Discrete Mathematics

MATH1064, Lecture 10

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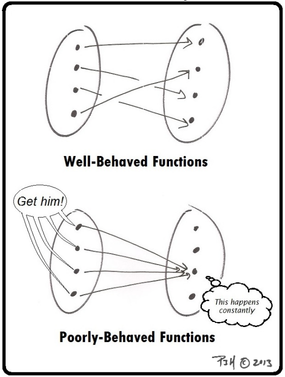
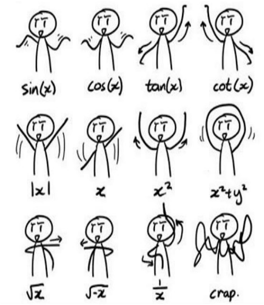


Extra exercises for Lecture 10

Section 2.2: Problems 50–53

Section 2.3: Problems 1–6, 8–10, 14, 26



If $f\colon X \to Y$ is a function, then:

1. X is called the domain of f.
2. Y is called the co-domain of f.
3. If $x \in X$, then $f(x)$ is called the image of x.
4. If $A \subseteq X$, then

   $$f(A) = \{f(x) \mid x \in A\} = \{y \mid \exists x \in A : f(x) = y\} \subseteq Y$$

   is called the image of A.
   The entire set $f(X)$ is called the range of f.
5. If $y \in Y$, then $f^{-1}(y) = \{x \in X \mid f(x) = y\} \subseteq X$
   is called the preimage of y.
6. If $B \subseteq Y$, then $f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X$
   is called the preimage of B.

Don't confuse f^{-1} with $\frac{1}{f}$!

Equality of functions

Functions $f, g\colon X \to Y$ are equal, written $f = g$, if and only if

$$f(x) = g(x) \qquad \text{for all } x \in X.$$

Example: Are the functions $f, g\colon \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n!$ and $g(n) = n \times (n-1)!$ equal?

Note: f denotes a function and $f(x)$ denotes an element of Y.

Example: sin is a function; $\sin(x)$ is just some number.

Floor and ceiling

Definition

Let $x \in \mathbb{R}$ be a real number. The floor of x, denoted $\lfloor x \rfloor$, is the unique integer n such that $n \leq x < n+1$.

If $x = \pi$, then $\lfloor x \rfloor =$

If $x = -0.9999$, then $\lfloor x \rfloor =$

If $x = e$, then $\lfloor x \rfloor =$

If $x = 5$, then $\lfloor x \rfloor =$

Definition

Let $x \in \mathbb{R}$ be a real number. The ceiling of x, denoted $\lceil x \rceil$, is the unique integer n such that $n - 1 < x \leq n$.

If $x = \pi$, then $\lceil x \rceil =$

If $x = -0.9999$, then $\lceil x \rceil =$

If $x = e$, then $\lceil x \rceil =$

If $x = 5$, then $\lceil x \rceil =$

Question

Definition

Let $x \in \mathbb{R}$ be a real number. The floor of x, denoted $\lfloor x \rfloor$, is the unique integer n such that $n \leq x < n + 1$.

True or false?

$\forall x \in \mathbb{R} : \lfloor x - 1 \rfloor = \lfloor x \rfloor - 1.$

True! In fact a more general statement is also true:

Lemma

For all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$, we have $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

Question

True or false?

$\forall x, y \in \mathbb{R} : \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor.$

Properties of functions

Let $f\colon X \to Y$. Then:

1. f is onto, or surjective, or a surjection, if:
   $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.
   Every element of Y is the image of something.
2. f is one-to-one, or injective, or an injection, if:
   $\forall x_1, x_2 \in X,\ f(x_1) = f(x_2) \to x_1 = x_2$.
   Different elements of X have different images.
3. f is a one-to-one correspondence, or bijective, or a bijection, if f is both one-to-one and onto.
   The elements of X are "paired off" with the elements of Y.

What is this function?

(1) one-to-one, (2) onto, (3) bijection, (4) none:

- $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 2n + 3$
- $g\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $g(\ (n, m)\ ) = n + m$
- $h\colon \mathbb{Z} \to \mathbb{Z}$ defined by $h(n) = n + 5$

- $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 2n + 3$

- $g\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $g(\ (n, m)\ ) = n + m$

- $h\colon \mathbb{Z} \to \mathbb{Z}$ defined by $h(n) = n + 5$

Proving that a function is onto

Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = \lfloor n/2 \rfloor$.

To show that f is onto, we need to show:

$$\forall y \in \mathbb{Z},\ \exists x \in \mathbb{Z} \text{ such that } f(x) = y.$$

Proof: Consider any $y \in \mathbb{Z}$. Let $x = 2y$.
Then:

- $x \in \mathbb{Z}$; that is, x belongs to the domain;
- $\lfloor x/2 \rfloor = \lfloor 2y/2 \rfloor = \lfloor y \rfloor = y$, and so $f(x) = y$.

Therefore f is onto. $\square$

Proving that a function is one-to-one

Let $f\colon \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$.

Is f one-to-one? Yes!

We need to show: $\forall x_1, x_2 \in X,\ f(x_1) = f(x_2) \ \to\ x_1 = x_2$.
Proof: Suppose $n, m \in \mathbb{N}$ such that $f(n) = f(m)$.

Then $n^2 = m^2$.

This implies $n = m$ or $n = -m$.

But $-m \notin \mathbb{N}$ unless $m = 0$. So we make two cases:

Case 1: If $m \neq 0$, then $-m \notin \mathbb{N}$ and so $n = m$.

Case 2: If $m = 0$, then $n = 0$, and so $n = m$.

Since in either case $n = m$, it follows that f is one-to-one. $\square$

Finite sets

Suppose X and Y are finite sets.

1. If $|X| > |Y|$, then there is no injective function $X \to Y$.
2. If $|X| < |Y|$, then there is no surjective function $X \to Y$.
3. There is a bijective function $X \to Y$ if and only if $|X| = |Y|$.

Put differently:

1. If $f\colon X \to Y$ is injective, then $|X| \leq |Y|$.
2. If $f\colon X \to Y$ is surjective, then $|X| \geq |Y|$.
3. If $f\colon X \to Y$ is bijective, then $|X| = |Y|$.

For finite sets with $|X| = |Y|$, the following statements are equivalent:

1. $f\colon X \to Y$ is injective
2. $f\colon X \to Y$ is bijective
3. $f\colon X \to Y$ is surjective

Composition of functions

If $f\colon X \to Y$ and $g\colon Y \to Z$, then the composition

$$g \circ f\colon X \to Z$$

is defined by:

$$(g \circ f)(x) = g(f(x)) \qquad \text{for all } x \in X.$$

We sometimes write gf instead of $g \circ f$.

Example:

$f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x \cdot y$

$g\colon \mathbb{R} \to \mathbb{R}^2$ defined by $g(x) = (\sin(x), x)$

Then $f \circ g$:

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