MATH1002 LECTURE 2-B

DOT PRODUCT

<u>Definition</u>: Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ be two vectors in $\mathbb{R}^n$.

The dot product of $\vec{u}$ and $\vec{v}$ is a scalar, defined by

$$\vec{u} \bullet \vec{v} := u_1v_1 + u_2v_2 + \dots + u_nv_n$$

Notation.

<u>Example #1</u>

$$\begin{bmatrix} 1/2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 1/2 \cdot (-1) + 3 \cdot 2$$

$$= -1/2 + 6 = 11/2.$$

<u>Example #2</u>

<u>Exercise</u>: What is $\begin{bmatrix} -1 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 2 \end{bmatrix}$?

<u>Solution</u>: $0 + 8 = 8$.

<u>Example #3</u>

$$\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = 2 \times (-1) + 1 \times 3 + (-4) \times 2$$

$$= -2 + 3 - 8 = -7.$$

<u>Example #4</u>

<u>Exercise</u>: Find $\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$.

<u>Solution</u> $(-1) \times 2 + 3 \times 1 + 2 \times (-4) = -2 + 3 - 8 = -7.$

(Compare to #3!)

Example # 5 $\begin{bmatrix} -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ ← this is not defined!

Properties of dot products:

Theorem: Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then:

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ [commutativity]

(b) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ [distributivity]

(c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

(d) $\vec{u} \cdot \vec{u} \geq 0$; and $\vec{u} \cdot \vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$.

Remark:
- Combining (a) & (b), we obtain also
$$(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$$
- Combining (a) & (c), we obtain also
$$\vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v}).$$

proof of property (a):

From the definition of dot product, we have

$$\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n.$$
$$\vec{v} \cdot \vec{u} = v_1 \cdot u_1 + v_2 \cdot u_2 + \cdots + v_n \cdot u_n$$

Since multiplication of real numbers is commutative, we have $u_i v_i = v_i u_i$ and hence

$$u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$$

i.e. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, as claimed. $\square$ ← end of proof.

proof of property (d)

Exercise: There are three things to check:

- for any $\vec{u}$, $\vec{u}\cdot\vec{u} \geq 0$
- If $\vec{u}\cdot\vec{u} = 0$, then $\vec{u} = \vec{0}$
- If $\vec{u} = \vec{0}$, then $\vec{u}\cdot\vec{u} = 0$

Solution
- take $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, so

$$\vec{u}\cdot\vec{u} = u_1u_1 + u_2u_2 + \cdots + u_nu_n$$
$$= u_1^2 + u_2^2 + \cdots + u_n^2$$

- we know that $u_i^2 \geq 0$ for $i = 1, 2, \ldots, n$, so we conclude that $\vec{u}\cdot\vec{u} \geq 0$.

- furthermore, if $\vec{u}\cdot\vec{u} = 0$, this must mean that $u_i^2 = 0$ for each i

$$\Rightarrow u_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, n$$

$$\Rightarrow \vec{u} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$$

- finally, if $\vec{u} = \vec{0}$, $\vec{u}\cdot\vec{u} = 0 + 0 + \cdots + 0 = 0$.

$\square$

Back to the remark:

Assume we know that $\cdot$ satisfies commutativity (a)

and (b) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$.

Now consider $(\vec{v}+\vec{w})\cdot\vec{u} = \vec{u}\cdot(\vec{v}+\vec{w})$ (commutativity)

$$= \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} \qquad (a)$$

$$= \vec{v}\cdot\vec{u} + \vec{w}\cdot\vec{u}$$

as claimed in the remark.

Exercise Assuming (a) & (c), deduce that
$$\vec{u}\cdot(c\vec{v}) = c(\vec{u}\cdot\vec{v}).$$

Solution
$$\vec{u}\cdot(c\vec{v}) = (c\vec{v})\cdot\vec{u} \qquad (a)$$
$$= c(\vec{v}\cdot\vec{u}) \qquad (c)$$
$$= c(\vec{u}\cdot\vec{v}) \qquad (a)$$
$\square$

Summary of the lecture

The dot product of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n. \quad \in \mathbb{R}$$

It satisfies nice properties with respect to vector addition and scalar multiplication:

(a) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

(b) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

(c) $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v})$

(d) $\vec{u}\cdot\vec{u} \geq 0$; and $\vec{u}\cdot\vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$.