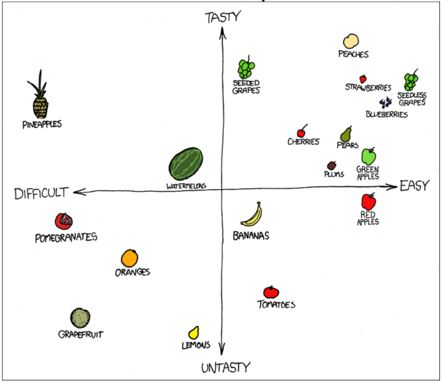
Discrete Mathematics

MATH1064, Lecture 32

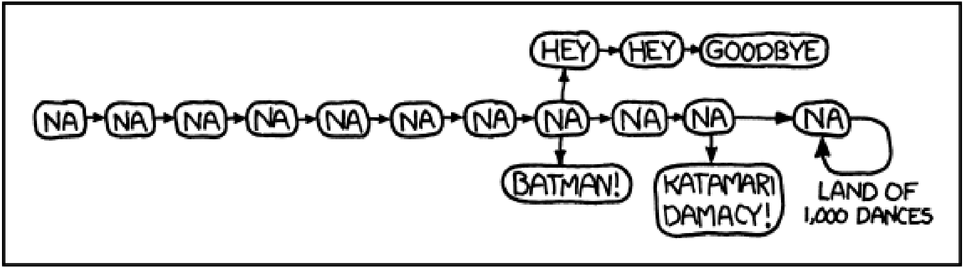
Jonathan Spreer



Extra exercises for Lecture 32

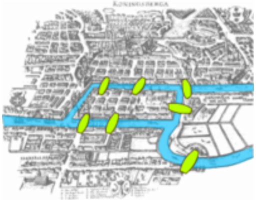
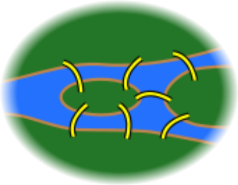
Section 10.4: Problems 1–9

Section 10.5: Problems 1–10, 13–15, 26, 27

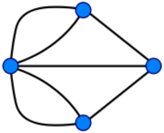


The bridges of Königsberg

Find a route through the city that crosses each bridge exactly once and takes you back to the starting place.
(No turning around on bridges or swimming, please!)

We can represent the city as a graph!



Land masses are vertices, and bridges are edges.

Paths

Let G be a graph, and let $x, y \in V(G)$. A path in G from x to y is an alternating sequence of vertices and edges

$$v_0,\ e_1,\ v_1,\ e_2,\ v_2,\ \dots,\ v_{n-1},\ e_n,\ v_n,$$

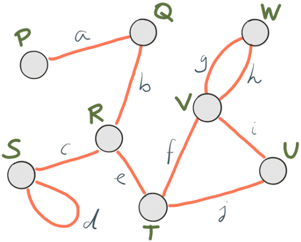
where $v_0 = x$, $v_n = y$, and each edge e_i has endpoints $\{v_{i-1}, v_i\}$.

Informally, we "walk" through the graph from vertex x to vertex y, following edges $e_1, e_2, \dots, e_n$ in turn.

Remark: A path in a digraph has the extra requirement that e_i is directed from v_{i-1} to v_i.

Paths

Example: Consider the following graph G.



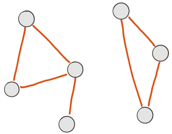
Vertices: $V(G) = \{P, Q, R, S, T, U, V, W\}$
Edges: $E(G) = \{a, b, c, d, e, f, g, h, i, j\}$

An example of a path from T to P is:

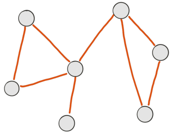
More on paths

A graph G is called connected if, for all vertices $x, y \in V(G)$, there is a path from x to y.
Otherwise G is called disconnected.

An example of a disconnected graph:

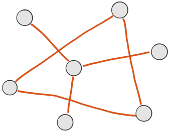


An example of a connected graph:



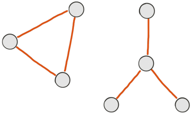
Question

Is the following graph connected?



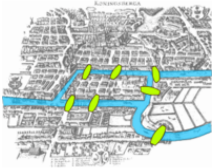
No! This graph is disconnected.
For example, there is no path from the top-left vertex to the bottom-right vertex.

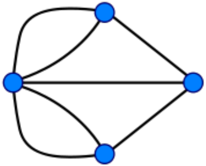
You could equally well draw the same graph like this:



Back to Königsberg

A path is called a circuit if it does not repeat any edge,
and it starts and ends at the same vertices.





In Königsberg, we are looking for a circuit that uses every edge.
That is, we are looking for a path that starts and ends at the same vertex, and uses every edge exactly once.

Such a path is called a Eulerian circuit.

Eulerian circuits

A Eulerian circuit is a path that starts and ends at the same vertex, and that uses every edge exactly once.

What are some necessary conditions for a Eulerian circuit to exist?

- If we ignore any isolated vertices (vertices with degree 0), then the remaining graph must be connected. Why?

- The degree of every vertex must be even. Why?

Lemma

If a graph has a Eulerian circuit, then every vertex degree is even.

Proof: Consider any vertex v.

- If v does not appear on the Eulerian circuit, then v cannot meet any edges. Therefore $\deg(v) = 0$, which is even.
- If v is on the Eulerian circuit but is not the start/end point:

  Consider each time the Eulerian circuit passes through v. The circuit comes in along one edge, and out along another. This contributes $+2$ to the degree of v. In the example above:

  $$\deg(v) = 2 + 2 + 2 + 2$$

  In general: $\deg(v) = 2 + \ldots + 2 = 2k$ for some $k \in \mathbb{Z}$, which is even.

- If v is the start/end point of the Eulerian circuit:

Consider each time the Eulerian circuit passes through v.
The circuit starts by exiting v, contributing $+1$ to $\deg(v)$.
Then it may pass through v several times, each time entering and exiting v, and each time contributing $+2$ to $\deg(v)$.
Finally, the circuit enters v and stops, contributing $+1$ to $\deg(v)$.
So: $\deg(v) = 1 + 2 + 2 + \ldots + 2 + 1 = 2\ell + 2$, $\ell \in \mathbb{Z}$, which is even.

$\square$

Are these conditions sufficient?

Let G be a connected graph. Is "every vertex has even degree" enough to ensure that there must be a Eulerian circuit?

What do you reckon?

Theorem

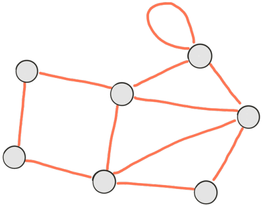
Let G be a connected graph. Then G has a Eulerian circuit if and only if every vertex of G has even degree.

We have just proven the "only if" direction: if G has a Eulerian circuit, then every vertex of G has even degree.

What remains to prove is the "if" direction: if every vertex has even degree, then G has a Eulerian circuit.

Proof for the "if" direction

Let G be a graph with at least one edge, and where every vertex of G has even degree.



We build a circuit in G as follows.

- Choose any vertex v with positive degree. This will be the start of our circuit.
- Continue walking along edges that we have not yet used, until reach v again.

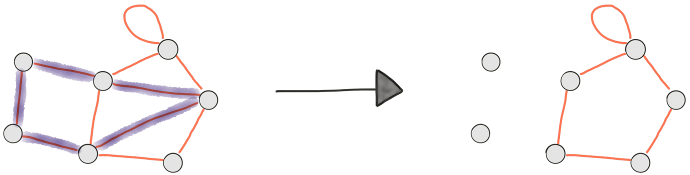
The result is a circuit, starting and ending at v.

How do we know we will reach v again?

- Let's say we are stuck at some vertex w, $v \neq w$.
- We have passed through w a total of $k \geq 0$, and we have arrived at w again.
- $\rightarrow$ we have "used up" $\deg(w) - 2k - 1$ of the edges around w.
- $deg(w)$ is even $\rightarrow \deg(w) - 2k - 1$ is odd $\rightarrow \deg(w) - 2k - 1 > 0$
- $\rightarrow$ There exists at least one edge around w that is still unused!
- We are not stuck, a contradiction

- As long as we do not reach v, we can keep walking
- But: we cannot keep walking forever! ($V(G)$ is finite)
- Eventually we always return to v

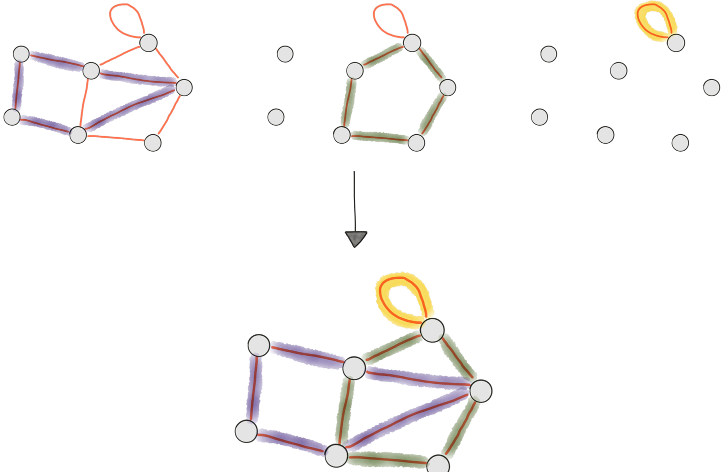
Now we delete the edges of this circuit from G.



Observation: In the resulting graph, every vertex has even degree!
Why? If the circuit passes through a vertex k times, its degree drops by $2k$. The degree was even before, so it is still even now!

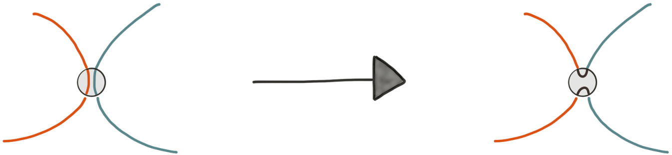
We continue to build circuits using this same procedure, until the resulting graph is empty.

The result: We have decomposed G into a collection of circuits, where every edge of G is used in exactly one of these circuits!



The last step is to combine these smaller circuits into a single Eulerian circuit.

This is simple enough: if two circuits pass through the same vertex v, we can “splice” them together at v.

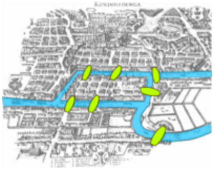


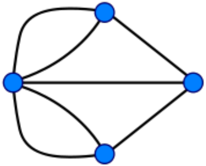
What if no two circuits have a vertex in common?
This would mean the graph is disconnected!

Since our theorem is about connected graphs, we know that we can continue splicing together circuits until they are all combined into a single large Eulerian circuit. □

Return to Königsberg

Can we find a route that crosses each bridge exactly once and takes us back to the starting place?





Yes or no?

No! The vertex degrees are 3, 5, 3 and 3. These are not even!

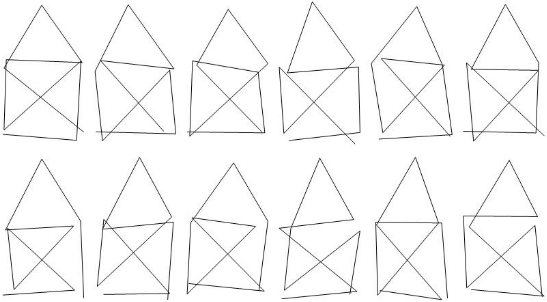
A Eulerian trail is a path using each edge exactly once, but whose start and end vertices can be different.

Questions

Does Königsberg have a Eulerian trail?
Which graphs have Eulerian trails?

An application of Eulerian trails

Do you know this game:



- We are looking for a Eulerian trail.
- Where do you have to start in order to succeed?
- Exercise: how many Eulerian trails are there?