# The chaotic properties of logistic growth

## The mathematics behind the computer programme "Simulator"

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*This document describes the chaotic properties of logistic growth using elementary accessible methods. The mathematical investigations are supported by experiments with the computer programme "Simulator".*

*The document is intended as a stimulus for extended maths lessons at middle school level, whether for courses outside the compulsory curriculum or for individual work by interested pupils.*

*The entire series of topics related to the "Simulator" includes:*

- *The chaotic properties of logistic growth*
- *The oval billiard table and periodic trajectories*
- *Newton iteration and the complex roots of unity*
- *Iteration of quadratic functions in the complex plane*
- *Numerical methods and coupled pendulums*
- *Planetary motion and the three-body problem*
- *Strange attractors and the weather forecast of Edward Lorenz*
- *Fractal sets and Lindenmayer systems*
- *The history of chaos theory*
- *Programming your own dynamic systems in the "simulator"*

*Each topic is dealt with in a separate document.*

*The computer programme "Simulator" enables the simulation of simple dynamic systems and experimentation with them. The code is publicly available on GitHub, as is a Microsoft Installer version. The corresponding link is: https://github.com/HermannBiner/Simulator. The following documentation is integrated into the "Simulator" in German and English:*

- *Mathematical documentation with examples and exercises*
- *Technical documentation with a detailed description of the functionality*
- *User manual with examples*
- *Version history*

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## 1. Logistic growth

When analysing the growth of a population, exponential growth is often chosen as the model. This means that a population grows in proportion to its current size. In the discrete model, the size of a generation is considered step by step and measured at the same time intervals.

If $x_n \in \mathbb{R}^+, n \in \mathbb{N}$ denotes the size of the nth generation, then exponential growth applies to the next generation:

$$x_{n+1} = ax_n, a > 1$$

Starting from the first generation $x_1 > 0$ then applies: $x_n = a^{n-1}x_1$. As the number of generations is in the *exponent*, this leads to a very strong growth rate after a few generations. As long as the resources for the population are unlimited, the population can theoretically grow to infinity.

Pierre-François Verhulst

In 1838, the Belgian mathematician Pierre Verhulst (1804 - 1849) analysed an alternative model in which it is assumed that the resources for the survival of the population are limited. He called this model *logistic* growth. "Logistic" in the sense of the Greek word $\lambda o\gamma o\sigma$ , which in this context can be translated as "reasonable" or "logical". In this model, a population can assume a value between 0% and 100%. 100% is the maximum possible size based on the resources.

The size of the nth generation is therefore a number between 0% and 100% or: $x_n \in [0,1]$

Verhulst's formula is:

$$x_{n+1} = ax_n(1 - x_n)$$

$1 - x_n$ is the growth-inhibiting term.

Here we iterate the function

$$f(x) = ax(1 - x), [0,1] \to [0,1]$$

where we refrain from marking the dependence of the function on$a$ with an index.

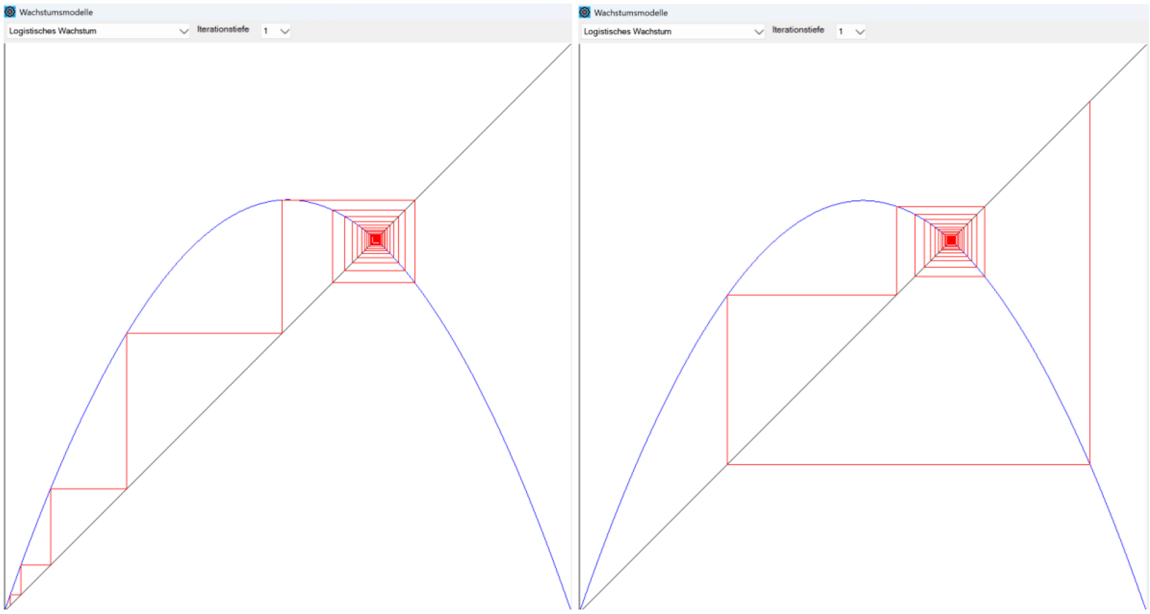
In which range may$a$ lie? On the one hand, $a > 0$ . On the other hand, the maximum of the function must be $\leq 1$ . It is:

$$f'(x) = a(1 - 2x)$$

The maximum is assumed to be at $x = \frac{1}{2}$ . The following must therefore apply: $f\left(\frac{1}{2}\right) = \frac{a}{4} \leq 1$

So $a \leq 4$ and $a \in ]0,4]$.

Now let's do some experiments with the simulator. The menu item "Growth models - Iteration" opens a window in which we select logistic growth as the iterated function and set the iteration depth to one. Now we can enter different values from$a$ and different starting values. On the left-hand side of the window, we can see the section of the iterated function$f$ in the interval $[0,1]$. You can see for yourself that the iteration is graphically equivalent to oscillating back and forth between the function graph and the 45° straight line. The detailed operation of the simulator in this window is described in the simulator manual.




Iteration for $a = 2.9$ and the start values $x_1 = 0.01$ (left) and $x_1 = 0.9$ (right)

The iteration apparently settles at a stable equilibrium position $x_\infty \approx 0.655$ regardless of the start value. This stable position is a fixed point of the function:

$$f(0.655) = 2.9 \cdot 0.655 \cdot (1 - 0.655) = 0.655$$

Obviously, the population no longer changes when it falls on a fixed point of the iterated function. The question therefore arises as to which fixed points there are and whether there are any periodic cycles in which the population oscillates between different states.

## 2. Fixed points

$\xi$ is a fixed point of logistic growth if: $f(\xi) = a\xi(1 - \xi) = \xi$

There are two solutions: $\xi_1 = 0$ and $\xi_2 = 1 - {}^{1}/_{a}$

The question is whether the iteration always moves towards one of these fixed points. In the first experiment, this was obviously not the case for the fixed point $0$. This was "repulsive" or repulsive. In contrast, the other fixed point 0.655 was "attracting" or attractive.

What determines whether a fixed point is attractive or repulsive? Apparently, it is attractive if an iteration value in its vicinity can no longer escape from its neighbourhood but tends towards the fixed point as $n$ increases. If you look at the graphs in the previous experiment, this seems to have something to do with the tangent gradient at the respective fixed point. In particular, (always for $a = 2.9$)

$$|f'(0)| = 2.9 > 1, |f'(0.655)| \approx 0.899 < 1$$

We now consider a fixed point $\xi$ with $|f'(\xi)| < 1$. Since $f'$ is continuous, there is a neighbourhood $U(\xi)$ in which

$$|f'(x)| \leq L < 1 \text{ if } x \in U(\xi)$$

If a point falls into this environment during the iteration, i.e. an $n \in \mathbb{N}$ exists for an initial value $x_0$ with $x_n = f^n(x_0) \in U(\xi)$, then according to the mean value theorem, there is a $\vartheta \in U(\xi)$ with

$$|x_{n+1} - \xi| = |f(x_n) - f(\xi)| = |f'(\vartheta)||x_n - \xi| \leq L|x_n - \xi|$$

Since $\vartheta \in U(\xi)$ applies $|f'(\vartheta)| \leq L$

The following applies for further iteration:

$$|x_{n+m} - \xi| \leq L^m |x_n - \xi| \longrightarrow 0 \; for \; m \longrightarrow \infty$$

Thus: $\lim_{n \to \infty} x_n = \xi$

You can be convinced that the fixed point is repulsive if $|f'(\xi)| > 1$

The value of the derivative in a fixed point therefore plays a decisive role in determining whether the fixed point is an *attractor* or a *repellor*. The *multiplier* of the fixed point is also called $\lambda \coloneqq |f'(\xi)|\xi$ .

In this investigation, we have only assumed that the iterated function$f$ is continuously differentiable. The result is therefore valid:

**Theorem 1**

Let $f(x)$ be continuously differentiable and $f(\xi) = \xi$

Assertion:

1) $|f'(\xi)| < 1 \implies \xi$ is an *attractor*. If a point of the iteration comes sufficiently close to the fixed point $\xi$ , it will move towards $\xi$ during the further iteration.
2) $|f'(\xi)| > 1 \implies \xi$ is a *repellor*. If a point in the iteration comes close to the fixed point $\xi$ without hitting it exactly, it moves away from $\xi$ again in the next iteration steps.

□

If $|f'(\xi)| = 1, \xi$ is *indifferent*. Here, an attractor often "flips" into a repellor or vice versa when $a$ changes.

What does this mean for logistic growth?

Whether the fixed points are attractive or repulsive depends on their multiplier or the value of the derivative in the absolute value at their position. It is:

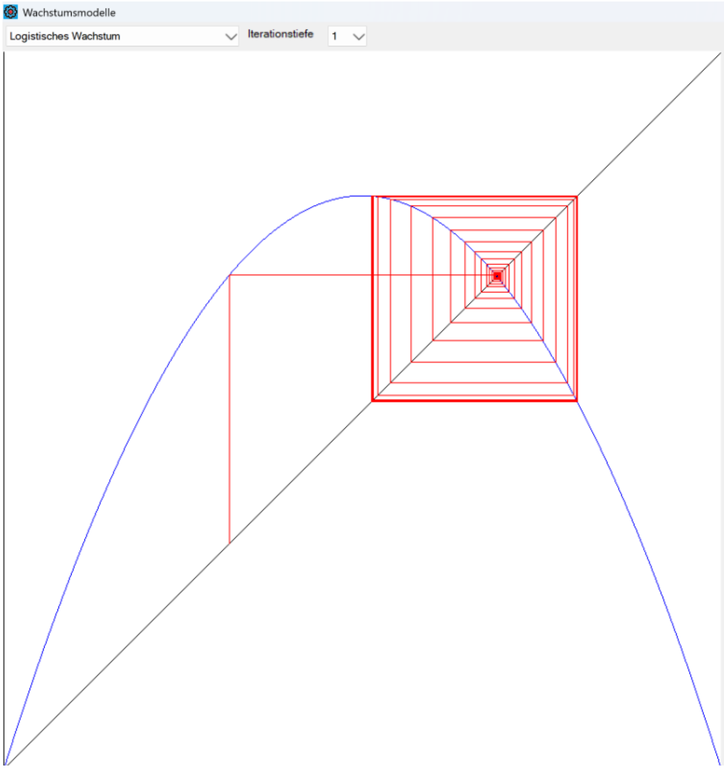
$$\lambda = |f'(x)| = |a(1 - 2x)|$$

The first multiplier is $\lambda_1 = |f'(0)| = a$ . This means that the first fixed point $\xi_1 = 0$ is attractive for $a < 1$ and repulsive for $a > 1$ . Furthermore is:

$$\lambda_2 = \left|f'(1 - {}^1\!/_a)\right| = |2 - a| < 1 \; for \; a \in ]1,3[$$

The fixed point $\xi_2 = 1 - {}^1\!/_a$ is attractive for $a \in ]1,3[$ and repulsive for $a \in ]3,4[$.

What happens for $a > 3$ ? Let's do an experiment:




Logistic growth for $a = 3.2$ and initial value $x_1 = 0.314159$

At the beginning of the iteration, its close to the fixed point $\xi_2 = 1 - \frac{1}{a} \approx 0.687$: $f(0.314159) \approx 0.689$ . The multiplier here is $\lambda = |f'(0.687)| \approx 1.197 > 1$ and the fixed point is repulsive: the iteration moves away from it.

Apparently, the iteration settles on a cycle with the values $\xi_3 \approx 0.79945$ and $\xi_4 \approx 0.51304$. It's in fact: $f(\xi_3) = \xi_4, f(\xi_4) = \xi_3$.

## 3. Cycles

For $a = 3.2$ we have come across a two-cycle logistic growth. Cycles of period 2 are fixed points of $f^2$. We want to determine these.

$f^2(\xi) = \xi$: The equation of period 2 provides a fourth-degree equation, whereby we already know two zeros, namely the fixed points of *f*: $\xi_1 = 0 \; and \; \xi_2 = 1 - {}^{1}/_{a}$.

To find the other two fixed points $\xi_3 und \; \xi_4$ we use the approach:

$$\begin{cases} \xi_3 = f(\xi_4) = a\xi_4(1 - \xi_4) \\ \xi_4 = f(\xi_3) = a\xi_3(1 - \xi_3) \end{cases}$$

If you subtract the lower equation from the upper one, you get:

$$\xi_3 - \xi_4 = -a(\xi_3 - \xi_4) + a(\xi_3^2 - \xi_4^2)$$

Since the fixed points we are looking for are truly 2-periodic, $\xi_3 \neq \xi_4$ applies and we obtain:

$$1 = -a + a(\xi_3 + \xi_4)$$

We insert $\xi_4 = \frac{1+a}{a} - \xi_3$ into the second equation and after a little calculation we get

$$a^2\xi_3^2 - a(1+a)\xi_3 + 1 + a = 0$$

This yields:

$$\xi_{3,4} = \frac{1 + a \pm \sqrt{(1+a)(a-3)}}{2a}$$

For $a > 3$ we therefore obtain a 2-cycle.

It is attractive if

$$|f^{2\prime}(\xi_3)| = |f'(f(\xi_3) \cdot f'(\xi_3)| = |f'(\xi_4) \cdot f'(\xi_3)| < 1$$

It is:

$$f'(x) = a(1-2x)$$

$$|f'(\xi_4) \cdot f'(\xi_3)| = a^2|1 - 2(\xi_3 + \xi_4) + 4\xi_3\xi_4|$$

With

$$\xi_3 + \xi_4 = \frac{1+a}{a} \; and \; \xi_3\xi_4 = \frac{1+a}{a^2}$$

we obtain the multiplier of the 2-cycle:

$$\lambda_{3,4} = \; |f'(\xi_4) \cdot f'(\xi_3)| = |a^2 - 2a - 4|$$

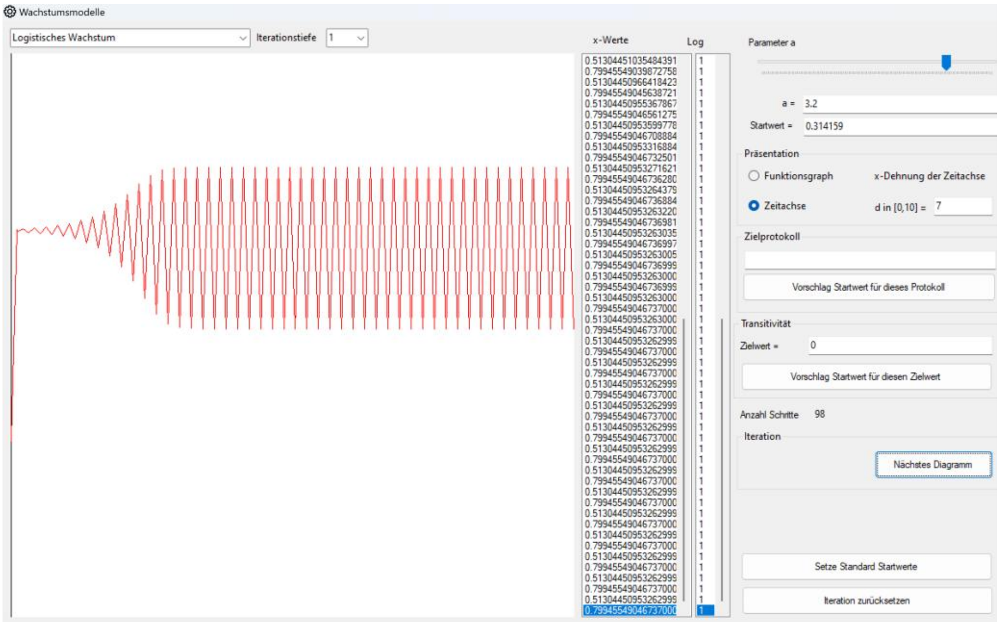
The 2-cycle is attractive for $\lambda_{3,4} < 1$ and we analyse the transition points:

$$a^2 - 2a - 4 = \pm 1$$

This equation has the solutions $a = 1 \pm \sqrt{6}$ where because of $a > 0$ only comes into question:

$$a = 1 + \sqrt{6} \approx 3.449499$$

The 2-cycle is therefore attractive in the interval $a \in ]3, 3.449499..[$

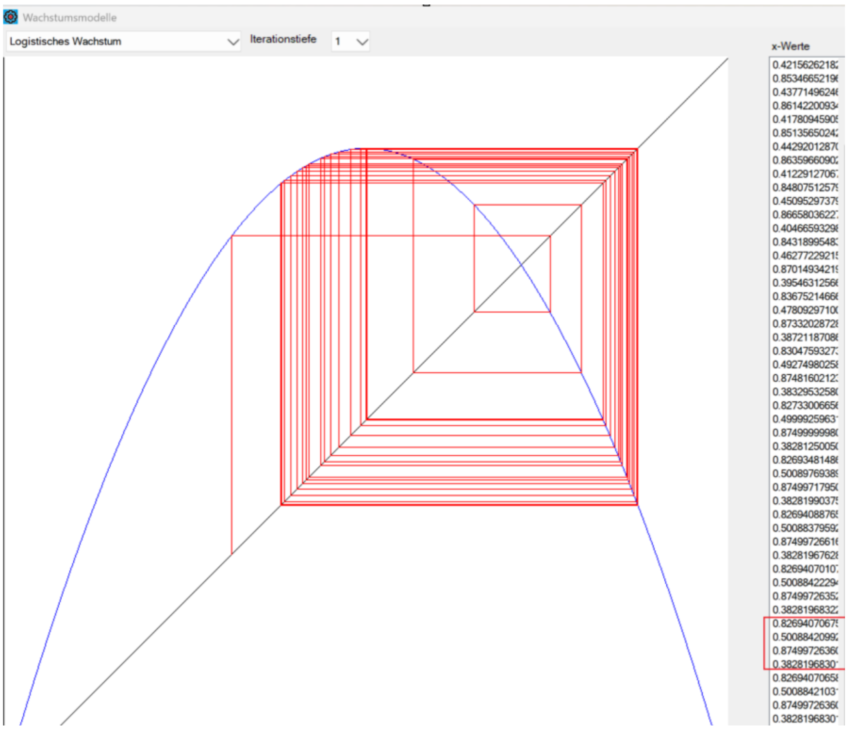



Logistic growth for $a = 3.2$ , plotted on the time axis with stretch 7

The image above shows $a = 3.2$ . The sequence soon settles into the attractive 2-cycle:

$$\xi_{3,4} = \begin{cases} 0.799455 \ldots \\ 0.513044 \ldots \end{cases}$$

What happens for $a > 3.449499$? Let's do an experiment:




The iteration for $a = 3.5$

Obviously, the iteration settles on a four-cycle:

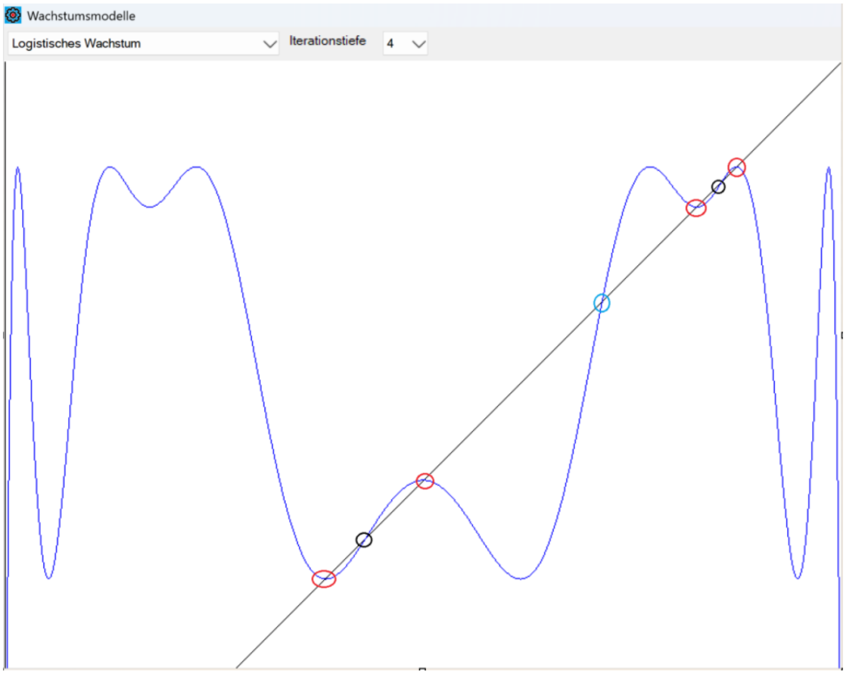
$$\{0.8269.., 0.5009.., 0.875.., 0.3828..\}$$

As you can easily do the maths, each of these points is a fixed point of $f^4$. In addition, the multiplier for $a = 3.5$ and this cycle is less than one. With the chain rule you have:

$$\left|f^{4'}(\xi_1)\right| = |f'(\xi_4)f'(\xi_3)f'(\xi_2)f'(\xi_1)| \approx 0.0217 < 1$$

In particular, the multiplier is the same for all elements of the cycle.

Any further cycles are fixed points of $f^k$, i.e. solutions of equations of degree $2^k$. The fixed points of $f^k$ are the intersections of the graph of $f^k$ with the 45° line. To be able to analyse this, the simulator also allows iterations with a depth of $k \leq 10$.




Graph of the function $f^4$

You can recognise the repulsive fixed point of $f$ (green), the repulsive two cycle or the repulsive fixed points of $f^2$ (black) and the attractive four cycle or the four fixed points of $f^4$ (red).

In general, $\xi$ is a *periodic point of order k or k-periodic point of f,* if this point is mapped back onto itself after at least k iteration steps. The following applies to such a point:

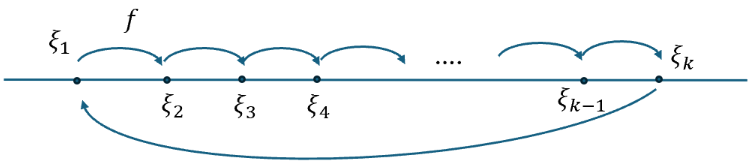
$$f^k(\xi) = \xi \text{ and } f^i(\xi) \neq \xi \text{ für } 0 < i < k$$

A k-periodic point of $f$ is a fixed point of $f^k$.

If $\xi$ is a k-periodic point of $f$, then the set

$$\{\xi, f(\xi), f^2(\xi), \dots, f^{k-1}(\xi)\} =: \{\xi_1, \xi_2, \dots, \xi_k\}$$

is called *k-cycle* of the iteration of $f$. Each point in this cycle is then a k-periodic point.




With a cycle, the question arises as to whether a cycle can consist of both attractors and repellors. Whether a point $\xi_i := f^i(\xi_1), 1 \leq i < k$ is attractive or not depends on the derivation of $f^k$ at this point. According to the chain rule:

$$|f^k(\xi_i)'| = \left| f'\left(f^{k-1}(\xi_i)\right) \cdot f'\left(f^{k-2}(\xi_i)\right) \cdots f'(\xi_i) \right|$$

The function arguments on the right-hand side run through the entire cycle.

This means that the derivative

$$\left|f^k(\xi_i)'\right| = \left|f'(\xi_k) \cdot f'(\xi_{k-1}) \cdots f'(\xi_1)\right| = \prod_{i=1}^{k} |f'(\xi_i)| = a^k \prod_{i=1}^{k} |1 - 2\xi_i|$$

is independent of *i* and assumes the same value for all points of the cycle. All points of the cycle therefore have *the same* multiplier and are all either *attractive* or *repulsive*.

Of course, this applies not only to logistic growth, but also to any continuously differentiable function $f$.

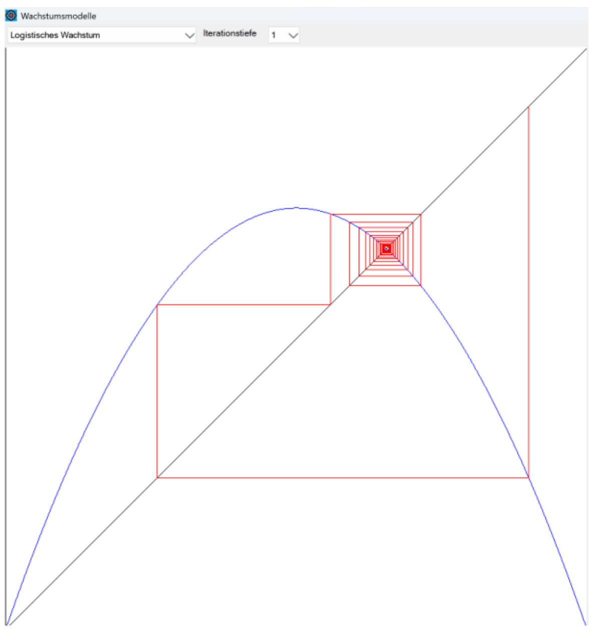
## 4. The basin of an attractor

We consider a continuously differentiable function $f$ with an attractive fixed point $\xi$. Here we are interested in the question of which starting points $x_1$ converge from the domain of definition of $f$ to $\xi$ during the iteration. These starting points form the *basin* of the attractor and we define:

$$B(\xi) := \{x \in \mathcal{D}_f / \lim_{n \to \infty} f^n(x) = \xi\}$$

Analogously, we can speak of the basin of an attractive cycle. If we have an attractive k-cycle of $f$, each element of the cycle is a fixed point of $g := f^k$ and we consider the set of starting points that converge to a fixed point of $g$ during the iteration. Since $g$ has k fixed points in a k-cycle (one per element of the cycle), there are also k different basin parts per fixed point of $g$, which can look quite intricate and nested within each other. However, their union is the basin of the attractive cycle of $f$. In this sense, the individual basins of the fixed points of $g$ belong to the same cycle of $f$ and are together the basin of an attractive cycle.

Let us return to the case of an attractive fixed point of logistic growth. For $a = 2.9$ the zero point was repulsive and the fixed point $\xi_2 \approx 0.687$ attractive. Since point 1 is mapped into the zero point, it does not belong to the basin of $\xi_2$. All other points in the interval ]0, 1[ obviously belong to it.




$a = 2.9$ and start value $x_1 = 0.9$

In this example, the basin was an open interval. In general, we have

**Theorem 2**

Let $f$ be a continuously differentiable function with an attractive fixed point $\xi$. Let the basin be contained entirely within the domain of definition of $f$, i.e. in particular $B(\xi) \cap \partial D_f = \emptyset$ .

Assertion:

The basin $B(\xi)$ is an open set.

□

Proof:

Let $x \in \mathcal{B}(\xi)$ , i.e.$|f^n(x) - \xi| < \frac{\varepsilon}{2}$ if n is large enough. Now we choose an $x'$ close enough to $x$ so that $|f^n(x') - f^n(x)| < \frac{\varepsilon}{2}$. Because of the continuity of $f^n$ this is possible and such an $x'$ exists, since $x$ does not belong to the boundary of $D_f$.

Then applies:

$$|f^n(x') - \xi| \leq |f^n(x') - f^n(x)| + |f^n(x) - \xi| < \varepsilon$$

For n large enough. Thus $x'$ also converges to $\xi$ and $x' \in \mathcal{B}(\xi)$.

□

We again consider the basin $B(\xi)$ of an attractive fixed point $\xi$ under the iterated function $f$. If $x \in B(\xi)$ , then also $f(x) \in B(\xi)$, because $\lim_{n\to\infty} f^n\big(f(x)\big) = \lim_{n\to\infty} f^{n+1}(x) = \xi$. Likewise, every preimage of x: $f^{-1}(x) \in B(\xi)$, because $\lim_{n\to\infty} f^n\big(f^{-1}(x)\big) = \lim_{n\to\infty} f^{n-1}(x) = \xi$.

Thus holds:

**Theorem 3**

Let$f$ be a continuously differentiable function with an attractive fixed point . $\xi$

Assertion:

The basin $B(\xi)$ is *fully invariant*. This means:$f(B(\xi)) \subseteq B(\xi)$ and $f^{-1}(B(\xi)) \subseteq B(\xi)$.

□

$f^{-1}(B(\xi))$ here in the sense of "the set of all preimages".

The *immediate basin* of an attractive fixed point plays an important role. This is the contiguous part of the basin that contains the fixed point $\xi$. We assume that the immediate basin lies entirely within the domain of definition of $f$, whereby $f$ is continuously differentiable. Then the immediate basin is an open interval $]a,b[\subset D_f$. The boundary points no longer belong to the basin: $a, b \notin \mathcal{B}(\xi)$.

Since $f$ is continuous, $f([a,b])$ is again connected and therefore an interval. Since $\xi$ is a fixed point of $f$, $\xi \in f([a,b])$. On the other hand, $]a,b[$ is fully invariant. Thus $f(]a,b[) \subseteq ]a,b[$. Because of the full invariance of the basin, $f(a), f(b) \notin ]a,b[$ . This leaves only the possibility $f(a), f(b) \in \{a,b\}$.

**Theorem 4**

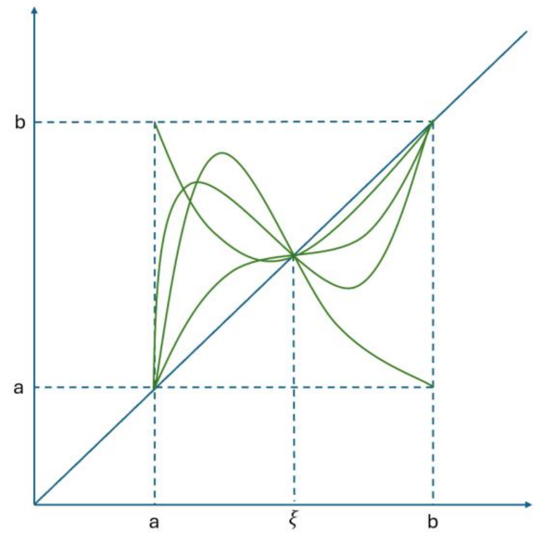
Let$f$ be a continuously differentiable function with an attractive fixed point $\xi$. Let the basin of $\xi$ be contained entirely within the domain of definition of $f$.

Assertion:

The immediate basin is an open interval $]a, b[$. The boundary points of this interval are again mapped to boundary points of the interval.

□

This sentence will be useful in the following section. The following picture shows what the immediate pool could look like. It is a combination of the cases $f(a) = a, f(a) = b, f(b) = a, f(b) = b$ and $f(]a, b[) \subseteq ]a, b[$.




The immediate basin of $f$

The question now is whether there can be several attractive cycles simultaneously for the logistic growth for a fixed $a$, so that you end up with one cycle or another depending on the starting point of the iteration. The question is also, which are suitable starting points for finding attractive cycles?

We will analyse this question in the next section.

## 5. The role of the critical point

Let $f$ be a continuously differentiable function. A *critical point* of $f$ is a point at which the derivative of $f$ vanishes. In logistic growth, the critical point is $\frac{1}{2}$.

The following theorem answers the question of how many attractive cycles there can be for a fixed value of$a$ in logistic growth.

**Theorem 5**

Let $f$ be the logistic growth.

Assertion: If there is an attractive cycle in the iteration of $f$, then the critical point lies in its basin.

□

The proof is elementarily accessible, but somewhat complex.

As preparation, we need the so-called Schwarz derivation (after the mathematician Hermann Schwarz 1843 - 1921). This is defined for a three times differentiable function $f$ with $f'(x) \neq 0$ as:

$$Sf(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}(\frac{f''(x)}{f'(x)})^2$$

In the case of logistic growth

$$Sf(x) = -\frac{3}{2}\left(\frac{-2a}{a(1-2x)}\right)^2 = \frac{-6}{(1-2x)^2}$$

We also need the following lemma:

**Lemma 6**

Let $f, g: [a, b] \to [a, b]$ be two triply differentiable functions with $f'(x), g'(x) \neq 0$ in $[a, b]$.

Assertion:

$$S(f \circ g)(x) = Sf\big(g(x)\big) \cdot g'(x)^2 + Sg(x)$$

□

Proof:

$$f\big(g(x)\big)' = f'\big(g(x)\big)g'(x)$$

$$f\big(g(x)\big)'' = f''\big(g(x)\big)g'(x)^2 + f'\big(g(x)\big)g''(x)$$

$$f\big(g(x)\big)''' = f'''\big(g(x)\big)g'(x)^3 + 3f''(g(x)g'(x)g''(x) + f'(g(x))g'''(x)$$

Now is:

$$\frac{f(g(x))'''}{f(g(x))'} = \frac{f'''(g(x))}{f'(g(x))} \cdot g'(x)^2 + \frac{3f''\big(g(x)\big)g''(x)}{f'\big(g(x)\big)} + \frac{g'''(x)}{g'(x)}$$

And:

$$-\frac{3}{2}\left(\frac{f\big(g(x)\big)''}{f\big(g(x)\big)'}\right)^2 = -\frac{3}{2}\left(\frac{f''\big(g(x)\big)g'(x)^2 + f'\big(g(x)\big)g''(x)}{f'\big(g(x)\big)g'(x)}\right)^2 =$$

$$-\frac{3}{2} \cdot \frac{f''(g(x))^2 g'(x)^4 + 2f''\big(g(x)\big)f'\big(g(x)\big)g'(x)^2 g''(x) + f'(g(x))^2 g''(x)^2}{f'(g(x))^2 g'(x)^2}$$

If you add the two equations, you get:

$$S(f \circ g)(x) = \left(\frac{f'''\big(g(x)\big)}{f'\big(g(x)\big)} - \frac{3}{2}\frac{f''\big(g(x)\big)^2}{f'\big(g(x)\big)^2}\right) \cdot g'(x)^2 + \left(\frac{g'''(x)}{g'(x)} - \frac{3}{2}\frac{g''(x)^2}{g'(x)^2}\right)$$

$$= Sf\big(g(x)\big) \cdot g'(x)^2 + Sg(x)$$

□

**Lemma 7**

Let $f$ be the logistic growth function and $g \coloneqq f^n$. $g$ have an attractive fixed point $\xi$ with $]a, b[$ as the immediate basin.

Assertion:

$g$ has a critical point and this lies in $]a, b[$.

□

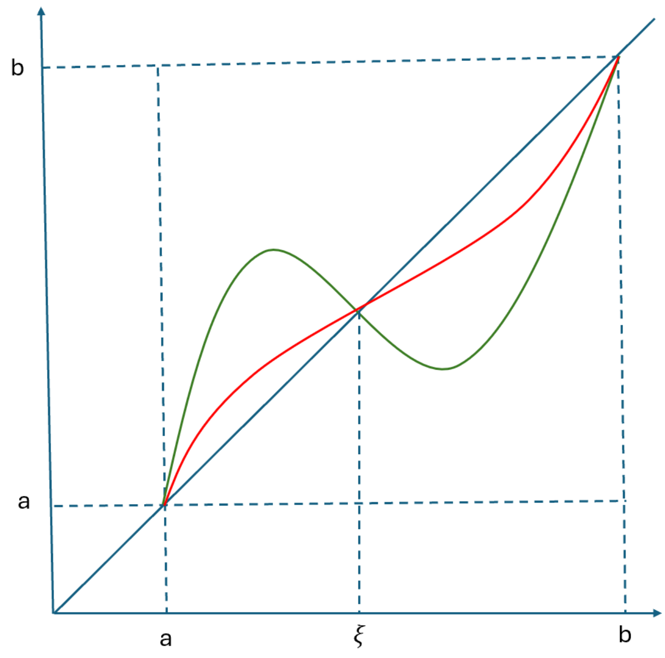
If $a = 0$ or $b = 1$ is, then the following proof is also conclusive with minor adjustments. We therefore consider the case in which $]a, b[$ lies completely inside $[0,1]$ .

According to Theorem 4, $a, b$ is mapped back to $a, b$ under the function $g$.

If $g(a) = a, g(b) = a$ or $g(a) = b, g(b) = b$ there is a point $c \in ]a, b[$ with $g'(c) = 0$ according to Rolle's theorem. In this case, the assertion holds.

The remaining cases are $g(a) = a, g(b) = b$ and $g(a) = b, g(b) = a$. We can trace the second case back to the first by looking at $g^2$. Then $g^2(a) = a, g^2(b) = b$. If there is a point $c \in ]a, b[$ with $g^{2\prime}(c) = g'(g(c)) \cdot g'(c) = 0$ then the assertion is valid.

So, the case $g(a) = a, g(b) = b$ remains to be analysed. The immediate basin of $\xi$ can look like this:




The immediate basin of $\xi$

In the case of the green curve, the assertion holds. We still must show that the red case is not possible. In this case, there is an inflection point of the curve $d \in ]a, b[$ with:

$$g'(d) > 0, g''(d) = 0, g'''(d) > 0$$

Since $g''$ has a minimum in $d$. Therefore, also applies:

$$Sg(d) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2 > 0$$

We now use complete induction on $n$ to show that this is not possible for the logistic growth $.f$

Base case: The following applies to all $x \in ]a, b[$ :

$$Sf(x) = \frac{-6}{(1-2x)^2} < 0$$

Induction step:

Let $Sf^n(x) < 0$ be for all $x \in ]a, b[$ . Then, according to Lemma 6:

$$Sf^{n+1}(x) = Sf\big(f^n(x)\big) \cdot f^{n\prime}(x)^2 + Sf^n(x) < 0$$

Thus $Sg > 0$ is not possible and so Lemma 7 is proved.

□

Proof of Theorem 5.

Let $d \in ]a, b[$ be the critical point of $g$ under the conditions of Theorem . Then it is:

$$0 = g'(d) = f'\big(f^{n-1}(d)\big) \cdot f'\big(f^{n-2}(d)\big) \cdots f'(d)$$

Thus $f'\left(f^{n-k}(d)\right) = 0$ holds for at least one of the factors and therefore $f^{n-k}(d) = c \in ]a, b[$ is the critical point of $f$ and this lies in the immediate basin.

□

This theorem has important consequences. If a function fulfils the conditions of the theorem, then the following applies:

**Corollary 8**

Let $f$ be the logistic growth.

Assertion:

1) The critical point $c = 0.5$ is the ideal starting point for finding an attractive cycle.
2) There is at most *one* attractive cycle. All other cycles are repulsive.
3) If the critical point falls on a repellor during iteration, then there is no attractive cycle.

□

1) and 3) follows directly from theorem 5.

2) applies because $f$ only has one critical point and this cannot belong to two different basins of attractive cycles at the same time.

## 6. The case a=4

If $a = 4$ the iterated function for logistic growth is:

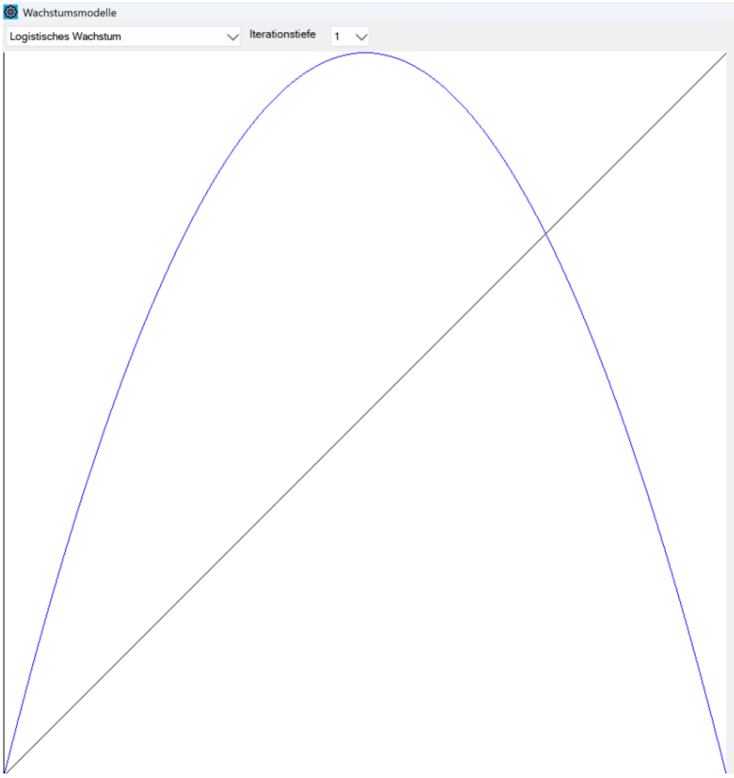
$$f(x) = 4x(1-x)$$

The critical point falls to zero after two iteration steps:

$$f^2(0.5) = f(1) = 0$$

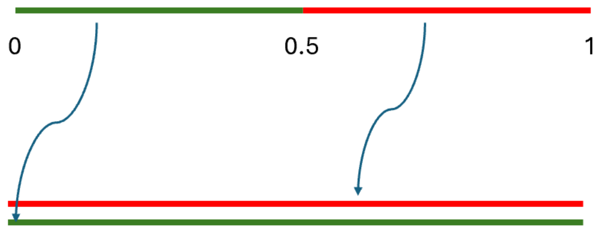
However, the zero point is a repellor. Therefore, there can be no attractive cycle in this case. What does this mean for the behaviour of the iteration?

Firstly, we look at the graph of $f$:



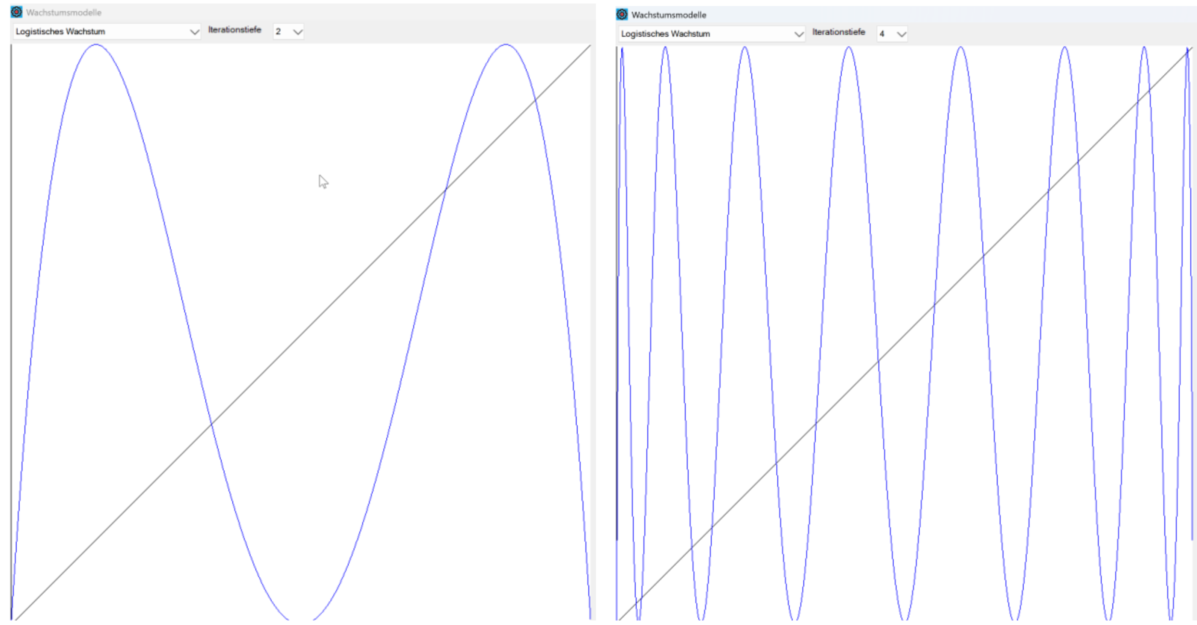

$$f(x) = 4x(1-x)$$

The interval $[0,0.5]$ is stretched and mapped to the whole of $[0,1]$ . The same applies to the interval $[0.5,1]$. The points in $[0,1]$ are therefore completely intermixed by the mapping without creating an attractive cycle.




Mixing of the points in $[0,1]$

In the graph, we can see that $f$ has a repulsive fixed point. If we are looking for cycles of $f$, we must analyse the fixed points of $f^n$. At each iteration step, the interval $[0,1]$ is halved, each half is stretched again to the full interval length and then placed on the interval. This results in the following graphs:




Left: the graph of $f^2$ , right: the graph of $f^4$

On the left you can see 4 fixed points, two of which belong to the fixed points of $f$ and the other two form a 2-cycle. You can see 16 fixed points. They belong to the fixed points and 2-cycle of $f^2$. Then you obviously have three different 4-cycle cycles of $f$.

Apparently, there are cycles for each $n \in \mathbb{N}$ , which fill the interval $[0,1]$ more and more densely. Since the tangent gradient at the points of intersection with the 45° straight line is always greater than 1, all cycles are repulsive.

**Theorem 9**

Let $f$ be the logistic growth with $a = 4$.

Assertion:

1) The polynomial $f^n(x) - x$ has exactly $2^n$ zeros and these are all real.
2) The function $f^n$ has exactly $2^n$ intersections with the 45° straight line in the interval$[0,1]$ .

□

Proof

First we establish: $1) \Leftrightarrow 2)$

Now we prove 2) with complete induction.

Base case:$f$ has exactly one maximum with the function value 1 and two minima with the function value 0 in the interval $[0,1]$ and is continuous. ⇒ $f$ has two intersection points with the 45° straight line.

Induction assumption: $f^n$ has in the interval $[0,1]$ exactly $2^n$ maxima with the function value 1 and $2^n + 1$ minima with the function value 0. ⇒ $f$ has $2^n$ intersections with the 45° straight line.

Then we look at $f^{n+1}$. Because of the convolution discussed earlier, the following applies:

If $x \in [0,0.5]$, then $f(x) \in [0,1]$ . This means that $f^{n+1}(x) = f^n(f(x))$ has exactly $2^n$ maxima with the function value 1 and $2^n + 1$ minima with the function value 0 in the interval $[0,0.5]$ . The same applies if $x \in [0.5,1]$ is $f(x) \in [0,1]$. Thus $f^{n+1}(x) = f^n(f(x))$ in the interval $[0.5,1]$ has exactly $2^n$ maxima with the function value 1 and $2^n + 1$ minima with the function value 0. Now we have counted the minimum at the interval boundary at $x = 0.5$ twice. This means that $f^{n+1}$ in the interval $[0.5,1]$ has exactly$2 \cdot 2^n$ maxima with the function value 1 and $2 \cdot 2^n + 1$ minima with the function value 0. ⇒ $f^{n+1}$ has $2^{n+1}$ intersections with the 45° straight line.

□

We will now carry out five experiments with the simulator. In the menu item "Growth models - two-dimensional", we move around the square $[0,1] \times [0,1]$ . Starting from a starting point$(x_1, y_1)$ , the next point in each component is calculated according to logistic growth with $a = 4$ , i.e:

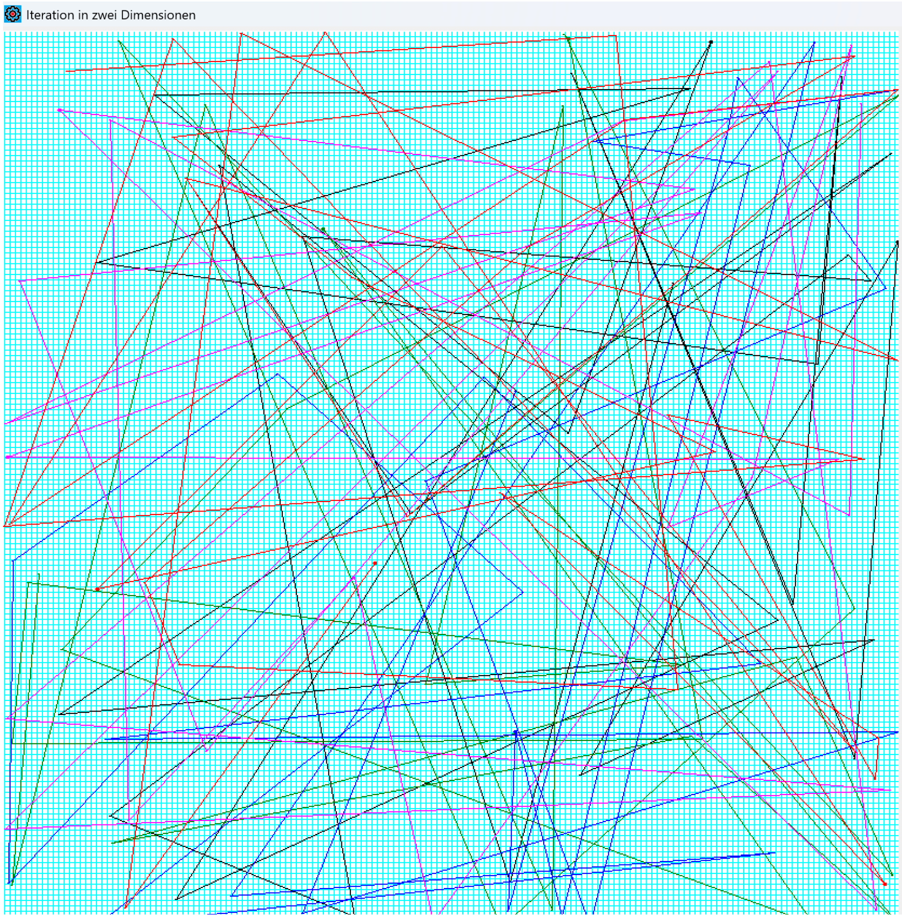
$$(x_{n+1}, y_{n+1}) \coloneqq (4x_n(1 - x_n), 4y_n(1 - y_n))$$

The two points are then connected with a straight line. You can therefore see the "orbit" of the starting point during iteration.

Firstly, we examine the "Sensitivity" option in the simulator window. We select the following five starting points and draw the first 30 steps of the orbit. Each experiment has a different colour. We notice that this orbit reacts sensitively to minimal changes in the start value.

| **Starting point x-coordinate** | **Starting point y-coordinate** | **Colour of the orbit** |
|---|---|---|
| 0.414 | 0.407 | Black colour |
| 0.4140000001 | 0.4070000001 | green |
| 0.41400000011 | 0.40700000011 | blue |
| 0.414000000111 | 0.407000000111 | violet |
| 0.4140000001111 | 0.4070000001111 | red |

Here is the result:

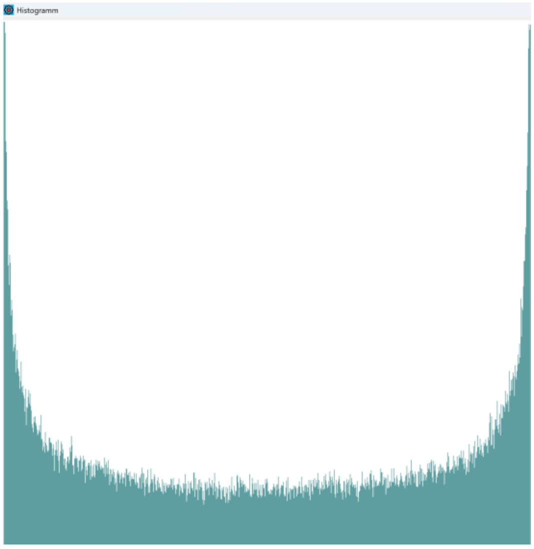



The five experiments

The starting point is always the same cell. In the simulator, the diagram has 600x600 pixels and the cell size is 5x5 pixels. So, there are 120 cells in one direction. With an interval width of 1 for logistic growth, this corresponds to a width of $\Delta x \approx 0.0083$.

Let us imagine an experimenter who is analysing the dynamic system presented here. He does not know the law of motion of the dynamic system (i.e. the logistic growth in each component), but would like to find it out. Sen's measuring device has an accuracy of around $0.01$ units. When setting the starting point, errors in this range can occur. When he carries out the above five experiments, he always chooses the same starting point from his point of view. However, the orbit is completely different for each experiment. Our experimenter would come to the conclusion that he is dealing with a random movement.

In the case of a random movement, a histogram would provide further information. The simulator offers this in the menu item "Growth models - Histogram": The interval $[0,1]$ is divided into as many parts as the display area has pixels. The number of hits per part is then totalled and displayed as a bar.




Histogram for logistic growth

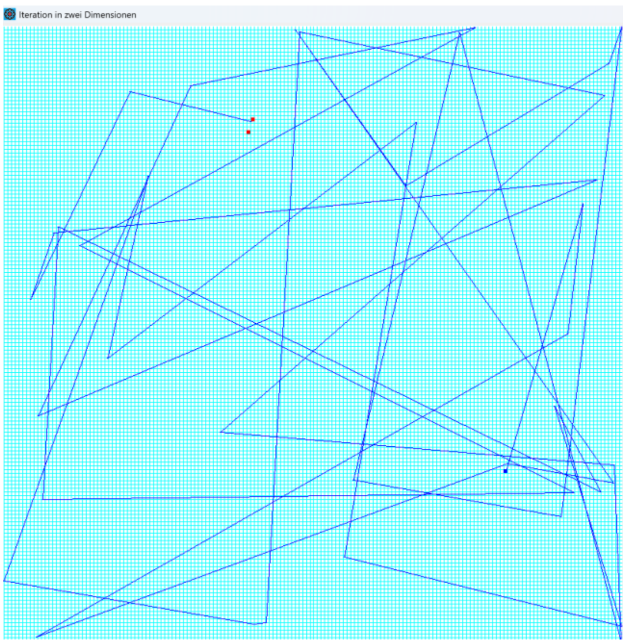
The curve in the histogram is not a uniform distribution. We will see shortly why the distribution function is $p(x) = \frac{1}{\pi\sqrt{x}\sqrt{1-x}}$.

Now the simulator has certain capabilities that we will only understand later. This is shown in the next experiment. We select the "Transitivity" option in the simulator window.

We select any starting point with the mouse and memorise its coordinates. For example $(0.811, 0.280)$ . We then select any end point with the mouse. The simulator now has the ability to minimise the starting point so that we can get as close as we like to the freely selected end point during the iteration. The following image shows the new starting point

$$(0.8109999490120963488132810481, 0.2799996838249315863022343866)$$

The start point is coloured blue in the image, the end point red. Then we let the iteration run. Here is the result:




After a few steps, you end up close to the specified end point

This experiment shows that, starting from an arbitrary starting point, any given end point can be almost reached if the starting point is changed just a little.

We will now try to "get to grips" with the logistic growth in the case of $a = 4$ in another way. We note the individual values of the iteration, i.e. the generated sequence of $x_n$ and make the following "protocol":

$$p(x_n) := \begin{cases} 0, x_n \in [0,0.5[ \\ 1, x_n \in [0.5,1] \end{cases}$$

At the starting point $x_1 = 0.314159$ , the simulator then returns the following values:

| x-Werte | Log |
|---|---|
| 0.314159 | 0 |
| 0.86185249087 | 1 |
| 0.47625109938 | 0 |
| 0.99774395887 | 1 |
| 0.00900380559 | 0 |
| 0.03569094833 | 0 |
| 0.13766841816 | 0 |
| 0.47486329921 | 0 |
| 0.99747258509 | 1 |
| 0.01008410831 | 0 |
| 0.03992967629 | 0 |
| 0.15334118898 | 0 |
| 0.51931067497 | 1 |
| 0.99850839132 | 1 |
| 0.00595753510 | 0 |
| 0.02368817152 | 0 |
| 0.09250816820 | 0 |
| 0.33580162809 | 0 |
| 0.89215557865 | 1 |
| 0.38485600852 | 0 |
| 0.94696744491 | 1 |
| 0.20088041275 | 0 |
| 0.64210989009 | 1 |
| 0.91921911654 | 1 |

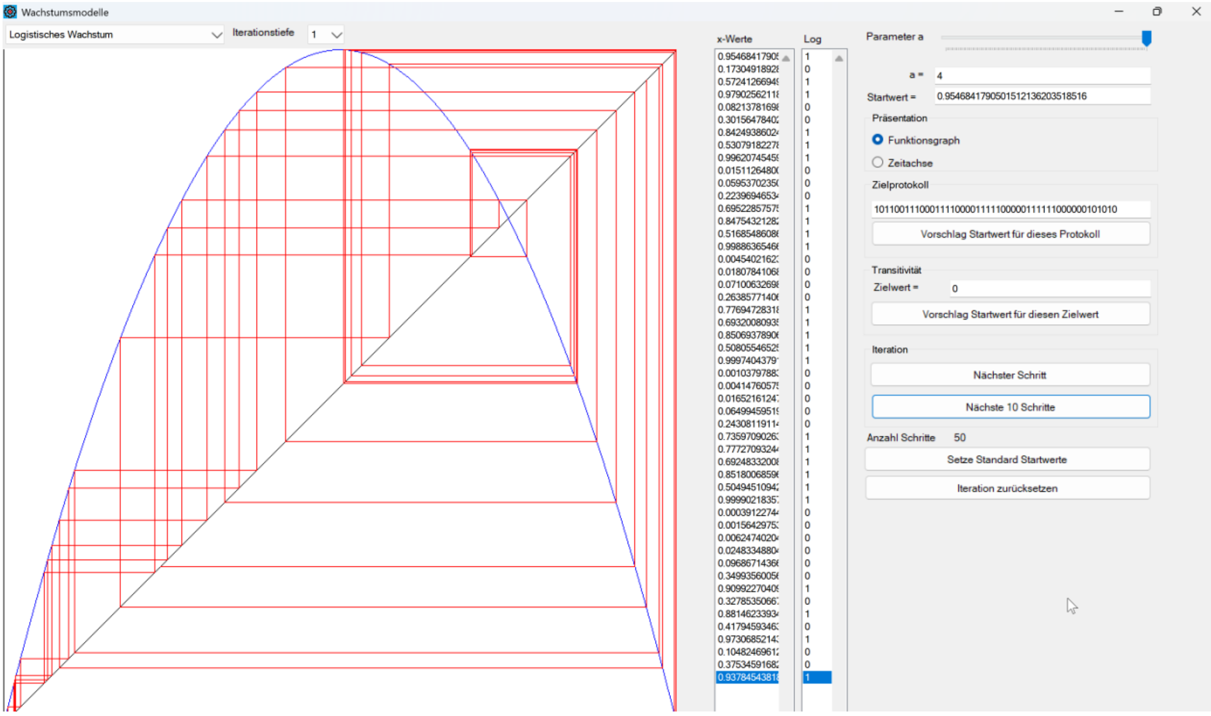
Log of the starting point $x_1 = 0.314159$ in the "log" column

The simulator now has a further capability. We can specify any log and the simulator will then find a starting point that provides this log. In the following example, we select the sequence as the target log:

$$1011001110001111000011111000001111110000000101010$$

The simulator then suggests as start value: $x_1 = 0.95468417905015121362035185 16$

Result:




The start value provides the specified protocol

In other words: With logistic growth, *any protocol* is *possible* in the case of $a = 4$ if the starting point is chosen appropriately.

We will understand later how the simulator "does this".

## 7. Cycles in the case a=4

We again consider the logistic growth $f$ with $a = 4$ . In Theorem 9, we saw that the iterated $f^n$ has exactly $2^n$ fixed points. This means that the same number of points belong to one cycle. Since a point cannot belong to two different cycles at the same time, all these cycles are *disjoint*.

We also note that if $\xi$ belongs to a p-cycle, then $f^n(\xi) = \xi \Leftrightarrow n$ is a multiple of $p$.

If $n$ is a prime number, then there is only the possibility for the fixed points of $f^n$ that they are either a fixed point of $f$ (of which there are two) or that they belong to a "real" cycle of length $n$. How many such different n-cycles are there then? In total, $2^n - 2$ points belong to the real cycles and each of them has $n$ points. Thus $n$ must be a divisor of $2^n - 2$ and there are $(2^n - 2)/n$ different cycles of length $n$.

**Theorem 10**

Let $f$ be the logistic growth with $a = 4$ . Furthermore, let $n$ be a prime number.

Assertion:

1) $n$ is a divisor of $2^n - 2$
2) There are $(2^n - 2)/n$ different cycles of length $n$

□

Remark: The fact that $n$ is a divisor of $2^n - 2$ also follows from Fermat's little theorem.

*Example: Let*

Let be $n = 3$. Then $2^3 - 2 = 6$ is and therefore there are two different 3-cycles.

□

If $p \neq q$ are two prime numbers, then there are $(2^p - 2)/p$ cycles of length $p$ and, analogously, $(2^q - 2)/q$ cycles of length $q$. This provides $2^p - 2$ fixed points for the p-cycles and $2^q - 2$ fixed points for the q-cycles. Together with the two fixed points of$f$ this provides $2^p + 2^q - 2$ fixed points.

This leaves $2^n - 2^p - 2^q + 2$ fixed points for the "real" n-cycles. Other cycles are not possible, because if $f^n(\xi) = \xi$ and $\xi$ were part of a k-cycle with $k < n$ , then $n$ would have to be a multiple of $k$. The following applies

**Theorem 11**

Let $f$ be the logistic growth with $a = 4$ . Let $p \neq q$ be prime numbers and $n = pq$.

Assertion:

1) $n$ is a divisor of $2^n - 2^p - 2^q + 2$
2) There are exactly $(2^n - 2^p - 2^q + 2)/n$ cycles of length $n$

□

*Example:*

$p = 3, q = 5, n = 15$.

$2^{15} - 2^3 - 2^5 + 2 = 32730$ and there are 2182 different cycles of length 15.

Furthermore, there are 2 different cycles of length 3 and 6 different cycles of length 5. Together with the two fixed points of $f$, this makes $15 \cdot 2182 + 6 + 30 + 2 = 32768 = 2^{15}$ fixed points of $f^{15}$.

□

*We will be able to determine all cycles explicitly in section 11*.

## 8. Chaotic dynamic systems

First, based on previous experience, we want to formulate in more general terms what is meant by a dynamic system. We will restrict ourselves to the discrete case. In a discrete dynamical system, we have a state space $X$ in which each point uniquely characterises the state of the system. In the case of logistic growth, $X = [0,1]$.

The dynamic system then obeys a *law of motion*. This is very important because one could assume that chaotic behaviour - however one defines it - arises because a law of motion is missing or only approximately describes the system. But this is not the case! Even chaotic systems obey a precise law of motion!

This law is generally given by a function $f: X \to X$. The system transitions from one state to the next via $f$. In the case of logistic growth, we had $f(x) = ax(1 - x)$ and $a \in ]0,4]$.

In a discrete system, this transition does not occur continuously, but step by step. You therefore count the step number in an index set $I$. Mostly (and with logistic growth), this index set is $I = \mathbb{N}$.

A dynamic system is therefore a triple $(X, f, I)$.

In the case of logistic growth, we have established in $a = 4$ that the system behaves in a certain way chaotically or pseudo-randomly. Let us make this more precise.

Let us assume that we have two intervals $U, V \subset X, U \cap V = \emptyset$ and that there are trajectories $(x_i)$ that run entirely in $U \cup V$. For such a trajectory, a log is created as follows:

$$p(x_i) = \begin{cases} 0, x_i \in U \\ 1, x_i \in \mathrm{V} \end{cases}$$

Logistic growth was $U = [0,0.5[, V = [0.5,1]$. This gives us a first simple definition that specifies chaotic behaviour.

**Definition 12**

A dynamic system is said to be *chaotic in the sense of the coin toss* if there are two sets U and V as described above, so that every 0-1 sequence occurs as a log of a trajectory if the starting value is chosen accordingly.

□

There are various other definitions, some of which are equivalent. Probably the most widely used definition dates back to 1989 and goes back to the mathematician *Robert Devaney* (1948 -..).

**Definition 13**

A dynamic system is said to be *chaotic in the sense of Devaney* if it fulfils the following properties:

1) It is *sensitive* to the initial conditions. This means that two starting values $x_1, x_2 \in X$ that are arbitrarily close to each other move arbitrarily far apart during iteration in the state space *X.*

2) It is *transitive*. This means that if an arbitrary start value $x_0 \in X$ and an arbitrary target value $z \in X$ are given, then there is an alternative start value in an arbitrarily small neighbourhood of $x_0$ that comes arbitrarily close to the target value *z* during the iteration.

3) The periodic orbits lie *densely* in the state space and are all repulsive.

□

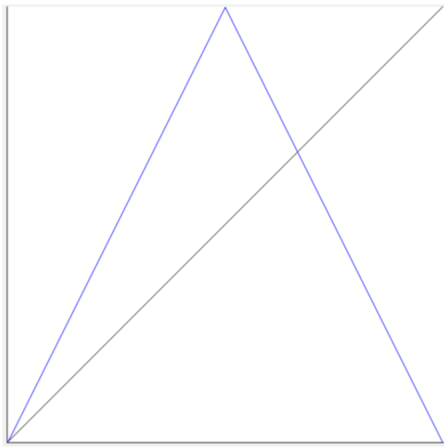
A connection between *"chaotic in the sense of the coin toss"* and the definition in the sense of Devaney is established in [2].

In the following, we consider an example of a dynamical system for which the chaotic properties are easy to prove.

## 9. The tent mapping

We consider the interval [0,1] and define the tent mapping as:

$$z: [0,1] \longrightarrow [0,1];\ z(u) = \begin{cases} 2u, u \in [0,0.5[ \\ 2(1-u), u \in [0.5,1] \end{cases}$$



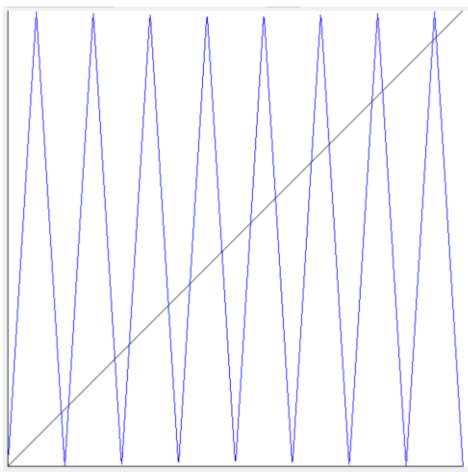
Graph of the tent mapping over the interval [0,1] created by the "simulator"

The tent mapping intersects the 45° straight line at the points 0 and 2/3. It therefore has the fixed points

$$\xi_1 = 0\ und\ \xi_2 = 2/3$$

We have $|f'(\xi_1)| = |f'(\xi_2)| = 2 > 1$ and both points are repulsive.

If you iterate the tent mapping, the unit interval is stretched and folded at each iteration. With four iterations, for example, you have the graph:



Graph of the quadruple iterated tent mapping $f^4: [0,1] \longrightarrow [0,1]$

The function $f^4$ shown above has a total of 8 fixed points (points of intersection with the 45° straight line) and all of them are repulsive (the derivative at all fixed points is > 1).

Now we analyse the chaotic properties of the tent mapping.

**Lemma 14**

Assertion: The tent mapping is sensitive.

□

Proof: If we represent a number $x \in [0,1]$ as a dual fraction, we can see the effect of the tent mapping. It is true in this representation and if $s_i \in \{0,1\}$ is the i-th dual digit after the decimal point:

$$f\colon \begin{cases} 0{,}0s_2s_3s_4 \ldots \longmapsto 0, s_2s_3s_4 \ldots \\ 0{,}1s_2s_3s_4 \ldots \longmapsto 0, \overline{s_2}\overline{s_3}\overline{s_4} \ldots \end{cases}$$

Where $\overline{s_i}$ denote the dual complement of $s_i$.

Now let an initial value be given in dual representation:

$$x_1 = 0.\, s_1s_2s_3s_4 \ldots.$$

We then truncate this dual fraction after a sufficiently large number of digits (or add "0" to missing digits) so that the initial value thus modified is close enough to the original. If it contains an odd number "1", we select $x'_1 = 0.\, s_1s_2s_3s_4..00\ldots s_1s_2s_3s_4 \ldots$ . This means that we then append the digits from $x_1$ to the end again. With each iteration step, these digits move one position to the left, whereby the dual complement of the remaining digits is formed for each "1" occurring in the first position after the decimal point. As this occurs an odd number of times, the number will be there after the corresponding number of iteration steps: $x_n = 0.\, \bar{s}_1\bar{s}_2\bar{s}_3 \ldots$ , i.e. a number with the maximum distance to $x_1$.

We proceed in the same way if the truncated dual fraction of$x_1$ , possibly supplemented with "0", contains an even number "1", and set$x'_1 = 0.\, s_1s_2s_3s_4..00\ldots \bar{s}_1\bar{s}_2\bar{s}_3\bar{s}_4 \ldots$ with the same result.

□

**Lemma 15**

Assertion: The tent mapping is transitive.

□

Proof:

As in the proof of sensitivity, we assume an initial value $x_1 = 0.\, s_1s_2s_3s_4 \ldots.$ and a target value $z_1 = 0.\, t_1t_2t_3 \ldots$ , both in dual fraction representation. We cut off the dual fraction of $x_1$ again after a sufficient number of digits (or add "0" to missing digits) so that the modified start value is close enough to the original. If the remaining dual fraction contains an even number "1", we append the digits of the target value to the end. If the number "1" is odd, then we append the dual complement to the end. In both cases, after a sufficient number of iteration steps, we obtain $z_n = 0.\, t_1t_2t_3 \ldots$

□

It is easy to see that the cycles are represented by periodic dual fractions and that these lie close together in [0,1]. The following therefore applies:

**Theorem 16**

Assertion: The tent mapping is chaotic in the sense of Devaney.

□

For the tent mapping we define a protocol *p* according to the rule:

$$p(x_n) = \begin{cases} 0, x_n \in [0,0.5[ \\ 1, x_n \in [0.5,1] \end{cases}$$

For each given protocol, the associated initial value that provides this protocol can be constructed: Take the given protocol as the initial value in dual representation and iterate it step by step. If the resulting protocol matches the specified one, this bit is left in the start value. Otherwise, it is replaced by its complement. Thus applies:

**Theorem 17**

Assertion: The tent mapping is chaotic in the sense of the coin toss.

□

## 10. Conjugates of the tent mapping

We have seen in the previous section that the tent mapping is chaotic both in the sense of the coin toss and in the sense of Devaney.

Now we consider the following transformation;

$$T: [0,1] \to [0,1], u \in [0,1] \longmapsto x = T(u) = sin^2 \frac{\pi}{2} u \in [0,1]$$

*T* is bijective, continuously differentiable and $T^{-1}$ is also continuously differentiable.

**Theorem 18**

Let $f$ be the logistic growth with $a = 4$ and $z$ the tent mapping. $T$ be the above transformation.

Assertion: The functions $f$ and $z$ are *conjugated*. This means that the following applies to $x \in [0,1]$ : . $f \equiv T^\circ z^\circ T^{-1}$

□

Proof: For $x \in [0,1]$ is also $u \in [0,1]$ . It holds:

$$f(T(u)) = f\left(sin^2 \frac{\pi}{2} u\right) = 4sin^2 \frac{\pi}{2} u \left(1 - sin^2 \frac{\pi}{2} u\right) = (2sin \frac{\pi}{2} u \cdot cos \frac{\pi}{2} u)^2 = sin^2 \frac{\pi}{2} \cdot 2u$$
$$= \begin{cases} sin^2 \frac{\pi}{2} \cdot 2u, u \in [0,0.5[ \\ sin^2 \frac{\pi}{2} \cdot 2(1-u), u \in [0.5,1] \end{cases} = \begin{cases} sin^2 \frac{\pi}{2} \cdot 2u \\ sin^2 (\pi - \frac{\pi}{2} \cdot 2u) \end{cases} = T(z(u))$$

Note: $\sin(\pi - \alpha) = sin\alpha$

□

A conjugate of the tent mapping adopts its properties regarding periodic and chaotic behaviour.

**Theorem 19**

Let $f$ be the logistic growth with $a = 4$ and $z$ the tent mapping. $T$ be the above transformation.

Assertion:

1) $\mathrm{f}^\mathrm{n} = \mathrm{T}^\circ \mathrm{z}^{\mathrm{n}\circ} \mathrm{T}^{-1}, \forall \mathrm{n} \in \mathbb{N}$. That is, $\mathrm{f}^\mathrm{n}$ is a conjugate of $\mathrm{z}^\mathrm{n}$
2) $\{\mathrm{u}_1, \mathrm{u}_2, \dots \mathrm{u}_\mathrm{k}\}$ is a k-cycle of z$\Leftrightarrow$ $\{\mathrm{T}(\mathrm{u}_1), \mathrm{T}(\mathrm{u}_2), \dots, \mathrm{T}(\mathrm{u}_\mathrm{k})\}$ is a k-cycle of f

3) The k-cycle of $z$ and the corresponding k-cycle of $f$ have the same multiplier

□

Proof:

1) $f^n = (T^\circ z^\circ T^{-1})^n = T^\circ z^\circ T^{-1\circ} T^\circ z^\circ T^{-1\circ} \ldots\ldots {}^\circ T^\circ z^\circ T^{-1} = T^\circ z^{n\circ} T^{-1}$

2) $u \in [0, 1]$ is a fixed point of $z^k$ $\Leftrightarrow T(u)$ is a fixed point of $f^k$:

$$" \Rightarrow ": \; z^k(u) = u \; \Rightarrow f^k(T(u)) = (T^\circ z^{k\circ} T^{-1})(T(u)) = T\left(z^k(u)\right) = T(u)$$

$$" \Leftarrow ": T\left(z^k(u)\right) = T(u) \Rightarrow z^k(u) = u$$

3) Let $u$ be a fixed point of $z^k$ for $u \in ]0,1[$ and $\xi = T(u)$, thus a cycle of z (the case u = 0 and u = 1 is trivial). Therefore $z^k(u) = u$ applies. Then $\xi$ is a fixed point of $f^k$: $f^k(\xi) = \xi$.

Note also: $T(T^{-1}(\xi)) = \xi$ , i.e. $[T(T^{-1}(\xi))]' = T'(u) \cdot T^{-1\prime}(\xi) = 1, u \in ]0, 1[$ according to the theorem on the derivative of the inverse mapping.

Then applies to the multipliers according to the chain rule:

$$\lambda_\xi = |f^{k\prime}(\xi)| = \left|(T^\circ z^{k\circ} T^{-1})'(\xi)\right| = |T'(z^k(T^{-1}(\xi))) \cdot z^{k\prime}(T^{-1}(\xi)) \cdot T^{-1\prime}(\xi)| =$$

$$\left|T'\left(z^k(u)\right) \cdot z^{k'}(u) \cdot T^{-1'}(\xi)\right| = |T'(u) \cdot z^{k'}(u) \cdot T^{-1'}(\xi)| = |z^{k'}(u)| = \lambda_u$$

□

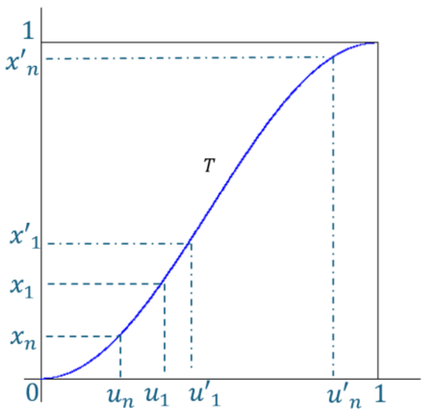
**Theorem 20**

Assertion: The logistic growth for $a = 4$ is chaotic in the sense of Devaney.

□

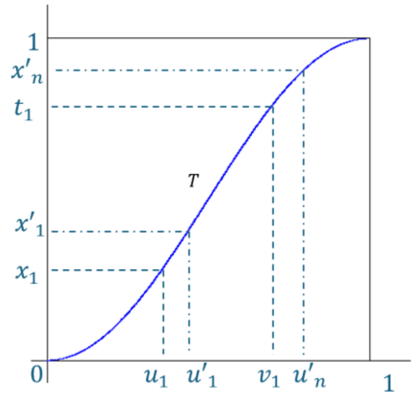
Proof:

Sensitivity: We consider the following image. The y-axis shows the iteration values of the logistic growth, and the x-axis shows the corresponding values of the tent mapping. $T$ is always the transformation that we defined at the beginning of the section.




The start value $x_1$ corresponds to a start value $u_1$. Then there is a start value $u_1{}'$ for the tent mapping, so that after n iteration steps the distance between $u_n$ and $u_n{}'$ becomes arbitrarily large. Then the distance between $x_n$ and$x_n{}'$ also becomes arbitrarily large.

Transitivity: We look at the image:




A start value $x_1$ and a target value $t_1$ are specified for the logistic growth. The corresponding values for the tent mapping are $u_1$ and $v_1$. Then, due to the transitivity of the tent mapping, there is a slightly modified start value $u_1{}'$, so that after n iteration steps $u_n{}'$ comes arbitrarily close to the target value $v_1$. The corresponding start value for logistic growth is $x_1{}'$ and $x_n{}'$ comes as close as possible to $t_1$.

Furthermore, it is easy to see from the images that - if the periodic points in the tent mapping lie close in the interval $[0,1]$ - the periodic points in logistic growth also lie close in this interval. Because of Theorem 16, 3), the cycles of logistic growth are also all repulsive.

□

**Theorem 21**

Assertion: Logistic growth is chaotic for $a = 4$ in the sense of the coin toss.

□

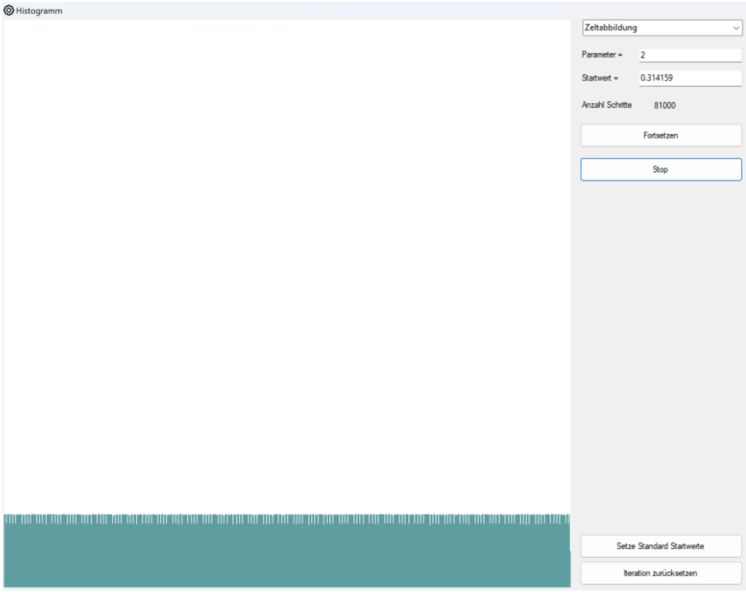
Proof: It is easy to check that the log of the tent mapping corresponds to the log of the logistic mapping. However, the protocol for the tent mapping can be specified arbitrarily. The transformation $T$ then provides the corresponding starting point for the logistic growth, which provides the specified protocol.

□

The simulator "works" exactly according to the methods used in the proof of Theorems 16 - 18 to find starting points for given logs or to prove the sensitivity and transitivity of logistic growth. A detailed description can be found in the mathematical documentation of the simulator.

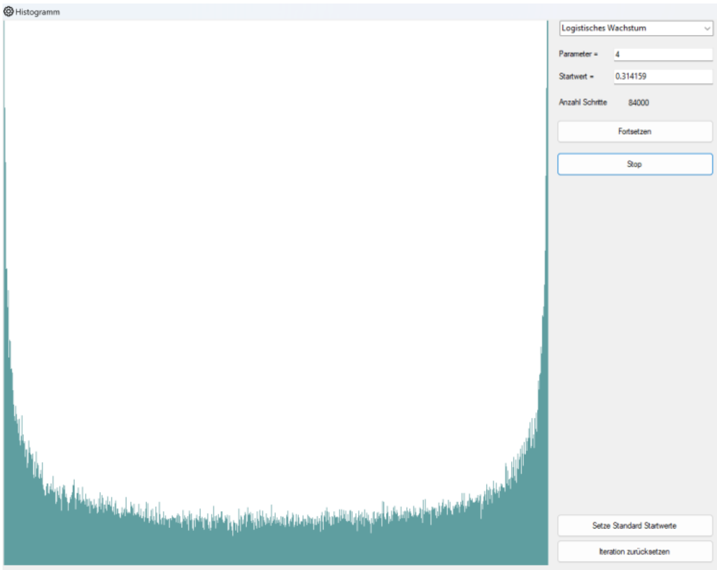
Finally, let's discuss the shape of the histogram we encountered in logistic growth for $a = 4$ .

Due to the dual representation of the numbers in the tent illustration, the distribution of the iteration values in a histogram is equally distributed. This is also confirmed by an experiment with the simulator.




Distribution of the intervals taken in the tent mapping after 81,000 steps

We found the following picture for the logistic mapping:




Histogram of logistic growth after 84,000 steps

Based on the uniformly distributed tent mapping, this distribution is an effect of the transformation to logistic growth given by: $x = T(u) = sin^2 \frac{\pi}{2} u$

If $p(x)$ is the probability distribution for logistic growth, then the probability that an iteration value falls into an interval $\Delta x$ is approximately $p(x)\Delta x$. However, this is equal to the probability that $u = T^{-1}(x)$ falls into the corresponding interval $\Delta u$. However, this probability is just $\Delta u$, because we have a uniform distribution in the tent mapping on the interval [0,1]. It is therefore $p(x)\Delta x \approx \Delta u$. The following applies in the limit:

$$p(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = {T^{-1}}'(u) = \frac{1}{T'(u)} = \frac{2}{\pi \sin(\pi u)}$$

With $u = \frac{2}{\pi} arcsin\sqrt{x}$ we get $p(x) = \frac{2}{\pi \sin(2 \arcsin\sqrt{x})} = \frac{1}{\pi\sqrt{x}\sqrt{1-x}}$.

This explains the symmetry of the histogram of logistic growth with respect to $x = 0.5$ or

$u = 0.5$ and also, the shape of the distribution at the edge of the iteration interval [0,1].

## 11. Explicit calculation of cycles

*First approach*: In the tent mapping, cycles can be determined by experimenting in the dual system. If you know these, you can use conjugation to arrive at the cycles of logistic growth for $a = 4$ .

For now, however, we use the following

**Theorem 22**

Assertion: The tent mapping has a true cycle of length n for each $\mathrm{n} \in \mathbb{N}$ .

□

Proof: Consider the initial value $x_1 = \frac{2}{2^n+1}$

This is multiplied by 2 at each step as long as the result is < 0.5. This is the case until the value

$$x_{n-1} = \frac{2^{n-1}}{2^n + 1}$$

How to easily check is still $x_{n-1} < {}^1/_2$ . In the next step, you get:

$$x_n = \frac{2^n}{2^n + 1} > \frac{1}{2}$$

Therefore, according to the iteration rule

$$x_{n+1} = 2(1 - x_n) = 2\frac{1}{2^n + 1} = x_1$$

All occurring fractions cannot be shortened, so the cycle effectively has the length n.

□

**Theorem 23**

Assertion: For all $\mathrm{n} \in \mathbb{N}, sin^2 \frac{\pi}{2^n+1}$ is an n-cycle of logistic growth with $a = 4$.

□

Proof: $\frac{2}{2^n+1}$ is an n-cycle of the tent mapping. Then, according to Theorem 19, the conjugation $sin^2 \frac{\pi}{2} \cdot \frac{2}{2^n+1}$ is an n-cycle of logistic growth.

□

In the following examples, we analyse the logistic growth with $a = 4$.

*Example*: $n = 2$

According to Theorem 19, there is exactly one 2-cycle for logistic growth with $a = 4$. In section three, we explicitly calculated the 2-cycles as:

$$\xi_{3,4} = \frac{1 + a \pm \sqrt{(1+a)(a-3)}}{2a} = \frac{5 \pm \sqrt{5}}{8} = \begin{cases} sin^2 \dfrac{\pi}{5} \\ sin^2 \dfrac{2\pi}{5} \end{cases}$$

*Example*: $n = 3$

According to Theorem 19, there are two different 3-cycles for logistic growth with $a = 4$. We know one of them by Theorem 23, but in this example, we want to find both by examining the tent mapping. Without this help, we would have to look for the zeros of the polynomial $f^3(x) - x$ , i.e. solve an equation of degree 8.

If we experiment with the tent mapping in the dual representation, we find the following 3-cycles:

$$0.110110 \ldots \rightarrow 0.0100100 \ldots \rightarrow 0.100100.. \rightarrow 0.11001100..$$

And

$$0.111000111000.. \rightarrow 0.00111000111.. \rightarrow 0.0111000 \ldots \rightarrow 0.111000 \ldots$$

The first initial value is identical to (we use the geometric series):

$$1 - \frac{1}{8} - \frac{1}{64} - \frac{1}{512} - \cdots = 1 - \frac{1}{8} \cdot \frac{1}{1 - {}^1\!/_8} = \frac{6}{7}$$

Control: In the tent mapping, we get it sequence: $\frac{6}{7} \rightarrow \frac{2}{7} \rightarrow \frac{4}{7} \rightarrow \frac{6}{7}$

Thus, according to Theorem 19, $sin^2 \frac{3\pi}{7} \approx 0.950484434 \ldots$ belongs to a 3-cycle of logistic growth for $a = 4$.

Control: We obtain the sequence in the simulator:

$$0.950484434 \rightarrow 0.18825509889 \rightarrow 0.61126046653 \rightarrow 0.950484434$$

Since the cycle is repulsive, we will move out of the cycle as we iterate further. For example, the value 0.989293719 is obtained after 30 iteration steps.

Now let's look at the second 3-cycle. It is:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + .. = \frac{7}{8} + \frac{1}{64}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = \frac{7}{8}(1 + \frac{1}{64} + \frac{1}{64^2} + \cdots)$$
$$= \frac{7}{8}\left(\frac{1}{1 - {}^1\!/_{64}}\right) = \frac{8}{9}$$

Check: In the tent mapping, this start value provides the sequence:

$$\frac{8}{9} \rightarrow \frac{2}{9} \rightarrow \frac{4}{9} \rightarrow \frac{8}{9}$$

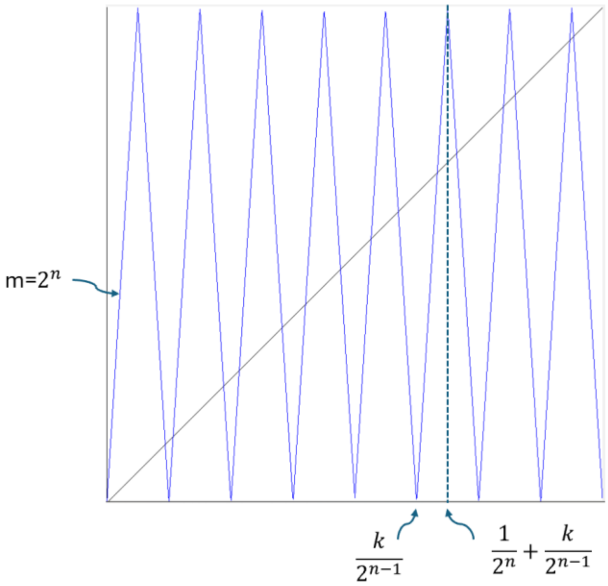
Thus, according to Theorem 19, $sin^2 \frac{4\pi}{9} \approx 0.9698463104 \ldots$ belongs to a 3-cycle of logistic growth for $a = 4$.

Control: We obtain the sequence in the simulator:

$$0.9698463104 \rightarrow 0.1169777784 \rightarrow 0.41317591108 \rightarrow 0.9698463103$$

For larger values of n, this procedure becomes increasingly cumbersome. We therefore want to try another approach to determine all cycles of logistic growth with $a = 4$.

*Second approach*: We consider the tent mapping and the nth iteration. Its graph looks like this (the image shows: $n = 4$)




Graph of the iterated tent mapping

The tangent slope of the individual line segments is $\pm 2^n$. The straight lines with a positive slope pass through the points $\frac{k}{2^{n-1}}$ on the x-axis for $k = 0, \dots, 2^{n-1} - 1$ . The straight lines with a positive slope therefore have the equation:

$$y = 2^n x - 2k$$

Their intersection points with the 45° straight line are $\xi = \frac{2k}{2^n - 1}, k = 0, \dots, 2^{n-1} - 1$.

The straight lines with a negative slope pass through the points $\frac{k+1}{2^{n-1}}$ on the x-axis, k = 0, ...,$2^{n-1} - 1$

You therefore have the equation:

$$y = -2^n x + 2(k + 1)$$

Their intersection points with the 45° line are $\xi = \frac{2(k+1)}{2^n + 1}, k = 0, \dots, 2^{n-1} - 1$.

However, all these intersection points provide all the points that belong to a cycle of the marquee mapping. Due to Theorem 19, we thus obtain all cycles of logistic growth without having to calculate the zeros of polynomials of degree $2^n$. The following applies:

**Theorem 24 (about the cycles of logistic growth and tent mapping)**

Assertion:

1) All cycles of the tent mapping have the form $\xi = \frac{2k}{2^n - 1}$ or $\xi = \frac{2(k+1)}{2^n + 1}$ with $k = 0, \dots, 2^{n-1} - 1$
2) All cycles of logistic growth for a=4 have the form

$$\xi = sin^2 \frac{\pi k}{2^n - 1} \; or \; \xi = sin^2 \frac{\pi (k + 1)}{2^n + 1}$$

with $k = 0, \ldots, 2^{n-1} - 1$

□

*Example*: $n = 4$

Let us consider the cycles of the tent diagram. The corresponding cycles for logistic growth are obtained by conjugation.

There are two fixed points: $\frac{0}{15}$ and $\frac{10}{15}$

There is a 2-cycle: $\frac{12}{15} \leftrightarrow \frac{6}{12}$

This leaves 12 points that belong to a 4-cycle. The corresponding three 4-cycles are:

$$\frac{8}{15} \longrightarrow \frac{14}{15} \longrightarrow \frac{2}{15} \longrightarrow \frac{4}{15}$$
$$\frac{16}{17} \longrightarrow \frac{2}{17} \longrightarrow \frac{4}{17} \longrightarrow \frac{8}{17}$$
$$\frac{14}{17} \longrightarrow \frac{6}{17} \longrightarrow \frac{12}{17} \longrightarrow \frac{10}{17}$$

Every point that belongs to a cycle has exactly one preperiodic predecessor:

$$\frac{5}{15} \to \frac{10}{15}; \frac{3}{15} \to \frac{6}{15}; \frac{9}{15} \to \frac{12}{15}; \frac{1}{15} \to \frac{2}{15}; \frac{13}{15} \to \frac{4}{15}; \frac{11}{15} \to \frac{8}{15}; \frac{7}{15} \to \frac{14}{15}$$
$$\frac{1}{17} \to \frac{2}{17}; \frac{15}{17} \to \frac{4}{17}; \frac{13}{17} \to \frac{8}{17}; \frac{9}{17} \to \frac{16}{17}; \frac{3}{17} \to \frac{6}{17}; \frac{11}{17} \to \frac{12}{17}; \frac{5}{17} \to \frac{10}{17}; \frac{7}{17} \to \frac{14}{17}$$

It can be shown that this is generally true.

□

It is up to the reader to calculate some cases or test them with the simulator.

## 12. Outlook: Period doubling

We have seen that the logistic growth for parameter values $a < 1$ has the attractive fixed point $\xi_1 = 0$. For $a \in ]1, 3[$ this fixed point becomes repulsive, but it is replaced by the attractive fixed point $\xi_2 = 1 - {}^1/_a$ . This is attractive for $a \in ]1, 3[$ . At the point $a = 3$ this fixed point becomes repulsive, but an attractive 2-cycle is created

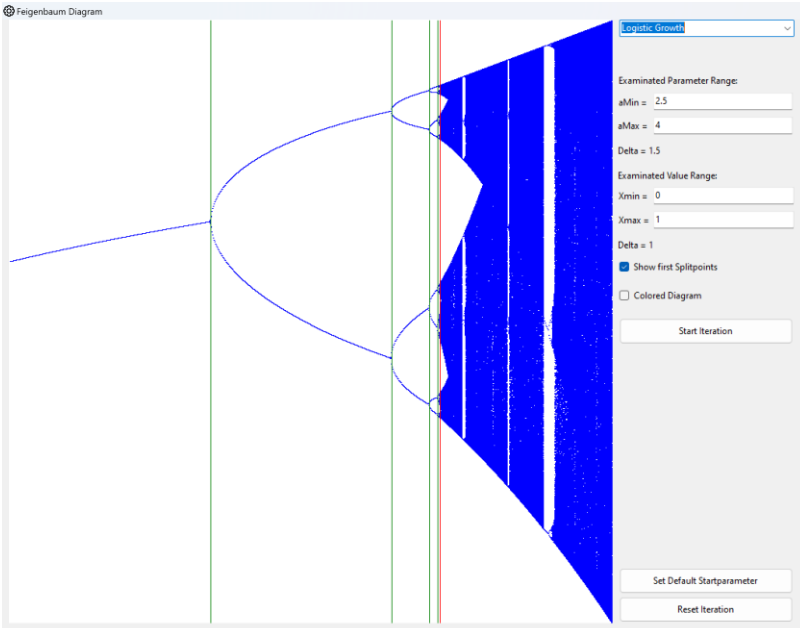
$$\xi_{3,4} = \begin{cases} 0.799455 \ldots \\ 0.513044 \ldots \end{cases}$$

This is attractive in the area $a \in ]3, 1 + \sqrt{6}[$ and becomes repulsive at the point $a = 1 + \sqrt{6}$ . The computer experiment then shows that an attractive 4-cycle is created here. For increasing *a,* an 8-cycle then arises, then a 16-cycle and the period of the cycles doubles continuously up to *a* certain limit value of *a* and then changes to chaotic behaviour.

The mathematician *Mitchell Feigenbaum* (1944 - 2019) investigated the phenomenon of period doubling and discovered the so-called Feigenbaum constant in 1975:

$$\delta \approx 4.669202 \ldots$$

Period doubling can be represented in the Feigenbaum diagram.



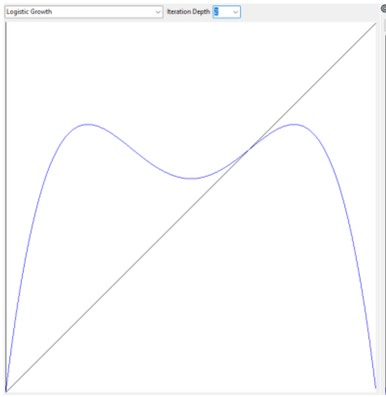

Feigenbaum diagram for logistic growth

The values of the parameter *a* are shown on the horizontal axis. The interval in which *a* move can be freely selected within ]0,4]. The figure above shows a section for $a \in [2.5, 4]$ . For each value of *a*, it is then iterated for so long that one can hope that the iteration either settles on one cycle or is chaotic. Further iteration steps are then carried out and the x-values of the iteration are entered vertically in the diagram.

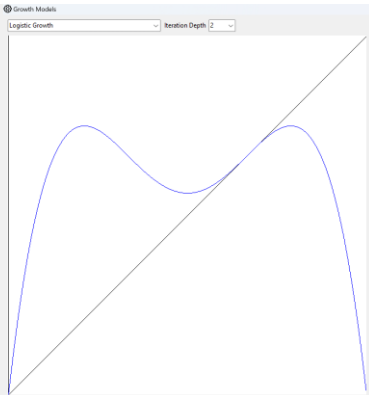
The green lines show the split points at which an attractive cycle becomes unstable, and a new cycle is created. The first split point is for $a_1 = 3$. An attractive 2-cycle is created there. At $a_2 = 3.449499 \ldots$ the period doubles and an attractive 4-cycle is created at the corresponding two split points. An attractive 8-cycle is created at $a_3 = 3.544090 \ldots$ with four split points. In the further course, the period doubles at ever shorter intervals and the sequence of $a_i$, i.e. the points at which further split points occur, tends towards the limit value $a_\infty = 3.569946 \ldots$ . This point is marked in the diagram with a red line.

The split points each belong to a $2^n$ cycle, which becomes attractive at this point. As all points belonging to a cycle have the same multiplier, as already stated earlier, they become attractive at the same time, i.e. they all lie on a straight line parallel to the vertical axis.

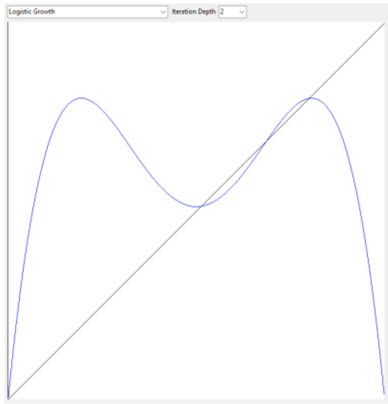
What happens, for example, when the fixed point $\xi_2 = 1 - {}^{1}/_{a}$ transitions into an attractive 2-cycle at the point *a=3* can be seen in the graph of $f^2$ at this point (generated by the simulator):



a = 2.9



a = 3



a = 3.2

The fixed point $\xi_2$ is the intersection of the graph with the 45° straight line. On the left, the curve has a tangent with a slope < 1, so the fixed point is attractive. For *a = 3*, the tangent slope is exactly 1 (centre image). If the curve becomes steeper for *a = 3.2*, the fixed point $\xi_2$ has become repulsive with a tangent slope > 1. Two new intersection points have been created with a tangent slope < 1. This is the newly created 2-cycle.

The first associated values of *a*, at which the respective cycles become unstable and split into a cycle of the double period, can be determined using numerical methods and are:

$$a_1 = 3, a_2 = 3.449499 \ldots, a_3 = 3.544090 \ldots, a_4 = 3.564407 \ldots,$$

$$a_5 = 3.568759 \ldots, a_6 = 3.569692 \ldots, a_7 = 3.569891 \ldots, a_8 = 3.569934 \ldots$$

The values of $a_k, k \in \mathbb{N}$ decrease geometrically and tend towards a limit value $a_\infty$ according to:

$$a_k \approx a_\infty - c \cdot \delta^{-k}, k \in \mathbb{N}$$

$\delta \approx 4.669202 \ldots$ is the so-called Feigenbaum constant. The following applies to logistic growth:

$$c \approx 2.6327 \ldots$$

For example:

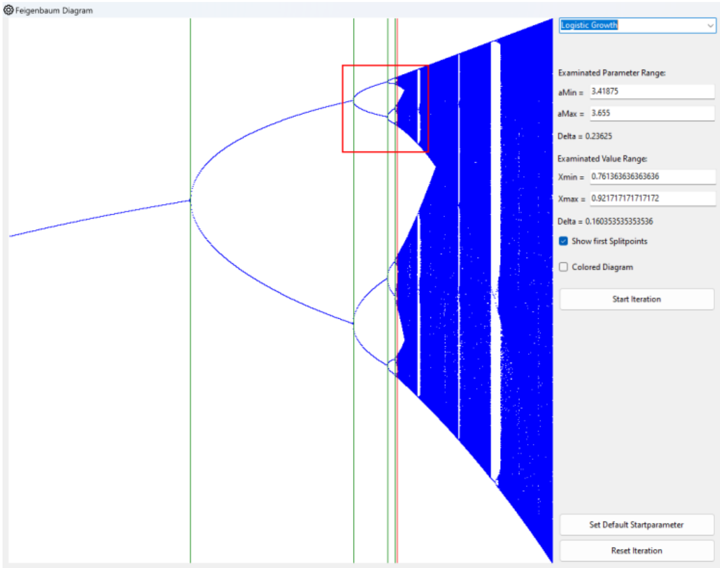
$$a_\infty - c \cdot \delta^{-5} = 3.568759 \approx a_5$$

An n-cycle fulfils the condition $f^n(\xi_i) = \xi_i,\ 1 \leq i < n$ . In logistic growth, these points are zeros of the polynomial $f^n(x) - x = 0$ , which has the degree $2^n$. Is it possible that some of these zeros are complex and the others are real? The answer is no. Suppose you know one real zero $\xi_1$. Then the other zeros result from the iteration $\xi_{i+1} = f(\xi_i), 1 \leq i < n$ and $\xi_1 = f(\xi_n)$ . This means that all other zeros are also real.

Feigenbaum discovered that the behaviour of period doubling is a universal phenomenon and occurs in many dynamic systems during the transition to chaos. The constant $\delta$ is always the same and appears to be universal.

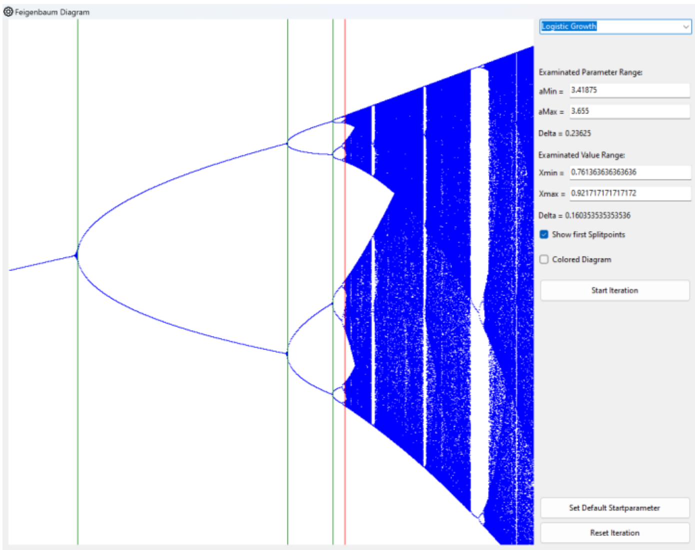
*Self-similarity*

Feigenbaum used the phenomenon of self-similarity of a quadratic function and its iterates in his investigation. A good description of this approach can be found in [1] and an elementary approach in [4].



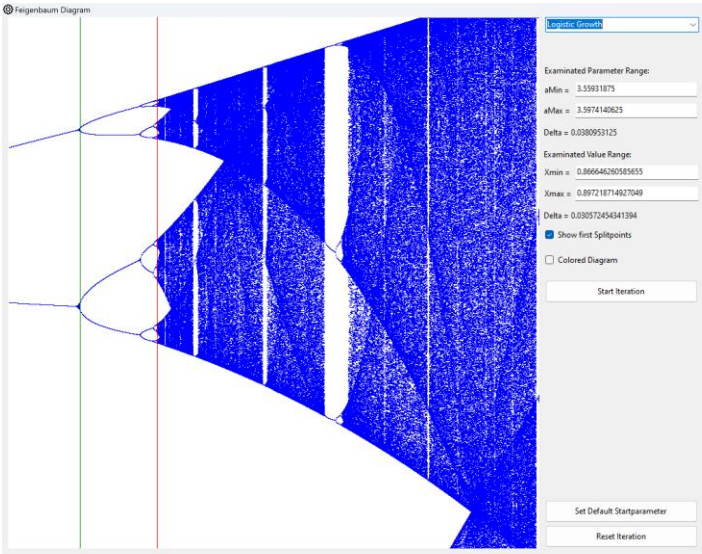

Self-similarity with period doubling

The simulator makes it possible to view the section marked with a red rectangle in a correspondingly scaled iteration. The following image is obtained for the section above:



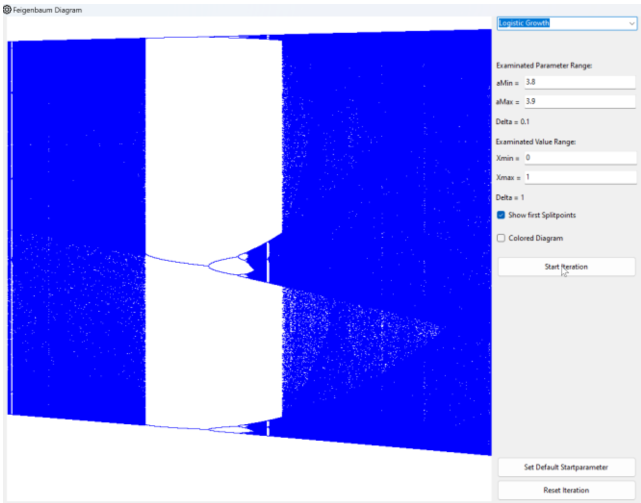

Red marked section in enlarged form

If *a* exceeds the value $a_\infty \approx 3.569946$ , the system becomes chaotic.




Transition to chaos - the red line marks the approximate value $a_\infty \approx 3.569946$

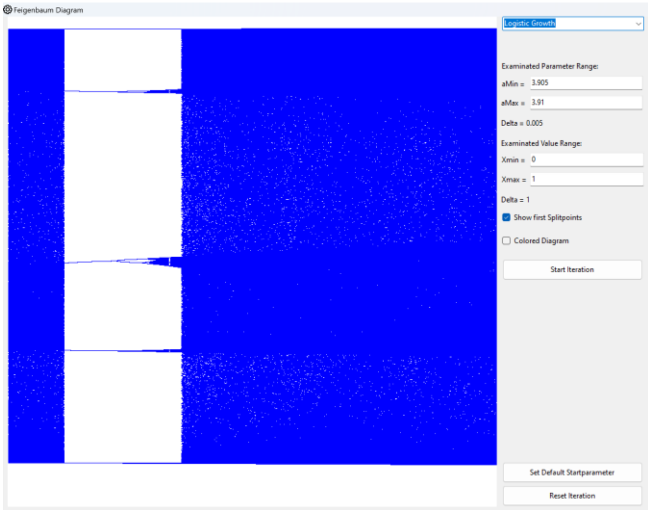
If *a* continues to increase, chaotic areas alternate with ever smaller windows containing attractive cycles. In the range $a \in [3.828427, 3.841499]$, for example, an attractive 3-cycle appears, which then becomes unstable at the right interval boundary. At this point, another cascade of period doublings takes place: The 3-cycle becomes a 6-cycle, then a 12-cycle and so on. Chaos is definitely reached at $a = 4$ .




Attractive 3-cycle in the range $a \in [3.8,3.9]$

In the image above, you can see how *a* stable 3-cycle appears out of nowhere at the point *a=3.82* after a chaotic area, which then changes into period doubling and later ends up in chaos again. The self-similarity can be seen slightly to the right of the period doubling: Windows with stable 9-cycles also appear here, similar to the 3-cycle.

Windows with stable cycles also appear in the $a \in [3.905,3.91]$ area, as the following image shows. For example, you can recognise a stable 5-cycle on the left-hand side of the diagram. If you look closely, you can faintly recognise that it changes to period doubling as soon as it becomes unstable.




Attractive 5-cycle in the range $a \in [3.905,3.91]$

## 13. Exercise examples

1. Examine the iteration

$$x_{n+1} = ax_n\sqrt{1 - x_n^2}, x_n \in [0,1]$$

   a) In which interval may the parameter $a$ lie?
   b) Determine fixed points and 2-cycles of the iteration
   c) For which values of $a$ are these attractive?
   d) Show that the iteration is conjugate to the tent mapping for *a* certain *a* (use trigonometric functions).

2. Investigate the sawtooth mapping (or the Bernoulli shift system):

$$f(x) = \begin{cases} 2x, x \in [0, 0.5[ \\ 2x - 1, x \in [0.5, 1] \end{cases}$$

   and the iteration defined by it.
   Using the dual fraction representation of x, show that it is chaotic both in the sense of the coin toss and in the sense of Devaney.

3. Investigate angle doubling on the unit circle regarding chaotic behaviour. Use the function $\varphi \to e^{i\varphi}$ in the complex plane.

4. Analyse the function:

$$(x) = ax^2(1 - x^2), [0,1] \to [0,1]$$

   a) In which interval may the parameter a lie?
   b) Determine fixed points and 2-cycles.
   c) For which a are these attractive?

5. Examine the tent mapping. Prove:
   Every rational number from the unit interval is periodic or pre-periodic (i.e. falls on a cycle after a

few iteration steps. The corresponding cycles are repulsive. They are close to the unit interval. (This task is somewhat more difficult. A solution can be found in the mathematical documentation for the simulator).

6. Examine the Feigenbaum diagram with the "Simulator". Select the red selection triangle so that the parameter differences of consecutive split points are hit. These differences are then displayed on the right. Use this to determine approximate values for the Feigenbaum constant.

7. The parabola is often analysed instead of logistic growth:

$$g(y) = 1 - \mu y^2, [-1,1] \longrightarrow [-1,1], \mu \in ]0,2]$$

Consider the transformation

$$x = T(y) := \frac{\mu}{a} y + \frac{1}{2}, [-1,1] \longrightarrow [0,1]$$

a) Let $f$ be the logistic growth with $a \in ]2,4]$ . Show that the following holds:

$$T^{-1} \circ f \circ T \equiv g$$

b) Show that the following holds for this transformation: $\mu = \frac{a(a-2)}{4}$

c) Show that this parabola for $\mu = 2$ is a conjugate of the tent mapping. Note: The conjugate is given by $T(u) = cos\pi((1-u)$.

8. Let $f$ be the logistic growth with $a = 4$ . Classify the cycles of $f^{21}$

9. Let $f$ be the logistic growth with $a = 4$ . Analyse the cycles in the case that $p$ is a prime number and $n = p^2$.

## Further literature

Listed here are a few sources where individual chapters are still within an elementary accessible framework. Based on these sources, you can find further literature references.


[1] Wolfgang Metzler: Nonlinear Dynamics and Chaos, Teubner Studienbücher 1998

[2] Urs Kirchgraber: Mathematics in Chaos, Mathematische Semesterberichte, Springer 1992

[3] Urs Kirchgraber: Chaotic behaviour in simple systems, Berichte über Mathematik und Unterricht, ETHZ, 1992

[4] Urs Kirchgraber, Niklaus Sigrist: Feigenbaum Universalität: Beschreibung und Beweisskizze, Berichte über Mathematik und Unterricht, ETHZ, 1995

[5] Pierre Collet, Jean-Pierre Eckmann: Iterated maps on the Interval as dynamical Systems, Birkhäuser 1986

[6] Moritz Adelmeyer: Theorem of Sarkovskii, Berichte über Mathematik und Unterricht, ETHZ, 1990

[16] An introduction to Chaotic Dynamical Systems, Robert L. Devaney, Addison Wesley, 1989