Week 1

Vectors — Magnitude & Direction

- A vector in 2D: a 2-component vector with components u_1 and u_2.

$$\mathbf{u} = (u_1, u_2)$$

or as a column vector

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Here the components u_1, u_2 are real numbers, $u_i \in \mathbb{R}$. This is an ordered tuple.
- In general, an n-component vector

$$\mathbf{u} = (u_1, u_2, \dots, u_n)^\top \in \mathbb{R}^n.$$

Example: Biological parameters

Suppose we measure n biological parameters (i.e. n components) for each subject. For example, for $n = 5$:

$$\begin{aligned} u_1 &\to \text{blood pressure}, \\ u_2 &\to \text{blood glucose level}, \\ u_3 &\to \text{pulse rate}, \\ u_4 &\to \text{heart rate}, \\ u_5 &\to \text{SpO}_2 \text{ level}. \end{aligned}$$

A single subject's measurement vector is

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} \in \mathbb{R}^5.$$

Collect these 5 parameters from 100 patients to form a data matrix $X \in \mathbb{R}^{100\times 5}$. Each row corresponds to a patient and each column to a parameter:

$$X = \begin{pmatrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} & u_4^{(1)} & u_5^{(1)} \\ u_1^{(2)} & u_2^{(2)} & u_3^{(2)} & u_4^{(2)} & u_5^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(100)} & u_2^{(100)} & u_3^{(100)} & u_4^{(100)} & u_5^{(100)} \end{pmatrix} \in \mathbb{R}^{100\times 5}.$$

Field F

A field F is a non-empty collection (set) of elements with two operations, addition (+) and multiplication ($\cdot$), such that the following properties hold.

(i) There exists $0 \in F$ such that for any element $a \in F$,

$$a + 0 = 0 + a = a.$$

(ii) For any two elements $a, b \in F$, the sum $a + b \in F$.

(iii) For each $a \in F$ there exists an additive inverse $(-a) \in F$ such that

$$a + (-a) = 0.$$

(iv) There exists an element $1 \in F$ with $1 \neq 0$ such that for all $a \in F$,

$$a \cdot 1 = 1 \cdot a = a.$$

(v) For any $a, b \in F$, the product $a \cdot b \in F$.

(vi) For every nonzero element $a \in F$ (i.e. $a \neq 0$) there exists a unique element $a^{-1} \in F$ such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

(Here a^{-1} is the multiplicative inverse of a.)

Examples:

- The set of real numbers $\mathbb{R}$ is a field; all the above properties hold.
- The set of natural numbers $\mathbb{N}$ is *not* a field. For example, $a = 2 \in \mathbb{N}$, but $a^{-1} = 1/2 \notin \mathbb{N}$.

- Can we make integers into a field?

Example: consider integers modulo 5.
- Let $\mathbb{Z}_5$ denote the set of remainders upon division by 5:

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}.$$

- Define addition and multiplication in $\mathbb{Z}_5$ by reducing modulo 5:

$$a + b \equiv a + b \pmod 5, \qquad a \cdot b \equiv ab \pmod 5.$$

Addition table (mod 5):

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Multiplication table (mod 5):

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Verify field axioms for $\mathbb{Z}_5$:

1. Closure under $+$ and $\cdot$ follows from reduction modulo 5.
2. Associativity and commutativity of $+$ and $\cdot$ inherit from integers.
3. Additive identity: 0 since $a + 0 \equiv a \pmod 5$.
4. Additive inverse: for every $a \in \mathbb{Z}_5$ there exists $-a$ with $a + (-a) \equiv 0 \pmod 5$.
5. Multiplicative identity: 1 since $a \cdot 1 \equiv a \pmod 5$.
6. Multiplicative inverses: for every $a \in \mathbb{Z}_5$ with $a \neq 0$ there exists a unique $a^{-1} \in \mathbb{Z}_5$ with $a \cdot a^{-1} \equiv 1 \pmod 5$. Concretely:

$$1^{-1} = 1, \quad 2^{-1} = 3, \quad 3^{-1} = 2, \quad 4^{-1} = 4 \pmod 5.$$

   Thus the nonzero elements form a multiplicative group.

•Remark: The natural numbers $\mathbb{N}$ fail to satisfy the existence of additive and multiplicative inverses (and closure under subtraction/division), so they are not a field.

•Conclusion: The set $\mathbb{Z}_5$ with addition and multiplication modulo 5 is a field.
Notes on element/inverse listing (mod 5): - Elements (nonzero): $1, 2, 3, 4$. - Their multiplicative inverses: $1, 3, 2, 4$ respectively. - Set of integers modulo n.

Define $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with addition and multiplication taken modulo n. We ask: for which n is $\mathbb{Z}_n$ a field?

- Example: $\mathbb{Z}_6$ is not a field.

Reason: some nonzero elements have no multiplicative inverse. For instance, $2 \cdot 3 \equiv 0 \pmod 6$, so 2 has no x with $2x \equiv 1 \pmod 6$. The units of $\mathbb{Z}_6$ (elements with multiplicative inverses) are those integers coprime to 6, namely $\{1, 5\}$.

- General fact.
$\mathbb{Z}_n$ is a field if and only if n is prime. If p is prime then every nonzero element of $\mathbb{Z}_p$ has a multiplicative inverse, so $\mathbb{Z}_p$ is a field.

- The case $\mathbb{Z}_2$.
$\mathbb{Z}_2 = \{0, 1\}$. Addition and multiplication tables:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Addition in $\mathbb{Z}_2$ is XOR (exclusive OR):

$$0 \oplus 0 = 0, \quad 0 \oplus 1 = 1, \quad 1 \oplus 0 = 1, \quad 1 \oplus 1 = 0.$$

- Notes about binary.
There are only two binary digits $\{0, 1\}$. Powers of two arise naturally in binary positional notation. When reducing integers modulo powers of two, remainders are determined by the lower bits (for example, dividing by 2^k removes the lowest k bits and the remainder is determined by those bits).

Vector Space

Let F be any field. Let V be a non-empty collection of objects called the *vectors*.

V is a *vector space* over F if rules for vector addition and scalar multiplication exist such that V is closed under vector addition and scalar multiplication.

Closure conditions:

1. For $u, v \in V$, we have $u + v \in V$.
2. For $\alpha \in F$ and $u \in V$, we have $\alpha u \in V$.

Properties (vector space axioms). For all $u, v, w \in V$ and all $\alpha, \beta \in F$:

1. (Commutativity of addition) $u + v = v + u$.
2. (Associativity of addition) $(u + v) + w = u + (v + w)$.
3. (Scalar multiplication associativity) $(\alpha\beta)u = \alpha(\beta u)$.
4. (Distributivity of scalar multiplication over vector addition) $\alpha(u + v) = \alpha u + \alpha v$.
5. (Distributivity of scalar addition over scalar multiplication) $(\alpha + \beta)u = \alpha u + \beta u$.
6. (Additive identity) There exists $0 \in V$ such that $u + 0 = u$ for all $u \in V$.
7. (Additive inverse) For each $u \in V$ there exists $-u \in V$ such that $u + (-u) = 0$.
8. (Multiplicative identity) $1 \cdot u = u$ for all $u \in V$, where 1 is the multiplicative identity in F.

Construction of an n-component vector space (example for $n = 2$) over the field $F = \mathbb{R}$.
Let $F = \mathbb{R}$.

Consider n copies of $\mathbb{R}$. An element

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}.$$

For the two-component case ($n = 2$):

$$u = (u_1, u_2), \qquad v = (v_1, v_2), \qquad u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

Vector addition (component-wise):

$$u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1,\ u_2 + v_2),$$

where the additions $u_i + v_i$ are the additions defined in the field $\mathbb{R}$.

Scalar multiplication:

$$\alpha \in \mathbb{R}, \qquad \alpha u = \alpha(u_1, u_2) = (\alpha u_1,\ \alpha u_2),$$

where the products αu_i are the multiplications defined in the field $\mathbb{R}$.

Hence $\mathbb{R}^n$ (and in particular $\mathbb{R}^2$) becomes a vector space over the field $\mathbb{R}$ with the above component-wise addition and scalar multiplication.

Examples for Vector Spaces

1. The field F itself is a vector space over F.
2. $\mathbb{R}^n$ (for any finite n) is a vector space over $\mathbb{R}$.
3. The set of all polynomials of degree $\leq n$ with real coefficients is a vector space over $\mathbb{R}$.
4. The set of all $m \times n$ matrices over $\mathbb{R}$ is a vector space over $\mathbb{R}$.
5. The set of all real symmetric matrices (of a fixed size) is a vector space over $\mathbb{R}$.
6. The set $C(\mathbb{R})$ of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ (functions of time t) is a vector space over $\mathbb{R}$.

Example. Consider $\mathbb{R}^2$. For $k \in \mathbb{R}$ define

$$S_k = \{(x, y) \in \mathbb{R}^2 : y = kx\}.$$

We examine S_0.

Note that

$$S_0 = \{(x, 0) : x \in \mathbb{R}\} = \text{span}\{(1, 0)\}.$$

Proof that S_0 is a vector space (subspace of $\mathbb{R}^2$):

- Let $u = (u_1, u_2),\ v = (v_1, v_2) \in S_0$. Then $u_2 = 0$ and $v_2 = 0$. Therefore

  $$u + v = (u_1 + v_1,\ 0) \in S_0,$$

  so S_0 is closed under vector addition.
- For any scalar $\alpha \in \mathbb{R}$,

  $$\alpha u = (\alpha u_1,\ 0) \in S_0,$$

  so S_0 is closed under scalar multiplication.
- The zero vector $(0, 0) \in S_0$.

Hence S_0 is a vector space (subspace) of $\mathbb{R}^2$ over $\mathbb{R}$.

- From the above observations:
  $\Rightarrow$ Any line passing through the origin is a vector space over $\mathbb{R}$.

$\Rightarrow$ Any subset of a vector space V which by itself is a vector space with the operations as defined in V is called a *vector subspace* of V.

Let V be a vector space. Some simple remarks about subspaces:

(i) Every subspace is a subset of the vector space itself. In particular, V is a subspace of V.

(ii) The set containing only the zero vector, $\{0\}$, is a vector subspace. Such a subspace (either $\{0\}$ or V itself) is called a *trivial subspace*.

(iii) In $\mathbb{R}^n$, any plane (or line) passing through the origin is a vector subspace.

Some questions:

1. What happens if we add multiple vectors in a vector space?
   Answer / remark: Combining vectors by addition and scalar multiplication leads to linear combinations; the set of all linear combinations of a given set of vectors is the *span* of those vectors (a subspace).
2. Suppose we want to understand (or span) the entire vector space. What are the strategies to study the effect of linear transformations on a vector space?
   Possible approaches: study bases and coordinates, compute images of basis vectors under the transformation, analyze invariant subspaces, determine rank and nullity (rank–nullity theorem), and use matrix representations to understand the transformation's action.

Linear combination of vectors

Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are k vectors with n components each. Thus

$$\mathbf{u}_i = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{in} \end{pmatrix} \quad \text{for } i = 1, 2, \ldots, k.$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be scalars. (In the present notes we take the vectors $\mathbf{u}_i$ and the scalars α_i to be real.) The vector

$$\boxed{\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_k \mathbf{u}_k}$$

is called the linear combination of the k vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$.

Examples:

1. Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are the standard unit vectors in $\mathbb{R}^n$:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad \ldots, \qquad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Any vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ can be written as a linear combination

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \cdots + v_n \mathbf{e}_n.$$

2. For $n = 2$, let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then

$$\mathbf{v} = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For instance,

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Linear combination of vectors

Suppose $\alpha_1, \alpha_2, \dots, \alpha_k$ are scalars and $u_1, u_2, \dots, u_k$ are vectors. A linear combination of the vectors $u_1, \dots, u_k$ is

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k.$$

Example. Let

$$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then

$$v = u_1 + u_2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix},$$

and for equal coefficients $\frac{1}{2}$,

$$v = \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix}.$$

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$, then

$$v = \tfrac{1}{n}(u_1 + u_2 + \cdots + u_n),$$

i.e. v is the average of the vectors $u_1, \dots, u_n$.

If the coefficients (the scalars) add up to 1, we call such a combination an affine combination.

- Suppose the coefficients in an affine combination are all non-negative. We call this combination
  - (i) a convex combination,
  - (ii) a weighted average.

Suppose $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative and

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1.$$

The linear (affine) combination is

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k.$$

Suppose we want to study the effect of a transformation on a vector space; what strategy do we adopt to do this?

Consider $u_1, u_2, \dots, u_n$ – n-component vectors. Look at all linear combinations of the u-vectors that result in the n-component zero vector. The linear combination of the u-vectors is given by

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = \mathbf{0}.$$

Example:

$$u_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

We solve for scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For example,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 \begin{pmatrix} 3 \\ 3 \end{pmatrix},$$

so $\alpha = (1, 1, -1)^\mathsf{T}$ is a nontrivial solution. In matrix form:

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Linear Independence of Vectors

Definition. Vectors $v_1, v_2, \ldots, v_n$ are said to be *linearly dependent* if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

If the only choice of scalars that gives the zero vector is $a_1 = a_2 = \cdots = a_n = 0$, then the vectors are *linearly independent*.

- Any set of vectors that contains the zero vector is a linearly dependent set.

  Example. Let $S = \{u_1, u_2, \ldots, u_k, 0\}$. Then

  $$0 = 0 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_k + 1 \cdot 0,$$

  so there exists a nontrivial linear combination equal to the zero vector; hence S is linearly dependent.

- (Redundancy) Linear dependence indicates that at least one vector in the set can be expressed in terms of the others.

- Let $v_1, v_2, \ldots, v_k$ be k vectors in $\mathbb{R}^n$ (i.e. n-component vectors), and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be scalars. Consider the linear combination

  $$\alpha_1v_1 + \alpha_2v_2 + \cdots + \alpha_kv_k = 0_n.$$

  If the only solution is $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$, then $v_1, \ldots, v_k$ are linearly independent. Otherwise they are linearly dependent.

Some observations:

1. A linearly independent set cannot contain the zero vector.
2. A single vector is always linearly independent unless it is the zero vector.
3. Any subset of a linearly independent set of vectors is always linearly independent.
4. A superset of a linearly dependent set is linearly dependent.
5. Two vectors are linearly independent if and only if none is a scalar multiple of the other.

   Span of a set of vectors:
   Let $v_1, v_2, \ldots, v_k$ be k vectors in a vector space V.
   Definition. The span of $\{v_1, v_2, \ldots, v_k\}$ is the set of all possible linear combinations:

   $$\operatorname{span}\{v_1, v_2, \ldots, v_k\} = \{\alpha_1v_1 + \alpha_2v_2 + \cdots + \alpha_kv_k \mid \alpha_i \in \mathbb{R} \text{ for } i = 1, \ldots, k\}.$$

Remarks:

- The span of a set of vectors is a subspace of V.
- It is the smallest subspace of V that contains the given set of vectors.

  More explicitly, for a finite set $\{v_1, \ldots, v_n\}$,

  $$\operatorname{span}\{v_1, \ldots, v_n\} = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;\middle|\; \alpha_i \in \mathbb{R} \right\}.$$

  Examples and basic properties:

1. The zero vector 0 always belongs to $\operatorname{span}\{v_1, \ldots, v_n\}$ (take all coefficients zero).
2. $\operatorname{span}\{v_1, \ldots, v_n\}$ is closed under vector addition and scalar multiplication.
3. Therefore $\operatorname{span}\{v_1, \ldots, v_n\}$ is a vector space (subspace of V).
4. For instance, if $v_1 = (1, 2)^\top$ and $v_2 = (1, 1)^\top$ in $\mathbb{R}^2$, then

   $$\operatorname{span}\{v_1, v_2\} = \{\alpha_1(1, 2)^\top + \alpha_2(1, 1)^\top \mid \alpha_1, \alpha_2 \in \mathbb{R}\}.$$

Basics

A set of n linearly independent n-component vectors is called a *basis* for the vector subspace that contains these n linearly independent n-component vectors.

(Span remark) The span of each basis for a given vector space is the vector space itself; consequently any two bases of the same vector space have the same span.

Uniqueness of coordinates with respect to a basis

Let $\{u_1, \ldots, u_k\}$ be a linearly independent set of vectors. Suppose

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k$$

and that x also has another representation in terms of the same vectors:

$$x = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_k u_k.$$

Subtracting the two expressions gives

$$0 = (\alpha_1 - \beta_1)u_1 + (\alpha_2 - \beta_2)u_2 + \cdots + (\alpha_k - \beta_k)u_k.$$

Since $u_1, \ldots, u_k$ are linearly independent, the only solution to the above equation is

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \cdots = \alpha_k - \beta_k = 0,$$

so $\alpha_i = \beta_i$ for all i.

Therefore:

- Any vector when expressed as a linear combination of a linearly independent set of vectors has a UNIQUE set of scalars.
- Equivalently, any vector in a vector space has a unique representation in terms of basis vectors.

Example

Let $V = \mathbb{R}^2$ and let the set

$$\mathcal{B} = \{(1,2), (1,1)\}.$$

Given $v = (2,3)$, seek scalars α_1, α_2 such that

$$\alpha_1(1,2) + \alpha_2(1,1) = (2,3).$$

This yields the linear system

$$\begin{cases} \alpha_1 + \alpha_2 = 2, \\ 2\alpha_1 + \alpha_2 = 3. \end{cases}$$

Solving gives $\alpha_1 = 1,\ \alpha_2 = 1$, which is the unique solution. Thus, the coordinates of v with respect to $\mathcal{B}$ are $(1,1)$.

There are infinitely many bases for R^n. However, all the basis will have exactly 'n' vectors.