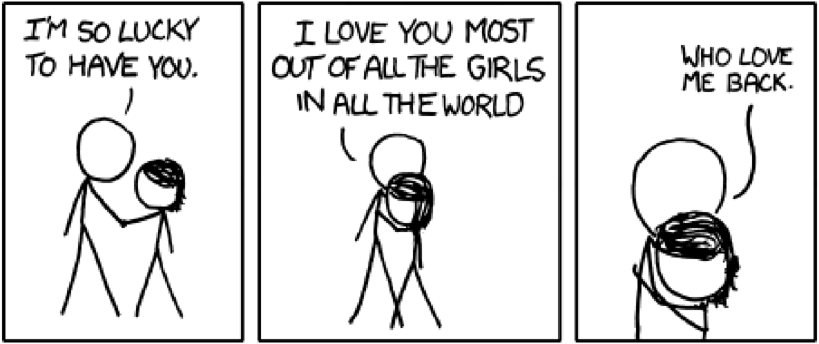
Discrete Mathematics

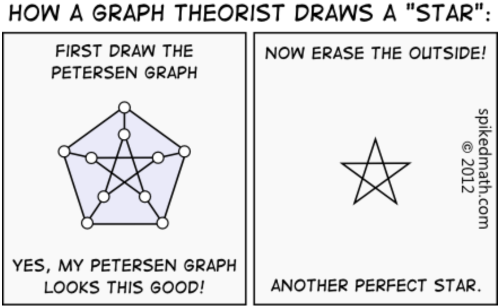
MATH1064, Lecture 34

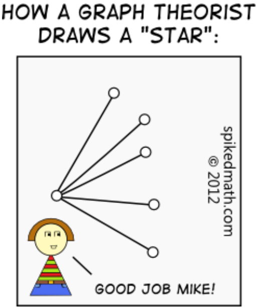
Jonathan Spreer



Extra exercises for Lecture 34

Section 10.2: Problems 21–30

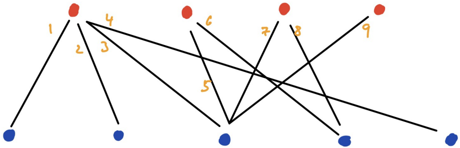




Bipartite graphs

The simple graph G is bipartite if it has at least two vertices and satisfies one (and hence all) of the following equivalent conditions:

1. The set of vertices $V(G)$ has a partition $\{V_1, V_2\}$ such that every edge is of the form $\{v_1, v_2\}$ where $v_k \in V_k$.
2. The vertices can be coloured with two colours such that no two adjacent vertices have the same colour.
3. Every circuit in G has even length.



Is a graph with no edges bipartite?

1 implies 2

- Partition $V(G)$ into $\{V_1, V_2\}$ such that all edges are in $V_1 \times V_2$
- Colour vertices in V_1 red and vertices in V_2 blue
- $\rightarrow$ 2-colouring of vertices such that no two adjacent vertices have the same colour

2 implies 3

- 2-colouring of vertices such that no two adjacent vertices have the same colour
- Along every circuit colours must alternate
- All circuits have even length

3 implies 1

Every circuit in G has even length $\rightarrow$ $V(G)$ has a partition $\{V_1, V_2\}$ such that every edge is of the form $\{v_1, v_2\}$ where $v_k \in V_k$

- Isolated verices and multiple connected components: easy
- Hence, assume G is connected (with at least one edge)
- Fix $v \in V(G)$

$$V_1 = \{w \in V(G) \mid \exists \text{ path of odd length between } v \text{ and } w\}$$

$$V_2 = \{w \in V(G) \mid \exists \text{ path of even length between } v \text{ and } w\}$$

3 implies 1

- $V_2 \neq \emptyset$ because $v \in V_2$
- $V_1 \neq \emptyset$ because G connected and has at least two vertices
- $V_1 \cup V_2 = V(G)$ since G is connected
- $V_1 \cap V_2 = \emptyset$ because $w \in V_1 \cap V_2 \to G$ contains odd length circuit
- Moreover: Edge with both endpoints in V_1 (V_2) implies G contains an odd length circuit

Applications of bipartite graphs

Text analysis: $V_1 =$ documents, $V_2 =$ terms of words, edge $\{v_1, v_2\}$ if word v_2 is in document v_1.

Movie Preferences: In 2009 Netflix gave a 1 Million dollar prize to the group that was best able to predict how much someone would enjoy a movie based on their preferences. $V_1 =$ viewers, $V_2 =$ movies. Edges are weighted by ratings given by viewers. The winning algorithm was BellKor's Pragmatic Chaos.

Timetabelling: $V_1 =$ students, $V_2 =$ units of study.

Matching problems: eg. $V_1 =$ graduating medical students, $V_2 =$ residences at hospitals, and we put an edge between a student and a hospital if and only if the student asks to be at the hospital and the hospital is interested in making the student an offer.

Hall's marriage theorem

Let G be a graph and $v \in V(G)$.
The neighbourhood $N(v)$ is the set of all vertices adjacent to v.
The neighbourhood of $A \subseteq V(G)$ is $N(A) = \bigcup_{v \in A} N(v)$.

A matching in the bipartite graph G is a subset $M \subseteq E(G)$ with the property that no two edges in M share a vertex. A matching is a complete matching from V_1 to V_2 if every vertex in V_1 is incident with an edge in M. Equivalently, if $|M| = |V_1|$.

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Let G be a bipartite graph with partition $\{V_1, V_2\}$ of the vertices. There is a complete matching from V_1 to V_2 if and only if $|A| \leq |N(A)|$ for all $A \subseteq V_1$.

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Let G be a bipartite graph with partition $\{V_1, V_2\}$ of the vertices. There is a complete matching from V_1 to V_2 if and only if $|A| \leq |N(A)|$ for all $A \subseteq V_1$.

Proof: $A \subset V_1$ such that $|N(A)| < |A|$ is called a Hall violater

"Easy direction": Complete matching M from V_1 to $V_2 \rightarrow$ no Hall violater

- Let $A \subset V_1$
- Let $M(A) \subset V_2$ be the set of all vertices in V_2 matched by M to A
- By definition of a matching: $|M(A)| = |A|$
- But $M(A) \subset N(A)$ since all elements of $M(A)$ are neighbours of A
- So: $|N(A)| \geq |M(A)| = |A|$ (and A is not a Hall violator)

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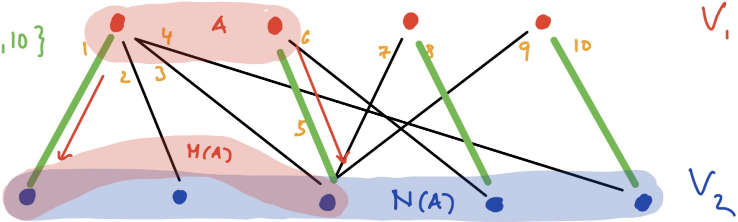
"Easy direction": Complete matching M from V_1 to V_2 $\rightarrow$ no Hall violater

$M = \{1, 5, 8, 10\}$



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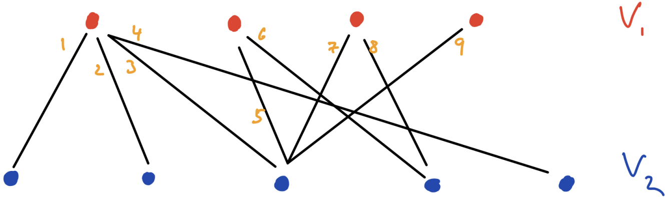
Let G be a bipartite graph with partition $\{V_1, V_2\}$ of the vertices. There is a complete matching from V_1 to V_2 if and only if $|A| \leq |N(A)|$ for all $A \subseteq V_1$.

"Hard direction": There exists a Hall-violator or a complete matching.

1. Start with $M = \emptyset$

$M = \emptyset$



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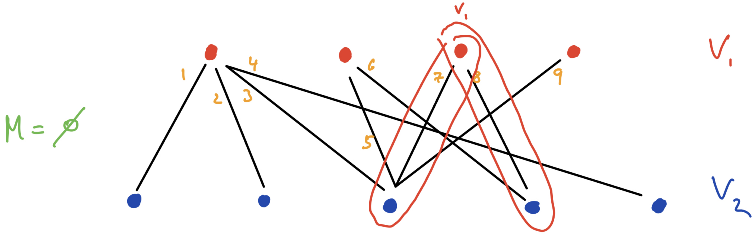
Let G be a bipartite graph with partition $\{V_1, V_2\}$ of the vertices. There is a complete matching from V_1 to V_2 if and only if $|A| \leq |N(A)|$ for all $A \subseteq V_1$.

"Hard direction": There exists a Hall-violator or a complete matching.

2. Pick $v_1 \in V_1 \setminus V(M) \neq \emptyset$ unmatched and build alternating paths from v



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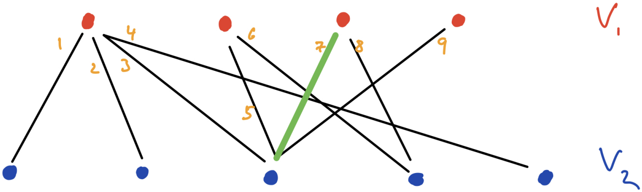
"Hard direction": There exists a Hall-violator or a complete matching.

3. If there exists a maximal alternating path $p = \{n_1, m_1, \ldots, n_{k-1}, m_{k-1}, n_k\}$ (n_i non-matched, m_j matched), then $M := M \setminus \{m_1, \ldots, m_{k-1}\} \cup \{n_1, \ldots, n_k\}$

$M = \{7\}$



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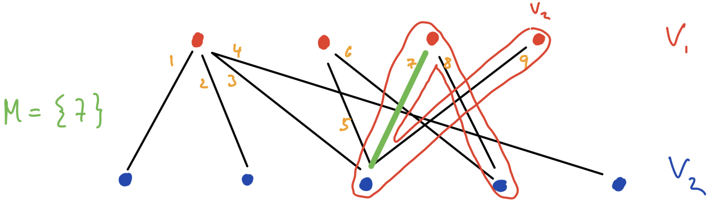
2. Pick $v_2 \in V_1 \setminus V(M) \neq \emptyset$ unmatched and build alternating paths from v



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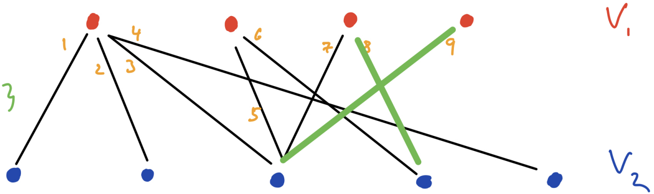
$M = \{8, 9\}$



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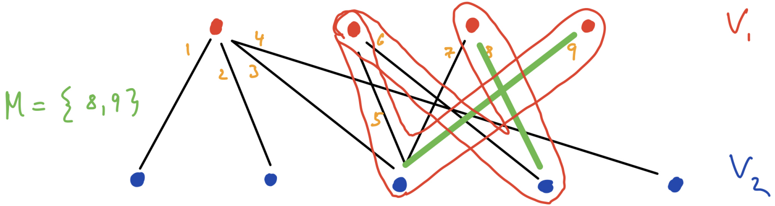

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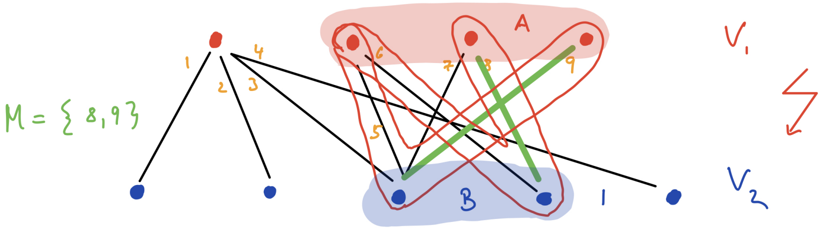
"Hard direction": There exists a Hall-violator or a complete matching.

4. If no maximal alternating path ends in V_2, let $A \subset V_1$ and $B \subset V_2$ be the vertices on these paths



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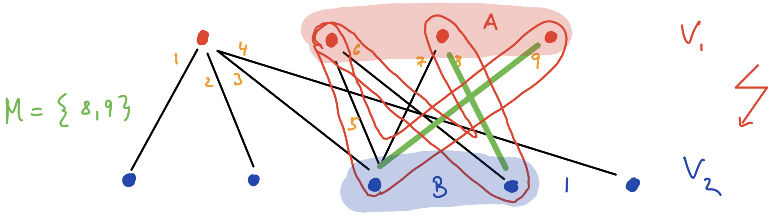
Let G be a bipartite graph with partition $\{V_1, V_2\}$ of the vertices. There is a complete matching from V_1 to V_2 if and only if $|A| \leq |N(A)|$ for all $A \subseteq V_1$.

"Hard direction": There exists a Hall-violator or a complete matching.

5. Paths give bijection between $A \setminus \{v\}$ and B and hence $|B| < |A|$



Hall's marriage theorem

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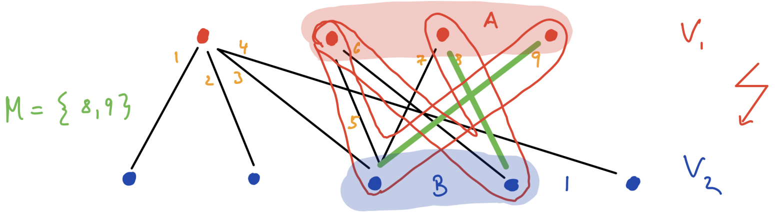
Let G be a bipartite graph with partition $\{V_1, V_2\}$ of the vertices. There is a complete matching from V_1 to V_2 if and only if $|A| \leq |N(A)|$ for all $A \subseteq V_1$.

"Hard direction": There exists a Hall-violator or a complete matching.

6. Since all maximal alternating paths end in V_1 we have $N(A) = B$, hence $|N(A)| < |A|$, return Hall violator A



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Let G be a bipartite graph with partition $\{V_1, V_2\}$ of the vertices. There is a complete matching from V_1 to V_2 if and only if $|A| \leq |N(A)|$ for all $A \subseteq V_1$.

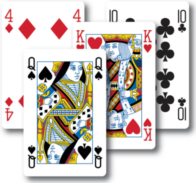
"Hard direction": There exists a Hall-violator or a complete matching.

- M grows by one edge each time until we find a Hall violator
- If no Hall violator exists, we must find a complete matching
- So, altogether: There always exists a Hall-violator $A \subset V_1$, or a complete matching from V_1 to V_2, but not both

Hall's marriage theorem

Let G be a bipartite graph with partition $\{V_1, V_2\}$ of the vertices. There is a complete matching from V_1 to V_2 if and only if $|A| \leq |N(A)|$ for all $A \subseteq V_1$.

Complete matching from V_1 to V_2 is an injection $V_1 \to V_2$



Take a standard deck of 52 cards, and deal them out into 13 piles of 4 cards each.

Question: Can one select exactly one card from each pile such that the 13 selected cards contain exactly one card of each rank?

Either:

$V_1 =$ the 13 piles

$V_2 = \{2, 3, \ldots,$ Queen, King, Ace $\} =$ the ranks

edges $=$ "this pile contains this rank"

or:

$V_1 = \{2, 3, \ldots,$ Queen, King, Ace $\} =$ the ranks

$V_2 =$ the 13 piles

edges $=$ "this rank appears in this pile"

Application

1	2	3	4	5	6	7	8
7	8	5	6	3	4	1	2
4	3	2	1	8	7	6	5
6	5	8	7	2	1	4	3
8	7	6	5	4	3	2	1
2	1	4	3	6	5	8	7
5	6	7	8	1	2	3	4
3	4	1	2	7	8	5	6

You are creating a Latin square of size $n \times n$. You have already written k rows of numbers and so far you have no obvious contradiction: each row contains each symbol exactly once, and each column contains each symbol at most once.

Can this always be extended to a Latin square?

It is enough to show that we can add one more row and then argue by induction!

$V_1 =$ empty cells of the next row
$V_2 = \{1, \dots, n\}$
edges $=$ "this symbol can still appear in this cell"