THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Discrete Probability – Week 11 Practice Class

MATH1064: Discrete Mathematics for Computing

Here is a list of **problems** for the practice class. Try to solve them before you go to class! There are more problems here than can be solved in the hour, so you should get started on them!

1. A fair die is rolled three times. What is the probability that at least one three was rolled given that at least one of the rolls was a six?

Solution: Let A be the event that at least one 6 is rolled. Let B be the event that at least one 3 is rolled. The question asks us to calculate $P(B \mid A)$. We do so using the following formula.

$$P(A \cap B) = P(B \mid A)P(A)$$

Therefore we need to find $P(A)$ and $P(A \cap B)$.

$P(A)$ is the probability that at least one 6 is rolled. The probability that no 6s are rolled is $(\frac{5}{6})^3$. Therefore the probability that at least one 6 is rolled is $1 - (\frac{5}{6})^3 = \frac{91}{216}$.

$P(A \cap B)$ is the probability that at least one 6 is rolled and at least one 3 is rolled. To find this we split the event into cases.

- The probability that exactly one 3 and one 6 are rolled is $3!(\frac{1}{6})^2 \frac{4}{6} = \frac{4}{36}$.
- The probability that exactly one 3 and two 6s are rolled is $\frac{3}{6^3} = \frac{1}{72}$.
- The probability that exactly one 6 and two 3s are rolled is also $\frac{1}{72}$.

Therefore $P(A \cap B) = \frac{4}{36} + \frac{1}{72} + \frac{1}{72} = \frac{5}{36}$.

Using the formula above we have

$$\frac{5}{36} = P(B \mid A)\frac{91}{216} \qquad \Rightarrow \qquad P(B \mid A) = \frac{30}{91}$$

2. An exhibition of 12 paintings features 10 originals and 2 forged paintings. A visitor selects a painting to buy at random, but consults an expert for their advice. This expert gives a correct judgement in 9 of 10 paintings, regardless of whether they are originals or fakes. When the expert declares the painting to be a fake, the visitor returns the painting and chooses a different one. With what probability is this second painting is an original?

Solution: Let E be the event that the first painting is an original. Let F be the event that the expert thinks that the first painting is an original. Let G be the event that the second painting is an original.

We want to know $P(G)$ (probability that the second painting is an original). We only pick an original, if the expert says, it is fake. There are two ways this can happen: the expert correctly says it is fake $P(\overline{E} \mid \overline{F})$ and the expert incorrectly says it is fake $P(E \mid \overline{F})$. So, we can split our probability into

$$P(G) = P(G \mid E)\, P\left(E \mid \overline{F}\right) + P\left(G \mid \overline{E}\right) P\left(\overline{E} \mid \overline{F}\right)$$

This can also be seen by writing a Venn diagram.

From the question statement we know that the expert is correct for 9 out of 10 paintings, regardless of original or fake. So, $P(E \mid F) = P(\overline{E} \mid \overline{F}) = \frac{9}{10}$.

We know that $P(G \mid E) = \frac{9}{11}$ since, if the first painting was an original, there are 9 original and 11 paintings left. We also know that $P(G \mid \overline{E}) = \frac{10}{11}$ since, if the first painting was fake, there are 10 original and 11 paintings left.

Now, we have $P\left(E \mid \overline{F}\right) = 1 - P\left(\overline{E} \mid \overline{F}\right)$, so we only need to calculate one of them, say $P\left(E \mid \overline{F}\right)$.

For this, we want to use Bayes' theorem (the easy variant), which says

$$P\left(E \mid \overline{F}\right) = \frac{P\left(\overline{F} \mid E\right) P(E)}{P(\overline{F})}.$$

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We know that $P(E) = \frac{10}{12}$, and we know that $P(\overline{F} \mid E) = 1 - P(F \mid E) = 1 - \frac{9}{10} = \frac{1}{10}$. So, the only thing we still need to calculate is $P(\overline{F})$. Here, we again, split $P(\overline{F}) = P(\overline{F} \cap E) + P(\overline{F} \cap \overline{E}) = P(\overline{F} \mid E)P(E) + P(\overline{F} \mid \overline{E})P(\overline{E})$ - and now we have everything we need $(P\overline{E}) = 1 - P(E) = \frac{2}{12})$ to calculate $P(G)$.

In detail:

$$P(\overline{F}) = P(\overline{F} \mid E)P(E) + P(\overline{F} \mid \overline{E})P(\overline{E}) = \frac{1}{10}\frac{10}{12} + \frac{9}{10}\frac{2}{12} = \frac{7}{30}$$

then

$$P\left(E \mid \overline{F}\right) = \frac{P\left(\overline{F} \mid E\right) P(E)}{P(\overline{F})} = \frac{\frac{1}{10}\frac{10}{12}}{\frac{7}{30}} = \frac{5}{14},$$

and

$$P(G) = P(G \mid E)\, P\left(E \mid \overline{F}\right) + P\left(G \mid \overline{E}\right) P\left(\overline{E} \mid \overline{F}\right) = \frac{9}{11}\frac{5}{14} + \frac{10}{11}\frac{9}{14} = \frac{135}{154}.$$

3. (Harder) Let $X : S \to \mathbb{N} \setminus \{0\}$ be a random variable on a countably infinite probability space S. Suppose that $p(X = k) = (1-p)^{k-1} p$ for each $k = 1, 2, \ldots$, where $p \in [0, 1]$.

1. Show that $E(X) = \frac{1}{p}$.
2. Show that $V(X) = \frac{1-p}{p^2}$.

Hint for the second part: Use the identities (1) $\sum_{k=1}^{\infty} k^2 q^{k-1} = \frac{1+q}{(1-q)^3}$ for all $0 < |q| < 1$; and (2) $V(X) = E(X^2) - E(X)^2$.

A random variable as in part (a) has a *geometric distribution with parameter p*.

Solution:

1. Recall that the definition of the expectation of a variable on a countably infinite probability space is
$$E(X) := \sum_{k=1}^{\infty} kp(X=k)$$
We are given that $p(S=k) = (1-p_0)^{k-1}p_0$ for some fixed $p_0 \in [0,1]$. Therefore
$$E(X) := \sum_{k=1}^{\infty} k(1-p_0)^{k-1}p_0$$
Now recall for a variable z we have the following statement about formal power series, where both sides converge to a real number if $z \in [0,1)$
$$\frac{1}{1-z} = 1+z+z^2+z^3+\dots$$
If we substitute $z = 1-p_0$ into this equation then we have
$$\frac{1}{p_0} = 1+(1-p_0)+(1-p_0)^2+(1-p_0)^3+(1-p_0)^4+\dots$$
Differentiating both sides with respect to p_0 yields
$$\begin{aligned}\frac{-1}{p_0^2} &= -1-2(1-p_0)-3(1-p_0)^2-4(1-p_0)^3-5(1-p_0)^4-\dots \\ \Rightarrow \frac{1}{p_0^2} &= 1+2(1-p_0)+3(1-p_0)^2+4(1-p_0)^3+5(1-p_0)^4+\dots \\ &= \sum_{k=1}^{\infty} k(1-p_0)^{k-1}\end{aligned}$$
Multiplying both sides by p_0 yields
$$\begin{aligned}\frac{1}{p_0} &= p_0\sum_{k=1}^{\infty} k(1-p_0)^{k-1} \\ &= \sum_{k=1}^{\infty} k(1-p_0)^{k-1}p_0 \\ &= E(X)\end{aligned}$$

2. Recall that $V(X) = E(X^2) - E(X)^2$. Using the previous part we have only to calculate $E(X^2)$.
$$\begin{aligned}E(X^2) &= \sum_{k=1}^{\infty} k^2(1-p_0)^{(k-1)}p_0 \\ &= p_0\sum_{k=1}^{\infty} k^2(1-p_0)^{(k-1)}\end{aligned}$$
To calculate $E(X^2)$ we first calculate $\sum_{k=1}^{\infty} k^2(1-p_0)^{(k-1)}$ then we multiply by p_0 at the end. Again we use the fact that
$$\frac{1}{1-z} = 1+z+z^2+z^3+z^4+\dots$$

Differentiating both sides with respect to z yields

$$\begin{aligned}\frac{1}{(1-z)^2} &= 1+2z+3z^2+4z^3+5z^4+\dots \\ &= \sum k = 1kz^{k-1}\end{aligned}$$

Multiplying both sides by z we have

$$\begin{aligned}\frac{z}{(1-z)^2} &= z+2z^2+3z^3+4z^4+5z^5+\dots \\ &= \sum_{k=1} kz^k\end{aligned}$$

Now we differentiate with respect to z again to get something of the form that we see in $E(X^2)$.

$$\begin{aligned}\frac{z}{(1-z)^2} &= \frac{1+z}{(1-z)^3} \\ &= 1+2^2z+3^2z^2+4^2z^3+5^2z^4+\dots \\ &= \sum_{k=1} k^2z^{k-1}\end{aligned}$$

Now we replace z with $1-p_0$ to obtain

$$\begin{aligned}\sum_{k=1} k^2(1-p_0)^{k-1} = \frac{1+(1-p_0)}{(1-(1-p_0))^3} \\ &= \frac{2-p_0}{p_0^3} \\ &\Rightarrow \sum_{k=1}^{\infty} k^2(1-p_0)^{k-1}p_0 = \frac{2-p_0}{p_0^2} = E(X^2)\end{aligned}$$

Now we use this identity, the answer from the previous part and the identity $V(X) = E(X^2) - E(X)^2$ to find $V(X)$.

$$\begin{aligned}V(X) &= E(X^2) - E(X)^2 \\ &= \frac{2-p_0}{p_0^2} - \frac{1}{p_0^2} \\ &= \frac{1-p_0}{p_0^2}\end{aligned}$$

4. Suppose you have a biased coin that shows heads with probability $\frac{1}{4}$ and tails with probability $\frac{3}{4}$. What is the expected number of flips until the coin comes up heads? What is the variance of this number of flips?

(Hint: Apply Question 3.)

Solution: If we can express our situation in terms of a geometric distribution with a parameter p_0 then we can express our answer using the formulas in Question 3. Let X be the random variable which counts the number of flips that must occur before we flip a 'heads'. The probability that a 'head' is flipped on the first attempt is $p(X=1) = \frac{1}{4}$. The probability that the first time a head is flipped is the second attempt is $p(X=2) = \frac{3}{4} \cdot \frac{1}{4}$. Similarly the probability that the first time a head is flipped is on the $k^t h$ attempt is

$$\begin{aligned} p(X=k) &= \frac{3}{4}^{(k-1)} \frac{1}{4} \\ &= (1-\frac{1}{4})^{(k-1)} \frac{1}{4} \end{aligned}$$

Therefore we can substitute the value $p_0 = \frac{1}{4}$ into the last question to find the expected value and variance of X. We have

$$\begin{aligned} E(X) &= \frac{1}{p_0} = 4 \\ V(X) &= \frac{1-p_0}{p_0^2} \\ &= \frac{3}{4} \cdot 16 = 12 \end{aligned}$$

5. A die is rolled until each of the values $1, \ldots, 6$ has appeared at least once.

1. Determine the expected value for the number of required rolls.
2. Determine the variance of the number of rolls that are required when the second distinct value has just been observed until the third distinct value appears. For example, in the sequence $1, 1, 2, 1, 3$ the second distinct value is observed on the 3rd roll and the third distinct value on the 5th roll.

Solution:

1. Let X_i be the number of rolls until the i^{th} distinct number has been rolled. We are interested in the expectation and variance of X_6. Now we define

$$Y_1 = 1 \quad \text{and} \quad Y_i = X_i - X_{i-1} \quad \forall i \in \{2,3,4,5,6\}$$

Therefore $X_6 = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6$ and we have

$$\begin{aligned} E(X_6) &= E(Y_1 + \cdots + Y_6) \\ &= E(Y_1) + \cdots + E(Y_6) \end{aligned}$$

Suppose that the $(i-1)^{th}$ distinct value has already been rolled. If $i = 2$ then 1 value has already been rolled so that probability of rolling a new value is $\frac{5}{6}$. If $i = 3$ then 2 distinct values have been rolled so the probability of rolling a new value is $\frac{4}{6}$. For an arbitrary i there is a $p_i := \frac{7-i}{6}$ chance of rolling a new value with the next roll. Now Y_i is the random variable that counts the difference between the number of rolls it takes to roll the $(i-1)^{th}$ distinct value and the i^{th} distinct value. Therefore

$$\begin{aligned} P(Y_i = 1) &= p_i \\ P(Y_i = 2) &= (1-p_i)p_i \\ P(Y_i = 3) &= (1-p_i)^2 p_i \\ &\vdots \\ P(Y_i = k) &= (1-p_i)^{k-1} p_i \end{aligned}$$

Therefore Y_i has a geometric distibution with parameter $\frac{7-i}{6}$. This means we can use the answer from Question 3 to find the expectation of Y_i.

$$\begin{aligned}
E(X_6) &= E(Y_1) + \cdots + E(Y_6) \\
&= \frac{6}{7-1} + \frac{6}{7-2} + \cdots + \frac{6}{7-6} \\
&= 14.7
\end{aligned}$$

2. This part is asking for the variance of the variable Y_3 from the previous part. We can calculate the variance of Y_i using the answer to Question 3. Recall that Y_3 is a geometric distribution with parameter $\frac{2}{3}$. We have

$$\begin{aligned}
V(Y_3) &= \frac{1-\frac{2}{3}}{\left(\frac{2}{3}\right)^2} \\
&= \frac{9-3\cdot 2}{4} \\
&= \frac{3}{4}
\end{aligned}$$

Puzzles on next page!

Here are two **puzzles** that you can think about during the week. Feel free to ask your tutors or lecturer for more hints!

O In an intervarsity chess competition, each university enters a team of two players. Each participant plays all other players except their team-mate.

For each game, the winner receives one point and the loser receives nothing. If the game is drawn, however, then both players receive half a point each. At the end of the competition, Sydney Uni has a total winning score of 39 points, and all other universities have an equal number of points. How many universities participated in the competition?

P "How old are your three children?" Kate asked her new neighbour, while she was leaning on his fence. "Their ages are positive integers, all different, whose product is 90," he replied, "and the sum of their ages is the same as the number of my house".

Kate thought for a while. "I'm sorry, but I still cannot tell how old they are."

"Their ages are all more than the age of your youngest child" he added. She had already told him the ages of her children. "Now I know your children's ages," she said.

What is the number of Kate's neighbour's house? How old is Kate's youngest child? Can you prove that your solution is the *only* solution possible?

Puzzle hints:

Stuck on the puzzles from the last sheet? Here are some hints!

M Can you find a relationship between $a_{n+1} - a_n$ and $a_n - a_{n-1}$?

N How many numbers in the list start with $1\ldots$? How many start with $2\ldots$? How many start with $10\ldots$? $12\ldots$?