MATH 1002 LECTURE 2-D

UNIT VECTORS

Recall: The length of a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is given by

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

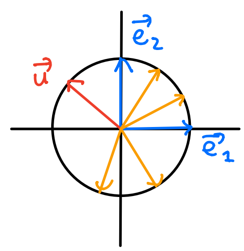
Definition: A unit vector is a vector of length 1.

Example In $\mathbb{R}^2$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$

are all unit vectors.

> **Exercise:** Check that $\|\vec{e}_1\| = \|\vec{e}_2\| = \|\vec{u}\| = 1$

A picture of all unit vectors in $\mathbb{R}^2$:



The unit circle!

Definition: $\vec{e}_1$ and $\vec{e}_2$ are called standard unit vectors.

Example: In $\mathbb{R}^n$, there are n standard unit vectors, given by $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$, where $\vec{e}_i$ has 1 as its i^{th} components and 0 for all other components:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

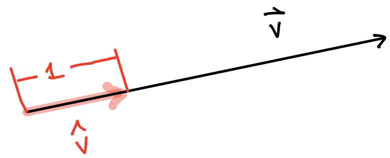
in i^{th} row

> **Exercise:** check that $\|\vec{e}_i\| = 1$.

Observation: Given a direction, there is exactly one unit vector pointing in that direction.

Definition: Let $\vec{v}$ be a non-zero vector.

The normalisation of $\vec{v}$ is the unique vector $\hat{v}$ with length 1 and direction the same as $\vec{v}$.



It's easy to see that $\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}$

positive scalar

$\Rightarrow$ $\hat{v}$ points in the same direction as $\vec{v}$.

$$\|\hat{v}\| = \left\| \frac{1}{\|\vec{v}\|}\vec{v} \right\|$$

$$= \left| \frac{1}{\|\vec{v}\|} \right| \|\vec{v}\| = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1.$$

Example: Normalise the following vectors

(a) $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (b) $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$

(a) The normalisation of $\vec{u}$ is $\hat{u} = \frac{1}{\|\vec{u}\|}\vec{u}$.

- $\|\vec{u}\| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$.

$$\Rightarrow \hat{u} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}.$$

(b) $\boxed{\text{Exercise: Find } \hat{v}.}$

Solution: $\|\vec{v}\| = \sqrt{2^2 + (-1)^2 + 0^2 + 3^2}$

$= \sqrt{4 + 1 + 0 + 9}$

$= \sqrt{14}$

$$\therefore \hat{v} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{14} \\ -1/\sqrt{14} \\ 0 \\ 3/\sqrt{14} \end{bmatrix}$$

Remark:
- The unit vector that points in the opposite direction to $\vec{v}$ is $-\hat{v}$.
- $\hat{0}$ doesn't make any sense!

Summary of the lecture

- A unit vector is a vector of length 1.
- We have some "favourite" unit vectors of length 1, called standard vectors.
- Given a non-zero vector $\vec{v}$, the process of finding the unique unit vector is called normalisation:

$$\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}.$$

You should be able to:

- be familiar with the terminology
- identify unit vectors, standard vectors
- normalise vectors.