MATH 1002 LECTURE 4-C

PLANES IN $\mathbb{R}^3$: COMPARING NORMAL & VECTOR FORM.

Recall: Lecture 4-A: A plane $\mathcal{P} \subset \mathbb{R}^3$ is determined by

(1) a point $P \in \mathcal{P}$ and a normal vector $\vec{n}$

↳ normal form $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

Lecture 4-B:

(2) 3 non-collinear points $P, Q, R \in \mathcal{P}$

(3) a point $P \in \mathcal{P}$ and two non-parallel direction vectors $\vec{u}, \vec{v}$

↳ vector form $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$, $s, t \in \mathbb{R}$

So far we know:

- normal form is great for determining if $X \in \mathcal{P}$.
  (plug $\vec{x}$ into the equation and see if it is satisfied)
- vector form is great for producing examples of $X \in \mathcal{P}$
  (plug in different values of $s, t \in \mathbb{R}$)
- we can translate from data (2) to data (3).

Question: How can we translate between

→ normal form & vector form

→ data (1) & data (2)/(3)

Answer: Yes

← is straightforward

→ is okay too.

⟵ Start with data (3) + vector form

- $P \in \mathcal{P}$, $\vec{u}, \vec{v}$ non-parallel direction vectors

To find the normal form, we need data (1)

- $P \in \mathcal{P}$ (have this already!)
- $\vec{n}$ orthogonal to $\mathcal{P}$

$\Leftrightarrow$ orthogonal to $\vec{u}$ & $\vec{v}$.

Exercise: How can we find a vector $\vec{n}$ orthogonal to both $\vec{u}$ & $\vec{v}$?

Answer: Let's take the cross product!

Example: Recall from last time the plane $\mathcal{P}$ passing through the points $P = (2,1,0)$, $Q = (-1,1,2)$, $R = (4,0,2)$.

Find a normal form for $\mathcal{P}$.

Solution: We can take P as our point and

$$\vec{u} = \vec{PQ} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{v} = \vec{PR} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$ as

our direction vectors.

So we should take

$$\vec{n} = \vec{u} \times \vec{v} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 3 \end{bmatrix}$$

$\therefore$ Normal form for $\mathcal{P}$ is $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$.

Here $\vec{n} \cdot \vec{p} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 10 \\ 3 \end{bmatrix} = 14$, so the normal form is

$\vec{n} \cdot \vec{x} = k.$

" $\rightsquigarrow$ " Start with data (1). Produce data (2) or (3).

Given $P \in \mathcal{P}$ and $\vec{n}$, we need to produce Q, R or $\vec{u}, \vec{v}$.

<u>Normal form</u> $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

<u>General form</u>: $ax + by + cz = d$ $\qquad \vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$

- Assume $c \neq 0$ (if it is 0, isolate for x or y instead of z)

Let $x = s \in \mathbb{R}$

and let $y = t \in \mathbb{R}$.

so $z = \frac{1}{c}(d - as - by)$

i.e. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d/c \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -a/c \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -b/c \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

$\hookrightarrow$ this gives a vector form for $\mathcal{P}$.

<u>Exercise</u>: So what can we take for direction vectors $\vec{u}$ & $\vec{v}$?

<u>Solution</u> $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -a/c \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -b/c \end{bmatrix}.$

<u>Exercise</u>: Consider the plane $\mathcal{P}$ passing through $P = (0,0,0)$ with normal vector $\vec{e}_1$.

Find a vector form and direction vectors.

Solution: we have $\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ so $b = c = 0$ and we'll have to isolate x later.

normal form: $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

general form: $x = 0$.

parametric equations: $y = s, \quad s \in \mathbb{R}$

$z = t, \quad t \in \mathbb{R}.$

$x = 0$ from above.

vector form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$

direction vectors.

Summary of the lecture

- If a plane has non-parallel direction vectors $\vec{u}, \vec{v}$, then it has normal vector $\vec{n} = \vec{u} \times \vec{v}$.

- If a plane has normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and general form

  $$ax + by + cz = d,$$

  then we can find parametric equations by letting two of the variables be s, t and expressing the third in terms of these

  coefficient of this one must not be 0.

You should be able to:

Use these two methods to translate between normal form and vector form for a plane in $\mathbb{R}^3$.