Discrete Mathematics

MATH1064, Lecture 2

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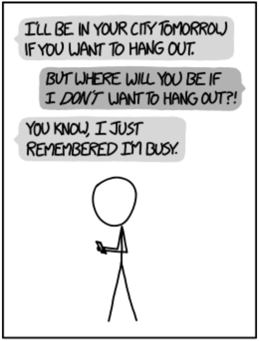

WHY I TRY NOT TO BE PEDANTIC ABOUT CONDITIONALS.

Propositions (aka Statements)

Definition (Proposition)

A proposition is a sentence that is true or false but not both.

1. Is it going to snow tomorrow?
2. He is sad.
3. Australia has exactly 4 states and 3 territories.
4. $a^2 + b^2 = c^2$.
5. $a^2 + b^2 = c^2$, $a = 3$, $b = 4$, $c = 5$.
6. Go to sleep.

Negation, Conjunction, Disjunction

Let p be a proposition.
Negation: $\neg p$ (read: not p), also written $\sim p$
Conjunction: $p \wedge q$ (read: p and q)
Disjunction: $p \vee q$ (read: p or q)

Defined by the truth tables:

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Note: $p \vee q$ is the "inclusive or"

Order of operation

$\neg$ comes first

$\wedge$ and $\vee$ are equal seconds

So:

$$\neg p \wedge q = (\neg p) \wedge q$$

But

$$p \wedge q \vee r$$

is ambiguous; one needs to use parentheses to say what is meant:

$$(p \wedge q) \vee r \quad \text{or} \quad p \wedge (q \vee r)$$

The "exclusive or"

Often written as $p \oplus q$

p	q	either p is true or q is true but not both
T	T	F
T	F	T
F	T	T
F	F	F

Claim: This can be written as $(p \lor q) \land \neg(p \land q)$

p	q	$p \lor q$	$p \land q$	$\neg(p \land q)$	$(p \lor q) \land \neg(p \land q)$
T	T				
T	F				
F	T				
F	F				

Variables and Compound Propositions

We've been thinking about p and q as propositions with some value (true or false). Now: think of them as variables (also called proposition variables or statement variables).

We use capital letters for compound propositions, which are made up from proposition variables (e.g. $p, q, r, ...$) and the symbols $\neg, \wedge, \vee$ with unambiguous parentheses.

Example: $P = (p \vee q) \ \wedge \neg(p \wedge q)$

If there are n variables in P, then the truth table for P has how many rows?

(1) n (2) $2n$ (3) n^2 (4) 2^n (5) $n!$

Definition (Logical equivalence)

Two compound propositions P and Q are logically equivalent,

$$P \equiv Q,$$

if they have identical truth values for every possible combination of truth values for their proposition variables.

Example from first lecture:

p	q	$p \wedge q$	$((p \wedge \neg q) \vee (p \wedge q)) \wedge q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Hence

$$((p \wedge \neg q) \vee (p \wedge q)) \wedge q \equiv p \wedge q$$

Which of the following propositions are logically equivalent?

1. $P = \neg(p \wedge q)$
2. $Q = \neg(p \vee q)$
3. $R = (\neg p) \wedge (\neg q)$
4. $S = (\neg p) \vee (\neg q)$

The correct answer is: $P \equiv S$ and $Q \equiv R$.

The equivalences are De Morgan's laws:

$$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$$

$$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$$

Contradictions and Tautologies

Definition (Contradiction)

A contradiction is a compound proposition which takes the value false for all possible truth values (true/false in all combinations) of its variables.

Example: $p \wedge \neg p$, $(q \wedge p) \wedge \neg p$

$p \wedge (\text{contradiction}) \equiv (\text{contradiction})$; $p \vee (\text{contradiction}) \equiv p$

Definition (Tautology)

A tautology is a compound proposition which takes the value true for all possible truth values (true/false in all combinations) of its variables.

Example: $p \vee \neg p$

$p \wedge (\text{tautology}) \equiv p$; $p \vee (\text{tautology}) \equiv (\text{tautology})$

Have a look at the table of logical equivalences on pages 27-28!

The statement

$$(p \wedge \neg q) \wedge (\neg p \vee q)$$

is a

1. contradiction
2. tautology
3. neither

The conditional

Let p and q be proposition variables; the conditional from p to q, $p \to q$, is defined by the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p is the hypothesis and q is the conclusion.

Example: *If you want to hang out, I will be in your city tomorrow.*

$p =$ "You want to hang out", $\quad q =$ "I will be in your city tomorrow"

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: *If you want to hang out, I will be in your city tomorrow.*

$p =$ "You want to hang out", $q =$ "I will be in your city tomorrow"

The following propositions are consistent with $p \to q$:

You want to hang out and I will be in your city tomorrow. $(p \land q)$

You don't want to hang out and I will be in your city tomorrow. $(\neg p \land q)$

You don't want to hang out and I won't be in your city tomorrow. $(\neg p \land \neg q)$

The following proposition is inconsistent with $p \to q$:

You want to hang out and I won't be in your city tomorrow. $(p \land \neg q)$

Different ways of saying $p \to q$:

1. p implies q
2. if p, then q
3. q if p
4. p only if q
5. p is sufficient for q
6. q is necessary for p

One way to remember it:

$p \to q$ is false if and only if it describes a counterexample;
that is, the hypothesis is true but the conclusion is false

In order of operation: third, after $\vee$ and $\wedge$ (which come after $\neg$)

The conditional

Expressing the conditional in terms of and/or/not

$p \to q \quad \equiv \quad \neg p \vee q$

Friday:
Proof by truth table:

p	q	$p \to q$	$\neg p$	q	$\neg p \vee q$
T	T	T			
T	F	F			
F	T	T			
F	F	T			

A final remark on truth tables

What do truth tables do?

- P and Q compound propositions (on n variables). Is $P \equiv Q$?
- Truth tables for P and Q are bit strings t_P and t_Q of length 2^n
- $P \equiv Q$ if and only if $t_P = t_Q$

What about the inverse problem? Given a bitstring t of length 2^n, produce a compound proposition with t as truth table.

Summary

A proposition is a sentence that is true or false but not both.

Negation: $\neg p$ (read: not p)
Conjunction: $p \wedge q$ (read: p and q)
Disjunction: $p \vee q$ (read: p or q)
Conditional: $p \to q$ (read: p implies q)

Two compound propositions P and Q are logically equivalent, $P \equiv Q$, if they have identical truth values for each possible combination of truth values for their proposition variables.

A contradiction is a compound proposition which takes the value false for all possible truth values of its variables.

A tautology is a compound proposition which takes the value true for all possible truth values of its variables.