Lean4 Project

Functional Analysis and Fredholm Operators

MingyuXia YuqingGao

January 14, 2025

1. Math
   - Functional Analysis
   - Fredholm Operators
   - What is done in our project

2. Formalization

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2. Formalization

- Functional Analysis is often described as "infinite-dimensional" Linear Algebra. Infinite-dimensional vector spaces naturally arise in Mathematics and Physics,
- Addressing their added complexity requires more than just the tools of Linear Algebra. We rely on concepts from Calculus and Topology, which provide the necessary framework to study continuity and related phenomena.

1 Math

Functional Analysis

Fredholm Operators

What is done in our project

2 Formalization

Definition Of Fredholm Operators

Definition (Fredholm Operator)

Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded linear operator. T is called Fredholm if the following hold:

- $\ker(T)$ is finite-dimensional.
- $\mathrm{Ran}(T)$ is closed.
- Coker (T) is finite-dimensional.

Motivation For Fredholm Operators

Fredholm operators arise from PDE. The use of Fredholm operators in partial differential equations is an abstract form of the parametrix method.

Main Theorem

Corollary (Characterization for Fredholm Operators)

$T : X \to Y$ is Fredholm if and only if there exists a bounded linear operator $R : Y \to X$ so that $RT - I$ and $TR - I$ are compact operators.

Roughly speaking, an operator is Fredholm iff it is invertible modulo compact operators.

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Statements of Lemmas and Some Proofs

- It is too much work for us to completely formalize the whole proof of the main theorem since there are many lemmas and propositions to be formalized before the main theorem.
- We only formalize definitions, statements of those propositions, and some proof of them.
- In next section we give some lemmas we have formalized.

1. Math

2. Formalization
   - Formalized Example 1
   - Formalized Example 2

A Criterion for Closeness of range

```
/-Lemma: Let T : X → Y be a operator so that the range admits a closed
complementary subspace. Then the range of T is closed.-/
lemma RangeClosedIfAdmittingRangeClosedCompletement    declaration uses 'sorry'
  {E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E]
  {F : Type*} [NormedAddCommGroup F] [NormedSpace ℝ F]
  (f : E →L[ℝ] F) [CompleteSpace F] [CompleteSpace E]
  (h : ∃ C : Subspace ℝ F, IsClosed (C : Set F) ∧ IsCompl (LinearMap.range f) C):
  IsClosed (LinearMap.range f : Set F):= by
```

We encounter two problems during this lemma's formalization:

- There is no direct way to deduce a continuous linear map $\bar{f} : E/\ker(f) \to F$ from a given continuous linear map $f : E \to F$
- This lemma compiles too slow.

A Criterion for Closeness of range(continued)

Here are the solutions:

- Mathlib has a function ($f \mapsto \bar{f}$) when f is a normed linear map(a notion in our situation equivalent to continuous linear map but realized differently in Lean 4)

```
/-Lemma: A continous linear map f:E →L[ℝ] F induces a continous linear map
f_bar:E/ker(f) →L[ℝ] F-/
noncomputable def QuotientOfContinuousLinearMap
  {E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E]
  {F : Type*} [NormedAddCommGroup F] [NormedSpace ℝ F]
  (f : E →L[ℝ] F): E ⧸ (LinearMap.ker f) →L[ℝ] F:=by
```

- We break the total lemma into several sub-lemmas.

A Criterion for Closeness of range(continued)

Outlines of the proof(the code consists of one hundred line):

1. Since C is a closed submodule of F, it inherits a complete normed space structure(we need this because in the latter we will apply the Banach mapping theorem).
2. By cosidering the induced map $\bar{f} : E/\ker(f) \to F$, we reduce to the case f is injective.
3. Define a morphism $S : (e, c) \in E \oplus C \mapsto (fe + c) \in F$, which we will show to be an isomorphism(where the Banach mapping theorem plays a role).
4. Notice that E in $E \oplus C$ is closed.

Invertible operator maintains invertibility under perturbation

Notation:
$\mathcal{L}(X, Y) := \{T : X \to Y : T \text{ is linear and continuous}\}$

Definition (Invertible)

T is an invertible operator if $T \in \mathcal{L}(X, Y)$ is bijective and $T^{-1} \in \mathcal{L}(Y, X)$.

Theorem

If $T : X \to Y$ is a bounded invertible operator then for all $p : X \to Y$ with sufficiently small norm $T + p$ is also invertible.

Sketch of the proof

- Goal: show $T + p \in \mathcal{L}(E, F)$ is invertible
- Special case: Let $T := \mathrm{Id} \in \mathcal{L}(E, E)$. Then the Neumann series converges as

$$\sum_{i=1}^{\infty} (-p)^i = (\mathrm{Id} - (-p))^{-1}$$

- Generalization: $T + p = T(\mathrm{Id} + p \cdot T^{-1})$ and let $q := p \cdot T^{-1}$

Lemma: Neumann series Theorem

Theorem (Neumann series)

Let X be a Banach space and $T \in \mathcal{L}(X, X)$ with $\|T\| < 1$. Then $\mathrm{Id} - T$ is bijective and $(\mathrm{Id} - T)^{-1} \in \mathcal{L}(X, X)$. Moreover the series $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{L}(X, X)$ and

$$(\mathrm{Id} - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Sketch of the proof

- Set $S_k := \sum_{n=0}^{k} T^n$
- Show S_k is Cauchy: By assumption, there exists $\theta := \|T\| < 1$ so that $\|T^n\| \leq \|T\|^n$ for all n. Then for all $\varepsilon > 0$, there exists a natural number N such that for $N \leq k < l$,

$$\|S_l - S_k\| = \|\sum_{n=k+1}^{l} T^n\| \leq \sum_{n=k+1}^{l} \|T^n\| \leq \sum_{k+1}^{\infty} \theta^n = \frac{\theta^{k+1}}{1-\theta} \to 0.$$

- Since $\mathcal{L}(X,X)$ is complete, $S_k \to S$ in $\mathcal{L}(X,X)$
- Then we show $(\mathrm{Id} - T)S = \mathrm{Id}$ by

$$(\mathrm{Id} - T)S \longleftarrow (\mathrm{Id} - T)S_k = \sum_{n=0}^{k}(T^n - T^{n+1}) = \mathrm{Id} - T^{k+1} \longrightarrow \mathrm{Id}$$

- $S(\mathrm{Id} - T) = \mathrm{Id}$ by $S_k * T = T * S_k$.

Thanks!