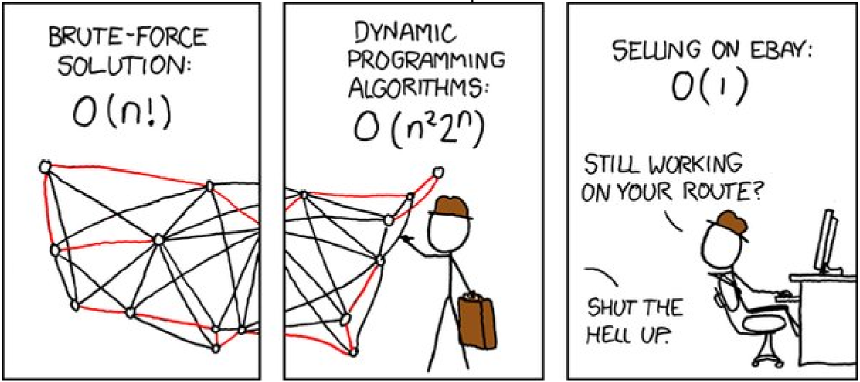
Discrete Mathematics

MATH1064, Lecture 15

Jonathan Spreer



Running times

Running time of an algorithm: number of steps required for it to finish

Depends on how large the input is

Running time is a function

$$f : \mathbb{N} \to \mathbb{N}; \text{Input size} \mapsto \text{Number of steps required}$$

Questions:

- When is an algorithm efficient?
- When is it better than a previous one?

Today, we'll introduce notation that helps to answer these questions!

O-notation

O-notation is a mathematical notation that describes how quickly a function f grows compared to some function g when both their arguments tend towards infinity.

For a given g it assigns a (large) set of functions which all grow at roughly the same rate, or slower. The comparison is done by checking whether f is an element of this set.

Definition [O-notation]

Let f and g be functions from a subset of $\mathbb{R}$ to $\mathbb{R}$. Then $f(x)$ is in $O(g(x))$ if there exist constants C and k such that for all $x \in A$, $x \geq k$:

$$|f(x)| \leq C \; |g(x)|$$

C and k are the witnesses of the statement "$f(x)$ is in $O(g(x))$"

O-notation

For this to make sense f must have the following properties:

- $f, g : A \subset \mathbb{R} \to \mathbb{R}$
- A is unbounded (for all $k \in A$ there exist infinitely many $x \in A$ such that $x \geq k$)

Example: $f(x) = 5x^2 + 3x + 1$ is in $O(x^2)$

Other notation: $f(x) \in O(g(x))$ or (caution!) $f(x) = O(g(x))$

Another example

$$f(n) = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}n(n+\frac{1}{2})(n+1) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Claim: $f(n) \in O(n^3)$

Need C and k such that for all $n > k : |f(n)| \leq C\ |n^3|$

Things you can do with the notation

$$f(n) = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}n(n+\frac{1}{2})(n+1) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Claim: $f(n) - \frac{1}{3}n^3 \in O(n^2)$

Notation: $f(n) \in \frac{1}{3}n^3 + O(n^2)$

More operations

- $f(n) \in O(f(n))$
- $O(c \cdot f(n)) = O(f(n))$ if c is a constant
- $O(f(n) + f(n)) = O(f(n))$
- $O(f(n)g(n)) = f(n) \cdot O(g(n))$

Note: Here, we use "$=$" to mean set equality.

Negative results

We can show that $n \in O(n^2)$: $C = k = 1 \rightarrow n < n^2$

Claim: $n^2 \notin O(n)$

Proof by contradiction:

Assume there are witnesses C and k such that

$$n^2 \leq Cn$$

for all $n > k$.

In particular, $n^2 \leq Cn$ for all $n > max(k, 1)$, and so

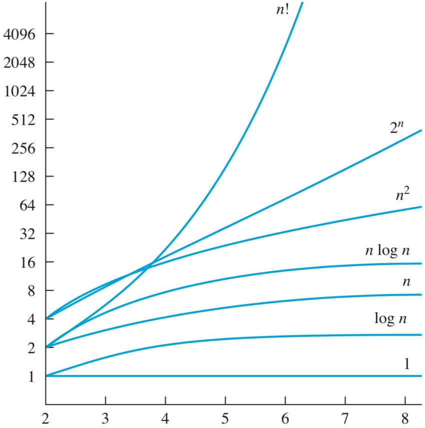
$$n \leq C$$

for all $n > max(1, k)$.

In particular, $n \leq C$ for all $n > max(1, k, C)$.

Then $C < n \leq C$. This is a contradiction! $\square$

Important examples



https://www.geogebra.org/graphing/qcyafxq4

Important examples

- $\log(n) \in O(n)$
- $\log_b(n) \in O(n)$
- $n^d \in O(b^n)$ if $d > 0$ and $b > 1$
- $b^n \in O(c^n)$ and $c^n \notin O(b^n)$ if $c > b > 1$
- $n! \in O(n^n)$
- $\log(n!) \in O(n \log(n))$

More Examples

- $x \in O(x^2)$
- $\frac{3x^2+5x}{x^2} \in O(1)$
- $\sqrt{x} + x \in O(x)$
- $2^n \in O(n!)$
- $n! \notin O(2^n)$
- $\log(x) \in O(x^\alpha)$ for all $\alpha > 0$
- $x^\alpha \in O(x^\beta)$ for all $\beta \geq \alpha$
- $3^n \notin O(2^n)$

Lower bounds and exact order

Definition [Ω-notation]

Let f and g be functions from a subset of $\mathbb{R}$ to $\mathbb{R}$. Then $f(x)$ is in $\Omega(g(x))$ if there are positive constants C and k such that for all $x > k$:

$$|f(x)| \geq C\ |g(x)|$$

Note: $f(x) \in \Omega(g(x))$ if and only if $g(x) \in O(f(x))$.

Definition [Θ-notation]

Let f and g be functions from a subset of $\mathbb{R}$ to $\mathbb{R}$. Then $f(x)$ is in $\Theta(g(x))$ if $f(x) \in O(g(x))$ and $f(x) \in \Omega(g(x))$.

$f(x)$ is $\Theta(g(x))$ means that $f(x)$ and $g(x)$ are of the same order!

Example

$$f(n) = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}n(n+\frac{1}{2})(n+1) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Claim: $f(n) \in \Theta(n^3)$

We already know that $f(n) \in O(n^3)$

$f(n) \in \Omega(n^3)$: Need C and k such that for all $n > k : |f(n)| \geq C\ |n^3|$

True or false?

$$\log_{17}(n) \in O(\log_2(n))$$

Summary - O, Ω, Θ

Let f and g be functions from a subset of $\mathbb{R}$ to $\mathbb{R}$.
Then $f(x) \in O(g(x))$ if there are constants C and k such that for all $x > k$:

$$|f(x)| \leq C\ |g(x)|$$

Let f and g be functions from a subset of $\mathbb{R}$ to $\mathbb{R}$.
Then $f(x) \in \Omega(g(x))$ if there are positive constants C and k such that for all $x > k$:

$$|f(x)| \geq C\ |g(x)|$$

Let f and g be functions from a subset of $\mathbb{R}$ to $\mathbb{R}$.
Then $f(x) \in \Theta(g(x))$ if $f(x) \in O(g(x))$ and $f(x) \in \Omega(g(x))$.

$f(x) \in \Theta(g(x))$ means that $f(x)$ and $g(x)$ are within a constant of each other.