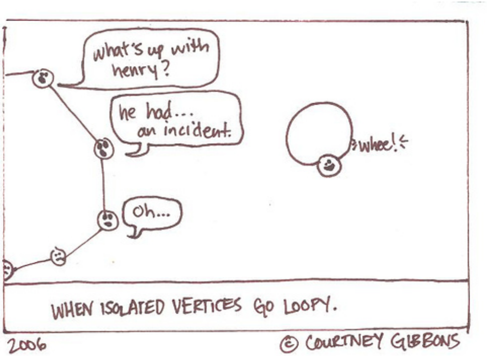
Discrete Mathematics

MATH1064, Lecture 33

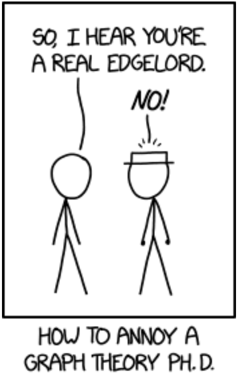
Jonathan Spreer



Extra exercises for Lecture 33

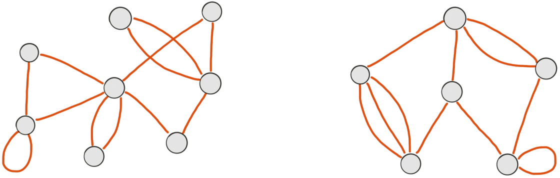
Section 10.3: Problems 5–8, 10–25

Section 10.5: Problems 30–40, 44



Question

Which of the two graphs below have a Eulerian circuit?

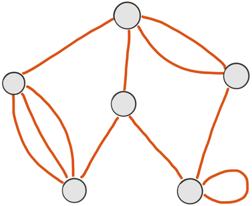


1. The first only
2. The second only
3. Both
4. Neither

Hamiltonian circuits

Eulerian circuit: circuit using every edge exactly once

Hamiltonian circuit (or Hamiltonian cycle): circuit using every vertex exactly once. (Except for start = end vertex, which must appear twice.)

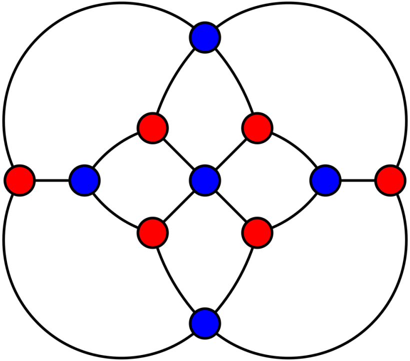


Testing for a Eulerian circuit: simple

Testing for a Hamiltonian circuit?

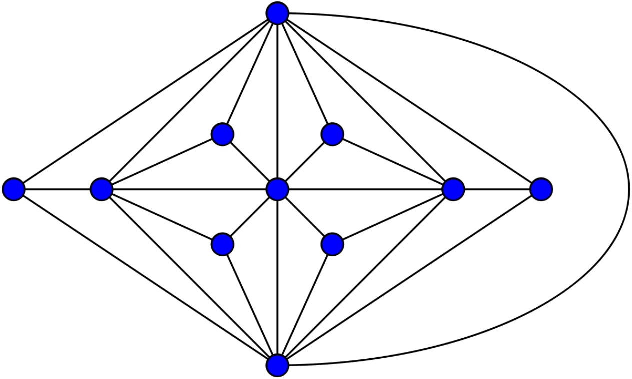
Hamiltonian circuits

Does this graph contain a Hamiltonian circuit?



1. Yes
2. No

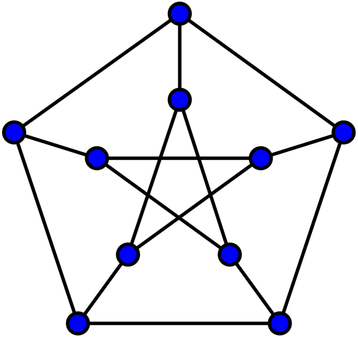
Does this graph contain a Hamiltonian circuit?



1. Yes
2. No

Hamiltonian circuits

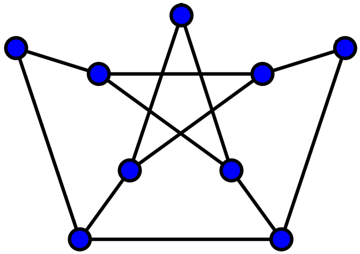
Does this graph contain a Hamiltonian circuit?



Watch: `https://youtu.be/AVe-OA-fcV0`

Hamiltonian circuits

Does this graph contain a Hamiltonian circuit?



Find a simple test for whether a Hamiltonian circuit exists—or prove that you cannot—and you can have \$1 000 000!

This would solve the famous P vs NP problem.

Reminder: P vs NP

A decision problem is a yes/no question, for which we wish to find an algorithm.

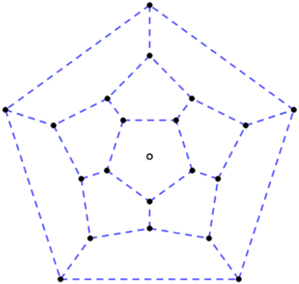
Examples:

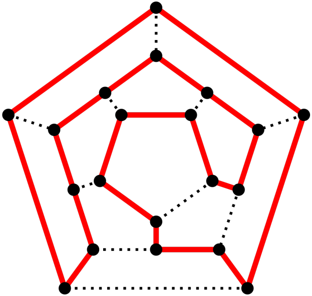
- Input: A graph G.
  Question: Does G have a Eulerian circuit?
- Input: A graph G.
  Question: Does G have a Hamiltonian circuit?

P vs NP is about which decision problems you can solve quickly (i.e., in running time a polynoial in the input size, eg. $O(n)$, $O(n^3)$, $O(n^{100})$), and which problems you can verify quickly.

Reminder: P vs NP

Here: Is finding a Hamiltonian circuit as difficult as verifying that a given circuit is a Hamiltonian circuit?



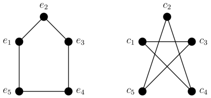


Look up the Icosian game: `https://www.geogebra.org/m/u3xggkcj`

More about P vs. NP:
https://www.win.tue.nl/~gwoegi/P-versus-NP.htm

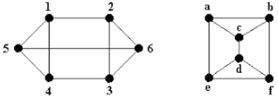
When are two graphs “the same”?

Are these graphs the same?



Remember: A graph G is defined purely in terms of $V(G)$ and $E(G)$. Its drawing does not matter.

What about these graphs?



And what about these?

G_1 G_2 G_3

Graph isomorphism

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic, written $G_1 \cong G_2$, if there exists a bijective function $\phi : V_1 \to V_2$ such that

$$\phi(E_1) = \{\{\phi(v_1), \phi(v_2)\} | \{v_1, v_2\} \in E_1\} = E_2.$$

Graph isomorphism

Given two graphs G_1 and G_2, are they isomorphic (does $G_1 \cong G_2$ hold)?

This question is known as the graph isomorphism problem.

It is famous for being "probably not difficult" to answer in general.

Matrices

Let $n \in \mathbb{N}$. An $n \times n$ matrix is just an $n \times n$ grid of numbers.

$$\text{Example } 3 \times 3 \text{ matrix:} \quad \mathbf{M} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 1 \\ -1 & 0 & -4 \end{bmatrix}$$

Let $\mathbf{M}$ be an $n \times n$ matrix. We write $\mathbf{M} = (m_{i,j})$, where for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, the symbol $m_{i,j}$ denotes the entry in row i, column j.

$m_{1,2} =$

$m_{3,1} =$

$m_{3,3} =$

Multiplying matrices

Let $\mathbf{A} = (a_{i,j})$ and $\mathbf{B} = (b_{i,j})$ be $n \times n$ matrices.
The product $\mathbf{AB}$ is an $n \times n$ matrix $\mathbf{AB} = (m_{i,j})$, with entries:

$$m_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} = a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \ldots + a_{i,n} b_{n,j}$$

Example:

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} =$$

Matrix multiplication has some (but not all) nice properties:

- It is associative:
- It has an identity:

An aside

Why this over-complicated formula for multiplication?

Why not just have $m_{i,j} = a_{i,j} \cdot b_{i,j}$?

The answer:

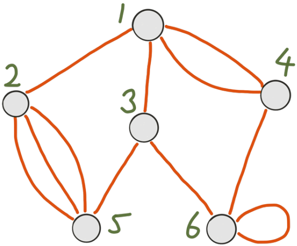
- $n \times n$ matrices essentially describe linear functions $f\colon \mathbb{R}^n \to \mathbb{R}^n$.
- Our formula is chosen so that multiplying matrices corresponds to the composition of functions $g \circ f$.

You will learn more about this in linear algebra!

Representing graphs using matrices

Let G a graph with n vertices, and suppose we label these vertices $V(G) = \{1, 2, \ldots, n\}$.

The adjacency matrix of G is the $n \times n$ matrix $\mathbf{A} = (a_{i,j})$, where each entry $a_{i,j}$ is the number of edges with endpoints $\{i, j\}$ (counted with multiplicity).



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{bmatrix}$$

Theorem

Let G be a graph with vertices $V(G) = \{1, 2, \ldots, n\}$, and adjacency matrix $\mathbf{A}$.

Then the number of paths of length k from vertex i to vertex j is the entry in row i, column j of the kth power $\mathbf{A}^k = \mathbf{A} \cdot \mathbf{A} \cdot \ldots \cdot \mathbf{A}$.

Proof: Let's use induction! Let $\mathbf{A} = (a_{i,j})$. Some more notation:

– Let $w_{i,j}^{(k)}$ denote the number of paths of length k from i to j.

– Let $a_{i,j}^{(k)}$ denote the entry in row i, column j of $\mathbf{A}^k$.

We must prove that $w_{i,j}^{(k)} = a_{i,j}^{(k)}$.

Basis step:

Consider $k = 1$.
A path of length 1 from i to j is just an edge with endpoints $\{i, j\}$!

Therefore the number of such paths is exactly the number of edges with endpoints $\{i, j\}$. That is: $w_{i,j}^{(1)} = a_{i,j} = a_{i,j}^{(1)}$.

Inductive step:

Let $k \geq 1$ and assume that, for all vertices i, j, $w_{i,j}^{(k)} = a_{i,j}^{(k)}$.

We must prove that, for all vertices i, j, $w_{i,j}^{(k+1)} = a_{i,j}^{(k+1)}$.

Consider a path of length $k+1$ from i to j.
What is the second-last vertex on the path?

This could be any of $1, 2, \ldots, n$. We take cases!

- The number of paths with second-last vertex 1 is $w_{i,1}^{(k)} \cdot w_{1,j}^{(1)}$, since we first path along k edges from i to 1, and then we path along one more edge from 1 to j.
- The number of paths with second-last vertex 2 is $w_{i,2}^{(k)} \cdot w_{2,j}^{(1)}$.
- ...
- The number of paths with second-last vertex n is $w_{i,n}^{(k)} \cdot w_{n,j}^{(1)}$.

Summing over all cases, the total number of paths of length $k+1$ from i to j is

$$w_{i,1}^{(k)} w_{1,j}^{(1)} + w_{i,2}^{(k)} w_{2,j}^{(1)} + \ldots + w_{i,n}^{(k)} w_{n,j}^{(1)}.$$

So: the total number of paths of length $k+1$ from i to j is

$$w_{i,1}^{(k)} w_{1,j}^{(1)} + w_{i,2}^{(k)} w_{2,j}^{(1)} + \ldots + w_{i,n}^{(k)} w_{n,j}^{(1)}.$$

By the inductive hypothesis, this is

$$a_{i,1}^{(k)} a_{1,j}^{(1)} + a_{i,2}^{(k)} a_{2,j}^{(1)} + \ldots + a_{i,n}^{(k)} a_{n,j}^{(1)}.$$

Recall that $a_{i,j}^{(k)}$ denotes the entry in row i, column j of $\mathbf{A}^k$,
and $a_{i,j}^{(1)}$ denotes the entry in row i, column j of $\mathbf{A}^1$.

The line above is the formula for matrix multiplication!
The number $a_{i,1}^{(k)} a_{1,j}^{(1)} + a_{i,2}^{(k)} a_{2,j}^{(1)} + \ldots + a_{i,n}^{(k)} a_{n,j}^{(1)}$ is just the entry in row i, column j of $\mathbf{A}^k \cdot \mathbf{A}^1 = \mathbf{A}^{k+1}$.

Therefore $w_{i,j}^{(k+1)} = a_{i,j}^{(k+1)}$ for all vertices i, j, as required!

By mathematical induction, it follows that, for all path lengths k, we have $w_{i,j}^{(k)} = a_{i,j}^{(k)}$ for all vertices i, j. $\square$

- Note that every loop edge contributes 2 to the respective diagonal entry of the adjacency matrix.
- In the above theorem this means we count paths with loop edges twice, once per orientation of the loop edge.
- $\rightarrow$ a single path with k loop edges is counted 2^k times.
- However: in our definition of a path, these paths are all considered to be the same.
- We can remove this ambiguity by only adding 1 per loop edge on the diagonal entry.