complex

I wear a chain complex now. Chain complexes are cool

27.01.2017

27 January 2017

Kristin Krogh Arnesen

Oystein Skartsaeterhagen

Kamal Saleh

Kristin Krogh Arnesen
Email: kristink@math.ntnu.no
Homepage: http://www.math.ntnu.no/~kristink
Address: Trondheim

Oystein Skartsaeterhagen
Email: oysteini@math.ntnu.no
Homepage: http://www.math.ntnu.no/~oysteini
Address: Trondheim

Kamal Saleh
Email: kamal.saleh@uni-siegen.de
Homepage: https://github.com/kamalsaleh/complex
Address: Siegen

Contents

Chapter 1

Complexes categories

1.1 Constructing chain and cochain categories

1.1.1 IsChainOrCochainComplexCategory (for IsCapCategory)

▷ `IsChainOrCochainComplexCategory(`*arg*`)` (filter)

Returns: `true` or `false`

bla bla

1.1.2 IsChainComplexCategory (for IsChainOrCochainComplexCategory)

▷ `IsChainComplexCategory(`*arg*`)` (filter)

Returns: `true` or `false`

bla bla

1.1.3 IsCochainComplexCategory (for IsChainOrCochainComplexCategory)

▷ `IsCochainComplexCategory(`*arg*`)` (filter)

Returns: `true` or `false`

bla bla

1.1.4 ChainComplexCategory (for IsCapCategory)

▷ `ChainComplexCategory(`*A*`)` (attribute)

Returns: a CAP category

Creates the chain complex category $\mathrm{Ch}_\bullet(A)$ an Abelian category A.

1.1.5 CochainComplexCategory (for IsCapCategory)

▷ `CochainComplexCategory(`*A*`)` (attribute)

Returns: a CAP category

Creates the cochain complex category $\mathrm{Ch}^\bullet(A)$ an Abelian category A.

1.1.6 UnderlyingCategory (for IsChainOrCochainComplexCategory)

▷ UnderlyingCategory(B) (attribute)

Returns: a CAP category

The input is a chain or cochain complex category $B = C(A)$ constructed by one of the previous commands. The outout is A

Let $\mathbb{Q}$ be the field of rationals and let $\mathrm{Vec}_{\mathbb{Q}}$ be the category of $\mathbb{Q}$-vector spaces. The cochain complex category of $\mathrm{Vec}_{\mathbb{Q}}$ can be constructed as follows

Example

```
gap> LoadPackage( "LinearAlgebraForCap" );;
gap> LoadPackage( "complex" );;
gap> Q := HomalgFieldOfRationals( );;
gap> matrix_category := MatrixCategory( Q );
Category of matrices over Q
gap> cochain_cat := CochainComplexCategory( matrix_category );
Cochain complexes category over category of matrices over Q
```

Chapter 2

Complexes

2.1 Categories and filters

2.1.1 IsChainOrCochainComplex (for IsCapCategoryObject)

▷ `IsChainOrCochainComplex(C)` (filter)
▷ `IsChainComplex(C)` (filter)
▷ `IsCochainComplex(C)` (filter)
▷ `IsBoundedBelowChainOrCochainComplex(C)` (filter)
▷ `IsBoundedAboveChainOrCochainComplex(C)` (filter)
▷ `IsBoundedChainOrCochainComplex(C)` (filter)
▷ `IsBoundedBelowChainComplex(C)` (filter)
▷ `IsBoundedAboveChainComplex(C)` (filter)
▷ `IsBoundedChainComplex(C)` (filter)
▷ `IsBoundedBelowCochainComplex(C)` (filter)
▷ `IsBoundedAboveCochainComplex(C)` (filter)
▷ `IsBoundedCochainComplex(C)` (filter)

Returns: `true` or `false`

bla bla

2.2 Creating chain and cochain complexes

2.2.1 ChainComplex (for IsCapCategory, IsZList)

▷ `ChainComplex(A, diffs)` (operation)
▷ `CochainComplex(A, diffs)` (operation)

Returns: a chain complex

The input is category `A` and an infinite list `diffs`. The output is the chain (resp. cochain) complex $M_\bullet \in \mathrm{Ch}(A)$ ($M^\bullet \in \mathrm{Ch}^\bullet(A)$) where $d_i^M = \mathrm{diffs}[i]$($d_M^i = \mathrm{diffs}[i]$).

2.2.2 ChainComplex (for IsDenseList, IsInt)

▷ `ChainComplex(diffs, n)` (operation)
▷ `CochainComplex(diffs, n)` (operation)

Returns: a (co)chain complex

The input is a finite dense list `diffs` and an integer `n` . The output is the chain (resp. cochain) complex $M_\bullet \in \mathrm{Ch}(A)$ ($M^\bullet \in \mathrm{Ch}^\bullet(A)$) where $d_n^M := \mathrm{diffs}[1](d_M^n := \mathrm{diffs}[1]), d_{n+1}^M = \mathrm{diffs}[2](d_M^{n+1} := \mathrm{diffs}[2])$, etc.

2.2.3 ChainComplex (for IsDenseList)

▷ `ChainComplex(diffs)` (operation)
▷ `CochainComplex(diffs)` (operation)

Returns: a (co)chain complex

The same as the previous operations but with $n = 0$.

2.2.4 StalkChainComplex (for IsCapCategoryObject, IsInt)

▷ `StalkChainComplex(diffs, n)` (operation)
▷ `StalkCochainComplex(diffs, n)` (operation)

Returns: a (co)chain complex

The input is an object $M \in A$. The output is chain (resp. cochain) complex $M_\bullet \in \mathrm{Ch}_\bullet(A)(M^\bullet \in \mathrm{Ch}^\bullet(A))$ where $M_n = M(M^n = M)$ and $M_i = 0(M^i = 0)$ whenever $i \neq n$.

2.2.5 ChainComplexWithInductiveSides (for IsCapCategoryMorphism, IsFunction, IsFunction)

▷ `ChainComplexWithInductiveSides(d, G, F)` (operation)

Returns: a chain complex

The input is a morphism $d \in A$ and two functions F, G. The output is chain complex $M_\bullet \in \mathrm{Ch}_\bullet(A)$ where $d_0^M = d$ and $d_i^M = G^i(d)$ for all $i \leq -1$ and $d_i^M = F^i(d)$ for all $i \geq 1$.

2.2.6 CochainComplexWithInductiveSides (for IsCapCategoryMorphism, IsFunction, IsFunction)

▷ `CochainComplexWithInductiveSides(d, G, F)` (operation)

Returns: a cochain complex

The input is a morphism $d \in A$ and two functions F, G. The output is cochain complex $M^\bullet \in \mathrm{Ch}^\bullet(A)$ where $d_M^0 = d$ and $d_M^i = G^i(d)$ for all $i \leq -1$ and $d_M^i = F^i(d)$ for all $i \geq 1$.

2.2.7 ChainComplexWithInductiveNegativeSide (for IsCapCategoryMorphism, IsFunction)

▷ `ChainComplexWithInductiveNegativeSide(d, G)` (operation)

Returns: a chain complex

The input is a morphism $d \in A$ and a functions G. The output is chain complex $M_\bullet \in \mathrm{Ch}_\bullet(A)$ where $d_0^M = d$ and $d_i^M = G^i(d)$ for all $i \leq -1$ and $d_i^M = 0$ for all $i \geq 1$.

2.2.8 ChainComplexWithInductivePositiveSide (for IsCapCategoryMorphism, IsFunction)

▷ `ChainComplexWithInductivePositiveSide(d, F)` (operation)

Returns: a chain complex

The input is a morphism $d \in A$ and a functions F. The output is chain complex $M_\bullet \in \mathrm{Ch}_\bullet(A)$ where $d_0^M = d$ and $d_i^M = F^i(d)$ for all $i \geq 1$ and $d_i^M = 0$ for all $i \leq 1$.

2.2.9 CochainComplexWithInductiveNegativeSide (for IsCapCategoryMorphism, IsFunction)

▷ `CochainComplexWithInductiveNegativeSide(d, G)` (operation)

Returns: a cochain complex

The input is a morphism $d \in A$ and a functions G. The output is cochain complex $M^\bullet \in \mathrm{Ch}^\bullet(A)$ where $d_M^0 = d$ and $d_M^i = G^i(d)$ for all $i \leq -1$ and $d_M^i = 0$ for all $i \geq 1$.

2.2.10 CochainComplexWithInductivePositiveSide (for IsCapCategoryMorphism, IsFunction)

▷ `CochainComplexWithInductivePositiveSide(d, F)` (operation)

Returns: a cochain complex

The input is a morphism $d \in A$ and a functions F. The output is cochain complex $M^\bullet \in \mathrm{Ch}^\bullet(A)$ where $d_M^0 = d$ and $d_M^i = F^i(d)$ for all $i \geq 1$ and $d_M^i = 0$ for all $i \leq 1$.

2.3 Attributes

2.3.1 Differentials (for IsChainOrCochainComplex)

▷ `Differentials(C)` (attribute)

Returns: an infinite list

The command returns the differentials of the chain or cochain complex as an infinite list.

2.3.2 Objects (for IsChainOrCochainComplex)

▷ `Objects(C)` (attribute)

Returns: an infinite list

The command returns the objects of the chain or cochain complex as an infinite list.

2.3.3 CatOfComplex (for IsChainOrCochainComplex)

▷ `CatOfComplex(C)` (attribute)

Returns: a Cap category

The command returns the category in which all objects and differentials of C live.

2.4 Operations

2.4.1 \[\] (for IsChainOrCochainComplex, IsInt)

▷ `\[\](C, i)` (operation)

Returns: an object

The command returns the object of the chain or cochain complex in index i.

2.4.2 \^ (for IsChainOrCochainComplex, IsInt)

▷ `\^(C, i)` (operation)

Returns: a morphism

The command returns the differential of the chain or cochain complex in index i.

2.4.3 CertainCycle (for IsChainOrCochainComplex, IsInt)

▷ `CertainCycle(C, n)` (operation)

Returns: a morphism

The input is a chain or cochain complex C and an integer n. The output is the kernel embedding of the differential in index n.

2.4.4 CertainBoundary (for IsChainOrCochainComplex, IsInt)

▷ `CertainBoundary(C, n)` (operation)

Returns: a morphism

The input is a chain (resp. cochain) complex C and an integer n. The output is the image embeddin of $i+1$'th (resp. $i-1$'th) differential of C.

2.4.5 DefectOfExactness (for IsChainOrCochainComplex, IsInt)

▷ `DefectOfExactness(C, n)` (operation)

Returns: a object

The input is a chain (resp. cochain) complex C and an integer n. The outout is the homology (resp. cohomology) object of C in index n.

2.4.6 IsExactInIndex (for IsChainOrCochainComplex, IsInt)

▷ `IsExactInIndex(C, n)` (operation)

Returns: true or false

The input is a chain or cochain complex C and an integer n. The outout is `true` if C is exact in i. Otherwise the output is `false`.

2.4.7 SetUpperBound (for IsChainOrCochainComplex, IsInt)

▷ `SetUpperBound(C, n)` (operation)

Returns: Side effect

The command sets an upper bound n to the chain (resp. cochain) complex C. This means $C_{i\geq n} = 0 (C^{\geq n} = 0)$. This upper bound will be called *active* upper bound of C. If C already has an active upper bound m, then m will be replaced by n only if n is better upper bound than m, i.e., $n \leq m$. If C has an active lower bound l and $n \leq l$ then the upper bound will set to equal l and as a consequence C will be set to zero.

2.4.8 SetLowerBound (for IsChainOrCochainComplex, IsInt)

▷ `SetLowerBound(C, n)` (operation)

Returns: Side effect

The command sets an lower bound n to the chain (resp. cochain) complex C. This means $C_{i\leq n} = 0 (C^{\leq n} = 0)$. This lower bound will be called *active* lower bound of C. If C already has an active lower bound m, then m will be replaced by n only if n is better lower bound than m, i.e., $n \geq m$. If C has an active upper bound u and $n \geq u$ then the lower bound will set to equal u and as a consequence C will be set to zero.

2.4.9 HasActiveUpperBound (for IsChainOrCochainComplex)

▷ `HasActiveUpperBound(C)` (operation)

Returns: true or false

The input is chain or cochain complex. The output is `true` if an upper bound has been set to C and `false` otherwise.

2.4.10 HasActiveLowerBound (for IsChainOrCochainComplex)

▷ `HasActiveLowerBound(C)` (operation)

Returns: true or false

The input is chain or cochain complex. The output is `true` if a lower bound has been set to C and `false` otherwise.

2.4.11 ActiveUpperBound (for IsChainOrCochainComplex)

▷ `ActiveUpperBound(C)` (operation)

Returns: an integer

The input is chain or cochain complex. The output is its active upper bound if such has been set to C. Otherwise we get error.

2.4.12 ActiveLowerBound (for IsChainOrCochainComplex)

▷ `ActiveLowerBound(C)` (operation)

Returns: an integer

The input is chain or cochain complex. The output is its active lower bound if such has been set to C. Otherwise we get error.

2.4.13 Display (for IsChainOrCochainComplex, IsInt, IsInt)

▷ `Display(C, m, n)` (operation)

Returns: nothing

The input is chain or cochain complex C and two integers m and n. The command displays all components of C between the indices m, n.

2.5 Truncations

2.5.1 GoodTruncationBelow (for IsChainComplex, IsInt)

▷ `GoodTruncationBelow(C, n)` (operation)

Returns: chain complex

Let $C_\bullet$ be chain complex. A good truncation of $C_\bullet$ below n is the chain complex $\tau_{\geq n}C_\bullet$ whose differentials are defined by

$$d_i^{\tau_{\geq n}C_\bullet} = \begin{cases} 0 : 0 \leftarrow 0 & \text{if} \quad i < n, \\ 0 : 0 \leftarrow Z_n & \text{if} \quad i = n, \\ \mathrm{KernelLift}(d_n^C, d_{n+1}^C) : Z_n \leftarrow C_{n+1} & \text{if} \quad i = n+1, \\ d_i^C : C_{i-1} \leftarrow C_i & \text{if} \quad i > n+1. \end{cases}$$

where Z_n is the cycle in index n. It can be shown that $H_i(\tau_{\geq n}C_\bullet) = 0$ for $i < n$ and $H_i(\tau_{\geq n}C_\bullet) = H_i(C_\bullet)$ for $i \geq n$.

$$\begin{tikzcd}
C_\bullet & \cdots & C_{n-1} \arrow[l] & C_n \arrow[l] & C_{n+1} \arrow[l] \arrow[dl] & C_{n+2} \arrow[l] & \cdots \arrow[l] \\
\tau_{\geq n}C_\bullet & \cdots & 0 \arrow[l] & Z_n \arrow[l] \arrow[u, hook] & & &
\end{tikzcd}$$

2.5.2 GoodTruncationAbove (for IsChainComplex, IsInt)

▷ `GoodTruncationAbove(C, n)` (operation)

Returns: chain complex

Let $C_\bullet$ be chain complex. A good truncation of $C_\bullet$ above n is the quotient chain complex $\tau_{<n}C_\bullet = C_\bullet/\tau_{\geq n}C_\bullet$. It can be shown that $H_i(\tau_{<n}C_\bullet) = 0$ for $i \geq n$ and $H_i(\tau_{<n}C_\bullet) = H_i(C_\bullet)$ for $i < n$.

2.5.3 GoodTruncationAbove (for IsCochainComplex, IsInt)

▷ `GoodTruncationAbove(C, n)` (operation)

Returns:

Let $C^\bullet$ be cochain complex. A good truncation of $C^\bullet$ above n is the cochain complex $\tau^{\leq n}C^\bullet$ whose differentials are defined by

$$d^i_{\tau^{\leq n}C^\bullet} = \begin{cases} 0 : 0 \rightarrow 0 & \text{if} \quad i > n, \\ 0 : Z^n \rightarrow 0 & \text{if} \quad i = n, \\ \mathrm{KernelLift}(d_C^n, d_C^{n-1}) : C^{n-1} \rightarrow Z^n & \text{if} \quad i = n-1, \\ d_C^i : C^i \rightarrow C^{i+1} & \text{if} \quad i < n-1. \end{cases}$$

where Z_n is the cycle in index n. It can be shown that $H^i(\tau^{\leq n}C^\bullet) = 0$ for $i > n$ and $H^i(\tau^{\leq n}C^\bullet) = H_i(C^\bullet)$ for $i \leq n$.

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-2} \arrow[r] & C^{n-1} \arrow[r] \arrow[dr] & C^n \arrow[r] & C^{n+1} \arrow[r] & \cdots & C^\bullet \\
 & & & Z^n \arrow[u, hook] \arrow[r] & 0 \arrow[r] & \cdots & \tau^{\leq n}C^\bullet
\end{tikzcd}$$

2.5.4 GoodTruncationBelow (for IsCochainComplex, IsInt)

▷ `GoodTruncationBelow(C, n)` (operation)

Returns: cochain complex

Let $C^\bullet$ be cochain complex. A good truncation of $C^\bullet$ above n is the quotient cochain complex $\tau^{>n}C^\bullet = C^\bullet/\tau^{\leq n}C^\bullet$. It can be shown that $H^i(\tau^{>n}C^\bullet) = 0$ for $i \leq n$ and $H^i(\tau^{>n}C^\bullet) = H_i(C^\bullet)$ for $i > n$.

2.5.5 BrutalTruncationBelow (for IsChainComplex, IsInt)

▷ `BrutalTruncationBelow(C, n)` (operation)

Returns: chain complex

Let $C_\bullet$ be chain complex. A brutal truncation of $C_\bullet$ below n is the chain complex $\sigma_{\geq n}C_\bullet$ where $(\sigma_{\geq n}C_\bullet)_i = C_i$ when $i \geq n$ and $(\sigma_{\geq n}C_\bullet)_i = 0$ otherwise.

2.5.6 BrutalTruncationAbove (for IsChainComplex, IsInt)

▷ `BrutalTruncationAbove(C, n)` (operation)

Returns: chain complex

Let $C_\bullet$ be chain complex. A brutal truncation of $C_\bullet$ above n is the chain quotient chain complex $\sigma_{<n}C_\bullet := C_\bullet/\sigma_{\geq n}C_\bullet$. Hence $(\sigma_{<n}C_\bullet)_i = C_i$ when $i < n$ and $(\sigma_{<n}C_\bullet)_i = 0$ otherwise.

2.5.7 BrutalTruncationAbove (for IsCochainComplex, IsInt)

▷ `BrutalTruncationAbove(C, n)` (operation)

Returns: chain complex

Let $C^\bullet$ be cochain complex. A brutal truncation of $C_\bullet$ above n is the cochain complex $\sigma^{\leq n}C^\bullet$ where $(\sigma^{\leq n}C^\bullet)_i = C_i$ when $i \leq n$ and $(\sigma^{\leq n}C^\bullet)_i = 0$ otherwise.

2.5.8 BrutalTruncationBelow (for IsCochainComplex, IsInt)

▷ `BrutalTruncationBelow(C, n)` (operation)

Returns: chain complex

Let $C^\bullet$ be cochain complex. A brutal truncation of $C^\bullet$ bellow n is the quotient cochain complex $\sigma^{>n}C^\bullet := C^\bullet/\sigma^{\leq n}C_\bullet$. Hence $(\sigma^{>n}C^\bullet)_i = C_i$ when $i > n$ and $(\sigma^{<n}C^\bullet)_i = 0$ otherwise.

2.6 Examples

Below we define the complex

$$\begin{array}{ccccccccc} \cdots & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{array}$$

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Q}^{1\times 1} \xrightarrow{\begin{pmatrix} 1 & 3 \end{pmatrix}} \mathbb{Q}^{1\times 2} \xrightarrow{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \mathbb{Q}^{1\times 1} \xrightarrow{\begin{pmatrix} 2 & 6 \end{pmatrix}} \mathbb{Q}^{1\times 2} \longrightarrow 0 \longrightarrow \cdots$$

Example

```
gap> A := VectorSpaceObject( 1, Q );
<A vector space object over Q of dimension 1>
gap> B := VectorSpaceObject( 2, Q );
<A vector space object over Q of dimension 2>
gap> f := VectorSpaceMorphism( A, HomalgMatrix( [ [ 1, 3 ] ], 1, 2, Q ), B );
<A morphism in Category of matrices over Q>
gap> g := VectorSpaceMorphism( B, HomalgMatrix( [ [ 0 ], [ 0 ] ], 2, 1, Q ), A );
<A morphism in Category of matrices over Q>
gap> C := CochainComplex( [ f, g, 2*f ], 3 );
<A bounded object in cochain complexes category over category of matrices over Q
with active lower bound 2 and active upper bound 7.>
gap> ActiveUpperBound( C );
7
gap> ActiveLowerBound( C );
2
gap> C[ 1 ];
<A vector space object over Q of dimension 0>
gap> C[ 3 ];
<A vector space object over Q of dimension 1>
gap> C^3;
<A morphism in Category of matrices over Q>
gap> C^3 = f;
true
gap> Display( CertainCycle( C, 4 ) );
[ [  1,  0 ],
  [  0,  1 ] ]

A split monomorphism in Category of matrices over Q
gap> diffs := Differentials( C );
<An infinite list>
gap> diffs[ 1 ];
<A zero, isomorphism in Category of matrices over Q>
gap> diffs[ 10000 ];
<A zero, isomorphism in Category of matrices over Q>
gap> objs := Objects( C );
<An infinite list>
gap> DefectOfExactness( C, 4 );
<A vector space object over Q of dimension 1>
gap> DefectOfExactness( C, 3 );
<A vector space object over Q of dimension 0>
gap> IsExactInIndex( C, 4 );
false
gap> IsExactInIndex( C, 3 );
true
gap> C;
```

```
<A not cyclic, bounded object in cochain complexes category over category of
matrices over Q with active lower bound 2 and active upper bound 7.>
gap> P := CochainComplex( matrix_category, diffs );
<An object in Cochain complexes category over category of matrices over Q>
gap> SetUpperBound( P, 15 );
gap> P;
<A bounded from above object in cochain complexes category over category of
matrices over Q with active upper bound 15.>
gap> SetUpperBound( P, 20 );
gap> P;
<A bounded from above object in cochain complexes category over category of
matrices over Q with active upper bound 15.>
gap> ActiveUpperBound( P );
15
gap> SetUpperBound( P, 7 );
gap> P;
<A bounded from above object in cochain complexes category over category of
matrices over Q with active upper bound 7.>
gap> ActiveUpperBound( P );
7
```

Chapter 3

Complexes morphisms

3.1 Categories and filters

3.1.1 IsChainOrCochainMorphism (for IsCapCategoryMorphism)

▷ `IsChainOrCochainMorphism(`*phi*`)` (filter)
▷ `IsBoundedBelowChainOrCochainMorphism(`*phi*`)` (filter)
▷ `IsBoundedAboveChainOrCochainMorphism(`*phi*`)` (filter)
▷ `IsBoundedChainOrCochainMorphism(`*phi*`)` (filter)
▷ `IsChainMorphism(`*phi*`)` (filter)
▷ `IsBoundedBelowChainMorphism(`*phi*`)` (filter)
▷ `IsBoundedAboveChainMorphism(`*phi*`)` (filter)
▷ `IsBoundedChainMorphism(`*phi*`)` (filter)
▷ `IsCochainMorphism(`*phi*`)` (filter)
▷ `IsBoundedBelowCochainMorphism(`*phi*`)` (filter)
▷ `IsBoundedAboveCochainMorphism(`*phi*`)` (filter)
▷ `IsBoundedCochainMorphism(`*phi*`)` (filter)

Returns: `true` or `false`

bla bla

3.2 Creating chain and cochain morphisms

3.2.1 ChainMorphism (for IsChainComplex, IsChainComplex, IsZList)

▷ `ChainMorphism(C, D, l)` (operation)

Returns: a chain morphism

The input is two chain complexes C,D and an infinite list l. The output is the chain morphism $\phi : C \to D$ defined by $\phi_i := l[i]$.

3.2.2 ChainMorphism (for IsChainComplex, IsChainComplex, IsDenseList, IsInt)

▷ `ChainMorphism(C, D, l, k)` (operation)

Returns: a chain morphism

The input is two chain complexes C,D, dense list l and an integer k. The output is the chain morphism $\phi : C \to D$ such that $\phi_k = l[1]$, $\phi_{k+1} = l[2]$, etc.

3.2.3 ChainMorphism (for IsDenseList, IsInt, IsDenseList, IsInt, IsDenseList, IsInt)

▷ `ChainMorphism(c, m, d, n, l, k)` (operation)

Returns: a chain morphism

The output is the chain morphism $\phi : C \to D$, where $C_m = c[1], C_{m+1} = c[2]$, etc. $D_n = d[1], D_{n+1} = d[2]$, etc. and $\phi_k = l[1]$, $\phi_{k+1} = l[2]$, etc.

3.2.4 CochainMorphism (for IsCochainComplex, IsCochainComplex, IsZList)

▷ `CochainMorphism(C, D, l)` (operation)

Returns: a cochain morphism

The input is two cochain complexes C,D and an infinite list l. The output is the cochain morphism $\phi : C \to D$ defined by $\phi_i := l[i]$.

3.2.5 CochainMorphism (for IsCochainComplex, IsCochainComplex, IsDenseList, IsInt)

▷ `CochainMorphism(C, D, l, k)` (operation)

Returns: a chain morphism

The input is two cochain complexes C,D, dense list l and an integer k. The output is the cochain morphism $\phi : C \to D$ such that $\phi^k = l[1]$, $\phi^{k+1} = l[2]$, etc.

3.2.6 CochainMorphism (for IsDenseList, IsInt, IsDenseList, IsInt, IsDenseList, IsInt)

▷ `CochainMorphism(c, m, d, n, l, k)` (operation)

Returns: a cochain morphism

The output is the cochain morphism $\phi : C \to D$, where $C^m = c[1], C^{m+1} = c[2]$, etc. $D^n = d[1], D^{n+1} = d[2]$, etc. and $\phi^k = l[1]$, $\phi^{k+1} = l[2]$, etc.

3.3 Attributes

3.3.1 Morphisms (for IsChainOrCochainMorphism)

▷ `Morphisms(phi)` (attribute)

Returns: infinite list

The output is morphisms of the chain or cochain morphism as an infinite list.

3.3.2 MappingCone (for IsChainOrCochainMorphism)

▷ `MappingCone(phi)` (attribute)

Returns: complex

The input a chain (resp. cochain) morphism $\phi : C \to D$. The output is its mapping cone chain (resp. cochain) complex Cone(ϕ).

3.3.3 NaturalInjectionInMappingCone (for IsChainOrCochainMorphism)

▷ `NaturalInjectionInMappingCone(phi)` (attribute)

Returns: chain (resp. cochain) morphism

The input a chain (resp. cochain) morphism $\phi : C \to D$. The output is the natural injection $i : D \to \mathrm{Cone}\phi)$.

3.3.4 NaturalProjectionFromMappingCone (for IsChainOrCochainMorphism)

▷ `NaturalProjectionFromMappingCone(phi)` (attribute)

Returns: chain (resp. cochain) morphism

The input a chain (resp. cochain) morphism $\phi : C \to D$. The output is the natural projection $\pi : \mathrm{Cone}(\phi) \to C[u]$ where $u = -1$ if ϕ is chain morphism and $u = 1$ if ϕ is cochain morphism.

3.4 Properties

3.4.1 IsQuasiIsomorphism_ (for IsChainOrCochainMorphism)

▷ `IsQuasiIsomorphism_(phi)` (property)

Returns: `true` or `false`

The input a chain (resp. cochain) morphism $\phi : C \to D$. The output is *true* if ϕ is quasi-isomorphism and *false* otherwise. If ϕ is not bounded an error is raised.

3.5 Operations

3.5.1 SetUpperBound (for IsChainOrCochainMorphism, IsInt)

▷ `SetUpperBound(phi, n)` (operation)

Returns: a side effect

The command sets an upper bound to the morphism ϕ. An upper bound of ϕ is an integer u with $\phi_{i \geq u} = 0$. The integer u will be called *active* upper bound of ϕ. If ϕ already has an active upper bound, say u', then u' will be replaced by u only if $u \leq u'$.

3.5.2 SetLowerBound (for IsChainOrCochainMorphism, IsInt)

▷ `SetLowerBound(phi, n)` (operation)

Returns: a side effect

The command sets an lower bound to the morphism ϕ. A lower bound of ϕ is an integer l with $\phi_{i \leq l} = 0$. The integer l will be called *active* lower bound of ϕ. If ϕ already has an active lower bound, say l', then l' will be replaced by l only if $l \geq l'$.

3.5.3 HasActiveUpperBound (for IsChainOrCochainMorphism)

▷ `HasActiveUpperBound(phi)` (operation)

Returns: true or false

The input is chain or cochain morphism ϕ. The output is `true` if an upper bound has been set to ϕ and `false` otherwise.

3.5.4 HasActiveLowerBound (for IsChainOrCochainMorphism)

▷ `HasActiveLowerBound(phi)` (operation)

Returns: true or false

The input is chain or cochain morphism ϕ. The output is `true` if a lower bound has been set to ϕ and `false` otherwise.

3.5.5 ActiveUpperBound (for IsChainOrCochainMorphism)

▷ `ActiveUpperBound(phi)` (operation)

Returns: an integer

The input is chain or cochain morphism. The output is its active upper bound if such has been set to ϕ. Otherwise we get error.

3.5.6 ActiveLowerBound (for IsChainOrCochainMorphism)

▷ `ActiveLowerBound(phi)` (operation)

Returns: an integer

The input is chain or cochain morphism. The output is its active lower bound if such has been set to ϕ. Otherwise we get error.

3.5.7 \[\] (for IsChainOrCochainMorphism, IsInt)

▷ `\[\](phi, n)` (operation)

Returns: an integer

The input is chain (resp. cochain) morphism and an integer n. The output is the component of ϕ in index n, i.e., ϕ_n(resp. ϕ^n).

3.5.8 Display (for IsChainOrCochainMorphism, IsInt, IsInt)

▷ `Display(phi, m, n)` (operation)

Returns:

The command displays the components of the morphism between m and n.

3.6 Examples

Let us define a morphism

$$
\begin{tikzcd}
\cdots & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Q}^{1\times 1} \arrow[r, "\begin{pmatrix} 1 & 3 \end{pmatrix}"] & \mathbb{Q}^{1\times 2} \arrow[r, "\begin{pmatrix} 0 \\ 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ 0 \end{pmatrix}"'] & \mathbb{Q}^{1\times 1} \arrow[r, "\begin{pmatrix} 2 & 6 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 10 \end{pmatrix}"] & \mathbb{Q}^{1\times 2} \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Q}^{1\times 1} \arrow[r, "\begin{pmatrix} 5 \end{pmatrix}"'] & \mathbb{Q}^{1\times 1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

Example
```
gap> h := VectorSpaceMorphism( A, HomalgMatrix( [ [ 5 ] ], 1, 1, Q ), A );
<A morphism in Category of matrices over Q>
gap> phi4 := g;
<A morphism in Category of matrices over Q>
gap> phi5 := 2*h;
<A morphism in Category of matrices over Q>
gap> D := CochainComplex( [ h ], 4 );
<A bounded object in cochain complexes category over category of matrices
over Q with active lower bound 3 and active upper bound 6.>
gap> phi := CochainMorphism( C, D, [ phi4, phi5 ], 4 );
<A bounded morphism in cochain complexes category over category of matrices
 over Q with active lower bound 3 and active upper bound 6.>
gap> Display( phi[ 5 ] );
[ [ 10 ] ]

A morphism in Category of matrices over Q
gap> ActiveLowerBound( phi );
3
gap> IsZeroForMorphisms( phi );
false
gap> IsExact( D );
true
gap> IsExact( C );
false
```

Now lets define the previous morphism using the command `CochainMorphism(c, m, d, n, l, k)`.

Example
```
gap> psi := CochainMorphism( [ f, g, 2*f ], 3, [ h ], 4, [ phi4, phi5 ], 4 );
<A bounded morphism in cochain complexes category over category of matrices
over Q with active lower bound 3 and active upper bound 6.>
```

In some cases the morphism can change its lower bound when we apply the function `IsZeroForMorphisms` .

Example

```
gap> IsZeroForMorphisms( psi );
false
gap> psi;
<A bounded morphism in cochain complexes category over category of matrices
over Q with active lower bound 4 and active upper bound 6.>
```

In the following we compute the mapping cone of ψ and its natural injection and projection.

$$
\begin{tikzcd}
\cdots & 2 & 3 & 4 & 5 & 6 & \cdots \\
C\colon \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] & \mathbb{Q}^{1\times 1} \arrow[r, "\begin{pmatrix} 1 & 3 \end{pmatrix}"] & \mathbb{Q}^{1\times 2} \arrow[r, "\begin{pmatrix} 0 \\ 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ 0 \end{pmatrix}"'] & \mathbb{Q}^{1\times 1} \arrow[r, "\begin{pmatrix} 2 & 6 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 10 \end{pmatrix}"] & \mathbb{Q}^{1\times 2} \arrow[r] & \cdots \\
D\colon \arrow[r] \arrow[d, "i"'] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Q}^{1\times 1} \arrow[r, "\begin{pmatrix} 5 \end{pmatrix}"'] \arrow[d, "\begin{pmatrix} 0 & 1 \end{pmatrix}"'] & \mathbb{Q}^{1\times 1} \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}"] & 0 \arrow[r] & \cdots \\
Cone(\psi)\colon \arrow[r] \arrow[d, "p"'] & \mathbb{Q}^{1\times 1} \arrow[r, "\begin{pmatrix} -1 & -3 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 \end{pmatrix}"'] & \mathbb{Q}^{1\times 2} \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1,0 \\ 0,1 \end{pmatrix}"'] & \mathbb{Q}^{1\times 2} \arrow[r, "\begin{pmatrix} -2 & -6 & -10 \\ 0 & 0 & 5 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & \mathbb{Q}^{1\times 3} \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}"] & 0 \arrow[r] & \cdots \\
C[1]\colon \arrow[r] & \mathbb{Q}^{1\times 1} \arrow[r, "\begin{pmatrix} -1 & -3 \end{pmatrix}"] & \mathbb{Q}^{1\times 2} \arrow[r, "\begin{pmatrix} 0 \\ 0 \end{pmatrix}"] & \mathbb{Q}^{1\times 1} \arrow[r, "\begin{pmatrix} -2 & -6 \end{pmatrix}"] & \mathbb{Q}^{1\times 2} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

Example

```
gap> cone := MappingCone( psi );
<A bounded object in cochain complexes category over category of matrices over
Q with active lower bound 1 and active upper bound 6.>
gap> cone^4;
<A morphism in Category of matrices over Q>
gap> Display( cone^4 );
[ [   -2,   -6,  -10 ],
  [    0,    0,    5 ] ]

A morphism in Category of matrices over Q
gap> i := NaturalInjectionInMappingCone( psi );
<A bounded morphism in cochain complexes category over category of matrices over
Q with active lower bound 3 and active upper bound 6.>
gap> p := NaturalProjectionFromMappingCone( psi );
<A bounded morphism in cochain complexes category over category of matrices over
Q with active lower bound 1 and active upper bound 6.>
```

Chapter 4

Functors

4.1 Basic functors for complex categories.

4.1.1 HomologyFunctor (for IsChainComplexCategory, IsCapCategory, IsInt)

▷ `HomologyFunctor(`*`Ch(A), A, n`*`)` (operation)

▷ `CohomologyFunctor(`*`Coch(A), A, n`*`)` (operation)

Returns: a functor

The first argument in the input must be the chain (resp. cochain) complex category of an abelian category A, the second argument is `A` and the third argument is an integer `n`. The output is the n'th homology (resp. cohomology) functor : $\mathrm{Ch}(A) \to A$.

4.1.2 ShiftFunctor (for IsChainOrCochainComplexCategory, IsInt)

▷ `ShiftFunctor(`*`Comp(A), n`*`)` (operation)

Returns: a functor

The inputs are complex category $\mathrm{Comp}(A)$ and an integer. The output is a the endofunctor $T[n]$ that sends any complex C to $C[n]$ and any complex morphism $\phi : C \to D$ to $\phi[n] : C[n] \to D[n]$. The shift chain complex $C[n]$ of a chain complex C is defined by $C[n]_i = C_{n+i}, d_i^{C[n]} = (-1)^n d_{n+i}^C$ and the same for chain complex morphisms, i.e., $\phi[n]_i = \phi_{n+i}$. The same holds for cochain complexes and morphisms.

4.1.3 UnsignedShiftFunctor (for IsChainOrCochainComplexCategory, IsInt)

▷ `UnsignedShiftFunctor(`*`Comp(A), n`*`)` (operation)

Returns: a functor

The inputs are complex category $\mathrm{Comp}(A)$ and an integer. The output is a the endofunctor $T[n]$ that sends any complex C to $C[n]$ and any complex morphism $\phi : C \to D$ to $\phi[n] : C[n] \to D[n]$. The shift chain complex $C[n]$ of a chain complex C is defined by $C[n]_i = C_{n+i}, d_i^{C[n]} = d_{n+i}^C$ and the same for chain complex morphisms, i.e., $\phi[n]_i = \phi_{n+i}$. The same holds for cochain complexes and morphisms.

4.1.4 ChainToCochainComplexFunctor (for IsCapCategory)

▷ `ChainToCochainComplexFunctor(`*`A`*`)` (operation)

Returns: a functor

The input is a category A. The output is the functor $F : \mathrm{Ch(A)} \to \mathrm{Coch(A)}$ defined by $C_\bullet \mapsto C^\bullet$ for any for any chain complex $C_\bullet \in \mathrm{Ch}(A)$ and by $\phi_\bullet \mapsto \phi^\bullet$ for any map ϕ where $C^i = C_{-i}$ and $\phi^i = \phi_{-i}$.

4.1.5 CochainToChainComplexFunctor (for IsCapCategory)

▷ `CochainToChainComplexFunctor(A)` (operation)

Returns: a functor

The input is a category A. The output is the functor $F : \mathrm{Coch(A)} \to \mathrm{Ch(A)}$ defined by $C^\bullet \mapsto C_\bullet$ for any cochain complex $C^\bullet \in \mathrm{Coch}(A)$ and by $\phi^\bullet \mapsto \phi_\bullet$ for any map ϕ where $C_i = C^{-i}$ and $\phi_i = \phi^{-i}$.

4.1.6 ExtendFunctorToChainComplexCategoryFunctor (for IsCapFunctor)

▷ `ExtendFunctorToChainComplexCategoryFunctor(F)` (operation)

Returns: a functor

The input is a functor $F : A \to B$. The output is its extention functor $F : \mathrm{Ch}(A) \to \mathrm{Ch}(B)$.

4.1.7 ExtendFunctorToCochainComplexCategoryFunctor (for IsCapFunctor)

▷ `ExtendFunctorToCochainComplexCategoryFunctor(F)` (operation)

Returns: a functor

The input is a functor $F : A \to B$. The output is its extention functor $F : \mathrm{Coch}(A) \to \mathrm{Coch}(B)$.

4.2 Examples

The theory tells us that the composition $i\psi$ is null-homotopic. That implies that the morphisms induced on cohomologies are all zero.

Example
```
gap> i_o_psi := PreCompose( psi, i );
<A bounded morphism in cochain complexes category over category of matrices
over Q with active lower bound 4 and active upper bound 6.>
gap> H5 := CohomologyFunctor( cochain_cat, matrix_category, 5 );
5-th cohomology functor in category of matrices over Q
gap> IsZeroForMorphisms( ApplyFunctor( H5, i_o_psi ) );
true
```

Next we define a functor $\mathbf{F} : \mathrm{Vec}_{\mathbb{Q}} \to \mathrm{Vec}_{\mathbb{Q}}$ that maps every $\mathbb{Q}$-vector space A to $A \oplus A$ and every morphism $f : A \to B$ to $f \oplus f$. Then we extend it to the functor $\mathbf{Coch}_{\mathbf{F}} : \mathrm{Coch}(\mathrm{Vec}_{\mathbb{Q}}) \to \mathrm{Coch}(\mathrm{Vec}_{\mathbb{Q}})$ that maps each cochain complex C to the cochain complex we get after applying the functor $\mathbf{F}$ on every object and differential in C and maps any morphism $\phi : C \to D$ to the morphism we get after applying the functor $\mathbf{F}$ on every object, differential or morphism in C, D and ϕ.

Example
```
gap> F := CapFunctor( "double functor", matrix_category, matrix_category );
double functor
gap> u := function( obj ) return DirectSum( [ obj, obj ] ); end;;
gap> AddObjectFunction( F, u );
gap> v := function( s, mor, r ) return DirectSumFunctorial( [ mor, mor ] ); end;;
gap> AddMorphismFunction( F, v );
gap> Display( f );
[ [  1,  3 ] ]
```

```
A morphism in Category of matrices over Q
gap> Display( ApplyFunctor( F, f ) );
[ [  1,  3,  0,  0 ],
  [  0,  0,  1,  3 ] ]

A morphism in Category of matrices over Q
gap> Coch_F := ExtendFunctorToCochainComplexCategoryFunctor( F );
Extended version of double functor from cochain complexes category over category
of matrices over Q to cochain complexes category over category of matrices over Q
gap> psi;
<A bounded morphism in cochain complexes category over category of matrices
over Q with active lower bound 4 and active upper bound 6.>
gap> Coch_F_psi := ApplyFunctor( Coch_F, psi );
<A bounded morphism in cochain complexes category over category of matrices
over Q with active lower bound 4 and active upper bound 6.>
gap> Display( psi[ 5 ] );
[ [  10 ] ]

A morphism in Category of matrices over Q
gap> Display( Coch_F_psi[ 5 ] );
[ [  10,   0 ],
  [   0,  10 ] ]

A morphism in Category of matrices over Q
```

Next we will compute the shift $C[3]$. As we know the standard shift functor may change the sign of the differentials since $d^i_{C[n]} = (-1)^n d^{i+n}_C$. Hence if we don't want the signs to be changed we may use the unsigned shift functor.

Example

```
gap> T := ShiftFunctor( cochain_cat, 3 );
Shift (3 times to the left) functor in cochain complexes category over category
 of matrices over Q
gap> C;
<A not cyclic, bounded object in cochain complexes category over category of
matrices over Q with active lower bound 2 and active upper bound 7.>
gap> C_3 := ApplyFunctor( T, C );
<A not cyclic, bounded object in cochain complexes category over category of
matrices over Q with active lower bound -1 and active upper bound 4.>
gap> Display( C^3 );
[ [  1,  3 ] ]

A morphism in Category of matrices over Q
gap> Display( C_3^0 );
[ [  -1,  -3 ] ]

A morphism in Category of matrices over Q
gap> S := UnsignedShiftFunctor( cochain_cat, 3 );
Unsigned shift (3 times to the left) functor in cochain complexes category over
category of matrices over Q
gap> C_3_unsigned := ApplyFunctor( S, C );
<A bounded object in cochain complexes category over category of matrices over
Q with active lower bound -1 and active upper bound 4.>
```

```
gap> Display( C_3_unsigned^0 );
[ [  1,  3 ] ]

A morphism in Category of matrices over Q
```

Chapter 5

Resolutions

5.1 Definitions

Let A be an abelian category and is $C^\bullet$ is a complex in $\mathrm{Ch}^\bullet(A)$. A *projective resolution* of $C^\bullet$ is a complex $P^\bullet$ together with cochain morphism $\alpha : P^\bullet \to C^\bullet$ of complexes such that

- We have $P^n = 0$ for $n \gg 0$, i.e., $P^\bullet$ is bounded above.
- Each P^n is projective object of A.
- The morphism α is quasi-isomorphism.

5.2 Computing resolutions

5.2.1 QuasiIsomorphismFromProjectiveResolution (for IsBoundedAboveCochainComplex)

▷ `QuasiIsomorphismFromProjectiveResolution(C)` (attribute)
▷ `QuasiIsomorphismFromProjectiveResolution(C)` (attribute)

Returns: a (co)chain epimorphism

The input is an above bounded cochain complex $C^\bullet$. The output is a quasi-isomorphism $q : P^\bullet \to C^\bullet$ such that $P^\bullet$ is upper bounded and all its objects are projective in the underlying abelian category. In the second command the input is a below bounded chain complex $C_\bullet$. The output is a quasi-isomorphism $q : P_\bullet \to C_\bullet$ such that $P_\bullet$ is lower bounded and all its objects are projective in the underlying abelian category.

5.2.2 QuasiIsomorphismInInjectiveResolution (for IsBoundedBelowCochainComplex)

▷ `QuasiIsomorphismInInjectiveResolution(C)` (attribute)
▷ `QuasiIsomorphismInInjectiveResolution(C)` (attribute)

Returns: a (co)chain epimorphism

The input is a below bounded cochain complex $C^\bullet$. The output is a quasi-isomorphism $q : C^\bullet \to I^\bullet$ such that $I^\bullet$ is below bounded and all its objects are injective in the underlying abelian category. In the second command the input is an above bounded chain complex $C_\bullet$. The output is a quasi-isomorphism

$q : C_\bullet \to I_\bullet$ such that $I_\bullet$ is below bounded and all its objects are injective in the underlying abelian category.

Index