THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to O-Notation, Induction and Recursion – Week 7 Tutorials

MATH1064: Discrete Mathematics for Computing

1. For each of the following functions $f : \mathbb{N} \to \mathbb{R}$ determine $g(n)$ such that $f(n) \in O(g(n))$. Can you also determine $g(n)$ such that $f(n) \in \Theta(g(n))$?

(a) $f(n) = 3n+7$

Solution: First note that $f(n) > 0$ and $n > 0$ for all $n \in \mathbb{N}$. We therefore don't need to take absolute values of the functions involved.

$f(n) \in \Theta(n)$ follows from $n \leq 3n+7 \leq 4 \cdot n$ for all $n > 6$.

(b) $f(n) = 3 + \sin(1/n)$

Solution: First note that $f(n) > 0$ for all $n \in \mathbb{N}$. We therefore don't need to take absolute values of the functions involved.

$f(n) \in \Theta(1)$ follows from $2 \cdot 1 = 2 \leq 3 + \sin(1/n) \leq 4 = 4 \cdot 1$ for all $n > 0$ since $-1 \leq \sin(1/n) \leq 1$ in this case.

(c) $f(n) = 5n^2 + 3n\log_2(n)$

Solution: First note that $f(n) > 0$ and $n^2 > 0$ for all $n \in \mathbb{N}$. We therefore don't need to take absolute values of the functions involved.

$f(n) \in \Theta(n^2)$ follows by using $1 \leq \log_2(n) \leq n$ for all $n \geq 2$, so

$$5 \cdot n^2 \leq 5n^2 + 3n\log_2(n) \leq 5n^2 + 3n^2 \leq 8 \cdot n^2$$

for all $n > 1$.

(d) $f(n) = \sum_{k=1}^{n} 2k$

Solution: First note that $f(n) > 0$ and $n^2 > 0$ for all $n \in \mathbb{N}$. We therefore don't need to take absolute values of the functions involved.

We have $f(n) = \sum_{k=1}^{n} 2k = 2 \cdot \sum_{k=1}^{n} k = n(n+1) = n^2 + n$, hence $n^2 \leq n^2 + n \leq n^2 + n^2 = 2 \cdot n^2$ for all $n > 0$. Hence $f(n) \in \Theta(n^2)$.

2. Can you find functions f and g such that $f(x) \notin O(g(x))$ and $g(x) \notin O(f(x))$?

Solution: Take your favourite function f. Choose a function g_1 that grows slower than f and a function g_2 that grows faster. Try to combine g_1 and g_2.

Example: $f(n) = n$, define $g(2k) = 1$ and $g(2k+1) = (2k+1)^2$.

3. Prove that $1 + 3 + 5 + \cdots + (2n-1) = n^2$, for all positive integers n.

Solution: $P(1)$ is the proposition $1 = 1^2$, which is clearly true.

Suppose that $P(n)$ is true for some $n \geq 1$, i.e. $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for some

$n \geq 1$. Then

$$\begin{aligned}
&1+3+5+\cdots+(2n-1)+\big(2(n+1)-1\big)\\
&=1+3+5+\cdots+(2n-1)+(2n+1)\\
&=[1+3+5+\cdots+(2n-1)]+(2n+1)\\
&=n^2+(2n+1) \quad \text{(induction hypothesis)}\\
&=n^2+2n+1\\
&=(n+1)^2,
\end{aligned}$$

so that $P(n+1)$ is true.

Hence $P(n)$ is true for all positive integers n.

4. Prove that $2+5+8+\cdots+(3n-1)=\dfrac{n(3n+1)}{2}$, for all positive integers n.

Solution: $P(1)$ is the proposition $2=\frac{1(3\cdot 1+1)}{2}$, which is clearly true.

Suppose that $P(n)$ is true for some $n \geq 1$. That is, suppose that

$$2+5+8+\cdots+(3n-1)=\frac{n(3n+1)}{2}$$

for some $n \geq 1$. Then

$$\begin{aligned}
&2+5+8+\cdots+(3n-1)+\big(3(n+1)-1\big)\\
&=[2+5+8+\cdots+(3n-1)]+(3n+2)\\
&=\frac{n(3n+1)}{2}+(3n+2) \quad \text{(induction hypothesis)}\\
&=\frac{3n^2+7n+4}{2}\\
&=\frac{(n+1)(3n+4)}{2}\\
&=\frac{(n+1)\big(3(n+1)+1\big)}{2},
\end{aligned}$$

which shows that $P(n+1)$ is true.

Hence $P(n)$ is true for all positive integers n.

5. Prove that for any integer $n \geq 1$, $\dfrac{(2n)!}{2^n}$ is an integer.

Solution: For $n=1$, we see that $\dfrac{2!}{2^1}=1$ and so $P(1)$ is true.

Suppose that $P(n)$ is true for some $n \geq 1$; that is, suppose that for some $n \geq 1$:

$$\frac{(2n)!}{2^n}=\ell$$

for some integer $\ell > 0$. Then

$$\begin{aligned}\frac{[2(n+1)]!}{2^{n+1}} &= \frac{2(n+1)(2n+1)(2n)!}{2^{n+1}}\\ &= (n+1)(2n+1)\cdot\frac{(2n)!}{2^n}\\ &= (n+1)(2n+1)\ell \qquad \text{(induction hypothesis),}\end{aligned}$$

and so $P(n+1)$ is true.

Hence $P(n)$ is true for all positive integers n.

6. Prove that 6 divides $n(n^2+5)$ for all positive integers n.

Solution: When $n=1$, $n(n^2+5) = 1(1+5) = 6$, which is divisible by 6. Hence $P(1)$ is true.

Suppose that $P(n)$ is true. That is suppose that $n(n^2+5) = 6\ell$, for some integer ℓ. Then

$$\begin{aligned}(n+1)\big((n+1)^2+5\big) &= (n+1)(n^2+5+2n+1)\\ &= n(n^2+5)+3n^2+3n+6\\ &= 6\ell+3n^2+3n+6 \quad \text{(induction hypothesis)}\\ &= 6\ell+3n(n+1)+6.\end{aligned}$$

For each positive integer n, either n or $n+1$ is even so that each term on the right-hand side (of the last equality) is divisible by 6. Thus $(n+1)\big((n+1)^2+5\big)$ is divisible by 6 and so $P(n+1)$ is true. Hence $P(n)$ is true for all positive integers n.

7. Prove that $5^n - 4n - 1$ is divisible by 16 for all positive integers n.

Solution: When $n=1$, $5^n-4n-1 = 0$, which is clearly divisible by 16. Hence $P(1)$ is true.

Suppose that $P(n)$ is true. That is, suppose that $5^n-4-1 = 16\ell$, for some integer ℓ. Then

$$\begin{aligned}5^{(n+1)}-4(n+1)-1 &= 5(5^n-4n-1)+5(4n)+5-4n-4-1\\ &= 5(16\ell)+4(4n) \quad \text{(induction hypothesis)}\\ &= 16(5\ell+n),\end{aligned}$$

which shows that $5^{(n+1)}-4(n+1)-1$ is divisible by 16, and so $P(n+1)$ is true.

Hence $P(n)$ is true for all positive integers n.

8. Describe each of the following sets as a recursively defined sequence.

(a) The set of all positive even integers, and

Solution: $a_0 = 2, \qquad a_n = a_{n-1}+2$ for all $n \geq 1$

(b) The set of all positive odd integers

Solution: $a_0 = 1, \qquad a_n = a_{n-1} + 2$ for all $n \geq 1$

9. Recall the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Let $\phi = (1+\sqrt{5})/2$.

(a) Prove that $\phi^2 = \phi + 1$.

Solution: $\phi^2 = (1+\sqrt{5})^2/2^2 = (1+2\sqrt{5}+\sqrt{5}^2)/4 = (6+2\sqrt{5})/4 = (3+\sqrt{5})/2 = (1+\sqrt{5})/2+1 = \phi+1$.

(b) Prove that $F_n \geq \phi^{n-2}$ for all $n \geq 2$.

Solution: We use strong induction, with base cases $n = 2$ and $n = 3$:

- $F_2 = 1$ and $\phi^{2-2} = \phi^0 = 1$. Therefore $F_2 \geq \phi^{2-2}$.
- $F_3 = 2$ and $\phi^{3-2} = \phi^1 = (1+\sqrt{5})/2$. We now observe:

  $$F_3 \geq \phi^{3-3} \longleftrightarrow 2 \geq (1+\sqrt{5})/2 \longleftrightarrow 4 \geq (1+\sqrt{5}) \longleftrightarrow 3 \geq \sqrt{5} \longleftrightarrow 9 \geq 5,$$

  which is clearly true.
- Now let $n \geq 4$, and suppose that $F_i \geq \phi^{i-2}$ for all $i = 2, \ldots, n-1$. Then

  $$F_n = F_{n-1} + F_{n-2} \geq \phi^{n-3} + \phi^{n-4},$$

  using the inductive hypothesis and the fact that $n-2 \geq 2$. Therefore:

  $$F_n \geq \phi^{n-3} + \phi^{n-4} = \phi^{n-4} \cdot (\phi+1) = \phi^{n-4} \cdot \phi^2 = \phi^{n-2},$$

  using the result from part (a).

By mathematical induction, it follows that $F_n \geq \phi^{n-2}$ for all $n \geq 2$.

(c) Show that the Fibonacci series has an exponentially fast growth rate.

Solution: This follows from Part (b) since $\phi > 1$. Specifically, $f(n) = F_n$ and $g(n) = \phi^{n-2}$ satisfy $f(n) \geq g(n)$ for all $n \geq 3$, and so $f(n) \in \Omega(g(n))$.

10. Consider the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_1 = 1, a_2 = 2, a_3 = 3$ and

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

for all $n \in \mathbb{N}$ satisfying $n \geq 3$. Use induction on n to show that $a_n \leq 3^n$ for all $n \in \mathbb{N}$.

Solution: Let $P(n)$ be the predicate "$a_n \leq 3^n$". This is defined for all $n \in \mathbb{N}$.

$\boxed{\textit{Base cases}}$

- $P(1): \quad a_1 = 1 \leq 3^1 = 3$ is true.
- $P(2): \quad a_2 = 2 \leq 3^2 = 9$ is true.
- $P(3): \quad a_3 = 3 \leq 3^3 = 27$ is true.

$\boxed{\textit{Inductive Hypothesis}}$ Assume $P(i)$ is true for every integer i such that, $1 \leq i \leq k$, where $k \geq 3$ is some integer. We want to prove that this implies $P(k+1)$ is true.

$\boxed{\textit{Inductive Step}}$ Now

$$
\begin{aligned}
a_{k+1} &= a_k + a_{k-1} + a_{k-2} && \text{(by definition)} \\
&\leq 3^k + 3^{k-1} + 3^{k-2} && \text{(by I.H.)} \\
&\leq 3^k + 3^k + 3^k \\
&= 3 \cdot 3^k \\
&= 3^{k+1}
\end{aligned}
$$

Hence, by (strong) mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$.