Strange attractors and the weather forecast of Edward Lorenz

The mathematics behind the computer programme "Simulator"

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This document describes strange attractors that were discovered in the context of a weather simulation by Edward Lorenz. We start from elementary accessible mathematics and extend the concept of linear approximation in one dimension to the $\mathbb{R}^3$. The mathematical investigations are supported by experiments with the computer program "Simulator", which demonstrates the behaviour of the Lorenz system very well.

The document is intended as a stimulus for extended maths lessons at intermediate level, whether for courses outside the compulsory curriculum or for individual work by interested pupils. It is particularly suitable as an introduction to the methods of linear algebra and multidimensional differential calculus.

The entire series of topics related to the "Simulator" includes:

- *The chaotic properties of logistic growth*
- *The oval billiard table and periodic orbits*
- *Newton iteration and the complex roots of unity*
- *Iteration of quadratic functions in the complex plane*
- *Numerical methods and coupled pendulums*
- *Planetary motion and the three-body problem*
- ***Strange attractors and the weather forecast of Edward Lorenz***
- *Fractal sets and Lindenmayer systems*
- *The history of chaos theory*
- *Programming your own dynamic systems in the "simulator"*

Each topic is dealt with in a separate document.

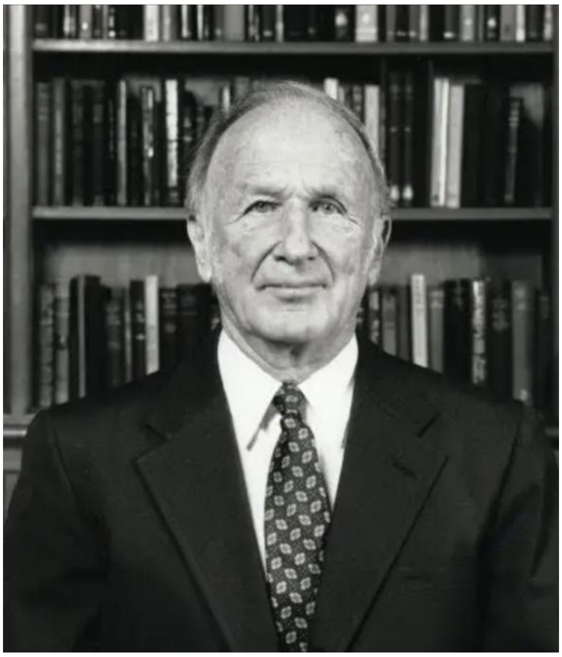
The computer programme "Simulator" enables the simulation of simple dynamic systems and experimentation with them. The code is publicly available on GitHub, as is a Microsoft Installer version. The corresponding link is: https://github.com/HermannBiner/Simulator. The following documentation is integrated into the "Simulator" in German and English:

- *Mathematical documentation with examples and exercises*
- *Technical documentation with a detailed description of the functionality*
- *User manual with examples*
- *Version history*

Contents

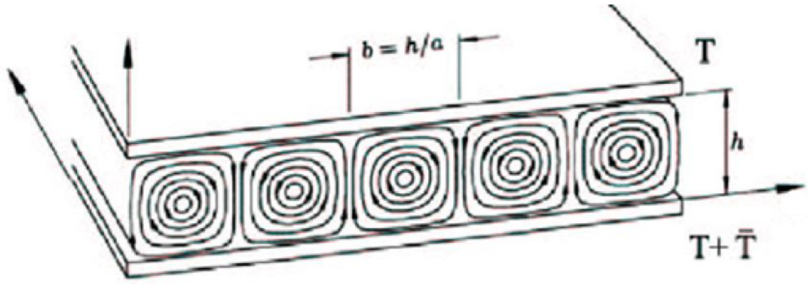
1. The weather forecast of Edward Lorenz

Thunderstorms can be quite turbulent, especially in the mountains or at sea. It is also difficult to predict exactly where and when they will occur. The earth's atmosphere is a complex system and long-term weather forecasting seems even more impossible. It is thought that this is mainly because many parameters play a role and that an exact physical description of what happens in the atmosphere is therefore unattainable. If the processes in extreme weather situations appear chaotic, then this seems to be because we do not have a handle on all the factors that influence the weather.



Edward Lorenz (1917 - 2008), mathematician and meteorologist, investigated a highly simplified weather model in which all parameters are known, and which is described by a simple system of differential equations. His aim was to be able to produce a long-term weather forecast. To his own surprise, he realised that even this simple model was unpredictable: he came across a chaotic dynamic system. He published his results in a meteorological journal in 1963 [2]. For this reason, his work remained unnoticed in mathematics for a long time. Today he is regarded as one of the forerunners of chaos theory.

For his investigations, Edward Lorenz used a model that had already been theorised in 1916 by the British Nobel Prize winner Lord Rayleigh (1842 - 1919).



Lord Rayleigh's convection model (source: [1])

A viscous (i.e. deformable) and incompressible fluid is located between two plates at a distance h. There is a temperature difference between the top and bottom $\overline{T}$, which leads to the formation of convection rolls above a critical value. Liquid elements heated from below rise due to their lower density and colder liquid volumes sink from above. The system is homogeneous along the longitudinal axis (i.e. towards the rear in the sketch).

This model for atmospheric flows is so simplified that it hardly reflects reality. Nevertheless, even this simple model harbours surprises, as we will soon see.

In his investigations, Edward Lorenz used the following system of differential equations, which is known today as the Lorenz system:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = \varrho x - y - {\color{red}xz} \\ \dot{z} = -\beta z + {\color{red}xy} \end{cases}$$

This system is "almost" linear. The non-linear parts are highlighted in red above.

We also write for the Lorenz system:

$$\dot{\vec{x}} = \mathcal{L}(\vec{x})$$

With

$$\mathcal{L}(\vec{x}) = \begin{pmatrix} -\sigma x + \sigma y \\ \varrho x - y - xz \\ -\beta z + xy \end{pmatrix}$$

We want to look at the system from a purely mathematical point of view without interpreting its physical meaning. We consider three real functions $x(t), y(t), z(t)$ as a function of time t. In the simulator, the functions have the value range $x, y \in [-20,20], z \in [0,50]$.

σ, β, ϱ are real constants. Lorenz worked with the values $\sigma = 10, \beta = {}^{8}/_{3}$ and ϱ can be varied in the interval$[0,30]$.

If a starting point $x(0), y(0), z(0)$ is specified, the further course of the system over time is *uniquely determined*. This progression is described by a "path" or "trajectory" $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \in \mathbb{R}^3$.

The Lorenz system does not have an analytical closed-form solution, i.e. a "formula" that describes the corresponding trajectory for each starting point. Lorenz therefore analysed the system using numerical methods. For example, instead of the infinitesimal quantity dt, one can consider a very small "time step" Δt and rewrite the system:

$$\begin{cases} \frac{\Delta x}{\Delta t} = -\sigma x + \sigma y \\ \frac{\Delta y}{\Delta t} = \varrho x - y - xz \\ \frac{\Delta z}{\Delta t} = -\beta z + xy \end{cases}$$

Then, starting from a starting point x_1, y_1, z_1, the next point of the trajectory approximated in this way is obtained by the recursion formula:

$$\begin{cases} x_{n+1} = x_n + (-\sigma x_n + \sigma y_n)\Delta t \\ y_{n+1} = y_n + (\varrho x_n - y_n - x_n z_n)\Delta t \\ z_{n+1} = z_n + (-\beta z_n + x_n y_n)\Delta t \end{cases}$$

However, this approximation method, also known as the Euler method (after the Swiss mathematician Leonhard Euler 1707 - 1783), is too imprecise. Lorenz worked with a refined method, the so-called implicit midpoint rule, whereby he had to consider the computing power of computers at the time. The simulator works with an even better method, namely the fourth-order Runge-Kutta method. These numerical methods are dealt with as part of the school project *Numerical Methods and Coupled Pendulums* and are not explained further here.

As part of his investigation, Lorenz came to the surprising conclusion that the behaviour of this simple system is *unpredictable* in the long term and that the smallest changes in the initial value cause qualitatively large changes in the associated trajectories. Lorenz is credited with coining the term "butterfly effect". He says that the fluttering of a butterfly in Brazil can trigger a hurricane in Texas.

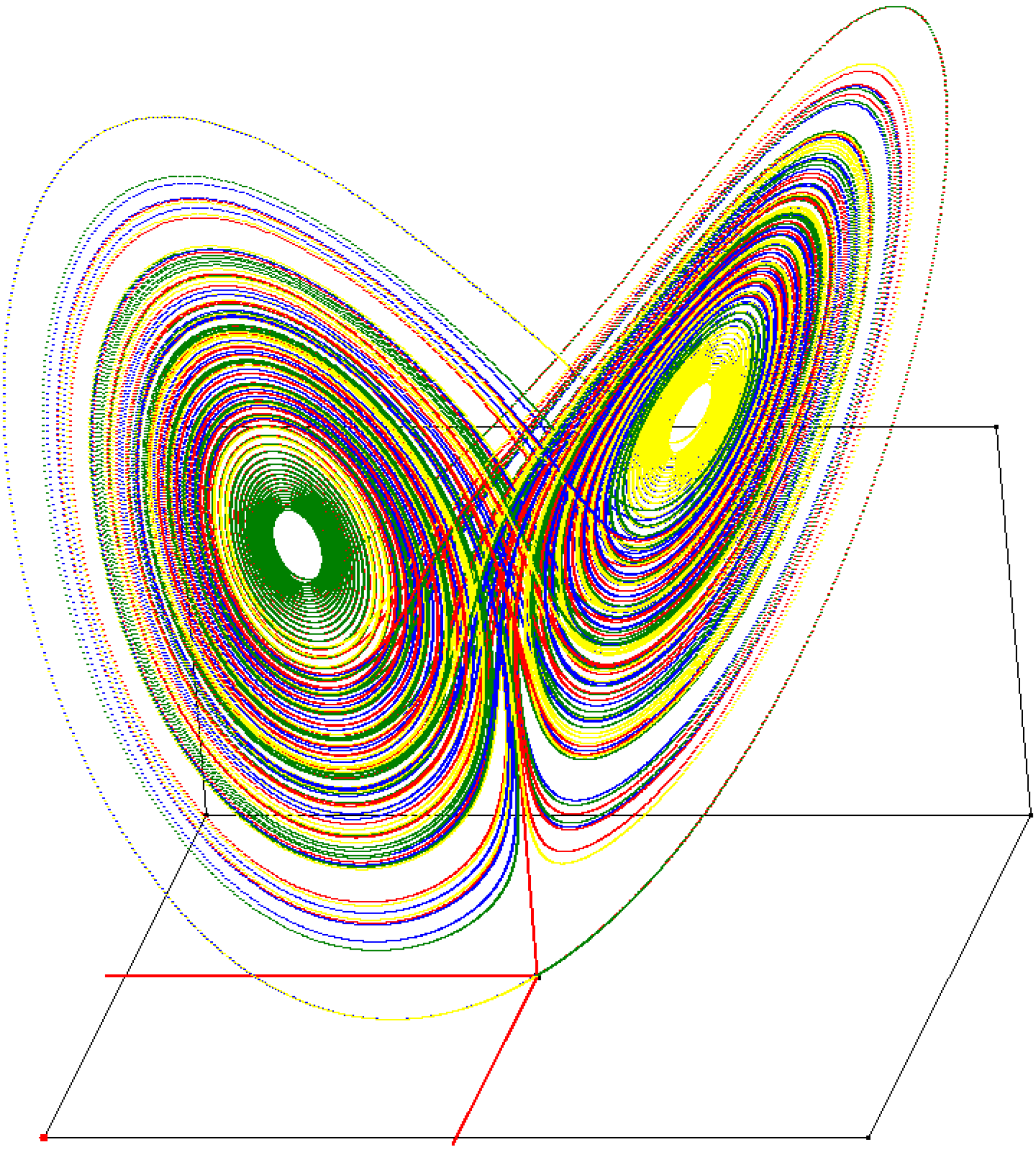
Lorenz states in [2]:

"When our results concerning the instability of non-periodic flow are applied to the atmosphere, which is ostensibly non-periodic, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly".

Here we can experiment with the simulator to reproduce Lorenz's results. For example, we select $\varrho = 28$ and four starting points close to each other with different colours:

$$\begin{cases} (0.01, 0.01, 0.1)\ blue \\ (-0.01, -0.01, 0.1) red \\ (-0.01, 0.01, 0.1) green \\ (0.01, -0.01, 0.1) yellow \end{cases}$$

and run the simulation. The trajectories associated with the starting points are drawn in the same colours. We perform around 200,000 steps in the simulator in the shortest possible time and with a calculation accuracy of 29 decimal places. These are numbers that Lorenz could only dream of with the computing capacity of the time. The result is known as the Lorenz attractor:



The Lorenz attractor in our experiment

The starting points are all close to zero. Their trajectories then tilt away to the right or left side of the attractor. Obviously, there is a point in the green and yellow centre that corresponds to an unstable equilibrium position: the trajectories near these points move away from them. Over time, the trajectories on the left and right sides of the attractor become intermixed. Although the starting points were close to each other, the long-term evolution of the system is unpredictable.

We will analyse this result in the following sections using elementary methods. But first a remark on the physics behind the system.

The parameters have the following physical meaning:

σ is the so-called "Prandtl number", named after the engineer Ludwig Prandtl (1875 - 1953). It describes the properties of the fluid under consideration in terms of viscosity and thermal diffusivity. For his investigations of atmospheric flows, Lorenz worked with the fixed value $\sigma = 10$.

$\beta = \frac{4}{1+a^2}$ is a measure of the cell geometry. Lorenz worked with the fixed value $a = \frac{h}{b} = \sqrt{2}$, i.e. $\beta = \frac{8}{3}$.

ϱ is the so-called Rayleigh number, which sets the buoyant and decelerating forces in relation to each other. This number is positive, and the behaviour of the convection current depends on it. In the "Simulator", this number can be varied in the interval [0, 30].

x is proportional to the intensity of the convection flow. y is proportional to the temperature difference between the rising and falling flow. z is proportional to the deviation from the linear temperature profile along the vertical axis.

The differential equations result from three conservation laws:

- Conservation of mass (the continuity equation and the concept of divergence are used here)
- Conservation of momentum (the Navier-Stokes differential equations are needed here)
- Conservation of energy (the heat transport equation is needed here)

An exact derivation can be found in [1]. However, it exceeds the mathematical tools available to us.

How will we proceed to analyse the Lorenz system? Here is an outline of the method:

In the project *The chaotic properties of logistic growth* we investigate the behaviour of an iteration in the neighbourhood of a fixed point. In the Lorenz system, the yellow and green centres in the previous image each appear to be a fixed point, albeit an unstable one. In the one-dimensional case and with logistic growth, it has been proven for a continuously differentiable function f that a fixed point ξ is attractive if $|f'(\xi)| < 1$. In the deduction, it has been used that

$$f(\xi + h) \approx f(\xi) + f'(\xi) \cdot h \text{ if } h \approx 0$$

We use the same approach to analyse the Lorenz system. However, everything now takes place in $\mathbb{R}^3$. The function f is no longer a function $\mathbb{R} \to \mathbb{R}$, but $\mathbb{R}^3 \to \mathbb{R}^3$. The function $f(\xi) + f'(\xi) \cdot h$ is a linear function in h for a fixed ξ. So, the first question to be answered is,

- *what do linear functions in $\mathbb{R}^3$ look like?*

We will answer this question in section 2.

In the one-dimensional case, the value of $|f'(\xi)|$ determines the character of the fixed point. In the three-dimensional case, we often speak of equilibrium positions instead of fixed points. The question that we must then pursue is

- *what types of equilibrium positions are there in $\mathbb{R}^3$?*

We will investigate this question in section 3.

Finally, in the one-dimensional case we have used the derivative of the function f to arrive at the approximation $f(\xi + h) \approx f(\xi) + f'(\xi) \cdot h$. So, the question is

- *what takes the place of the derivative f' in the three-dimensional case?*

We will pursue this question in section 4.

Clarifying these questions may seem time-consuming, but it is feasible using elementary methods. Once we have achieved this, we are ready to analyse the Lorenz system in section 5.

2. Linear mappings $\mathbb{R}^3 \to \mathbb{R}^3$

In this section, we look at how the linear functions in the one-dimensional case are transferred to linear mappings in $\mathbb{R}^3$. All considerations could also be applied to the general case of $\mathbb{R}^n$.

Definition 2.1

Linear mappings $\mathbb{R}^3 \to \mathbb{R}^3$ are given by a mapping rule

$$f(x_1, x_2, x_3) = \begin{cases} a_1x_1 + b_1x_2 + c_1x_3 \\ a_2x_1 + b_2x_2 + c_2x_3 \\ a_3x_1 + b_3x_2 + c_3x_3 \end{cases}$$

Where all coefficients are $a_i, b_i, c_i \in \mathbb{R}, i = 1,2,3$. □

We introduce the matrix notation for these mappings. The mapping matrix A is given by:

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

The image of a vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is then defined as:

$$A\vec{x} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} := \begin{pmatrix} a_1x_1 + b_1x_2 + c_1x_3 \\ a_2x_1 + b_2x_2 + c_2x_3 \\ a_3x_1 + b_3x_2 + c_3x_3 \end{pmatrix}$$

Each row of the image vector is the scalar product of the corresponding matrix row with the vector $\vec{x}$.

The mapping matrix A is often identified with the corresponding linear mapping $A: \mathbb{R}^3 \to \mathbb{R}^3$. As can be easily verified, the following applies to a linear mapping:

Theorem 2.2

Let $\mathrm{A}: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping.

Assertion: The following applies for any vectors $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $\lambda, \mu \in \mathbb{R}$:

$$\mathrm{A}(\lambda\vec{x} + \mu\vec{y}) = \lambda \mathrm{A}\vec{x} + \mu \mathrm{A}\vec{y}$$

□

The proof follows by direct recalculation.

Definition 2.3

Let be $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $\lambda, \mu \in \mathbb{R}$. Then $\lambda\vec{x} + \mu\vec{y} \in \mathbb{R}^3$ is called a *linear combination* of $\vec{x}, \vec{y}$. □

Note: The inverse is also true for this theorem 2.2: A mapping f (somehow defined, perhaps not by a mapping matrix) is linear if for any vectors $\vec{r}, \vec{s} \in \mathbb{R}^3$ and $\lambda, \mu \in \mathbb{R}$ holds: $f(\lambda\vec{r} + \mu\vec{s}) = \lambda f(\vec{r}) + \mu f(\vec{s})$.

The linearity of A has a first important consequence: *You can work with the basis vectors*. If $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are basis vectors of $\mathbb{R}^3$ (usually given by an orthonormalised system of unit vectors in the direction of the three coordinate axes), then each vector can be written as a linear combination of these basis vectors:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$

Because of the linearity of A then applies:

$$A\vec{x} = x_1 A\vec{e}_1 + x_2 A\vec{e}_2 + x_3 A\vec{e}_3$$

The image of a vector $\vec{x}$ is therefore uniquely determined if the images of the basis vectors $A\vec{e}_1, A\vec{e}_2, A\vec{e}_3$ are known.

If we now consider the first basis vector $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, for example, the following applies, as can be easily calculated:

$$A\vec{e}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

The same applies to the other two basis vectors and we have the

Theorem 2.4

Let A: $\mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping.

Assertion: The corresponding mapping matrix A is given by the images of the basis vectors in their columns.

□

Example

We consider a rotation D of 120° (clockwise) around the axis in space, which is generated by the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. This mapping is linear because it does not matter whether you first rotate vectors and then combine them linearly or first combine them linearly and then rotate them: $D(\lambda\vec{r} + \mu\vec{s}) = \lambda D\vec{r} + \mu D\vec{s}$. D swaps the basis vectors cyclically: $D\vec{e}_1 = \vec{e}_3, D\vec{e}_3 = \vec{e}_2, D\vec{e}_2 = \vec{e}_1$. Thus, the mapping matrix of D is given by:

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

In this example, you can also see that the axis of rotation remains "stationary" or is invariant:

$$D\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For a linear mapping to be defined, it is therefore sufficient to know the images of the basis vectors!

The question of which vectors are transferred in a linear mapping to$\vec{0}$ will prove helpful. The set of these vectors is called *the kernel* of the linear mapping A.

Definition 2.5

Let A: $\mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping. Then: $Ker\ A := \{\vec{x}: A\vec{x} = \vec{0}\}$. □

It is $\vec{0} \in Ker\ A$. If there is another vector $\vec{x} \neq \vec{0}$ in the kernel of A, then $\lambda\vec{x} \in Ker\ A$ for $\lambda \in \mathbb{R}$ because $A(\lambda\vec{x}) = \lambda A\vec{x} = \vec{0}$. This means that the entire straight line spanned by $\vec{x}$ lies in the kernel. If there is another vector $\vec{y} \neq \vec{x}, \vec{y} \neq \vec{0}$ in the kernel of A, then every linear combination of $\vec{x}, \vec{y}$ is also in the kernel due to linearity, because $A(\lambda\vec{x} + \mu\vec{y}) = \lambda A\vec{x} + \mu A\vec{y} = \vec{0}$ for $\lambda, \mu \in \mathbb{R}$. This means that the entire plane spanned by $\vec{x}, \vec{y}$ is in the kernel. If there is a third vector in the kernel that does not lie in this plane, then the kernel is even the entire space and A is the zero matrix.

Theorem 2.6

Let $A: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping.

Assertion: The kernel of A is either the origin, a line through the origin, a plane through the origin or the whole space (if A is the zero matrix). □

Example

We consider the orthogonal projection of the space onto the plane α given by the equation

$$x + y + z = 0$$

Then the projection direction is given by the normal vector of the plane:

$$\vec{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We first determine the mapping matrix. To do this, it is sufficient to know the images of the basis vectors. For the first basis vector, this is the intersection of the line $\vec{x}(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ with the plane α. This provides the solution $t = -\frac{1}{3}$ and the image $A\vec{e}_1 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$. Similarly, $A\vec{e}_2 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$ and $A\vec{e}_3 = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ are obtained. The mapping matrix is therefore given by: $A = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$.

In this projection, the normal vector is mapped onto the plane at the origin. It is therefore (and you can also do the maths): $KerA = \{\lambda\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}\}$.

□

How can you easily check whether the kernel is different from the origin? If $Ker\ A \neq \vec{0}$, then there is at least one vector $\vec{x} \neq \vec{0}$ with $A\vec{x} = \vec{0}$. If $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$, then $A\vec{x} = x_1 A\vec{e}_1 + x_2 A\vec{e}_2 + x_3 A\vec{e}_3 = \vec{0}$. The images of the basis vectors or the columns of the mapping matrix are therefore linearly dependent and lie in one plane. For example, if $x_3 \neq 0$, the third column vector can be represented as a linear combination of the first two: $A\vec{e}_3 = -\frac{1}{x_3}(x_1 A\vec{e}_1 + x_2 A\vec{e}_2)$.

The scalar triple product of the column vectors $[A\vec{e}_1, A\vec{e}_2, A\vec{e}_3]$ is equal to the volume of the parallelepiped spanned by the three vectors, as we know from vector geometry. If these three vectors are linearly dependent, i.e. lie in a plane, then this product is equal to zero. This provides the desired criterion. In the case of a linear mapping, we now speak of the determinant rather than the scalar triple product, as this definition can be generalised to higher dimensions. The following definition is sufficient for us:

Definition 2.7

Let A: $\mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping. Then the *determinant* of$A$ is defined by:

$$\det A := [A\vec{e}_1, A\vec{e}_2, A\vec{e}_3] = (A\vec{e}_1 \times A\vec{e}_2) \cdot A\vec{e}_3$$

□

Theorem 2.8

Let A be a linear mapping.

Assertion : $Ker\ A \neq \vec{0} \Leftrightarrow \det A = 0$

□

Proof:

"$\Longrightarrow$ " See above.

"$\Longleftarrow$ " The images of the basis vectors are linearly dependent, e.g.$A\vec{e}_3 = x_1 A\vec{e}_1 + x_2 A\vec{e}_2$. Then$\vec{z} \coloneqq x_1\vec{e}_1 + x_2\vec{e}_2 - \vec{e}_3 \in Ker\ A$ and $\vec{z} \neq \vec{0}$.

□

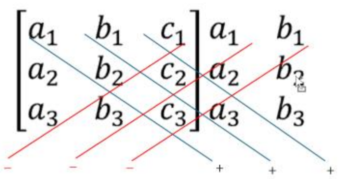
Example

For the normal projection in the previous example

$$det\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = \left(\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \times \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \right) \cdot \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = 0$$

□

We can also use the following method to calculate the determinant. Write down the first two columns of the matrix again to the right of it and then form the products of the diagonals. At the end, add everything up. The products of the main diagonals with a positive sign and the other products with a negative sign. You can easily check that the result corresponds to definition 2.7.



Calculating the determinant

$$detA = a_1 b_2 c_3 + b_1 b_2 a_3 + c_1 a_2 b_3 - c_1 b_2 a_3 - a_1 c_2 b_3 - b_1 a_2 c_3$$

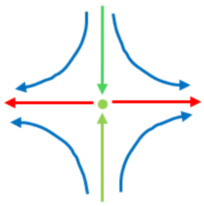
3. Equilibrium positions

We are now ready to investigate the different types of fixed points or equilibrium positions in $\mathbb{R}^3$.

In the school project *The chaotic properties of logistic growth*, we saw when a fixed point is attractive or repulsive in the one-dimensional case. In contrast to the one-dimensional case, it can happen in higher dimensions that a fixed point is repulsive in certain directions and attractive in other directions. In two dimensions, for example, the following cases may exist:

An attractive fixed point (sink) on the left and a repulsive fixed point (source) on the right

In contrast to the one-dimensional case, there can be a "mixture" here, namely when the fixed point is attractive in certain directions and repulsive in others. This is referred to as a *homoclinic fixed point*.



Homoclinic fixed point (saddle point)

Things can become correspondingly more complicated in three dimensions.

The zero point is always the fixed point of a linear mapping. If there are certain directions in the mapping that are invariant, we can analyse whether the mapping is contracting or dilating in these directions. A direction is invariant if a vector is merely stretched or compressed in this direction. If A is a linear mapping, we are looking for solutions to the equation:

$$A\vec{x} = \lambda\vec{x}$$

for$\lambda \in \mathbb{R}$ and a vector $\vec{x} \in \mathbb{R}^3$.

Definition 3.1

Let $A: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping.

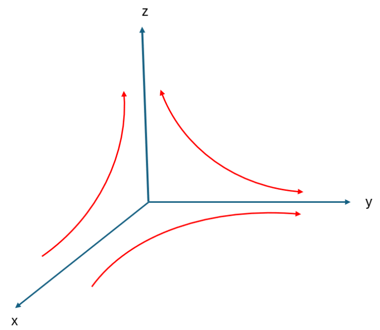
If the equation $A\vec{x} = \lambda\vec{x}$ has a solution for $\lambda \in \mathbb{R}$ and a vector $\vec{x} \in \mathbb{R}^3$, then λ *is the eigenvalue* of A and $\vec{x}$ is the corresponding *eigenvector*. □

If $\lambda = 1$, then every point on the line spanned by $\vec{x}$ is a fixed point. For $\lambda > 1$, the mapping in the direction of the eigenvector is dilating, because the corresponding eigenvector is stretched. For $\lambda < 1$ it is contracting.

Example

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The fixed point of this mapping is the zero point. The eigenvectors are just the basis vectors, as you can easily check. This mapping is contracting in the direction of the first basis vector and dilating in the direction of the second and third. There are no other fixed points, because the condition $A\vec{x} = \vec{x}$ leads to $(A - \mathbb{E})\vec{x} = \vec{0}$, where $\mathbb{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the unit matrix. Thus $\vec{x} \in Ker\ (A - \mathbb{E})$, but $\det(A - \mathbb{E}) = det \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -0.25 \neq 0$, therefore $Ker\ (A - \mathbb{E}) = \vec{0}$ according to theorem 2.8.



The behaviour of the above mapping near the zero point

How to find eigenvalues and eigenvectors? It is:

$$A\vec{x} = \lambda\vec{x} \Leftrightarrow (A - \lambda\mathbb{E})\vec{x} = \vec{0}$$

According to Theorem 2.8, there is only one solution if

$$Ker\ (A - \lambda\mathbb{E}) \neq \vec{0} \Leftrightarrow det(A - \lambda\mathbb{E}) = 0$$

To ensure that the equation $A\vec{x} = \lambda\vec{x}$ has a solution with $\vec{x} \neq \vec{0}$, we look for a λ such that A applies with the previous terms:

$$det(A - \lambda\mathbb{E}) = det \begin{bmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{bmatrix} = 0$$

Definition 3.2

Let $A: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping.

The polynomial

$$p(\lambda) \coloneqq det \begin{bmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{bmatrix}$$

is called *the characteristic polynomial* of A. □

In our case, it has degree 3, from which follows:

Theorem 3.3

Let $A: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping.

Assertion: This mapping has either three real eigenvalues or one real eigenvalue and two conjugate complex eigenvalues. □

Since there is always a real eigenvalue with an associated eigenvector, there is always *at least one direction* that is *invariant* in the mapping.

Example

If you rotate a circle around the centre point, no point on the circle line generally remains fixed. Does this also apply to a sphere in $\mathbb{R}^3$? A rotation of a sphere around the centre point is a linear mapping. No matter how tricky you make this rotation, according to theorem 3.3 there is always one direction that remains invariant and therefore there are always two points on the sphere that remain fixed.

Example

Be

$$A = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{3}{2} \\ 1 & \frac{3}{2} & \frac{3}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

We investigate the behaviour of the mapping near the zero point. We are looking for invariant directions, i.e. the eigenvalues and eigenvectors of the mapping. To do this, we calculate the characteristic polynomial and set it equal to zero:

$$p(\lambda) := det \begin{bmatrix} -\lambda & -\frac{1}{2} & -\frac{3}{2} \\ 1 & \frac{3}{2} - \lambda & \frac{3}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} - \lambda \end{bmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

This polynomial has the zeros $\lambda_{1,2,3} = 1, 2, -1$. The eigenvector for eigenvalue 1 is in the kernel of

$$(A - 1 \cdot \mathbb{E}) = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{3}{2} \\ 1 & \frac{1}{2} & \frac{3}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

If we look at the first and third lines, for example, the eigenvector $\vec{x}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ must apply:

$$\begin{cases} -x_1 - \frac{y_1}{2} - \frac{3z_1}{2} = 0 \\ -x_1 + \frac{y_1}{2} - \frac{z_1}{2} = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 = y_1 \\ z_1 = -y_1 \end{cases}$$

Since only the direction of $\vec{x}_1$ is sought, you can freely choose a parameter, e.g. $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

You check: $A\vec{x}_1 = 1 \cdot \vec{x}_1$. The other two eigenvectors are obtained in the same way. For the eigenvalue $\lambda_2 = 2$ you have the eigenvector $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and for the eigenvalue $\lambda_3 = -1$ the eigenvector $\vec{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. With these eigenvectors as the basis, the matrix has the following form:

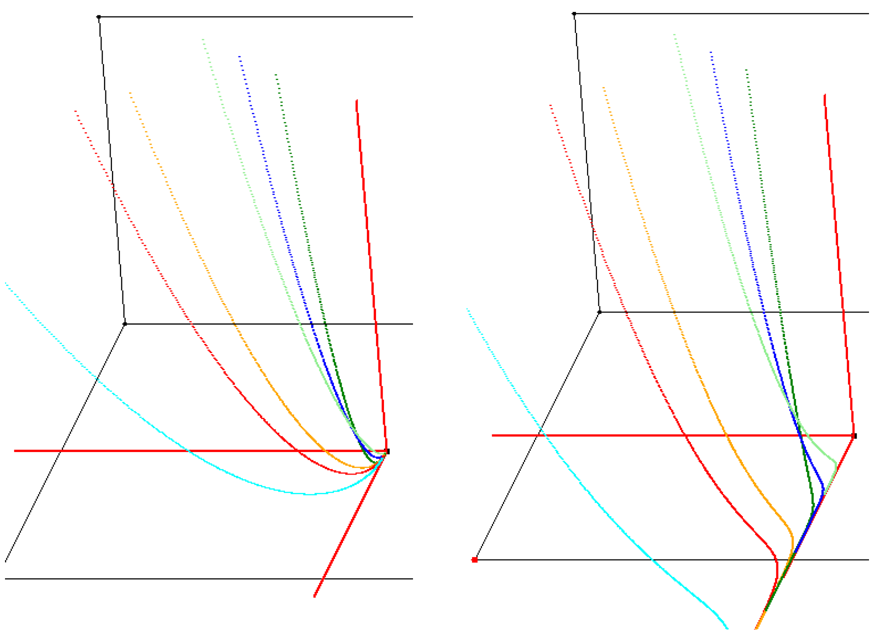
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Each point on the straight line spanned by $\vec{x}_1$ is a fixed point. Each point on the line spanned by $\vec{x}_3$ is cyclic of order two: It is mirrored at the zero point during the mapping. The mapping is dilating in the direction of $\vec{x}_2$.

Note that the eigenvectors generally do not form an orthonormal basis.

□

We can carry out corresponding experiments in the simulator. The exact parameters are described in the manual. An example is shown below. In this experiment, we select 6 starting points at the top left of the image and then consider their trajectories under a linear mapping which has three real eigenvalues. The x-axis points to the left, the y-axis points forwards and the z-axis points upwards.



Left: The zero point is attractive in the x,y,z direction. Right: The zero point is repulsive in the y-direction.

□

Now we will analyse the case where a mapping has one real eigenvalue and two conjugate complex eigenvalues. Let the eigenvalues be $\lambda_3 \in \mathbb{R}$ and $\lambda_{1,2} = \alpha \pm i\beta \in \mathbb{C}$. Let the eigenvector associated with λ_3 be $\vec{e}_3 \in \mathbb{R}^3$. The eigenvectors belonging to $\lambda_{1,2}$ are conjugate complex. Suppose they have the form: $\vec{e}_{1,2} = \vec{u} \pm i\vec{v}$ with $\vec{u}, \vec{v} \in \mathbb{R}^3$. If we choose $\vec{e}_1, \vec{e}_2, \vec{e}_3$ as basis and calculate with complex numbers, the matrix has diagonal form:

$$A = \begin{bmatrix} \alpha + i\beta & 0 & 0 \\ 0 & \alpha - i\beta & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

However, we can also choose $\vec{u}, \vec{v}, \vec{e}_3$ as basis, because $\vec{e}_{1,2}$ can be represented as a (complex) linear combination of $\vec{u}, \vec{v}$. What form does the matrix A have in relation to this base?

Since $\vec{e}_1 = \vec{u} + i\vec{v}$ is an eigenvector, the following applies:

$$A\vec{u} + iA\vec{v} = A\vec{e}_1 = (\alpha + i\beta)\vec{e}_1 = (\alpha + i\beta)(\vec{u} + i\vec{v}) = (\alpha\vec{u} - \beta\vec{v}) + i(\beta\vec{u} + \alpha\vec{v})$$

If you compare the real parts and imaginary parts, you can see:

$$A\vec{u} = \alpha\vec{u} - \beta\vec{v}, A\vec{v} = \beta\vec{u} + \alpha\vec{v}$$

The image of the first basis vector $\vec{u}$ under the mapping A is therefore $\begin{pmatrix} \alpha \\ -\beta \\ 0 \end{pmatrix}$ and the image of $\vec{v}$ is $\begin{pmatrix} \beta \\ \alpha \\ 0 \end{pmatrix}$. Thus, the matrix with respect to the basis $\vec{u}, \vec{v}, \vec{e}_3$ has the form

$$A = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

The submatrix $\tilde{A} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ is a rotational dilation in the $\vec{u}, \vec{v}$ plane.

Justification:

Each basis vector is stretched by the same amount, namely by $\sqrt{\alpha^2 + \beta^2}$. This means that every linear combination of these, i.e. every vector, is also stretched by this amount. The basis vectors are generally not an orthonormalised system. However, we want to show that the angle between $\vec{u}, \vec{v}$ and $\tilde{A}\vec{u}, \tilde{A}\vec{v}$ is maintained. To do this, we introduce an orthogonal system.

If, for example, $\vec{u}, \vec{v}$ forms a right-hand system, we can normalise $\vec{u}$ to length 1 and select this normalised vector in the direction of $\vec{u}$ as the first basis vector $\vec{e}_1$. We then select the second basis vector in the $\vec{u}, \vec{v}$ plane perpendicular to $\vec{e}_1$ so that it forms a right-hand system with this vector. In this orthogonal system, $\vec{u}, \vec{v}$ then has the components:

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v} = \begin{pmatrix} v \\ 1 \end{pmatrix}$$

For a suitable $v \in \mathbb{R}$.

Then the angle φ between $\vec{u}$ and $\vec{v}$ is given by:

$$cos\varphi = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{v}{\sqrt{1 + v^2}}$$

The angle φ' between $\tilde{A}\vec{u}$ and $\tilde{A}\vec{v}$ is given by:

$$cos\varphi' = \frac{\tilde{A}\vec{u} \cdot \tilde{A}\vec{v}}{|\tilde{A}\vec{u}||\tilde{A}\vec{v}|} = \frac{\binom{\alpha}{-\beta} \cdot \binom{\alpha v+\beta}{-\beta v+\alpha}}{\sqrt{\alpha^2+\beta^2} \cdot \sqrt{(\alpha^2+\beta^2)(1+v^2)}} = \frac{v}{\sqrt{1+v^2}}$$

Finally, the cross product is$\binom{\alpha}{-\beta} \times \binom{\beta}{\alpha} > 0$, so the sense of rotation is preserved in the mapping.

To summarise:

1) In the mapping, each vector is stretched by the same factor.
2) The mapping is angle-preserving.
3) The direction of rotation is retained in the mapping.

The mapping is therefore a rotational dilatation.

If $\sqrt{\alpha^2+\beta^2} < 1$, then the zero point in the $\vec{u}, \vec{v}$ plane is attractive, and the rotation causes the orbit of a point in this plane to spiral around the zero point. If $\sqrt{\alpha^2+\beta^2} > 1$, then such a point moves away from the origin on a spiral.

We summarise the considerations in this section in

Theorem 3.4

Let A: $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear mapping.

Assertion: Then there are the following two cases for the behaviour near the origin:

1) The mapping has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$. With respect to the associated eigenvectors as (not necessarily orthogonal) basis, the mapping has the form

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

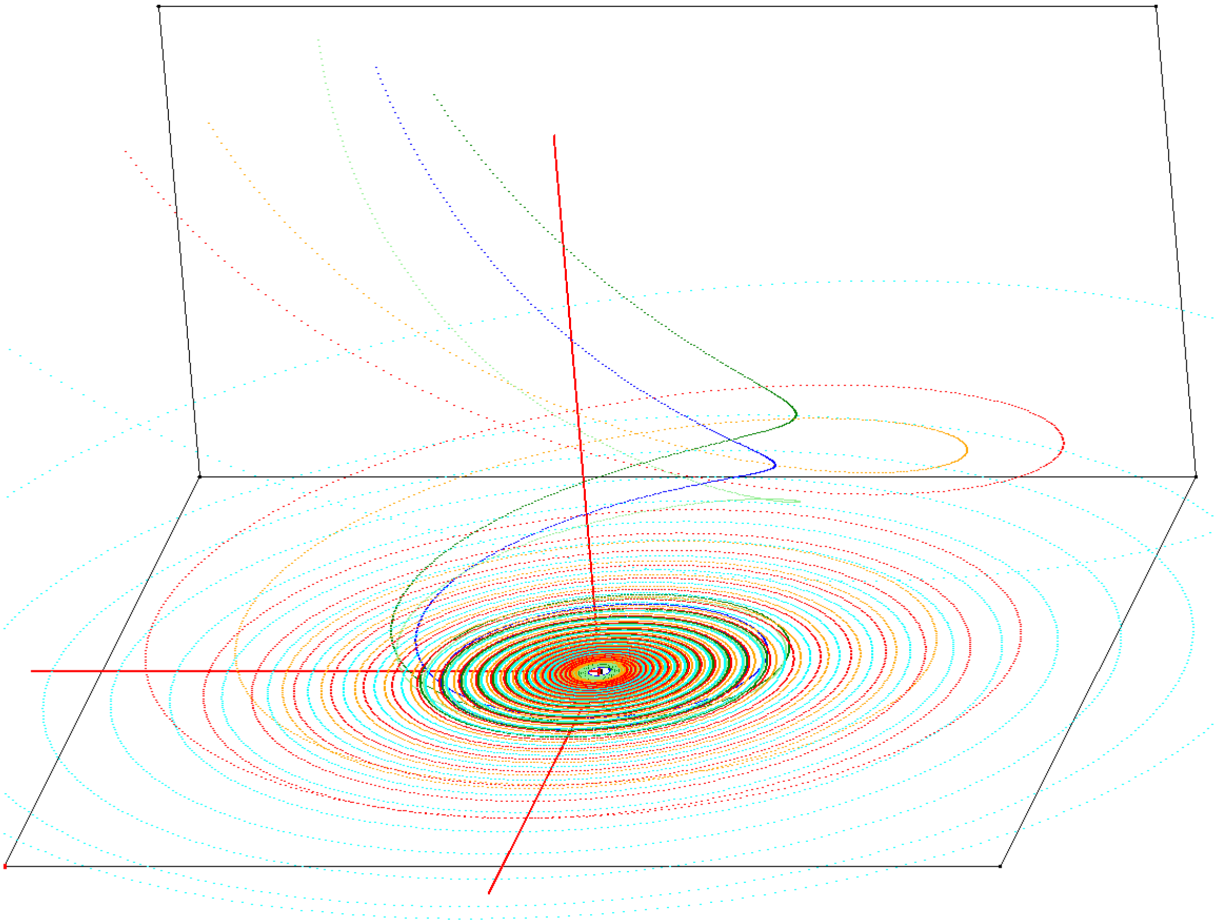
The mapping is contracting in the direction of an eigenvector $\vec{x}_i$ if and only if $|\lambda_i| < 1$

2) The mapping has two complex eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$ with eigenvectors $\vec{u} \pm i\vec{v}$ and one real eigenvalue λ_3 with eigenvector $\vec{x}_3$. Regarding $\vec{u}, \vec{v}, \vec{x}_3$ as a (not necessarily orthogonal) basis, the mapping has the following form:

$$A = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

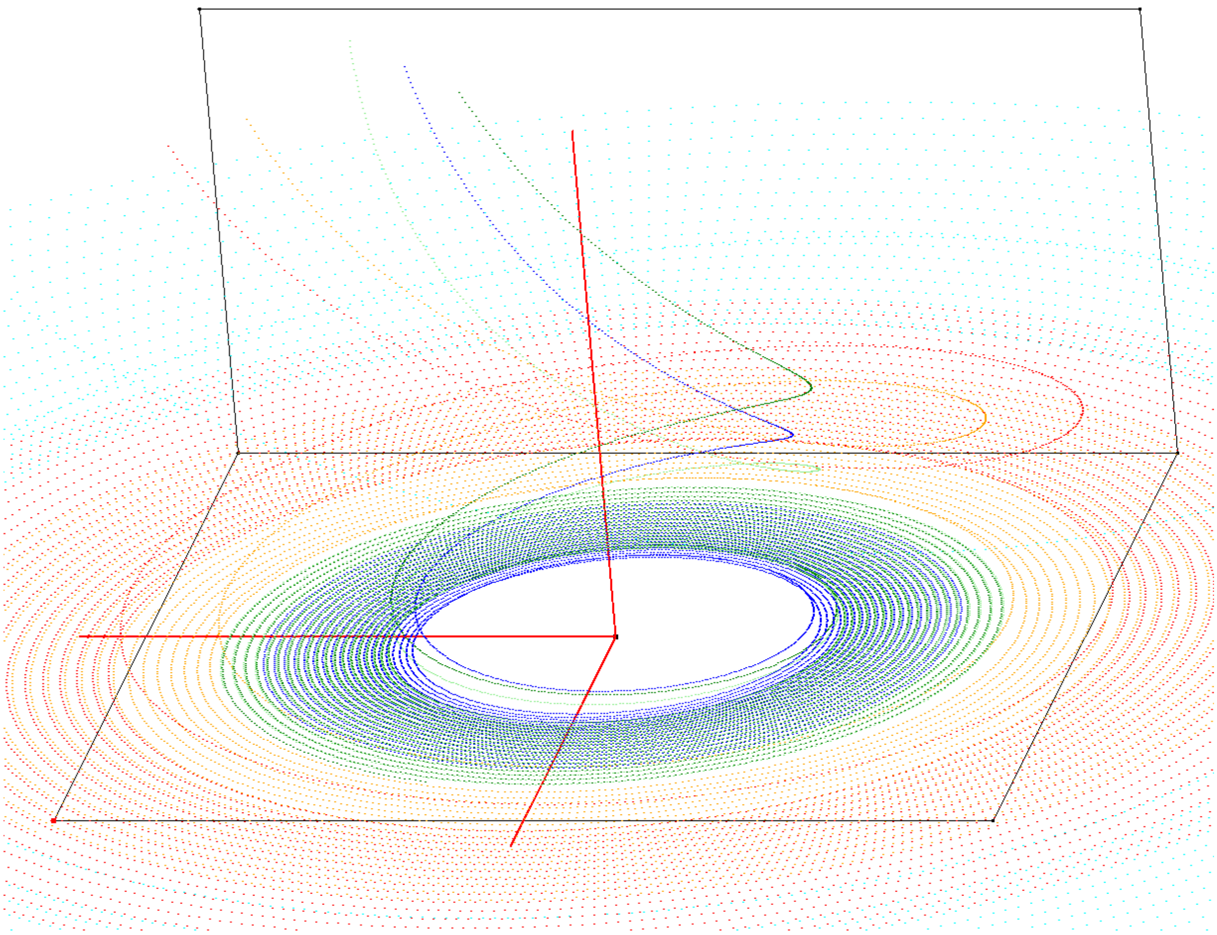
Near the origin, the mapping in the $\vec{u}, \vec{v}$ plane is a rotational dilatation and contracting exactly when $\sqrt{\alpha^2+\beta^2} < 1$.
□

We can carry out corresponding experiments in the simulator. Details are described in the manual. Here are two examples where the mapping has two complex eigenvalues:



Trajectories for $\varrho = 17.4$

The zero point is attractive and the trajectories spiral towards it.



Trajectories for $\varrho = 17.6$

The zero point is only attractive in the z-direction and repulsive in the y-x plane. The trajectories spiral away from the zero point.

4. Continuously differentiable mappings $\mathbb{R}^3 \to \mathbb{R}^3$

To investigate the behaviour of a real and continuously differentiable function f, which is iterated, in the neighbourhood of a fixed point ξ, we approximated this function in the one-dimensional case with the aid of the derivative of this function by a linear function:

$$f(\xi + h) \approx f(\xi) + f'(\xi) \cdot h, h \approx 0$$

The result was relatively simple: the fixed point is attractive exactly when $|f'(\xi)| < 1$.

In the multidimensional case, we try the same thing. We consider a function

$$f: \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix}$$

which is differentiable in every component and every variable. This is also called f *partially differentiable.* Furthermore, the derivatives should be continuous. If f has a fixed point $\vec{\xi} \in \mathbb{R}^3$, i.e. a point with $f(\vec{\xi}) = \vec{\xi}$, then we want to investigate the behaviour of f in the neighbourhood of $\vec{\xi}$. Like in the one-dimensional case, we do this by replacing f in the neighbourhood of $\vec{\xi}$ with a linear function.

In preparation for this, we have analysed the properties of a linear mapping in the neighbourhood of a fixed point (the zero point) in the previous section.

To find the appropriate linear approximation at the fixed point $\vec{\xi}$, we must first estimate how small changes in the variables affect the function value. This leads us to the concept of the *total differential*.

We look at this component by component. We set $\vec{\xi} = (x, y, z)$ and estimate the change in the function $f_1(x, y, z)$ if the variables (x, y, z) change a little. If (dx, dy, dz) denotes this (arbitrarily small) change in the corresponding variables and df_1 denotes the resulting change in the function value, then we can write:

$$df_1 \coloneqq f_1(x + dx, y + dy, z + dz) - f_1(x, y, z)$$

$$= f_1(x + dx, y + dy, z + dz) - f_1(x, y + dy, z + dz)$$

$$+f_1(x, y + dy, z + dz) - f_1(x, y, z + dy)$$

$$+f_1(x, y, z + dz) - f_1(x, y, z)$$

For the differences, only one variable changes at a time and we can use the approximation for the one-dimensional case: $f_1(x + dx) - f_1(x) \approx f_1{}'(x) \cdot dx$. This then results in:

$$df_1 \approx \frac{\partial f_1}{\partial x}(x, y + dy, z + dz) \cdot dx + \frac{\partial f_1}{\partial y}(x, y, z + dz) \cdot dy + \frac{\partial f_1}{\partial x}(x, y, z) \cdot dz$$

Due to the continuity of the derivative: $\frac{\partial f_1}{\partial x}(x, y + dy, z + dz) \approx \frac{\partial f_1}{\partial x}(x, y, z)$ and $\frac{\partial f_1}{\partial x}(x, y, z + dz) \approx \frac{\partial f_1}{\partial x}(x, y, z)$. Then, as a result and in the limit case, when dx, dy, dz becomes arbitrarily small, we have the so-called *total differential*

:

$$df_1 = \frac{\partial f_1}{\partial x}(x, y, z) \cdot dx + \frac{\partial f_1}{\partial y}(x, y, z) \cdot dy + \frac{\partial f_1}{\partial x}(x, y, z) \cdot dz$$

The analogue equation also applies to the other components f_2, f_3. At the fixed point $\vec{\xi}$ then applies:

$$\begin{cases} df_1 = \frac{\partial f_1}{\partial x}(\vec{\xi})dx + \frac{\partial f_1}{\partial y}(\vec{\xi})dy + \frac{\partial f_1}{\partial z}(\vec{\xi})dz \\ df_2 = \frac{\partial f_2}{\partial x}(\vec{\xi})dx + \frac{\partial f_2}{\partial y}(\vec{\xi})dy + \frac{\partial f_2}{\partial z}(\vec{\xi})dz \\ df_3 = \frac{\partial f_3}{\partial x}(\vec{\xi})dx + \frac{\partial f_3}{\partial y}(\vec{\xi})dy + \frac{\partial f_3}{\partial z}(\vec{\xi})d \end{cases}$$

We can also write this in matrix form:

$$df = \begin{bmatrix} \frac{\partial f_1}{\partial x}(\vec{\xi}) & \frac{\partial f_1}{\partial y}(\vec{\xi}) & \frac{\partial f_1}{\partial z}(\vec{\xi}) \\ \frac{\partial f_2}{\partial x}(\vec{\xi}) & \frac{\partial f_2}{\partial y}(\vec{\xi}) & \frac{\partial f_2}{\partial z}(\vec{\xi}) \\ \frac{\partial f_3}{\partial x}(\vec{\xi}) & \frac{\partial f_3}{\partial y}(\vec{\xi}) & \frac{\partial f_3}{\partial z}(\vec{\xi}) \end{bmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

We have thus found our linear approximation function:

$$df = \mathfrak{J}_f(\vec{\xi}) \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \mathfrak{J}_f(\vec{\xi}) d\vec{x}$$

Definition 4.1

Be

$$f: \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix}$$

be partially differentiable. Then the *Jacobi matrix* at a point $\vec{\xi}$ is defined as:

$$\mathfrak{J}_f(\vec{\xi}) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(\vec{\xi}) & \frac{\partial f_1}{\partial y}(\vec{\xi}) & \frac{\partial f_1}{\partial z}(\vec{\xi}) \\ \frac{\partial f_2}{\partial x}(\vec{\xi}) & \frac{\partial f_2}{\partial y}(\vec{\xi}) & \frac{\partial f_2}{\partial z}(\vec{\xi}) \\ \frac{\partial f_3}{\partial x}(\vec{\xi}) & \frac{\partial f_3}{\partial y}(\vec{\xi}) & \frac{\partial f_3}{\partial z}(\vec{\xi}) \end{bmatrix}$$

□

The Jacobi matrix is named after Carl Gustav Jacobi (1804 - 1851).

If the partial derivatives of f are continuous, the following applies (without us proving this here):

$$\lim_{|d\vec{x}| \to 0} \left| df - \mathfrak{J}_f(\vec{\xi}) d\vec{x} \right| = 0$$

f is then also called *totally differentiable*.

We summarise this:

Theorem 4.2

Let

$$f: \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix}$$

A totally differentiable function (all partial derivatives exist and are continuous).

Assertion:

$$f(\vec{x} + \vec{h}) \approx f(\vec{x}) + \mathfrak{J}_f(\vec{x})\vec{h} \text{ if } \vec{h} \approx \vec{0}$$

□

Example

$$f: \mathbb{R}^{\mathbf{3}} \to \mathbb{R}^{\mathbf{3}}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{3}x^2y + z + \frac{1}{3} \\ xy - yz - 1 \\ x + y^2z - 1 \end{pmatrix}$$

You can easily do the maths: $\vec{\xi} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a fixed point of f.

The Jacobian matrix is:

$$\mathfrak{J}_f(\vec{x}) = \begin{bmatrix} \frac{2}{3}xy & \frac{1}{3}x^2 & 1 \\ y & x - z & -y \\ 1 & 2yz & y^2 \end{bmatrix}$$

$$\mathfrak{J}_f(\vec{\xi}) = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

To investigate the behaviour of the mapping in the vicinity of the fixed point, we analyse the linear mapping $\mathfrak{J}_f(\vec{\xi})$ and look for its eigenvalues and eigenvectors. The characteristic polynomial is then:

$$p(\lambda) = Det \begin{bmatrix} -\frac{2}{3} - \lambda & \frac{1}{3} & 1 \\ -1 & -\lambda & 1 \\ 1 & -2 & 1 - \lambda \end{bmatrix} = -\lambda^3 + \frac{1}{3}\lambda^2 - \frac{2}{3}\lambda + \frac{4}{3}$$

And we look for its zeros.

Obviously, $\lambda_1 = 1$ is a zero. If we divide $p(\lambda)$ by $\lambda - 1$, the condition for the remaining two zeros remains:

$$\lambda^2 + \frac{2}{3}\lambda - 1 = 0$$

This gives us the other two zeros:

$$\lambda_{2,3} = -\frac{1}{3} \pm \frac{2\sqrt{2}}{3}$$

f is therefore indifferent in the direction of the first eigenvector (neither attractive nor repulsive). In the direction of the second eigenvector, f is attractive, since $|\lambda_2| \approx 0.609 < 1$ and in the direction of the third eigenvector it is repulsive, since $|\lambda_3| \approx 1.276 > 1$.

5. Equilibrium positions of the Lorenz system

After the preparations in the last three sections, we are now ready to examine the Lorenz system in more detail.

The Lorenz system was given by:

$$\dot{\vec{x}} = \mathcal{L}(\vec{x})$$

With

$$\mathcal{L}(\vec{x}) = \begin{pmatrix} -\sigma x + \sigma y \\ \varrho x - y - xz \\ -\beta z + xy \end{pmatrix}$$

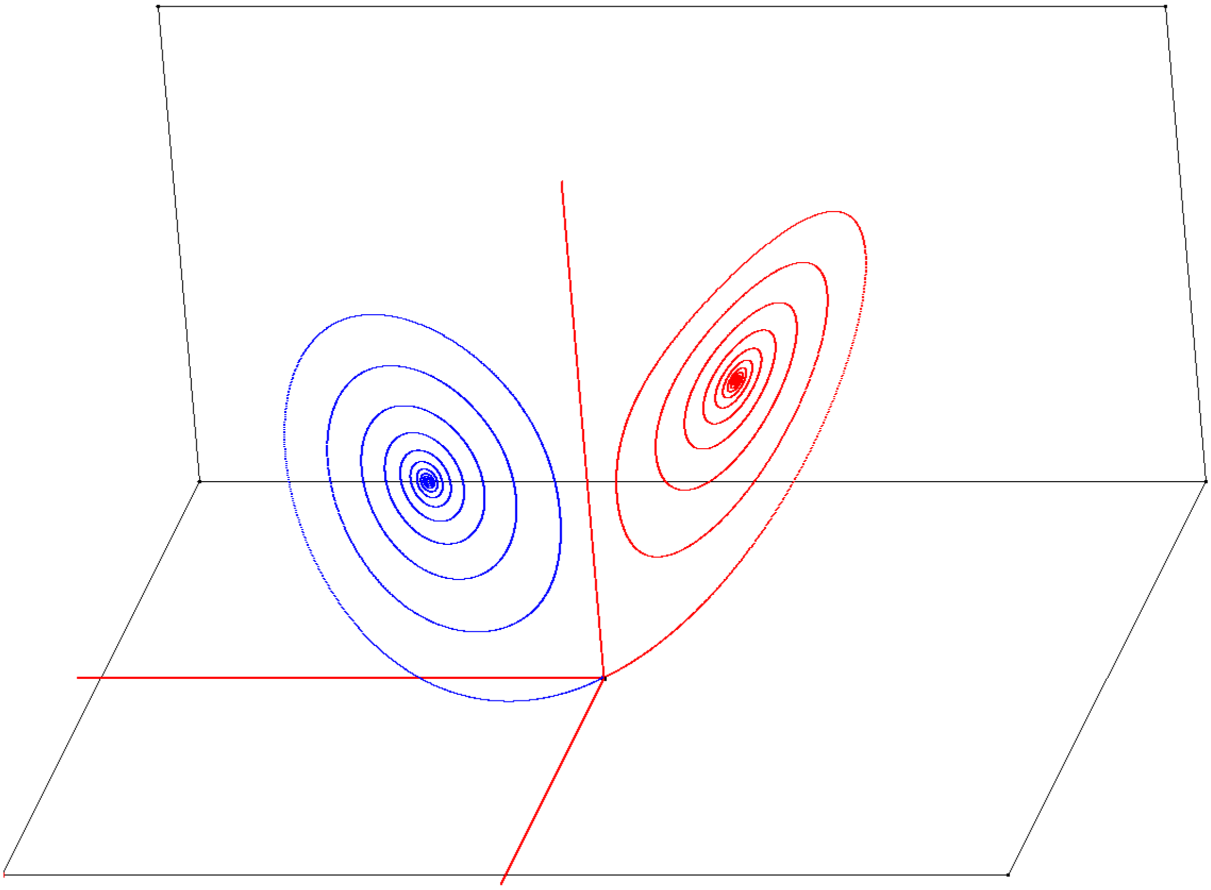
Mirror symmetry relative to the z-axis

If in the system of equations

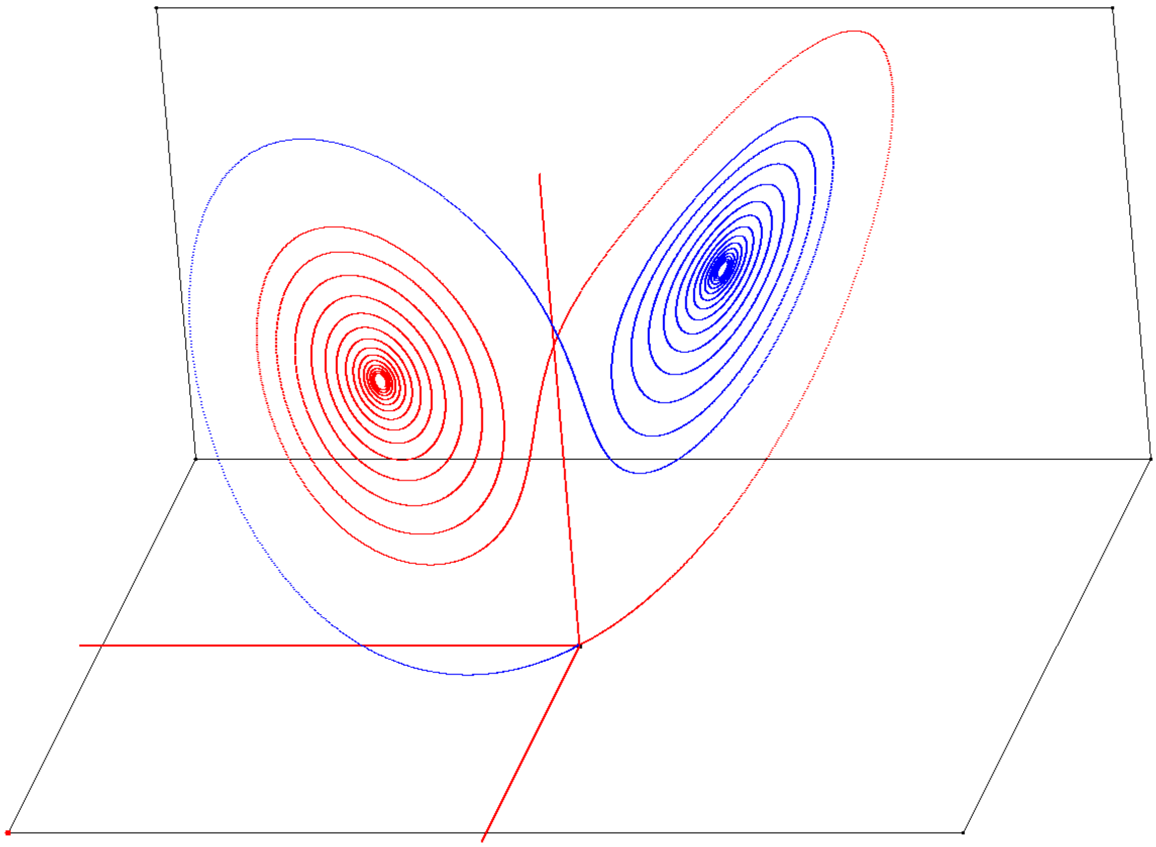
$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = \varrho x - y - xz \\ \dot{z} = -\beta z + xy \end{cases}$$

x is replaced by $-x$ and y by $-y$, it does not change. This means that starting points that are mirrored on the z-axis produce an orbit that is also mirror-symmetrical.

We consider two starting points that are very close to each other: $(0.01, 0.01, 0.1)$ (blue) and $(-0.01, -0.01, 0.1)$ (red). These are symmetrical to the z-axis and produce symmetrical orbits. We set $\varrho = 12.03$. Due to the symmetry, these starting points provide symmetrical orbits and converge towards different symmetrical fixed points.



Symmetrical orbits for two starting points that are close to each other



For $\varrho = 16.08$, the orbits of the same starting points are "swapped"

This already indicates that the system reacts sensitively to changes: If the starting point changes minimally, a significantly different orbit can be generated depending on the situation.

Invariance of the z-axis

A starting point on the z-axis $(0, 0, z)$ remains on the z-axis, because

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \\ \dot{z} = -\beta z \end{cases}$$

The z-axis is invariant. Furthermore, if t is used to denote time: $z = Ce^{-\beta t}, C \in \mathbb{R}$. For $t \to \infty$, orbits on the z-axis run towards the zero point, as $\beta > 0$.

This gives us:

Theorem 5.1

Assertion: In the Lorenz system, all trajectories are mirror-symmetrical to the z-axis. The z-axis itself is invariant.

□

Equilibrium positions

Equilibrium positions are points at which the system no longer changes. They correspond to the fixed points in the one-dimensional case. In this case, this means that all derivatives to the three variables disappear. They are solutions of the system of equations:

$$\dot{\vec{x}} = \vec{0} = \mathcal{L}(\vec{x})$$

or

$$\begin{cases} 0 = -\sigma x + \sigma y \\ 0 = \varrho x - y - xz \\ 0 = -\beta z + xy \end{cases}$$

Obviously, the zero point $(0,0,0)$ is a solution of this system of equations and therefore an equilibrium position.

Now we exclude the zero point. The first equation leads to $x = y$. If $x \neq 0$ the second equation leads to $z = \varrho - 1$ and the third equation to $x = \pm\sqrt{\beta(\varrho - 1)}$ for $\varrho > 1$. This gives us two further equilibrium positions (or fixed points):

$$C^{\pm} = (\pm\sqrt{\beta(\varrho - 1)}, \pm\sqrt{\beta(\varrho - 1)}, \varrho - 1)$$

These two are only defined for $\varrho > 1$. The symmetry with respect to the z-axis can be seen here again.

If a starting point close to a fixed point tends towards it, then this equilibrium position is *stable*. This corresponds to an *attractive fixed point*.

If a starting point near a fixed point leads to an orbit that moves away from the fixed point, then this equilibrium position *is unstable*. This corresponds to a *repulsive fixed point*.

Theorem 5.2

Assertion:

1) The Lorenz system has the equilibrium positions $\vec{0}$, and if $\varrho > 1$ in addition $C^{\pm} = (\pm\sqrt{\beta(\varrho - 1)}, \pm\sqrt{\beta(\varrho - 1)}, \varrho - 1)$.
2) The following applies: $\mathcal{L}(\vec{0}) = \vec{0}$ and $\mathcal{L}(C^{\pm}) = \vec{0}$

□

The zero point

We now examine the behaviour of the Lorenz system in the neighbourhood of the zero point, i.e. for the function $\mathcal{L}: \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\mathcal{L}(x, y, z) = \begin{cases} -\sigma x + \sigma y \\ \varrho x - y - xz \\ -\beta z + xy \end{cases}$$

For which $\vec{0}$ is a fixed point.

The Jacobian matrix from the previous section is

$$\mathfrak{J}_{\mathcal{L}}(\vec{x}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \varrho - z & -1 & -x \\ y & x & -\beta \end{bmatrix}$$

And at the fixed point:

$$\Im_{\mathcal{L}}(\vec{0}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \varrho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

Near the zero point, i.e. for $\vec{x} \approx \vec{0}$ applies according to Theorem 4.2:

$$\dot{\vec{x}} = \mathcal{L}(\vec{x}) = \mathcal{L}(\vec{0} + \vec{x}) \approx \mathcal{L}(\vec{0}) + \Im_{\mathcal{L}}(\vec{0})\vec{x}$$

And because $\mathcal{L}(\vec{0}) = \vec{0}$ is the equation for the Lorenz system near the zero point:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = \varrho x - y \\ \dot{z} = -\beta z \end{cases}$$

In the third component, we have the equation $\dot{z} = -\beta z$ with the solution $z = Ce^{-\beta t}, C \in \mathbb{R}$. So, for $t \to \infty$ orbits on the z-axis run towards the zero point because $\beta > 0$.

Now we look at the x-y plane near the zero point. This is independent of z and invariant. In the neighbourhood of the zero point, the system in this plane is determined by the (linear) mapping $A: \mathbb{R}^2 \to \mathbb{R}^2$ with the matrix

$$A = \begin{bmatrix} -\sigma & \sigma \\ \varrho & -1 \end{bmatrix}$$

The corresponding characteristic polynomial is

$$p(\lambda) = \lambda^2 + (1 + \sigma)\lambda - \varrho\sigma$$

With the zeros

$$\lambda_{1,2} - \frac{1+\sigma}{2} \pm \frac{1}{2}\sqrt{(1+\sigma)^2 + 4\sigma(\varrho - 1)} = -\frac{1+\sigma}{2} \pm \frac{1}{2}\sqrt{(\sigma - 1)^2 + 4\sigma\varrho}$$

Since $\varrho > 0$ the expression under the root is always positive. The corresponding eigenvectors are:

$$\vec{e}_{1,2} = \begin{pmatrix} \sigma \\ \frac{\sigma - 1}{2} \pm \frac{1}{2}\sqrt{(\sigma - 1)^2 + 4\sigma\varrho} \end{pmatrix}$$

The question now is for which $\lambda_{1,2}$ the system of equations leads to a contraction in the neighbourhood of the zero point. In this case, the zero point would be a sink or attractive.

We choose the eigenvectors $\vec{e}_{1,2}$ as the basis for our coordinate system. This is possible because $\vec{e}_1 \nparallel \vec{e}_2$. Then we move along $\vec{e}_1$. We therefore choose a starting point that has the coordinates $\binom{x}{0}$ with respect to the new coordinate system with $x \approx 0$. Then the mapping equation reads:

$$\begin{pmatrix} \dot{x} \\ 0 \end{pmatrix} = A \begin{pmatrix} x \\ 0 \end{pmatrix} = xA\vec{e}_1 = x\lambda_1\vec{e}_1 = \begin{pmatrix} \lambda_1 x \\ 0 \end{pmatrix}$$

Or $\dot{x} = \lambda_1 x$. This differential equation has the solution $x(t) = C_1 e^{\lambda_1 t}, C_1 \in \mathbb{R}$. This means that the mapping in this direction is contracting if $\lambda_1 < 0$ is and dilating for $\lambda_1 > 0$.

For a point that moves along $\vec{e}_2$, i.e.$\binom{0}{y} = y \cdot \vec{e}_2$, you also get: $\dot{y} = \lambda_2 y$ with the solution: $y(t) = C_2 e^{\lambda_2 t}, C_2 \in \mathbb{R}$. The mapping is contracting in this direction if $\lambda_2 < 0$ is and dilating for $\lambda_2 > 0$.

It is now easy to do the maths:

$$\lambda_1 = -\frac{1+\sigma}{2} + \frac{1}{2}\sqrt{(\sigma-1)^2 + 4\sigma\varrho} < 0 \iff \varrho < 1$$

$$\lambda_2 = -\frac{1+\sigma}{2} - \frac{1}{2}\sqrt{(\sigma-1)^2 + 4\sigma\varrho} < 0, \forall\varrho$$

This gives us the result:

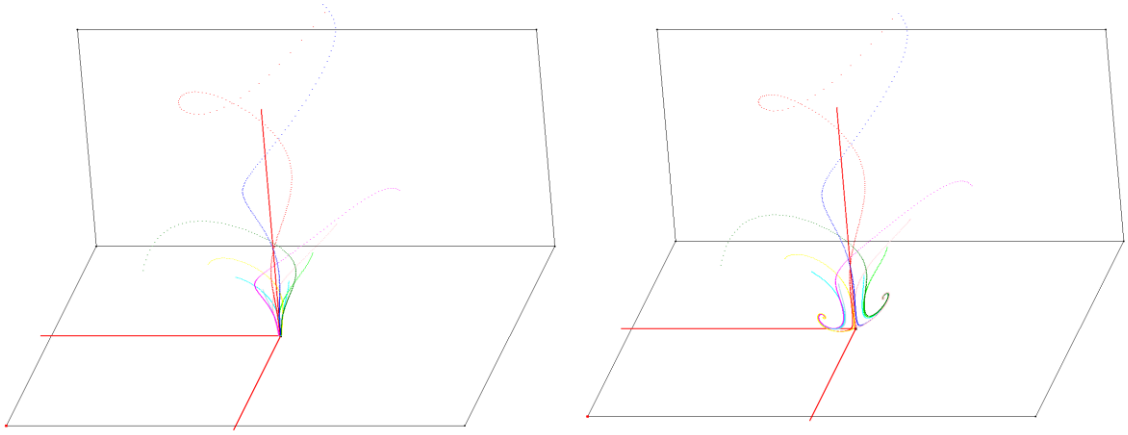
Theorem 5.3

We consider the Lorenz system and the equilibrium position at the zero point.

Assertion:

1) The zero point is always contracting in the z-direction.
2) In the x-y plane it is a sink for $\varrho < 1$ and a saddle point for $\varrho > 1$.

□



Left Orbits of a point cloud for $\varrho = 0.93$ and attractive zero point. On the right the corresponding orbits for $\varrho = 3.12$. Here the zero point is only attractive in the z-direction. In the x-y plane, the orbits first move towards the zero point, but then away from it in the x-y plane and then remain at the equilibrium positions $C^{\pm}$ according to Theorem 5.2.

The equilibrium positions $C^{\pm}$

In addition to the zero point, $\varrho > 1$ has two further equilibrium positions according to Theorem 5.2:

$$C^{\pm} = (\pm\sqrt{\beta(\varrho-1)}, \pm\sqrt{\beta(\varrho-1)}, \varrho - 1)$$

In the following, we always consider $\varrho > 1$ and analyse the Lorenz system in the neighbourhood of these equilibrium positions. For this purpose, we use the results from section 4.

The original system of equations had the associated Jacobi matrix

$$\mathfrak{J}_{\mathcal{L}} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \varrho - z & -1 & -x \\ y & x & -\beta \end{bmatrix}$$

And at the points $C^{\pm} = (\pm\sqrt{\beta(\varrho-1)}, \pm\sqrt{\beta(\varrho-1)}, \varrho - 1)$ this matrix becomes:

$$\mathfrak{J}_{\mathcal{L}}(C^{\pm}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{\beta(\varrho-1)} \\ \pm\sqrt{\beta(\varrho-1)} & \pm\sqrt{\beta(\varrho-1)} & -\beta \end{bmatrix}$$

We write shorter: $\mathfrak{J}^{\pm} \coloneqq \mathfrak{J}_{\mathcal{L}}(C^{\pm})$

According to Theorem 4.2, it applies in a neighbourhood of a fixed point $C^{\pm}$, i.e. for $\vec{x} \approx \vec{0}$:

$$\mathcal{L}(\vec{c}^{\pm} + \vec{x}) \approx \mathcal{L}(\vec{c}^{\pm}) + \mathfrak{J}^{\pm}\vec{x} = \vec{0} + \mathfrak{J}^{\pm}\vec{x}$$

For the Lorenz system in the neighbourhood of the equilibrium positions $C^{\pm}$, i.e. for $\vec{x} \approx \vec{0}$:

$$\dot{\vec{x}} = \mathcal{L}(\vec{c}^{\pm} + \vec{x}) \approx \mathfrak{J}^{\pm}\vec{x}$$

To analyse the effect of the mapping $\mathfrak{J}^{\pm}$, we again look for the eigenvalues and eigenvectors of the matrix $\mathfrak{J}^{\pm}$. In an exercise it is shown that the corresponding characteristic polynomial for both equilibrium positions is $C^{\pm}$:

$$p(\lambda) = \lambda^3 + (\sigma + \beta + 1)\lambda^2 + \beta(\sigma + \varrho)\lambda + 2\sigma\beta(\varrho - 1)$$

The values of λ we are looking for are the zeros of this polynomial.

To investigate the characteristic polynomial further, we set

$$\begin{cases} a = \sigma + \beta + 1 = \dfrac{41}{3} \\ b = \beta(\sigma + \varrho) = \dfrac{8}{3}(\varrho + 10) \\ c = 2\sigma\beta(\varrho - 1) = \dfrac{160}{3}(\varrho - 1) \end{cases}$$

And have:

$$p(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c$$

Since the degree of the polynomial is odd, there is at least one real zero. As all the coefficients of the polynomial are positive ($\varrho > 1$) *, this real zero* must *be negative.*

If this zero point λ_1 exists, there are the eigenvectors $\vec{e}_1^{\,\pm}$ belonging to λ_1 depending on the sign of $\vec{c}^{\pm}$. This allows us to investigate the behaviour of $\mathfrak{J}^{\pm}$ when a starting point moves from the equilibrium position in the direction of $\vec{e}_1^{\,+}$, e.g. a point $\vec{x} = \vec{c}^{+} + h\,\vec{e}_1^{\,+}$ with a small $h \in \mathbb{R}$.

The equation for the Lorenz system is then as follows:

$$\dot{\vec{x}} = \dot{h}\vec{e}_1^{\,+} = \mathfrak{J}^{+}(\vec{c}^{+} + h\,\vec{e}_1^{\,+}) = \mathfrak{J}^{+}\vec{c}^{+} + h\mathfrak{J}^{+}\vec{e}_1^{\,+} = \vec{0} + h\lambda_1\vec{e}_1^{\,+}$$

And we have the solution: $h(t) = hCe^{\lambda_1 t}, C \in \mathbb{R}$ or $\vec{x}(t) = \vec{c}^{+} + \ \vec{e}_1^{\,+}Ce^{\lambda_1 t}$.

For $\lambda_1 < 0$ and increasing t the starting point in the direction of $\vec{e}_1^{\,+}$converges to C^{+}.

This means *that the mapping* is *contracting in* the direction of $\vec{e}_1^{\,+}$. There is always such a direction, as there is always *at least one negative real* eigenvalue, and this explains why the orbits are "flattened" into a plane spanned by $\vec{e}_{2,3}^{\,+}$.

The following applies:

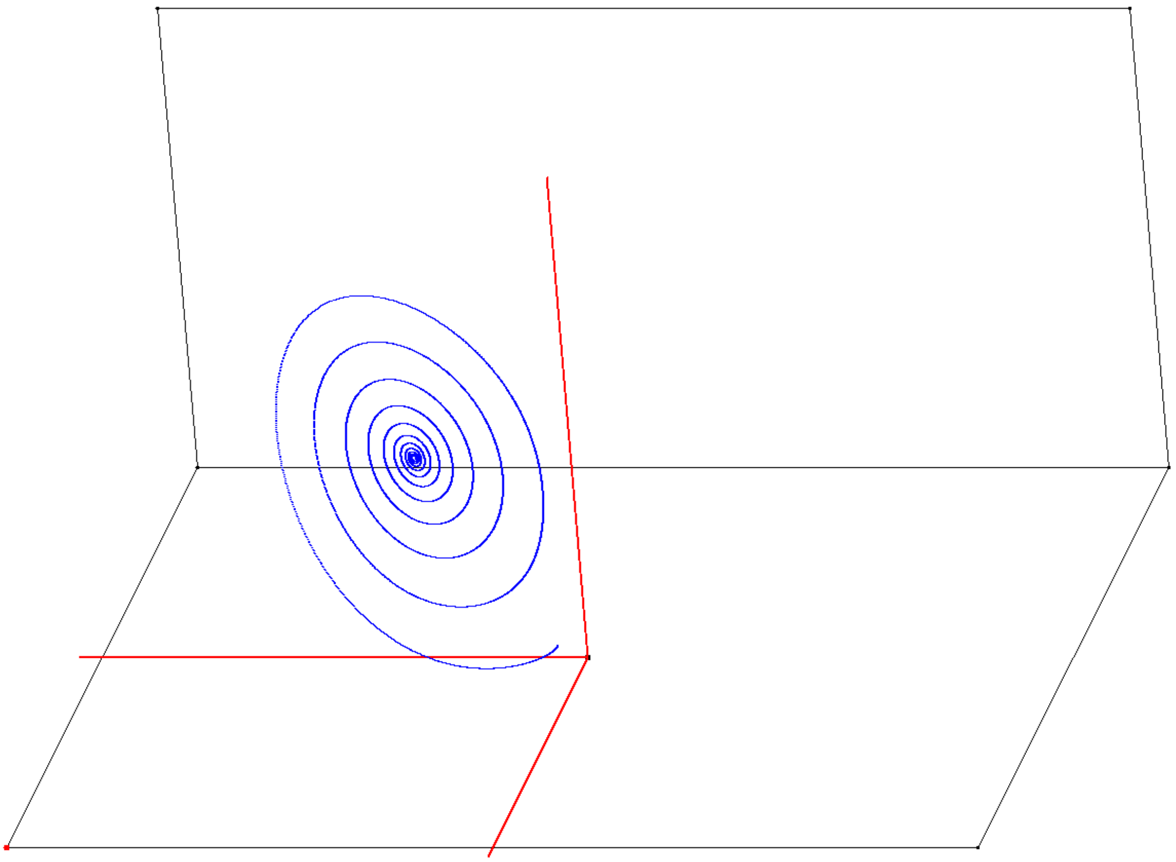
Theorem 5.4

We consider the Lorenz system with $\varrho > 1$.

Assertion:

- There is a real eigenvalue $\lambda_1 < 0$ and an associated eigenvector $\vec{e}_1{}^{\pm}$ for each fixed point $C^{\pm}$, in the direction of which the mapping is contracting.
- Each trajectory settles in a plane spanned by $\vec{e}_{2,3}{}^{+}$ which passes through C^{+} or C^{-}.

□



"Flattened" orbit for $\varrho = 12.51$ that oscillates in a plane

Now we consider the behaviour of the system in the plane spanned by $\vec{e}_{2,3}{}^{+}$according to Theorem 5.4 and consider only the fixed point C^{+} as a proxy. To simplify the characteristic polynomial, we apply the usual transformation $\mu := \lambda + \frac{a}{3}$. The polynomial is then

$$\left(\mu - \frac{a}{3}\right)^3 + a\left(\mu - \frac{a}{3}\right)^2 + b\left(\mu - \frac{a}{3}\right) + c =: \mu^3 + p\mu + q$$

where

$$\begin{cases} p = b - \dfrac{a^2}{3} = \dfrac{8}{3}\varrho - \dfrac{961}{27} \approx \dfrac{8}{3}\varrho - 35.5926 \\ q = \dfrac{2a^3}{27} - \dfrac{ab}{3} + c = \dfrac{1112}{27}\varrho + \dfrac{10402}{27^2} \approx 41.1852\varrho + 14.2689 \end{cases}$$

as a recalculation shows.

And the following applies:

$$p_\lambda(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c = 0 \Leftrightarrow p_\mu(\mu) = \mu^3 + p\mu + q = 0$$

The formulae of Gerolamo Cardano (1501 - 1576) exist for the zeros of the transformed polynomial. We do not want to calculate these zeros explicitly but merely analyse their character. According to the Cardano formulae, the so-called determinant is determined:

$$D = (\frac{q}{2})^2 + (\frac{p}{3})^3 \approx 0.7023\varrho^3 + 395.9327\varrho^2 + 669.1938\varrho - 1619.0954$$

And the following case distinction applies:

$$\begin{cases} D > 0: There\ is\ a\ real\ and\ two\ conjugate\ complex\ solutions \\ D = 0\ and\ p = 0: This\ case\ doesn't show\ up\ here \\ D = 0\ and\ p \neq 0: There\ are\ two\ real\ solutions \\ D < 0: There\ are\ three\ real\ solutions \end{cases}$$

The case $D < 0$

For $\varrho = 1$ is $D < 0$. $D(\varrho)$ is monotonically increasing. As you can check, $D = 0$ for $\varrho = 1.346 \ldots$. *So, there are three different real zeros or eigenvalues in the interval* $\varrho \in [1, 1.346 \ldots]$ *, all of which are negative. Thus,* $C^{\pm}$ *for* ϱ *are attractive equilibrium positions in this range*. However, these are so close to zero that they are not visible in the "simulator".

We will not investigate the case $D = 0\ and\ p \neq 0$ any further. We then have two real zeros and one of them is a double zero.

The case $D > 0$

If ϱ continues to grow, it becomes $D > 0$. Then there is a real zero of p_μ, let's call it μ_1. This is not necessarily negative, because only the corresponding $\lambda_1 = \mu_1 + a/3$ is necessarily negative. Furthermore, there are two conjugate complex zeros for $D > 0$.

Now we will analyse the case $D > 0$ further and define:

$$\begin{cases} u := \sqrt[3]{-\frac{q}{2} + \sqrt{D}} \\ v := \sqrt[3]{-\frac{q}{2} - \sqrt{D}} \end{cases}$$

In this case, Cardano's formulae for the desired zeros $\mu_{1,2,3}$ of p_μ are as follows:

$$\begin{cases} \mu_1 = u + v \\ \mu_{2,3} = -\frac{1}{2}(u + v) \pm i\frac{\sqrt{3}}{2}(u - v) \end{cases}$$

The corresponding eigenvalues of the original characteristic polynomial p_λ are:

$$\begin{cases} \lambda_1 = u + v - \frac{a}{3} \\ \lambda_{2,3} = -\frac{1}{2}(u + v) \pm i\frac{\sqrt{3}}{2}(u - v) - \frac{a}{3} \end{cases}$$

$\lambda_{2,3}$ therefore, has the form $\lambda_{2,3} = \alpha \pm i\beta$. We again consider the approximation according to Theorem 4.2:

$$\dot{\vec{x}} = \mathcal{L}(\vec{c}^+ + \vec{x}) \approx \mathfrak{J}^+\vec{x}$$

Where $\vec{x}$ is small and in the plane spanned by ${\vec{e}_{2,3}}^{+}$. Then $\vec{x}$ is a linear combination of the complex eigenvectors in this plane. These have the form:

$$\vec{e}_{2,3} = \vec{w} \pm i\vec{z}$$

For suitable $\vec{w}, \vec{z} \in \mathbb{R}^3$. If we take $\vec{e}_1$ and $\vec{e}_{2,3}$ as a basis, $\mathfrak{J}^+$ has diagonal form:

$$\mathfrak{J}^+ = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \alpha + i\beta & 0 \\ 0 & 0 & \alpha - i\beta \end{bmatrix}$$

If we choose the vectors $\vec{w}, \vec{z}$ as the basis instead of $\vec{e}_{2,3}$, then we obtain the form with respect to this basis with an analogous calculation as in Theorem 3.4 for $\mathfrak{J}^+$:

$$\mathfrak{J}^+ = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}$$

And thus, with respect to this basis, we obtain for a small $\vec{x} = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}$ in the plane $\perp \vec{e}_1^{+}$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \approx \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix} \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha x_2 + \beta x_3 \\ -\beta x_2 + \alpha x_3 \end{pmatrix}$$

And we must solve the system of equations:

$$\begin{cases} \dot{x}_2 = \alpha x_2 + \beta x_3 \\ \dot{x}_3 = -\beta x_2 + \alpha x_3 \end{cases}$$

The approach: $x_2(t) = Ce^{\alpha t} sin\beta t$ and $x_3(t) = Ce^{\alpha t} cos\beta t$ with $C \in \mathbb{R}$ provides a solution, because:

$$\begin{cases} \dot{x}_2(t) = \alpha Ce^{\alpha t} sin\beta t + \beta Ce^{\alpha t} cos\beta t = \alpha x_2(t) + \beta x_3(t) \\ \dot{x}_3(t) = -\beta Ce^{\alpha t} sin\beta t + \alpha Ce^{\alpha t} cos\beta t = -\beta x_2(t) + \alpha x_3(t) \end{cases}$$

We can now see that the trajectory moves around the fixed point C^+ and β only has an influence on the rotational speed. On the other hand, $x_2(t)$ and $x_3(t)$ become smaller if $\alpha < 0$. So, we have the result: *C^+ is attractive for $\alpha < 0$ and repulsive for $\alpha > 0$.*

The question now is where this "tipping point" is, for which C^+ becomes unstable. This is at the transition when the real part of the complex zero changes from negative to positive. At the critical point, the real part is zero, i.e. the complex zeros are purely imaginary.

We would therefore have to solve the following equation for ϱ:

$$Re(\lambda_2) = -\frac{1}{2}(u + v) + \frac{a}{3} = 0$$

This seems time-consuming, as this expression contains cubic and quadratic roots and the variable ϱ is in the determinant D in the form of a third-degree polynomial. We prefer to try an approach $\lambda_2 = i\eta$ and compare the coefficients. It is:

$$p_\lambda(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c = 0$$

So

$$p_\lambda(\lambda_2) = -i\eta^3 - a\eta^2 + bi\eta + c = 0$$

$$\Longrightarrow i\eta(-\eta^2 + b) = 0\, und\ - a\eta^2 + c = 0$$

$$\Longrightarrow \eta^2 = b\, und\, \eta^2 = \frac{c}{a}$$

$$\Longrightarrow ab = c$$

$$\Longrightarrow \frac{41}{3} \cdot \frac{8}{3}(\varrho + 10) = \frac{160}{3}(\varrho - 1)$$

This gives us the solution $\varrho \approx 24.7368$.

We have thus derived the following theorem:

Theorem 5.5

We consider the Lorenz system with the previous notations.

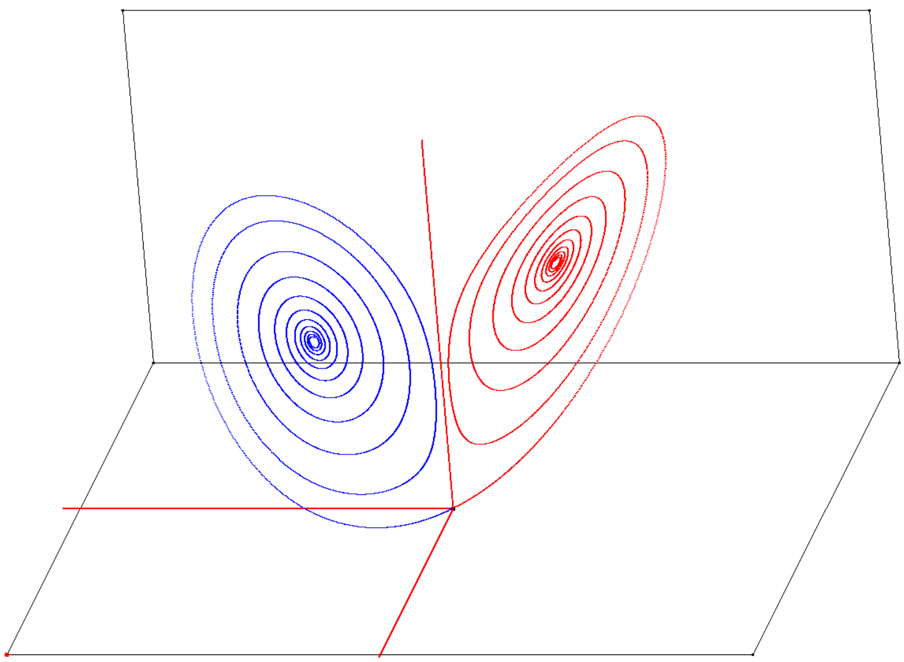
Assertion:

1) For $\varrho \in [1, 1.346 \ldots]$, $C^{\pm}$ are attractive equilibrium positions. A trajectory that comes close to one of the equilibrium positions strives directly towards it.
2) For $\varrho \in]1.346 \ldots, 24.7368 \ldots[$, $C^{\pm}$ are attractive equilibrium positions. The trajectories that come close to them move towards them in a circle.
3) For $\varrho > 24.7368 \ldots$, $C^{\pm}$ are repulsive equilibrium positions. Trajectories that come close to them move away from them in a circle.

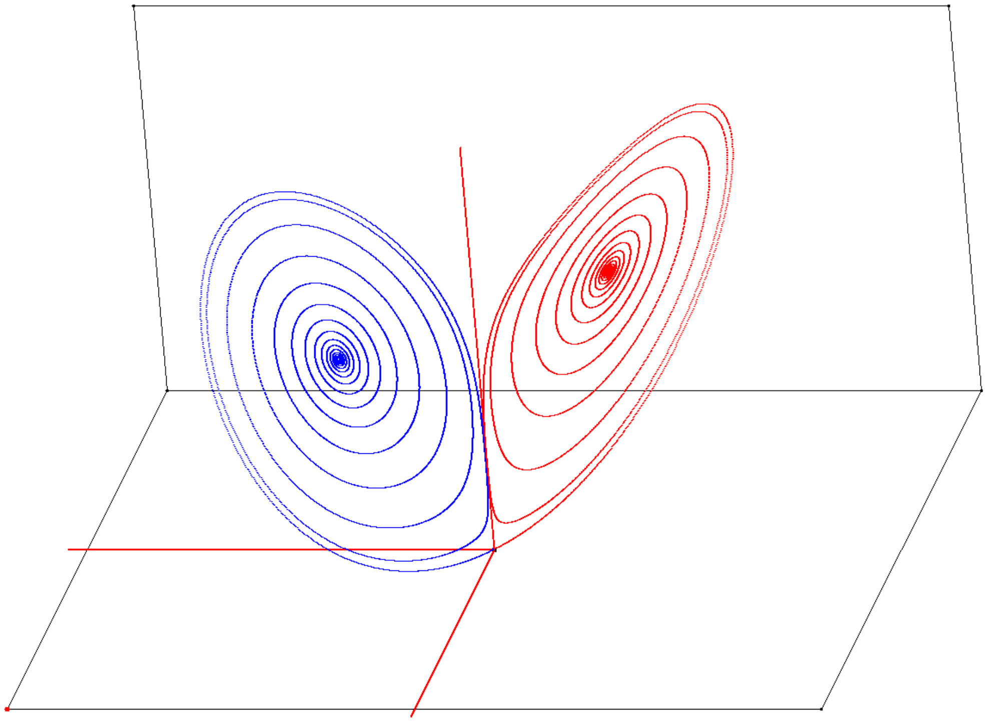
□

Higher methods can be used to show that the system always moves in a finite section of the $\mathbb{R}^3$ (see [1]).

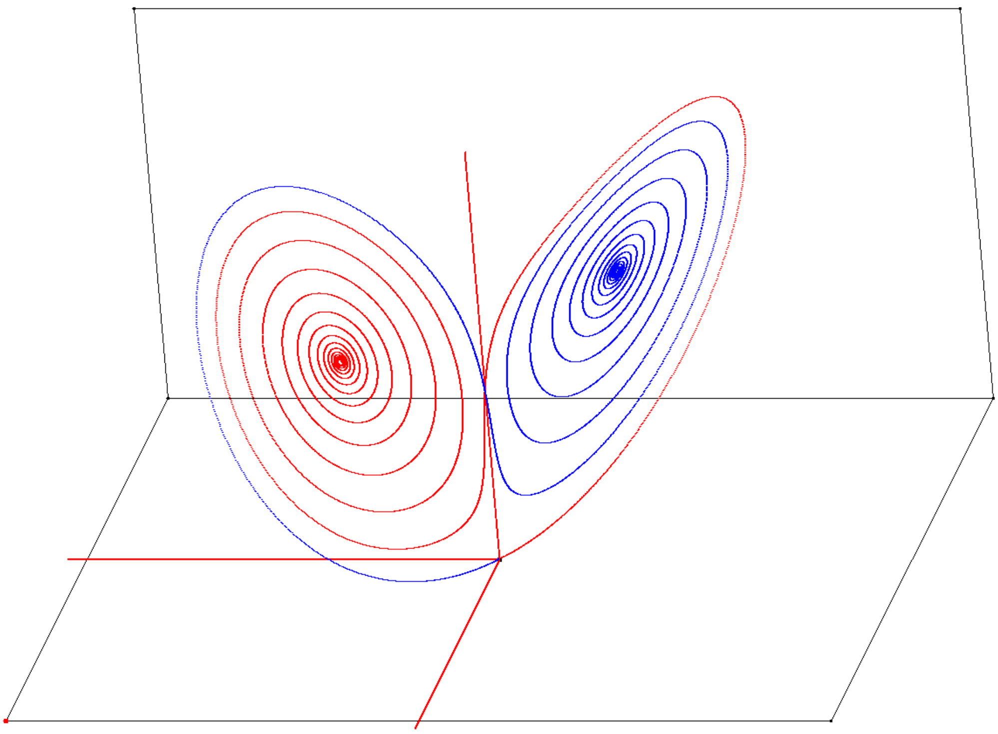
We show a few more examples with the simulator.



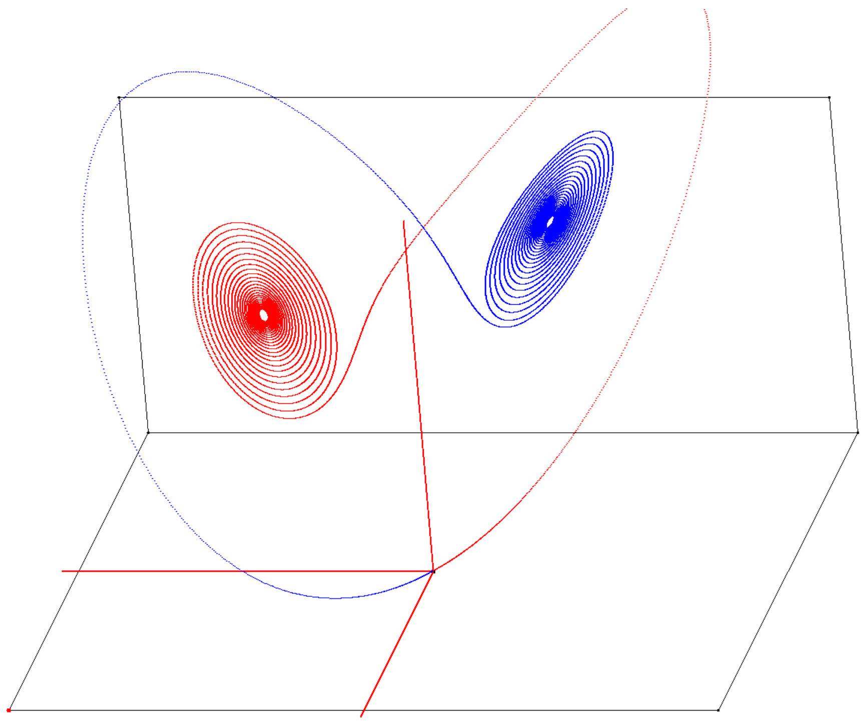
Attractive equilibrium positions for $\varrho = 13.44$



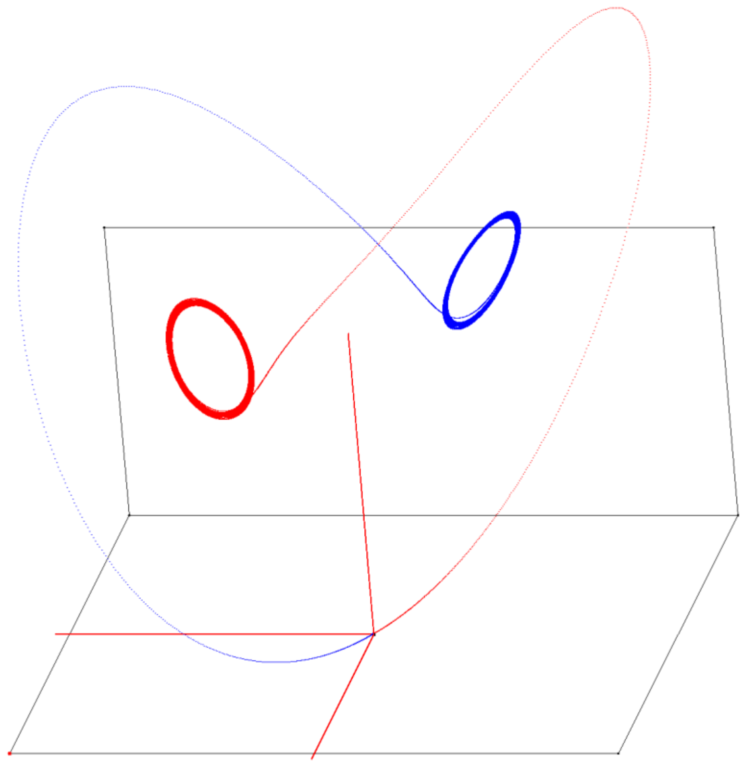
When ϱ approaches the value 13.926 , the orbits almost touch each other



For $\varrho > 13.926$ the same starting points run to the respective other fixed point

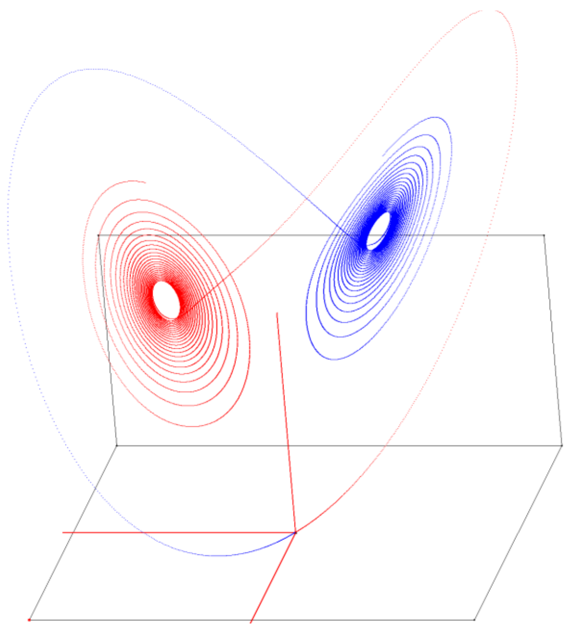


$\varrho = 20.016$: $C^{\pm}$ are still attractive



$\varrho = 24.69$: The orbits each remain almost on a stable circle around $C^{\pm}$

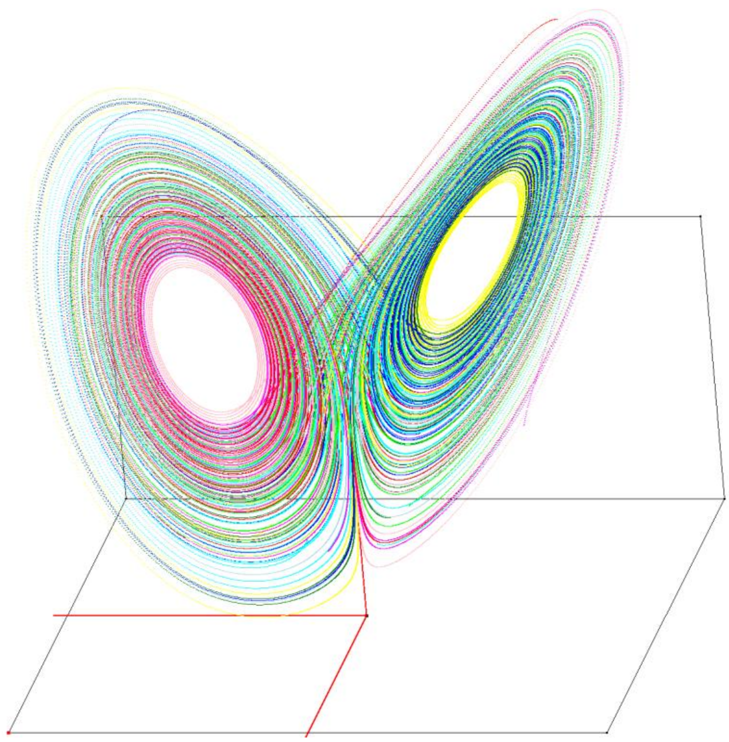
At this point, the equilibrium positions $C^{\pm}$ in the plane perpendicular to the eigenvector, which belongs to the (negative) real eigenvalue, become indifferent. The iteration moves on a circle around the respective equilibrium position.



$\varrho = 27.96$: The orbits move away from $C^{\pm}$

If ϱ continues to grow, the equilibrium positions in this plane become repulsive. However, no other new attractive equilibrium positions are created, but the orbits approach a complicated-looking set, the Lorenz attractor, which is called a *strange attractor*.

If we choose a "point cloud" as the starting set, we can see roughly what the attractor looks like. All trajectories are flattened and wind alternately around the unstable equilibrium positions $C^{\pm}$ for $\varrho > 24.7368 \ldots$ without reaching them.



The "strange" Lorenz attractor

6. Outlook: Strange attractors

In the school project *The chaotic properties of logistic growth*, the attractors were either fixed points or cycles, i.e. always a finite number of points. In the Lorenz system, however, a set of points in phase space appears to be attractive. This set of points appears as the final state of the Lorenz system: Any starting point within the defined area of the phase space gradually fills up the entire set of points of the attractor with its orbit if the behaviour is chaotic, i.e. if $\varrho > 24.7368$.

The definition of an attractor from the one-dimensional case must therefore be generalised here. A precise mathematical definition of a strange attractor can be found in [1]. Here we want to sketch the idea of the definition using the example of the Lorenz attractor.

Firstly, the set of points belonging to the attractor must be invariant. This means that once points have landed in this set, they should not move out of the set. Then sets of points that lie in the vicinity of the attractor should be "attracted" by it. We can thus express the definition of an attractive set of points as follows:

Definition 6.1

Let(X, f) be a discrete dynamic system, and *A* a closed subset of$X$, which is invariant under *f*. Thus:$f(A) \subseteq A$. *A* is called *attractive* if a neighbourhood *U* of *A* exists with the property: For every (however small) neighbourhood *V* of *A* the following applies: $f^n(U) \subset V$ for sufficiently large $n \in \mathbb{N}$. □

All points from *U* therefore come arbitrarily close to the set *A* during the iteration.

Now this does not mean that an attractive point set has chaotic behaviour or something like sensitivity and transitivity. For transitivity, the points in the neighbourhood of *A* or *A* itself should be arbitrarily "intermixed". We therefore define

Definition 6.2

Let(X, f) be a discrete dynamical system, and *A* an *f-invariant* subset of X. *f* is called *topologically transitive* on *A* if for every pair of open sets $\emptyset \neq U, V \subset A$ there is a $n \in \mathbb{N}$ with $f^n(U) \cap V \neq \emptyset$. □

In the one-dimensional case, we have required for the transitivity that during the iteration one moves from the neighbourhood of any starting point to the neighbourhood of any target point. Here, *U* replaces the role of the starting point and *V* the role of the destination point. So, if I choose *U* arbitrarily and iterate long enough, then $f^n(U)$ will intersect with any given target set *V*.

Definition 6.3

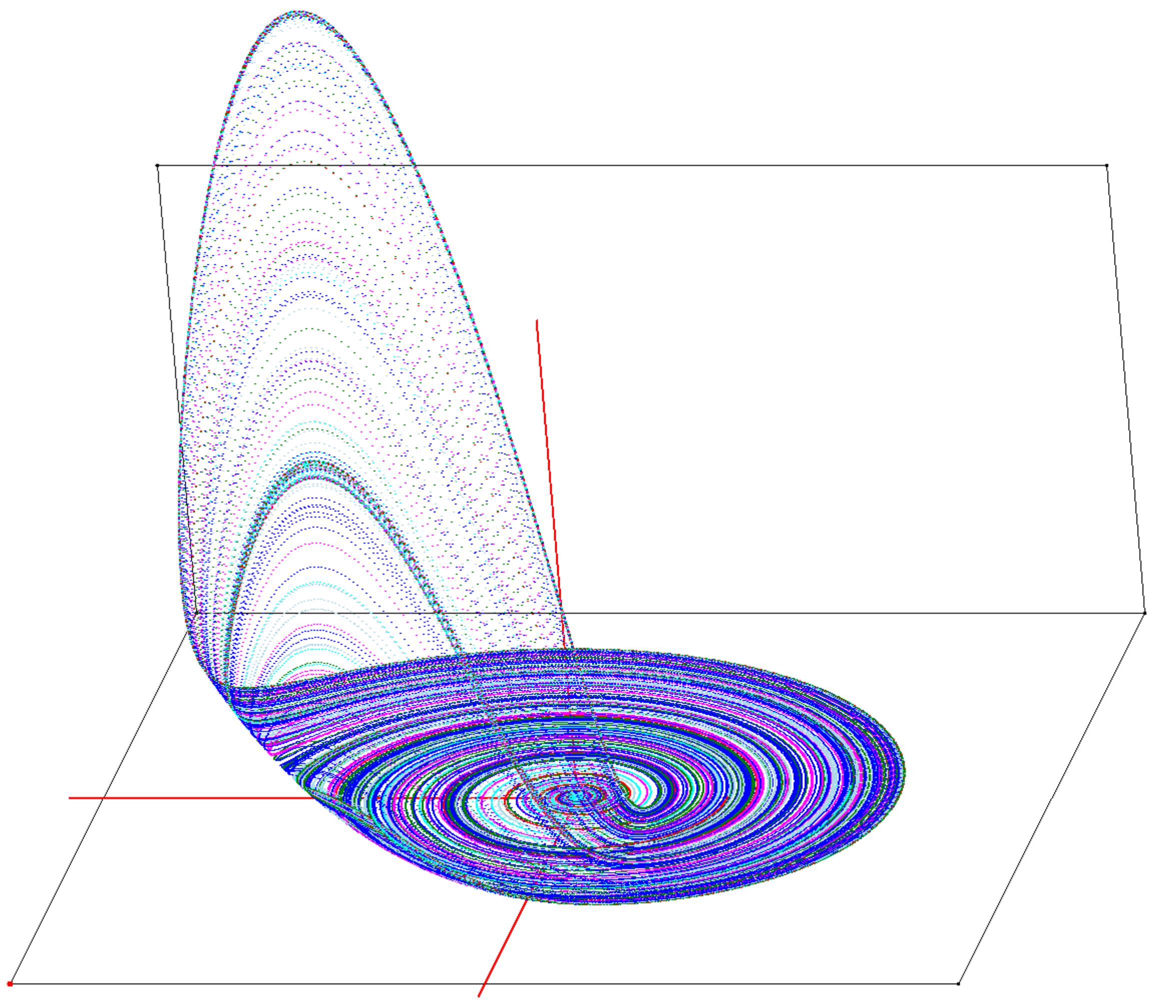
If an *f-invariant* subset $A \subset X$ is attractive and topologically transitive at the same time, it is also sensitive to initial conditions. In this case, *A* is called a *strange attractor*. □

For an in-depth study of the Lorenz attractor as a strange attractor, see [1].

In addition to the Lorenz attractor, two other strange attractors are implemented in the "simulator". The first is the so-called *Roessler attractor*, named after the biochemist and chaos researcher Otto Rössler (1940 -). The Roessler system is defined by the differential equation system:

$$\begin{cases} \dot{x} = -(y+z) \\ \dot{y} = x + ay \\ \dot{z} = b + xz - cz \end{cases}$$

Where $a = b = 0.2$ and $c \in [1,10]$. For $c > 2$ one has chaotic behaviour.

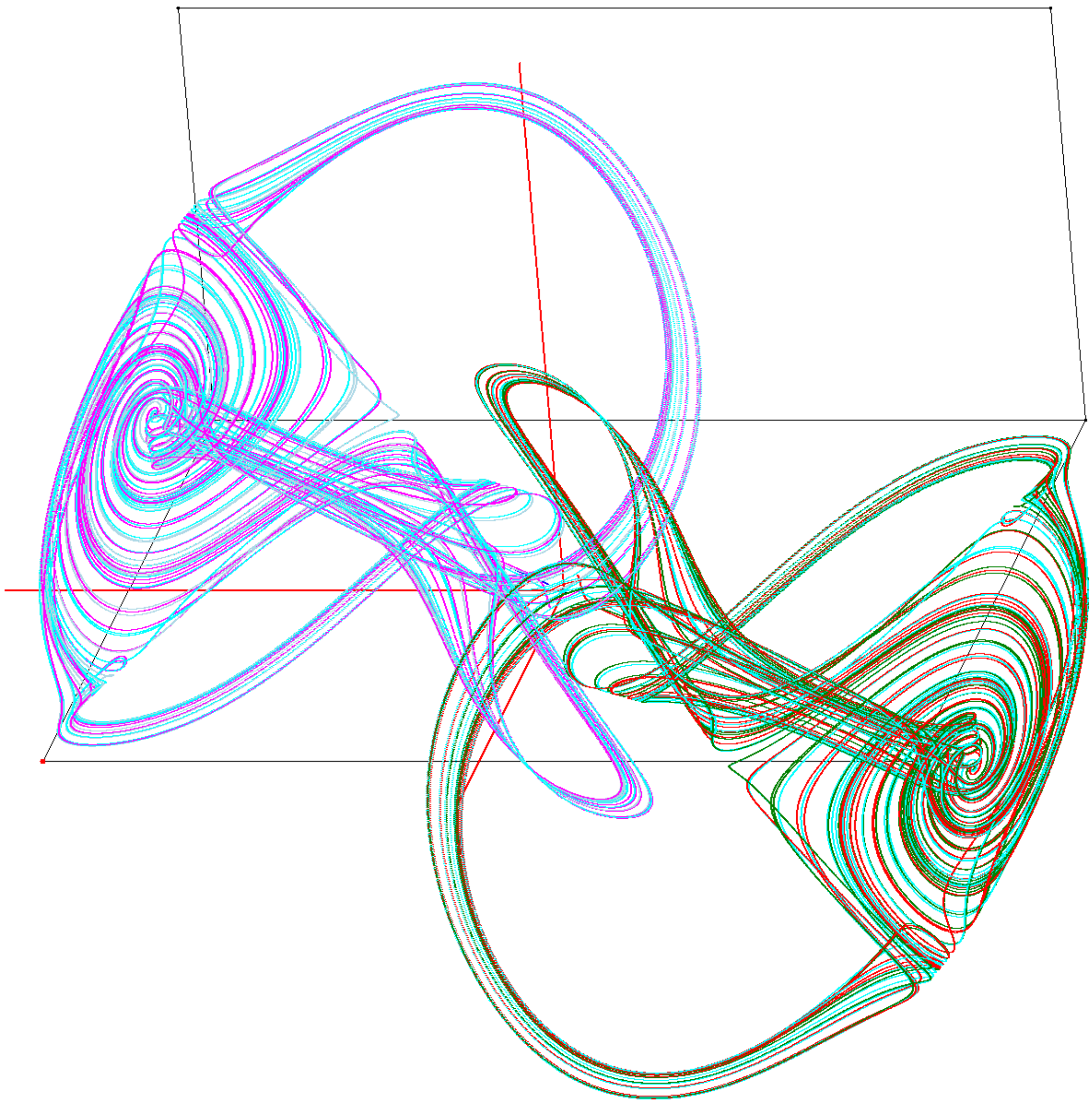


The Roessler attractor for $c = 6$

Another implemented attractor is the *Thomas attractor*. It is named after the French mathematician and physicist René Thomas (1928 - 2017). He proposed the system for modelling feedback mechanisms in biological networks. The system is defined by the differential equation system:

$$\begin{cases} \dot{x} = siny - bx \\ \dot{y} = sinz - by \\ \dot{z} = sinx - bz \end{cases}$$

Where the parameter is $b \in]0,1]$ and exhibits chaotic behaviour for $b < 0.215$.



The Thomas attractor for $b = 0.1998$

7. Exercise examples

1. Given a point reflection at the origin in the space $\mathbb{R}^3$. Show that this mapping is linear and determine the corresponding mapping matrix.

2. Given a rotation around the z-axis and the angleφ in the space $\mathbb{R}^3$. Show that this mapping is linear and determine the corresponding mapping matrix.

3. Determine the kernel of the mapping $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$

4. Determine $\lambda \in \mathbb{R}$ so that $KerA$ with $A = \begin{bmatrix} 1-\lambda & -1 & 2 \\ -1 & 2-\lambda & 0 \\ 2 & 0 & 3 \end{bmatrix}$ is different from $\vec{0}$.

5. The mapping $A = \begin{bmatrix} 2 & -1/2 & 1/2 \\ -1 & 3/2 & -1/2 \\ 1 & 1/2 & 5/2 \end{bmatrix}$ is given. Determine its eigenvalues and eigenvectors. Discuss the behaviour of the mapping near the zero point.

6. Consider the Lorenz system with the zero point as the fixed point. Investigate different paths to the zero point and under which conditions the zero point is attractive.

7. Given is the mapping

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{\beta(\varrho-1)} \\ \pm\sqrt{\beta(\varrho-1)} & \pm\sqrt{\beta(\varrho-1)} & -\beta \end{bmatrix}$$

   at the location of the equilibrium positions $C^{\pm}$ for the Lorenz system. We are looking for eigenvalues λ and eigenvectors $\vec{e}$ such that: $A\vec{e} = \lambda\vec{e}$. Show that the condition for their existence is given by: $p(\lambda) = \lambda^3 + (\sigma+\beta+1)\lambda^2 + \beta(\sigma+\varrho)\lambda + 2\sigma\beta(\varrho-1) = 0$.

8. Examine the zeros of the transformed characteristic polynomial of the Lorenz system $p(\mu) = \mu^3 + p\mu + q$ for $\varrho > 1$ using Cardano's formulae. To do this, create an Excel table for the calculations of the zeros when different values of ϱ are entered. Then calculate the original eigenvalues $\lambda = \mu + a/3$ and discuss the stability.

9. Check that in the Lorenz system for $\varrho = 24.7368$ the real part of the complex zero of $p_\lambda(\lambda)$, i.e. the expression $-\frac{1}{2}(u+v) + \frac{a}{3}$, is ≈ 0.

10. Analyse the Roessler attractor regarding fixed points and their properties.

11. Analyse the Thomas attractor regarding fixed points and their properties.

Further reading

[1] The Lorenz system, Seminar on ordinary differential equations, University of Hamburg, Uwe Jönck and Florian Prill, February 2003

[2] Deterministic Nonperiodic Flow, Edward N. Lorenz, Journal of the Atmospheric Sciences, MIT, 1963