MATH 1002

LECTURE 8-E

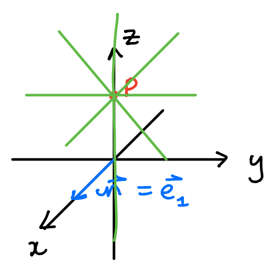
LINES IN $\mathbb{R}^2$ & $\mathbb{R}^3$: VECTOR FORM

Last lecture: We saw that to specify a line ℓ in $\mathbb{R}^2$, it's enough to give :
- a point $P \in \ell$
- a normal vector $\vec{n}$.

This doesn't work in $\mathbb{R}^3$.

Exercise: Why not?

Answer: in $\mathbb{R}^3$, the condition of being normal to $\vec{n}$ doesn't uniquely determine the direction of the line.



any line in the yz plane will be orthogonal to $\vec{n} = \vec{e}_1$

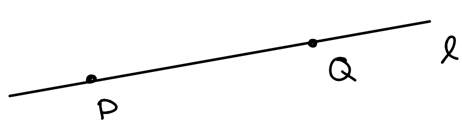
So the normal vector actually determines a plane.

Remember this for next week!

Question So how can we specify a line in $\mathbb{R}^3$?

Answer In $\mathbb{R}^3$ (or in $\mathbb{R}^n$) we can uniquely specify a line ℓ by giving

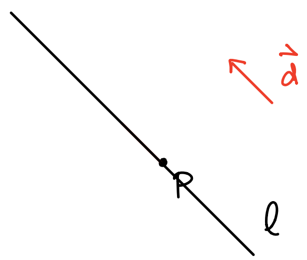
(1) two distinct points $P, Q \in \ell$.



in this case we could take $\vec{d} = \vec{PQ}$

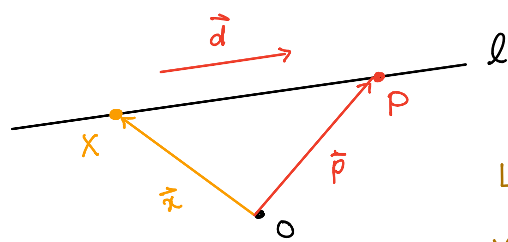
OR

(2) a point $P \in \ell$ and a direction vector $\vec{d}$



<u>Definition</u>: A direction vector for a line ℓ is a non-zero vector $\vec{d}$ which is parallel to the line ℓ.

<u>Goal</u>: Given a direction vector $\vec{d}$ for ℓ and a point $P \in \ell$, write down an equation that will quickly let us see which points are in ℓ.



Let X be any point. $\left(\begin{matrix} X = (x, y) \\ \text{or} \\ X = (x, y, z) \end{matrix} \right)$

$X \in \ell \iff \overrightarrow{PX}$ is parallel to $\vec{d}$

$\iff$ there is some scalar $t \in \mathbb{R}$ such that $\overrightarrow{PX} = t\vec{d}$.

- <u>Note</u>: $\overrightarrow{PX} = \vec{x} - \vec{p}$.

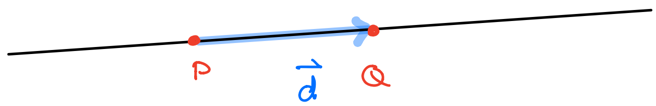
So $X \in \ell \iff \boxed{\vec{x} = \vec{p} + t\vec{d} \text{ for some } t \in \mathbb{R}}$ (*)

this is called the vector form of a line ℓ in $\mathbb{R}^n$

This tells us: $\ell = \{ X \mid \text{we can find } t \in \mathbb{R} \text{ s.t. } \vec{x} = \vec{p} + t\vec{d} \}$.

Definition: $t \in \mathbb{R}$ is called a parameter.

Remark • If instead of $(P, \vec{d})$ we're given two points P, Q on ℓ, we can take $\vec{d} = \vec{PQ}$, and use this to write down a vector equation



• different choices of $(P, \vec{d})$ or (P, Q) give us different looking equations that still represent the same line.

Parametric form for lines in $\mathbb{R}^2$

We start from the vector form: $\vec{x} = \vec{p} + t\vec{d}, \ t \in \mathbb{R}$. (*)

If we're working in $\mathbb{R}^2$, we can write

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad \text{and} \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

⇨ Then (*) becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad t \in \mathbb{R}.$$

This gives two equations:

$$\boxed{\begin{aligned} x &= p_1 + td_1 \\ y &= p_2 + td_2, \end{aligned} \quad t \in \mathbb{R}}$$

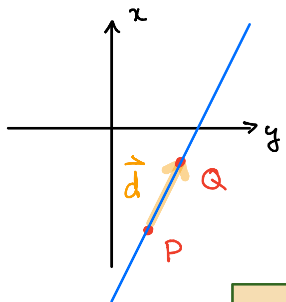
parametric equations for the line $\ell \subset \mathbb{R}^2$ passing through the point $P = (p_1, p_2)$

with direction vector $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

<u>Example</u>: Find a vector form and parametric equations for the line ℓ passing through the points $P = (1, -3)$
$Q = (2, -1)$

<u>Solution</u>: We take $P = (1, -3) \Rightarrow \vec{p} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

and $\vec{d} = \overrightarrow{PQ} = \begin{bmatrix} 2 & -1 \\ -1 & +3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



So the vector form is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}.$$

<u>Exercise</u>: What about the parametric equations?

<u>Solution</u>
$$\begin{cases} x = 1 + t \\ y = -3 + 2t \end{cases}, \quad t \in \mathbb{R}.$$

<u>Exercise</u>: Use the vector or parametric equations to write down 2 points on ℓ other than P, Q.

<u>Solution</u>: We just plug in different values of t to get points $(x, y) \in \ell$.

- $t = 0 \rightsquigarrow P$, $\quad t = 1 \rightsquigarrow Q$

Many choices for t, but e.g.

$t = 2 \rightsquigarrow (x, y) = (3, 1)$

$t = -1 \rightsquigarrow (x, y) = (0, -5)$.

<u>Recall</u>: These equations are not unique.

For example, we can take $(2,-1)$ as our point

& $\sqrt{7}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}\sqrt{7}\\2\sqrt{7}\end{bmatrix}$ as our direction vector

Exercise: What vector equation do you get from these choices?

Solution: $\vec{x} = \begin{bmatrix}2\\-1\end{bmatrix} + t\begin{bmatrix}\sqrt{7}\\2\sqrt{7}\end{bmatrix}, \quad t \in \mathbb{R}.$

Since t is a variable, we can give it a new name:

$$\vec{x} = \begin{bmatrix}2\\-1\end{bmatrix} + s\begin{bmatrix}\sqrt{7}\\2\sqrt{7}\end{bmatrix}, \quad s \in \mathbb{R}.$$

Compare to first equation:

$$\vec{x} = \begin{bmatrix}1\\-3\end{bmatrix} + t\begin{bmatrix}1\\2\end{bmatrix}, \quad t \in \mathbb{R}.$$

$\Rightarrow$ $x = 2 + s\sqrt{7}$ $\qquad$ $y = -1 + 2s\sqrt{7}$

$= 1 + t$ $\qquad$ $= -3 + 2t$

$\Rightarrow t = 1 + s\sqrt{7}$, so the parameters are related.

Parametric equations for lines in $\mathbb{R}^3$:

We start from the vector form: $\vec{x} = \vec{p} + t\vec{d}, \quad t \in \mathbb{R}$. $(*)$

If we're working in $\mathbb{R}^3$, we can write

$$\vec{p} = \begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}, \quad \vec{d} = \begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}, \quad \text{and} \quad \vec{x} = \begin{bmatrix}x\\y\\z\end{bmatrix}.$$

Then $(*)$ becomes:

$$\boxed{\begin{aligned} x &= p_1 + td_1 \\ y &= p_2 + td_2 \qquad t \in \mathbb{R} \\ z &= p_3 + td_3 \end{aligned}}$$

parametric equations for the line ℓ in $\mathbb{R}^3$ passing through the point $P = (p_1, p_2, p_3)$ and with direction vector $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$.

Summary of the lecture:

- To write down a line in $\mathbb{R}^n$ we can specify a point $P \in \ell$ and a direction vector $\vec{d}$ parallel to ℓ.

  ⇒ vector equation $\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$

  ⇒ if we know the components of $\vec{p}$ & $\vec{d}$ we can write down a list of parametric equations for ℓ.

You should be able to:

- given two points ($P, Q \in \ell$) or a point & direction vector ($P, \vec{d}$), write down vector and parametric equations.
- use vector or parametric equations for a line ℓ to write down new examples of points on ℓ.

Next time: overview: comparing normal, general, vector, parametric forms.