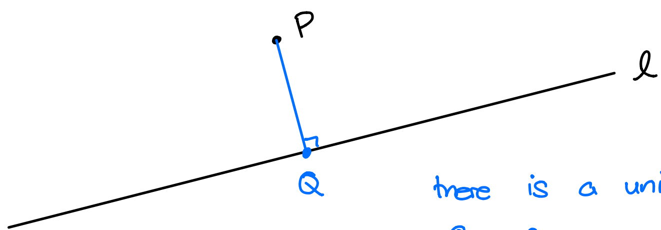
MATH1002 LECTURE 2-F

PROJECTIONS

Warm-up : **Exercise**: Given a point P and a line ℓ, what is the "distance" from P to ℓ?

Solution

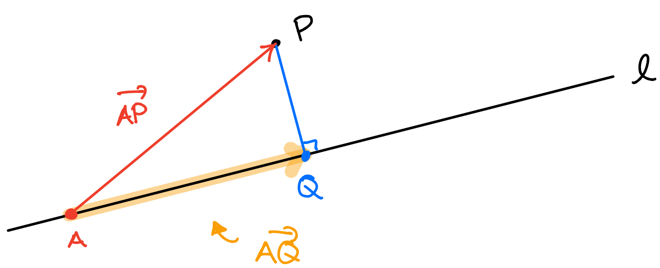


there is a unique point $Q \in \ell$ such that the distance between P & Q is minimal.

- This is the distance from P to ℓ
- $\overleftrightarrow{PQ}$ is perpendicular to ℓ.
- We say that Q is the projection of P onto ℓ.

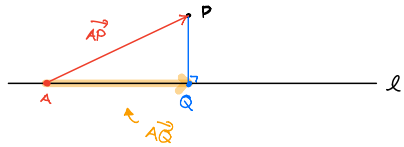
Projection of vectors:

- Choose any point $A \in \ell$ and consider the vector $\overrightarrow{AP}$



Then $\overrightarrow{AQ}$ is the projection of $\overrightarrow{AP}$ onto ℓ.

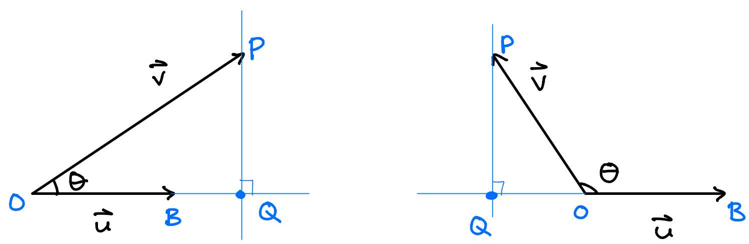
Think of a big light shining onto ℓ from far away



Then $\vec{AQ}$ is the shadow of $\vec{AP}$ on ℓ.

- In general, we can define the projection of any vector $\vec{v} \in \mathbb{R}^n$ onto a non-zero vector $\vec{u} \in \mathbb{R}^n$

Two cases:



$$\vec{OP} = \vec{v}$$

$$\vec{OB} = \vec{u}$$

$\vec{OQ}$ is the projection of $\vec{v}$ onto $\vec{u}$.

$$=: \text{proj}_{\vec{u}}(\vec{v}).$$

In both cases, $\|\vec{OQ}\| = \|\vec{OP}\| \, |\cos\theta| = \|\vec{v}\| \, |\cos\theta|$

→ so we know the length of $\vec{OQ}$

and its direction is either the same or the negative of $\vec{u}$.

$\therefore$ this information determines $\vec{OQ}$ completely.

If $0 \leq \theta \leq \pi/2$,

$\vec{OQ} = \|\vec{OQ}\| \cdot$ unit vector in direction $\vec{u}$

$$= (\|\vec{v}\| \cos\theta) \left(\frac{1}{\|\vec{u}\|} \vec{u} \right)$$

$$= \|\vec{v}\| \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} \right) \frac{1}{\|\vec{u}\|} \vec{u}$$

If $\frac{\pi}{2} < \theta \leq \pi$:

$\vec{OQ} = \|\vec{OQ}\| \cdot$ unit vector in opposite direction to $\vec{u}$

$$= (-\|\vec{v}\| \cos\theta) \left(- \frac{1}{\|\vec{u}\|} \vec{u} \right)$$

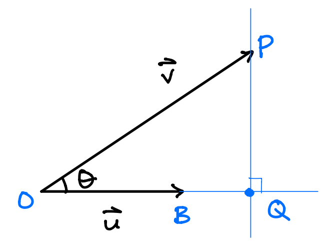
$$= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}.$$

$$= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}$$

Definition: For $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{u} \neq \vec{0}$, the projection of the vector $\vec{v}$ onto $\vec{u}$ is denoted by $\text{proj}_{\vec{u}}(\vec{v})$ and defined by

$$\text{proj}_{\vec{u}}(\vec{v}) := \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \in \mathbb{R}^n.$$

scalar vector.



Another method to find $\text{proj}_{\vec{u}}(\vec{v}) = \overrightarrow{OQ}$

We need to find Q.

- We know that $\overrightarrow{OQ}$ will be parallel to $\vec{u}$

  $\Rightarrow$ there is some $\lambda \in \mathbb{R}$ such that $\overrightarrow{OQ} = \lambda \vec{u}$.

- We know that $\overrightarrow{QP}$ will be orthogonal to $\overrightarrow{OB} = \vec{u}$.

  $\Rightarrow \overrightarrow{QP} \cdot \vec{u} = 0$

we can write $\overrightarrow{QP} = \overrightarrow{QO} + \overrightarrow{OP} = \overrightarrow{OP} - \overrightarrow{OQ} = \vec{v} - \lambda \vec{u}$.

So $\overrightarrow{QP} \cdot \vec{u} = (\vec{v} - \lambda \vec{u}) \cdot \vec{u} = \vec{v} \cdot \vec{u} - \lambda \vec{u} \cdot \vec{u} = 0$

$\Rightarrow \lambda \vec{u} \cdot \vec{u} = \vec{u} \cdot \vec{v}$

$\Rightarrow \lambda = \dfrac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}.$

We conclude that $\overrightarrow{OQ} = \lambda \vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}$ as seen above.

Example Find the projection of $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$.

$$\vec{u}\cdot\vec{v} = (-1)\times 1 + (-1)\times 2 + (-2)\times 4 = -1-2-8 = -11$$

$$\vec{u}\cdot\vec{u} = (-1)^2 + (-1)^2 + (-2)^2 = 1+1+4 = 6$$

so $$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u} = \frac{-11}{6}\begin{bmatrix}-1\\-1\\-2\end{bmatrix} = \begin{bmatrix}11/6\\11/6\\11/3\end{bmatrix}$$

Proposition: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$, $\vec{u} \neq \vec{0}$.

We can always write $\vec{v}$ as a sum

$$\vec{v} = \vec{v}_p + \vec{v}_o$$

where $\vec{v}_p$ is parallel to $\vec{u}$ and $\vec{v}_o$ is orthogonal to $\vec{u}$.

proof: we can write

$$\vec{v} = \vec{v} - \text{proj}_{\vec{u}}(\vec{v}) + \text{proj}_{\vec{u}}(\vec{v})$$

- $\text{proj}_{\vec{u}}(\vec{v}) = \dfrac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}$ is parallel to $\vec{u}$ by construction.

- Claim: $\vec{v} - \text{proj}_{\vec{u}}(\vec{v})$ is orthogonal to $\vec{u}$.

Exercise: prove the claim, by showing that the dot product is 0.

Solution: $$(\vec{v} - \text{proj}_{\vec{u}}(\vec{v}))\cdot\vec{u} = \vec{v}\cdot\vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}\right)\cdot\vec{u}$$

$$= \vec{v}\cdot\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}\cdot\vec{u}$$

$$= \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} = 0$$

This completes the proof of the proposition. $\square$.

Example Let $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

Express $\vec{v}$ as the sum of a vector parallel to $\vec{u}$ and a vector orthogonal to $\vec{u}$.

Solution: we write $\vec{v} = \underbrace{\mathrm{proj}_{\vec{u}}(\vec{v})}_{\text{parallel}} + \underbrace{[\vec{v} - \mathrm{proj}_{\vec{u}}(\vec{v})]}_{\text{perpendicular}}$.

Exercise: Find $\mathrm{proj}_{\vec{u}}(\vec{v})$ and $\vec{v} - \mathrm{proj}_{\vec{u}}(\vec{v})$.

- $\mathrm{proj}_{\vec{u}}(\vec{v}) = \dfrac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}$

$$= \frac{(1 \times 3 + (-1) \times (-1) + (-1) \times 2)}{(1^2 + (-1)^2 + (-1)^2)} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$= \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

- $\vec{v} - \mathrm{proj}_{\vec{u}}(\vec{v}) = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -1/3 \\ 8/3 \end{bmatrix}$

Always check your work

- $\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ ← parallel to $\vec{u}$ ✓

- $\begin{bmatrix} 7/3 \\ -1/3 \\ 8/3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 7/3 + 1/3 - 8/3 = 0$ ← orthogonal to $\vec{u}$ ✓.

Summary of the lecture

- Given two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{u} \neq \vec{0}$, the projection of $\vec{v}$ onto $\vec{u}$ is

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

  - It is the unique vector such that
    - $\mathrm{proj}_{\vec{u}}(\vec{v})$ is parallel to $\vec{u}$
    - $\vec{v} - \mathrm{proj}_{\vec{u}}(\vec{v})$ is orthogonal to $\vec{u}$.

You should be able to:

- calculate $\mathrm{proj}_{\vec{u}}(\vec{v})$ and understand its geometric meaning.
- not confuse the roles of $\vec{u}$ & $\vec{v}$.
- express $\vec{v}$ as a sum of a vector parallel to $\vec{u}$ and a vector orthogonal to $\vec{u}$.