Discrete Mathematics

MATH1064, Lecture 30

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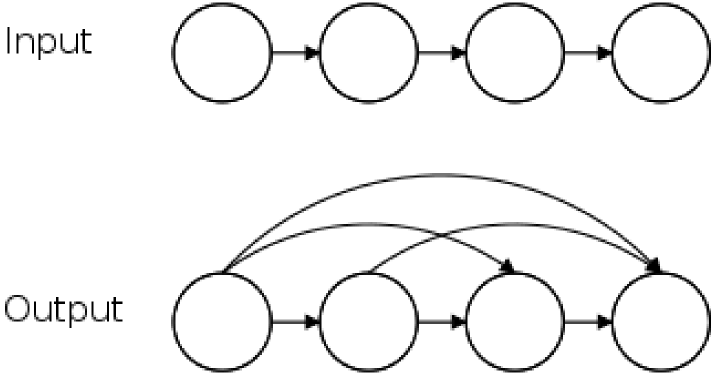
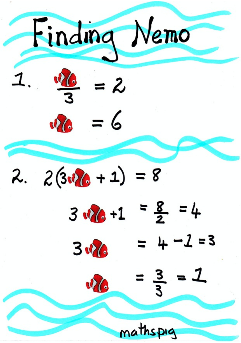


Extra exercises for Lecture 30

Section 9.3: Problems 1–4, 9, 10, 18–28

Section 9.4: Problems 1–11, 27, 28



Representing relations

Most relations (especially for infinite sets) are defined implicitly.

For small sets, we can describe them using digraphs.

Example: $X = \{1, 2, 3\}$, and xRy if and only if $x > y$.

Example: $X = \{1, 2, 3\}$, and xRy if and only if $x \geq y$.

Making a relation reflexive

Recall that R is reflexive if $\forall x \in X$, $(x, x) \in R$.

Suppose $S \subseteq X \times X$ is some relation.

What is the smallest reflexive relation containing S?

Answer: $S \cup \{(x, x) \mid x \in X\}$.

This is the reflexive closure $\mathrm{ref}(S)$ of S.
$\Delta = \{(x, x) \mid x \in X\}$ is the diagonal relation.

Example: $x > y$ on $\mathbb{R}$ is not reflexive.
What is the reflexive closure?

Making a relation reflexive

Reflexive closure: $\text{ref}(R) = R \cup \{(x,x) \mid x \in X\} = R \cup \Delta$

For a digraph representation: add a loop at each vertex!

Making a relation symmetric

Recall that R is symmetric if $(x, y) \in R$ implies $(y, x) \in R$.

Suppose $S \subseteq X \times X$ is some relation.

What is the smallest symmetric relation containing S?

Answer: $S \cup \{(y, x) \mid (x, y) \in S\}$.

This is the symmetric closure $\mathrm{sym}(S)$ of S.
$S^{-1} = \{(y, x) \mid (x, y) \in S\}$ is the inverse relation to S.

Example: $x > y$ on $\mathbb{R}$ is not symmetric.
What is the symmetric closure?

Making a relation symmetric

Symmetric closure: $\mathrm{sym}(R) = R \cup \{(y, x) \mid (x, y) \in R\} = R \cup R^{-1}$

For a digraph representation: add edges in the opposite direction!

Making a relation transitive

For a digraph representation:

If there is a path from x to y, add an edge from x to y.

Making a relation transitive

Let R be a relation on a set X.

There is a path of length k from x to y in R if there are $x = x_0, \ldots, x_k = y$ such that xRx_1, x_0Rx_1, $\ldots$, $x_{k-1}Ry$.

Claim: There is a path of length k from x to y if and only if xR^ky.

How to you prove this? Induction!

Making a relation transitive

Define $R^\star = \bigcup_{k=1}^{\infty} R^k$. This is the connectivity closure.

Fact 1: The connectivity closure $R^\star$ is the transitive closure $\mathrm{tra}(R)$.

Fact 2: If X has only n elements, then $R^\star = \bigcup_{k=1}^{n} R^k$.

Warshall's algorithm to compute transitive closure

Input: Relation R on set with n elements
Output: Transitive closure $\mathrm{tra}(R)$

Main idea: A path exists between vertices x and y if and only if:

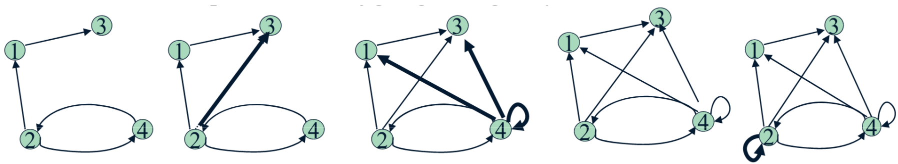
- there is an edge from x to y; or
- there is a path from x to y going through vertex 1; or
- there is a path from x to y going through vertices 1 and/or 2; or
- there is a path from x to y going through vertices 1 and/or 2 and/or 3; or
- ...
- there is a path from x to y going through any of the other $n - 2$ vertices.

Warshall's algorithm

Example:

Warshall's algorithm

Example:



Things to ponder

Does it matter in which order you take closures?

Which of the following equalities hold?

$$\mathrm{ref}(\mathrm{sym}(R)) = \mathrm{sym}(\mathrm{ref}(R))$$

$$\mathrm{ref}(\mathrm{tra}(R)) = \mathrm{tra}(\mathrm{ref}(R))$$

$$\mathrm{tra}(\mathrm{sym}(R)) = \mathrm{sym}(\mathrm{tra}(R))$$

More difficult questions:

Is $\mathrm{tra}(\mathrm{sym}(\mathrm{ref}(R)))$ an equivalence relation?

Is $\mathrm{ref}(\mathrm{sym}(\mathrm{tra}(R)))$ an equivalence relation?