THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Graphs and Relations – Week 12 Practice Class

MATH1064: Discrete Mathematics for Computing

Here is a list of **problems** for the practice class. Try to solve them before you go to class! There are more problems here than can be solved in the hour, so you should get started on them!

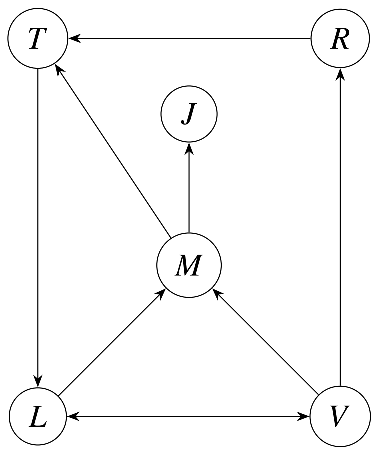
1. Your lecturer is under investigation for spreading mathematical gossip. After investigation of sources of the gossip, we have the following evidence.

Prof. Maher heard the gossip from your lecturer and Dr. Vine. Prof. Jaco heard the gossip from Prof. Maher. Prof. Tillmann heard the rumor from Prof. Maher and Prof. Rubinstein. Your lecturer heard it from Dr. Vine and Prof. Tillmann. Prof. Rubinstein heard it from Dr. Vine. Dr. Vine heard it from your lecturer.

1. Can one conclude that your lecturer is the true culprit?
2. If Dr. Vine incorrectly stated that your lecturer was the source of the gossip, then what can one conclude?

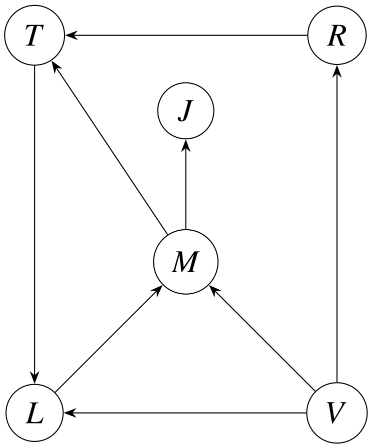
Solution:

1. We draw a directed graph to try to find the culprit.



We can't conclude that the lecturer is the true culprit. The only thing we can conclude is that Professor Jaco is not the culprit.

2. We draw a new directed graph for this question.

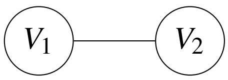


Now we can conclude that the only possible source of the gossip is Dr. Vine.

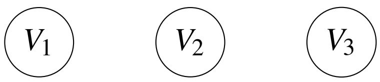
2. How many non-isomorphic simple graphs are there with n vertices, when $n \in \{2,3,4\}$?

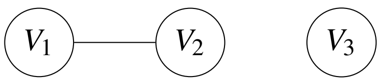
Solution: The following are the isomorphism types for graphs with two vertices.

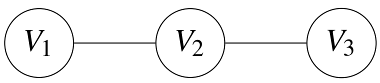
1. 

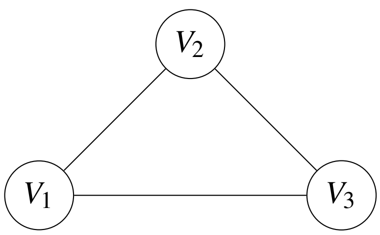
2. 

The isomorphism classes of graphs with three vertices are:

1. 

2. 

3. 

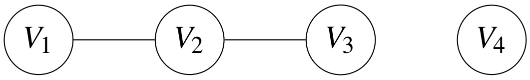
4. 

The isomorphism classes of graphs with four vertices are:

1.

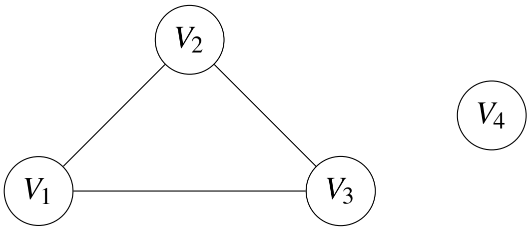
2.

3.

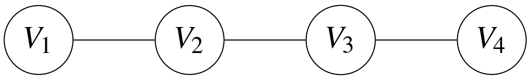


4.

5.



6.



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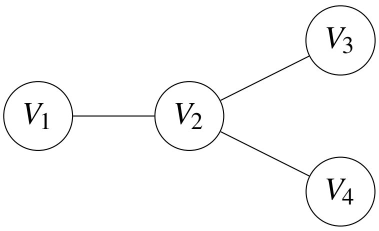

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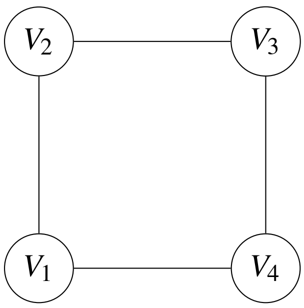

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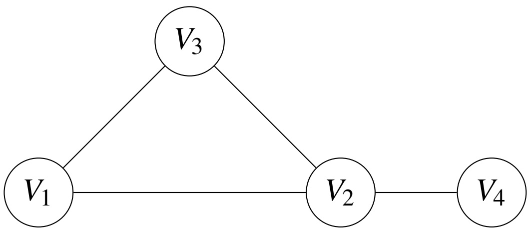
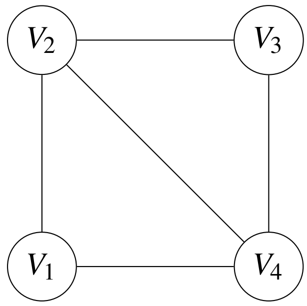
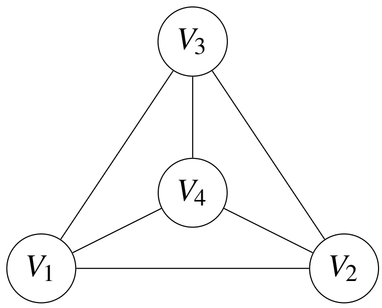


10.



11.



3. Are the simple graphs with the following adjacency matrices isomorphic?

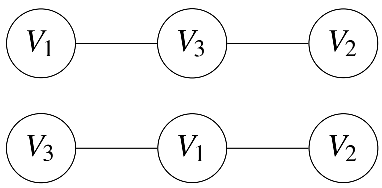
1. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

2. $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

Solution:

1. Let's draw the graphs of these two adjacency matrices.



Therefore we obtain an isomorphism between these two graphs by swapping the labels on V_1 and V_3.

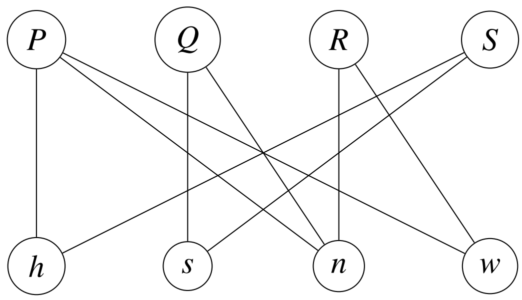
2. The second graph is obtained from the first by adding an edge between the first vertex and the third (i.e. the vertices corresponding to the first and third row/column of the adjacency matrices). In particular the second graph has more edges than the first. Therefore they cannot be isomorphic.
3. While we can't read off from the graph the fact that we obtain the first graph from the second by adding a particular edge, we can still immediately read off from the graph that there are three edges in the graph represented by the second adjacency matrix and four edges in the first. That is sufficient information to determine that the graphs represented by these adjacency matrices are not isomorphic.

4. Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.

1. Use a bipartite graph to model the four employees and their qualifications.
2. Use Hall's theorem to determine whether there is an assignment of employees to support areas so that each employee is assigned one area to support.
3. If an assignment of employees to support areas so that each employee is assigned to one support area exists, find one.

Solution:

1.



2. We need to check that $|N(A)| \geq |A|$ for all subsets $A \subset \{P, Q, R, S\}$ where $N(A)$ is the set of neighbours of A. This is clear for the empty set and the set $\{P, Q, R, S\}$ as they have zero and four neighbours respectively. We show that the other inequalities are satisfied as follows.
   - The subsets having one element are given in the question. We need to check that they have at least one neighbour each. This is clearly true, P has three neighbours, while Q, R and S each have two neighbours.
   - The subsets having two elements must also satisfy the required condition since every subset having one element already has two neighbours.
   - The subsets having three elements have at least three neighbours if they contain P because P has three neighbours by itself. The only remaining subset is $\{Q, R, S\}$ which has four neighbours.

Therefore we have satisfied Hall's Theorem in that $|N(A)| \leq |A|$ for all subsets $A \subset \{P,Q,R,S\}$.

3. One such assignment is:
   - Ping is assigned to Hardware;
   - Quiggley is assigned to Networking;
   - Ruiz is assigned to Wireless;
   - Sitea is assigned to Software;

5. Show that graph isomorphism is an equivalence relation amongst simple graphs.

Solution: We need to check that the relation "is graph isomorphic to" is reflexive, symmetric and transitive.

1. We show that the relation is reflexive. Let $\mathscr{G} = (V,E)$ be a graph. We show that $\mathscr{G}$ is isomorphic to itself. We define a bijection
$$\phi : V \to V$$
by sending every $v \in V$ to itself. The statement $\{v,w\} \in E \iff \{\phi(v),\phi(w)\} \in E$ is trivial.

2. We show that the relation is symmetric. Suppose we have two graphs $\mathscr{G}_0 = (V_0,E_0)$ and $\mathscr{G}_1 = (V_1,E_1)$ and an isomorphism
$$\phi : \mathscr{G}_0 \to \mathscr{G}_1$$
between them. In defining such an isomorphism we must have a bijection $\phi_V : V_0 \to V_1$ such that for all pairs of distinct vertices $v,w \in E_0$
$$\{v,w\} \in E_0 \iff \{\phi_V(v),\phi_V(w)\} \in E_1$$
Therefore we have a bijection $\phi_V^{-1} : V_1 \to V_0$. It remains to show that for all pairs of distinct vertices $x,y \in E_1$ we have
$$\{x,y\} \in E_1 \iff \{\phi_V^{-1}(x),\phi_V^{-1}(y)\} \in E_0$$
However since ϕ is an isomorphism we know that $\{\phi_V \circ \phi_V^{-1}(x), \phi_V \circ \phi_V^{-1}(y)\} \in E_1 \iff \{\phi_V^{-1}(x),\phi_V^{-1}(y)\} \in E_0$. However $\phi_V \circ \phi_V^{-1}(x) = x$ and $\phi_V \circ \phi_V^{-1}(y) = y$ so the required result holds.

3. We show that the relation is transitive. Suppose we have three graphs $\mathscr{G}_0, \mathscr{G}_1$ and $\mathscr{G}_2$. Moreover suppose we have isomorphisms:
$$\phi : \mathscr{G}_0 \to \mathscr{G}_1$$
$$\psi : \mathscr{G}_1 \to \mathscr{G}_2$$
We want to show that there is an isomorphism $\Phi : \mathscr{G}_0 \to \mathscr{G}_2$. I claim that $\psi \circ \phi : \mathscr{G}_0 \to \mathscr{G}_2$ is an isomorphism.Since ϕ and ψ are isomorphisms they each define

bijections $\phi_V : V_0 \to V_1$ and $\psi : V_1 \to V_2$. Therefore $\psi_V \circ \phi_V : V_0 \to V_2$ is a bijection because it is the composition of two bijections. It remains to show that

$$\{v,w\} \in E_0 \iff \{\psi_V \circ \phi_V(v), \psi_V \circ \phi_V(w)\} \in E_2$$

Since ϕ is a bijection we know that

$$\{v,w\} \in E_0 \iff \{\phi_V(v), \phi_v(w)\} \in E_1$$

Similarly, since ψ is a bijection we know that

$$\{\phi_V(v), \phi_V(w)\} \in E_1 \iff \{\psi \circ \phi(v), \psi \circ \phi(w)\} \in E_2$$

Therefore $\{v,w\} \in E_0 \iff \{\psi_V \circ \phi_V(v), \psi_V \circ \phi_V(w)\} \in E_2$ and the required result holds.

6. Suppose that the function $f : V_1 \to V_2$ is an isomorphism of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Show that the number of comparisons needed to verify that f is an isomorphism is at most a polynomial in terms of the number of vertices of the graph G_1.

Solution: First we need to check that f is a bijection. Let $n := |V_1|$ and $m = |V_2|$. We assume that n and m are finite otherwise the answer is trivially true given the information in the question. Let the vertices of G_1 be $x_1, x_2, \ldots, x_n$. We check that f is a bijection as follows. The value of B will tell us whether f is a bijection.

$B = \text{TRUE}$
if $n \neq m$
  $B = \text{FALSE}$
  return B
for $i = 1$ **to** $n - 1$ **do:**
  for $j = i + 1$ **to** n **do:**
    if $f(x_i) = f(x_j)$
      then $B = \text{FALSE}$
return B

We perform one check to see if $n = m$ then we perform

$$(n-1) + (n-2) + (n-3) + \cdots + 2 + 1 = \frac{n(n-1)}{2}$$

checks to see if f is injective. If f is injective and $n = m$ then f must also be surjective. Therefore it takes $\frac{n(n-1)}{2} + 1 \in O(n^2)$ checks to see if f is a bijection.

Suppose that f is a bijection. In particular suppose that $n = m$. Denote the vertices of G_2 by $y_1, \ldots, y_n$ so that $f(x_k) = y_k$. It takes $O(n)$ comparisons to perform this renaming. We now need to check that

$$\{x_i, x_j\} \in E_1 \iff \{y_i, y_j\} \in E_2$$

Now denote $p = |E_1|$ and $q = |E_2|$. As above we need to check that $p = q$. After doing so it is sufficient to check that

$$\{x_i, x_j\} \in E_1 \Rightarrow \{y_i, y_j\} \in E_2$$

We do so as follows.

$B = \text{TRUE}$
if $p \neq q$
    $B = \text{FALSE}$
    return B
for $i = 1$ **to** $n - 1$ **do:**
    for $j = i + 1$ **to** n **do:**
        if $\{x_i, x_j\} \in E_1$
            if $\{y_i, y_j\} \notin E_2$
                $B = FALSE$
return B

This requires one comparison to check that $p = q$. This requires $O(p)$ comparisons to check if f is an isomorphism of graphs. Since $p \leq \frac{n(n-1)}{2}$ it takes $O(n^2)$ comparisons to check that f is an isomorphism of graphs after we know that it is an isomorphism of vertices and we have renamed the vertices in G_2. In total we require $O(n^2)$ comparisons to check that f is an isomorphism of graphs.

Puzzles on next page!

Here are two **puzzles** that you can think about during the week. Feel free to ask your tutors or lecturer for more hints!

Q Prove that, for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$,

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor nx \rfloor.$$

R Recall the complete graph K_n, which is defined in Lecture 28. You can draw K_3 and K_4 on a plane (i.e., a sheet of paper) without any edges crossing, but not K_5.

What happens if you draw your graphs on the surface of a doughnut instead of a plane? Can you draw K_5 without any edges crossing? How about K_6? K_7? K_8?

Puzzle hints:

Stuck on the puzzles from the last sheet? Here are some hints!

O Suppose there are n universities, and that the other universities all score k points each. Try to find an equation involving n and k. Remember that n and k are integers!

P There are not many ways in which you can express 90 as a product of natural numbers. What are they?

Q It's induction, but not as you know it! Try *fixing* n, and proving the result for all $x \in [\frac{k-1}{n}, \frac{k}{n})$ by induction on k.

R Which K_n can we draw on the surface of an n-holed doughnut without any edges crossing?