Discrete Mathematics

MATH1064, Lecture 7

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You guys are both my witnesses... He insinuated that ZFC set theory is superior to Type Theory!

Set theory

Developed by Georg Cantor (1874)

Together with logic, set theory provides the fundamental "atoms" from which all of mathematics is built

A set S is a collection of objects, which are called the elements of S

If x is in S, we write $x \in S$
If not, we write $x \notin S$

We can list the elements of S with curly braces: $S = \{x_1, x_2, \ldots\}$

Example: $S = \{2, 3, 4, 5\}$
$3 \in S$, but $1 \notin S$
This is a finite set

Example: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
$2 \in \mathbb{N}$, but $-17 \notin \mathbb{N}$ and $\pi \notin \mathbb{N}$
This is an infinite set

When are two sets equal?

Two sets are equal if they contain the same elements:

$$S = T \text{ means } \forall x,\ x \in S \leftrightarrow x \in T$$

Order does not matter:

$$\{2, 3, 5, 8\} = \{3, 8, 5, 2\}$$

Repetition is ignored:

$$\{1, 3, 4\} = \{1, 1, 3, 4\} = \{4, 3, 1, 4, 1, 3, 4, 4, 3, 1\}$$

What matters is whether one set contains some element that the other does not:

- $\{1, 3, 4\} \neq \{1, 4, 4\}$, since 3 is in the first set but not the second
- $\mathbb{N} \neq \mathbb{Z}$, since $-1 \in \mathbb{Z}$ but $-1 \notin \mathbb{N}$

The empty set, written $\emptyset$, contains no elements at all:

$$\emptyset = \{\}$$

Formally, when we say that $\emptyset$ is the empty set, we mean:

$$\forall x,\ x \notin \emptyset$$

A one-element set $\{x\}$ is not the same as x: $\quad 2 \neq \{2\}$

Question time!

- Let $A = \{1, 2, 3\}$, $B = \{3, 2, 1\}$, and $C = \{2, \{1\}, 3\}$
  Does $A = B$?
  Does $B = C$?
- Does $\{1, 1, 2\} = \{1, 2\}$?
- Does $1 \in \{1\}$?
- Does $1 \in \{\{1\}\}$?

Defining sets using predicates

You can define a set by a property that its elements must satisfy:

$A = \{x \in S \mid P(x)\}$ means that the elements of A are precisely those elements $x \in S$ for which the predicate $P(x)$ is true.

Example: The set of all even numbers is

$$\{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z} \text{ such that } n = 2k\}$$

Example: $\{x \in \mathbb{N} \mid 3 < x < 7\} =$

Example: $\{k \in \mathbb{Z} \mid k \geq 0\} =$

Verbally: $\{x \in S \mid P(x)\}$ is

Defining sets using predicates

Sets can contain other sets: $\{1, \{2,3\}, \{1, \{\{4\}\}\}\}$

What about:

$$S = \{x \mid x \notin x\}$$

Is $S \in S$?

- If $S \in S$, then (by definition of S) we have $S \notin S$.
- If $S \notin S$, then (by definition of S) we have $S \in S$.

This is Russell's paradox (Bertrand Russell, 1901)

What went wrong?

We are arguing about sets without having defined what a set is!

But where to start?

Recall the empty set $\emptyset$, which has no elements at all:

$$\forall x,\ x \notin \emptyset$$

Observation

There is only one empty set.
That is, if both S and T are empty sets, then $S = T$.

Proof: By the definition of an empty set, $x \in S$ is always false (i.e., a contradiction). Likewise, $x \in T$ is always false.

Therefore $\forall x,\ x \in S \leftrightarrow x \in T$, and so $S = T$. $\square$

So... we can use "the" empty set $\emptyset$ as a starting point.

From here we can build other sets:

$\{\emptyset\} = \{\{\}\}$
$\{\emptyset, \{\emptyset\}\} = \{\{\},\ \{\{\}\}\}$
$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
...and so on.

We can, if we like, give these sets names:

$0 = \emptyset$
$1 = \{\emptyset\} = \{0\}$
$2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$
$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}$
...

In this way we can, beginning with just the empty set, construct the entire natural numbers!

To avoid problems such as Russell's paradox, we can attempt to define all sets recursively:

- Base: $\emptyset$ is a set
- Recursion: We define several operations that build new sets from old

This is very delicate, and is another topic for another course.
However, we will see some of these operations in a moment.

If you are interested in formalising set theory, read about ZFC: the Zermelo-Fraenkel axioms plus the axiom of choice, which rigorously pin down what set theory does and does not allow.

Union

For sets S and T, their union is written $S \cup T$.

It contains all elements that belong to S or T (possibly both):

$$\forall x,\ x \in S \cup T \leftrightarrow (x \in S \vee x \in T)$$

Example:

$$\{1, 2, 3\} \cup \{2, 5\} = \{1, 2, 3, 5\}$$

Example:

$$\{0, 1, 2, 3, \ldots\} \cup \{0, -1, -2, -3, \ldots\} = \mathbb{Z}$$

Intersection

For sets S and T, their intersection is written $S \cap T$.

It contains all elements that belong to both S and T:

$$S \cap T = \{x \mid x \in S \wedge x \in T\}$$

Example:

$$\{1, 2, 3\} \cap \{2, 5\} = \{2\}$$

Example:

$$\{0, 1, 2, 3, \ldots\} \cap \{0, -1, -2, -3, \ldots\} = \{0\}$$

Example:

$$\{1, 2, 3, \ldots\} \cap \{-1, -2, -3, \ldots\} = \emptyset \quad \text{(the empty set)}$$

You can take the union or intersection of many sets at once:

$$\bigcup_{i=1}^{5} \{i, 2i\} =$$

$$\bigcap_{i=1}^{3} \{i, i+1, i+2\} =$$

You can even use infinitely many sets:

$$\bigcup_{i=0}^{\infty} \{i, -i\} = \{0\} \cup \{1, -1\} \cup \{2, -2\} \cup \ldots =$$

How do we formally define this infinite union?

$$\bigcup_{i=0}^{\infty} A_i =$$

$[a, b]$ denotes the set of all real numbers x for which $a \leq x \leq b$:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Example: $[0, 1]$ contains all real numbers between 0 and 1 inclusive.

An example of an infinite intersection:

$$\bigcap_{i=1}^{\infty} [-i, i] = [-1, 1] \cap [-2, 2] \cap [-3, 3] \cap \ldots = [-1, 1]$$

Formally:

$$\bigcap_{i=1}^{\infty} A_i = \{x \mid \forall i \geq 1,\ x \in A_i\}$$

Example:

$$\bigcap_{n=1}^{\infty} [0, \tfrac{1}{n}] = \{0\}$$

Why? Only $x = 0$ satisfies $\forall n \in \mathbb{N}$, $0 \leq x \leq \frac{1}{n}$.

Subsets

For sets S and T, we say that S is a subset of T if every element of S belongs to T also.

We write this as $S \subseteq T$. Formally:

$$S \subseteq T \text{ means } \forall x,\ x \in S \rightarrow x \in T$$

Compare this to set equality:

$$S = T \text{ means } \forall x,\ x \in S \leftrightarrow x \in T$$

Example:

$$\{2, 3, 5, 7\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$$

S is a proper subset of T if $S \subseteq T$ and $S \neq T$.
For proper subsets we use notation $S \subset T$

Questions!

$$S \subseteq T \text{ means } \forall x,\ x \in S \rightarrow x \in T$$

True or false?

Let $A = \{a, b, c, d\}$, $B = \{a, b, e\}$ and $C = \{a, b, c, d, e\}$.

- Is $B \subseteq A$?
- Is $A \subseteq C$?
- Is $B \subseteq B$?
- Is $(A \cup B) \subseteq C$?
- Is $C \subseteq (A \cup B)$?
- Is $C \subseteq (A \cap B)$?

Proving subset relationships

To prove $S \subseteq T$, we need to show that $\forall x,\ x \in S \rightarrow x \in T$.

Method: Choose an arbitrary $x \in S$, and prove that $x \in T$ also.

Example

Let $S = \{2^n \mid n \in \mathbb{N}, n > 0\}$ and $T = \{2m \mid m \in \mathbb{N}\}$.
Show that $S \subseteq T$.

Proof: Choose any $x \in S$.
Then there exists $n > 0$ such that $x = 2^n$.
Hence $x = 2 \cdot 2^{n-1}$, where $n - 1 \in \mathbb{N}$.
Hence $2^{n-1} \in \mathbb{N}$, and so $x = 2 \cdot 2^{n-1} \in T$.
Since $x \in S$ was arbitrary, we have $S \subseteq T$.

Is this a proper subset? Yes! For example, $6 \in T$ but $6 \notin S$.

Proving equality relationships

$$S \subseteq T \quad \text{means} \quad \forall x,\ x \in S \to x \in T$$
$$S = T \quad \text{means} \quad \forall x,\ x \in S \leftrightarrow x \in T$$

Since $p \leftrightarrow q$ is equivalent to $(p \to q) \wedge (q \to p)$,
it follows that $S = T$ is equivalent to $(S \subseteq T) \wedge (T \subseteq S)$!

Method to prove that $S = T$:

- Choose an arbitrary $x \in S$, and prove that $x \in T$ also;
- Choose an arbitrary $x \in T$, and prove that $x \in S$ also.

Exercise

Let $A = \{a, d\}$, $B = \{b, c\}$ and $C = \{a, c\}$.

What is $A \cap (B \cup C)$?

1. $\emptyset$
2. $\{a\}$
3. $\{a, b, c\}$
4. $\{a, c, d\}$
5. $\{a, d\}$
6. $\{b, c\}$