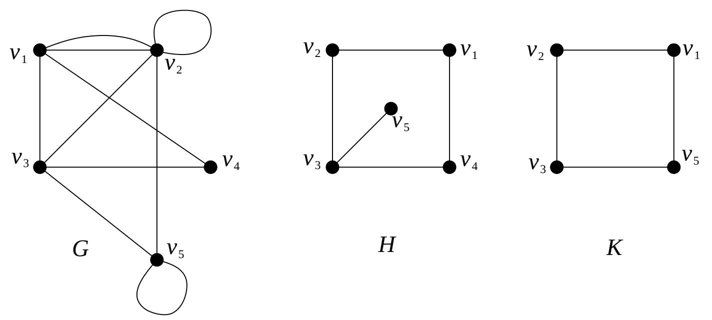
THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Graph Theory – Week 12 Tutorials

MATH1064: Discrete Mathematics for Computing

1. Let G, H and K be the graphs below.



(a) Which of these are simple graphs?

Solution: xercises for the Tutorial in H and K are simple graphs, since they do not contain any loops or parallel edges. G is not simple, since it contains both loops and parallel edges.

(b) Which of these have an Eulerian circuit?

Solution: G has an Eulerian circuit, since it is connected with all even vertex degrees (4, 6, 4, 2 and 4). H does not have an Eulerian circuit, since vertices v_3 and v_5 have odd degrees (3 and 1). K has an Eulerian circuit, since it is connected with all vertex degrees even (2, 2, 2 and 2).

Give a brief explanation for each of your answers.

2. For each of the following, state whether or not there exists a *simple* graph with vertices having the given degrees. If your answer is "yes", then draw such a graph. If your answer is "no", then explain why no such graph exists.

(a) Five vertices with degrees $3,3,2,2,2$.

Solution: Yes (draw a pentagon with exactly one diagonal).

(b) Seven vertices with degrees $4,2,3,1,1,1,1$.

Solution: No. The handshake theorem implies that the sum of all vertex degrees must be even, but $4+2+3+1+1+1+1 = 13$ which is odd.

(c) Five vertices with degrees $5,4,3,1,1$.

Solution: No. In a simple graph with 5 vertices, the largest possible vertex degree is 4 (since each vertex has at most four possible adjacent neighbours). However, the given list includes a vertex of degree 5.

3. For each $n \in \mathbb{N}$, the *complete graph* K_n is the simple graph with n vertices and an edge between every pair of vertices.

(a) Draw K_3, K_4, K_5 and K_6.

Solution: See the diagrams at:

`http://en.wikipedia.org/wiki/Complete_graph`

(b) What is the degree of each vertex in K_n?

Solution: In K_n, each vertex has degree $n-1$ (since it is adjacent to all $n-1$ other vertices).

(c) For which n does K_n contain an Eulerian circuit?

Solution: K_n has an Eulerian circuit if and only if n is odd (since every K_n is connected with all vertex degrees $n-1$, and $n-1$ is even if and only if n is odd).

4. Consider the graph G from Question 1.

(a) Write down the adjacency matrix $\mathbf{A}$ of this graph.

Solution: The adjacency matrix of G is:

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}$$

(b) Compute the matrices $\mathbf{A}^2$ and $\mathbf{A}^3$.

Solution: The second and third powers of $\mathbf{A}$ are:

$$\mathbf{A}^2 = \begin{bmatrix} 6 & 5 & 3 & 1 & 3 \\ 5 & 10 & 5 & 3 & 5 \\ 3 & 5 & 4 & 1 & 3 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 5 & 3 & 1 & 6 \end{bmatrix}$$

$$\mathbf{A}^3 = \begin{bmatrix} 14 & 28 & 15 & 9 & 14 \\ 28 & 40 & 23 & 10 & 25 \\ 15 & 23 & 12 & 7 & 15 \\ 9 & 10 & 7 & 2 & 6 \\ 14 & 25 & 15 & 6 & 20 \end{bmatrix}$$

(c) In G, how many walks are there of length 3 from v_2 to v_5?

Solution: The number of walks of length 3 from v_2 to v_5 is the entry in row 2, column 5 of $\mathbf{A}^3$, which is 25.

5. Let G be a simple graph with at least two vertices. Show that G has two vertices of the same degree.

Solution: This is an application of the pigeonhole principle. Suppose G has n vertices. Mapping each vertex to its degree gives a map $V(G) \to \{0, \ldots, n-1\}$. The codomain also has n elements, so we need to add an extra argument—in fact, we've done something similar in the past, remember?

If there is a vertex of degree $n-1$, then it is connected to all other $n-1$ vertices (since the graph is simple). Hence no vertex has degree 0, and the range of the "degree map" is $\{1,\ldots,n-1\}$. So we are done by the pigeonhole principle in this case.

The remaining case is that there is no vertex of degree $n-1$. In this case, the range of the "degree map" is $\{0,\ldots,n-2\}$. So we are done by the pigeonhole principle in this case also.

Since in each case we conclude that there are two vertices of the same degree, we are done.

6. Prove that if v is a vertex of odd degree in a graph G, then there is a path in G from v to another vertex of odd degree.

Solution: First proof: Vertex v lies in some connected component. Since the connected component is a graph, the handshaking lemma implies that there is a different odd degree vertex in the component, and clearly there is a path between these vertices by connectedness.

Second proof: First note that by the handshake theorem, there is at least one other vertex of odd degree. Now we repeat part of the argument given in the proof of the existence of Eulerian circuits in a graph where all vertex degrees are even.

Since the degree of v is odd, there is at least one edge $e=\{v,v_1\}$ incident with v. We have a number of cases. If $v_1 \neq v$ and v_1 has odd degree, then we are done. If $v_1 \neq v$ and v_1 has even degree, then there is another edge $\{v_1,v_2\}$. If $v_1 = v$, then e is a loop and since the degree of v is odd, there is another edge $\{v_1,v_2\}$. We repeat the same argument with v_2 in place of v_1. Since there are only finitely many edges in G and everytime we visit v again, we can also exit v, it follows that this procedure must terminate at a vertex of odd degree distinct from v.

Remark: If there are exactly 2 vertices of odd degree, then the proof of the already mentioned theorem can also be used to show that there is an Eulerian circuit: a path that ends in one of the vertices of odd degree, terminates in the other vertex of odd degree and passes through every edge in the graph exactly once. Namely, first find a path between the two edges of odd degree. Delete the associated edges from the graph. The remaining graph has all vertices of even degree, and can be decomposed into circuits as in the proof of the theorem. In the end, these circuits and the initial path can be spliced together.