MATH1002 - LECTURE 1-E

SCALAR MULTIPLICATION

Recall: In the last lecture, we discussed vector addition

Take two vectors $\vec{a}, \vec{b}$

Produce a new vector $\vec{a}+\vec{b}$.

Today: Scalar multiplication

Take a scalar $c \in \mathbb{R}$ and a vector $\vec{a}$

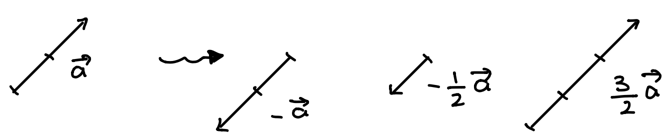
and produce a new vector $c\vec{a}$.

Geometric definition: $c\vec{a}$ is the vector that we get by stretching or shrinking ("scaling") $\vec{a}$ until it has length $|c|$ times its original length.

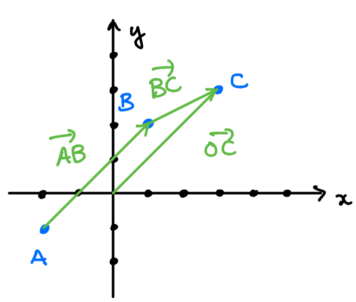
- if $c>0$, $c\vec{a}$ must point in the same direction as $\vec{a}$
- if $c<0$, $c\vec{a}$ must point in the opposite direction as $\vec{a}$.

Remark: if $c=0$, then $c\vec{a}=\vec{0}$ ← vector with all components 0

Examples



Example Recall the exercise from lecture 1-C:



$$\vec{AB}=\begin{bmatrix}3\\3\end{bmatrix},\quad \vec{BC}=\begin{bmatrix}2\\1\end{bmatrix},\quad \vec{OC}=\begin{bmatrix}3\\3\end{bmatrix}$$

We saw that $\vec{BC}$ and $\vec{OA}$ were similar, but not equal, because they point in opposite directions.

Now we can make this precise:

$$\vec{BC} = (-1)\cdot\vec{OA} \underset{\uparrow\ \text{Notation}}{=} -\vec{OA} \underset{\uparrow\ \text{Fact.}}{=} \vec{AO}$$

Algebraic definition of scalar multiplication:

$$\text{if } \vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \text{ then } c\vec{a} = \begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix}.$$

Example: if $c = 0$, $c\vec{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$

Exercise: if $\vec{a} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, what is $-3\vec{a}$?

(Answer geometrically and algebraically)

- $\downarrow \vec{a} \rightsquigarrow \downarrow 3\vec{a} \rightsquigarrow \uparrow -3\vec{a}$

answers match :)

$$-3\vec{a} = -3\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ (-3)(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Definitions:

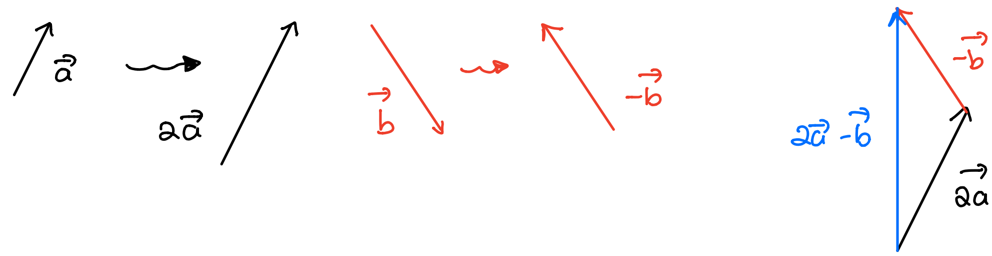
- $-1\cdot\vec{a} = -\vec{a}$ is called the negative of $\vec{a}$.
- Given vectors $\vec{a}, \vec{b}$, we define vector subtraction as follows:

$$\vec{a} - \vec{b} := \vec{a} + (-\vec{b})$$

Exercise: if $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, what is $2\vec{a} - \vec{b}$?

Answer geometrically and algebraically.

Geometrically



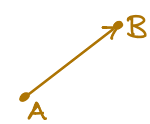
Algebraically: $2\vec{a}-\vec{b} = 2\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix} - \begin{bmatrix}2\\-3\end{bmatrix}$

$= \begin{bmatrix}0\\7\end{bmatrix}$ (matches drawing ☺)

Recall: • if $A = (a_1, a_2)$, we have the position vector $\vec{OA} = \begin{bmatrix}a_1\\a_2\end{bmatrix}$

• if $B = (b_1, b_2)$, we have the displacement vector

$\vec{AB} = \begin{bmatrix}b_1 - a_1\\b_2 - a_2\end{bmatrix}$ vector from A to B



Remark: $\vec{AB} = \begin{bmatrix}b_1\\b_2\end{bmatrix} - \begin{bmatrix}a_1\\a_2\end{bmatrix}$

$= \vec{OB} - \vec{OA}$

Vector addition and scalar multiplication in $\mathbb{R}^3$

• geometrically and algebraically, it works the same way. (just harder to draw!)

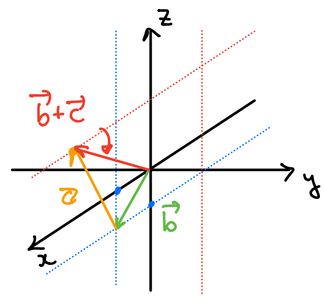
$$\boxed{\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} + \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = \begin{bmatrix}a_1+b_1\\a_2+b_2\\a_3+b_3\end{bmatrix} \;;\quad c\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}ca_1\\ca_2\\ca_3\end{bmatrix}}.$$

Examples: Let $\vec{a} = \begin{bmatrix}2\\4\\6\end{bmatrix}$, $\vec{b} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$, $\vec{c} = \begin{bmatrix}2\\1\\3\end{bmatrix}$

• What is $\vec{b} + \vec{c}$?

algebraically: $\vec{b} + \vec{c} = \begin{bmatrix} 1+2 \\ 0+1 \\ -1+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

geometrically:



(hard to do correctly!)

Exercise: Use the algebraic definitions to find $\vec{a} + \vec{b} - 2\vec{c}$.

Solution: $\vec{a} + \vec{b} - 2\vec{c} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1-4 \\ 4+0-2 \\ 6-1-6 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

Summary of the lecture

scalar multiplication stretches or shrinks the original vector $\vec{a}$ until its length is $|c|$ times the original length.

- if $c < 0$ the new vector points in the opposite direction.

$$c \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix}$$

negative: $-\vec{a} = (-1)\vec{a}$

vector subtraction $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

- We also defined vector addition and scalar multiplication in $\mathbb{R}^3$.