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2017
MCM/ICM
Summary Sheet
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Type a summary of your results on this page. Do not include the name of your school, advisor, or team members on this page.

Summary

This paper establishes the macroscopic model and the microscopic model of traffic flow to determine a better shape, size and merging pattern of the merging area in the toll plaza.

Firstly, according to Greenshield's macroscopic stream model, we simplify the relation between speed and density of traffic flow as a linear relationship and derive the Density-Flow Model and the Speed-Flow Model. The relationship of the three models above implies that the maximum throughput of a freeway corresponds to half of the jam density and half of the free-flow speed. Applying this conclusion to develop our Length-Control Model, we seek to find out the relationship between the density and the total length of the merging area. We eventually get the range of the total length corresponding to the maximum throughput of the merging area.

Secondly, we analyze the drawbacks of existing toll plaza shape and develop our Binary-Tree shape design (we call it BT shape later in our paper) with a four-to-one BT merging pattern (i.e., cars leaving from four tollbooths merge back to one highway) in the microscopic model. Based on the above design, we modify the merging model of vehicle on freeway acceleration from other paper to obtain the probability distribution of unsuccessful merging. Applying this probability distribution and the range of total length we get in the macroscopic model into the merging model, we can get the appropriate total length of the merging area. When the traffic flow is 2000 veh/h, the total length is around 700m.

Thirdly, we study the performance of our solution in light and heavy traffic, how would our solution change as more autonomous vehicles are added to the traffic mix and how our solution is affected by the proportions of different collection booths. In the light traffic, our model works well with safety. In the heavy traffic, the congestion time in our solution that is more than 43.2 minutes has the probability of 0.03. As more autonomous vehicles are added to the traffic mix, our solution become more reasonable and stable because there are less subjective factors. Since we assume the number of vehicles leaving the tollbooths obeys the Poisson distribution, the proportions of different collection booths have little influence on our model. Thus, we eventually conclude that our model is robust.

Finally, we provide a one-page letter to the *New Jersey Turnpike Authority* to introduce our solution.

Key words: macroscopic stream model, microscopic traffic model, Binary-Tree shape design, merging model, Poisson distribution

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1 Introduction

1.1 Background

A barrier toll, with a row of tollbooths placed across the highway, is a system used to collect tolls on highways (see Figure 1[1]).

Figure 1: A Tollbooth in the UK

The area that facilitates the barrier toll is called toll plaza, consisting of the fan-out area before the barrier toll (i.e., the area where the incoming traffic flow arrives), the toll barrier itself, and the fan-in area after the toll barrier (i.e., the merging area where cars merge back to highways). Since there are usually more tollbooths than there are incoming lanes of traffic, congestions often occur in the merging area (see Figure 2)[2].

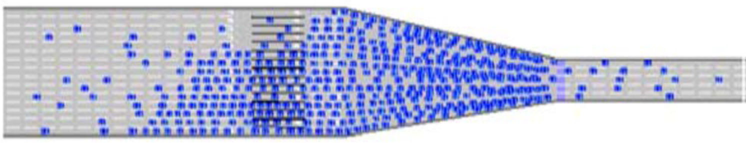


Figure 2: Congestions in the merging area.

Because the lanes of the fan-in area following tollbooths are more than that of highways, sometimes the quantity of vehicles in fan-in area is larger than the quantity of lanes in highways. As lanes decrease, that means, the vehicles enter the traffic bottleneck area, vehicles on the both sides of the fan-in area have to change lane to merge into highways. But when the traffic is heavy, it is troublesome for vehicles to merge into the main road. Meanwhile, the velocity of following vehicles will be affected. That's why highways usually become congested in fan-in area. What's worse, congestion will greatly increase the incidence of accidents.

To improve the traffic efficiency of the toll plaza, the shape, size, and merging pattern of the fan-in area must be well designed.

1.2 Our Work

Our work is divided into two main parts, the macroscopic part and the microscopic part. We use our macroscopic model to determine the range of the length of the merging area in order to achieve the maximum throughput of the toll plaza. Based on the result of our macroscopic model, we later develop the microscopic model to help design the desired shape, size and merging pattern of the merging area.

In the macroscopic model, we first introduce Greenshields macroscopic stream model[3]. This model illustrates a linear relationship between speed and density of traffic. The Density-Flow Model and Speed-Flow Model[4] derived from Greenshields Speed-Density Model provide us with the specific relationship between real density and jam density corresponding to the maximum throughput of the highway. We apply this relationship in our Length-Control Model and eventually get the desired range of the length of the merging area.

Since our target is to find a better solution to design the shape, size and merging pattern of the merging area, we seek to get the fine result in our microscopic model. Firstly, we design the binary-tree shape of the merging lanes which we believe can minimum the probability of congestion. Secondly, we develop our merging model[7] by moderately modifying the *Merging model of vehicle on freeway acceleration* proposed by LI Wen-quan, etc. Finally, we apply some specific data to the merging model to get the ideal size and complete the merging pattern of the merging area.

2 General Assumptions

- All the drivers are rational. When lining up into tollbooths, they will select the shortest queue. Based on this assumption, the traffic flow at each toll gate is uniform.
- Drivers will obey the traffic rules very well.
- All vehicles are the same in length and width.
- The weather condition is suitable for driving.
- The property of all vehicles is the same.

3 Model

3.1 Macroscopic Model

3.1.1 Local Symbols

Table 1: Frequently Used Symbols

Symbols	Definition	Units
k	the traffic density on a freeway	veh/km
k_j	the jam density on a freeway	veh/km
K	the density of the merging area	veh/km
K_j	the jam density of the merging area	veh/km
v_f	the free-flow speed when k = 0	km/h
v	the mean speed at density k	km/h
q	the traffic flow	veh/h
q_m	the maximum traffic flow	veh/h
k_m	the traffic density of the maximum traffic flow	veh/km
v_m	the mean speed of the maximum traffic flow	km/h
T	the largest time for reaching the merging points from a tollbooth	h, min, s
S	the total length of all roads in the merging area	m
Q	the largest quantity of vehicles	veh
	the mean number of cars arriving at the toll plaza in a time step	
B	the number of tollbooths	

3.1.2 Local Assumptions

- The serving interval time of every tollbooth is Δt.
- The relationship between T and S is linear, because the longer the S is, the more time will be cost for a car to reach the merging point.
- There is a linear relationship between the jam density K of the merging area and its total length S, since the jam density K will certainly get larger when the total length of the merging area becomes longer.
- The number of vehicles which enter the merging area from every tollbooth obeys the Poisson probability distribution.

3.1.3 The Density-Flow Model and Speed-Flow Model[3, 4]

We introduce the linear relationship between speed and traffic density proposed by Greenshields in 1935:

$$v = v_f \cdot (1 - \frac{k}{k_j}) \ , \tag{1}$$

where: k is the traffic density (veh/km); k_j is the jam density (veh/km); v_f is the free-flow speed (km/h) when $k = 0$; v is the mean speed (km/h) at density k.

Furthermore, we also know that:

$$q = v \cdot k \ , \tag{2}$$

where: q is the traffic flow (veh/h).
Substituting equation (1) in equation (2), we get the Density-Flow Model:

$$q = v_f \cdot k \cdot (1 - \frac{k}{k_j}) \ . \tag{3}$$

Similarly we get the Speed-Flow Model:

$$q = k_j \cdot v \cdot (1 - \frac{v}{v_j}) \ . \tag{4}$$

Figure 3 exactly illustrates the above equations:

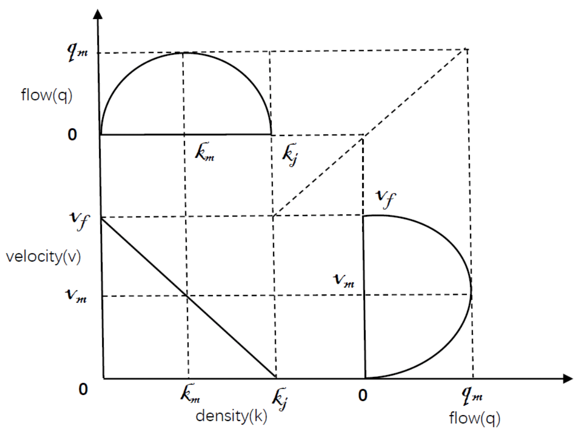


Figure 3: The relationship of q, v, k

Since the Density-Flow Model and the Speed-Flow Model are parabolic shape, a maximum traffic flow can be gained when:

$$k_m = \frac{k_j}{2} \ , \tag{5}$$

$$v_m = \frac{v_f}{2} \ , \tag{6}$$

Substituting equation (5) in equation (3), we get the maximum traffic flow:

$$q_m = \frac{v_f \cdot k_j}{4} \ . \tag{7}$$

This conclusion will be used as the control algorithm in the length control model introduced later.

3.1.4 Limitations of the Density-Flow Model and Speed-Flow Model

Although the Density-Flow Model and Speed-Flow Model greatly simplify the traffic flow problem, it is based on the assumption of linear relationship between speed and density. In reality, we can hardly get such a relationship between speed and density. However, simulation shows that this drawback has little influence on our model since we adjust the control algorithm with restriction $0 \leq k \leq \frac{k_j}{2}$.

3.1.5 The Length-Control Model

In this model, we seek to find the best total length of the merging area in order to get the maximum throughput of the toll plaza. According to the Density-Flow Model and the Speed-Flow Model described above, the maximum throughput of the merging area can be achieved when its density K is nearly half of its jam density K_j(i.e., $K = \frac{K_j}{2}$). Considering the relationship between the total length of the merging area and its density, we finally develop our Length-Control Model.

Before introduction of our Length-Control Model, we define an important variable S, indicating the total length of all paths in the merging area. To explain this variable clearly, we make use of a graph. In Figure 4, $S = S_1 + S_2$.

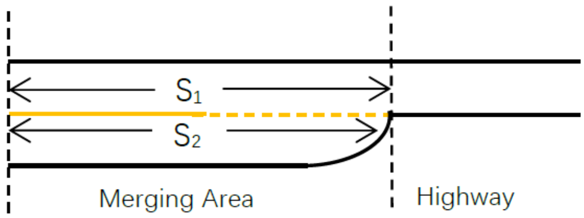


Figure 4: The Graph for explaining variable S

Firstly, we define the longest time T when a car reaches the merging points from one of the tollbooths (i.e., the last car of cars leaving the tollbooths at the time interval Δt to reach the merging points). Then we simply assume a linear relationship between T and S:

$$T = aS + c_1 \ , \ \ with \ a > 0, \ \ c_1 > 0 \ . \tag{8}$$

Where a and c_1 are constant. Such an assumption is reasonable because the longer the S is, the more time will be cost for a car to reach the merging point. Taking constantly changing velocity of vehicles into consideration, we use this equation to approximately simulate the real condition.

Then, we assume that the serving interval time is Δt, and we can easily get that the largest quantity of vehicles(Q) in the merging area:

$$Q = \frac{B \cdot T}{\Delta t} = \frac{B \cdot (aS + c_1)}{\Delta t} \ . \tag{9}$$

Once we get the quantity of vehicles and the total length of the merging area, we can get the density of the area:

$$K = \frac{Q}{S} = \frac{B \cdot T}{S \cdot \Delta t} = \frac{B \cdot (aS + c_1)}{S \cdot \Delta t} \ . \tag{10}$$

Given that the jam density of a road is related to its length, we can simply assume a linear relationship between the jam density of the merging area and its total length:

$$K_j = b \cdot S + c_2, \ \ with \ b > 0, \ \ c_2 \leq 0 \ . \tag{11}$$

Where b and c_2 are constant. We believe such an assumption is also rational since the jam density will certainly get larger when the total length of the merging area becomes longer.

In order to maintain the maximum throughput of the toll plaza and avoid congestion, the density of the merging area must satisfy:

$$0 \le K \le \frac{K_j}{2} \ . \tag{12}$$

Substituting equation (10) and equation (11) in inequality (12), we finally get the size control algorithm:

$$\begin{cases} B \cdot (aS + c_1)/(S \cdot \Delta t) \ge 0 \\ S \ge 0 \\ b \cdot \Delta t \cdot S^2 + (c_2 \cdot \Delta t - 2B \cdot a)S - 2B \cdot c_1 \ge 0 \end{cases} \quad with\ c_1 \ge 0,\ c_2 \le 0\ , \tag{13}$$

Solving the inequalities (13),

First we get:

$$\Delta = (c_2 \cdot \Delta t - 2B \cdot a)^2 + 8B \cdot b \cdot \Delta t \cdot c_1 > 0\ . \tag{14}$$

It's obvious that the inequalities (13) has the solution, and it is also obvious that

$$\Delta > (c_2 \cdot \Delta t - 2B \cdot a)^2 \Leftrightarrow \sqrt{\Delta} > |c_2 \cdot \Delta t - 2B \cdot a| = 2B \cdot a - c_2 \cdot \Delta t\ . \tag{15}$$

Then,

$$S = \frac{-(c_2 \cdot \Delta t - 2B \cdot a) - \sqrt{\Delta}}{2b \cdot \Delta t} < 0\ . \tag{16}$$

Finally, the solution of the inequalities (13) is:

$$S \ge \frac{-(c_2 \cdot \Delta t - 2B \cdot a) + \sqrt{\Delta}}{2b \cdot \Delta t}\ . \tag{17}$$

Thus we obtain the range of the total length S of the merging area that is needed to meet the maximum throughput.

The equation (9) implies that Q is the largest traffic flow of the merging area. However, the largest traffic flow doesn't occur all the time. Thus, we improve our model:

We first assume that the number of vehicles which enter the merging area from every tollbooth obeys the Poisson probability distribution. The number of vehicles leaving each tollbooth is independent from each other. We then get the probability of n vehicles that leave one of the tollbooths at an interval time[5]:

$$P(N = n) = \frac{e^{-\lambda \Delta t} \cdot \lambda \Delta t}{n!}\ . \tag{18}$$

where: N is a random variable describing the number of vehicles departing one of the tollbooths. Δt is the time interval. λ is the mean number of cars that arrive at the toll plaza in a time step, which can be gained by using queuing model to analyze the arriving cars at the tollbooths (see the proof in MCM paper ***For Whom the Booth Tolls***[5] in 2005).

According to the characteristics of Poisson probability distribution, the average number of vehicles that leave each tollbooth at an interval time Δt is:

$$E(N) = \lambda\ . \tag{19}$$

So, we can get that:

$$Q = \frac{T \cdot B \cdot E(N)}{\Delta t} = \frac{(aS + c_1)B \cdot \lambda}{\Delta t} \ . \tag{20}$$

Then density k changes into:

$$K = \frac{Q}{S} = \frac{T \cdot B \cdot E(N)}{S \cdot \Delta t} = \frac{(aS + c_1)B \cdot \lambda}{S \cdot \Delta t} \ . \tag{21}$$

The inequalities (13) turn into:

$$\begin{cases} (aS + c_1)B \cdot \lambda/(S \cdot \Delta t) \geq 0 \\ S \geq 0 \\ b \cdot S^2 + (c_2 - 2aB \cdot \lambda/\Delta t) \cdot S - 2B \cdot c_1 \cdot \lambda/\Delta t \geq 0 \end{cases} \quad \textit{with } c_1 \geq 0, \ c_2 \leq 0 \ , \tag{22}$$

Solving the inequalities (22):

Firstly, we get:

$$\Delta = (c_2 - \frac{2a \cdot B \cdot \lambda}{\Delta t})^2 + \frac{8b \cdot B \cdot c_1 \cdot \lambda}{\Delta t} > 0 \ . \tag{23}$$

It's obvious that the inequalities (22) has the solution, and it is also obvious that:

$$\Delta > (c_2 - \frac{2a \cdot B \cdot \lambda}{\Delta t})^2 \ \Leftrightarrow \ \sqrt{\Delta} > |c_2 - \frac{2a \cdot B \cdot \lambda}{\Delta t}| = -c_2 + \frac{2a \cdot B \cdot \lambda}{\Delta t} \ . \tag{24}$$

Then,

$$S = \frac{-(c_2 - 2a \cdot B \cdot \lambda/\Delta t) - \sqrt{\Delta}}{2b} < 0 \ . \tag{25}$$

Finally, the solution of the inequalities (22) is:

$$S \geq \frac{-(c_2 - 2a \cdot B \cdot \lambda/\Delta t) + \sqrt{\Delta}}{2b} \ . \tag{26}$$

3.1.6 Limitations of the Length-Control Model

Although the Length-Control Model is helpful for our design, it oversimplifies the problem and thus is incomplete. The assumptions we make seem a little unrealistic. In other words, the result we obtain from this model can only be used as a reference data when applied to the construction of the toll plaza.

3.2 Microscopic Model

3.2.1 Local Symbols

Table 2: Frequently Used Symbols

Symbols	Definition	Units
B	the number of tollbooths	
L_a	the number of lanes	
L	the total length of the acceleration lane	m
l	the total length of the acceleration lane	m
q_1	the traffic flow of the upstream main lane	veh/h
q_2	the traffic flow of the side main lane	veh/h
q_3	the traffic flow of the acceleration lane	veh/h
T_c	the time gap that a car can merge into main lane successfully	s
S_0	the distance a car drive before merging	m
u	the initial speed of the car at the starting point	km/h
a_0	the acceleration of the car	m/s^2
Δt	the time interval	h, min, s
V_{max}	the largest velocity of vehicles in the merging area	km/h
U	the mean velocity of vehicles on the main lane	km/h
U'	the speed a car need to successfully merge into the main lane	km/h

3.2.2 Local Assumptions

- The vehicles on the main road continue to accelerate, but for safety, their acceleration will decrease when they arrive the merging subarea. And the vehicles on merging lane keep their velocity. The speed difference between them will not be so large that they are failed to merge.

- The vehicles on the main lane will continue to accelerate with acceleration $a_1 = 1\ m/s^2$ in the No Lane-Changing Subarea, while $a_1 = 0.5\ m/s^2$ in the merging subarea, until the speed reach the maximum velocity $V_{max} = 60\ km/h$ (i.e., 16.7 m/s). Because for safety, the drivers would not hold a high speed.

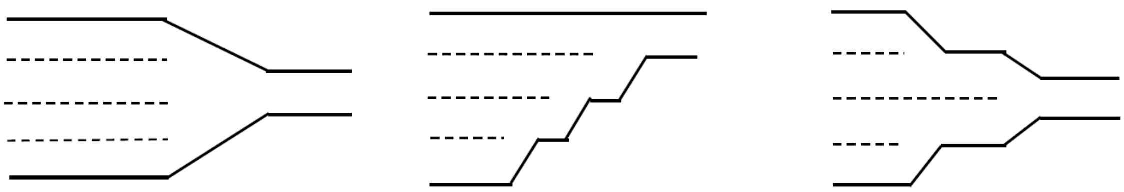
- The time gap between two cars on the side lane obeys the Erlang distribution of first order.

- The velocity of vehicles must arrive at merging velocity U' when they are merging into the main lane.

- When in the merging area, vehicles will merge into one lane once there exists enough space.

3.2.3 Analysis of Existing Toll Plazas Design

We search the papers of former MCM Problem B and many literatures. There are mainly three kinds of designs presented in them:



(a) The First Design (b) The Second Design (c) The Third Design

Figure 5

The following is the performance analysis of the design showed above:

- **The First Design:** The advantage of this design is that the acceleration lanes can be designed to be relatively short, which saves land cost and road construction cost. But owing to only one lane of the highway in the merging area, it is very likely to cause road congestion so that the efficiency of ramp merging will be very low. The simulation and proof refer to the literature *Toll Plaza Merging Traffic Control for Throughput Maximization*[2].

- **The Second Design:** The lane of the highway is on the one side of the merging area. This results that the vehicles on the other side of the merging area have no choice but constantly change lanes before arriving at the merging point to enter the main road. This leads to the problem of low efficiency of ramp merging. Meanwhile, the fan-in area needs designing to be very long to satisfy the need of lane-charging.

- **The Third Design:** This design merges traffic flow into the main lanes step by step. It can reduce congestion. But it only produces two merging points so the efficiency is too low. Thus, it needs larger merging area. If the quantity of tollbooths is close to the quantity of the lanes, this design is quite rational. But when the quantity gap between tollbooths and lanes becomes larger, the length of merging area increases rapidly, needing more land to construct fan-in area than the first design. Otherwise, the merging area is likely to become congested when the traffic is busy. To construct such merging area needs much cost.

From the above, the third design maybe the best one of the three for it splits the merging area and have high efficiency. But we have to take the land cost and road construction cost into consideration. Besides, we found that unsafe lane-changing behavior is one of the main reasons that causes car crashes in highways by Google. Therefore, we intend to reduce the costs and the possibility of accidents at the same time to further improve efficiency with the divide-and-conquer method.

3.2.4 Overview of Our Model

To optimize the model of Figure 5(c), making the length of merging area would not increase rapidly when the quantity gap between tollbooths and lanes is large. We refer to the Binary-Tree structure in computer science and put forward a model with similar structure, named as **Binary-Tree Merging Model**, abbreviated as **BT Merging Model**(see Figure 6). Meanwhile, we introduce the concept of the **level** of merging area. The level of merging area indicates the number of lane-changing required for a car which enters the fan-in area from the outermost tollbooths of the toll plaza. Its magnitude corresponds to the height of Binary-Tree structure in computer science. Take Figure 6 for example, its number of level is 2.

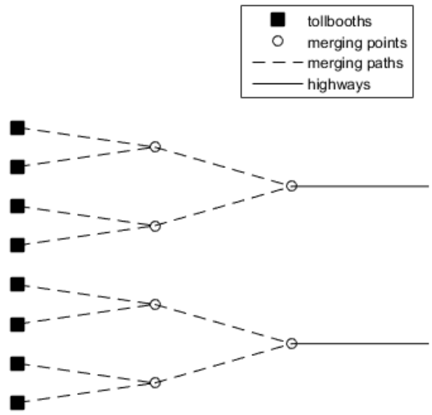


Figure 6: A Legend of BT Merging Model

Because our model needs to consider the relationship between tollbooths B and the number of lanes L, we refer to the conclusion of one of the paper on MCM Problem B[6], and find that the ratio of B to $L_a$is no more than 4. Then, for the universality of our model, we use the maximum of the ratio of B to L_a, that is 4. Thus, we design a prototype of the merging area. We will explain and demonstrate its superiority.

3.2.5 The Merging Model

In order to describe the BT Merging Pattern mentioned above in mathematical model, We develop our merging model based on the *Merging model of vehicle on freeway acceleration*[7] proposed by LI Wen-quan, Wang Wei, etc. By moderately changing the initial conditions of the original merging model, we make the merging model quietly fit the BT Merging Pattern.

3.2.6 Merging Model on Highways[7]

We first introduce the original merging model of vehicle on freeway acceleration:

Where: L is the total length of the acceleration lane; l is the driving distance on the acceleration lane.

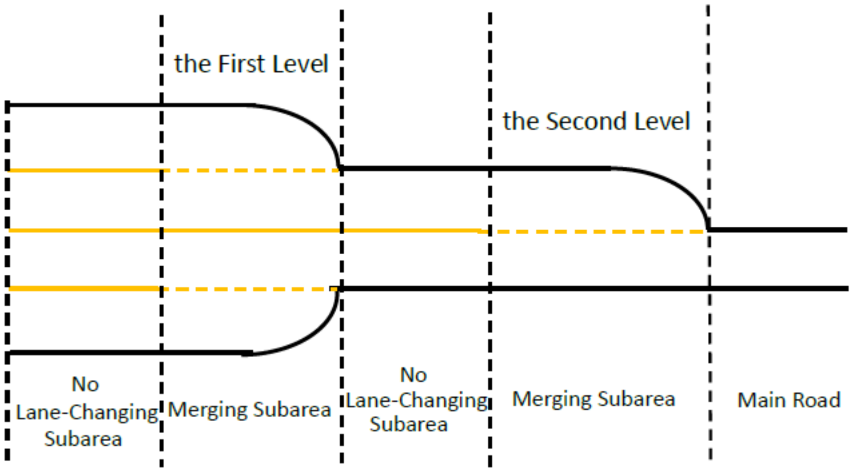


Figure 7: Our Prototype of the Merging Area

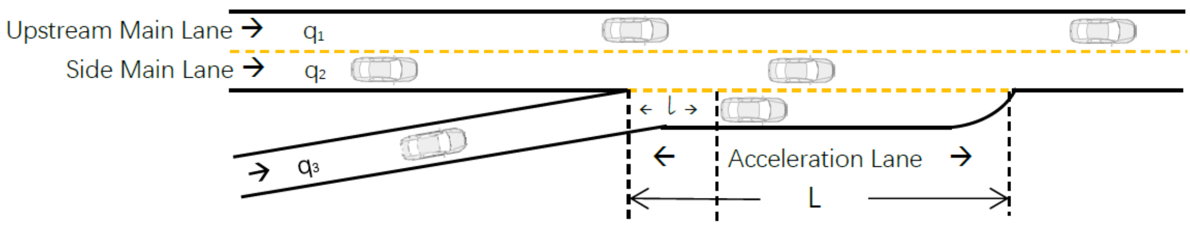


Figure 8: The Original Merging Model

The formula recommended by American HCM1985[8] properly describes the relationship of the traffic flow of the upstream main lane, the side main lane and the acceleration lane:

$$q_2 = 136 + 0.345q_1 - 0.115q_3\ . \tag{27}$$

Where: q_1 is the traffic flow of the upstream main lane; q_2 is the traffic flow of the side main lane; q_3 as the traffic flow of the acceleration lane.

Assuming that the time gap between two cars on the side lane obeys the Erlang distribution of first order, we then get the cumulative distribution function (cdf) of the time gap:

$$F(t) = P(h \geq t) = e^{-q_2 \cdot t/3600}\ . \tag{28}$$

We define U (m/s) as the mean velocity of vehicles on the main lane and U' (m/s) as the speed a car need to successfully merge into the main lane. Then we simply suppose that:

$$U' = U - 5.5\ . \tag{29}$$

Once we get the merging speed U', we can find the distance a car drives before merging:

$$S_0 = \frac{(U'^2 - u^2)}{2a_0}\ . \tag{30}$$

Where: S_0 (m) is the distance a car drives before merging; u (m/s) is the initial speed of the car at the starting point; $a_0(m/s^2)$ is the acceleration of the car.

Next, we define T_c as the time gap a car on the acceleration lane needed to merge into the main lane. Thus, $F(T_c(l))$ is the probability of the occurrence of the time gap after a car drives l distance on the acceleration lane. We then get the difference equation of the probability that a car successfully merges into the main lane:

$$P(l + \Delta l) = P(l) + [1 - P(l)]\Delta t \cdot F(T_c(l)),\ with\ l > S_0\ . \tag{31}$$

Where: $P(l)$ is the probability that a car successfully merges into the main lane after driving l distance on the acceleration lane. Δt is the time for a car to drive Δl distance on the acceleration lane.

Since Δt in equation (31) is only a very short time, we can assume that:

$$\Delta t = \frac{\Delta l}{U'}\ . \tag{32}$$

Substituting equation (32) in equation (29), we get:

$$(P(l + \Delta l) - \frac{P(l))}{\Delta l} = -\frac{P(l) \cdot F(T_c(l))}{U'} + \frac{F(T_c(l))}{U'},\ with\ l > S_0\ . \tag{33}$$

When $\Delta l \to 0$, we get the corresponding differential equation:

$$P'(l) = -Q(l) \cdot P(l) + Q(l)\ , \tag{34}$$

where $Q(l) = F(T_c(l))/U'$. Then we get the expected value of $Q(l)$:

$$Q = E(Q(I)) = \frac{E(F(T_c(l)))}{U'} = \frac{e^{-q_2 \cdot T_c(l)/3600}}{U'}\ . \tag{35}$$

The general solution of the equation (34) is:

$$P(l) = e^{-Q \cdot l}(e^{-Q \cdot l} + C)\ , \tag{36}$$

Note that $P(l)$ is the probability that a car successfully merges into the main lane after driving l distance on the acceleration lane. It is apparent that $P(0) = 0$. By substituting $P(0)$ in equation (35) we get $C = -1$. Then a specific equation can be obtained:

$$P(l) = 1 - e^{-Q \cdot l}\ , \tag{37}$$

In other words, we get the probability that a car fail to merge into the main lane after driving l distance:

$$P_d(l) = 1 - P(l) = e^{-Q \cdot l},\ with\ l > S_0\ , \tag{38}$$

Considering all the equations above, we finally get:

$$P_d(l) = \begin{cases} 1 & 0 < l \leq S_0 \\ e^{-Q(l - S_0)} & S_0 < l \leq L \end{cases} \tag{39}$$

3.2.7 Our Modification

Considering our BT merging pattern, we modify the original merging model and use it to calculate the length of the merging lane. We believe that such a length can ensure the rate of successful merging and thus reduce the probability of congestion.

Since we remove the upstream lane of the main load, we can only define q_2 as the traffic flow of the side lane of the main load and q_3 as the traffic flow of the merging lane (which we previously define as acceleration lane in equation (40)). In addition, since the lanes in No Lane-Changing Area have the same fixed length, we can find that the velocity of vehicles is the same when they are going to arrive the merging subarea. Thus, we assume that vehicles on the main road are continue to accelerate, but for safety, their acceleration will decrease when they arrive the merging subarea. And the vehicles on merging lane keep their velocity. The speed difference between them will not be so large that they are failed to merge. Then, we get that:

$$a_0 = 0 \ and \ S_0 = 0 \ . \tag{40}$$

Thus we develop our merging model:

$$P_d(l) = e^{-Q \cdot l}, \ where \ Q = e^{-\frac{q_2 \cdot T_c(l)}{3600}} / U' \ . \tag{41}$$

Next, we seek to figure out the length of the merging lane.

Firstly, we assume that the vehicles on the main lane will continue to accelerate with acceleration $a_1 = 1 \ m/s^2$ in the No Lane-Changing Subarea, while $a_1 = 0.5 \ m/s^2$ in the merging subarea, until the speed reach the maximum velocity $V_{max} = 60 \ km/h$ (i.e., $16.7 \ m/s$). Then, we let the traffic flow on the main lane $q = 2000 \ veh/h$ in accordance with the traffic flow in peak period (see in the paper *Merging model of vehicle on freeway acceleration*[7]). Since our model describes the four-to-one BT merging pattern, the traffic flow of each merging lane of the first level is $2000/4 = 500 \ veh/h$. As a result, for the merging points of the first level, the traffic flow of the side lane is $500 \ veh/h$.

We design that the length of each no-changing lane is 50m. Then the velocity of a car reaching the merging point of the second level is $10 \ m/s$, then we let $T_c(l) = 5 \ s$. Substituting this velocity in our merging model, we get:

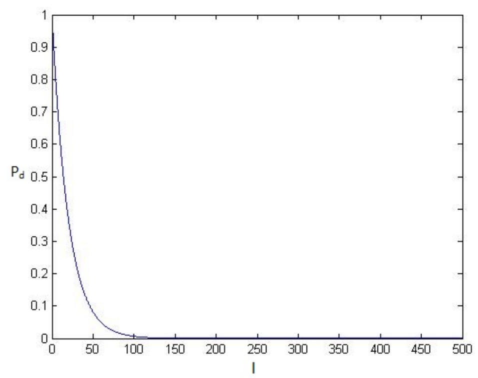


Figure 9: The relationship of $P_d(l)$ and l on the first level.

From Figure 9 we get:

When $l_1 > 50\ m$ (l_1 is the distance between the starting point and the merging point of the first-level merging lane), the probability of unsuccessful merging is so small that we can ignore it. Thus, we design that $L = 50\ m$ (L is the length of the first-level merging lane).

When cars reach the merging points of the second level, their velocity $U = V_{max} = 16.7\ m/s$, which can be easily calculated with Newtonian Classical Mechanics[1]. And the traffic flow of each merging lane of the second level is $500 + 500 = 1000\ veh/h$. In general, the merging speed U' is less than the U, we let $U' = U - 2 = 14.7\ m/s$, and $T_c(l) = 3\ s$. Substituting this velocity in our merging model, we get:

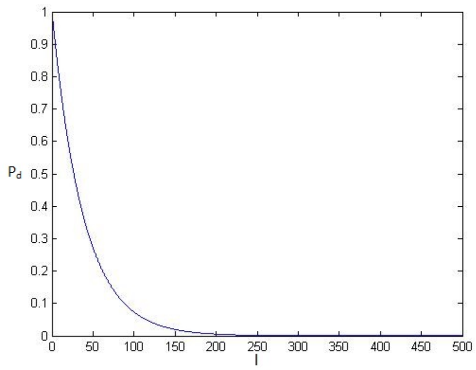


Figure 10: The relationship of $P_d(l)$ and l on the second level.

When $l_2 > 100\ m$ (l_2 is the distance between the starting point and the merging point of the second-level merging lane), the probability of unsuccessful merging is so small that we can ignore it. Thus, we design that $L_2 = 100\ m$ (L_2 is the length of the second-level merging lane).

Hence, we finally get the specific design of the merging area:

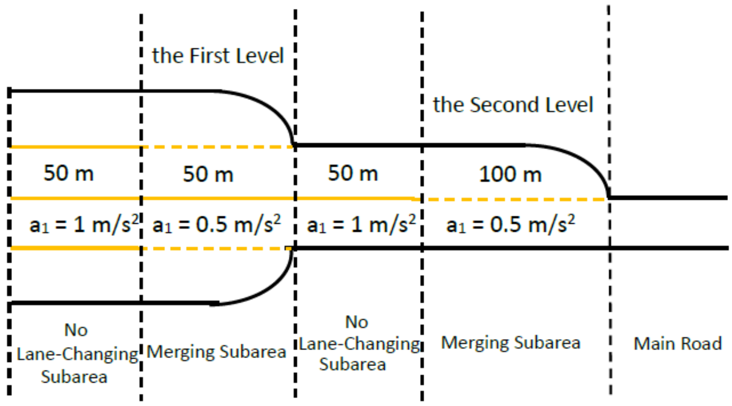


Figure 11: The Four-To-One BT Merging Model

In this design, we calculate that the magnitude of S is $700\ m$.

Through the calculation of the above sub-model, we have a deeper understanding of the model. In the actual plan, we can make good use of the above

[1] $v(t) = v_0 + a \cdot t$, a is the acceleration.

sub-model and its methods to design the merging area. However, the value of the q_2 should be recalculate, because we use the q_{max} as q_2 above, which will increase land and road construction cost.

To get the appropriate value of q_2, we assume that the internal of the vehicle flow is $[q_{min}, q_{max}]$, and $f(q)$ is the probability density function of the vehicle flow. But in most cases, the traffic flow doesn't reach the maximum. If we use the maximum traffic flow in the actual design, S will be larger, resulting in low road utilization rate and the waste of cost.

For the random value of q', the possibility of the vehicle flow less than q' is:

$$P(q') = \int_{q_{min}}^{q'} f(q)dq. \tag{42}$$

Then, the possibility of the vehicle flow not less than q' is $1 - P(q')$. Considering the land and road construction cost and the problem of traffic congestion, we can assign the value of 97% to $P(q')$. That's means, there are only

$$24(h) \times 3\% = 43.2\ (min) \tag{43}$$

that the designed merging area becomes congestion.

Assume in the process of designing the merging area, there are a roads in the No Lane-Changing Area and b levels in the total merging area. Then, we can get the total distance S:

$$S = 50a + \sum_{i=1}^{b} L(i)\ , \tag{44}$$

where $L(i)$ is the length of the merging lane with i-th level.

A last, our two sub-models have been already established. As a result, we obtain the following formula:

$$S \geq \frac{-(\Delta t \cdot c_2 - 2a \cdot \lambda \cdot B) + \sqrt{(\Delta t \cdot c_2 - 2a \cdot \lambda \cdot B)^2 8b \cdot c_1 \lambda \cdot B \cdot \Delta t}}{2b\Delta t} = S_1\ , \tag{45}$$

$$S = 50a + \sum_{i=1}^{b} L(i) = S_2\ . \tag{46}$$

When $S_2 \geq S_1$, $S = S_2$.

When $S_2 < S_1$, we can adjust the length of the lanes in the second sub-model to make it satisfy $S_2 \geq S_1$.

4 Sensitivity Analysis

To test the robustness of our model and obtain improvements, we now analyze the performance of our solution in light and heavy traffic, how would our solution change as more autonomous vehicles are added to the traffic mix and how is our solution affected by the proportions of different collection booths.

4.1 The performance of our solution in light and heavy traffic

In light traffic, our model works well with safety. In heavy traffic, from the equation (43), we know that there are only 43.2 minutes that the designed merging area may become congestion in one day, which is still better than the current toll plaza design. Thus, our model is relatively stable in both light and heavy traffic.

4.2 How would our solution change with more autonomous vehicles

Since the autonomous vehicles are under the control of computers, they can be well designed to strictly obey the rules of our model. With less subjective factors that may cause safety problems, more autonomous vehicles added to the traffic mix make our model more reasonable and stable.

4.3 How is our solution affected by the proportions of d-ifferent collection booths

Different collection booths only affect the service time of the tollbooths. Since we assume that the number of vehicles leaving the tollbooths obeys the Poisson distribution, the proportions of different collection booths have little influence on our model.

5 Strengths and Weaknesses

5.1 Strengths

- **Simplification.** By making proper assumption, we simplify the problem and concentrate on the design of the shape, size and merging pattern of the merging area.
- **New merging pattern.** We create the Binary-Tree merging pattern for vehicles to merge back to the highway from the tollbooths, which efficiently solve the congestion problem.
- **Comprehensiveness.** We comprehensively establish both the macroscopic and microscopic models to analyze the problem and develop the solution.

5.2 Weaknesses

- **No specific solution.** Lack of data, we cannot get a solution of the problem in detail.
- **Incomplete.** Our model can be improved by taking other specific factors such as the subjective factor of drivers, the weather condition and so on into consideration.

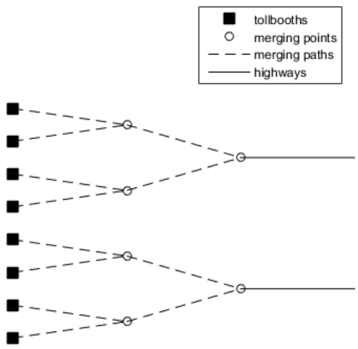
A A Letter to New Jersey Turnpike Authority

January 23, 2017

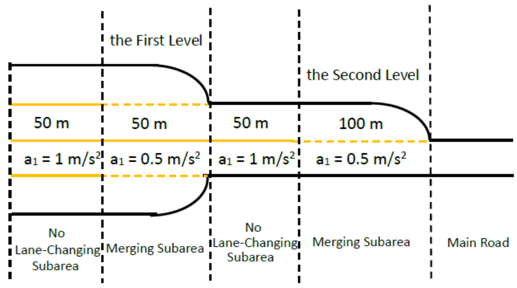
Dear governor of New Jersey Turnpike Authority,

We all know that the design of the toll plaza on toll highways is very important for the traffic efficiency. Hearing that improvements should be applied to the design of the shape, size and merging pattern of the area following the toll barrier in which vehicles fan in from tollbooths down to the lanes of traffic (we call it merging area in this letter), we are pleasure to introduce our solution to you. Different from general highway planning methods, we develop some mathematical models to help our design.

For the design of merging pattern of the merging area, in order to reduce the congestion when vehicles merge back to the lanes of traffic, we design the Binary-Tree merging pattern illustrated in Figure 12(a). Based on our Binary-Tree merging pattern, we devise the corresponding shape of the merging area which is shown in Figure 12(b).



(a) The Binary-Tree merging pattern



(b) The shape of the merging area

Figure 12

For the size of the merging area, we apply our Length-Control model and Merging model to calculate the proper total length of the merging area. The total length we get is helpful to maintain the maximum throughput of the merging area even in heavy traffic and doesnt cost much. Comparing our design of toll plaza with existing ones, we believe our solution is better.

Yours sincerely,

Team #55671

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