THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Relations – Week 11 Tutorials

MATH1064: Discrete Mathematics for Computing

1. Consider the following relations on the set $\{1,2,3,4\}$.

- $R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$
- $R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$
- $R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$
- $R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$
- $R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$
- $R_6 = \{(1,1),(2,2),(3,3),(4,4)\}$.

For each of the relations R_1, R_2, R_3, R_4, R_5 and R_6, determine whether it is reflexive, symmetric, or transitive. If it is an equivalence relation, determine the equivalence classes.

You may find it helpful to draw *arrow diagrams*: draw four points labelled $1,2,3,4$, and draw an arrow from x to y whenever $x R y$.

Solution: The relations R_2 and R_6 are equivalence relations:

- R_2 has equivalence classes $\{1,2\}$, $\{3\}$ and $\{4\}$;
- R_6 is the relation $=$ and has equivalence classes $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$.

The relations R_4 and R_5 are order relations:

- R_4 is $>$ (not reflexive, not symmetric, but transitive);
- R_5 is $\leq$ (reflexive and transitive, not symmetric).

The relation R_1 is not reflexive, not symmetric, not transitive.

The relation R_3 is reflexive and symmetric, but not transitive.

2. Define the relation R on $\mathbb{N} \times \mathbb{N}$ as follows: for all $(a,b) \in \mathbb{N} \times \mathbb{N}$ and all $(c,d) \in \mathbb{N} \times \mathbb{N}$,

$$(a,b)\,R\,(c,d) \quad \text{if and only if} \quad a+d = c+b\,.$$

(a) Give three distinct elements of R.

Solution: Examples: $(\,(1,2),(3,4)\,)$, $(\,(6,2),(8,4)\,)$, $(\,(3,3),(4,4)\,)$, etc.

(b) Is R reflexive?

Solution: Let $(a,b) \in \mathbb{N} \times \mathbb{N}$. Then $a+b = a+b$, and so $(a,b)\,R\,(a,b)$. Hence R is reflexive.

(c) Is R symmetric?

Solution: Suppose $(a,b)\ R\ (c,d)$. Then $a+d=c+b$. Hence $c+b=a+d$ and so $(c,d)\ R\ (a,b)$. Therefore R is symmetric.

(d) Is R transitive?

Solution: Suppose $(a,b)\ R\ (c,d)$ and $(c,d)\ R\ (e,f)$. Hence $a+d=c+b$ and $c+f=e+d$. Solving the second equation for c gives $c=e+d-f$. Substituting this into the first gives $a+d=(e+d-f)+b$. Hence $a+f=e+b$. Hence $(a,b)\ R\ (e,f)$ and the relation is transitive.

(e) If R is an equivalence relation, determine a representative for each equivalence class.

Solution: What are the equivalence classes? Suppose $(a,b)\in\mathbb{N}\times\mathbb{N}$ with $a>b$. Then $(a,b)R(a,b)$ gives $a+b=a+b$, and hence $(a-b)+b=a+0$, where $a-b\in\mathbb{N}$. Hence letting $n=a-b$ we have $(n,0)R(a,b)$. We choose $(n,0)$ as a representative for the equivalence class of (a,b) and now determine the whole equivalence class. Note that

$$(a,b)\ R\ (c,d) \quad \text{if and only if} \quad a+d=c+b \text{ if and only if } \quad a-b=c-d$$

and so the equivalence class is precisely $[(n,0)]=\{(c,d)\in\mathbb{N}\times\mathbb{N}\mid c-d=n\}$.

Similarly, if $(a,b)\in\mathbb{N}\times\mathbb{N}$ with $a<b$, the equivalence class has representative $(0,n)$ and is given by $[(0,n)]=\{(c,d)\in\mathbb{N}\times\mathbb{N}\mid c-d=-n\}$.

Last, if $(a,b)\in\mathbb{N}\times\mathbb{N}$ with $a=b$, then the equivalence class has representative $(0,0)$ and is given by $[(0,0)]=\{(c,d)\in\mathbb{N}\times\mathbb{N}\mid c-d=0\}$.

From the description of the equivalence classes, one can see that this equivalence relation gives a way of defining the integers $\mathbb{Z}$ using the natural numbers $\mathbb{N}$ by mapping $[(a,b)]\mapsto a-b$. In particular, using the representatives, we have $[(0,0)]\mapsto 0$, $[(n,0)]\mapsto n$ and $[(0,n)]\mapsto -n$.

3. Let R be the relation on $\mathbb{Z}$ defined by $x\ R\ y \iff x^2-y^2\equiv 0 \pmod 2$.

Show that R is an equivalence relation.

Solution: We need to show that the relation is reflexive, symmetric and transitive.

Let $x\in\mathbb{Z}$. Then $x^2-x^2=0\equiv 0 \pmod 2$, and so xRx. Hence R is reflexive.

Suppose xRy. Then $x^2-y^2\equiv 0 \pmod 2$. Multiplying by -1 gives $y^2-x^2\equiv 0 \pmod 2$, and so yRx. Hence R is symmetric.

Suppose xRy and yRz. Then $x^2-y^2\equiv 0 \pmod 2$ and $y^2-z^2\equiv 0 \pmod 2$. Using modular arithmetic, this gives $x^2-z^2\equiv (x^2-y^2)-(y^2-z^2)\equiv 0-0\equiv 0 \pmod 2$. Hence xRz and the relation is transitive.

In summary, since the relation is reflexive, symmetric and transitive, it is an equivalence relation.

4. Let R be the relation on $\mathbb{Z}$ defined by

$$x \, R \, y \iff x^3 - 1 < y^3.$$

(a) Is R reflexive?

Solution: Yes, since $x^3 - 1 < x^3$ is equivalent to $-1 < 0$.

(b) Is R symmetric or antisymmetric?

Solution: Note that $(0,1) \in R$, but $(1,0) \notin R$, so the relation is not symmetric. Now suppose xRy and yRx. Then $x^3 - 1 < y^3$ and $y^3 - 1 < x^3$. This gives $x^3 - y^3 < 1$ and $-1 < x^3 - y^3$. Hence $-1 < x^3 - y^3 < 1$. But $x^3 - y^3$ is an integer, and so $x^3 - y^3 = 0$. This implies $x^3 = y^3$ and hence $x = y$. So R is antisymmetric.

(c) Is R transitive?

Solution: Suppose xRy and yRz. Then $x^3 - 1 < y^3$ and $y^3 - 1 < z^3$. Hence $z^3 > y^3 - 1 > x^3 - 2$. Now we again remember that we are working with integers. Since $y^3 - 1$ is an integer lying strictly between the two integers z^3 and $x^3 - 2$, we must have $z^3 > x^3 - 1$. (If you don't believe this, prove it by contradition!) Hence $x^3 - 1 < z^3$ and so xRz. It follows that the relation is transitive.

(d) For any two integers x and y, show that we either have xRy or yRx.

Solution: Let x and y be two arbitrary integers. Without loss of generality, we may assume that $y \geq x$. This implies $y^3 \geq x^3 > x^3 - 1$. Hence xRy.

5. Let X be a set with exactly seven elements. How many equivalence relations with precisely four equivalence classes are there on X?

Solution: There are four equivalence classes and each contains at least one element. The partition sizes for the equivalence classes are 1, 1, 1, 4 with 35 options; 1, 1, 2, 3 with 210 options and 1, 2, 2, 2 with 105 options. Hence a total of 350 such equivalence relations!