (Modus ponens (5))
   (g) $\neg s$ (Modus ponens (6) $\equiv$ Conclusion)

   This is an invalid formal derivation.

   Second part: We now have a conjecture that this argument is invalid. If we assign false to m, true to t, and true to s, then premises $m \to t$, $\neg t \to \neg s$, and $\neg m$ are certainly true, but the conclusion $\neg s$ is false. So $(a) \wedge (b) \wedge (c) \to (g)$ is not a tautology and our argument is invalid.

2. We first mention three inference rules for quantified statements. These rules are known as existential instantiation, existential generalization, and universal modus ponens. Please see pages 75–78 of Rosen for a thorough discussion.

   Next we define five predicates which will be used in our solution:

   - $E(x)$ means "x eats".
   - $M(x)$ means "x makes a mess".
   - $D(x)$ means "x drools".
   - $S(x)$ means "x smiles".
   - $C(x)$ means "x screams".

   Finally we present a valid formal argument which shows that the desired conclusion follows from the premises.

1.	$\forall x(E(x) \to M(x) \land D(x))$	Premise
2.	$\forall x(D(x) \to S(x))$	Premise
3.	$\exists x(E(x) \land C(x))$	Premise
4.	$E(b) \land C(b)$	Exist. instant. of 3
5.	$E(b)$	Simplific. of 4
6.	$M(b) \land D(b)$	Univ. modus ponens 1,5
7.	$D(b)$	Simplific. of 6
8.	$S(b)$	Univ. modus ponens 2,7
9.	$\exists x S(x)$	Exist. general. of 8

2. Write the statement "if an integer is a multiple of 6, then it also is a multiple of 3" in symbolic form, using the quantifiers $\forall$ and/or $\exists$. Then determine the negation of this statement.

Solution: We talk about the set of all integers (written $\mathbb{Z}$). The above statement can be written as follows:

$$\forall x \in \mathbb{Z}, (\exists y \in \mathbb{Z} : x = 6y) \ \to \ (\exists z \in \mathbb{Z} : x = 3z)$$

Negation (we use that $p \to q \equiv \neg p \lor q$):

$$\begin{aligned} \neg(\forall x, (\exists y : x = 6y) \ \to \ (\exists z : x = 3z)) &\equiv \neg(\forall x, (\neg(\exists y : x = 6y) \lor (\exists z : x = 3z))) \\ &\equiv \exists x : ((\exists y : x = 6y) \land \neg(\exists z : x = 3z)) \\ &\equiv \exists x : ((\exists y : x = 6y) \land (\forall z : x \neq 3z)) \end{aligned}$$

3. Let $Q(x,y)$ be the predicate "x praises y", with domain "people". Write the following sentences using the existential and universal quantifiers.

1. Someone praises someone.
2. Everyone praises everyone.
3. Everybody praises somebody.
4. There is someone who everybody praises.

5. Everyone is praised by someone.
6. There is someone who praises everyone.

Solution:

1. $\exists x, \exists y : Q(x,y)$
2. $\forall x, \forall y : Q(x,y)$
3. $\forall x, \exists y : Q(x,y)$ (note, here y can be different for every x)
4. $\exists y, \forall x : Q(x,y)$ (note, here x is a fixed person)
5. $\forall y, \exists x : Q(x,y)$
6. $\exists x, \forall y : Q(x,y)$

4. Let $P(x)$ be the predicate "x is a multiple of 10", and let $S(x)$ be the predicate "x is a multiple of 5". Suppose the domain of x is $\{1,2,\ldots,99\}$ (that is, $x \in \{1,2,\ldots,99\}$). Determine the truth sets for $P(x)$ and $S(x)$ and indicate the relationship between $P(x)$ and $S(x)$ using some of the symbols $\forall$, $\exists$, $\to$ and $\leftrightarrow$.

Solution: Truth set of $P(x)$ is $\{10,20,30,40,50,60,70,80,90\}$.

Truth set of $S(x)$ is $\{5,10,15,20,25,30,35,40,45,50,55,60,65,70,75,80,85,90,95\}$.

We have

$$\forall x \in \{1,2,\ldots,99\}, (P(x) \to S(x))$$

5. For each of the following sets of statements, determine whether it is consistent or inconsistent.

1. $\{p \vee (\neg p \wedge q),\ \neg p \wedge \neg q\}$
2. $\{p \vee (\neg p \wedge q),\ p \to q\}$

Solution:

1. We show that the first set of statements is inconsistent by showing, that their conjunction is a contradiction.

$$\begin{aligned}(p \vee (\neg p \wedge q)) \wedge (\neg p \wedge \neg q) &\equiv ((p \vee \neg p) \wedge (p \vee q)) \wedge (\neg p \wedge \neg q)\\ &\equiv (p \vee q) \wedge (\neg p \wedge \neg q)\\ &\equiv (p \vee q) \wedge \neg(p \vee q)\end{aligned}$$

   Where in the first line we use the distributive law, in the second line we remove the tautology $(p \vee \neg p)$ and in the third line we use one of De Morgan's laws.

2. We show that the second set of statements is consistent by identifying truth values of p and q which satisfy all statements simultaneously.

p	q	$(p \vee (\neg p \wedge q)$	$(p \rightarrow q)$
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

It follows that the set of statements $\{p \vee (\neg p \wedge q),\ p \rightarrow q\}$ is consistent (see rows two and four of the truth table).

6. If a and b are integers such that $a = 4x + 1$ and $b = 4y + 1$ for some $x, y \in \mathbb{Z}$, then prove that the product ab is of the the form $4m + 1$, for some integer m. Can you generalise this?

Solution: Using logic, what do we want to do?

Consider the predicate

$$P(c) = \text{“}\exists x : c = 4x + 1\text{”}$$

We want to show that

$$\forall a, b \in \mathbb{Z} : ((P(a) \wedge P(b)) \rightarrow P(ab))$$

Reminder, with $p = (P(a) \wedge P(b))$ and $q = P(ab)$ we can now show, for instance,

1. $p \rightarrow q$ by logic inferences (direct proof)
2. $(\neg(p \rightarrow q) \equiv$ (contradiction) (proof by contradiction)
3. $\neg q \rightarrow \neg p$ (proof by contraposition)

We use a direct proof.

Assume $P(a) \wedge P(b)$, so $a = 4x + 1$ and $b = 4y + 1$ for $x, y \in \mathbb{Z}$. We have

$$\begin{aligned} ab &\equiv (4x + 1)(4y + 1) \\ &\equiv 16xy + 4x + 4y + 1 \\ &\equiv 4(4xy + x + y) + 1 \end{aligned}$$

It follows that $ab = 4z + 1$ for $z = 4xy + x + y \in \mathbb{Z}$ and hence $P(ab)$ (that is, ab is in the truth set of P).

7. Prove that there is no rational number r for which $r^3 + r + 1 = 0$.

(Hints: Use proof by contradiction, and consider cases depending on whether the denominator or numerator of your rational number are odd or even.)

Solution:

Proof by contradiction: Assume that there exists a rational number $r \in \mathbb{Q}$ such that $r^3 + r + 1 = 0$. That is, there exist integers $p, q \in \mathbb{Z}$, $q \neq 0$, such that $r = p/q$.

It follows that $(p/q)^3 + p/q + 1 = 0$ and after multiplying by q^3 we have

$$p^3 + pq^2 + q^3 = 0.$$

If both p and q are even, then $r = (p/2)/(q/2)$ and we can proceed with $p/2$ and $q/2$. Iterate this step until at least one of p and q is odd. (In other words: Without loss of generality we can assume that at least one of p and q is odd.)

Now we have three cases:

1. p is even and q is odd. Then p^3 and pq^2 are even and q^3 is odd. Hence $p^3 + pq^2 + q^3$ is odd, a contradiction to $p^3 + pq^2 + q^3 = 0$.
2. p is odd and q is even. Now p^3 is odd, but pq^2 and q^3 are even, again a contradiction.
3. Both p and q are odd. But now p^3, pq^2 and q^3 are odd, and thus $p^3 + pq^2 + q^3$ is, once agaon, odd. A contradiction.

It follows that now $p, q \in \mathbb{Z}$, $q \neq 0$ can exist such that $r = p/q$ and hence $r \notin \mathbb{Q}$ is irrational.

8. Prove that for any two distinct rational numbers x and y, there is a rational number r that is strictly greater than one of them and strictly less than one of them.

Solution: Without loss of generality assume that $x < y$. We want to prove that there exists a rational number r such that $x < r < y$.

Since both $x, y \in \mathbb{Q}$ are rational we have integers $p, q, r, s \in \mathbb{Z}$, $q, s \neq 0$ such that $x = p/q$ and $y = r/s$. It follows that we have for the average

$$\frac{x+y}{2} = \frac{sp+qr}{2qs}.$$

But $sp + qr, 2qs \in \mathbb{Z}$ and $2qs \neq 0$ and so $\frac{sp+qr}{2qs} = \frac{x+y}{2} \in \mathbb{Q}$ is rational.

It remains to show that $x < \frac{x+y}{2} < y$.

For the first inequality observe the following sequence of equivalent (biconditional) statements

$$\begin{aligned} x &< \frac{x+y}{2} \\ 2x &< x+y \\ x &< y, \end{aligned}$$

which is true by assumption.

Similarly, for the second inequality we have

$$\begin{aligned} \frac{x+y}{2} &< y \\ x+y &< 2y \\ x &< y, \end{aligned}$$

which, again, is true be assumption. In summary, we have $x < r < y$ for $r = (x+y)/2$ and $r \in \mathbb{Q}$.

Puzzles on next page!

Here are two **puzzles** that you can think about during week 3. Feel free to ask your tutors or lecturer for more hints!

C Show that in any group of six people, either three people all know each other, or three people all don't know each other. We assume that whenever A knows B then B also knows A.

D What is the smallest integer n with the following property: for every choice of n integers, there exist at least two whose sum or difference is divisible by 2015? Can you prove this?

Puzzle hints:

Stuck on the puzzles from week 2? Here are some hints!

A Drawing one common perpendicular gives you *two* points of the same colour, but not *four* points at the corners of a rectangle. What happens if you draw two perpendiculars? What happens if you draw many?

B You can't say much about any three consecutive points in particular, since you don't know where the labels $1, \ldots, 10$ have been placed. Can you say something about the *sum* of *all ten* sets of three consecutive points?