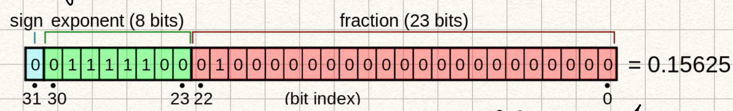
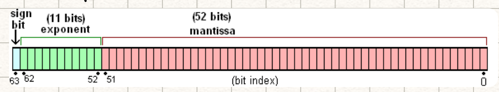
A2.

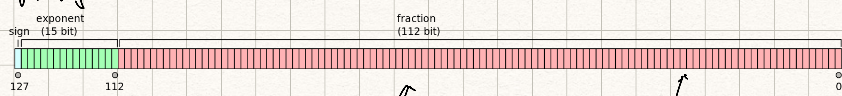
1. einfacher



zweifacher + 00000000000000000000001



vierfache



exponent e mantissa p Base b.

$$b^{e-(p-1)} \overset{e=0}{=} b^{-(p-1)}$$

$$x \in [1+\varepsilon, 1-\varepsilon] \rightarrow x \overset{!}{=} 1$$

$$\varepsilon = \frac{\overset{2\to}{b}{}^{-(p-1)}}{2} = 2^{-(p-1)-1} = 2^{-p}$$

einfacher: $p=24$ $\quad \varepsilon = 2^{-24} \approx 5{,}96 \cdot 10^{-8}$

zweifacher: $p=53$ $\quad \varepsilon = 2^{-53} \approx 1.11 \cdot 10^{-16}$

vierfacher: $p=113$ $\quad \varepsilon = 2^{-113} \approx 9{,}63 \cdot 10^{-35}$.

Numerische bestimmen:

```
ε = 1.0
while (1.0 + 0.5·ε) != 1.0:
    ε = 0.5·ε.
```

Numerische Differentiation.

Def: Ableitung $\quad f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$.

$\Rightarrow$ Numerisch $\quad f'_N(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$ $\quad$ für $h \ll 1$.

Besten? $\quad f(x_0+h) \underset{\text{bzg. } h}{\overset{\text{Taylor}}{=}} f(x_0) + h f'(x_0) + O(h^2)$

$$\rightarrow f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} + O(h^2).$$

Verbesser: zu $\mathcal{O}(h^3)$

$$f(x_0 \pm h) = f(x_0) \pm h f'(x_0) + \underbrace{\frac{h^2}{2} f''(x_0)} + \mathcal{O}(h^3).$$

$$f(x_0+h) - f(x_0-h) = 2h f'(x_0) + \mathcal{O}(h^3) \quad \Big| \cdot \frac{1}{2h}$$

$$\Rightarrow \quad f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + \mathcal{O}(h^3)$$

besser!

⋮

n-te Ableitung?

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$f''(x_0) = \lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \left(\frac{f(x_0+2h) - f(x_0+h)}{h} - \frac{f(x_0+h) - f(x_0)}{h} \right)$$

$$= \lim_{h\to 0} \frac{1}{h^2} \left(f(x_0+2h) - 2 f(x_0+h) + f(x_0) \right).$$

binominal Koeff.

$$\rightarrow f^{(n)}(x_0) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} f(x+kh).$$

Numerisch:

$$f_N^{(n)}(x_0) = \frac{1}{h^n} \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} f(x+kh)$$

Mit 3-Punkt-Form?

$$f_N'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$f_N''(x_0) = \frac{f'(x_0+h) - f'(x_0-h)}{2h} = \frac{1}{(2h)^2} \left(f(x_0+2h) - f(x_0) - f(x_0) + f(x_0-2h) \right)$$

$$= \frac{1}{(2h)^2} \left(f(x_0+2h) - 2f(x_0) + f(x_0-2h) \right)$$

$$f_N^{(3)}(x_0) = \frac{f''(x_0+h) - f''(x_0-h)}{2h}$$

$$= \frac{1}{(2h)^3}\big(f(x_0+3h) - 2f(x_0+h) + f(x_0-h)$$
$$- f(x_0+h) + 2f(x_0-h) - f(x_0-3h)\big)$$

$$= \frac{1}{(2h)^3}\big(f(x_0+3h) - 3f(x_0+h) + 3f(x_0-h) - f(x_0-3h)\big)$$

Order

$$f_N'(x_0) \longrightarrow f_N''(x_0) \longrightarrow f_N'''(x_0) \longrightarrow \cdots$$

$\uparrow$ Fehler $\quad\uparrow$ Fehler $\quad\uparrow$ Fehler ......

Optimal Wert von h:

$$f'(x_0) - \frac{f_N(x_0+h) - f_N(x_0)}{h}$$

$f_N(x_0+h) = f(x_0+h)(1+\varepsilon_2)$

$f_N(x_0) = f(x_0)(1+\varepsilon_1)$.

$$= f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h} - \frac{f(x_0+h)\varepsilon_2 - f(x_0)\varepsilon_1}{h}$$

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(\xi_h) \quad \xi_h \in (x_0, x_0+h)$$

$$= -\frac{h}{2}f''(\xi_h) - \frac{f(x_0+h)\varepsilon_2 - f(x_0)\varepsilon_1}{h}$$

$| h \ll 1 \rightarrow f(x_0+h) \approx f(x_0) \quad \xi_h \approx x_0$

$$\approx -\frac{h}{2}f''(x_0) - \frac{\varepsilon_2 - \varepsilon_1}{h}f(x_0)$$

$$|E(f, x_0, h)| \approx \left|-\frac{h}{2}f''(x_0) - \frac{\varepsilon_2 - \varepsilon_1}{h}f(x_0)\right|$$

$$\le \frac{h}{2}|f''(x_0)| + \frac{|\varepsilon_2 - \varepsilon_1|}{h}|f(x_0)| \qquad \varepsilon(h) := \max(|\varepsilon_2|, |\varepsilon_1|)$$

$$\le \frac{h}{2}|f''(x_0)| + \frac{\overbrace{|\varepsilon_2| + |\varepsilon_1|}^{:= 2\varepsilon(h)}}{h}|f(x_0)|$$

$$\le \frac{h}{2}|f''(x_0)| + \frac{2\varepsilon(h)}{h}|f(x_0)|$$

$$\varepsilon^* = \varepsilon(h)$$

$$e(h) = \frac{h}{2}|f''(x_0)| + \frac{2\varepsilon^*}{h}|f(x_0)|$$

$$e'(h) = \frac{|f''(x_0)|}{2} - \frac{2\varepsilon^*}{h^2}|f(x_0)| \overset{!}{=} 0$$

$$\Rightarrow |f''(x_0)| \overset{!}{=} \frac{4\varepsilon^*}{h_{min}^2}|f(x_0)|$$

$$\Rightarrow h_{min} = 2\sqrt{\frac{\varepsilon^*|f(x_0)|}{|f''(x_0)|}}$$

$$\varepsilon^* = \varepsilon$$

$x_0 = 3$

$f(x_0) = 1.0986 \leftarrow e=0$

$f''(x_0) = -\frac{1}{x_0^2} = -\frac{1}{9}$

$$h_{min} = \begin{cases} 1.53\cdot 10^{-3} & \text{float32} \\ 6.63\cdot 10^{-8} & \text{float64} \\ 6.17\cdot 10^{-17} & \text{float128.} \end{cases}$$

$$e(h) = \begin{cases} 1.71\cdot 10^{-4} & \text{float32} \\ 7.36\cdot 10^{-9} & \text{float64} \\ 6.86\cdot 10^{-18} & \text{float128.} \end{cases}$$

2. $f(x) = \log(x)^{\frac{1}{x}}$

$F(x) = f'(x)$

⓪ Maximum existiert, ob nur ein Extremum in $[1,6]$

① Brechnen $F(x)$ durch numerisch Diff.

② Sekanten-Verfahren für $F(x)$, $x \in [1,6]$

$$\downarrow x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

$$\leftarrow F'(x_n) \approx \frac{F(x_n) - F(x_{n-1})}{x_n - x_{n-1}}$$

~~– cross-check: Bisektion.~~ $\Rightarrow x_{n+1} = x_n - \frac{x_n - x_{n-1}}{F(x_n) - F(x_{n-1})} \cdot F(x_n)$

analytisch: $x_0 = 5,8312$

3. $\Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1}$ $\quad z \in \mathbb{R}^+$

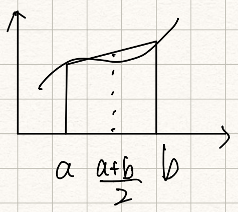
⓪ $z \to 0^+ \to \Gamma(z) \to \infty$

① $z \in \mathbb{N}^+ \to \Gamma(z) = (z-1)!$

↑ unsigned int

② $\Gamma(z+1) = z\Gamma(z)$

Trapez - Integration.



$$\int_a^b f(x)\,dx \approx \underbrace{(b-a)}_{=h} \cdot f\left(\frac{a+b}{2}\right)$$

$$= h \cdot f\left(\frac{x_n + x_{n+1}}{2}\right)$$

mit $x_{n+1} = x_n + h$.

Verbesser?

→ Analytische Form?

→ andere Methode?

z.B. mont carlo?