Philosophy of Mathematics

Herlock(SeyedAbolfazl) Rahimi

Spring 2025

Contents

Abstract

This paper explores the landscape of mathematical structuralism as a foundational and philosophical perspective. Structuralism redefines mathematical ontology by prioritizing interrelations over intrinsic identity [5, 6], offering a unified account of mathematical objects, semantics, and applications. We analyze the three canonical variants—*ante rem*, *in re*, and *modal* structuralism—drawing from the work of Shapiro [5], Resnik [7], and Hellman [3], and evaluate their metaphysical, epistemological, and semantic commitments. Further, we examine lesser-known forms such as relativist, abstractionist, universalist, and formalist structuralism, integrating insights from Reck and Price's comprehensive taxonomy [8]. Structuralism's strengths include solutions to the identity problem [9], explanatory power for applicability, and alignment with modern mathematics [4]. Yet it faces open challenges concerning the epistemology of abstract structures, semantic reformulations, and the ontological status of uninstantiated entities. Through rigorous comparison and philosophical critique, the paper situates structuralism not as a monolith but as a rich, evolving framework in the philosophy of mathematics.

1 Introduction

The philosophy of mathematics has long been concerned with foundational questions about the identity, existence, and applicability of mathematical objects. What is the number 3? Does it exist independently of our theories or representations? And how can abstract entities such as sets, functions, or points figure meaningfully in empirical science? In response to such questions, *mathematical structuralism* has emerged as a dominant framework that reconceives mathematics not as the study of particular objects but of *structures*—configurations of positions interconnected by formal relations [1, pp. 3–5].

According to structuralism, mathematical objects derive their identity and epistemic role from their place within a structure, rather than from any intrinsic nature [2, Ch. 4, pp. 95–97]. The number 2, for instance, is not a specific set or entity, but a position within the structure of the natural numbers [6, §1]. This reorientation, from objects to positions within structures, parallels how a square on a chessboard gains meaning only from its location and potential interactions, or how a role in a bureaucracy is defined by its relational function.

The structuralist perspective is motivated not only by conceptual clarity but by historical and methodological shifts. In the nineteenth century, Richard Dedekind characterized the natural numbers as any system satisfying a certain set of relational properties, emphasizing structure over implementation [4, pp. 31–39]. David Hilbert formalized this insight in his *Grundlagen der Geometrie* (1899), where he defined geometric terms like "point" and "line" entirely through their relations, famously suggesting that one could replace them with "tables," "chairs," or "beer mugs" as long as the axioms are preserved [4, pp. 59–67]. Emmy Noether then pushed this shift further in abstract algebra, focusing not on elements but on invariants under homomorphisms [4, pp. 183–187].

The structuralist vision found its clearest articulation in the Bourbaki group's mid-twentieth century project, which explicitly defined mathematics as the study of structures [4, pp. 190–195]. This view was reinforced by developments in model theory, which emphasized structures and their isomorphisms over foundational substance, and in category theory—introduced by Eilenberg and Mac Lane—which redefined objects in terms of morphisms and functorial relations [8, p. 345]. In category theory, identity is often replaced with equivalence, and structures are studied via their external relations rather than internal constitution.

Philosophically, structuralism offers elegant resolutions to long-standing problems. One is Benacerraf's dilemma [9], which highlights the conflict between the semantics of mathematical language and the identity conditions of abstracta: multiple, non-equivalent set-theoretic constructions can all represent "3," raising questions about which object "3" really is. Structuralism sidesteps this problem by treating "3" not as an object, but as a position in a structure, invariant under isomorphism [2, pp. 100–104].

It also explains the striking applicability of mathematics to the physical world. Structural patterns—such as symmetry groups, vector spaces, and topologies—are instantiated in physical systems, and mathematics captures these invariants independently of the materials involved [1, pp. 45–48]. This account, which underlies the mathematical modeling of physical laws, depends not on the substance of entities, but on the structural relations they instantiate.

Despite its unifying appeal, structuralism is not a monolithic view [8, p. 347]. Philosophers have articulated multiple versions of it, each with different metaphysical and semantic commitments. In *ante rem structuralism*, structures and their positions are treated as abstract entities existing independently of any instantiations [2, pp. 95–104]. In *in re structuralism*, structures exist only insofar as they are instantiated in particular systems [2, pp. 105–110]. *Modal structuralism*, as developed by Hellman, avoids any commitment to mathematical objects by interpreting statements modally: what is true is what must hold in all possible structures satisfying a given theory [3, Ch. 2, pp. 18–29].

Beyond these three canonical views, additional structuralist approaches have been proposed. *Relativist structuralism*, rooted in model-theoretic semantics, understands structures as model-relative frameworks without a global ontology [8, p. 345]. *Structural abstractionism*, inspired by Frege and neo-Fregeans, defines structures as abstractions over equivalence classes of systems [8, p. 342]. *Universalist structuralism*, associated with category theory, treats mathematics as the study of the entire universe of morphism-linked structures. *Formalist structuralism*, following Hilbert and Bourbaki, reduces mathematics to the derivation of theorems from axioms in purely syntactic terms [5, pp. 259–261].

Each variant addresses foundational tensions differently: the problem of identity in abstracta [9]; the semantics of reference in mathematics [5, pp. 267–272]; the epistemology of abstract entities [5, pp. 275–283]; and the alignment between foundational philosophy and contemporary mathematical practice [8, pp. 345–348].

This paper presents a systematic and critical survey of these structuralist views. We trace their historical development, compare their metaphysical assumptions and semantic frameworks, and assess their philosophical strengths and vulnerabilities. Our goal is not merely to catalogue structuralist positions but to argue that structuralism is best understood as a family of views—diverse, evolving, and deeply embedded in the foundations of modern mathematics.

2 Historical Foundations: Dedekind, Hilbert, and Noether

The intellectual foundations of mathematical structuralism were laid well before its explicit philosophical articulation in the late twentieth century. Three figures stand out for their pivotal roles in this transformation: Richard Dedekind, David Hilbert, and Emmy Noether. Their respective contributions—Dedekind's logical analysis of number systems, Hilbert's axiomatic formalism, and Noether's abstract algebraic methodology—together inaugurated the structuralist turn in mathematics.

2.1 Dedekind: Numbers as Structured Systems

Dedekind's conception of number was revolutionary in its focus on relational properties rather than intrinsic features. In his seminal work *Was sind und was sollen die Zahlen?* (1888), Dedekind defined the natural numbers via systems possessing a specified structure: a distinguished element (interpreted as 0 or 1), and a successor function satisfying particular logical conditions. Crucially, Dedekind insisted that what matters are not the individual elements per se, but the structure they collectively instantiate. This move toward characterizing mathematical objects by their place within a relational system is a paradigmatic precursor to modern structuralism [4, pp. 31–35].

Dedekind's notion of "simply infinite systems" formalized this insight. He showed that any system with the required properties could serve as a model of the natural numbers, emphasizing that the identity of the number 2, for example, is determined not by any internal feature but by its role in such a system. In this way, Dedekind offered a methodologically precise, logic-based, and inherently structural approach to number theory [4, pp. 39–42].

2.2 Hilbert: Axiomatic Method and Structural Form

Hilbert's work pushed Dedekind's vision further by explicitly divorcing the meaning of mathematical terms from any contentful or intuitive interpretation. In his *Grundlagen der Geometrie* (1899), Hilbert treated concepts such as "point", "line", and "plane" as undefined terms whose meanings are fixed entirely by the axioms that govern their relations. He famously claimed that one could just as well replace these terms with "tables", "chairs", and "beer mugs", as long as the axioms remain satisfied [4, pp. 59–60].

This radical formalism embodied a structuralist ethos avant la lettre. Hilbert's approach treated mathematical theories as frameworks—relational systems whose components gain meaning through their role in a network of axioms. Moreover, Hilbert advanced the view that the validity of a mathematical theory lies in its internal consistency and deductive structure, rather than in any extrinsic or ontological commitment [4, pp. 64–67].

2.3 Noether: Structural Invariance in Abstract Algebra

Emmy Noether's contribution was transformative in consolidating structural thinking in algebra. She moved away from a focus on concrete calculations and toward the investigation of invariant properties under homomorphisms and isomorphisms. In doing so, she helped redefine algebra as the study of abstract systems—groups, rings, fields—understood in terms of their structural relations rather than specific representations [4, pp. 183–186].

Noether's emphasis on mappings and invariant properties laid the groundwork for the modern category-theoretic perspective, where morphisms and structural interrelations become central. As Reck notes, her approach "shifted the primary mathematical interest from objects to structure-preserving transformations among objects," a shift that would deeply influence both the mathematics and the philosophy of structure [4, p. 187].

2.4 Toward Structuralism Proper

Together, Dedekind, Hilbert, and Noether articulated a vision of mathematics in which relational form supersedes ontological substance. Dedekind anticipated the idea that numbers are positions in a structure; Hilbert formalized the principle that mathematical meaning is axiomatic; and Noether transformed entire fields by focusing on structural invariance. Their collective legacy set the stage for the twentieth-century structuralist philosophies of mathematics, notably those of Resnik, Shapiro, Hellman, and others.

This historical trajectory also underpins the foundational shift toward category theory, model theory, and structural abstraction in modern mathematics. As Reck and Price observe, "what started as a methodological orientation gradually became a metaphysical and semantic thesis about the nature of mathematical objects and truths" [8, p. 346].

2.5 Bourbaki and the Codification of Structuralism

The structuralist outlook found its most explicit and influential formulation in the work of the Bourbaki group, a collective of French mathematicians writing under the pseudonym "Nicolas Bourbaki". Beginning in the 1930s and continuing through the twentieth century, Bourbaki sought to reconstruct all of mathematics on a unified foundation explicitly grounded in structures. They defined mathematics as "the study of structures," and organized their treatises accordingly: group theory, topology, set theory, and algebra were all presented not in terms of objects per se, but as systems governed by sets of structural axioms [4, p. 191].

The Bourbaki approach emphasized the identification of "mother structures" (structures génériques) from which other theories could be derived. For example, their treatment of algebra did not begin with numbers or operations, but with the general concept of a group, understood purely through its axioms and morphisms. As Reck and Schiemer recount, this methodological program was "rigorously formal, systematically structural, and strikingly abstract," setting a new standard for mathematical exposition and development [4, pp. 190–194].

Crucially, Bourbaki eschewed any appeal to intuition or spatial imagery. Geometry, for instance, was treated in terms of topological and algebraic structures. Even classical subjects were reframed: instead of studying "the" real numbers as a specific construction (e.g., Dedekind cuts or Cauchy sequences), they focused on the properties that make a system a complete ordered field, up to isomorphism. Thus, Bourbaki's foundational orientation reinforced a principle central to structuralism: the identity of mathematical entities lies in their role within a structure, not in any intrinsic nature or particular construction.

Bourbaki's project also popularized the rigorous use of set theory as a universal foundation—not in the naive sense of asserting that all mathematical objects are sets, but in the methodological sense that any mathematical structure could be encoded as a system of sets satisfying axioms. Yet even this reliance on sets was tempered by a structuralist viewpoint: sets were not the subject of mathematics, but a language for expressing structural relationships.

2.6 Category Theory and the Relational Turn

If the Bourbaki school marked the formal codification of structuralism in mathematical exposition, then the emergence of category theory in the 1940s signaled a conceptual transformation of the very notion of mathematical structure. Introduced by Samuel Eilenberg and Saunders Mac Lane [4, p. 195], category theory provided a radically new framework for conceptualizing mathematical entities—not as isolated systems defined by their elements, but as nodes within a web of interrelations mediated by structure-preserving morphisms.

In categorical language, the fundamental objects of study are no longer sets or the internal constituents of a system, but rather *categories*, consisting of objects and morphisms (arrows) that preserve relevant structure. Between these categories, one studies *functors*, which map objects and morphisms from one category to another while preserving composition and identity, and *natural transformations*, which provide higher-order coherence between functors. This apparatus foregrounds not the internal composition of mathematical objects, but the ways in which they interact and transform.

From a structuralist perspective, this shift is philosophically significant: identity becomes relational, not intrinsic. What it is to be a group, a space, or a sheaf, is no longer specified by an underlying set equipped with operations or relations, but by the object's place in a network of morphisms within a category. In the category **Grp**, for instance, the structure of a group is entirely characterized by the group homomorphisms it admits to and from other groups. The category **Top**, of topological spaces and continuous maps, functions analogously. This morphism-based conception of structure aligns deeply with the structuralist emphasis on interrelations over internal essence [8, pp. 345–346].

Central to this transformation is the notion of *universal properties*, which define mathematical objects not in terms of their components but in terms of how they relate uniquely to all other objects. A product, limit, or initial object is specified not by its elements but by its interactional role in a diagram. Thus, objects become individuated by their relational roles—fulfilling the structuralist vision perhaps more completely than any prior formalism.

However, this heightened abstraction brings with it new philosophical tensions. While category theory embodies the structuralist spirit in its purest form, it also challenges some traditional foundational assumptions. For instance, in category theory, set-theoretic membership is replaced by morphisms, and the very notion of identity is weakened from equality to isomorphism or even equivalence. This undermines the classical logic of identity and forces a reevaluation of the metaphysics of mathematical objects [5, pp. 265–266]. In higher category theory, where morphisms between morphisms are studied, the notion of identity becomes even more nuanced—suggesting a kind of structural fluidity that defies simple foundational capture.

Moreover, categorical approaches invert the typical foundational hierarchy. In standard set-theoretic foundations, logic precedes and governs mathematical theory. But in categorical logic and topos theory, logical structures can be internal to categories—meaning that logic itself becomes a kind of structure to be studied, rather than a fixed backdrop [8, p. 348]. This challenges the traditional view that logic undergirds mathematics, instead suggesting that logic is emergent from the structural relationships between objects.

Philosophically, category theory thus brings structuralism to its apex: it enacts a view where mathematical content is entirely relational, and where identity, logic, and even existence are defined structurally. Yet, it also poses a danger of collapsing into pure formalism or hyper-abstract algebraism, potentially losing contact with the concrete intuitions and semantic clarity that many structuralists, especially those of the *ante rem* variety, have sought to preserve.

As Reck and Price observe, category theory reflects "a new level of abstraction in which the relationships between theories become more important than the content of any single theory" [8, p. 346]. It realizes the structuralist ideal that mathematical objects are defined by their structural roles, not by intrinsic nature. But it also raises questions about how much abstraction philosophy of mathematics can accommodate before structure ceases to be anchored in intelligible foundations.

In sum, category theory both fulfills and transcends traditional structuralism. It offers a rigorous mathematical embodiment of the idea that *structure is all there is*, but it also destabilizes older philosophical frameworks by relativizing logic, weakening identity, and prioritizing transformation over substance. As such, it demands not only admiration but critical philosophical scrutiny.

2.7 Historical Continuity and Conceptual Shift

Together, the Bourbaki program and the rise of category theory mark the full emergence of structuralism as both a methodological and philosophical orientation. The former made structural methods explicit in the organization of mathematics, while the latter transformed the very language in which mathematics is formulated. As Reck and Schiemer argue, these developments do not merely represent a change in mathematical practice; they signal a shift in how mathematicians conceive the nature of mathematical existence and identity [4, pp. 195–198].

This historical evolution thus supports the claim that structuralism is not an extrinsic philosophical interpretation imposed on mathematics, but a reflective articulation of deep tendencies within the mathematical tradition itself. From Dedekind's relational conception of number, through Hilbert's formal axioms, to Noether's invariant mappings and Bourbaki's structural taxonomy, the path to category theory reveals an increasing emphasis on abstraction, relationality, and form over substance—a trajectory that structuralist philosophers have sought to codify and clarify. In what follows I try to discuss most dominant variants of Structuralism.

3 Ante Rem Structuralism: Structures as Abstract Entities

Ante rem structuralism constitutes the most metaphysically committed variant of structuralist philosophy. Developed and defended prominently by Stewart Shapiro and Michael Resnik, this position holds that mathematical structures exist independently of their instantiations, and that positions within structures—such as the natural number 2—are ontologically real objects [5, pp. 263–270], [6, pp. 1–3]. Unlike other accounts that define mathematical objects through their instantiations, *ante rem* structuralism posits a realm of abstract, free-standing structures analogous to Platonic universals.

3.1 Ontology: Structures Exist Independently of Systems

According to Shapiro, a structure is not a property of a system, but a genuine abstract entity that systems instantiate. In this view, a system such as the sequence of von Neumann ordinals is merely one instantiation of the natural number structure. The structure itself exists independently and prior to any such representation [5, pp. 263–264].

Shapiro articulates this view through the analogy of offices and office-holders: just as the role of "vice president" exists independently of any person who fills it, so too do structural positions such as "the successor of 2" exist independently of any particular object playing that role in a system [5, p. 265]. Resnik similarly affirms that "we have only structures" in mathematics, and "mathematical objects are structureless points or positions in structures" [5, p. 259], originally cited from Resnik [7].

3.2 Semantics: Literal Reference and Truth-Value Realism

One of the key virtues of *ante rem* structuralism is that it allows for a face-value semantics for mathematical language. On this view, singular terms in mathematical statements refer directly to structural positions, and quantifiers range over these places. Hence, statements such as $3 \times 8 = 24$ are literally about objects—the places "3", "8", and "24"—in the abstract natural number structure [6, p. 6], [5, pp. 267–269].

This yields a form of truth-value realism: every well-formed mathematical sentence is determinately true or false, independently of empirical or linguistic context. Such semantic transparency is one of the strongest attractions of the *ante rem* position, as emphasized throughout Shapiro's systematic treatment in *Thinking About Mathematics* [5, pp. 267–270].

3.3 Epistemological Challenges

Despite its semantic clarity, *ante rem* structuralism faces a notable epistemological challenge: how do we come to know facts about abstract structures and their positions, especially when they are uninstantiated and causally inert? Shapiro acknowledges this issue and attempts to mitigate it by appealing to our ability to recognize and work with patterns. Mathematical practice, on this view, involves the intellectual apprehension of structural configurations rather than sensory access [5, pp. 275–283]. However, the epistemic gap between abstract objects and human cognition remains a point of contention, especially for critics skeptical of non-empirical knowledge.

As Reck and Price emphasize, this problem is central to the viability of any metaphysically realist structuralism. If structures and their positions exist independently, then a satisfactory account is needed of how such entities can be epistemically accessed without appealing to concrete instances [8, p. 347].

3.4 Benacerraf's Problem and the Motivation for Structuralism

A key motivation for adopting *ante rem* structuralism is its elegant solution to Paul Benacerraf's identity dilemma. Benacerraf famously noted that multiple equally valid set-theoretic reductions of arithmetic (e.g., Zermelo and von Neumann constructions) yield conflicting identity claims—such as whether $3 \in 17$—and that no purely mathematical consideration suffices to privilege one representation over another [9].

By positing that numbers are positions in an abstract structure, *ante rem* structuralism avoids this dilemma. Since these positions are invariant across isomorphic systems, identity claims are grounded in structural role rather than particular set-theoretic representation [5, pp. 264–265], [8, p. 344]. This effectively dissolves the problem of conflicting encodings and underwrites the stability of mathematical reference.

3.5 Contrast with Traditional Platonism

Although it shares Platonism's commitment to abstract existence, *ante rem* structuralism differs significantly from classical Platonism. Rather than positing independently existing mathematical objects with intrinsic properties, it defines mathematical entities relationally: their identity and existence are determined by their position within a structure [6, p. 2]. As Reck and Price summarize, "mathematical objects have no more to them than can be expressed in terms of the basic relations of the structure" [8, p. 342].

This distinguishes the view from object-oriented Platonism and from in re or modal variants of structuralism. The objects of mathematics are not sets, nor do they possess any internal nature beyond the relations defined by the structure they inhabit.

3.6 Open Questions and Philosophical Trade-offs

Despite its strengths, *ante rem* structuralism is not without difficulties:

- **Epistemic access:** How do mathematicians acquire justified beliefs about abstract structures that are causally disconnected from experience?
- **Individuation:** Without intrinsic properties, how are positions within a structure uniquely identified?
- **Overgeneration:** Do we have reason to believe that every consistent axiom system corresponds to a real, existing abstract structure?

These tensions are noted by both Shapiro [5, pp. 283–286] and Reck & Price [8, p. 346], who argue that any version of structuralism must navigate carefully between metaphysical inflation and epistemological implausibility. While *ante rem* structuralism remains one of the most coherent and powerful structuralist philosophies, its viability depends on whether these deep philosophical challenges can be addressed without undermining its foundational commitments.

4 In Re Structuralism: Structures in the World

A second major strand within the structuralist philosophy of mathematics is *in re* structuralism. In contrast to the *ante rem* variant, which posits structures as independently existing abstract entities, *in re* structuralism is more Aristotelian in spirit: it holds that structures exist only insofar as they are instantiated. That is, a structure exists only when some system of objects instantiates it, and the positions within the structure are simply relational roles played by the elements of that system [5, pp. 270–275], [6, p. 2].

This position aims to retain the relational ontology that motivates structuralism while avoiding some of the epistemological and metaphysical difficulties that plague *ante rem* realism. However, this deflationary move comes at a cost: it raises questions about the generality of mathematical discourse and the status of uninstantiated but mathematically important structures.

4.1 Ontology: Structures Dependent on Systems

The core metaphysical commitment of *in re* structuralism is that structures are not self-standing entities. They do not exist in a Platonic realm but are immanent in the systems that instantiate them. As Shapiro puts it, "the in re structuralist holds that structures exist only in the systems that instantiate them" [5, p. 270]. A natural number structure exists only insofar as there is a system—a sequence of strokes, sets, or physical markers—that exhibits the Peano structure. No system, no structure.

In this framework, the number 3 is not a position in an abstract structure, but a role played by some object (say, a particular set or token) in a system satisfying the relevant axioms. This understanding makes the structural roles ontologically posterior to the system itself, as opposed to the *ante rem* view in which the system is metaphysically posterior to the structure it instantiates [8, pp. 344–345].

4.2 Semantic Consequences: Shifting from Abstract to Concrete Reference

Semantically, *in re* structuralism avoids treating numerals as referring to abstract objects. Instead, they are interpreted contextually, relative to an instantiating system. Mathematical discourse is then read as asserting generalizations about all such systems.

For instance, the sentence "$2 + 2 = 4$" is not taken to be a statement about specific abstract entities, but rather a generalization: in any system that satisfies the Peano axioms, the object in the '2-place' added to the object in the '2-place' gives the object in the '4-place'. In Shapiro's formal regimentation:

> *For any system S that satisfies the axioms of arithmetic, the object playing the '2' role plus the object playing the '2' role is the object playing the '4' role in S.* [5, p. 271]

This approach closely resembles the structuralist paraphrases introduced in Paul Bernays's reflections on Hilbert's formalism, where theories are understood as schematic structures applicable to any domain satisfying the axioms [8, p. 345].

4.3 Epistemological Modesty and Its Consequences

One of the key motivations for adopting *in re* structuralism is epistemological: by grounding mathematical knowledge in instantiated structures, the view sidesteps the problematic question of how we can access uninstantiated, abstract entities. If structures exist only when instantiated, and we interact with these systems empirically or symbolically (e.g., through marks on paper, or mental constructions), then epistemic access is straightforward: we know mathematical truths by studying the systems that instantiate them.

As Shapiro notes:

> "We interact with systems that instantiate mathematical structures. We explore those systems. Thus, we explore the structures in re." [5, p. 272]

However, this epistemological benefit is not without cost. By restricting existence to instantiated structures, the *in re* structuralist potentially excludes large swaths of mathematics that seem meaningful and indispensable but have no known physical or symbolic instantiation. Examples include large cardinal axioms in set theory, nonstandard models of arithmetic, and ideal geometrical spaces in analysis and topology.

Reck and Price emphasize this limitation:

> "The cost is a diminished capacity to account for general and idealized structures that are routinely used in mathematics." [8, p. 346]

4.4 Generality and Mathematical Practice

A central appeal of mathematics lies in its capacity to describe not just actual systems, but possible and even idealized ones. The appeal to uninstantiated structures is often indispensable: we reason about Hilbert spaces of infinite dimension, higher-order categories, or large cardinals not because they are realized physically or cognitively, but because they form the backbone of major mathematical theories.

From the perspective of *in re* structuralism, however, these structures lack ontological status unless they are instantiated. This risks rendering large portions of higher mathematics unintelligible or nonfactual—a consequence many mathematicians and philosophers find unacceptable.

Shapiro acknowledges this limitation and suggests that *in re* structuralism is best suited for *applied* mathematics, where structures are more likely to be instantiated, whereas pure mathematics might require an *ante rem* or hybrid account [5, p. 274]. Nevertheless, this concession underscores a core tension within the view.

4.5 Structural Identity and Indeterminacy

Another challenge arises in the individuation of mathematical objects. In the absence of a global abstract structure, identity conditions are always local to a system. Thus, the number 3 in one model may not be identical (in any robust metaphysical sense) to the number 3 in another. This leads to a form of referential relativity, where mathematical terms do not denote across systems in a uniform way.

While this may seem tolerable in applications where only one model is considered at a time, it becomes problematic for comparative and higher-order reasoning, where multiple systems are discussed simultaneously. The lack of a shared abstract referent leads to semantic and inferential complications.

4.6 Summary: Modest Metaphysics, Limited Reach

In sum, *in re* structuralism offers a metaphysically modest and epistemologically accessible form of structuralism. It grounds mathematical knowledge in the study of actual systems, avoids positing a realm of abstract entities, and supports a generalist, model-theoretic interpretation of mathematics. However, its modesty comes with philosophical trade-offs:

- It struggles to account for uninstantiated but widely used mathematical structures.
- It faces challenges in preserving cross-model identity and generality.
- It potentially fragments mathematical discourse into model-relative truths.

As Reck and Price note, while *in re* structuralism is a coherent philosophical position, it may ultimately fall short of capturing the full scope and ambition of mathematical practice [8, pp. 346–348]. Whether the trade-off between metaphysical economy and mathematical generality is justified remains a subject of ongoing debate.

5 Modal Structuralism: Mathematics Without Ontology

The most ontologically austere form of structuralism is known as *modal structuralism*, or sometimes *eliminative structuralism*. It was developed and most extensively articulated by Geoffrey Hellman, particularly in his influential book *Mathematics Without Numbers* [3]. Unlike *ante rem* and *in re* structuralism, modal structuralism denies the existence of mathematical objects altogether—whether abstract or instantiated. Instead, it interprets mathematical discourse as a kind of modal logic: mathematical assertions are seen as conditional statements about what would hold in any possible structure satisfying a given set of axioms.

5.1 Ontology: No Numbers, No Structures

The guiding metaphysical motivation for modal structuralism is to preserve the power and utility of mathematical reasoning while avoiding ontological commitment to mathematical objects. Hellman frames this project as a response to the nominalist challenge: can we account for mathematical practice without postulating abstract entities like numbers, sets, or functions?

The answer offered by modal structuralism is affirmative, but it requires a substantial reinterpretation of mathematical claims. Instead of asserting that "$2 + 2 = 4$" is true about abstract numbers, the modal structuralist reads this as a conditional:

> *Necessarily, in any system satisfying the axioms of arithmetic, the object playing the '2' role added to the object playing the '2' role yields the object playing the '4' role.*

This replaces commitment to specific objects with commitment to modal truths about systems—what must be true in all admissible interpretations [3, pp. 18–20], [5, pp. 272–273].

5.2 Formal Strategy: Second-Order Modal Logic

To formalize this approach, Hellman develops a system of second-order modal logic with plural quantification. This logic allows one to speak conditionally about any possible structure that satisfies a given axiom system, while quantifying over relations and functions without requiring them to refer to actual entities. For example, rather than asserting "There exists a function f such that $f(0) = 1$," the modal structuralist asserts:

$$\Box\exists f \forall x\, A(f, x)$$

where $\Box$ is a modal necessity operator and $A(f, x)$ expresses the relevant structural condition. The quantification over f does not commit us to the actual existence of a function; rather, it states that in all admissible systems, some object behaves like such a function [3, Ch. 2–3].

This formal apparatus allows Hellman to reconstruct substantial portions of mathematics —particularly arithmetic and analysis— without invoking any ontological commitment to mathematical objects. The modal framework acts as a *semantic shield*, letting us make meaningful statements *as if* about numbers or sets, while actually speaking only about hypothetical scenarios.

5.3 Epistemology: Structural Knowledge Without Objects

One of the chief virtues of modal structuralism is that it avoids the epistemological problems associated with access to abstract entities. Since the theory makes no ontological commitments, it sidesteps the question of how we can know about causally inert and unobservable objects. Instead, knowledge of mathematics becomes knowledge of what follows necessarily from a set of axioms under a structural constraint [5, pp. 272–274].

Moreover, this approach supports a kind of *structural necessity*: statements like "There are infinitely many primes" are not about any particular instantiation but are true in all possible models satisfying the axioms of arithmetic. This aligns closely with the modal logic of necessity and possibility, and replaces metaphysical realism with logical compulsion.

5.4 Metaphysical Parsimony and Nominalist Appeal

Modal structuralism is particularly attractive to philosophers with nominalist leanings who are skeptical of abstract entities. By grounding mathematics in modality rather than ontology, it offers a rigorous pathway to reconcile nominalism with classical mathematical practice. As Hellman puts it, the goal is "to make room for mathematics without making room for numbers" [3, p. ix].

In the philosophical landscape, this places modal structuralism alongside other attempts at nominalist reconstructions of mathematics (e.g., Hartry Field's work), but it retains a closer affinity to the structuralist tradition in that it retains the formal role of structure, albeit reinterpreted modally.

5.5 Limitations: Modal Logic, Informal Practice, and Structural Generality

Despite its elegance and parsimony, modal structuralism faces significant challenges, both technical and philosophical.

1. Dependence on Modal Logic. The framework presupposes a rich and philosophically contentious modal logic, including second-order quantification and a well-behaved necessity operator. Critics question whether the appeal to such a powerful modal ontology is truly ontologically innocent. As Shapiro notes, "the modalist must still tell us what the modality is and how we come to know truths about it" [5, p. 273]. In some sense, modal structuralism trades ontological commitment to mathematical entities for commitment to abstract modal facts.

2. Disconnection from Informal Mathematical Practice. Another concern is that modal structuralism does not reflect how working mathematicians typically reason. Mathematicians do not normally express theorems in modal terms or regard axioms as merely hypothetical. The translation of ordinary mathematical statements into modal logic is non-trivial and philosophically loaded, which can make the approach feel artificial or externally imposed [8, p. 347].

3. Generality and Uninstantiated Structures. While modal structuralism allows us to talk about uninstantiated structures (since possibility and necessity are defined over arbitrary models), it lacks the expressive ease and conceptual clarity of ante rem structuralism. For example, category theory and set theory often involve reasoning about whole classes of structures, or higher-order entities such as functor categories, which are difficult to encode purely in a modal framework without extensive reconstruction [8, p. 348].

5.6 Summary: Mathematics as Modal Necessity

Modal structuralism provides a philosophically rigorous and ontologically deflationary account of mathematics. It preserves structural reasoning and mathematical truth while eliminating commitment to mathematical objects. Its metaphysical minimalism is both its strength and its liability: it appeals to nominalists and logical empiricists, but may alienate those who seek a more intuitive or practice-aligned foundation for mathematics.

As Reck and Price observe, modal structuralism is "the most radically eliminative among structuralist views," and while it achieves remarkable formal clarity, "it comes at the price of detachment from actual mathematical language and methodology" [8, p. 347]. Whether this trade-off is worth it depends on one's philosophical priorities: ontological parsimony or fidelity to mathematical practice.

6 Other Variants of Structuralism: Philosophical Alternatives and Hybrids

Beyond the now-canonical distinctions among *ante rem*, *in re*, and *modal* structuralism, the philosophical literature reveals a broader landscape of structuralist positions—some historically foundational, others emerging as hybrid or alternative approaches. These variants serve to refine or extend the structuralist project and often seek to navigate tensions between semantic adequacy, ontological modesty, and fidelity to mathematical practice.

In this section, we explore several such alternatives: *relativist structuralism*, *structural abstractionism*, *universalist structuralism*, and views rooted in the formalist or model-theoretic tradition. Drawing especially on Reck and Price's typology [8, pp. 345–349], we evaluate the philosophical strengths and limitations of these perspectives and compare them critically to the three major forms discussed earlier.

6.1 Relativist Structuralism: Semantics without Ontology

Relativist structuralism takes cues from model theory and semantic anti-realism. It is rooted in the idea that mathematical objects derive meaning not from any metaphysical underpinning but from their position within a structure relative to a chosen model. This view is implicit in the practice of much twentieth-century logic, especially in the model-theoretic turn influenced by Tarski and Hilbert.

On this approach, mathematical statements are not about any specific structure but about what holds in *any model* satisfying a given theory. It resembles modal structuralism semantically, but avoids its modal apparatus by staying within first-order frameworks. Reck and Price identify this view as an "epistemologically cautious" structuralism [8, p. 345].

Relativist structuralism resembles in re structuralism in avoiding abstract ontology, but diverges by offering no unified ontological stance—only semantic generality. Unlike modal structuralism, it avoids modal logic but may fall short in accounting for cross-model identity. Compared to ante rem structuralism, its greatest weakness lies in its inability to preserve the literal truth and determinacy of mathematical statements across systems [8, p. 346].

6.2 Structural Abstractionism: Abstracting Over Systems

Another hybrid form is what some have called *structural abstractionism*. This approach treats structures not as objects but as abstractions over classes of isomorphic systems. In this view, a mathematical object (like the number 2) is not a position in a freestanding entity, but the abstraction of all entities playing the "2-role" across models of arithmetic.

This perspective builds on the philosophical tradition of abstraction principles—akin to Frege's Hume Principle or neo-Fregean logicism. It allows one to define numbers as equivalence classes (or higher-order abstracts) over systems or configurations. While compatible with structuralist motivations, abstractionism changes the semantic emphasis: mathematical terms denote abstracted roles, not concrete or modal positions [8, p. 342].

This view occupies a middle ground between ante rem and relativist structuralism. Like ante rem realism, it aims to preserve cross-model identity, but avoids commitment to independently existing structures. Unlike modal structuralism, it does not rely on modality, and unlike in re views, it permits reference to idealized uninstantiated roles. Its critical weakness lies in its dependence on abstraction principles, which may lack clear epistemological or metaphysical foundations.

6.3 Universalist Structuralism: All Structures at Once

A more recent perspective explored in Reck and Price's taxonomy is *universalist structuralism* [8, p. 345]. Rather than focusing on individual structures, this view takes mathematics to be about the *totality of structures* conforming to given axioms. Here, mathematics is seen as a meta-theoretical exploration of the entire universe of models of a theory.

This aligns with perspectives in category theory, homotopy type theory, and structural set theory (e.g., Lawvere's ETCS), where focus shifts from individual mathematical objects to morphisms and functorial relations across structures. In such frameworks, identity is structural up to isomorphism, and mathematics becomes a science of structural transformation.

Universalist structuralism expands the structuralist agenda beyond individual instantiation, aligning well with modern practice in categorical logic and abstract algebra. However, it departs from traditional semantics: it no longer treats mathematical truth as located in a *particular* model or structure. Instead, it replaces ontological commitment with systematic generality. This makes it powerful for mathematics-as-practice, but challenging for grounding semantics or truth-value realism [8, pp. 348–349].

6.4 Formalist Structuralism: Structures as Inferential Schemas

A further historical strand—what we may call *formalist structuralism*—can be found in the legacy of Hilbert and the Bourbaki school. Here, structures are not ontological entities or modal abstractions, but inferential schemas governed by axioms. Mathematics is seen as the development of consequences from specified formal systems.

This view offers ontological neutrality, epistemological tractability, and alignment with mathematical rigor. However, it is often considered too deflationary for foundational purposes. If structures are mere syntactic frameworks, what grounds the applicability, objectivity, and consistency of mathematics?

Formalist structuralism is ontologically minimal like modal and relativist views but differs by avoiding even modal vocabulary. It is methodologically fruitful but lacks a semantic or metaphysical theory. As Shapiro argues, it cannot account for the content of mathematics if all meaning is reduced to derivability [5, pp. 259–261].

6.5 Comparative Evaluation: Dimensions of Structuralist Commitment

We may summarize the differences among these lesser-known variants by organizing them along key philosophical dimensions:

- **Ontological Commitment:** Ante rem and universalist structuralism assert structures as entities; modal and formalist views reject this.
- **Epistemological Orientation:** In re and relativist structuralism emphasize epistemic contact with instantiated systems; abstractionism and modal structuralism seek epistemic neutrality.
- **Semantic Realism:** Ante rem and abstractionist views preserve direct reference; modal and relativist views favor conditional or schematic reinterpretation.
- **Alignment with Practice:** Universalist and abstractionist approaches are increasingly reflected in modern mathematics; modal and formalist approaches diverge from typical mathematical language.

Each position negotiates a trade-off between mathematical generality, philosophical rigor, and ontological clarity. As Reck and Price caution, the complexity of structuralism lies not only in its metaphysical choices but in how these choices interrelate with semantics, logic, and methodology [8, p. 348]. Structuralism is not a monolith, but a philosophical landscape, and each variant must be evaluated with sensitivity to its conceptual commitments and explanatory ambitions.

7 Comparative Reflections on Structuralist Philosophies

The comparative table above (Table 1) illustrates the richness and internal diversity of structuralism as a philosophical stance. While all forms of structuralism agree on the primacy of relational roles over intrinsic objecthood, they diverge fundamentally in their commitments to ontology, semantics, and epistemology.

Ante rem structuralism stands out for its robust metaphysical realism. By positing structures and their positions as independently existing entities, it provides a literal semantics and a solution to Benacerraf's identity problem [9]. However, this comes at the price of deep epistemological concerns: how do we access entities that are abstract, causally inert, and uninstantiated? Shapiro's attempts to ground such access in mathematical practice and abstraction remain controversial [5, pp. 275–283].

In contrast, *in re* structuralism avoids these metaphysical burdens by restricting structures to instantiated systems. This offers a more empirically grounded epistemology [5, p. 272], yet it undermines the generality of mathematics and risks marginalizing large swaths of pure mathematics dealing with idealized or uninstantiated structures [8, p. 346]. It aligns well with mathematical applications but struggles with theoretical universality.

Modal structuralism goes further, eliminating mathematical ontology altogether in favor of modal logic [3]. It achieves philosophical parsimony and sidesteps the access problem entirely.

Yet, its reliance on second-order modal logic and its departure from how mathematicians actually think and work makes it a hard sell to both philosophers of mathematics and mathematicians themselves [8, p. 347].

Among the more flexible variants, *structural abstractionism* and *universalist structuralism* offer alternative paths forward. The former interprets mathematical entities as abstractions over equivalence classes of roles, preserving cross-system identity without strong ontological claims [8, p. 342]. The latter mirrors category-theoretic and homotopical methods, treating mathematics as the study of networks of structures and transformations. These approaches are well-aligned with contemporary mathematical methodologies but often come at the cost of clear metaphysical grounding and semantic transparency.

Finally, *formalist structuralism* offers a minimalist, rule-governed conception that is appealing from a methodological standpoint but philosophically weak in explaining the content, reference, or applicability of mathematics [5, pp. 259–261].

In short, each version of structuralism negotiates a trade-off among metaphysical commitment, semantic realism, epistemic plausibility, and practical alignment with mathematics. No variant dominates all others. Rather, each reflects a distinct philosophical priority: ontological realism (ante rem), empirical modesty (in re), nominalist rigor (modal), structural generality (universalist), or inferential discipline (formalist). The future of structuralism may lie not in choosing one over the others, but in articulating new hybrid positions that balance these tensions more holistically.

Variant of Structuralism	**Ontological Commitment**	**Semantic Interpretation**	**Epistemological Strategy**	**Strengths and Weaknesses**
Ante Rem Structuralism (Shapiro, Resnik)	Structures and positions exist independently of systems [5, pp. 263–270]	Mathematical terms refer to places in abstract structures [5, pp. 267–269]	Knowledge of structures through abstraction and practice [5, pp. 275–283]	**Strengths:** Truth-value realism; direct semantics; solves Benacerraf's dilemma [9] **Weaknesses:** Epistemic access problem; abstract ontology [8, p. 347]
In Re Structuralism (Shapiro)	Structures exist only when instantiated [5, pp. 270–275]	Terms refer to positions in existing systems; statements interpreted as generalizations over such systems [5, p. 271]	Direct empirical or symbolic interaction with instantiated systems [5, p. 272]	**Strengths:** Avoids abstract ontology; epistemically modest [8, p. 346] **Weaknesses:** Struggles with uninstantiated structures; undermines generality [8, p. 346]
Modal Structuralism (Hellman)	No mathematical objects; modal quantification over possible structures [3, Ch. 1–2]	Mathematical truths are necessary conditionals about possible models [3, pp. 18–20]	Avoids metaphysical access problem by reducing mathematics to modal logic [3, pp. 21–24]	**Strengths:** Ontologically parsimonious; aligns with nominalism [3, p. ix] **Weaknesses:** Requires rich modal logic; detached from informal practice [8, pp. 346–347]
Relativist Structuralism (implicit in model theory)	Structures relative to chosen systems; no universal ontology [8, p. 345]	Terms are contextually interpreted within a model; no fixed referents [8, p. 345]	Epistemology tied to interpretability within formal semantics	**Strengths:** Technically tractable; avoids abstract metaphysics **Weaknesses:** No unified reference; lacks metaphysical grounding [8, p. 346]
Structural Abstractionism (inspired by Frege, neo-Fregeans)	Structures as abstractions over equivalence classes of systems [8, p. 342]	Reference to abstracted roles, not ontological entities [8, p. 342]	Abstraction over practice, pattern recognition, or logical form	**Strengths:** Preserves cross-model identity; avoids robust ontology **Weaknesses:** Depends on controversial abstraction principles
Universalist Structuralism (category-theoretic)	Mathematics is about the universe of all structures and morphisms [8, p. 345]	Semantic focus on transformations, functors, and universal properties	Reasoning over large-scale relational networks; diagrammatic and axiomatic knowledge	**Strengths:** Matches modern mathematical methods; supports structural generality [8, p. 348] **Weaknesses:** Difficult to ground classical semantics; metaphysically diffuse
Formalist Structuralism (Hilbert, Bourbaki)	No commitment to structures; only symbolic systems and derivations [5, pp. 259–261]	Meaning arises from formal derivability; no external reference	Epistemic access through rules and inference schemes	**Strengths:** Methodologically rigorous; ontologically neutral **Weaknesses:** Semantically empty; fails to explain applicability or content [5, pp. 259–261]

Table 1: Comparative Evaluation of Structuralist Philosophies of Mathematics

8 Conclusion: Philosophical Significance and Challenges of Structuralism

Across its many variants, mathematical structuralism has emerged as one of the most philosophically fruitful frameworks for understanding the nature of mathematical objects, truth, and practice. By shifting attention away from intrinsic natures and toward relational configurations, structuralism reorients foundational debates around identity, reference, and applicability.

One of structuralism's greatest achievements is its elegant solution to the identity problem for abstracta. Rather than identifying mathematical entities with specific set-theoretic constructions—as criticized by Benacerraf [9]—structuralism defines objects in terms of their roles within structures. This relational conception avoids arbitrariness and aligns more closely with how mathematical reasoning actually functions across different domains [5, pp. 263–270], [8, pp. 342–344].

Moreover, structuralism offers a compelling account of the *objectivity* and *applicability* of mathematics. Its emphasis on structural invariance across systems provides a philosophical bridge between pure mathematics and its real-world applications. As Resnik and Shapiro emphasize, what allows mathematical descriptions to map onto the physical world is precisely their capacity to describe interrelations and constraints—regardless of the nature of the objects instantiating them [5, pp. 257–260]. This view is strongly reinforced by developments in category theory and model theory, where relational structure and morphism take precedence over substance [8, p. 345].

In addition, structuralism reflects contemporary mathematical methodology. The structural turn initiated by Dedekind, Hilbert, and Noether, and developed further by Bourbaki and the categorical tradition, reshaped mathematics into the study of invariant properties under transformation. Today, mathematical logic, algebraic geometry, and homotopy type theory are thoroughly structuralist in outlook. The success of these disciplines gives weight to the philosophical claim that mathematics is fundamentally about structures and their interrelations [4, pp. 190–198].

Yet, structuralism is not without internal tensions and unresolved questions. The epistemology of *ante rem* structuralism remains contested: how do we access abstract structures and their positions if they are causally and cognitively remote? While Shapiro defends abstraction and pattern recognition as routes to structural knowledge [5, pp. 275–283], the justification for such access remains philosophically fraught.

Modal structuralism, for its part, solves this problem by eliminating abstract entities altogether, but it introduces new complications. Its reliance on second-order modal logic raises questions about the metaphysical grounding of modality and its interpretative distance from everyday mathematical language [3, pp. 18–24], [8, p. 347].

Similarly, *in re* structuralism provides a grounded, empirical alternative by requiring instantiated structures. But this comes at the expense of mathematical generality: idealized structures such as large cardinals, nonstandard models, or infinite-dimensional spaces resist such instantiation [8, pp. 345–346]. The status of these uninstantiated yet indispensable structures remains unresolved.

Furthermore, as Reck and Price warn, structuralism risks conceptual dilution unless carefully distinguished from other philosophical frameworks [8, p. 348]. There is a danger of terminological ambiguity—particularly where set-theoretic realism masquerades as structuralism, or where formalism is redescribed as structuralist without substantive reorientation. This calls for more rigorous taxonomic clarity and philosophical refinement in future work.

As we reflect on the many dimensions of structuralism in the philosophy of mathematics—its metaphysical ambitions, semantic nuances, and methodological reach—it is fitting to close with a remark that captures its spirit with simplicity and boldness.

> *"One must be able to say at all times — instead of points, straight lines, and planes — tables, chairs, and beer mugs."*

This oft-cited declaration by **David Hilbert** encapsulates the structuralist ethos avant la lettre: that the meaning of mathematical entities lies not in what they are, but in how they relate. It affirms that substance is secondary to form, and that mathematics gains its universality not by tethering to specific content, but by abstracting to invariant structure [6, p. 343].

References

[1] Hellman, G., & Shapiro, S. (2018). *Mathematical Structuralism*. Cambridge University Press. Chapters 1–3 introduce the motivations and variants of structuralism, and Chapter 4 discusses metaphysical and semantic evaluations of different views.

[2] Shapiro, S. (2000). *Philosophy of Mathematics: Structure and Ontology*. Oxford University Press. See especially Chapter 4 ("Structure and Ontology"), pp. 95–122, where Shapiro elaborates on the distinction between *ante rem* and *in re* structuralism.

[3] Hellman, G. (1989). *Mathematics Without Numbers: Towards a Modal-Structural Interpretation*. Oxford University Press.

[4] Reck, E. H., & Schiemer, G. (2020). *The Prehistory of Mathematical Structuralism*. Oxford University Press.

[5] Shapiro, S. (1997). *Thinking About Mathematics: The Philosophy of Mathematics*. Oxford University Press.

[6] Shapiro, S. (n.d.). *Mathematical Structuralism*. In *The Internet Encyclopedia of Philosophy*. Retrieved from https://iep.utm.edu/m-struct/

[7] Resnik, M. (1981). *Mathematics as a Science of Patterns: Ontology and Reference*. Noûs, 15(4), 529–550.

[8] Reck, E. H., & Price, M. P. (2000). *Structures and Structuralism in Contemporary Philosophy of Mathematics*. Synthese, 125(3), 341–383.

[9] Benacerraf, P. (1965). *What Numbers Could Not Be*. The Philosophical Review, 74(1), 47–73.