Discrete Mathematics

MATH1064, Lecture 24

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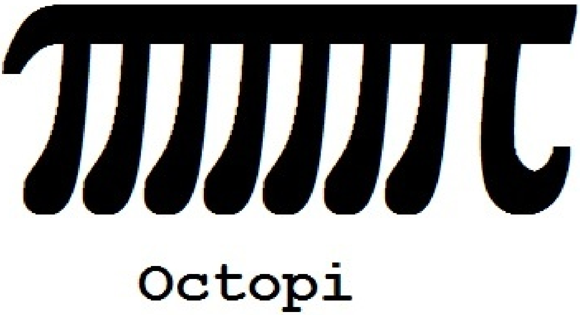
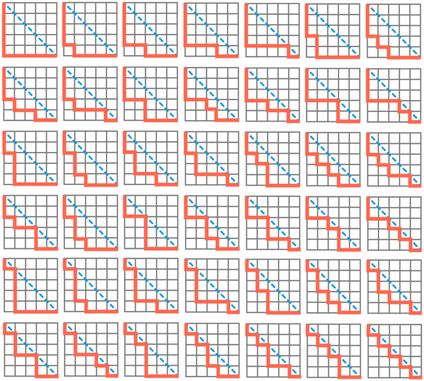


Extra exercises for Lecture 24

Section 8.1: Problems 2–5, 8, 9



Recurrences revisited

A different view

Resolving recurrences is for the computer scientist what is solving differential equations for the engineer

Consider a linear homogeneous recurrence relation of order 2:

$$a_n = \alpha a_{n-1} + \beta a_{n-2}$$

We can solve it using the following method:

(1) Factor $x^2 - \alpha x - \beta = (x - \lambda_1)(x - \lambda_2)$

(2a) If $\lambda_1 \neq \lambda_2$, then $a_n = A\lambda_1^n + B\lambda_2^n$ for some constants A and B.

(2b) If $\lambda_1 = \lambda_2$, then $a_n = C\lambda^n + Dn\lambda^n$, where $\lambda = \lambda_1 = \lambda_2$ and C and D are some constants.

Solutions in (2a) and (2b) are the general solution of the rec. relation.

Initial conditions (values of a_0 and a_1) determine constants A, B or C, D.

A similar method works for higher degree! See pages 518-519.

Example

Let $f : \mathbb{N} \to \mathbb{Z}$ be a function defined recursively by $f(0) = -1$, $f(1) = 5$ and for all $n \geq 2$ by

$$f(n) = 10\ f(n-1) - 25\ f(n-2).$$

Fibonacci (again)

$F_0 = F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$

Some recipes for solving recurrences!

Consider a linear non-homogeneous recurrence relation of order 2:

$$a_n = \alpha a_{n-1} + \beta a_{n-2} + F(n)$$

We can solve it using the following method:

(1) Find one particular solution $a_n^{(p)}$ by poking around.

(2) Determine the general solution $a_n^{(h)}$ to the homogeneous equation

$$a_n = \alpha a_{n-1} + \beta a_{n-2}$$

The general solution of the non-homogeneous recurrence relation is then given by $a_n^{(p)} + a_n^{(h)}$.

When we are given initial conditions, i.e. values of a_0 and a_1, then these determine the constants A, B or C, D.

A similar method works for higher degree! See pages 521-522.

Example

$$a_n = 10\ a_{n-1} - 25\ a_{n-2} + 3^n.$$

From before, the homogeneous equation $a_n = 10\ a_{n-1} - 25\ a_{n-2}$ has the general solution $a_n^{(h)} = C \cdot 5^n + D \cdot n5^n$.

Sums of squares

How many ways can we write an integer as a sum of two squares?

$$n = a^2 + b^2, \quad \text{where } a, b \in \mathbb{Z}$$

For $n = 13$, there are eight solutions:

$$\begin{aligned} 13 &= 3^2 + 2^2 = (-3)^2 + 2^2 = 3^2 + (-2)^2 = (-3)^2 + (-2)^2 \\ &= 2^2 + 3^2 = (-2)^2 + 3^2 = 2^2 + (-3)^2 = (-2)^2 + (-3)^2 \end{aligned}$$

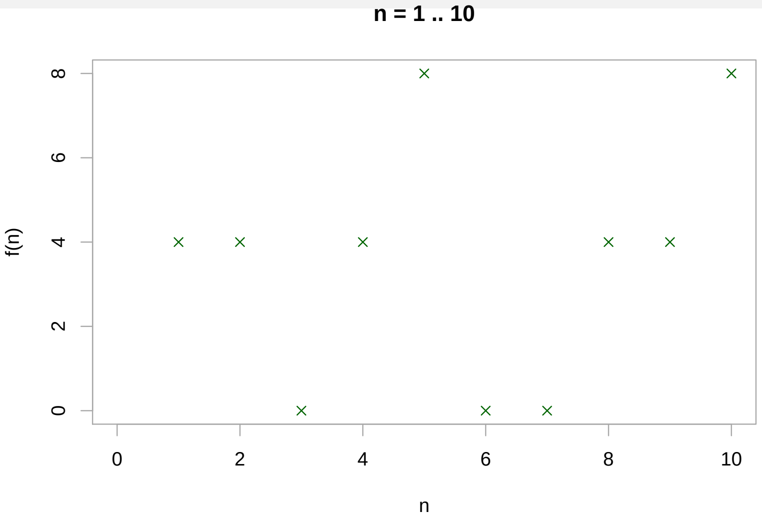
For $n = 16$, there are four solutions:

$$16 = 4^2 + 0^2 = (-4)^2 + 0^2 = 0^2 + 4^2 = 0^2 + (-4)^2$$

For $n = 11$, there are no solutions.

Define a function $f\colon \mathbb{N} \to \mathbb{N}$, where $f(n)$ denotes the number of solutions for n. So $f(13) = 8$, $f(16) = 4$, and $f(11) = 0$.
What do you think happens to $f(n)$ on average, as n grows large?

Looking at $f(n)$



Average $f(n) \simeq 3.6$

n = 1 .. 100

Theorem

As n grows, the average of $f(1), \dots, f(n)$ approaches π!

Proof: Consider pairs of integers a, b as lattice points.

:Marking the solutions to $a^2 + b^2 = 1$:Marking the solutions to $a^2 + b^2 = 2$:Marking the solutions to $a^2 + b^2 = 3$:Marking the solutions to $a^2 + b^2 = 4$:Marking the solutions to $a^2 + b^2 = 5$:Marking the solutions to $a^2 + b^2 = 6$:Marking the solutions to $a^2 + b^2 = 7$:Marking the solutions to $a^2 + b^2 = 8$:Marking the solutions to $a^2 + b^2 = 9$:Marking the solutions to $a^2 + b^2 = 10$:Marking the solutions to $a^2 + b^2 = 11$:Marking the solutions to $a^2 + b^2 = 12$:Marking the solutions to $a^2 + b^2 = 13$:Marking the solutions to $a^2 + b^2 = 14$:Marking the solutions to $a^2 + b^2 = 15$:Marking the solutions to $a^2 + b^2 = 16$:Marking the solutions to $a^2 + b^2 = 17$:Marking the solutions to $a^2 + b^2 = 18$:Marking the solutions to $a^2 + b^2 = 19$:Marking the solutions to $a^2 + b^2 = 20$:The solutions to $a^2 + b^2 \leq n$ fill a circle of radius $\sqrt{n}$!

The average of $f(1), \dots, f(n)$ is

$$\frac{f(1) + f(2) + \ldots + f(n)}{n}.$$

But $f(1) + f(2) + \ldots + f(n)$ is the number of integer solutions to the equation $a^2 + b^2 \leq n$ (excluding the trivial case $a = b = 0$).

This is the number of lattice points inside the circle of radius $\sqrt{n}$ (excluding the origin), which is roughly the area of this circle! (Because each unit square contains on average one lattice point.)

So, as n becomes large, $f(1) + f(2) + \ldots + f(n) \simeq \pi \sqrt{n}^2 = \pi \cdot n$.

The average then becomes

$$\frac{f(1) + f(2) + \ldots + f(n)}{n} \simeq \frac{\pi \cdot n}{n} = \pi.$$

This completes the proof! $\square$