THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Truth Tables – Week 2 Tutorials

MATH1064: Discrete Mathematics for Computing

1. Use De Morgan's laws to find the negation of each of the following propositions.

(a) Kwame will take a job in industry or do a masters.

Solution: Let p be the proposition "Kwame will take a job in industry" and q be the proposition "Kwame will do a masters". Then the proposition is equivalent to $p \vee q$. De Morgan's laws give $\neg(p \vee q) \equiv \neg p \wedge \neg q$. So the negation is equivalent to "Kwame will not take a job in industry and not do a masters."

(b) Yoshiko knows Java and calculus.

Solution: Do this as above!

(c) James is young and strong.

Solution: Do this as above!

(d) Rita will move to Melbourne or Brisbane.

Solution: Do this as above!

2. Draw truth tables for the following propositions.

(a) $\neg(p \vee q)$.

Solution:

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(b) $\neg p \wedge \neg q$.

Solution:

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(c) $p \wedge (p \rightarrow q)$.

Solution:

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

3. Show that the following pairs of propositions are equivalent:

(a) $(p \vee q) \rightarrow r$; $(\neg p \wedge \neg q) \vee r$.

Solution: This can be done with logical eqln -s ../header.tex header.texuivalences or truth tables. Here is a solution showing that the two given propositions have the same truth table. The truth tables for the two given propositions are:

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

p	q	r	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$(\neg p \wedge \neg q) \vee r$
T	T	T	F	F	F	T
T	T	F	F	F	F	F
T	F	T	F	T	F	T
T	F	F	F	T	F	F
F	T	T	T	F	F	T
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	T	T

Since the given propositions have the same truth tables, they are equivalent.

(b) $p \rightarrow \neg q$; $q \rightarrow \neg p$.

Solution: This can be done with logical equivalences or truth tables. Here is a solution showing that the two given propositions have the same truth table.

p	q	$\neg q$	$p \rightarrow \neg q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

p	q	$\neg p$	$q \rightarrow \neg p$
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	T

Since the given propositions have the same truth tables, they are equivalent.

4. Show that

(a) $(p \wedge q) \wedge \neg(p \vee q)$ is a contradiction.

Solution: We have

p	q	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	F	T	F
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	T	F	F

Since every entry in the last column is F, the given proposition is a contradiction.

(b) $((p \vee q) \wedge \neg q) \to p$ is a tautology.

Solution: We have

p	q	$(p \vee q)$	$\neg q$	$((p \vee q) \wedge \neg q)$	$((p \vee q) \wedge \neg q) \to p$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	T

Since every entry in the last column is T, the given proposition is a tautology. Again, this can also be shown using logical equivalences.

5. For each of the following propositions, write down its truth table and determine whether or not it is a tautology or contradiction.

(a) $(p \wedge \neg q) \to \neg(p \to q)$

Solution: Let $s = (p \wedge \neg q) \to \neg(p \to q)$. Then the truth table for the proposition s is

p	q	$\neg q$	$p \wedge \neg q$	$(p \to q)$	$\neg(p \to q)$	s
T	T	F	F	T	F	T
T	F	T	T	F	T	T
F	T	F	F	T	F	T
F	F	T	F	T	F	T

Since every entry in the last column is T, the given proposition is a tautology.

(b) $(p \wedge \neg q) \wedge (\neg p \vee q)$

Solution: We have

p	q	$\neg p$	$\neg q$	$(p \wedge \neg q)$	$(\neg p \vee q)$	$(p \wedge \neg q) \wedge (\neg p \vee \neg q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

Since every entry in the last column is F, the given proposition is a contradiction.

(c) $(p \to q) \to (\neg p \vee q)$,

Solution: The truth table for the given proposition is

p	q	$(p \to q)$	$\neg p$	$(\neg p \vee q)$	$(p \to q) \to (\neg p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since every entry in the last column is T, the proposition is a tautology.

(d) $(p \wedge q) \vee \neg(p \to q)$.

Solution: The truth table for the given proposition is

p	q	$(p \wedge q)$	$(p \to q)$	$\neg(p \to q)$	$(p \wedge q) \vee \neg(p \to q)$
T	T	T	T	F	T
T	F	F	F	T	T
F	T	F	T	F	F
F	F	F	T	F	F

Since not every entry in the last column is true, it is not a tautology; and since not every entry in the last column is false, it is not a contradiction.

6. (*) Dirichlet's approximation theorem states that for any real numbers $\alpha, \beta, \beta \geq 1$, there exist integers p and q such that $1 \leq q \leq \beta$ and

$$|q\alpha - p| \leq \frac{1}{\lfloor \beta \rfloor + 1} < \frac{1}{\beta}$$

($\lfloor \beta \rfloor$ denotes the integer part of β).

Prove Dirichlet's approximation theorem with the help of the pigeon hole principle.

Solution: For simplicity, denote the integer part of β by $b = \lfloor \beta \rfloor$. Moreover, define $0 \leq \alpha_q = \alpha q - \lfloor \alpha q \rfloor < 1$ for all integers $1 \leq q \leq b$.

Now, consider the disjoint real intervals $I_k = \left[\frac{k-1}{b+1}, \frac{k}{b+1}\right)$ for all integers $1 \leq k \leq b+1$. Observe that any real number between 0 and up to, but not including, 1 is contained in one of those intervals. In particular, every α_q is contained in one of these intervals.

Suppose there exists an integer q such that $\alpha_q \in \left[0, \frac{1}{b+1}\right)$, then

$$0 \leq \alpha_q = \alpha q - \lfloor \alpha q \rfloor < \frac{1}{b+1} = \frac{1}{\lfloor \beta \rfloor + 1} < \frac{1}{\beta}$$

and we are done by taking $p = \lfloor \alpha q \rfloor$.

Suppose that this is not the case, but suppose that there exists q such that $\alpha_q \in \left[\frac{b}{b+1}, 1\right)$, then

$$\frac{b}{b+1} \leq \alpha_q = \alpha q - \lfloor \alpha q \rfloor < 1$$

and hence

$$-\frac{1}{b+1} \leq \alpha q - \lfloor \alpha q \rfloor - 1 < 0$$

and after multiplying by (-1) we obtain

$$0 < |\alpha q - (\lfloor \alpha q \rfloor + 1)| \leq \frac{1}{b+1}$$

and again we are done (by taking $p = (\lfloor \alpha q \rfloor + 1)$).

If neither of the above cases holds, then the $b-1$ remaining intervals I_k contain the b values of α_q. Therefore, by the Pigeonhole Principle, there are $k_0 \in \{2,\dots,b\}$ and $q_1 < q_2$ such that $\alpha_{q_1}, \alpha_{q_2} \in I_{k_0}$.
Then for $i \in \{1,2\}$

$$\frac{k_0-1}{b+1} \leq \alpha q_i - \lfloor \alpha q_i \rfloor < \frac{k_0}{b+1}.$$

In particular,

$$|\alpha q_2 - \alpha q_1 - \lfloor \alpha q_2 \rfloor + \lfloor \alpha q_1 \rfloor| < \frac{1}{b+1}$$

and hence

$$\alpha(q_2 - q_1) - (\lfloor \alpha q_2 \rfloor - \lfloor \alpha q_1 \rfloor) < \frac{1}{b+1} < \frac{1}{\beta}$$

and taking $q = (q_2 - q_1)$ and $p = \lfloor \alpha q_2 \rfloor - \lfloor \alpha q_1 \rfloor$ completes the construction.