Formalization of Basic Properties of Fredholm Operators

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1 Introduction

In this project, we formalized the basic theory of Fredholm Operators in Lean, including their fundamental definitions and key lemmas. Our work consists of three main sections:

- `section FredholmOperatorsDef` – where we formalized the basic definitions of Fredholm operators.
- `section RangeClosednessIsUnnecessary` – where we formalized the lemma that if an operator's range admits a closed complementary subspace, then its range is closed.
- `section InvertibilityIsALocalProperty` – here we formalized the lemma that invertible operators remain invertible under sufficiently small perturbations.

2 Main Results

2.1 Fundamental Definitions

In this section, we formalized the definitions related to Fredholm operators in Lean. The main definition is the `FredholmOperators` class, which defines a Fredholm operator as a continuous linear operator between complete normed spaces that satisfies three key properties:

- Its kernel is finite-dimensional
- Its range is closed
- Its cokernel is finite-dimensional

We then defined `Fred(X, Y)` as the set of all Fredholm operators between spaces X and Y. To work with Fredholm operators more effectively, we also formalized several important related concepts:

- `ker`: the kernel of a Fredholm operator
- `ran`: the range of a Fredholm operator
- `coker`: the cokernel of a Fredholm operator
- `ind`: the index of a Fredholm operator, defined as the difference between the dimension of the kernel and the dimension of the cokernel

In the document, we used identity operator as an example of a Fredholm operator.

2.2 Lemmas

Our main work focuses on formalizing two fundamental lemmas from the reference. These lemmas play important roles in the theory of Fredholm operators. The first lemma states that if an operator's range admits a closed complementary subspace, then its range is closed:

```
lemma RangeClosedIfAdmittingRangeClosedCompletement
{E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E]
{F : Type*} [NormedAddCommGroup F] [NormedSpace ℝ F]
(f : E →L[ℝ] F) [CompleteSpace F] [CompleteSpace E]
(h : ∃ C : Subspace ℝ F, IsClosed (C : Set F) ∧ IsCompl
 (LinearMap.range f) C):
IsClosed (LinearMap.range f : Set F):= by ...
```

The second lemma shows that invertible operators remain invertible under sufficiently small perturbations:

```
theorem BoundedInvertibleOperatorPlusεIsInvertible
{E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E]
{F : Type*} [NormedAddCommGroup F] [NormedSpace ℝ F]
(f : E →L[ℝ] F) [CompleteSpace E] [Nontrivial E] [CompleteSpace
 F](hf : IsInvertible f) :
∃ (ε : ℝ), ε > 0 ∧ ∀ (p : E →L[ℝ] F), ‖ p ‖ < ε → IsInvertible (f +
  p) := by ...
```

Both lemmas were formalized in their respective sections as mentioned in Section 1.

3 Unfinished Parts

Unfortunately we did not finish all the things in our reference and leave some of them with `sorry` in the end of the document. We initially formalized the statements of these theorems, but during the proof process, we found that working with lemmas that do with analysis required more complicated techniques than we expected. As a result, some of the proofs remain incomplete in our current work.

4 Reference

The formalization mainly based on
https://ocw.mit.edu/courses/18-965-geometry-of-manifolds-fall-2004/8a7e4dd837d1bdd6988e0330babb8c5e_lecture16_17.pdf