THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Functions, Sequences, and Number Theory – Week 5 Practice Class

MATH1064: Discrete Mathematics for Computing

Here is a list of **problems** for the practice class. Try to solve them before you go to class! There are more problems here than can be solved in the hour, so you should get started on them!

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 1$. Write down:

1. the domain, codomain and range of f;
2. the image of $[-1,2]$ under f;
3. the preimage of $[0,3]$ under f.

Solution:

1. Domain and codomain are both $\mathbb{R}$. Range is $\{x \in \mathbb{R} \mid x \geq -1\} = [-1,\infty) \subset \mathbb{R}$;
2. The image of $[-1,2]$ under f is $[-1,3]$;
3. The preimage of $[0,3]$ under f is $[-2,-1] \cup [1,2]$.

2. Determine whether the following functions are one-to-one, onto, bijections, or none of the above:

1. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \lceil x \rceil$;
2. $f : \mathbb{R} \to \mathbb{Z}$ where $f(x) = \lceil x \rceil$;
3. $f : \mathbb{Z} \to \mathbb{R}$ where $f(x) = \lceil x \rceil$;
4. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = \lceil x \rceil$.

Solution:

1. None;
2. Onto (surjective);
3. One-to-one (injective);
4. Bijective (onto and one-to-one);

3. Let $X = \{1,2\}$ and $Y = \{1,2,3\}$. Count the number of:

1. functions from X to Y;
2. injective functions from X to Y;

3. surjective functions from X to Y;
4. functions from Y to X;
5. injective functions from Y to X;
6. surjective functions from Y to X;
7. bijections from Y to Y.

Solution:

1. Number of functions from X to Y is $|Y|^{|X|} = 3^2 = 9$;
2. Three choices for image of 1, two choices for image of 2, $3 \cdot 2 = 6$;
3. Zero, since $|Y| > |X|$;
4. $2^3 = 8$;
5. Zero since $|Y| > |X|$;
6. All functions but the constant functions are surjective: $8 - 2 = 6$;
7. Number of bijections from Y to Y are $|Y|! = 3! = 6$.

4. Find:

1. a one-to-one function $f : \mathbb{N}^3 \to \mathbb{N}$;
2. an onto function $g : [0,1] \to \mathbb{R}$;
3. a bijection between $\mathscr{P}(\mathbb{N})$ and $\mathscr{P}(\mathbb{Z})$.

Solution:

1. Construct an ordering of the elements of $\mathbb{N}^3$. Then we can define a function $f : \mathbb{N}^3 \to \mathbb{N}$ by assigning each triple $(a,b,c) \in \mathbb{N}$ its position in the ordering.

   One may construct such an ordering on $\mathbb{N}^3$ in two steps.

   First we (partially) order triples in $\mathbb{N}^3$ by their sum. That is, for $(a_1,b_1,c_1),(a_2,b_2,c_2) \in \mathbb{N}^3$ we consider (a_1,b_1,c_1) to be smaller than (a_2,b_2,c_2) if $a_1+b_1+c_1 < a_2+b_2+c_2$. Only triples with distinct sum of entries can be ordered this way, but for every sum $k \in \mathbb{N}$ we observe that at most a finite number of triples have entries summing to k. Hence, the sets of triples which pairwise cannot be compared in this (partial) order are finite.

   For such pairs triples with the same sum we then use what is called the *lexicographic order*. Given $(a_1,b_1,c_1),(a_2,b_2,c_2) \in \mathbb{N}^3$, we say that (a_1,b_1,c_1) is considered smaller than (a_2,b_2,c_2) if $a_1 < a_2$. If $a_1 = a_2$, then (a_1,b_1,c_1) is considered smaller than (a_2,b_2,c_2) if $b_1 < b_2$ – and if $a_1 = a_2$ and $b_1 = b_2$, then (a_1,b_1,c_1) is considered smaller than (a_2,b_2,c_2) if $c_1 < c_2$.

   Combining the partial order above with the lexicographic order, with the partial order being dominant, the first elements of $\mathbb{N}^3$ (considerung sum of entries up to 5) are

$$
\begin{aligned}
&(1,1,1);\\
&(1,1,2),(1,2,1),(2,1,1);\\
&(1,1,3),(1,2,2),(1,3,1),(2,1,2),(2,2,1),(3,1,1);\\
&\dots
\end{aligned}
$$

We then define $f : \mathbb{N}^3 \to \mathbb{N}$ accordingly. That is, $f((1,1,1)) = 1$, $f((1,1,2)) = 2$, etc etc.

2. The idea here is to take the interval $[0,1]$ and "stretch" it accross $\mathbb{R}$.
   Let $g : [0,1] \to \mathbb{R}$. Define $g(1/2) = 0$, $g(x) < 0$ for $x < 1/2$ and $g(y) > 0$ for $y > 1/2$.
   This could look like $g(x) = 2 - \frac{1}{x}$ for $x < 1/2$ and $g(y) = \frac{1}{1-y} - 2$ for $y > 1/2$:

$$g : [0,1] \to \mathbb{R}; \quad x \mapsto \begin{cases} 2 - \frac{1}{x} & \text{if } x \leq \frac{1}{2} \\ \frac{1}{1-x} - 2 & \text{else.} \end{cases}$$

3. The idea here is to first construct a bijection between $\mathbb{N}$ and $\mathbb{Z}$. For instance, consider

$$f : \mathbb{N} \to \mathbb{Z}; \quad n \mapsto \begin{cases} -\frac{x-1}{2} & \text{if } x \text{ is odd} \\ \frac{x}{2} & \text{else.} \end{cases}$$

   Now define

$$F : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{Z}); \quad \{x_i \in \mathbb{N} \mid i \in I, I \subset \mathbb{N}\} \mapsto \{f(x_i) \in \mathbb{Z} \mid i \in I, I \subset \mathbb{N}\}.$$

   Since f is a bijection, every subset of $\mathbb{Z}$ corresponds to exactly one subset of $\mathbb{N}$ and hence F must be a bijection.

5. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^3 - 1$, and $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = x + 5$.

1. Write each of $f \circ g$ and $g \circ f$ as a polynomial.
2. Plot $f(x)$ (with x is on the horizontal axis, and $f(x)$ on the vertical axis).
3. Determine which of the following functions is a bijection: f, g, $g \circ g$, $g \circ f$.
4. For each bijection in (c), determine its inverse.
5. Answer questions (c) and (d) again, but this time with f and g defined on the *reals*; that is, with $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.

Solution:

1. The function $f \circ g$ is defined as follows.

$$
\begin{aligned}
f \circ g(x) =& f(g(x)) \\
=& f(x+5) \\
=& (x+5)^3 - 1 \\
=& x^3 + 15x^2 + 75x + 124
\end{aligned}
$$

The function $g \circ f$ is defined as follows.

$$\begin{aligned} g \circ f(x) =& g(f(x)) \\ =& g(x^3 - 1) \\ =& x^3 + 4 \end{aligned}$$

2. The function is depicted in Figure **??** below.

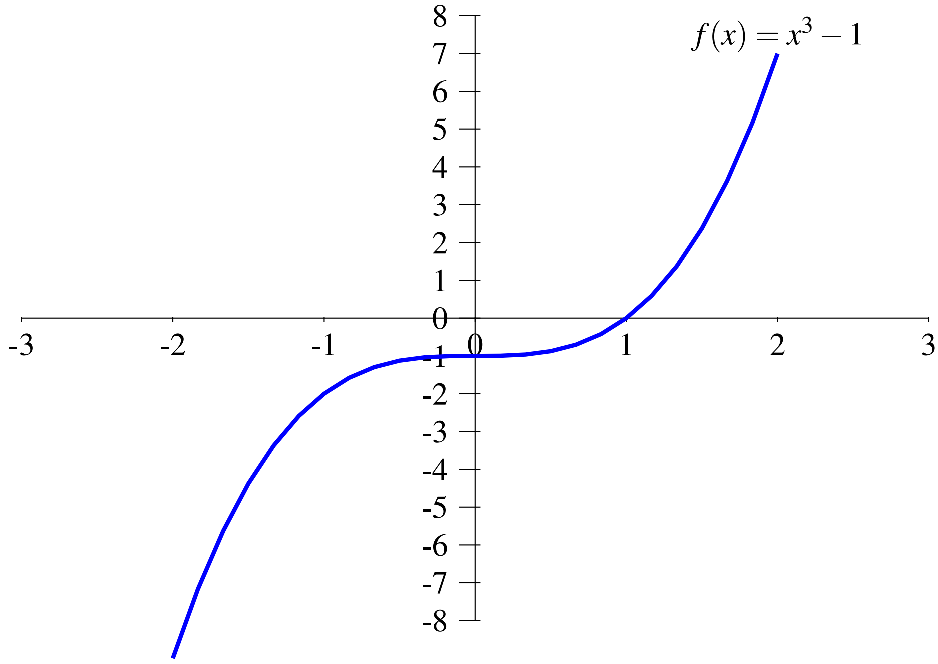


Figure 1:

3. The function f is not a bijection because it is not surjective. For example there is no $x \in \mathbb{Z}$ such that $f(x) = 2$.

   The function g is a bijection.

   The composite function $g \circ g(x) = g(x+5) = x+10$ is also a bijection. The composition of two bijections is always a bijection.

   Since f is not surjective and g is bijective (hence injective), it follows that $g \circ f$ is not bijective. For example, as there is no $x \in \mathbb{Z}$ such that $f(x) = 2$, there is no $x \in \mathbb{Z}$ such that $g \circ f(x) = g(2) = 7$ as this would violate the injectivity of g.

4.

$$\begin{aligned} g^{-1}(x) &= x - 5 \\ (g \circ g)^{-1}(x) &= x - 10 \end{aligned}$$

5. This time g and f are both bijections. It follows immediately that any composition of g, f or their inverses is also a bijection. This includes $g \circ g$ and $g \circ f$. We define

the respective inverse functions as follows

$$
\begin{aligned}
f^{-1}(x) &= \sqrt[3]{x+1} \\
g^{-1}(x) &= x-5 \\
(g \circ g)^{-1}(x) &= x-10 \\
(g \circ f)^{-1}(x) &= \sqrt[3]{x-4}
\end{aligned}
$$

6. Consider the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = (-1)^n(n+1)^2$.

1. Write the sequence as a function $f : \mathbb{N} \to \mathbb{Z}$.
2. Find an injective function $g : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ which has the same image as f.

Solution:

1. $f : \mathbb{N} \to \mathbb{Z}; \quad n \mapsto (-1)^n(n+1)^2$.
2. There is a bijection $h : \mathbb{N} \cup \{0\} \to \mathbb{N}$ defined by

$$h(n) = n+1$$

Since f is injective and h is bijective, $f \circ h$ is injective and has the same image as f. Therefore the answer to the question is given by the function $g = f \circ h : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ which can be defined by

$$g : \mathbb{N} \cup \{0\} \to \mathbb{Z}; \quad n \mapsto (-1)^{n+1}(n+2)$$

7. For each of the following sequences, conjecture a general formula for a_n and prove that your formula is correct.

1. $a_1 = 2$, $a_2 = 3$, $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$
2. $a_1 = 3$, $a_n = a_{n-1} + 3^n$ for $n \geq 2$
3. $a_1 = 5$, $a_n = a_{n-1} + (n+1)^2 - n^2$ for $n \geq 2$
4. $a_1 = 8$, $a_n = 3a_{n-1} - 10$ for $n \geq 2$

Solution:

1. The sequence begins like:

$$2, 3, 5, 9, 17 \ldots$$

Note that the differences between consecutive terms are $2^0, 2^1, 2^2, 2^3, 2^4, \ldots$.
Claim: $a_n = 1 + 2^{n-1}$ for $n \in \mathbb{N}$. By direct calculation we can see that using this definition we have $a_1 = 2$ and $a_2 = 3$ as required. For $n \geq 3$ we have

$$
\begin{aligned}
3a_{n-1} - 2a_{n-2} &= 3(1+2^{n-2}) - 2(1+2^{n-3}) \\
&= 3 + 3 \cdot 2^{n-2} - 2 - 2 \cdot 2^{n-3} \\
&= 1_2^{n-1} + 2^{n-2} - 2^{n-2} \\
&= 1 + 2^{n-1} \\
&= a_n
\end{aligned}
$$

This proves the required recurrence relation.

2. Claim: The sequence is given by

$$a_n = \sum_{i=1}^{n} 3^i$$

We prove this as follows. It is immediate that $a_1 = 3$ for $n \geq 2$ we have

$$\begin{aligned} a_{n-1} + 3^n &= 3^1 + 3^2 + 3^3 + 3^4 + \cdots + 3^n \\ &= a_n \end{aligned}$$

This proves the required recurrence relation.

3. We can rewrite the recurrence relation as follows

$$a_n = a_{n-1} + (n+1)^2 - n^2 = a_{n-1} + 2n + 1$$

The sequence starts like

$$5, 10, 17, 26, 37, 50, \ldots$$

The sequence of differences between successive terms is

$$5, 7, 9, 11, 13, \ldots$$

Therefore we claim that the required closed form is given as follows.

$$a_n = 2 + \sum_{i=1}^{n} (2i+1)$$

First note that using this definition $a_1 = 5$ as required. It remains to verify the recurrence relation in the question.

$$\begin{aligned} a_{n-1} + 2n + 1 = 2 + 2n + 1 + \sum_{k=1}^{n-1} (2k+1) \\ = 2 + \sum_{k=1}^{n} (2k+1) \\ = a_n \end{aligned}$$

This verifies the required recurrence relation.

4. List the first few terms of the sequence to see if we can establish a pattern.

$$\begin{aligned} a_1 &= 8 \\ a_2 &= 3 \cdot 8 - 10 \\ a_3 &= 3 \cdot (3 \cdot 8 - 10) - 10 \\ &= 3^2 \cdot 8 - (3+1) \cdot 10 \\ a_4 &= 3 \cdot (3^2 \cdot 8 - (3+1) \cdot 10) - 10 \\ &= 3^3 \cdot 8 - (3^2 + 3) \cdot 10 - 10 \\ &= 3^3 \cdot 8 - (3^2 + 3 + 1) \cdot 10 \end{aligned}$$

I claim that a_n is given by

$$a_n = 3^{n-1} \cdot 8 - (3^{n-2} + 3^{n-3} + 3^{n-4} + \cdots + 3 + 1) \cdot 10$$

Using this definition we have

$$a_1 = 3^0 \cdot 8 - 0 \cdot 10 = 8$$

We verify the recurrence relation as follows.

$$\begin{aligned}
3 \cdot a_{n-1} - 10 &= 3 \cdot (3^{n-2} \cdot 8 - (3^{n-3} + \cdots + 1) \cdot 10) - 10 \\
&= 3^{n-1} \cdot 8 - (3^{n-2} + 3^{n-3} + \cdots + 3^2 + 3) \cdot 10 - 10 \\
&= 3^{n-1} \cdot 8 - (3^{n-2} + 3^{n-3} + \cdots + 3^2 + 3 + 1) \cdot 10 \\
&= a_n
\end{aligned}$$

8. Prove that if a is an integer other than 0, then

1. $1 \mid a$.
2. $a \mid 0$.

Solution:

1. By assumption we have that $a \neq 0$. For $1 \mid a$, there must exist an integer $k \in \mathbb{Z}$ such that $1 \cdot k = a$. For $k = a$ this equation is always satisfied and hence we must always have $1 \mid a$ for non-zero $a \in \mathbb{Z}$.
2. Again, we must find an integer $k \in Z$ such that $a \cdot k = 0$. This equation is always satisfied for $k = 0$ and thus we must have $a \mid 0$ for all $a \in Z$ (including $a = 0$).

9. What are the quotient and remainder when

1. 19 is divided by 7?
2. -111 is divided by 11?
3. 789 is divided by 23?
4. 1001 is divided by 13?
5. 0 is divided by 19?
6. 3 is divided by 5?
7. -1 is divided by 3?
8. 4 is divided by 1?

Solution:

1. $19 = 2 \cdot 7 + 4$. Quotient $q = 2$ and remainder $r = 4$.
2. $-111 = -11 \cdot 11 + 10$. Quotient $q =$ and remainder $r = 10$.
3. $789 = 34 \cdot 23 + 7$. Quotient $q = 34$ and remainder $r = 7$.
4. $1001 = 77 \cdot 13 + 0$. Quotient $q = 77$ and remainder $r = 0$.

5. $0 = 0 \cdot 19 + 0$. Quotient $q = 0$ and remainder $r = 0$.
6. $3 = 0 \cdot 5 + 3$. Quotient $q = 0$ and remainder $r = 3$.
7. $-1 = -1 \cdot 3 + 2$. Quotient $q = -1$ and remainder $r = 2$.
8. $4 = 4 \cdot 1 + 0$. Quotient $q = 4$ and remainder $r = 0$.

10. Find counterexamples to each of these statements about congruences, or show that they are true.

1. If $ac \equiv bc(\mod m)$, where $a, b, c, m \in \mathbb{Z}$ with $m \geq 2$, then $a \equiv b(\mod m)$.
2. If $a \equiv b(\mod m)$ and $c \equiv d(\mod m)$, where $a, b, c, d, m \in \mathbb{Z}$, $m \geq 2$, then $ac \equiv bd(\mod m)$.

Solution:

1. $a = c = m = 2$, $b = 3$. Then $ac = 4 \equiv bc = 6(\mod 2)$. But $2 \not\equiv 3(\mod 2)$.
2. Look at quotient and remainder for all of a, b, c, d under division by m. Let $a = mq_a + r_a$, $b = mq_b + r_b$, $c = mq_c + r_c$, and $d = mq_d + r_d$, $0 \leq r_a, r_b, r_c, r_d < m$. By assumption we then have $r_a = r_b$ and $r_c = r_d$.
   We have
   $$ac = (mq_a + r_a)(mq_c + r_c) = m(mq_a q_c + q_a r_c + q_c r_a) + r_a r_c,$$
   and
   $$bd = (mq_b + r_b)(mq_d + r_d) = m(mq_b q_d + q_b r_d + q_d r_b) + r_b r_d.$$
   But this means that we have $ac \equiv r_a r_c(\mod m)$ and $bd \equiv r_b r_d(\mod m)$, and since $r_a = r_b$ and $r_c = r_d$ we have $r_a r_c = r_b r_d$ and thus $ac \equiv bd(\mod m)$.

11. Show that if n is an integer then $n^2 \equiv 0$ or $1(\mod 4)$.

Solution: Let n be even. That is, there exists an integer $k \in Z$ such that $n = 2k$. Hence $n^2 = (2k)^2 = 4k^2$ and hence $n^2 = 0(\mod 4)$.

Let n be odd, that is, $n = 2k + 1$ for some $k \in Z$. Hence $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$ and hence $n^2 1(\mod 4)$.

The statement now follows from the fact that all integers are either even or odd.

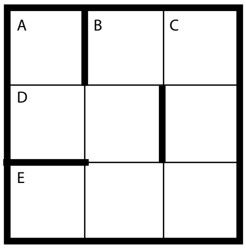
Puzzles on next page!

Here are two **puzzles** that you can think about during week 3. Feel free to ask your tutors or lecturer for more hints!

G Below is a cross-number puzzle! Each square holds one digit, and the answer to every clue is a positive integer. Solve the puzzle, and show that there is only one solution.

Across: **[B]** The sum of the digits of B down. **[E]** A down + B across + C down.

Down: **[B]** A multiple of 99. **[C]** The square of D across.



H Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ that have the property

$$f(x+y) = f(x) + f(y) + 2xy \quad \text{for all } x, y \in \mathbb{Q}.$$

Can you *prove* that you answer(s) are the only such functions possible?

Puzzle hints:

Stuck on the puzzles from week 3? Here are some hints!

E 4096 – that's an unusual number. Have you seen it somewhere before?

F What happens to the *sum* of the slips remaining in the hat after each step?