Discrete Mathematics

MATH1064, Lecture 11

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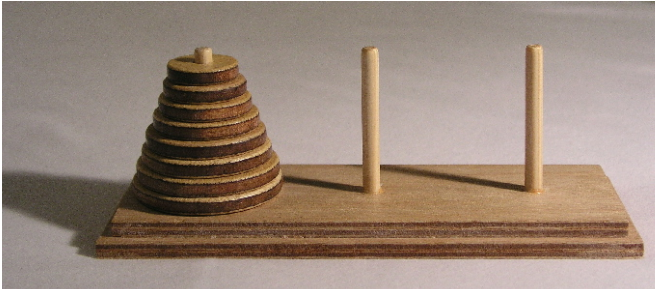
The Tower of Hanoi



The Tower of Hanoi

Given: A tower of 8 discs in decreasing size on one of three pegs.
Problem: Transfer the entire tower to one of the other pegs.
Rule 1: Move only one disc at a time.
Rule 2: Never move a larger disc onto a smaller disc.

Generalising to n discs

Let T_n =minimal number of moves. From week two:

$$T_0 = 0,\ T_1 = 1,\ T_2 = 3,\ T_3 = 7,\ T_4 = 15,\ T_5 = 31,\ T_6 = 63,\ \ldots$$

This is a sequence: $T_0, T_1, T_2, \ldots$

We write this sequence as $(T_n)_{n\geq 0}$, or $(T_n)_{n=0}^{\infty}$, or $\{T_n\}$.

Each number T_i in the sequence is called a term.

- We conjectured that:

$$T_0 = 0 \quad \text{and} \quad T_n = 2T_{n-1} + 1 \text{ for } n \geq 1$$

  This is a recursive definition for (T_n).

- We also conjectured:

$$T_n = 2^n - 1$$

  This is an explicit formula, or closed formula, for T_n.

What is the next term in this sequence?

$$0,\ 1,\ 3,\ 7,\ 15,\ 31,\ 63,\ \ldots$$

Working out the first few terms is good for spotting patterns... but this is not rigorous.

To prove things rigorously, you need something more precise, like:

- $T_n =$ the minimal number of moves for n discs; or
- $T_0 = 0$ and $T_n = 2T_{n-1} + 1$ for $n \geq 1$; or
- $T_n = 2^n - 1$.

Converting between these different descriptions is sometimes very difficult!

Types of sequences

Sequences can be finite:

> 5, 5, 20, 5, 5, 60 can be written as
> $(a_n)_{n=1}^6$, where $a_1 = 5$, $a_2 = 5$, $\ldots$, $a_6 = 60$

or infinite:

> 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, $\ldots$ can be written as
> $(g_n)_{n\geq 0}$, where $g_0 = 0$, $g_1 = 1$, $g_2 = 3$, $\ldots$

The index (the subscript) does not need to start at 0, and does not need to be called n:

> 1, 5, 15, 35, 70, 126, 210, $\ldots$ can be written as
> $(f_i)_{i\geq 4}$, where $f_4 = 1$, $f_5 = 5$, $f_6 = 15$, and so on

An alternating sequence alternates between positive and negative:

$$\left((-\tfrac{1}{2})^n\right)_{n\geq 0} = 1,\ -\tfrac{1}{2},\ \tfrac{1}{4},\ -\tfrac{1}{8},\ \tfrac{1}{16},\ -\tfrac{1}{32},\ \ldots$$

A famous sequence

The Fibonacci sequence:

$$(F_n)_{n \geq 0} = 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots$$

is defined recursively as:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$

This turns up in all sorts of unexpected fields of study!

To define a sequence recursively, we need:

- Initial conditions, which directly specify one or more terms that begin the sequence:
  $$F_0 = 0, \quad F_1 = 1$$
- A recurrence relation, which defines every other term using earlier terms:
  $$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$

Another famous sequence

The factorials:

$$(n!)_{n\geq 0} = 1,\ 1,\ 2,\ 6,\ 24,\ 120,\ \ldots$$

defined recursively as:

$$0! = 1, \quad n! = (n-1)! \cdot n \text{ for } n \geq 1$$

This means that:

$$\begin{aligned}
0! &= 1 \\
1! &= 1 \cdot 1 = 1 \\
2! &= 1 \cdot 2 = 2 \\
3! &= 1 \cdot 2 \cdot 3 = 6 \\
4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24 \\
n! &= n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1
\end{aligned}$$

Factorials appear in many types of problems,
from counting to statistics to analysis of algorithms.

Sequences can be silly

$$1, 11, 21, 1211, 111221, 312211, \ldots$$

Even more famous sequences

You can find many, many more interesting sequences online!

See the Online Encyclopaedia of Integer Sequences at `oeis.org`.

Notation for sums

For a sequence (a_i), we can add some or all of its terms:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n$$

For instance:

$$\sum_{i=3}^{6} i! = 3! + 4! + 5! + 6! = 6 + 24 + 120 + 720 = 870$$

If $m = n$, then the sum has only one term:

$$\sum_{i=5}^{5} i^2 = 5^2 = 25$$

If $m > n$ then the sum is empty, and we define it to be zero:

$$\sum_{i=6}^{5} i! = 0$$

Remember Gauss?

We thought about $1 + 2 + \ldots + n$, and saw:

$$\begin{array}{ccccccccccc} & & 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 & & \\ \hline n & + & n & + & n & + & \cdots & + & n & + & n \end{array}$$

In other words, we considered the sequence $(g_n)_{n \geq 1}$ where

$$g_n = \sum_{i=1}^{n} i,$$

and obtained an explicit formula:

$$g_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Dummy variables

The i in $\sum_{i=m}^{n}$ is a dummy variable, like the x in $\forall x$.

You can use any letter here (as long as it is not already taken):

$$\sum_{i=1}^{n} i = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

The dummy variable is only relevant inside the sum, which means you can reuse it outside the sum:

$$\sum_{i=1}^{3} i + \sum_{i=1}^{4} i^2 = 1 + 2 + 3 + 1 + 4 + 9 + 16$$

You can also perform a change of variable. If $k = i + 1$, then:

$$3! + 4! + 5! = \sum_{i=3}^{5} i! = \sum_{k=4}^{6} (k-1)! = \sum_{i=4}^{6} (i-1)!$$

Question

The sum $\displaystyle\sum_{k=1}^{6} \frac{(-1)^{k+1}}{k^2}$ is:

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$
2. $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6}$
3. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$
4. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$
5. $-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36}$
6. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$

Arithmetic with finite sums

Infinite sums are dangerous—what seems obvious is not always correct.
You can find closed formulae for some finite and infinite sums on page 166.

There are many ways to manipulate finite sums.

Adding / subtracting over the same range:

$$\sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i = \sum_{i=m}^{n} (a_i \pm b_i)$$

For example:

$$\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2 = (1 + \ldots + n) - (1^2 + \ldots + n^2)$$

$$= (1 - 1^2) + \ldots + (n - n^2) = \sum_{i=1}^{n} (i - i^2)$$

Taking out a common factor:

$$\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$$

For example:

$$\sum_{i=1}^{n} \frac{1}{2} \cdot i! = \frac{1}{2} \cdot 1! + \ldots + \frac{1}{2} \cdot n! = \frac{1}{2}(1! + \ldots + n!) = \frac{1}{2} \sum_{i=1}^{n} i!$$

Combining consecutive indices:

$$\sum_{i=p}^{q} a_i + \sum_{i=q+1}^{r} a_i = \sum_{i=p}^{r} a_i \quad \text{if } p \leq q \leq r$$

For example:

$$\sum_{i=1}^{3} \frac{1}{i} + \sum_{i=4}^{6} \frac{1}{i} = (\frac{1}{1} + \frac{1}{2} + \frac{1}{3}) + (\frac{1}{4} + \frac{1}{5} + \frac{1}{6}) = \sum_{i=1}^{6} \frac{1}{i}$$

Index shift:

$$\sum_{i=m}^{n} a_i = \sum_{i=m+p}^{n+p} a_{i-p} = \sum_{i=m-q}^{n-q} a_{i+q}$$

For example:

$$\sum_{i=1}^{n} 2^{i-1} = 2^0 + 2^1 + \ldots + 2^{n-1} = \sum_{i=0}^{n-1} 2^i$$

Telescoping sums:

$$\sum_{i=m}^{n} (a_i - a_{i+1}) = a_m - a_{n+1} \text{ if } m \leq n$$

For example:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$