THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Counting I – Week 8 Tutorials

MATH1064: Discrete Mathematics for Computing

1. You are given an alphabet of 20 consonants and 6 vowels.

(a) In how many ways can we select a consonant and then a vowel?

Solution: The number of ways of choosing a consonant from 20 consonants is 20 and the number of ways of choosing a vowel from 6 vowels is 6. Hence the number of ways of selecting a consonant and then a vowel is $20 \times 6 = 120$.

(b) In how many ways can we make a two-letter string consisting of one consonant and one vowel?

Solution: The number of ways of making a two-letter word consisting of one consonant and one vowel is $20 \times 6 + 6 \times 20 = 240$.

2. In a town of 18,000 people everyone has three initials (each consisting of one of the 26 letters of the alphabet). Must there be two people with the same initials?

Solution: The number of ways of getting three initials is $26 \times 26 \times 26 = 17576$. Since there are 18,000 people, there must be two people with the same initials.

3. (a) How many strings of four digits are there if 0 is never used?

Solution: Since 0 is never used, we see that there are 9 ways to choose each digit. Hence the number of strings of four digits, if 0 is never used, is $9^4 = 6561$.

(b) How many six digit numbers are there which do not repeat a digit and do not begin with 0?

Solution: Since the digits do not begin with 0, there are 9 ways of choosing the first digit of the six digit numbers. Since the numbers do not repeat a digit, there are 9 ways of choosing the second digit and then 8 ways of choosing the third, 7 ways the fourth, 6 ways the fifth and 5 ways the last digit. Hence the total number of six digit numbers, which do not repeat a digit and do not begin with 0, is $9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136080$.

(c) How many strings of length 3 start with 2 digits and end with one of the 26 capital letters of the alphabet?

Solution: There are 10 ways to choose the first digit and second digit, and there are 26 ways of choosing the capital letter. Hence the number of strings of length 3 start with 2 digits and one capital letter is $10 \times 10 \times 26 = 2600$.

4. You have a deck of fifty-two cards. As with all word problems, but with this in particular, there is some chance of differing interpretations!

(a) How many ways are there of choosing a hand of five cards?

Solution: The number of hands of five cards is $\binom{52}{5} = \dfrac{52!}{47!5!}$.

(b) How many of the hands in part(a) contain the queen of hearts?

Solution: If one of the five cards is the queen of hearts, then the problem is the same as choosing 4 cards from 51. The answer is thus

$$\binom{51}{4} = \frac{51!}{47!4!} = 249900.$$

(c) In how many ways can four hands of five cards each be given to four players, if we care which hand goes to which player?

Solution: There are $\binom{52}{5}$ ways of giving the first hand of five cards to the first player. Then there are 47 cards left so that there are $\binom{47}{5}$ ways of giving the second hand to the second player. Similarly, there are $\binom{42}{5}$ ways of giving the third hand to the third player and $\binom{37}{5}$ ways of giving the fourth hand to the fourth player. Hence the number of ways that four hands of five cards can be given to four players is

$$\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5} = \frac{52!}{32!5!5!5!5!}.$$

Alternatively, this can be written as a multinomial coefficient, where the left-over cards (32 of them) are considered as belonging to a fifth hand. The answer is then $\binom{52}{5,5,5,5,32}$.

(d) In how many ways can four hands of five cards be selected from the deck? (This time we *don't* care which hand goes to which player.)

Solution: The difference between this and the previous part is that now we don't care which hand goes to which player. So what counts as 4! different ways of distributing hands in part (*iii*) only counts as one way here. Thus the answer is that for part (*iii*) divided by 4!, i.e.

$$\frac{52!}{32!5!5!5!5!4!}.$$

5. How many distinguishable arrangements are there of the letters in the words

(a) imperseverant,

Solution: Since there are 3 e's, 2 r's and 1 of each of the other 8 letters, the number of distinguishable arrangements of the letters in the word *imperseverant* is $\frac{13!}{3!2!}$.

(b) myristicivorous,

Solution: Since there are 2 r's, 3 i's, 2 s's, 2 o's and 1 of each of the other 6 letters, the number of distinguishable arrangements of the letters in the word *myristicivorous* is $\frac{15!}{2!3!2!2!}$.

(c) indistinguishable,

Solution: There are 4 i's, 2 n's, 2 s's and 1 of each of the other 9 letters and so the number of distinguishable arrangements of the letters in the word *indistinguishable* is $\frac{17!}{4!2!2!}$.

(d) sociological?

Solution: There are 1 s, 3 o's, 2 c's 2 i's, 2 l's, 1 g and 1 a so that the number of distinguishable arrangements of the letters in the word *sociological* is $\frac{12!}{3!2!2!2!}$.

Note: All of these answers can be written as multinomial coefficients. For example, the answer to part (a) is $\binom{13}{3,2,1,1,1,1,1,1,1,1}$.

Note: "Imperseverant" does mean "not persevering". It is in the big Oxford dictionary (available online through the Uni Library), but there is only one recorded mention of the word.

"Myristicivorous" is only doubtfully a word; very likely no-one has used it seriously; it is supposed to mean "nutmeg-eating". It is not in the big Oxford dictionary. Search Google for `myristicivorous Chambers` for a comment on this word.

6. Suppose 20 people are divided into 6 different committees (labelled C_1 to C_6). Suppose that committee C_1 is to have 3 people, committee C_2 is to have 4 people, committee C_3 is to have 4 people, committee C_4 is to have 2 people, committee C_5 is to have 3 people and committee C_6 is to have 4 people. How many arrangements are there?

Solution: The total number of arrangements is $\frac{20!}{3!4!4!2!3!4!} = \binom{20}{3,4,4,2,3,4}$.

7. In how many ways can 15 distinct balls be placed in 4 boxes so that the first box contains 5 balls, the second box contains 3 balls, the third box contains 4 balls and the fourth box contains 3 balls?

Solution: The total number of ways is

$$\binom{15}{5}\binom{10}{3}\binom{7}{4}\binom{3}{3} = \frac{15!}{5!3!4!3!} = \binom{15}{5,3,4,3}.$$

8. (a) How many ways of arranging six a's and ten b's with no consecutive a's?

Solution: We must separate a by b. Since there are 10 b's, there are 11 positions one can choose for the six a's. Hence the number of possible ways is

$$\binom{11}{6} = 462.$$

(b) In how many ways can 8 identical pens be distributed among 4 students?

Solution: This is equivalent to selecting among the students 8 times. So we are selecting 8 times from 4 things with repetition. The answer is

$$\binom{8+4-1}{8} = \binom{11}{8} = \binom{11}{3} = 165.$$

(c) In how many ways can 8 identical pens be distributed among 4 students so that each student receives at least one pen?

Solution: Since each student receives at least one pen, let us begin by giving one pen to each student. We then distribute the 4 remaining pens arbitrarily. The answer is

$$\binom{4+4-1}{4} = \binom{7}{4} = 35.$$

(d) In how many ways can 8 identical pens and 10 identical pencils be distributed among 4 students?

Solution: Let us distribute the pens first and the pencils second. As above, we see that the number of ways of distributing the pens is $\binom{8+4-1}{8} = \binom{11}{8}$ and similarly the number of ways of distributing the pencils is $\binom{10+4-1}{10} = \binom{13}{10}$. Hence the number of ways that 8 pens and 10 pencils can be distributed among 4 students is

$$\binom{11}{8} \times \binom{13}{10} = 165 \times 286 = 47190.$$

(e) In how many ways can 8 identical pens and 10 identical pencils be distributed among 4 students if each student receives at least 1 pen and 1 pencil?

Solution: If each student receives at least 1 pen and 1 pencil, then the possible ways is

$$\binom{4+4-1}{4} \times \binom{6+4-1}{6} = \binom{7}{4} \times \binom{9}{6} = 35 \times 84 = 2940.$$

(f) Twelve students are awaiting enrolment in some courses (each student is to enrolled in just one course). If 3 of them are to be enrolled in Mathematics, 4 in Computer Science and 5 in Physics, in how many ways can the enrolment be made?

Solution: There are $\binom{12}{3}$ ways of choosing 3 students for Mathematics. Then there are $\binom{9}{4}$ ways of choosing 4 from the remaining 9 students for Computer Science and finally there is $\binom{5}{5}$ ways to choose the remaining 5 for Physics. Hence the possible ways the enrolment can be made is

$$\binom{12}{3} \times \binom{9}{4} \times \binom{5}{5} = \frac{12!}{3!4!5!} = 27720.$$

This is equal to the multinomial coefficient $\binom{12}{3,4,5}$.