MATH 1002 LECTURE 3-A.

CROSS PRODUCT

Recall: Taking the dot product of two vectors in $\mathbb{R}^n$ produces a scalar.

In $\mathbb{R}^3$: there is another operation that produces a vector.

Definition: The cross product of two vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$ is the vector

$$\boxed{\vec{u} \times \vec{v} := \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}}$$

Input: two vectors in $\mathbb{R}^3$

$\downarrow$

Output: one vector in $\mathbb{R}^3$

How are we supposed to remember this?

(a) Practice!

(b) Trick # 1:
- Memorise the first component $u_2 v_3 - u_3 v_2$.
- Now you can deduce the second & third components by permuting the subscripts:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 1$$

second: $u_3 v_1 - u_1 v_3$

(c) Trick # 2:

$$\begin{array}{cc} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \\ u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{array} \quad \begin{array}{c} \\ \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \\ \end{array}$$

(solid line) − (dashed line)

e.g. 3rd component

$u_1 v_2 - u_2 v_1$.

(d) <u>Trick # 3</u>: we'll see a mnemonic later on using **determinants**.

<u>Example # 1</u> Let $\vec{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$.

Then $\vec{a} \times \vec{b} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} 1\times 4 - 2\times 0 \\ 2\times 5 - 3\times 4 \\ 3\times 0 - 1\times 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -5 \end{bmatrix}$$

$$\begin{array}{cc} 3 & 5 \\ 1 & 0 \\ 2 & 4 \\ 3 & 5 \\ 1 & 0 \\ 2 & 4 \end{array}$$

<u>Example # 2</u>: Recall the standard vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

$\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

<u>Exercise</u>: Find $\vec{e}_1 \times \vec{e}_2$, $\vec{e}_2 \times \vec{e}_3$, $\vec{e}_3 \times \vec{e}_1$.

<u>Solution</u>: $\vec{e}_1 \times \vec{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 - 0 \\ 0 - 0 \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{e}_3.$$

$$\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}$$

Similarly, $\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$; $\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$.

Exercise: For $\vec{a}, \vec{b} \in \mathbb{R}^3$, is it true that $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$?

Solution: No!

$$\left.\begin{array}{l} \vec{a} \times \vec{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \\[2em] \vec{b} \times \vec{a} = \begin{bmatrix} b_2a_3 - b_3a_2 \\ b_3a_1 - b_1a_3 \\ b_1a_2 - b_2a_1 \end{bmatrix} \end{array}\right\} \Rightarrow \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

Properties of the cross product

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, $c \in \mathbb{R}$. Then

(a) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$! anti-commutativity. The order is important.

(b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ distributivity.

(c) $(c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$

(d) $\vec{u} \times \vec{u} = \vec{0}$

(e) $\vec{u} \times \vec{0} = \vec{0}$; $\vec{0} \times \vec{u} = \vec{0}$

(f) $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ & $\vec{v}$.

Exercise: prove (d): $\vec{u} \times \vec{u} = \vec{0}$

Solution: Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.

So $\vec{u} \times \vec{u} = \begin{bmatrix} u_2u_3 - u_3u_2 \\ u_3u_1 - u_1u_3 \\ u_1u_2 - u_2u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\square$.

Proof of (f): We need to show $(\vec{u}\times\vec{v})\cdot\vec{u} = 0$

$(\vec{u}\times\vec{v})\cdot\vec{v} = 0.$

- $$(\vec{u}\times\vec{v})\cdot\vec{u} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3$$

$$= \underline{u_1u_2}v_3 - \underline{u_1v_2}u_3 + \underline{v_1u_2}u_3 - \underline{u_1u_2}v_3 + \underline{u_1v_2}u_3 - \underline{v_1u_2}u_3$$

$$= 0$$

- $(\vec{u}\times\vec{v})\cdot\vec{v}$ is similar. □

Remark: There is no associative law for the cross product.

Example: We can show that $(\vec{e}_1\times\vec{e}_1)\times\vec{e}_2 \neq \vec{e}_1\times(\vec{e}_1\times\vec{e}_2)$.

- $(\vec{e}_1\times\vec{e}_1)\times\vec{e}_2 = \vec{0}\times\vec{e}_2 = \vec{0}$

- $\vec{e}_1\times(\vec{e}_1\times\vec{e}_2) = \vec{e}_1\times\vec{e}_3 = -(\vec{e}_3\times\vec{e}_1) = -\vec{e}_2.$

Example: For $\vec{u}, \vec{v}\in\mathbb{R}^3$, simplify $(\vec{u}+\vec{v})\times(\vec{u}-\vec{v})$ using the above properties.

Solution: $(\vec{u}+\vec{v})\times(\vec{u}-\vec{v}) = \vec{u}\times(\vec{u}-\vec{v}) + \vec{v}\times(\vec{u}-\vec{v})$ [dist.]

$= \vec{u}\times\vec{u} - \vec{u}\times\vec{v} + \vec{v}\times\vec{u} - \vec{v}\times\vec{v}$ [dist]

$= \vec{0} - \vec{u}\times\vec{v} + \vec{v}\times\vec{u} - \vec{0}$

$= \vec{v}\times\vec{u} + \vec{v}\times\vec{u}$

$= 2\vec{v}\times\vec{u}$

Summary of the lecture:

- The cross product of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ is

  the vector $\vec{u} \times \vec{v} := \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \in \mathbb{R}^3.$

  Satisfies nice properties

  - anti-commutativity
  - distributivity
  - $\vec{u} \times \vec{u} = 0$
  - $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ & to $\vec{v}$.
  - No associativity law.

You should be able to:

- calculate cross products
- manipulate expressions involving cross products & linear combinations using the properties listed above.