test

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Box analysis by hand

$$\partial_s G_{1111} = -\int \frac{\partial_s D_2}{D_1 D_2^2 D_3 D_4} - \int \frac{\partial_s D_3}{D_1 D_2 D_3^2 D_4} - \int \frac{\partial_s D_4}{D_1 D_2 D_3 D_4^2} \tag{1}$$

By using the fact that,

$$2K \cdot p_1 = 0 \tag{2}$$

where

$$K = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 \tag{3}$$

one gets,

$$\partial_s D_2 = 2K \cdot k, \quad \partial_s D_3 = 2K \cdot k + 1, \quad \partial_s D_4 = 2K \cdot k \tag{4}$$

Plugging (4) into (1),

$$\begin{aligned}\partial_s G_{1111} = & -\beta_1 G_{1111} + s(\beta_2 - \beta_3) G_{1211} - (1 - s(\beta_2 - \beta_3))\, G_{1121} \\ & + s(\beta_2 - \beta_3) G_{1112} + triangles\end{aligned} \tag{5}$$

where,

$$2p_1 \cdot k = D_2 - D_1, \quad 2p_2 \cdot k = D_3 - D_2 - s, \quad 2p_3 \cdot k = D_4 - D_3 + s \tag{6}$$

are taken into consideration.
Since I am only interested in the homogeneous part of the DE, I will be dropping terms such as triangles that do not contribute to G_{1111} when reduced. Using the following three IBP relations for the box,

$$G_{1211} = \frac{5-d}{t} G_{1111} + triangles \tag{7}$$

$$G_{1121} = \frac{5-d}{s} G_{1111} + triangles \tag{8}$$

$$G_{1112} = \frac{5-d}{t} G_{1111} + triangles \tag{9}$$

one finally gets,

$$\partial_s G_{1111} = -\frac{s + t + \epsilon t}{s(s+t)} G_{1111} \tag{10}$$

which is the 33 component of the A_s matrix of pg. 10 in Henn's DE lectures. It can be easily seen, that for a change of basis,

$$g_3 = c\epsilon(-s)^\epsilon st G_{1111} \implies \partial_s g_3 = \frac{\epsilon}{s+t} g_3 \tag{11}$$

the DE transforms as,

$$\partial_x g_3 = -\epsilon \left(\frac{1}{x} - \frac{1}{1+x} \right) g_3 \tag{12}$$

where again only the homogeneous part is kept alive. This reproduces Henn's result of pg. 11. Of course, a variable change $x = t/s$ is implemented.

Box analysis via FIRE6[1]

SetDirectory["/home/herc/fire/FIRE6"];

(* set the UT master integrals *)

NewBasis =<< "UTNL2.m";

(*UTNL2.$m = \{\epsilon * ((-s)\hat{}\epsilon) * t * G[0, \{0, 1, 0, 2\}],$

$\epsilon * ((-s)\hat{}\epsilon) * s * G[0, \{1, 0, 2, 0\}], \epsilon\hat{}2 * ((-s)\hat{}\epsilon) * s * t * G[0, \{1, 1, 1, 1\}]\}$***)**

Rule for the action of IBP[x,1] in the integrals:

gtof1 $= \{G[0, \{\text{a\_\_}\}] :\rightarrow \text{FF}[a]\};$

ftog1 $= \{\text{FF}[\text{a\_\_}] :\rightarrow G[0, \{a\}]\};$

action1 $= \{Y[\text{j\_}]\text{FF}[\text{a\_\_\_}] :\rightarrow \text{FF}[\text{Table}[\text{If}[i == j,$

$\{a\}[[j]] + 1, \{a\}[[i]]], \{i, 1, \text{Length}[\{a\}]\}]/.\text{List} \rightarrow \text{Sequence}]\};$

action2 $= \{\text{Ym}[\text{j\_}]\text{FF}[\text{a\_\_\_}] :\rightarrow \text{FF}[\text{Table}[\text{If}[i == j,$

$\{a\}[[j]] - 1, \{a\}[[i]]], \{i, 1, \text{Length}[\{a\}]\}]/.\text{List} \rightarrow \text{Sequence}]\};$

The independent integrals of the NewBasis in FF[] notation:

(* change notation for the master integrals *)

mi =<< "MiFireOrdered1.m"//.$\{1, \{\text{a1\_\_}\}\} :\rightarrow G[0, \{\text{a1}\}];$

(*MiFireOrdered1.$m = \{\{1, \{0, 1, 0, 1\}\}, \{1, \{1, 0, 1, 0\}\}, \{1, \{1, 1, 1, 1\}\}\}$***)**

miF = mi/.gtof1;

Run of FIRE:

(* setting the family *)

Get["FIRE6.m"];

Internal $= \{k\};$

External $= \{\text{p1}, \text{p2}, \text{p3}\};$

Propagators $= \{k\hat{}2, (k + \text{p1})\hat{}2, (k + \text{p1} + \text{p2})\hat{}2, (k + \text{p1} + \text{p2} + \text{p3})\hat{}2\};$

Replacements $= \{\text{p1}\hat{}2 \rightarrow 0, \text{p2}\hat{}2 \rightarrow 0, \text{p3}\hat{}2 \rightarrow 0, \text{p1p2} \rightarrow s/2, \text{p2p3} \rightarrow t/2, \text{p1p3} \rightarrow (-s - t)/2\};$

PrepareIBP[];

FIRE, version 6.4.2
Current commit: a9cb8dc Merged in feature/improve_masters (pull request #2)
Prepared

The IBP relations when we differentiate with respect to x are (Y[a[n]] is a raising operator rasing by one the integer a[n] and Ym[a[n]] is a lowering operator lowering by one the integer a[n]):

(* multiply with 1 and differentiate with respect to p1 *)

[1] Special thanks to Jimaras are in order, without whom the only thing FIRE6 would have burnt, would be my brain.

```
IBP[p1, 1]//.Replacements//Expand;
```

$-2ka[2]Y[2]-2\mathrm{p1}a[2]Y[2]-2ka[3]Y[3]-2\mathrm{p1}a[3]Y[3]-2\mathrm{p2}a[3]Y[3]$
$-2ka[4]Y[4]-2\mathrm{p1}a[4]Y[4]-2\mathrm{p2}a[4]Y[4]-2\mathrm{p3}a[4]Y[4]$

```
(*Imultiplywith((2s + t)(p1)/(2s(s + t)) + (p2)/(2s) + (p3)/(2(s + t)))tofindd/ds, andsetrulesD → Ym*)
```

$\mathrm{Expand}\left[\left(\frac{\mathrm{p2}}{2s}+\frac{\mathrm{p3}}{2(s+t)}+\frac{\mathrm{p1}(2s+t)}{2s(s+t)}\right)*(-2ka[2]Y[2]-2\mathrm{p1}a[2]Y[2]-2ka[3]Y[3]-2\mathrm{p1}a[3]Y[3]\right.$
$\left.-2\mathrm{p2}a[3]Y[3]-2ka[4]Y[4]-2\mathrm{p1}a[4]Y[4]-2\mathrm{p2}a[4]Y[4]-2\mathrm{p3}a[4]Y[4])\right]/.$

```
{p1^2 → 0, p2^2 → 0, p3^2 → 0, kp1 → Ym[2]/2 − Ym[1]/2, kp2 → Ym[3]/2 − Ym[2]/2 − s/2,
kp3 → Ym[4]/2 − Ym[3]/2 + s/2, p1p2 → s/2, p2p3 → t/2, p1p3 → (−s − t)/2};
dex = %//Simplify
```

The derivatives of the Basis elements with respect to x before the IBP reduction (Deqx):

```
Clear[tmp1, tmp2, deqx]
Do[tmp1 = mi[[i]][[2]];
tmp2 = dex//.Table[a[i] → tmp1[[i]], {i, 1, Length[Propagators]}];
deqx[i] = Expand[tmp2 * mi[[i]]//.gtof1]//.action1//.action2; , {i, 1, Length[mi]}]

Deqx = Table[deqx[i], {i, 1, Length[mi]}]//.ftog1;

Cases[Deqx, _G, Infinity]//DeleteDuplicates;
redlist = DeleteCases[%, Alternatives@@mi];
%%//Length
%%//Length
```

27

24

```
(* redlist is the integrals that need to be reduced *)

utmis = Cases[NewBasis, _G, Infinity]//DeleteDuplicates;
%//Length
```

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```
Join[redlist, utmis]//DeleteDuplicates;
%//Length
%%/.G[0, {a__}] :→ {1, {a}};
% >> "difseedNL1.m"
```

Rules for the reduction of the above integrals to the independent integrals of the NewBasis:

(* result of the reduction *)

$$\text{rulered1} = \{G[0,\{-1,1,0,2\}] \to \tfrac{1}{2}(2-2*\epsilon)G[0,\{0,1,0,1\}],$$
$$G[0,\{-1,2,0,1\}] \to \tfrac{1}{2}(2-2*\epsilon)G[0,\{0,1,0,1\}], G[0,\{0,0,0,2\}] \to 0,$$
$$G[0,\{0,1,-1,2\}] \to \tfrac{1}{2}(2-2*\epsilon)G[0,\{0,1,0,1\}],$$
$$G[0,\{0,1,0,2\}] \to \tfrac{-(1-2*\epsilon)G[0,\{0,1,0,1\}]}{t},$$
$$G[0,\{0,2,-1,1\}] \to \tfrac{1}{2}(2-2*\epsilon)G[0,\{0,1,0,1\}],$$
$$G[0,\{0,2,0,0\}] \to 0, G[0,\{0,2,0,1\}] \to \tfrac{-(1-2*\epsilon)G[0,\{0,1,0,1\}]}{t},$$
$$G[0,\{0,0,2,0\}] \to 0, G[0,\{1,-1,2,0\}] \to \tfrac{1}{2}(2-2*\epsilon)G[0,\{1,0,1,0\}],$$
$$G[0,\{1,0,2,-1\}] \to \tfrac{1}{2}(2-2*\epsilon)G[0,\{1,0,1,0\}],$$
$$G[0,\{1,0,2,0\}] \to \tfrac{-(1-2*\epsilon)G[0,\{1,0,1,0\}]}{s},$$
$$G[0,\{0,1,1,2\}] \to \tfrac{2(1-2*\epsilon)G[0,\{0,1,0,1\}]}{t^2},$$
$$G[0,\{0,1,2,1\}] \to \tfrac{4(1-2*\epsilon)G[0,\{0,1,0,1\}]}{(-2-2*\epsilon)t^2},$$
$$G[0,\{0,2,1,1\}] \to \tfrac{2(1-2*\epsilon)G[0,\{0,1,0,1\}]}{t^2},$$
$$G[0,\{1,0,1,2\}] \to \tfrac{4(1-2*\epsilon)G[0,\{1,0,1,0\}]}{(-2-2*\epsilon)s^2},$$
$$G[0,\{1,0,2,1\}] \to \tfrac{2(1-2*\epsilon)G[0,\{1,0,1,0\}]}{s^2},$$
$$G[0,\{1,1,0,2\}] \to \tfrac{2(1-2*\epsilon)G[0,\{0,1,0,1\}]}{t^2},$$
$$G[0,\{1,1,1,2\}] \to \tfrac{4(-1-2*\epsilon)(1-2*\epsilon)G[0,\{1,0,1,0\}]}{(-2-2*\epsilon)s^2t} - \tfrac{(-1-2*\epsilon)G[0,\{1,1,1,1\}]}{t},$$
$$G[0,\{1,1,2,0\}] \to \tfrac{2(1-2*\epsilon)G[0,\{1,0,1,0\}]}{s^2},$$
$$G[0,\{1,1,2,1\}] \to \tfrac{4(-1-2*\epsilon)(1-2*\epsilon)G[0,\{0,1,0,1\}]}{(-2-2*\epsilon)st^2} - \tfrac{(-1-2*\epsilon)G[0,\{1,1,1,1\}]}{s},$$
$$G[0,\{1,2,0,1\}] \to \tfrac{2(1-2*\epsilon)G[0,\{0,1,0,1\}]}{t^2},$$
$$G[0,\{1,2,1,0\}] \to \tfrac{4(1-2*\epsilon)G[0,\{1,0,1,0\}]}{(-2-2*\epsilon)s^2},$$
$$G[0,\{1,2,1,1\}] \to \tfrac{4(-1-2*\epsilon)(1-2*\epsilon)G[0,\{1,0,1,0\}]}{(-2-2*\epsilon)s^2t} - \tfrac{(-1-2*\epsilon)G[0,\{1,1,1,1\}]}{t},$$
$$G[0,\{1,1,1,1\}] \to G[0,\{1,1,1,1\}]\};$$

Using the above rules to DerivativesOfMIs:

Der = Deqx/.rulered1;

Cases[Der, _G, Infinity]//DeleteDuplicates//Length

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From the above we find the matrix Am of the diferential equation $\frac{\partial}{\partial x}F = \mathrm{Am}F$:

(* matrix of the de for the fire master integrals *)

Am = Table[Table[Coefficient[Der[[*i*]], mi[[*j*]]], {*j*, 1, Length[mi]}], {*i*, 1, Length[mi]}];

```
Monitor[Amsim = Table[Am[[i, j]]//Cancel//ExpandAll//Together//Expand,
{i, 1, Length[mi]}, {j, 1, Length[mi]}], {i, j}];
```

$$\mathbf{MatrixForm}\left[\left\{\{0,0,0\},\left\{0,-\frac{\epsilon}{s},0\right\},\left\{-\frac{2}{st(s+t)}+\frac{4\epsilon}{st(s+t)},\frac{2}{s^2(s+t)}-\frac{4\epsilon}{s^2(s+t)},-\frac{1}{s+t}-\frac{t}{s(s+t)}-\frac{t\epsilon}{s(s+t)}\right\}\right\}\right]$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{s} & 0 \\ -\frac{2}{st(s+t)}+\frac{4\epsilon}{st(s+t)} & \frac{2}{s^2(s+t)}-\frac{4\epsilon}{s^2(s+t)} & -\frac{1}{s+t}-\frac{t}{s(s+t)}-\frac{t\epsilon}{s(s+t)} \end{pmatrix}$$

```
Newbasis = NewBasis//.rulered1;
Cases[Newbasis, _G, Infinity]//DeleteDuplicates//Length
```

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(************************************ * *)

We define the matrix T which conects the NewBasis with the independent master integrals (G=T F):

```
T = Monitor[Table[Table[Coefficient[Newbasis[[i]], mi[[j]]]//Cancel//ExpandAll//Together,
{j, 1, Length[mi]}], {i, 1, Length[mi]}], {j, i}];
Variables[T]
```

$\{(-s)^{\epsilon}, s, t, \epsilon\}$

```
Tinv = Inverse[T];
```

The matrix Amcf of the differential equation $\frac{\partial}{\partial x}G = \text{Amcf}G$ should be:

```
Apre = T.Amsim.Tinv − T.D[Tinv, s];

FullSimplify[Apre = T.Amsim.Tinv − T.D[Tinv, s]]
```

$$\left\{\left\{\frac{\epsilon}{s},0,0\right\},\{0,0,0\},\left\{\frac{2\epsilon}{s+t},-\frac{2t\epsilon}{s^2+st},\frac{\epsilon}{s+t}\right\}\right\}$$

(*nowjustchangevariables$x = t/s$andwefinalyget*)

$$\mathbf{MatrixForm}\left[\left\{\left\{-\frac{\epsilon}{x},0,0\right\},\{0,0,0\},\left\{-\frac{2\epsilon}{x+x^2},\frac{2\epsilon}{1+x},-\frac{\epsilon}{x+x^2}\right\}\right\}\right]$$

$$\begin{pmatrix} -\frac{\epsilon}{x} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2\epsilon}{x+x^2} & \frac{2\epsilon}{1+x} & -\frac{\epsilon}{x+x^2} \end{pmatrix}$$

By hand analysis for the f_5 DE

I tried following the same reasoning for Henn's paper where,

$$f_5 = \epsilon^3(-s)^{1+2e}tG_{111(\epsilon+1)}G_{12} \tag{13}$$

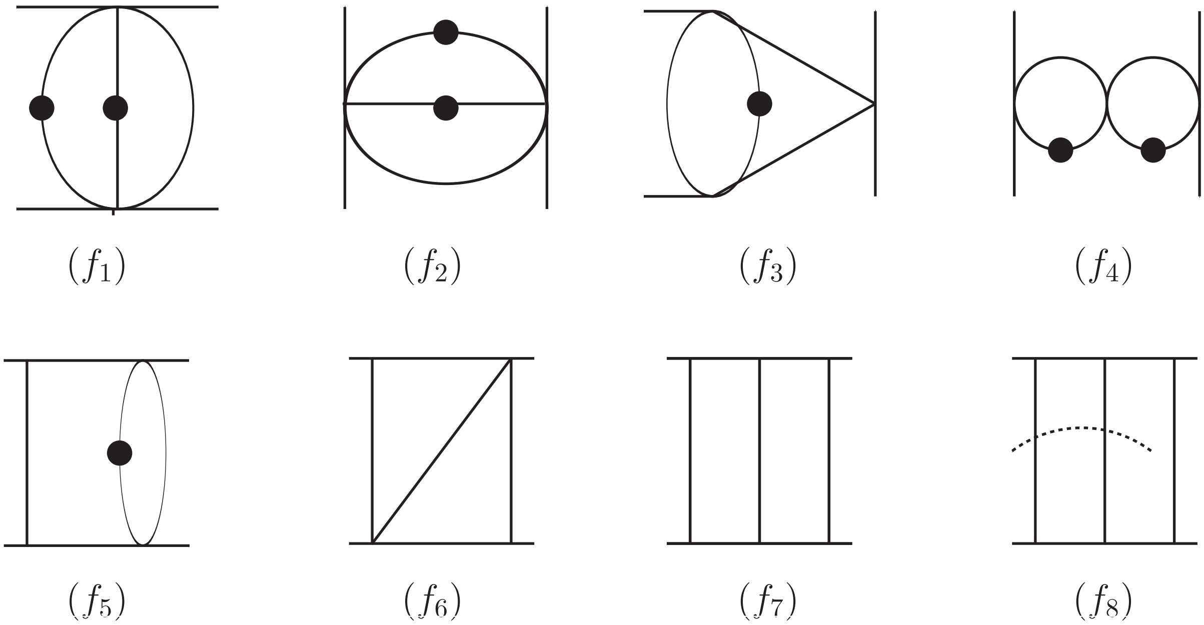

Figure 1: Integral basis corresponding to $f_1 ... f_4$ (first line) and $f_5 ... f_8$ (second line), up to overall factors. Fat dots indicate doubled propagators, and the dotted line an inverse propagator. The incoming momenta are labeled in a clockwise order, starting with p_1 in the lower left corner.

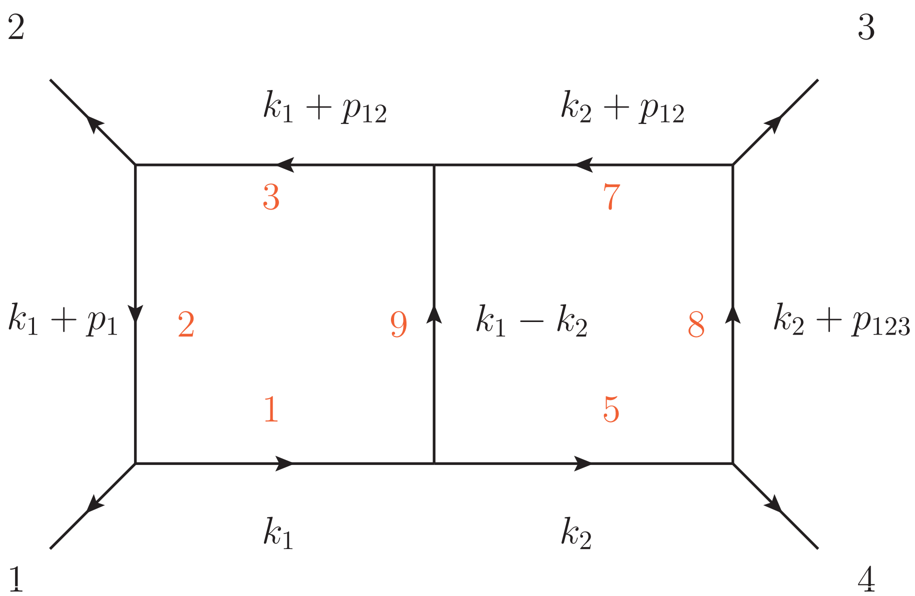

Figure 2: Main double box which provides the master integrals of Figure 1. Red numbers represent the ordering of propagators, when expressing an arbitrary integral, following the notation of Henn's paper.

where,

$$G_{111(\epsilon+1)} = \int \frac{d^d k}{k^2(k+p_1)^2(k+p_{12})^2((k+p_{123})^2)^{\epsilon+1}} \tag{14}$$

and G_{12} is just gamma functions independent of kinematic variables. Like before,

$$\begin{aligned}
\partial_s G_{111(\epsilon+1)} =& -\int \frac{\partial_s D_2}{D_1 D_2^2 D_3 D_4'} - \int \frac{\partial_s D_3}{D_1 D_2 D_3^2 D_4'} - \int \frac{\partial_s D_4'}{D_1 D_2 D_3 D_4'^2} \\
=& -\int \frac{\beta_1(D_2-D_1)+\beta_2(D_3-D_2-s)+\beta_3(D_4-D_3+s)}{D_1 D_2^2 D_3 D_4^{\epsilon+1}} \\
& -\int \frac{\beta_1(D_2-D_1)+\beta_2(D_3-D_2-s)+\beta_3(D_4-D_3+s)+1}{D_1 D_2 D_3^2 D_4^{\epsilon+1}} \\
& -(\epsilon+1)\int \frac{\beta_1(D_2-D_1)+\beta_2(D_3-D_2-s)+\beta_3(D_4-D_3+s)}{D_1 D_2 D_3 D_4^{\epsilon+2}}
\end{aligned} \tag{15}$$

where $D_4' = ((k+p_{123})^2)^{\epsilon+1}$. Now

$$\partial_s D_4' = (\epsilon+1)((k+p_{123})^2)^{\epsilon}\partial_s D_4 \tag{16}$$

where $\partial_s D_4$ is known from before and (15) is written,

$$\begin{aligned}
\partial_s G_{111(\epsilon+1)} =& (\epsilon+1)\left(s(\beta_2-\beta_3)G_{111(\epsilon+2)} - \beta_3 G_{111(\epsilon+1)}\right) + s(\beta_2-\beta_3)G_{121(\epsilon+1)} \\
& + (\beta_3-\beta_1)G_{111(\epsilon+1)} + \left(s(\beta_2-\beta_3)-1\right)G_{112(\epsilon+1)} \\
& - \beta_3(G_{112\epsilon} + G_{121\epsilon})
\end{aligned} \tag{17}$$

where triangles are again dropped. Like before, using the IBP relations (5.18)-(5.21) from Smirnov,

$$G_{121(\epsilon+1)} = \frac{1}{t}\left((1+4\epsilon)G_{111(\epsilon+1)} + G_{211\epsilon} + G_{121\epsilon} + G_{112\epsilon}\right) \tag{18}$$

$$G_{112(\epsilon+1)} = \frac{1+3\epsilon}{s}G_{111(\epsilon+1)} + triangles \tag{19}$$

$$G_{111(\epsilon+2)} = \frac{1+3\epsilon}{(1+\epsilon)t}G_{111(\epsilon+1)} + triangles \tag{20}$$

Now the problem lies in (18) where more terms than before are present and I don't have a clear way of expressing them as $G_{111(\epsilon+1)}$.

$$a_1 s 1^+ = a_1 + a_2 + 2a_3 + a_4 - d + (a_1 1^+ + a_2 2^+ + a_4 4^+)3^- \tag{21}$$

$$a_2 t 2^+ = a_1 + a_2 + a_3 + 2a_4 - d + (a_1 1^+ + a_2 2^+ + a_3 3^+)4^- \tag{22}$$

$$a_3 s 3^+ = 2a_1 + a_2 + a_3 + a_4 - d + (a_2 2^+ + a_3 3^+ + a_4 4^+)1^- \tag{23}$$

$$a_4 t 4^+ = a_1 + 2a_2 + a_3 + a_4 - d + (a_1 1^+ + a_3 3^+ + a_4 4^+)2^- \tag{24}$$

Form (21) $\times a_3 3^+ -$ (23) $\times a_1 1^+$ and (22) $\times a_4 4^+ -$ (24) $\times a_2 2^+$ to get,

$$(d - a_{2433} - 2)a_3 3^+ = (d - a_{2411} - 2)a_1 1^+ + (a_1 - a_3)(a_2 2^+ + a_4 4^+) \tag{25}$$

$$(d - a_{1322} - 2)a_2 2^+ = (d - a_{1344} - 2)a_4 4^+ + (a_2 - a_4)(a_1 1^+ + a_3 3^+) \tag{26}$$

Using (21) and (24) we get,

$$sG_{211\epsilon} = 3\epsilon G_{111\epsilon} + triangles \tag{27}$$

and

$$tG_{111(\epsilon+1)} = 3\epsilon G_{111\epsilon} + triangles \tag{28}$$

respectively.

Hence,

$$G_{211\epsilon} = G_{112\epsilon} = \epsilon x G_{111(\epsilon+1)} \tag{29}$$

Using (26)

$$G_{121\epsilon} = \frac{1}{1+\epsilon}\left(2\epsilon^2 + (\epsilon-1)\epsilon x\right)G_{111(\epsilon+1)} \tag{30}$$

From (29) and (30) we get into (18) that,

$$G_{121(\epsilon+1)} = \frac{1}{t}\left(1+4\epsilon+2\epsilon x+\frac{1}{1+\epsilon}\left(2\epsilon^2+(\epsilon-1)\epsilon x\right)\right)G_{111(\epsilon+1)} \tag{31}$$

Plugging (19), (20), (29), (30), (31) into (17) we get

$$\partial_s G_{111(\epsilon+1)} = -\frac{t+s+\epsilon t}{s(s+t)}G_{111(\epsilon+1)} \tag{32}$$

or

$$\partial_s f_5 = \epsilon\frac{2s+t}{s(s+t)}f_5 \implies \partial_x f_{5-homog} = \epsilon\left(-\frac{2}{x}+\frac{1}{1+x}\right)f_5 \tag{33}$$

which is idd the correct expression for f_5 up to non-homogeneous terms.

Now let us take care of the non-homogeneous terms. The terms that have been dropped from (15) read,

$$\begin{aligned}\partial_s G_{111(\epsilon+1)} = &+\beta_1 G_{021(\epsilon+1)} + (\beta_3-\beta_2)G_{120(\epsilon+1)} - \beta_1 G_{102(\epsilon+1)}\\ &+\beta_1 G_{012(\epsilon+1)} + \beta_2 G_{102(\epsilon+1)} - (\epsilon+1)(\beta_1 G_{101(\epsilon+2)}\\ &-\beta_1 G_{011(\epsilon+2)} + \beta_2 G_{110(\epsilon+2)} - \beta_2 G_{101(\epsilon+2)} - \beta_3 G_{110(\epsilon+2)})\end{aligned} \tag{34}$$

There are also triangle terms that are dropped from (19) and (20) which will contribute an extra,

$$\begin{aligned}&+\frac{s(\beta_2-\beta_3)}{t}(G_{201(\epsilon+1)} + G_{102(\epsilon+1)} + (\epsilon+1)G_{101(\epsilon+2)})\\ &+\frac{s(\beta_2-\beta_3)-1}{s}\left(G_{021(\epsilon+1)} + G_{012(\epsilon+1)} + (\epsilon+1)G_{011(\epsilon+2)}\right)\end{aligned} \tag{35}$$

Combing the two relations above,

$$\partial_s G_{111(\epsilon+1)} = \frac{1}{2(s+t)}\left(G_{201(\epsilon+1)} - x(2+3\epsilon)G_{210(\epsilon+1)}\right) \tag{36}$$

where we used (25) and (26) to reduce the terms with $a_4 = \epsilon+2$, in the following fashion,

$$G_{011(\epsilon+2)} = \frac{2+3\epsilon}{\epsilon+1}G_{012(\epsilon+1)} - \frac{1}{\epsilon+1}G_{021(\epsilon+1)} \tag{37}$$

$$G_{110(\epsilon+2)} = \frac{2+3\epsilon}{\epsilon+1}G_{210(\epsilon+1)} - \frac{1}{\epsilon+1}G_{120(\epsilon+1)} \tag{38}$$

$$G_{101(\epsilon+2)} = -\frac{1}{2(2\epsilon+1)}(G_{201(\epsilon+1)} + G_{102(\epsilon+1)}) \tag{39}$$

Note that f_1 and f_3 are written as,

$$f_1 = -\epsilon^2(-s)^{2\epsilon}tG_{12}G_{020(\epsilon+1)}, \quad f_3 = \epsilon^3(-s)^{1+2\epsilon}G_{12}G_{1(\epsilon+1)10} \tag{40}$$

What is left, is to rewrite (36) in terms of (40) to produce the non-homogeneous terms of the DE. Doing the necessary reductions, (36) is written

$$\partial_s G_{111(\epsilon+1)} = -\frac{3}{2s(s+t)}G_{020(\epsilon+1)} - \epsilon\frac{3}{s(s+t)}G_{1(\epsilon+1)10} \tag{41}$$

or w.r.t. x,

$$\partial_x G_{111(\epsilon+1)} = \frac{3}{2x(s+t)}G_{020(\epsilon+1)} + \frac{3\epsilon}{x(s+t)}G_{1(\epsilon+1)10} \tag{42}$$

Multiply[2] the last eqn with $G_{12}\epsilon^3(-s)^{1+2\epsilon}t$ to finally get the rest of the DE (the non-homogeneous terms),

$$\partial_x f_{5-nonhomog} = \frac{3\epsilon}{2}\left(\frac{1}{x}-\frac{1}{1+x}\right)f_1 + \frac{3\epsilon}{1+x}f_3 \tag{43}$$

This equation alongside (33) forms the full DE for f_5.

[2]There is a subtlety here, where one multiplies from the left side of the ∂_x operator. This happens so that the operator does not act on the aforementioned quantity. The operator has already acted on this quantity and provided with the homogeneous part that (33) is.

Analysis via FIRE6 for the Double-Box

SetDirectory["/home/herc/fire/FIRE6"];

NewBasis =<< "UTNL3.m";

(*NewBasis = $\{-(\epsilon)^\wedge 2 * (-s)^\wedge(2*\epsilon) * t * G[0, \{0,2,0,0,0,0,0,1,2\}],$

$(\epsilon)^\wedge 2 * (-s)^\wedge(2*\epsilon+1) * G[0, \{0,0,2,0,1,0,0,0,2\}],$

$(\epsilon)^\wedge 3 * (-s)^\wedge(2*\epsilon+1) * G[0, \{0,1,0,0,1,0,1,0,2\}],$

$-(\epsilon)^\wedge 2 * (-s)^\wedge(2*\epsilon+2) * G[0, \{2,0,1,0,2,0,1,0,0\}],$

$(\epsilon)^\wedge 3 * (-s)^\wedge(2*\epsilon+1) * t * G[0, \{1,1,1,0,0,0,0,1,2\}],$

$-(\epsilon)^\wedge 4 * (-s)^\wedge(2*\epsilon) * (s+t) * G[0, \{0,1,1,0,1,0,0,1,1\}],$

$-(\epsilon)^\wedge 4 * (-s)^\wedge(2*\epsilon+2) * t * G[0, \{1,1,1,0,1,0,1,1,1\}],$

$-(\epsilon)^\wedge 4 * (-s)^\wedge(2*\epsilon+2) * G[0, \{1,1,1,0,1,-1,1,1,1\}]\}$*)

Rule for the action of IBP[x,1] in the integrals:

gtof1 = $\{G[0, \{$a__$\}] :\to$ FF$[a]\}$;

ftog1 = $\{$FF[a__] $:\to G[0, \{a\}]\}$;

action1 = $\{Y$[j_]FF[a___] $:\to$ FF[Table[If[$i == j, \{a\}[[j]] + 1,$

$\{a\}[[i]]], \{i, 1,$ Length$[\{a\}]\}]$/.List $\to$ Sequence]};

action2 = {Ym[j_]FF[a___] $:\to$ FF[Table[If[$i == j, \{a\}[[j]] - 1,$

$\{a\}[[i]]], \{i, 1,$ Length$[\{a\}]\}]$/.List $\to$ Sequence]};

The independent integrals of the NewBasis in FF[] notation:

mi =<< "MiFireOrdered2.m"//.$\{1, \{$a1__$\}\} :\to G[0, \{$a1$\}]$;

(*mi = $\{\{1, \{0,1,0,0,0,0,0,1,1\}\},$

$\{1, \{0,0,1,0,1,0,0,0,1\}\},$

$\{1, \{0,1,0,0,1,0,1,0,1\}\},$

$\{1, \{1,0,1,0,1,0,1,0,0\}\},$

$\{1, \{1,1,1,0,0,0,0,1,1\}\},$

$\{1, \{0,1,1,0,1,0,0,1,1\}\},$

$\{1, \{1,1,1,0,1,0,1,1,1\}\},$

$\{1, \{1,1,1,-1,1,0,1,1,1\}\}\}$*)

miF = mi/.gtof1;

Run of FIRE:

Get["FIRE6.m"];

Internal = {k1, k2};

```
External = {p1, p2, p3};
Propagators = {k1^2, (k1 + p1)^2, (k1 + p1 + p2)^2, (k1 + p1 + p2 + p3)^2,
k2^2, (k2 + p1)^2, (k2 + p1 + p2)^2, (k2 + p1 + p2 + p3)^2, (k1 − k2)^2};
Replacements = {p1^2 → 0, p2^2 → 0, p3^2 → 0, p1p2 → s/2, p2p3 → t/2, p1p3 → (−s − t)/2};
PrepareIBP[];
```

FIRE, version 6.4.2
Current commit: a9cb8dc Merged in feature/improve_masters (pull request #2)
Prepared

The IBP relations when we differentiate with respect to x are (Y[a[n]] is a raising operator rasing by one the integer a[n] and Ym[a[n]] is a lowering operator lowering by one the integer a[n]):

```
IBP[p1, 1]//.Replacements//Expand;

%14
```

$-2\mathrm{k1}a[2]Y[2]-2\mathrm{p1}a[2]Y[2]-2\mathrm{k1}a[3]Y[3]-2\mathrm{p1}a[3]Y[3]-2\mathrm{p2}a[3]Y[3]-2\mathrm{k1}a[4]Y[4]-2\mathrm{p1}a[4]Y[4]-2\mathrm{p2}a[4]Y[4]-2\mathrm{p3}a[4]Y[4]-2\mathrm{k2}a[6]Y[6]-2\mathrm{p1}a[6]Y[6]-2\mathrm{k2}a[7]Y[7]-2\mathrm{p1}a[7]Y[7]-2\mathrm{p2}a[7]Y[7]-2\mathrm{k2}a[8]Y[8]-2\mathrm{p1}a[8]Y[8]-2\mathrm{p2}a[8]Y[8]-2\mathrm{p3}a[8]Y[8]$

$$\mathrm{Expand}\left[\left(\frac{\mathrm{p2}}{2s}+\frac{\mathrm{p3}}{2(s+t)}+\frac{\mathrm{p1}(2s+t)}{2s(s+t)}\right)*\right.$$

```
(−2k1a[2]Y[2] − 2p1a[2]Y[2] − 2k1a[3]Y[3] − 2p1a[3]Y[3] − 2p2a[3]Y[3] − 2k1a[4]Y[4] − 2p1a[4]Y[4]−
2p2a[4]Y[4] − 2p3a[4]Y[4] − 2k2a[6]Y[6] − 2p1a[6]Y[6] − 2k2a[7]Y[7] − 2p1a[7]Y[7] − 2p2a[7]Y[7]−
2k2a[8]Y[8] − 2p1a[8]Y[8] − 2p2a[8]Y[8] − 2p3a[8]Y[8])]/.
{p1^2 → 0, p2^2 → 0, p3^2 → 0, k1p1 → Ym[2]/2 − Ym[1]/2, k1p2 → Ym[3]/2 − Ym[2]/2 − s/2,
k1p3 → Ym[4]/2 − Ym[3]/2 + s/2, k2p1 → Ym[6]/2 − Ym[5]/2, k2p2 → Ym[7]/2 − Ym[6]/2 − s/2,
k2p3 → Ym[8]/2 − Ym[7]/2 + s/2, p1p2 → s/2, p2p3 → t/2, p1p3 → (−s − t)/2};
dex = %//Simplify
```

The derivatives of the Basis elements with respect to x before the IBP reduction (Deqx):

```
Clear[tmp1, tmp2, deqx]
Do[tmp1 = mi[[i]][[2]];
tmp2 = dex//.Table[a[i] → tmp1[[i]], {i, 1, Length[Propagators]}];
deqx[i] = Expand[tmp2 * mi[[i]]//.gtof1]//.action1//.action2; , {i, 1, Length[mi]}]

Deqx = Table[deqx[i], {i, 1, Length[mi]}]//.ftog1;

Cases[Deqx, _G, Infinity]//DeleteDuplicates;
redlist = DeleteCases[%, Alternatives@@mi];
%%//Length
```

%%//Length

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85

utmis = Cases[NewBasis, _G, Infinity]//DeleteDuplicates;

%//Length

8

Join[redlist, utmis]//DeleteDuplicates;

%//Length

%%/.G[0, {a__}] :→ {1, {a}};

% >> "difseedNL3.m"

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Rules for the reduction of the above integrals to the independent integrals of the NewBasis:

(*FindRules[MasterIntegrals[]]wasusedtoreducethenumberofbasisfrom12 → 8thatareusedinmi*)

rulered1 = Get["redfiMIs.m"]/.{G[0, {0, 1, 0, 0, 1, 0, 1, 1, 1}] → G[0, {1, 1, 1, 0, 0, 0, 0, 1, 1}],

G[0, {1, 1, 0, 0, 0, 0, 1, 1, 1}] → G[0, {0, 1, 1, 0, 1, 0, 0, 1, 1}],

G[0, {1, 0, 1, 0, 0, 0, 0, 1, 1}] → G[0, {0, 1, 0, 0, 1, 0, 1, 0, 1}],

G[0, {1, 0, 0, 0, 0, 0, 1, 0, 1}] → G[0, {0, 0, 1, 0, 1, 0, 0, 0, 1}], d → (4 − 2ϵ)};

Using the above rules to DerivativesOfMIs:

Der = Deqx/.rulered1;

Cases[Der, _G, Infinity]//DeleteDuplicates//Length

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From the above we find the matrix Am of the diferential equation $\frac{\partial}{\partial x}F = \mathrm{Am}F$:

Am = Table[Table[Coefficient[Der[[i]], mi[[j]]], {j, 1, Length[mi]}], {i, 1, Length[mi]}]

| show less | show more | show all | set size limit... |

Monitor[Amsim = Table[Am[[i, j]]//Cancel//ExpandAll//Together//Expand,

{i, 1, Length[mi]}, {j, 1, Length[mi]}], {i, j}];

Newbasis = NewBasis//.rulered1;

Cases[Newbasis, _G, Infinity]//DeleteDuplicates//Length

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We define the matrix T which conects the NewBasis with the independent master integrals (G=T F):

T = Monitor[Table[Table[Coefficient[Newbasis[[i]], mi[[j]]]//Cancel//ExpandAll//Together,

{j, 1, Length[mi]}], {i, 1, Length[mi]}], {j, i}];

Variables[T]

$\{(-s)^{\epsilon}, s, t, \epsilon\}$

Tinv = Inverse[T];

The matrix Amcf of the differential equation $\frac{\partial}{\partial x}G = \mathrm{Amcf}G$ should be:

Apre = T.Amsim.Tinv − T.D[Tinv, s];

FullSimplify[Apre = T.Amsim.Tinv − T.D[Tinv, s]]

MatrixForm[

FullSimplify[

$$-s\text{^}2/t\left\{\left\{\frac{2\epsilon}{s},0,0,0,0,0,0,0\right\},\{0,0,0,0,0,0,0,0\},\{0,0,0,0,0,0,0,0\},\{0,0,0,0,0,0,0,0\},\right.$$
$$\left\{\frac{3\epsilon}{2(s+t)},0,\frac{3t\epsilon}{s^2+st},0,\frac{(2s+t)\epsilon}{s(s+t)},0,0,0\right\},\left\{\frac{\epsilon}{2s},-\frac{\epsilon}{2s},0,0,0,\frac{2\epsilon}{s+t},0,0\right\},$$
$$\left\{\frac{3\epsilon}{s+t},\frac{3(s-t)\epsilon}{s(s+t)},-\frac{6t\epsilon}{s^2+st},\frac{2t\epsilon}{s^2+st},\frac{4\epsilon}{s+t},-\frac{12\epsilon}{s+t},\frac{2\epsilon}{s+t},\frac{2t\epsilon}{s^2+st}\right\},$$
$$\left.\left\{\frac{9\epsilon}{2(s+t)},\frac{3\epsilon}{s+t},-\frac{3\epsilon}{s+t},\frac{(s+2t)\epsilon}{s(s+t)},\frac{4\epsilon}{s+t},-\frac{18\epsilon}{s+t},\frac{\epsilon}{s+t},-\frac{\epsilon}{s+t}\right\}\right\}/.t\to s*x]]$$

$$\begin{pmatrix}
-\frac{2\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3\epsilon}{2x+2x^2} & 0 & -\frac{3\epsilon}{1+x} & 0 & -\frac{(2+x)\epsilon}{x(1+x)} & 0 & 0 & 0 \\
-\frac{\epsilon}{2x} & \frac{\epsilon}{2x} & 0 & 0 & 0 & -\frac{2\epsilon}{x+x^2} & 0 & 0 \\
-\frac{3\epsilon}{x+x^2} & \frac{3(-1+x)\epsilon}{x(1+x)} & \frac{6\epsilon}{1+x} & -\frac{2\epsilon}{1+x} & -\frac{4\epsilon}{x+x^2} & \frac{12\epsilon}{x+x^2} & -\frac{2\epsilon}{x+x^2} & -\frac{2\epsilon}{1+x} \\
-\frac{9\epsilon}{2x+2x^2} & -\frac{3\epsilon}{x+x^2} & \frac{3\epsilon}{x+x^2} & -\frac{\epsilon+2x\epsilon}{x+x^2} & -\frac{4\epsilon}{x+x^2} & \frac{18\epsilon}{x+x^2} & -\frac{\epsilon}{x+x^2} & \frac{\epsilon}{x+x^2}
\end{pmatrix}$$

MatrixForm[a]

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -2 & 0 & 0 \\ -3 & -3 & 0 & 0 & -4 & 12 & -2 & 0 \\ -\frac{9}{2} & -3 & 3 & -1 & -4 & 18 & -1 & 1 \end{pmatrix}$$

MatrixForm[*b*]

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 3 & 6 & 6 & -2 & 4 & -12 & 2 & -2 \\ \frac{9}{2} & 3 & -3 & -1 & 4 & -18 & 1 & -1 \end{pmatrix}$$

The reason some signs are slightly off in comparison to Henn's paper is caused by the fact that I used a plus sign when defining the propagators, instead of a minus.