Numerical methods and coupled pendulums

The mathematics behind the computer programme Simulator

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This document deals with different versions of coupled pendulums. An elementary introduction to ordinary differential equations serves as preparation. The differential equations describing the coupled pendulums are created using the Lagrange method. If these cannot be solved analytically, some numerical methods are presented and compared. The mathematical investigations are supported by experiments with the computer programme Simulator, which enables the simulation of all the pendulums discussed.

The document is intended as a stimulus for extended maths lessons at intermediate level, whether for courses outside the compulsory curriculum or for individual work by interested pupils. It is particularly suitable as an introduction to the topic of ordinary differential equations and numerical methods for their approximate solution.

The entire series of topics related to the Simulator includes:

- *The chaotic properties of logistic growth*
- *The oval billiard table and periodic orbits*
- *Strange attractors and the weather forecast of Edward Lorenz*
- ***Numerical methods and coupled pendulums***
- *Planetary motion and the three-body problem*
- *Fractal sets and Lindenmayer systems*
- *Newton iteration and the complex roots of unity*
- *Iteration of quadratic functions in the complex plane*
- *The history of chaos theory*
- *Programming your own dynamic systems in the Simulator*

Each topic is dealt with in a separate document.

The computer programme Simulator enables the simulation of simple dynamic systems and experimentation with them. The code is publicly available on GitHub, as is a Microsoft Installer version. The corresponding link is: https://github.com/HermannBiner/Simulator. The following documentation is integrated into the Simulator in German and English:

- *Mathematical documentation with examples and exercises*
- *Technical documentation with a detailed description of the functionality*
- *User manual with examples*
- *Version history*

Contents

1. The cyclist

When we lift a heavy weight, we need a lot of strength to do so. To prevent this weight from falling again, we must maintain this force. When we ride a bicycle, we also must pedal hard, depending on the gradient, so that the bicycle picks up speed. However, if the road is absolutely flat, we can relax a little once the bike is moving. Even if we stop pedalling, the bike will keep going at an almost constant speed, especially if it is well oiled and the tyre pressure is optimal.

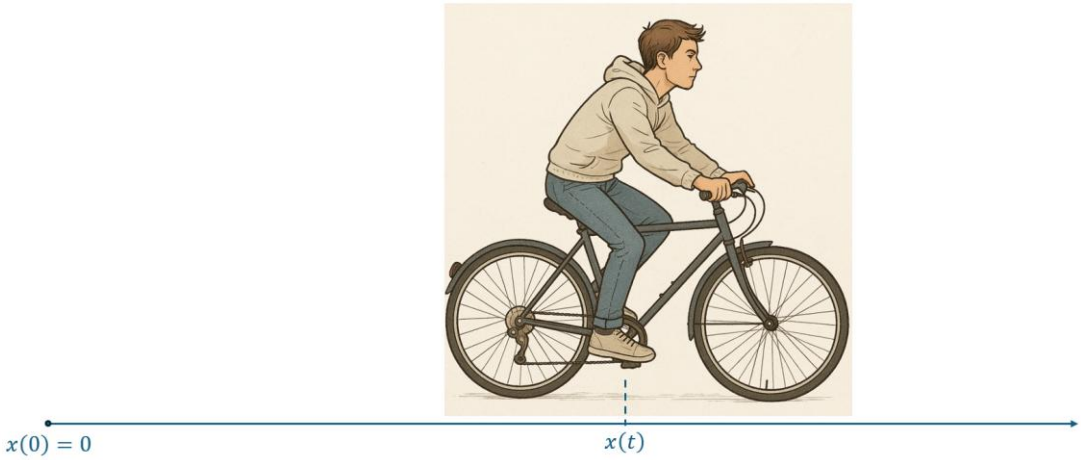
This was not so clear for a long time. The Greek philosopher Aristotle (384 - 322 BC) believed that a continuous force was necessary to keep a body in motion. This opinion persisted for centuries. Back then, if you wanted to move faster than on foot, the horse was probably the most common means of transport. And it is clear that to move the rider (and itself), the horse had to exert constant force. If it was travelling fast, it often had to be replaced after 20 miles. It was not until the Middle Ages that people began to think about whether movement was possible without continuous power.

The definitive answer to this question was provided by the English polymath and founder of mechanics, Sir Isaac Newton (1643 - 1727), without being a cyclist. In his first law of motion, he formulated the following:

- A body on which no force acts remain at rest or moves in a straight line at a constant speed.

This means *that no* continuous force is required to keep a body in motion. This also confirms that the bicycle - if it were frictionless - would continue to roll at a constant speed even if we stopped pedalling.

We look at the distance the cyclist is travelling. If he starts at the time$t = 0$ at the point$x = 0$, then we denote the distance travelled in the time$t$ by $x(t)$.



The cyclist

His speed will vary. It is equal to the distance travelled per unit of time, i.e. in the limiting case at the time t:

$$\lim_{h\to 0}\frac{x(t+h)-x(t)}{h} =: \dot{x}(t)$$

This is nothing other than the derivative of the function $x(t)$ with respect to the time t and, following Newton, we write the point for this in this paper: $\dot{x}(t)$.

If the speed remains constant without the application of force, what does the force do? When the cyclist starts pedalling hard again and thus exerting a force, *the speed changes*. The bike *accelerates*. The instantaneous acceleration at the time t is equal to the change in speed and in the limiting case:

$$\lim_{h\to 0}\frac{\dot{x}(t+h)-\dot{x}(t)}{h} =: \ddot{x}(t)$$

The acceleration is the second derivative of the function $x(t)$ with respect to the time t.

How are force and acceleration related? Newton provides the answer in his second law of motion:

- Force = mass x acceleration or: $\vec{F} = m\vec{a}$

The first thing to notice is that, in contrast to the previous law, the moving mass m appears in this law. For the bicycle to continue travelling at a constant speed, it does not matter whether the cyclist is sitting on the bicycle empty or loaded with a heavy rucksack. However, if he wants to accelerate the bike, he must exert more force with a heavy rucksack. This is not so clear, because whether with or without a rucksack, the bike always keeps the rider at a constant distance from the ground and exerts a counterforce on the rider that corresponds exactly to their total weight. This is Newton's third law, which can be summarised as follows:

- Force = counterforce

The greater weight of the rucksack, which pulls the rider to the ground, is therefore fully compensated for by the counterforce of the bike. Nevertheless, the rider with the rucksack needs more force to accelerate the bike. The mass is *inert* and counteracts the acceleration.

So, if the cyclist pushes the pedals with constant force and the total moving mass is m , this means that the rider's initial speed is zero:

$$\ddot{x} = \frac{F}{m} \Longrightarrow$$

$$\dot{x} = \frac{F}{m}t + C$$

If his initial speed was zero, $C = 0$. Therefore, in the time$t$ he covers the distance

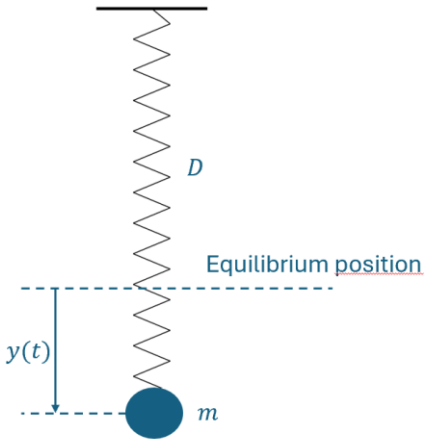
$$x = \frac{1}{2}\frac{F}{m}t^2$$

Since $x(0) = 0$. The question is how long the cyclist can keep this up. He must bear in mind that speeding is also a punishable offence on a bicycle!

The question now is: How do you know how great the force is? Without knowing it explicitly, you cannot reconstruct the acceleration and therefore the movement of a body. Among other things, Newton investigated planetary motion and found the formula for the gravitational force. This is dealt with in more detail in the school project *The planetary motion and the three-body problem.* In the following section, we will look at a simpler case.

2. The spring pendulum

Here we will examine the spring pendulum as we know it from physics.



Spring pendulum at the position $y(t)$ at time t

Gravity pulls the mass downwards; the spring force of the deflected spring pulls it upwards. According to Newton's third law of motion, the downward pull of gravity and the upward force of the spring cancel each other out at rest.

If the mass is deflected from its rest position by the distance $y(t)$, the (retracting) spring force increases in proportion to this deflection. This law was discovered by Robert Hooke (1635 - 1703) and can be easily demonstrated experimentally by hanging different weights on the spring and measuring how much it is deflected. The proportionality factor depends on the nature of the spring. We denote it by *D*. This means that the spring force acts on the mass at the position $y(t)$

$$F = -Dy(t)$$

The minus sign is because the *y-axis* is pointing downwards, but the spring force is pointing upwards.

Our experience with spring pendulums now tells us: If we release the mass at the deflected position, the spring pendulum begins to swing evenly. If $y(t)$ denotes the position of the pendulum at the time t, then its *speed* is the change in position per unit of time, i.e. $\dot{y}(t)$, as we saw with the cyclist.

This speed is obviously zero at the maximum deflection and changes continuously in between. It is maximum when passing through the equilibrium position. The change in speed per unit of time is the *acceleration*, i.e. $\ddot{y}(t)$, as we have also seen with the cyclist. This change in speed can also be understood as the *instantaneous tendency* of the system: We know how the system behaves at an infinitely small point in time. How can we now use this "*microscopic*" information about the system to reconstruct its *global* behaviour?

If we use Newton's second law of motion again, the following applies:

$$-Dy(t) = m\ddot{y}(t)$$

The instantaneous tendency of the system or $\ddot{y}(t)$ depends on the force acting at the position $y(t)$.

In this equation, the unknown quantity is not a number, but *the function* $y(t)$. It is called a *differential equation*. To find the solution to this equation, we have to determine the function value $y(t)$ for each value of t by "adding up" the microscopic effects of the accelerating force. Alternatively, we can look for the function $y(t)$ as a whole. This seems challenging.

The oscillation behaviour of the system or Newton's equation of motion only depends on the quotient $\frac{D}{m}$. If we set: $\omega^2 := \frac{D}{m}$, we get:

$$\ddot{y}(t) + \omega^2 y(t) = 0$$

Without explaining how we arrived at this idea, we can see that, for example, the function

$$y(t) = A\cos(\omega t)\,, A \in \mathbb{R}$$

is a solution to this differential equation. For the moment, we leave open whether this is the only solution and whether a solution exists for every differential equation.

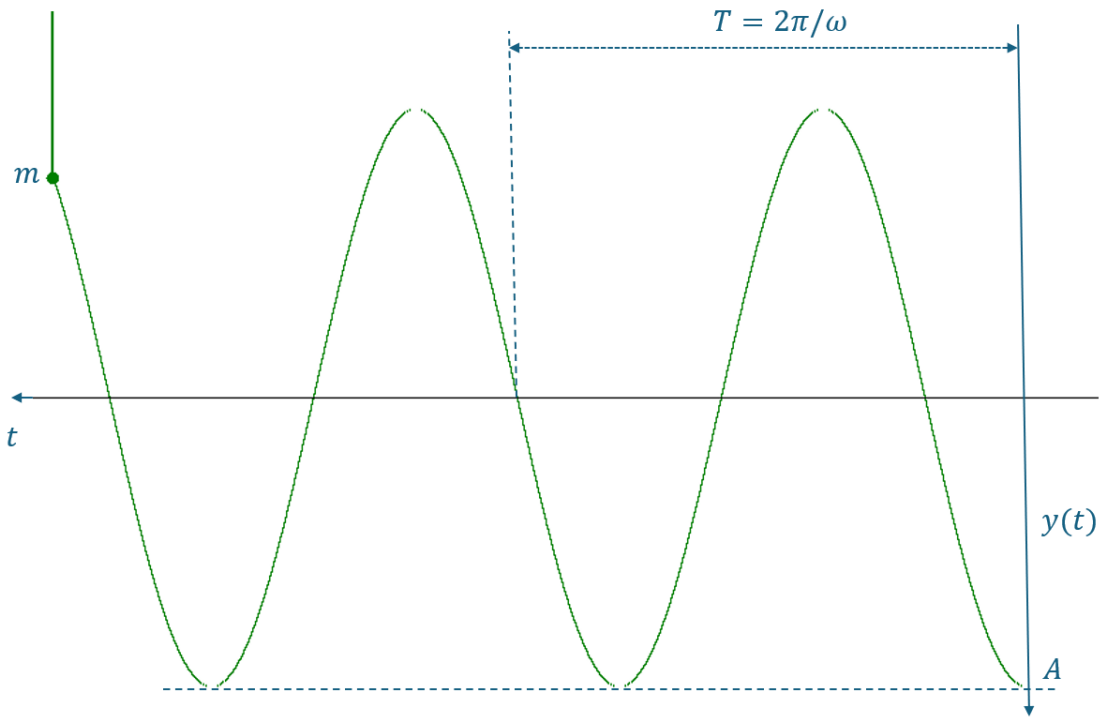
A is the *amplitude* of the oscillation andω is the *angular frequency*. If the pendulum needs the time T to perform an oscillation, then the pendulum will return to the same position after this time. It therefore applies because of the periodicity of the cosine:

$$\omega T = 2\pi$$

The equation is obviously fulfilled for any value of A. This constant could be chosen arbitrarily. However, if we specify an *initial condition*, e.g. how large the maximum deflection $y(0)$ is at which we release the pendulum at the time $t = 0$, then A is determined, namely:

$$A = y(0)$$

The pendulum oscillation can be observed in the simulator. There, the pendulum swings up and down and the movement is plotted at the same time, with the plot moving to the right. The time axis is then orientated to the left:



Swinging spring pendulum in the simulator

In the case of coupled pendulums, we will also come across differential equations that describe these systems. We will therefore look at some of the basics in the next section.

3. Ordinary differential equations

In a differential equation, in addition to the function being searched for, there are also derivatives of this function. If the function you are looking for only depends on *one* variable, then the differential equation is called *ordinary*. For the investigation of coupled pendulums, this variable will be the time t.

In the case of the spring pendulum, we had the differential equation

$$\omega^2 y(t) + \ddot{y}(t) = 0$$

It is therefore an ordinary differential equation, and we can also write it in the form

$$f(y, \ddot{y}) = 0$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}, (u, v) \longmapsto \omega^2 u + v$

A differential equation can contain further derivatives of $y(t)$ or the parameter t itself. This gives us an equation in the general case:

$$f\left(t, y(t), \dot{y}(t), \ddot{y}(t), \dots\ y^{(n)}(t)\right) = 0$$

f depends on $n + 1$ parameters and has a certain domain of definition $D_f \subseteq \mathbb{R}^{n+1}$. This leads to the definition:

Definition 2.1

Let f be a function $f: D_f \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$. Then is called

$$f\left(t, y(t), \dot{y}(t), \ddot{y}(t), \dots, y^{(n)}(t)\right) = 0$$

Ordinary differential equation of order n. □

In this equation, we are looking for a function $y(t)$, which fulfils the equation for t in a certain range. In our case, $t \in \mathbb{R}^+$ represents the time that starts at $t = 0$. □

Example

$$f\big(t, \dot{y}(t)\big) = \dot{y}(t) - g(t)$$

Where g is a real function. If g is not integrable, this differential equation has *no solution*. In the other case, the solution of the differential equation is the integral

$$y(t) = \int_0^t g(\tau)d\tau + C, C \in \mathbb{R}$$

This means that the solution is only determined up to an arbitrarily selectable constant $C \in \mathbb{R}$.

□

What conditions must be formulated for the required solution function $y(t)$ of a differential equation of order n? For all derivatives to exist, $y(t)$ must be differentiable *n times*. Then $y(t)$ must fulfil the differential equation. However, the tuple $(t, y(t), \dot{y}(t), \ddot{y}(t), \dots\ y^{(n)}(t))$ must lie within the domain of definition of f. Perhaps all of this is only fulfilled for$t$ in a certain range$I \subseteq \mathbb{R}$. This leads to

Definition 2.2

Given an ordinary differential equation of order n.

If there is a function $y: I \subseteq \mathbb{R}, t \longmapsto y(t)$ which fulfils the following conditions:

1) y is differentiable *n times* on I
2) For all $t \in I$ is$(t, y(t), \dot{y}(t), \ddot{y}(t), \dots\ y^{(n)}(t)) \subseteq D_f$, i.e. in the domain of definition of f
3) For all $t \in I, y(t)$ fulfils the differential equation

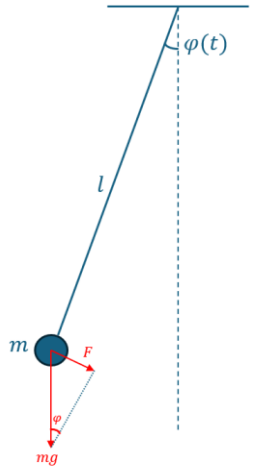
Then this function is called the *solution of the differential equation*. □

Remark

Without being able to justify this with the means available to us, we note that

1) A differential equation does not need to have a solution.
2) The solution of a differential equation is generally not uniquely determined. In the case of the spring pendulum, this was only the case when an *initial condition* for the pendulum was defined.
3) Whether an existing solution is unambiguous apart from the initial conditions is generally a difficult question. This will be the case with coupled pendulums, as we will investigate.
4) Even if a solution exists, it can only be described *analytically* by a formula in a few cases. In this case, it can be approximated more or less accurately by numerical methods. We will discuss some of these in the next section.

Example of a thread pendulum



A mass m oscillates on a thread of length l

The force of gravity mg acts on the mass m , where g is the acceleration due to gravity. If the pendulum is deflected by the angle φ, then the restoring force is $F = -mg \cdot \sin\varphi$, as can be seen from the sketch above. The mass m moves on a circular path at a distance l from the suspension point. The movement is only in a tangential direction to the circular path. The speed is $l\dot{\varphi}$ and the acceleration $l\ddot{\varphi}$. This is Newton's law of motion:

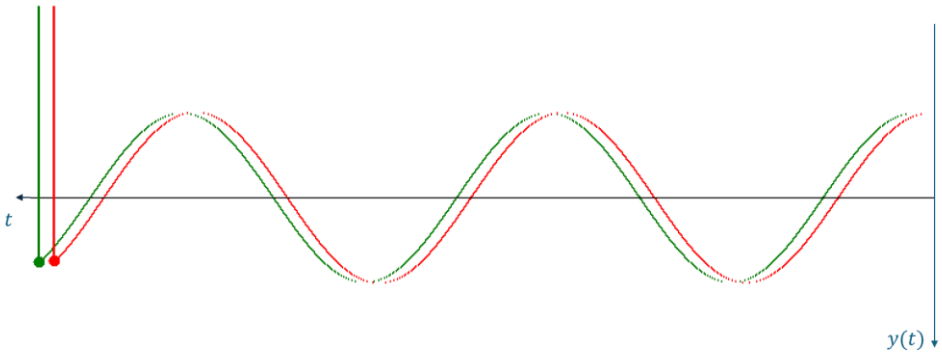
$$-mg\sin\varphi = ml\ddot{\varphi}$$

The mass is cancelled out. The oscillation is therefore independent of m. If we introduce a circular frequency $\omega^2 := \frac{g}{l}$ again, we have:

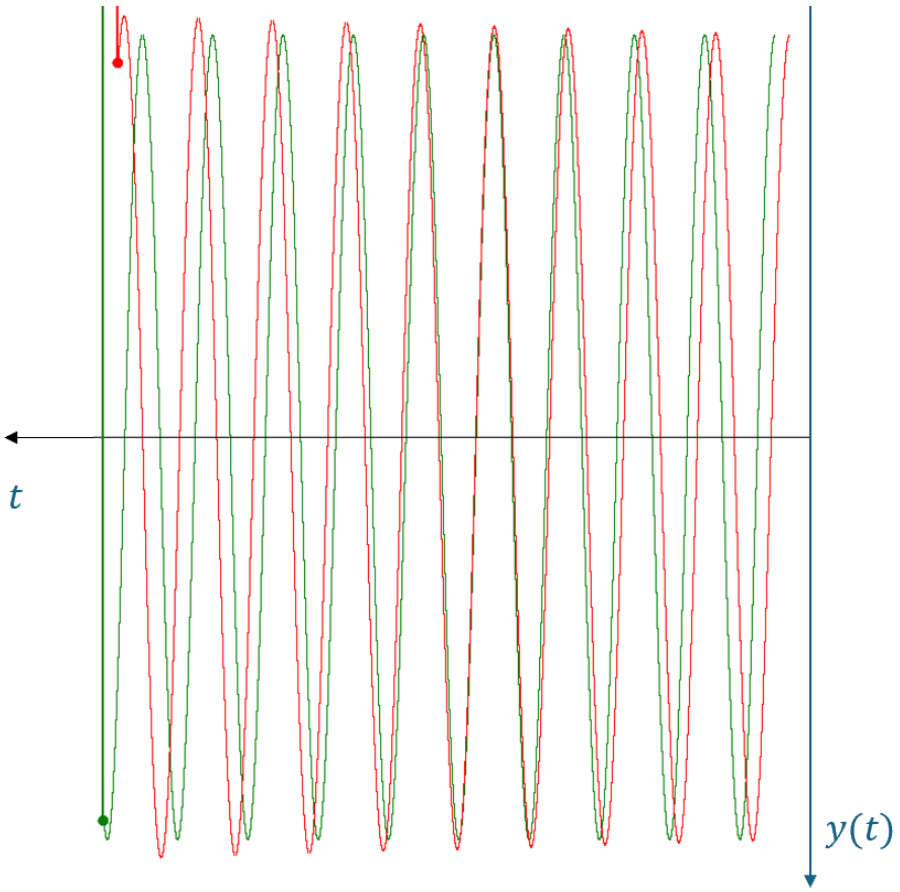
$$\omega^2\sin\varphi + \ddot{\varphi} = 0$$

This equation has *no* analytical solution. For $\varphi \approx 0, \sin\varphi \approx \varphi$ applies and we again have the same differential equation as for the spring pendulum with the general solution $\varphi(t) = A\cos(\omega t)$ and A is determined by the starting point $A = \varphi(0)$.

The simulator approximates the solution for the differential equation using numerical methods and the oscillation of the thread pendulum can be compared with that of the spring pendulum. The spring pendulum is shown in green and the (real and approximated) thread pendulum in red:



Spring pendulum and (real) thread pendulum at low amplitude



As the deflection increases, the thread pendulum begins to run behind the spring pendulum. In the simulation, its period of oscillation is slightly longer than that of the spring pendulum.

When analysing pendulums, we will be dealing with ordinary second-order differential equations. Furthermore, we will always be able to convert the equation so that $\ddot{y}(t)$ is explicitly on the left-hand side of the equation. The equation then has the form:

$$\ddot{y} = f(t, y, \dot{y})$$

We now want to show that such an equation can be converted into a system of two first-order differential equations.

Given an explicit second-order differential equation of the form $\ddot{y} = f(t, y, \dot{y})$. We now introduce two new functions: $u(t) = y(t)$ and $v(t) = \dot{y}(t)$. As a result, we immediately have

Theorem 2.3

Assertion: The explicit second-order differential equation $\ddot{y} = f(t, y, \dot{y})$ can be transformed into a system of two explicit first-order differential equations by transforming $u(t) = y(t)$ and $v(t) = \dot{y}(t)$

$$\begin{cases} \dot{u} = v \\ \dot{v} = f(t, u, v) \end{cases}$$

□

Example

In the case of the spring pendulum, the differential equation $\ddot{y} = -\omega^2 y$ is equivalent to the system through the transformation $u = y$ and $v = \dot{y}$

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\omega^2 u \end{cases}$$

This transformation will be important in the next section when it comes to finding approximate solutions to second-order differential equations using numerical methods.

4. Euler method

Because of Theorem 2.3, it is sufficient in this section to consider ordinary differential equations of the first order, i.e. equations of the form

$$\dot{y} = f(t, y)$$

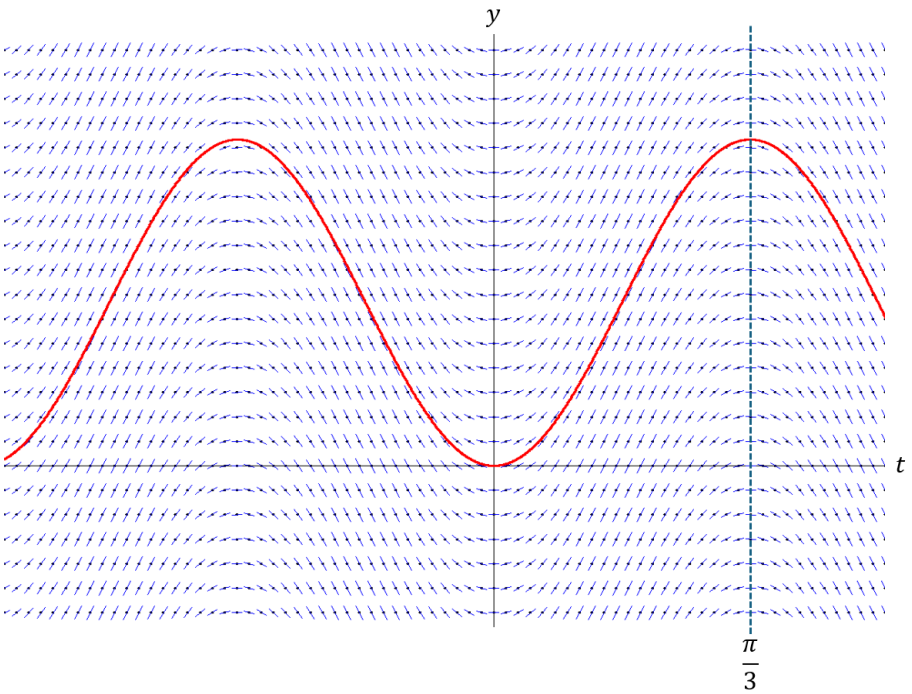
These can also be interpreted in such a way that each point $(t, y) \in \mathbb{R}^2$ is assigned the *slope of a tangent* which lies at this point on the function being searched for. This is also referred to as a *direction field* $f: \mathbb{R}^2 \to \mathbb{R}, (t, y) \to \dot{y} \in \mathbb{R}$.

Example

The differential equation is given

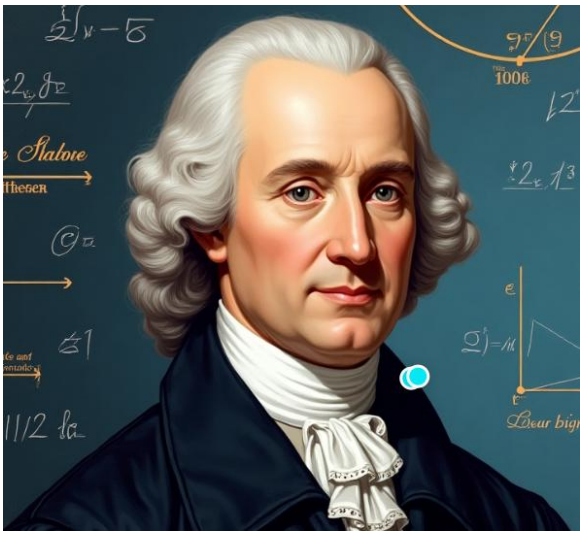
$$\dot{y} = \sin(3t)$$

As you can easily calculate, $y(t) = C - \frac{\cos(3t)}{3}, C \in \mathbb{R}$ is the general solution of this equation. The associated direction field only depends on t. The following figure shows this direction field with the specific solution $y(t) = (1 - \cos(3t))/3$.



Direction field $\dot{y} = \sin(3t)$ and function $y(t) = (1 - \cos(3t))/3$

□



Leonhard Euler 1707 - 1783 was one of the greatest mathematicians of his time. He was involved in all areas of mathematics and made important and pioneering contributions everywhere. The Euler method for the numerical approximation of ordinary differential equations discussed in this section originates from him. Euler had neither a calculator nor a computer at his disposal and had to carry out all calculations "by hand".

Given a differential equation $\dot{y} = f(t, y)$. The following applies:

$$\lim_{h \to 0} \frac{y(t+h) - y(t)}{h} = \dot{y}(t) = f(t, y)$$

Euler therefore used the approximation:

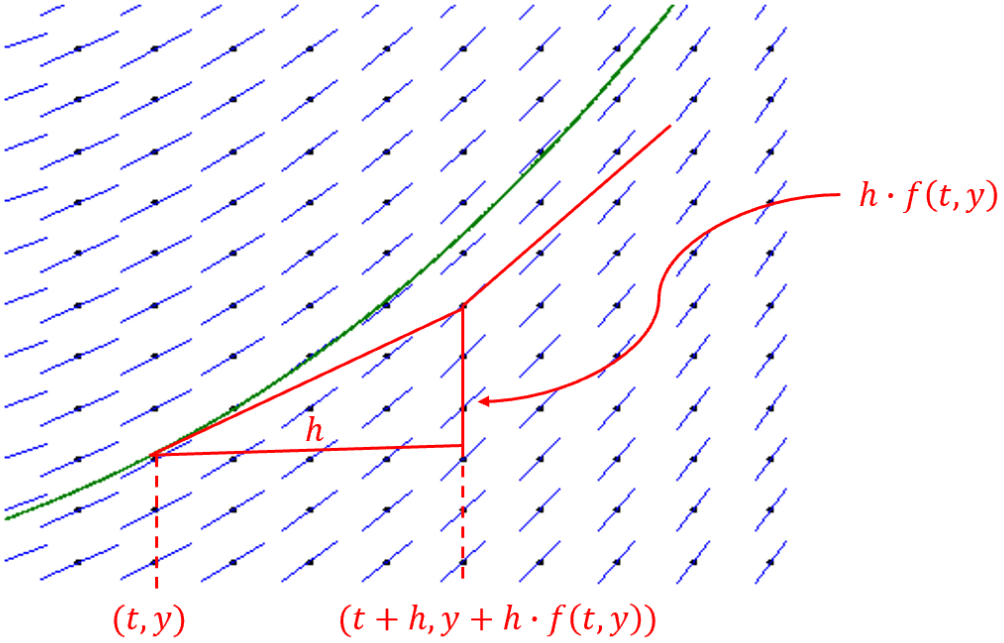
$$y(t+h) \approx y(t) + h \cdot f(t, y)$$

to determine the next function value $y(t+h)$ from a starting point (t_1, y_1). The smaller the "step size" h is chosen, the more accurate this approximation is.

It generates a sequence of function values (t_n, y_n) according to the iteration formula:

$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + h \cdot f(t_n, y_n) \end{cases}$$

Note that $f(t, y)$ represents the tangent slope to the function $y(t)$ at the point (t, y). The following diagram shows an iteration step:



Green: Actual solution curve. Red: Iteration step of the Euler method

Since the next pair of points (t_{n+1}, y_{n+1}) is on the left-hand side of the iteration formula, this method is called *the explicit* Euler method.

Definition 3.1

Given a differential equation of the form $\dot{y} = f(t,y)$. The desired function $y(t)$ is approximated by a sequence of points (t_n, y_n), starting from a starting point (t_1, y_1) and a (small) *increment* h. In the *explicit* Euler method, the corresponding iteration formula is

$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + h \cdot f(t_n, y_n) \end{cases}$$

□

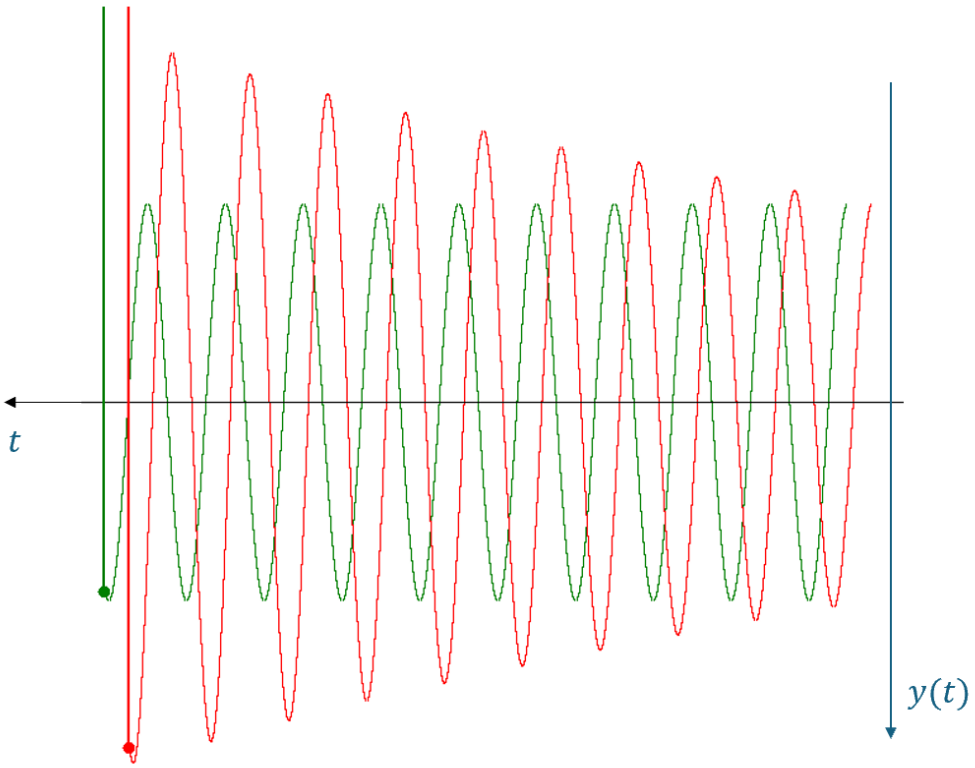
In the simulator, the oscillating spring pendulum can be approximated using the explicit Euler method. For the spring pendulum, the differential equation $\ddot{y} = -\omega^2 y$ was used, which was converted into the system by the transformation $u = y$ and $v = \dot{y}$

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\omega^2 u \end{cases}$$

system. The explicit Euler method is now applied to each equation. In the first equation, the tangent slope at the point (t, u, v) is equal to v. For the second equation, it is $-\omega^2 u$ at this point. This gives the iteration formulae:

$$\begin{cases} t_{n+1} = t_n + h \\ u_{n+1} = u_n + h v_n \\ v_{n+1} = v_n - h\omega^2 u_n \end{cases}$$

The following diagram shows the comparison between the real pendulum and the approximated pendulum.



Green: Actual spring pendulum, red: numerical approximation with Euler explicit

As you can see from this example, the method is only useful for the first few steps. Subsequently, the numerical approximation "overshoots".

An alternative approximation would be the following: You do not know the next point you are looking for (t_{n+1}, y_{n+1}), but use the gradient at the *point you are looking for* in the iteration step. The iteration formula is then as follows:

$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + h \cdot f(t_{n+1}, y_{n+1}) \end{cases}$$

Now the unknown y_{n+1} *is implicit* in the second equation. This leads to

Definition 3.2

Given a differential equation of the form $\dot{y} = f(t, y)$. The desired function $y(t)$ is approximated by a sequence of points (t_n, y_n) starting from a starting point (t_1, y_1) and a (small) *increment* h. In the *implicit* Euler method, the corresponding iteration formula is

$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + h \cdot f(t_{n+1}, y_{n+1}) \end{cases}$$

□

To calculate y_{n+1}, the second equation must be solved to y_{n+1} if possible (depending on the function f).

Let us look again at the spring pendulum and the corresponding differential equations:

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\omega^2 u \end{cases}$$

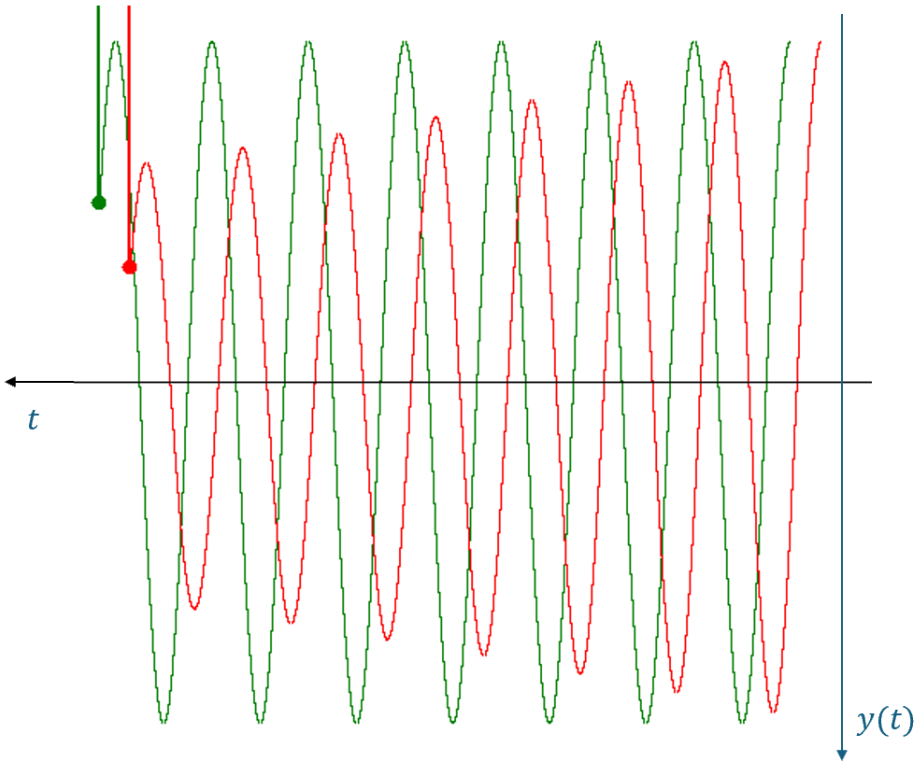
The recursion formula of the explicit method must now be slightly modified:

$$\begin{cases} t_{n+1} = t_n + h \\ u_{n+1} = u_n + v_{n+1}h \\ v_{n+1} = v_n - \omega^2 u_{n+1}h \end{cases}$$

You must therefore solve the second and third equations to (u_{n+1}, v_{n+1}). As you can easily recalculate, this is what the recursion formula provides:

$$\begin{cases} t_{n+1} = t_n + h \\ u_{n+1} = (u_n + v_n h)/(1 + h^2) \\ v_{n+1} = (v_n - u_n h)/(1 + h^2) \end{cases}$$

We can run another simulation in the simulator and compare the real pendulum with the alternative approximation:



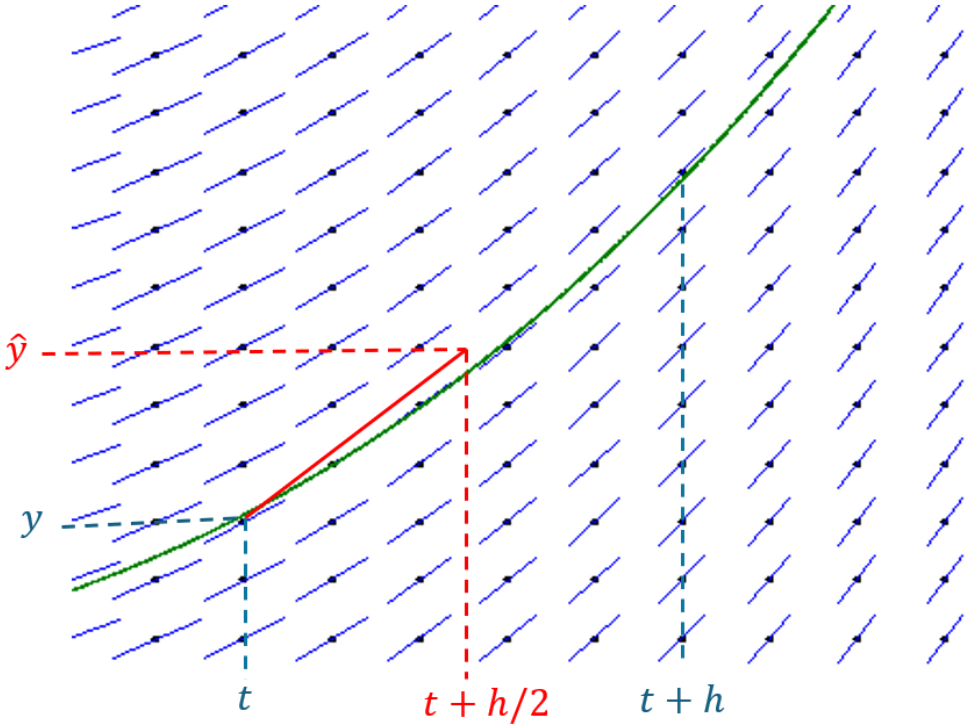
Green: real spring pendulum, red: approximation with Euler implicit

As you can see, the approximation is "damped" compared to the real pendulum.

With the explicit Euler method, the amplitude of the approximation has increased too much. With the implicit method, it was damped too much. The question is whether you can find "something in the middle" of both methods. This means that neither the tangent slope at the point (t_n, y_n) nor at the point (t_{n+1}, y_{n+1}) is used for the approximation, but rather the tangent slope at a "centre point" $\left(t_n + \frac{h}{2}, \hat{y}_n\right)$, whereby $\hat{y}_n$ still has to be determined. The *implicit* Euler method is used to arrive at $\hat{y}_n$. The equation for $\hat{y}_n$ is then as follows:

$$\hat{y}_n = y_n + \frac{h}{2} f(t_n + \frac{h}{2}, \hat{y}_n)$$

The following sketch shows the procedure.



First find the tangent slope at the point $(t_n + \frac{h}{2}, \hat{y}_n)$

After you have calculated $\hat{y}_n$ using the implicit formula, you obtain the tangent slope at the point $\left(t_n + \frac{h}{2}, \hat{y}_n\right)$, namely $f(t_n + \frac{h}{2}, \hat{y}_n)$. This is used to get to the next point:

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, \hat{y}_n)$$

Definition 3.3

Given a differential equation of the form $\dot{y} = f(t, y)$. The required function $y(t)$ is approximated by a sequence of points (t_n, y_n) starting from a starting point (t_1, y_1) and a (small) *increment* h. With the *implicit centre point rule*, you first calculate a point $\hat{y}$ using the implicit equation:

$$\hat{y}_n = y_n + \frac{h}{2} f(t_n + \frac{h}{2}, \hat{y}_n)$$

The next point is then obtained from the iteration formula:

$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + h \cdot f(x_n + \frac{h}{2}, \hat{y}_n) \end{cases}$$

□

Now we apply this method to the spring pendulum. The starting point is again the system of differential equations:

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\omega^2 u \end{cases}$$

First, we determine $\hat{u}_n$ and $\hat{v}_n$. The corresponding implicit equations are as follows:

$$\begin{cases} \hat{u}_n = u_n + \frac{h}{2}\hat{v}_n \\ \hat{v}_n = v_n - \omega^2 \frac{h}{2}\hat{u}_n \end{cases}$$

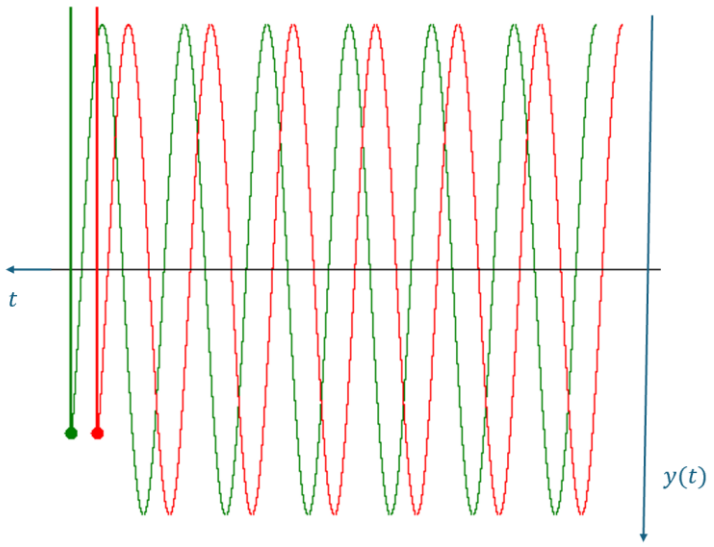
Solving the system of equations according to $(\hat{u}_n, \hat{v}_n)$, we obtain:

$$\begin{cases} \hat{u}_n = (u_n + \frac{h}{2}v_n)/(1 + {}^{h^2}/_4) \\ \hat{v}_n = (v_n - \frac{h}{2}u_n)/(1 + {}^{h^2}/_4) \end{cases}$$

We use these values to calculate the next iteration value:

$$\begin{cases} t_{n+1} = t_n + h \\ u_{n+1} = u_n + h\hat{v}_n = u_n + h(v_n - \frac{h}{2}u_n)/(1 + {}^{h^2}/_4) \\ v_{n+1} = v_n - h\hat{u}_n = v_n - h(u_n + \frac{h}{2}v_n)/(1 + {}^{h^2}/_4) \end{cases}$$

This procedure is implemented in the Simulator, and we compare the result with the real spring pendulum. To the right of the graph, the differences between the real pendulum and the approximated pendulum are recorded for each iteration step. The list shows the iteration steps 200 to 230. The differences remain small.



00200: -0.0001576132
00201: -0.0001666827
00202: -0.0001741612
00203: -0.0001799571
00204: -0.0001839949
00205: -0.0001862163
00206: -0.0001865809
00207: -0.0001850669
00208: -0.0001816713
00209: -0.0001764103
00210: -0.0001693192
00211: -0.0001604523
00212: -0.0001498825
00213: -0.0001377007
00214: -0.0001240149
00215: -0.0001089496
00216: -0.0000926445
00217: -0.000075253
00218: -0.0000569411
00219: -0.0000378856
00220: -0.0000182727
00221: 0.0000017043
00222: 0.0000218465
00223: 0.0000419518
00224: 0.0000618162
00225: 0.0000812367
00226: 0.0001000128
00227: 0.0001179489
00228: 0.0001348561
00229: 0.0001505543
00230: 0.0001648742

Green: Real pendulum, red: Approximation using the implicit centre point rule

As the picture shows, this looks quite good. However, this has nothing to do with the fact that the implicit centre point rule is particularly good and suitable for practical applications. It is simply that this rule provides a good approximation in the case of the spring pendulum or similar problems. As can be seen on the right in the list of differences between the real (green) and approximated (red) pendulum at each iteration step, these are small but not zero.

5. Runge Kutta method

Again, we consider an ordinary differential equation of the first order:

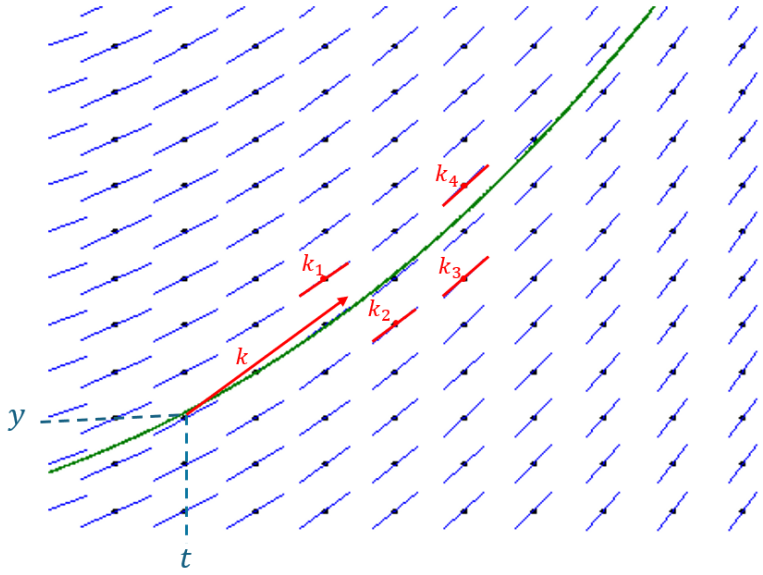
$$\dot{y} = f(t, y)$$

and approximate the solution step by step by a sequence of points (t_n, y_n) from a starting point (t_1, y_1). To get from one point to the next, we again use an iteration formula:

$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + h \cdot k \end{cases}$$

In it, k is a "tangent slope" yet to be determined. The idea here is to analyse several points to the right of (t_n, y_n) in the direction field of the differential equation and note their tangent slope k_j. The effective slope, which is used for the iteration step, is then a weighted average of these slopes, namely $k = \sum_{j=1}^{n} b_j k_j$ with $\sum_{j=1}^{n} b_j = 1$.

The following diagram outlines the procedure:



Runge Kutta method

This method goes back to the mathematicians Carl Runge (1865 - 1927) and Martin Wilhelm Kutta (1867 - 1944), who developed it at the beginning of the 20th century. It is important to bear in mind that every calculation back then was painstaking manual labour.

For our purposes, we will analyse four points. Then $n = 4$ is the *fourth-order* Runge Kutta method.

The question now is how to choose the points to be analysed to arrive at the slopes k_1, k_2, k_3, k_4.

Definition 4.1

Given a differential equation of the form $\dot{y} = f(t, y)$. The desired function $y(t)$ is approximated by a sequence of points (t_n, y_n) starting from a starting point (t_1, y_1) and a (small) *increment* h. In the *fourth-order Runge Kutta method*, the tangent slope is first calculated at four points:

$$\begin{cases} k_1 = f(t_n, y_n) \\ k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right) \\ k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right) \\ k_4 = f(t_n + h, y_n + h k_3) \end{cases}$$

The averaged slope is then used for the iteration step

$$k = (k_1 + 2k_2 + 2k_3 + k_4)/6$$

And you have the iteration formula:

$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + hk \end{cases}$$

□

Now let's apply this to the spring pendulum again. The corresponding system of differential equations was given by:

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\omega^2 u \end{cases}$$

Since we have two components, we need two sets of coefficients for the tangent slopes. We denote them by k_j for the first component and l_j for the second.

This gives us first:

$$\begin{cases} k_1 = f_1(t_n, u_n, v_n) = v_n \\ l_1 = f_2(t_n, u_n, v_n) = -\omega^2 u_n \end{cases}$$

$$\begin{cases} k_2 = f_1\left(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_1, v_n + \frac{h}{2}l_1\right) = v_n + \frac{h}{2}l_1 \\ l_2 = f_2\left(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_1, v_n + \frac{h}{2}l_1\right) = -\omega^2(u_n + \frac{h}{2}k_1) \end{cases}$$

$$\begin{cases} k_3 = f_1\left(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_2, v_n + \frac{h}{2}l_2\right) = v_n + \frac{h}{2}l_2 \\ l_3 = f_2\left(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_2, v_n + \frac{h}{2}l_2\right) = -\omega^2(u_n + \frac{h}{2}k_2) \end{cases}$$

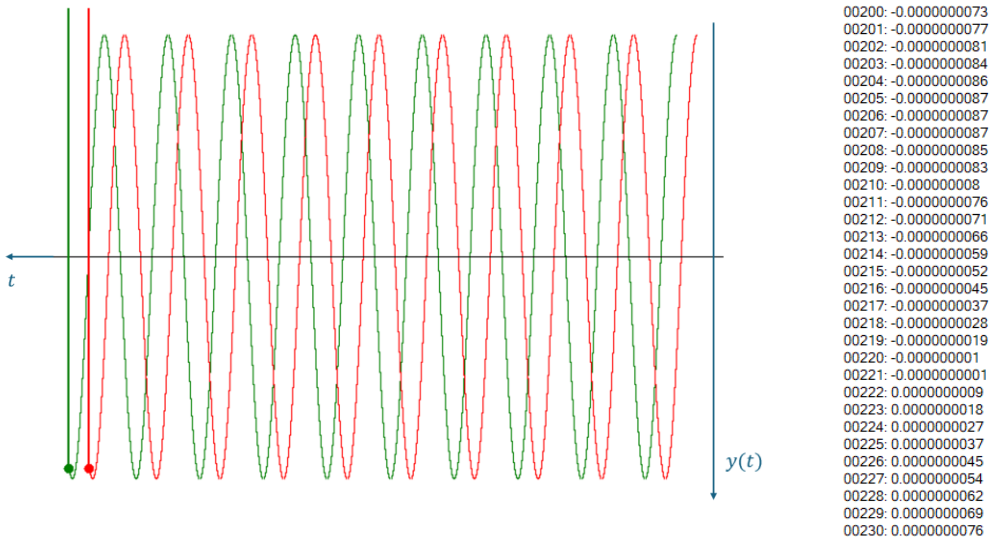
$$\begin{cases} k_4 = f_1(t_n + h, u_n + hk_3, v_n + hl_3) = v_n + hl_3 \\ l_4 = f_2(t_n + h, u_n + hk_3, v_n + hl_3) = -\omega^2(u_n + hk_3) \end{cases}$$

This gives us the recursion formulae:

$$\begin{cases} t_{n+1} = t_n + h \\ u_{n+1} = u_n + h(k_1 + 2k_2 + 2k_3 + k_4)/6 \\ v_{n+1} = v_n + h(l_1 + 2l_2 + 2l_3 + l_4)/6 \end{cases}$$

As you can see, this is an explicit method.

It is implemented in the Simulator. The result is as follows:



The red pendulum is approximated by the Runge Kutta method of order four

This approximation is also visually quite good. On the right-hand side, the list of differences between the real pendulum and the approximated pendulum shows that they have become significantly smaller at each step. The list again shows the iteration steps 200 to 230.

We will use this Runge Kutta method for the approximation in the next chapter for the simulation of different types of pendulums, because it is easy to implement and at least the idea is elementary understandable. It has a good balance between accuracy and computational effort. It does not require higher derivatives of the function than the first one. In addition, the error is proportional to the fifth power of the step size, which is sufficient for our purposes.

In practice, depending on the type of differential equation, there are more efficient methods, in some cases multi-step methods or adaptive methods with an adapted step size per step. If you are interested, please refer to the extensive literature. MATLAB also offers a whole range of practical methods, which are often used by engineers, with its ODE (= Ordinary Differential Equation) suite.

6. Partial derivative and total differential

In mechanical systems involving several masses, such as coupled pendulums, we will encounter equations of motion with functions that depend on several parameters. The terms partial derivative and total differential are used here. If you are familiar with these terms, you can skip this section. For others, we offer a brief introduction in preparation for the Lagrangian formalism in the next section.

Definition 5.1

Given a function in several variables:

$$f: (q_1, q_2, \dots, q_n) \in D_f \subseteq \mathbb{R}^n \to f(q_1, q_2, \dots, q_n) \in \mathbb{R}$$

Then the *partial derivative* with respect to a variable q_i is defined as:

$$\frac{\partial f}{\partial q_i} := \lim_{h \to 0} \frac{f(q_1, q_2, \dots, q_i + h, \dots, q_n) - f(q_1, q_2, \dots, q_n)}{h}$$

if this limit exists. If this is the case for all i, then f is called partially differentiable. □

In practice, only q_i is regarded as a variable and derived according to q_i as usual, while all other variables $q_k, k \neq i$ are regarded as constant.

Example

Given: $f(x,y,z) \coloneqq 2xy + xz^2$. Then is:

$$\frac{\partial f}{\partial x} = 2y + z^2, \frac{\partial f}{\partial y} = 2x, \frac{\partial f}{\partial z} = 2xz$$

□

To investigate the behaviour of a real and continuously differentiable function f, we have approximated this function in the one-dimensional case using the derivative of this function by a linear function:

$$f(x+h) \approx f(x) + f'(x) \cdot h, h \approx 0$$

Now we try the same for a function

$$f\colon (q_1, q_2, \dots, q_n) \in D_f \subseteq \mathbb{R}^n \to f(q_1, q_2, \dots, q_n) \in \mathbb{R}$$

which is partially continuously differentiable. This means that the partial derivatives with respect to each variable exist and are continuous in each variable.

We therefore want to estimate a difference:

$$df \coloneqq f(q_1 + dq_1, q_2 + dq_2, \dots., q_n + dq_n) - f(q_1, q_2, \dots., q_n)$$

Where $dq_i \approx 0, \forall i$. The following applies:

$$\begin{aligned} df \coloneqq\ & f(q_1 + dq_1, q_2 + dq_2, \dots., q_n + dq_n) - f(q_1, q_2 + dq_2, \dots., q_n + dq_n) \\ & + f(q_1, q_2 + dq_2, \dots., q_n + dq_n) - f(q_1, q_2, q_3 + dq_3, \dots., q_n + dq_n) \\ & + f(q_1, q_2, q_3 + dq_3, \dots., q_n + dq_n) - \cdots \\ & + f(q_1, q_2, q_3, \dots, q_{n-1}, q_n + dq_n) - f(q_1, q_2 + dq_2, \dots., q_n + dq_n) \end{aligned}$$

If we estimate each of these differences as in the one-dimensional case, the following applies:

$$\begin{aligned} & f(q_1 + dq_1, q_2 + dq_2, \dots., q_n + dq_n) - f(q_1, q_2 + dq_2, \dots., q_n + dq_n) \\ & \approx \frac{\partial f}{\partial q_1}(q_1, q_2 + dq_2, \dots., q_n + dq_n) dq_1 \end{aligned}$$

and so on for all i. Since the partial derivatives are continuous, the following also applies:

$$\frac{\partial f}{\partial q_1}(q_1, q_2 + dq_2, \dots., q_n + dq_n) \approx \frac{\partial f}{\partial q_1}(q_1, q_2, \dots., q_n)$$

This gives us the so-called total differential:

Definition 5.2

Let $f\colon (q_1, q_2, \dots, q_n) \in D_f \subseteq \mathbb{R}^n \to f(q_1, q_2, \dots, q_n) \in \mathbb{R}$ be partially continuously differentiable.

Then is called:

$$df\colon = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \cdots + \frac{\partial f}{\partial q_n} dq_n$$

Total differential of f. □

The total differential describes the infinitesimal change of a function $f(q_1, q_2, \dots, q_n)$ if each of its variables q_i changes by an infinitesimal value dq_i.

Example

Given: $f(x, y, z) \coloneqq 2xy + xz^2$. Then is:

$$df = (2y + z^2)dx + 2xdy + 2zxdz$$

□

It may be that the parameters q_i are again functions depending on another parameter, in particular depending on the time t. So: $q_i = q_i(t)$, whereby all q_i are continuously differentiable. This does not change the partial derivative of f. On the other hand, you can replace the total differential in Definition 5.2:

$$dq_i(t) = \frac{dq_i}{dt}dt = \dot{q}_i dt$$

where the point is the derivative with respect to time. In this case, you have in Definition 5.2:

$$df = \frac{\partial f}{\partial q_1}\dot{q}_1 dt + \frac{\partial f}{\partial q_2}\dot{q}_2 dt + \cdots + \frac{\partial f}{\partial q_n}\dot{q}_n dt$$

Or:

$$\dot{f} = \frac{df}{dt} = \frac{\partial f}{\partial q_1}\dot{q}_1 + \frac{\partial f}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial f}{\partial q_n}\dot{q}_n$$

The following lemma will be useful for the next chapter.

Lemma 5.3

Let $f: (q_1, q_2, \dots, q_n) \in D_f \subseteq \mathbb{R}^n \to f(q_1, q_2, \dots, q_n) \in \mathbb{R}$ be partially continuously differentiable.

Assertion:

$$\frac{\partial \dot{f}}{\partial \dot{q}_i} = \frac{\partial f}{\partial q_i}$$

and

$$\frac{\partial \dot{f}}{\partial q_i} = \frac{d}{dt}\frac{\partial f}{\partial q_i}$$

□

Proof:

The first equation follows directly from

$$\dot{f} = \frac{df}{dt} = \frac{\partial f}{\partial q_1}\dot{q}_1 + \frac{\partial f}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial f}{\partial q_n}\dot{q}_n$$

In the second equation you have

$$\frac{\partial \dot{f}}{\partial q_i} = \frac{\partial}{\partial q_i}\frac{df}{dt} = \frac{\partial^2 f}{\partial q_i \partial q_1}\dot{q}_1 + \frac{\partial^2 f}{\partial q_i \partial q_2}\dot{q}_2 + \cdots + \frac{\partial^2 f}{\partial q_i \partial q_n}\dot{q}_n$$

and

$$\frac{d}{dt}\frac{\partial f}{\partial q_i} = \frac{\partial^2 f}{\partial q_1 \partial q_i}\dot{q}_1 + \frac{\partial^2 f}{\partial q_2 \partial q_i}\dot{q}_2 + \cdots + \frac{\partial^2 f}{\partial q_n \partial q_i}\dot{q}_n$$

Now there is the Schwarz-Clairaut theorem, which we cannot explain further here. It says that you can swap the order of the partial derivatives if they are continuous, i.e.:

$$\frac{\partial^2 f}{\partial q_i \partial q_k} = \frac{\partial^2 f}{\partial q_k \partial q_i}$$

This results in the assertion.

□

Example:

Let the following be given: $f(x, y, z) \coloneqq 2xy + xz^2$

Furthermore, let x, y, z be dependent on time, namely:

$$\begin{cases} x(t) = r\cos t \\ y(t) = r\sin t \\ z(t) = e^{-rt} \end{cases}$$

Then $f(t) = 2r^2\cos t \cdot \sin t + re^{-2rt}\cos t$ and

$$\dot{f}(t) = -2r^2\sin^2 t + 2r^2\cos^2 t - 2r^2 e^{-2rt}\cos t - re^{-2rt}\sin t$$

If we use the total differential, then

$$\dot{f}(t) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} = (2y + z^2)\dot{x} + 2x\dot{y} + 2xz\dot{z}$$

With

$$\begin{cases} \dot{x}(t) = -r\sin t \\ \dot{y}(t) = r\cos t \\ \dot{z}(t) = -re^{-rt} \end{cases}$$

we have:

$$\dot{f}(t) = -(2r\sin t + e^{-2rt})r\sin t + 2r^2\cos^2 t - 2r\cos t \cdot re^{-2rt}$$

This gives us as in Lemma 5.3:

$$\frac{\partial \dot{f}}{\partial \dot{x}} = 2y + z^2 = \frac{\partial f}{\partial x}$$

and:

$$\frac{\partial \dot{f}}{\partial x} = 2\dot{y} + 2z\dot{z} = \frac{d}{dt}\frac{\partial f}{\partial x}$$

7. The Lagrangian formalism

In the following sections we will look at examples of coupled pendulums and simulate their dynamics in the simulator. The derivation of the corresponding equations of motion based on Newton's laws of force can be difficult and time-consuming. We will use the so-called Lagrangian formalism for such cases. To do this, we need the concept of the partial derivative and the total differential from the previous section.

Joseph-Louis Lagrange (1736 - 1813) was a French mathematician who produced profound work in all areas of mathematics and especially in theoretical physics. In this section, we will learn about the Lagrange function named after him. It is based on the so-called calculus of variations, which was mainly developed by Leonhard Euler and Lagrange.

We assume that the laws of conservation and thus terms such as momentum, angular momentum, kinetic and potential energy are known from physics. Furthermore, we restrict ourselves to force fields that originate from a *potential field*. Such force fields are called *conservative*. We will now define this in more detail.

Definition 6.1

A *potential field*, or "*potential*" for short, in $\mathbb{R}^3$ is a (in each variable) continuously differentiable map:

$$V: \vec{x} \in \mathbb{R}^3 \to V(\vec{x}) \in \mathbb{R}$$

□

The term "*field*" means that the domain of definition of this map lies in space (or generally in $\mathbb{R}^n$). If the image is a scalar, i.e. lies in $\mathbb{R}$, the image is also called a *scalar field.*

We want to distinguish between potential and potential energy. A potential is *independent* of a mass that lies in its domain.

Example

The *potential* of the Earth's gravitational field at a point $\vec{x}$ relative to the centre of the Earth is

$$V(\vec{x}) = -MG\frac{1}{|\vec{x}|}$$

Where $M \approx 5.97 \cdot 10^{24}\ kg$ is the mass of the Earth and $G \approx 6.67 \cdot 10^{-11}\ \frac{Nm^2}{kg^2}$ is the gravitational constant.

The *potential energy* of a mass m, which is located in this potential field, is then

$$E_{pot} = mV(\vec{x}) = -mMG\frac{1}{|\vec{x}|}$$

□

Definition 6.2

If we say in mechanics that a force $\vec{F}(\vec{x})$, which acts on a mass m at a point $\vec{x}$, *originates from a potential*, then this means that a potential

$$V\colon \vec{x} \in \mathbb{R}^3 \to V(\vec{x}) \in \mathbb{R}$$

exists, so that applies:

$$\vec{F} = -m\begin{pmatrix}\frac{\partial V}{\partial x_1}\\ \frac{\partial V}{\partial x_2}\\ \frac{\partial V}{\partial x_3}\end{pmatrix} =: -m\nabla V$$

In this case, we call the force *conservative*.

□

$\nabla V = \begin{pmatrix}\frac{\partial V}{\partial x_1}\\ \frac{\partial V}{\partial x_2}\\ \frac{\partial V}{\partial x_3}\end{pmatrix}$ is also known as the *gradient of* V.

Example

The potential was in the Earth's gravitational field:

$$V(\vec{x}) = -MG\frac{1}{|\vec{x}|}$$

It is now

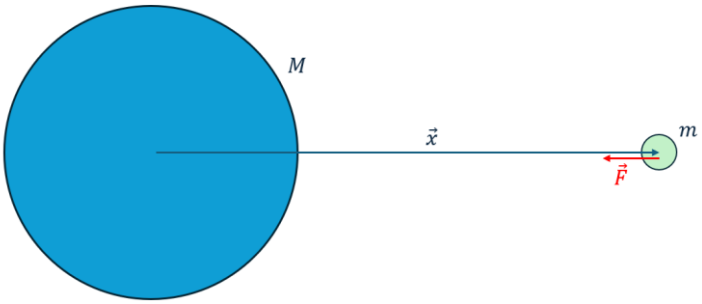
$$\frac{\partial V}{\partial x_1} = -MG\frac{\partial}{\partial x_1}\frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = MG\frac{x_1}{|\vec{x}|^3}$$

Therefore, the associated force is

$$\vec{F} = -m\nabla V = -mMG\frac{1}{|\vec{x}|^2}\cdot\frac{\vec{x}}{|\vec{x}|}$$

This is nothing other than *Newton's law of gravity*.

$-\frac{\vec{x}}{|\vec{x}|}$ is a unit vector in the direction of the centre of the earth. The force points in this direction and decreases with the square of the distance from the centre of the earth.



A mass m at the position $\vec{x}$ relative to the centre of the earth

If V is changed by a constant $C \in \mathbb{R}$, this obviously produces the same force, since the constant disappears in the derivative ∇V. In practice, this constant is "normalised" by setting the zero level of the potential energy, so that calculations with the potential become "more manageable".

Example

We consider a mass m near the earth at a distance h from the earth's surface. If $R \approx 6.37\ km$ is the Earth's radius, then $|\vec{x}| = R + h$. The corresponding potential is then

$$V(h) = -MG\frac{1}{R+h} + C$$

Now we choose $C \in \mathbb{R}$ so that $V(0) = 0$ is that the potential energy on the earth's surface disappears. Then $C = MG\frac{1}{R}$ and since h is very small compared to the Earth's radius, the following applies:

$$V(h) = MG\frac{1}{R} - MG\frac{1}{R+h} = MG\frac{h}{R(R+h)} \approx \frac{MG}{R^2} \cdot h$$

If you set $g = \frac{MG}{R^2} \approx 9.81\ \frac{m}{s^2}$, you have the familiar formula for the potential energy of a mass m near the Earth:

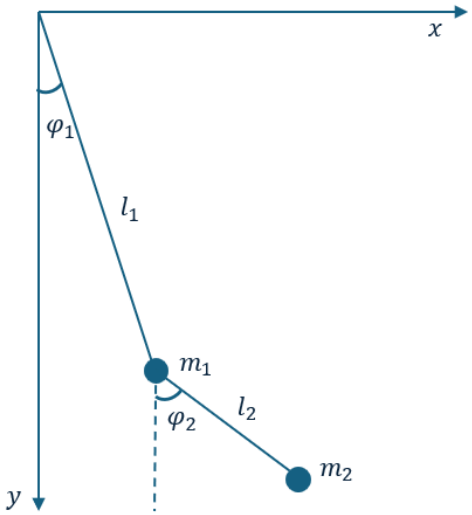
$$E_{pot}(h) = mgh$$

The gravitational force is then constant:

$$F = -\frac{dE_{pot}}{dh} = -mg$$

□

Now consider a system of N masses $m_1, m_2, \dots, m_N$. The current positions of the masses in the system are determined by their Cartesian coordinates $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N$. However, these coordinates often do not represent the system in a suitable form, namely when the masses are "coupled" in such a way that the position coordinates are not independent of each other. Let us look at an example:



Double pendulum

In the double pendulum above, the position of the masses is clearly determined by the parameters φ_1, φ_2. It is said that the *degree of freedom* of the system is two. If you use four Cartesian coordinates (x_1, y_1) and(x_2, y_2) for the position of the masses instead of these parameters, you need two further equations to obtain the degree of freedom of the system again, namely:

$$\begin{cases} x_1^2 + y_1^2 = l_1^2 \\ x_2^2 + y_2^2 = l_2^2 \end{cases}$$

These equations are also called *constraints*. Let us summarise:

Definition 6.3

The minimum number of parameters required to uniquely describe the positions of the masses in a system is called *the degree of freedom* of the system. If the number of selected parameters is greater than the degree of freedom, then further restrictive equations are required. These are called *constraints*. The parameters that describe the system and do not need to be Cartesian are called *generalised coordinates.* □

Newton's law of force works with Cartesian coordinates and reads for a mass m with (Cartesian) position vector $\vec{x}$:

$$m\ddot{\vec{x}} = \vec{F}(\vec{x})$$

The question now is how to formulate a corresponding law of motion with the help of generalised coordinates. This is where the Lagrange function comes into play. For our purposes, it is sufficient to consider the following situation:

Precondition *

- We consider masses $m_1, m_2, \dots, m_j, \dots, m_K$ in space or in the plane
- The degree of freedom of the system is N and the state of the system at a point in time t is described by generalised coordinates $q_1(t), q_2(t), \dots, q_i(t), \dots, q_N(t)$.
- The corresponding Cartesian coordinates of the masses $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_j, \dots, \vec{x}_K$ depend only on these generalised coordinates (and not also on the time) t.

- You therefore have formulae: $\vec{x}_j = \vec{x}_j(q_1, q_2, \dots, q_N), j = 1..K$ and in these the time t does not appear explicitly.
- Note that $\dot{\vec{x}}_j = \dot{\vec{x}}_j(q_1, q_2, \dots, q_N, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_N), j = 1..K$ also depends on $\dot{\vec{q}}$ because of the chain rule and the inner derivatives.
- The coordinate transformations $\vec{x}_j = \vec{x}_j(q_1, q_2, \dots, q_N), j = 1..K$ should *be regular*, i.e. the following applies: The vectors $\frac{\partial \vec{x}_j}{\partial q_i}, i = 1 \dots N$ are defined for each j and are linearly independent. They span the entire tangential space.
- The masses move in a conservative force field which exerts the force $\vec{F}_j = -m_j \nabla V(\vec{x}_j)$ on the mass m_j at the position $\vec{x}_j$. Here $V: \vec{x} \in \mathbb{R}^3 \to V(\vec{x}) \in \mathbb{R}$ is the associated potential.

□

As the force field is conservative, the potential energy E_{pot} of all masses is defined in addition to the kinetic energy E_{kin}. We can now define the Lagrange function for this scenario:

Definition 6.4

Assumption: Precondition *. Then the Lagrange function is defined as:

$$L: \mathbb{R}^{2N} \to \mathbb{R}: L(q_1, q_2, \dots, q_N, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_N) \coloneqq E_{kin} - E_{pot}$$

Where

$$E_{kin} = \frac{1}{2} \sum_{j=1}^{K} m_j \left| \dot{\vec{x}}_j(\vec{q}, \dot{\vec{q}}) \right|^2$$

is the total kinetic energy of the system and

$$E_{pot} = \sum_{j=1}^{K} m_j V\left(\vec{x}_j(\vec{q})\right)$$

is the total potential energy.

□

Theorem 6.5

Assumptions: Precondition *. L is the Lagrange function.

Assertion: The equations of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \; ; \; i = 1, \dots, N$$

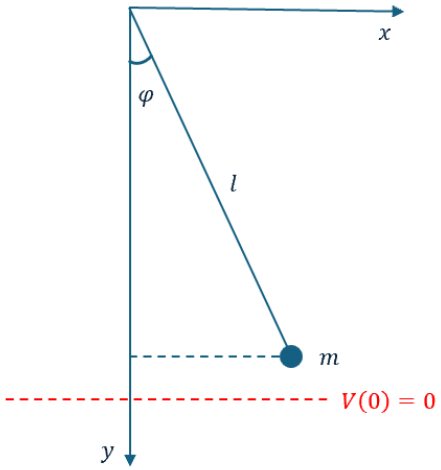
are equivalent to Newton's law of force

□

Before we prove the theorem with the help of Newton's law of force, let us look at an example.

Example: Thread pendulum

In a thread pendulum, the position of the mass m is determined by the angle φ and the degree of freedom is one.



Thread pendulum with generalised coordinate φ

The Cartesian coordinates then depend on φ as follows:

$$\vec{x}(\varphi) = l\begin{pmatrix}\sin\varphi\\-\cos\varphi\end{pmatrix}$$

The potential is the gravitational field near the earth and thus given by

$$V(\varphi) = mgl(1-\cos\varphi)$$

if we normalise $V(0) = 0$. Note: The y-axis points downwards, hence the minus sign for the cosine.

This gives the Lagrange function (the first term is the kinetic energy):

$$L(\varphi,\dot{\varphi},t) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgl(1-\cos\varphi) = \frac{m}{2}l^2\dot{\varphi}^2 - mgl(1-\cos\varphi)$$

The left-hand side of the Lagrange equation is:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} = \frac{d}{dt}ml^2\dot{\varphi} = ml^2\ddot{\varphi}$$

The right-hand side of the Lagrange equation is:

$$\frac{\partial L}{\partial\varphi} = -mgl\cdot\sin\varphi$$

If we equate both sides, we get:

$$ml^2\ddot{\varphi} = -mgl\sin\varphi$$

and thus again the familiar equation from chapter 2:

$$l\ddot{\varphi} + g\sin\varphi = 0$$

□

Proof of theorem 6.5.

We first make some preparations. We consider a mass m in a conservative force field in three-dimensional space. Let its real position in Cartesian coordinates be

$$\vec{x} = \vec{x}(q_1, q_2, \dots, q_N) = \begin{pmatrix} x(q_1, q_2, \dots, q_N) \\ y(q_1, q_2, \dots, q_N) \\ z(q_1, q_2, \dots, q_N) \end{pmatrix}$$

And let condition* apply.

With the notation

$$\frac{\partial \vec{x}}{\partial q_i} \coloneqq \begin{pmatrix} \frac{\partial x}{\partial q_i} \\ \frac{\partial y}{\partial q_i} \\ \frac{\partial z}{\partial q_i} \end{pmatrix}$$

we mean that each of the three components of $\vec{x}$ is partially derived from q_i.

We now make some preparations. According to Lemma 5.3:

$$\frac{\partial \dot{\vec{x}}}{\partial \dot{q}_i} = \frac{\partial \vec{x}}{\partial q_i}, i = 1 \dots N$$

and:

$$\frac{\partial \dot{\vec{x}}}{\partial q_i} = \frac{d}{dt}\frac{\partial \vec{x}}{\partial q_i}$$

Furthermore, according to the chain rule:

$$\frac{\partial V(\vec{x}(\vec{q}))}{\partial q_i} = \frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial q_i} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial q_i} + \frac{\partial V}{\partial x_3}\frac{\partial x_3}{\partial q_i} = \nabla V \cdot \frac{\partial \vec{x}}{\partial q_i}$$

The Lagrange function is:

$$L = \frac{1}{2}m\left|\dot{\vec{x}}(\vec{q}, \dot{\vec{q}})\right|^2 - mV(\vec{q}) = \frac{1}{2}m\dot{\vec{x}}\left(\vec{q}, \dot{\vec{q}}\right) \cdot \dot{\vec{x}}\left(\vec{q}, \dot{\vec{q}}\right) - mV(\vec{q})$$

We compare both sides of the Lagrange equation of motion. First we calculate the left-hand side and use Lemma 5.3.

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}m\dot{\vec{x}} \cdot \dot{\vec{x}}\right) = m\dot{\vec{x}} \cdot \frac{\partial \dot{\vec{x}}}{\partial \dot{q}_i} = m\dot{\vec{x}} \cdot \frac{\partial \vec{x}}{\partial q_i} \Longrightarrow$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = m\ddot{\vec{x}} \cdot \frac{\partial \vec{x}}{\partial q_i} + m\dot{\vec{x}} \cdot \frac{d}{dt}\frac{\partial \vec{x}}{\partial q_i}$$

On the right hand side we have using Lemma 5.3 and the chain rule:

$$\frac{\partial L}{\partial q_i} = m\dot{\vec{x}} \cdot \frac{\partial \dot{\vec{x}}}{\partial q_i} - m\frac{\partial V}{\partial q_i} = m\dot{\vec{x}} \cdot \frac{d}{dt}\frac{\partial \vec{x}}{\partial q_i} - m\frac{\partial V(\vec{x})}{\partial q_i} = m\dot{\vec{x}} \cdot \frac{d}{dt}\frac{\partial \vec{x}}{\partial q_i} - m\nabla V \cdot \frac{\partial \vec{x}}{\partial q_i}$$

Thus $\forall i$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \Leftrightarrow m\ddot{\vec{x}} \cdot \frac{\partial \vec{x}}{\partial q_i} = -m\nabla V \cdot \frac{\partial \vec{x}}{\partial q_i} \Leftrightarrow m\ddot{\vec{x}} = -m\nabla V = \vec{F}$$

However, the right-hand equation is just Newton's law of force.

" ⇐ " is clear.

" ⇒ "The vectors $\frac{\partial \vec{x}}{\partial q_i}$ are linearly independent by assumption and span the entire tangential space.

We have also seen that the theorem applies regardless of the choice of generalised coordinates as long as their number corresponds to the degree of freedom of the system.

□

Example: Momentum theorem

We consider a mass on which no force acts. In this case, V is constant and, with appropriate normalisation, $V \equiv 0$.

The Lagrange function then consists only of the kinetic energy:

$$L = \frac{1}{2} m \left| \dot{\vec{x}}(\vec{q}, \dot{\vec{q}}) \right|^2$$

Then applies (use the formulae in the proof of the theorem):

$$m\ddot{\vec{x}} \cdot \frac{\partial \vec{x}}{\partial q_i} + m\dot{\vec{x}} \cdot \frac{d}{dt}\frac{\partial \vec{x}}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} = m\dot{\vec{x}} \cdot \frac{d}{dt}\frac{\partial \vec{x}}{\partial q_i}$$

Thus $m\ddot{\vec{x}} \cdot \frac{\partial \vec{x}}{\partial q_i} = \vec{0}, \forall i$. However, the vectors $\frac{\partial \vec{x}}{\partial q_i}$ are linearly independent by assumption and span the entire tangential space.

From this follows: $m\ddot{\vec{x}} = \vec{0}$ and $m\dot{x} = \text{konstant}$.

□

Example: Energy conservation

We consider a mass m at the position $\vec{x}(\vec{q})$ in a conservative force field with potential $V(\vec{x})$. The Lagrange function is:

$$L(\vec{q}, \dot{\vec{q}}) = \frac{1}{2} m\dot{\vec{x}} \cdot \dot{\vec{x}} - mV(\vec{x})$$

In preparation, we calculate:

$$\sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_{i=1}^{N} \dot{q}_i \frac{\partial}{\partial \dot{q}_i}\left(\frac{m}{2}\dot{\vec{x}} \cdot \dot{\vec{x}}\right) = m\sum_{i=1}^{N} \dot{q}_i \dot{\vec{x}} \cdot \frac{\partial \dot{\vec{x}}}{\partial \dot{q}_i}$$

This is according to Lemma 5.3:

$$= m\sum_{i=1}^{N} \dot{q}_i \dot{\vec{x}} \cdot \frac{\partial \vec{x}}{\partial q_i} = m\dot{\vec{x}} \cdot \sum_{i=1}^{N} \frac{\partial \vec{x}}{\partial q_i}\dot{q}_i = m\dot{\vec{x}} \cdot \frac{d\vec{x}}{dt} = m\,\dot{\vec{x}} \cdot \dot{\vec{x}}$$

The total energy is , where we use the above result:

$$E = E_{kin} + E_{pot} = \frac{1}{2} m\dot{\vec{x}} \cdot \dot{\vec{x}} + mV(\vec{x}) = m\dot{\vec{x}} \cdot \dot{\vec{x}} - L = \sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

Thus, using the product and chain rule and since L depends on both q_i and $\dot{q}_i$, we have

$$\frac{dE}{dt} = \sum_{i=1}^{N} \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \sum_{i=1}^{N} \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \left(\sum_{i=1}^{N} \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right) = \sum_{i=1}^{N} \dot{q}_i\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right) = 0$$

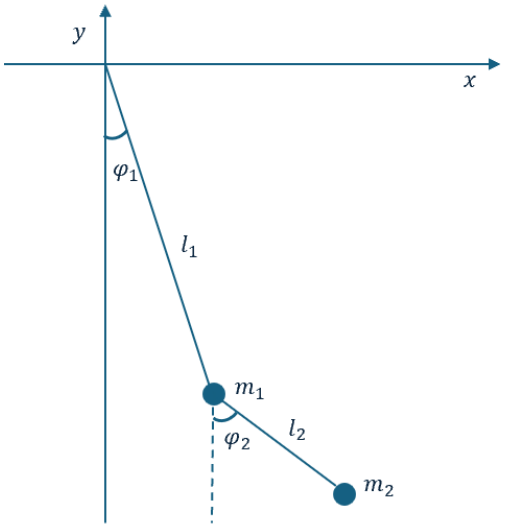
according to Lagrange's equations of motion.

Remark

Since we only need the Lagrangian formalism as an aid for the coupled pendulums, we have reduced it to Newton's law of force. The derivation of the Lagrangian formalism from Hamilton's principle of stationary action is theoretically possible on the basis of secondary school maths with additional effort. It can be interesting to immerse yourself in the calculus of variations using simple examples. A good introduction can be found in [2], for example.

8. Coupled pendulums

The first example of a coupled pendulum is the double pendulum.



Double pendulum

The generalised coordinates are $\begin{pmatrix}\varphi_1(t)\\ \varphi_2(t)\end{pmatrix}$. They completely describe the state of the system at a point in time t. $l_1 > 0, l_2 > 0$ remain constant during the movement.

Now we will have to do some calculations to derive the equations of motion with the help of Lagrange.

The position vector of the mass m_1 is:

$$\vec{r}_1 = \begin{pmatrix}x_1\\ y_1\end{pmatrix} = l_1\begin{pmatrix}\sin\varphi_1\\ -\cos\varphi_1\end{pmatrix}, \dot{\vec{r}}_1 = l_1\dot{\varphi}_1\begin{pmatrix}\cos\varphi_1\\ \sin\varphi_1\end{pmatrix}$$

And the position vector of the mass m_2:

$$\vec{r}_2 = \begin{pmatrix}x_2\\ y_2\end{pmatrix} = l_1\begin{pmatrix}\sin\varphi_1\\ -\cos\varphi_1\end{pmatrix} + l_2\begin{pmatrix}\sin\varphi_2\\ -\cos\varphi_2\end{pmatrix}$$

$$\dot{\vec{r}}_2 = l_1\dot{\varphi}_1\begin{pmatrix}\cos\varphi_1\\ \sin\varphi_1\end{pmatrix} + l_2\dot{\varphi}_2\begin{pmatrix}\cos\varphi_2\\ \sin\varphi_2\end{pmatrix}$$

The kinetic energy is given by:

$$E_{kin} = \frac{1}{2}m_1\left|\dot{\vec{r}}_1\right|^2 + \frac{1}{2}m_2\left|\dot{\vec{r}}_2\right|^2$$

$$= \frac{1}{2}m_1 l_1^2\dot{\varphi}_1^2 + \frac{1}{2}m_2\{l_1^2\dot{\varphi}_1^2 + 2l_1l_2\dot{\varphi}_1\dot{\varphi}_2(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2) + l_2^2\dot{\varphi}_2^2\}$$

$$= \frac{1}{2}m_1 l_1^2\dot{\varphi}_1^2 + \frac{1}{2}m_2\{l_1^2\dot{\varphi}_1^2 + 2l_1l_2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2) + l_2^2\dot{\varphi}_2^2\}$$

And the potential energy:

$$E_{pot} = m_1gy_1 + m_2gy_2 = -m_1gl_1\cos\varphi_1 - m_2gl_1\cos\varphi_1 - m_2gl_2\cos\varphi_2$$

This gives the Lagrange function:

$$L = E_{kin} - E_{pot}$$

$$= \frac{1}{2}(m_1 + m_2)l_1^2\dot{\varphi}_1^2 + m_2l_1l_2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2) + \frac{1}{2}m_2l_2^2\dot{\varphi}_2^2$$

$$+g(m_1 + m_2)l_1\cos\varphi_1 + gm_2l_2\cos\varphi_2$$

The Lagrange equation for the first generalised coordinate is:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}_1} = \frac{\partial L}{\partial\varphi_1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}_1} = \frac{d}{dt}\{l_1^2(m_1 + m_2)\dot{\varphi}_1 + m_2l_1l_2\cos(\varphi_1 - \varphi_2)\cdot\dot{\varphi}_2\}$$

$$= l_1^2(m_1 + m_2)\ddot{\varphi}_1 + m_2l_1l_2\cos(\varphi_1 - \varphi_2)\cdot\ddot{\varphi}_2 - m_2l_1l_2\sin(\varphi_1 - \varphi_2)\cdot\dot{\varphi}_2(\dot{\varphi}_1 - \dot{\varphi}_2)$$

$$\frac{\partial L}{\partial\varphi_1} = -m_2l_1l_2\dot{\varphi}_1\dot{\varphi}_2\sin(\varphi_1 - \varphi_2) - g(m_1 + m_2)l_1\sin\varphi_1$$

We insert these expressions into the Lagrange equation and bring everything to the left. It can also be reduced by l_1. This provides:

$$l_1(m_1 + m_2)\ddot{\varphi}_1 + m_2l_2\cos(\varphi_1 - \varphi_2)\cdot\ddot{\varphi}_2$$

$$-m_2l_2\sin(\varphi_1 - \varphi_2)\cdot\dot{\varphi}_2(\dot{\varphi}_1 - \dot{\varphi}_2) + m_2l_2\dot{\varphi}_1\dot{\varphi}_2\sin(\varphi_1 - \varphi_2) + g(m_1 + m_2)\sin\varphi_1 = 0$$

If we multiply out the third summand, a part cancels out with the fourth summand, and it remains:

$$l_1(m_1 + m_2)\ddot{\varphi}_1 + m_2l_2\{\cos(\varphi_1 - \varphi_2)\cdot\ddot{\varphi}_2 + \sin(\varphi_1 - \varphi_2)\cdot\dot{\varphi}_2^2\} + g(m_1 + m_2)\sin\varphi_1 = 0$$

We set:

$$\Delta\varphi := \varphi_1 - \varphi_2, \mu := \frac{m_2}{m_1 + m_2}$$

If $m_1 > 0$ is $\mu < 1$.

This gives us:

$$(1)\quad l_1\ddot{\varphi}_1 + \mu l_2\cos\Delta\varphi\cdot\ddot{\varphi}_2 + \mu l_2\sin\Delta\varphi\cdot\dot{\varphi}_2^2 + g\sin\varphi_1 = 0$$

The Lagrange equation for the second generalised coordinate is analogous:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_2} = \frac{\partial L}{\partial \varphi_2}$$

A corresponding calculation, which we are happy to leave to the reader as an exercise, provides the equation:

$$(2) \qquad l_2\ddot{\varphi}_2 + l_1\cos\Delta\varphi \cdot \ddot{\varphi}_1 - l_1 \sin\Delta\varphi \cdot \dot{\varphi}_1^2 + g\sin\varphi_2 = 0$$

Both equations have a high degree of symmetry: if the indices $1 \leftrightarrow 2$ are swapped, one almost merge into the other. They then only differ by a factor of μ and $\Delta\varphi$ becomes $-\Delta\varphi$.

If we multiply the second equation (2) by $\mu\cos\Delta\varphi$, we obtain the equations:

$$\begin{cases} l_1\ddot{\varphi}_1 + \mu l_2\cos\Delta\varphi \cdot \ddot{\varphi}_2 + \mu l_2\sin\Delta\varphi \cdot \dot{\varphi}_2^2 + g\sin\varphi_1 = 0 \\ l_1\mu\cos^2\Delta\varphi \cdot \ddot{\varphi}_1 + \mu l_2\cos\Delta\varphi \cdot \ddot{\varphi}_2 - \mu l_1\sin\Delta\varphi\cos\Delta\varphi \cdot \dot{\varphi}_1^2 + \mu\cos\Delta\varphi \cdot g\sin\varphi_2 = 0 \end{cases}$$

Now we subtract the lower equation from the upper equation and obtain when we solve for $\ddot{\varphi}_1$:

$$(3) \quad \ddot{\varphi}_1 = \frac{-\mu\sin\Delta\varphi \cdot (l_2\dot{\varphi}_2^2 + l_1\cos\Delta\varphi \cdot \dot{\varphi}_1^2) + g(\mu\cos\Delta\varphi \cdot \sin\varphi_2 - \sin\varphi_1)}{l_1(1 - \mu\cos^2\Delta\varphi)}$$

If we multiply the first equation (1) by $\cos\Delta\varphi$, we get the equations:

$$\begin{cases} l_1\cos\Delta\varphi \cdot \ddot{\varphi}_1 + \mu l_2\cos^2\Delta\varphi \cdot \ddot{\varphi}_2 + \mu l_2\sin\Delta\varphi \cdot \cos\Delta\varphi \cdot \dot{\varphi}_2^2 + g\cos\Delta\varphi \cdot \sin\varphi_1 = 0 \\ l_1\cos\Delta\varphi \cdot \ddot{\varphi}_1 + l_2\ddot{\varphi}_2 - l_1\sin\Delta\varphi \cdot \dot{\varphi}_1^2 + g\sin\varphi_2 = 0 \end{cases}$$

Again, we subtract the lower equation from the upper one:

$$l_2(\mu\cos^2\Delta\varphi - 1)\ddot{\varphi}_2 + \mu l_2\sin\Delta\varphi \cdot \cos\Delta\varphi \cdot \dot{\varphi}_2^2 + g\cos\Delta\varphi \cdot \sin\varphi_1 + l_1\sin\Delta\varphi \cdot \dot{\varphi}_1^2 - g\sin\varphi_2 = 0$$

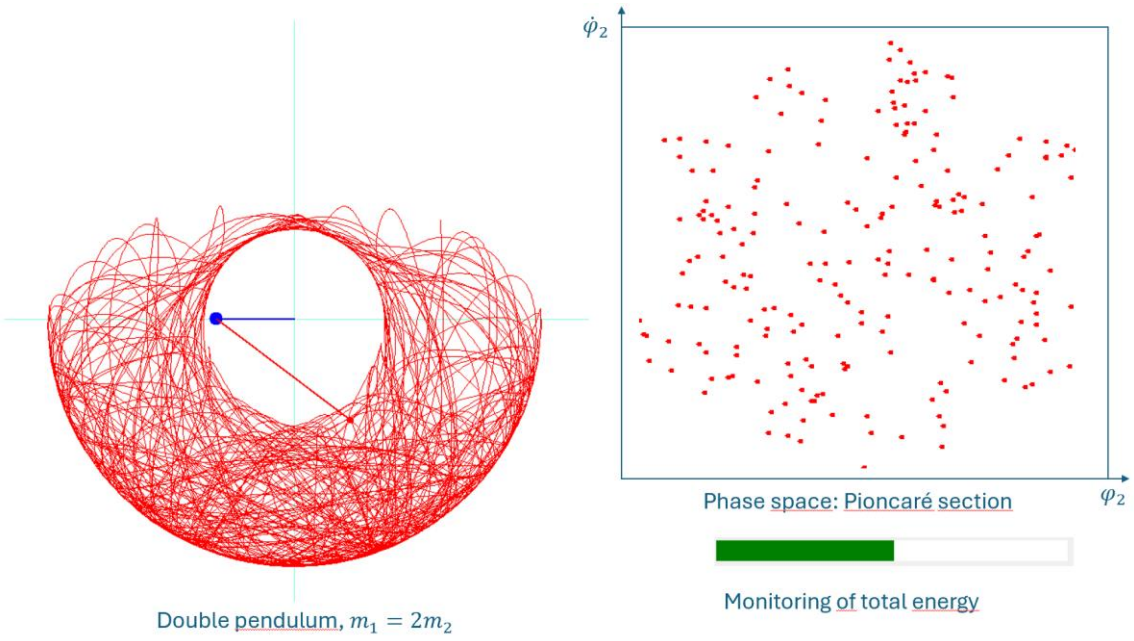
Solving for $\ddot{\varphi}_2$ yields:

$$(4) \quad \ddot{\varphi}_2 = \frac{\sin\Delta\varphi \cdot (l_1\dot{\varphi}_1^2 + \mu l_2\cos\Delta\varphi \cdot \dot{\varphi}_2^2) + g(\cos\Delta\varphi \cdot \sin\varphi_1 - \sin\varphi_2)}{l_2(1 - \mu\cos^2\Delta\varphi)}$$

If $m_1 > 0$ is $\mu < 1$ and therefore the denominator is greater than 0 in both cases.

Equations (3) and (4) describe the double pendulum but have no analytical solution. In the simulator, these equations are the starting point for the numerical approximation of the required movement using the fourth-order Runge Kutta method. Details of this implementation can be found in the mathematical documentation for the simulator.

Now we will show some experiments with the simulator.



Double pendulum

In the simulator, you can specify the length of the pendulum arms, the ratio of the masses and the starting points of the individual pendulums. Other parameters, such as the step length of the Runge Kutta method, can also be specified.

During the movement of the pendulums, the total energy of the system is continuously calculated, monitored and displayed as a bar. It is green as long as the total energy at this step deviates by less than 1% from the starting energy. If the zero level of the potential energy is normalised to $y = -1$, this is

$$E_{tot} = E_{pot} = g\{(1 - l_1 \cos\varphi_1)(m_1 + m_2) - m_2 l_2 \cos\varphi_2\}$$

$$\varphi_i = u_i, \dot{\varphi}_i = v_i, i \in \{1, 2\}$$

The total energy for each position of the pendulum is:

$$E_{tot} = \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 \{l_1^2 \dot{\varphi}_1^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + l_2^2 \dot{\varphi}_2^2\} + g\{(1 - l_1 \cos\varphi_1)(m_1 + m_2) - m_2 l_2 \cos\varphi_2\}$$

In the school project *The oval billiard table and periodic orbits* you learn about phase space. We will therefore refrain from a detailed explanation here. There are various ways of visualising the phase space:

- The states of the pendulums $(\varphi_1, \dot{\varphi}_1)$ and $(\varphi_2, \dot{\varphi}_2)$ can be displayed separately in the respective pendulum colour.
- Enter (φ_1, φ_2) as a point on the torus $[-\pi, \pi[\times [-\pi, \pi[$ and $(\dot{\varphi}_1, \dot{\varphi}_2)$ as a tangent at this point
- Make a Poincaré section (after Henri Poincaré 1854 - 1912). This is explained below.

In the double pendulum, the phase space is four-dimensional, and the coordinates of a point are labelled $(\varphi_1, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2)$. During the movement of the double pendulum, this point moves in phase space along a trajectory, i.e. a curve in phase space.

If you now place a hyperplane (this is an (n-1) dimensional subspace) in the phase space, you can analyse the points at which the trajectory of the path intersects this hyperplane.

In our case, we want to define the hyperplane by the condition $\varphi_1 = 0$. We therefore consider the points in the phase space at the time when the upper pendulum passes straight through the *y-axis*. This gives us a *discrete* dynamic system for the double pendulum. A point of intersection with the hyperplane is mapped to the point at which the trajectory intersects the hyperplane on the next pass:

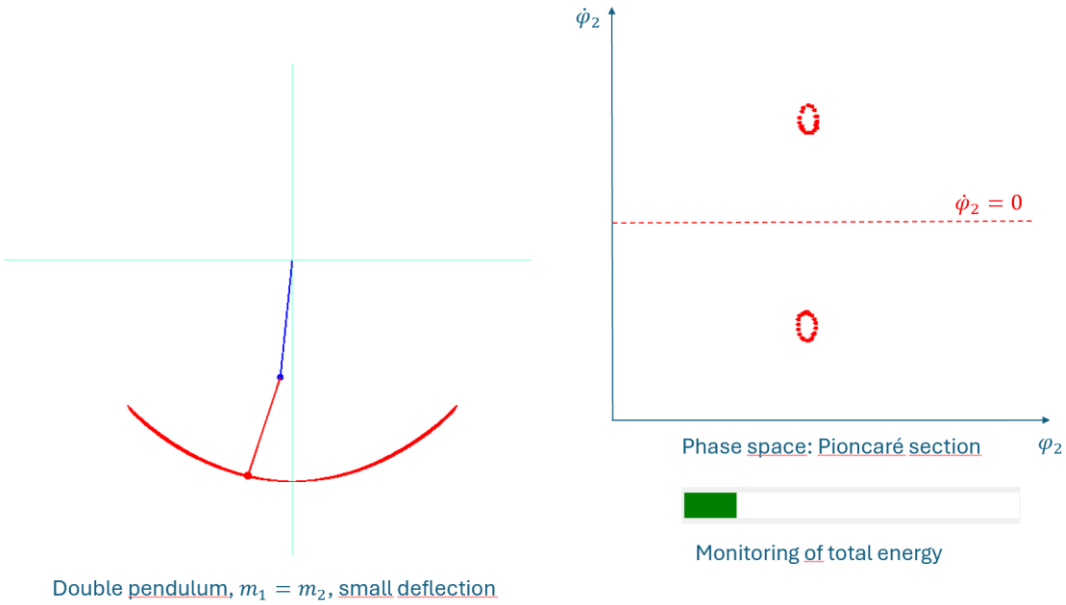
$$(0, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2)_n \mapsto (0, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2)_{n+1}$$

The total energy along a trajectory is constant. For $\varphi_1 = 0$ it is given by:

$$E_{tot} = \frac{m_1 + m_2}{2} l_1^2 \dot{\varphi}_1^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos\varphi_2 + \frac{m_2}{2} l_2^2 \dot{\varphi}_2^2 + g((1 - l_1)(m_1 + m_2) - m_2 l_2 \cos\varphi_2)$$

Now we project the hyperplane $(\dot{\varphi}_1, \varphi_2, \dot{\varphi}_2)$ onto $(\varphi_2, \dot{\varphi}_2)$, i.e. we only enter the coordinates of the lower pendulum into the phase space. If $(\varphi_2, \dot{\varphi}_2)$ is given, then $\dot{\varphi}_1$ is also determined by the above energy equation. This means that $\dot{\varphi}_1$ is not an independent parameter, as the energy is constant.

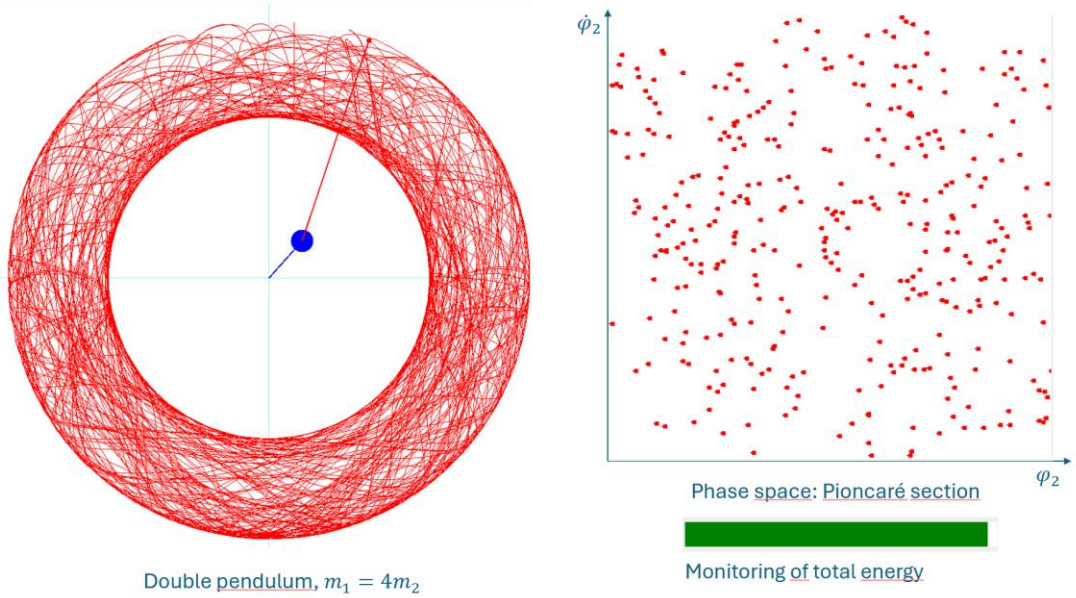
Here is another experiment. The deflection at the start is small and $\varphi_1 \approx \varphi_2$,



Double pendulum with small deflection

the double pendulum then oscillates approximately like a normal thread pendulum. In phase space, φ_2 and $\dot{\varphi}_2$ vary in a small range. Depending on whether the zero crossing in the Poincaré section occurs from the right or from the left, $\dot{\varphi}_2$ is less or greater than zero, but symmetrical to the zero axis. This is a consequence of the pendulum symmetry with respect to the *y-axis*.

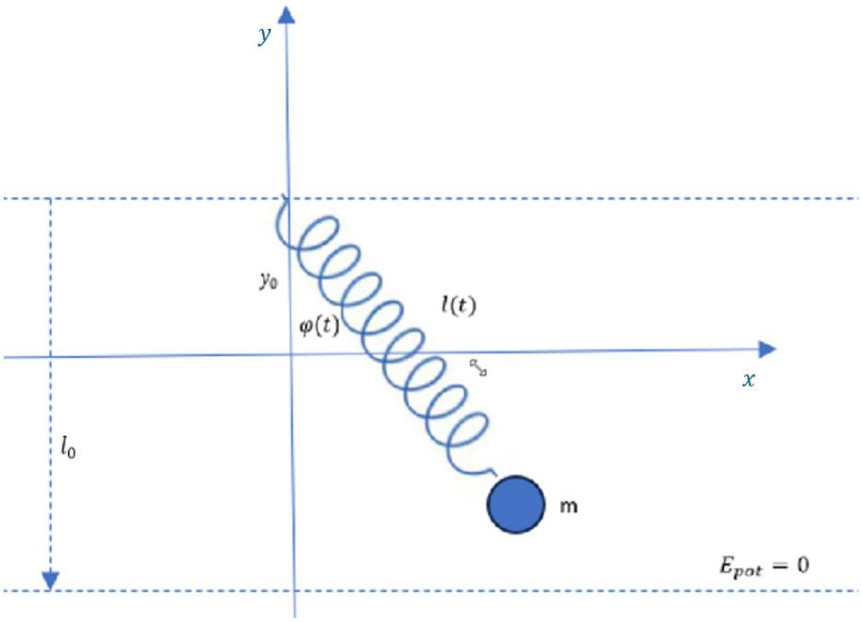
If the total energy is sufficiently large and the pendulum has strong deflections, then the movement appears chaotic. The phase space with the Poincaré section shows a random-looking pattern, as shown in another experiment with the simulator.



Another experiment with a high total energy

9. The oscillating spring pendulum

In this section we look at a spring pendulum which can also oscillate back and forth like a thread pendulum.



Oscillating spring pendulum

A spring is attached to the point $(0, y_0)$ of the coordinate system, with a mass *m* hanging from its other end. This mass also swings back and forth. The deflection angle $\varphi(t)$ and the spring length $l(t) > 0$ depend on the time t.

Let l_0 be the length of the spring when the spring force and the weight of the mass m are in equilibrium. Let the zero level of potential energy in relation to gravity be at the height $y = y_0 - l_0$ or at $\varphi = 0, l = l_0$. We denote the spring constant by *D*.

The position vector of the mass *m* is:

$$\vec{r}(t) = l(t)\begin{pmatrix}\sin\varphi(t)\\ -\cos\varphi(t)\end{pmatrix} + \begin{pmatrix}0\\ y_0\end{pmatrix}$$

$$\dot{\vec{r}} = \dot{l}\begin{pmatrix}\sin\varphi\\ -\cos\varphi\end{pmatrix} + l\dot{\varphi}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$$

$$\left|\dot{\vec{r}}\right|^2 = \dot{l}^2 + 2l\dot{l}\dot{\varphi}[\sin\varphi\cos\varphi - \cos\varphi\sin\varphi] + l^2\dot{\varphi}^2$$

$$E_{kin} = \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\varphi}^2\right)$$

Then the following applies to the potential energy (spring energy plus gravitational energy):

$$E_{pot} = \frac{1}{2}D(l - l_0)^2 + mg(l_0 - l\cos\varphi)$$

This gives us the Lagrange function:

$$L = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\varphi}^2\right) - \frac{D}{2}(l - l_0)^2 - mg(l_0 - l\cos\varphi)$$

For the first coordinate $l(t)$ applies:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{l}} = \frac{d}{dt}\left(m\dot{l}\right) = m\ddot{l}$$

$$\frac{\partial L}{\partial l} = ml\dot{\varphi}^2 - D(l - l_0) + mg\cos\varphi$$

This provides the Lagrange equation:

$$\ddot{l} = l\dot{\varphi}^2 - \frac{D}{m}(l - l_0) + g\cos\varphi$$

For the second coordinate $\varphi(t)$ we have:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{d}{dt}(ml^2\dot{\varphi}) = 2ml\dot{l}\dot{\varphi} + ml^2\ddot{\varphi}$$

$$\frac{\partial L}{\partial \varphi} = -mgl\sin\varphi$$

This gives the Lagrange equation:

$$ml^2\ddot{\varphi} = -2ml\dot{l}\dot{\varphi} - mgl\sin\varphi$$

$$\ddot{\varphi} = -(2\dot{\varphi}\dot{l} + g\sin\varphi)/l$$

Only the ratio $\omega^2 \coloneqq \frac{D}{m}$ is relevant for the movement. This gives us the equations of motion:

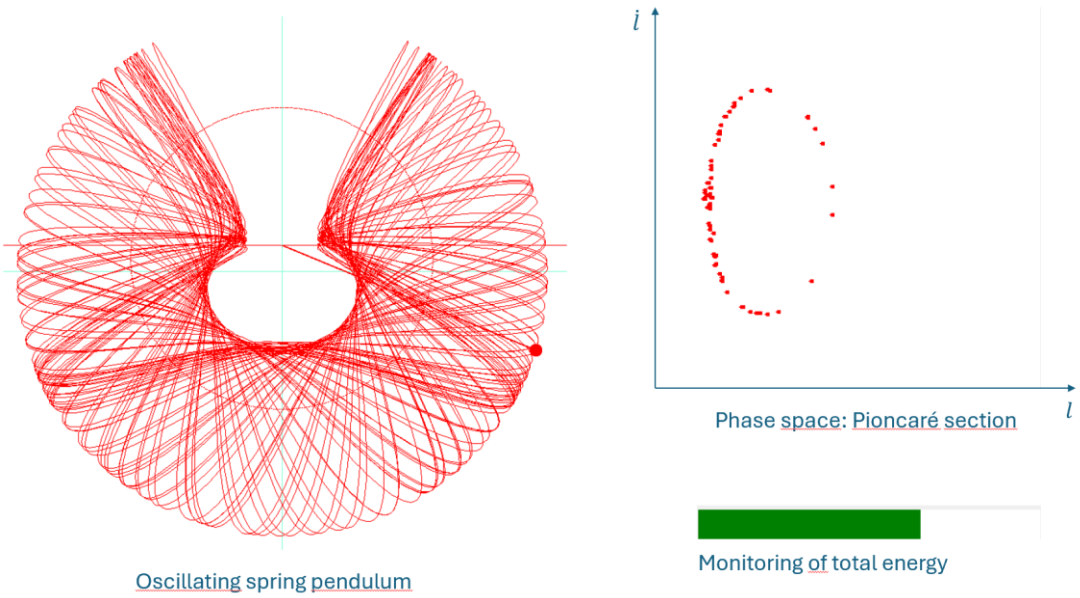
$$\begin{cases}\ddot{l} = l\dot{\varphi}^2 - \omega^2(l - l_0) + g\cos\varphi\\ \ddot{\varphi} = -\dfrac{2\dot{\varphi}\dot{l} + g\sin\varphi}{l}\end{cases}$$

To convert the equations into a system of first-order differential equations, we introduce the following variables:

$$u_1 = l, v_1 = \dot{l}, u_2 = \varphi, v_2 = \dot{\varphi}$$

This gives us the system of equations for the Runge Kutta method:

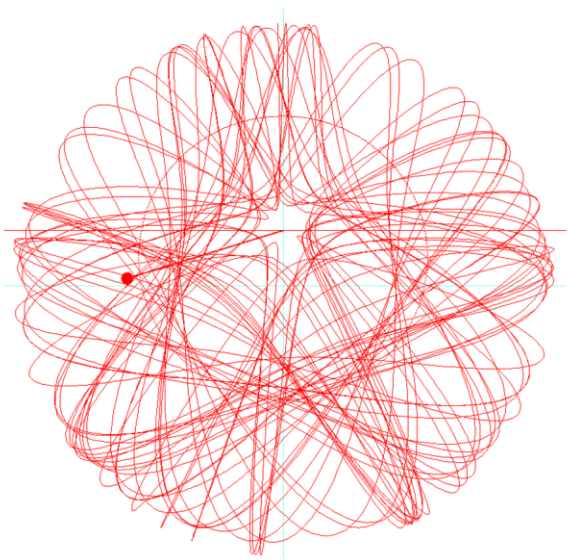
$$\begin{cases} \dot{u}_1 = v_1 \\ \dot{v}_1 = u_1 v_2^2 - \omega^2(u_1 - l_0) + g\cos u_2 \\ \dot{u}_2 = v_2 \\ \dot{v}_2 = -\dfrac{2v_2 v_1 + g\sin u_2}{u_1} \end{cases}$$



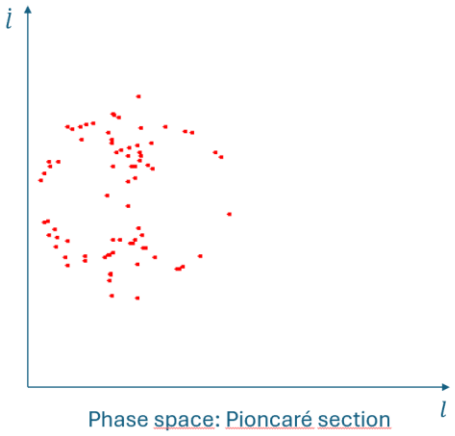
Oscillating spring pendulum

Since u_1, i.e. the length of the spring pendulum, is in the denominator of the last equation, $\dot{v}_2$ can become very large if the pendulum swings in the direction of the point $(0, y_0)$ and comes very close to it. In the case of a physical pendulum, the spring, whose length has a minimum dimension, blocks this effect. In the implementation, we therefore assume that the length of the spring pendulum should not fall below a value l_{min}. This value is defined during implementation.

If $u_1 < l_{min}$ is now changed during the iteration, this is caught and set in place of the Runge Kutta formulae $u_1 = l_{min}, v_1 = -v_1$. The total energy does not change significantly at most. Visually, this looks as if the mass *m* is reflected from the sphere with radius l_{min} and centre $(0, y_0)$ without friction.



Oscillating spring pendulum



Phase space: Pioncaré section

Monitoring of total energy

Here, the mass comes very close to the suspension point of the pendulum and is then reflected

For any angle φ there is a starting point of the pendulum at which the spring force just compensates for the gravitational force. In an exercise, the reader can show that this is the case if

$$l(\varphi) = l_0 + \frac{g\cos\varphi}{\omega^2}$$

The corresponding locus line is a cardioid and is drawn in red in the simulator, depending on the starting position of the pendulum.

If we analyse the total energy of the system, where the zero level of the gravitational potential energy is again at the height $y = y_0 - l_0$, then the following applies to the energy:

$$E_{tot} = E_{kin} + E_{Feder} + E_{pot} = \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\varphi}^2\right) + \frac{D}{2}(l - l_0)^2 + mg(l_0 - l\cos\varphi) \geq$$

$$\frac{\omega^2}{2}(l - l_0)^2 + g(l_0 - l) =: \tilde{E}(l)$$

For which$l$ is the total energy minimal? $\tilde{E}(l)$ is an upwardly open parabola and we are looking for its minimum:

$$\frac{d\tilde{E}}{dl} = \omega^2(l - l_0) - g = 0$$

The minimum is assumed to be just at the lower equilibrium state: $l_{min} = l_0 + \frac{g}{\omega^2}$ and thus applies:

$$E_{tot} \geq -\frac{g^2}{2\omega^2}, \forall l, \varphi$$

10. The shaked pendulum

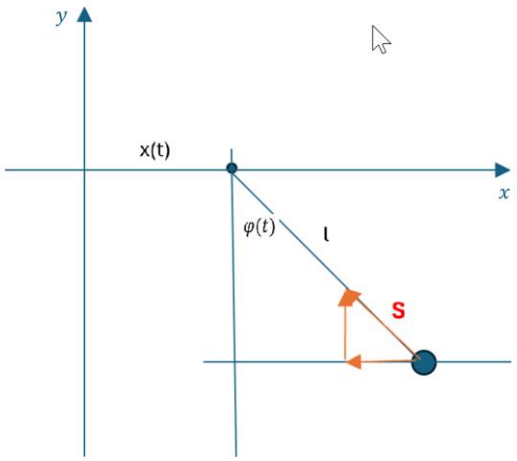
We consider a vibrator that moves back and forth horizontally in the form of a cosine oscillation. At the time t it is at the position

$$x(t) = A\cos(\omega t)$$

A is the amplitude of its oscillation and ω is its frequency.

A pendulum with a rigid and massless axis is attached to its end point. This pendulum can swing freely like a thread pendulum and is stimulated to swing by the vibrator.

Since in this case an external force acts on the suspension point of the pendulum, we cannot use the Lagrangian formalism in its previous form. Instead, we use Newton's law: force = mass x acceleration.



Shaking pendulum

Let the length of the pendulum axis be *l*. At time *t*, it forms the angle $\varphi(t)$ with the vertical *y*-axis. Then the pendulum mass is at the location at time *t*:

$$\vec{r}(t) = \begin{pmatrix} A\cos(\omega t) + l\sin\varphi(t) \\ -l\cos\varphi(t) \end{pmatrix}$$

And its velocity

$$\dot{r}(t) = \begin{pmatrix} -A\omega\sin(\omega t) + l\dot{\varphi}\cos\varphi \\ l\dot{\varphi}\sin\varphi \end{pmatrix}$$

The acceleration of the pendulum mass is then

$$\ddot{\vec{r}}(t) = \begin{pmatrix} -A\omega^2\cos(\omega t) + l\ddot{\varphi}\cos\varphi - l\dot{\varphi}^2\sin\varphi \\ l\ddot{\varphi}\sin\varphi + l\dot{\varphi}^2\cos\varphi \end{pmatrix}$$

The force $\vec{S}$, which acts on the pendulum mass *m*, is directed towards the suspension point. If it is broken down into x and y directions, it is:

$$\vec{S} = \begin{pmatrix} -S\cdot\sin\varphi \\ S\cdot\cos\varphi \end{pmatrix}$$

Where *S* is its magnitude. Together with the force of gravity $-mg$ this gives the two equations according to Newton:

$$\begin{cases} -\frac{S}{m}\sin\varphi = -A\omega^2\cos(\omega t) + l\ddot{\varphi}\cos\varphi - l\dot{\varphi}^2\sin\varphi \\ \frac{S}{m}\cos\varphi - mg = l\ddot{\varphi}\sin\varphi + l\dot{\varphi}^2\cos\varphi \end{cases}$$

We multiply the upper equation by $\cos\varphi$, the lower equation by $\sin\varphi$ and add the equations. Then $\frac{S}{m}$ is omitted and we obtain the equation after a short transformation:

$$\ddot{\varphi} = \frac{1}{l}(A\omega^2 \cos(\omega t)\cos\varphi - g\sin\varphi)$$

This equation determines the movement of the pendulum.

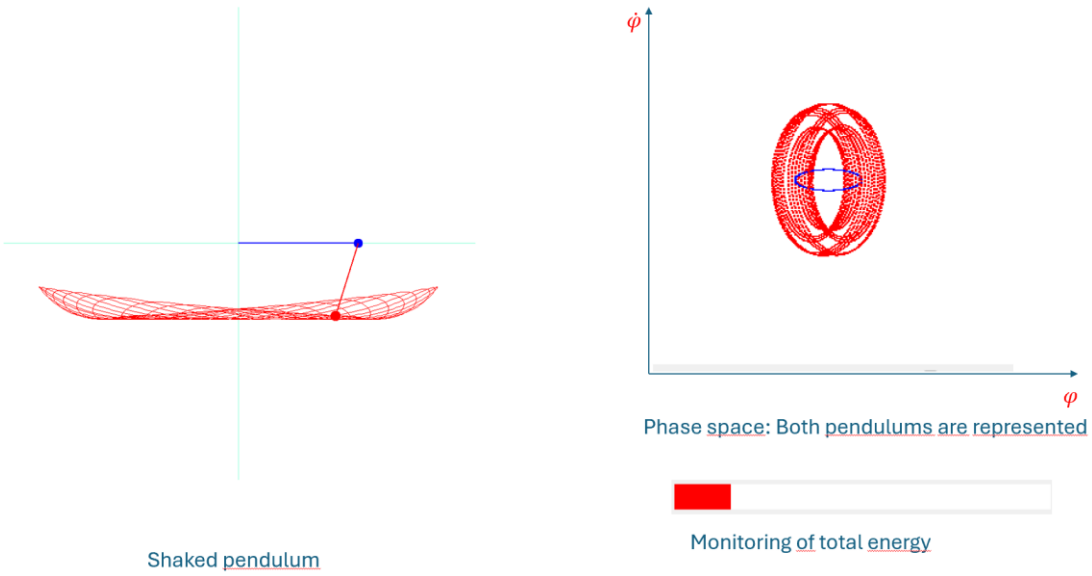
We convert it into a first-order system of differential equations by setting:

$$u = \varphi, v = \dot{\varphi}$$

And obtain:

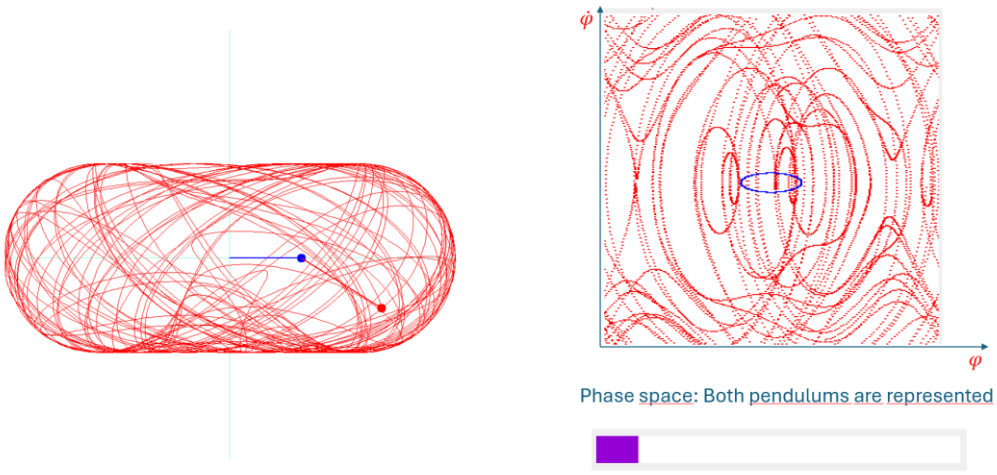
$$\begin{cases} \dot{u} = v \\ \dot{v} = \frac{1}{l}(A\omega^2 \cos(\omega t)\, cosu - gsinu) \end{cases}$$

The formulae for the explicit Runge Kutta method in the implementation can be found in the mathematical documentation for the simulator. We will now carry out some experiments.



Shaked pendulum at low shaking frequency

If the shaking pendulum has a significantly lower frequency than the excited pendulum, the latter will only oscillate moderately. In phase space, the blue curve of the shaking pendulum shows an ellipse. The excited pendulum is similar to a group of ellipses. The total energy is sometimes exceeded and sometimes fallen short of. In the first case the bar is red, in the second case it is purple.

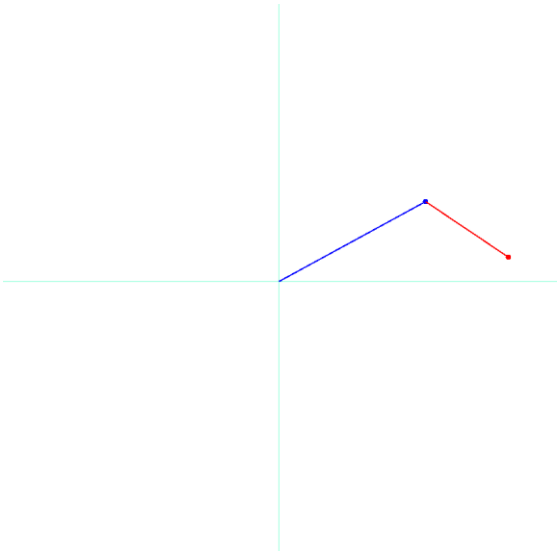


Here the frequency of the shaking pendulum is higher

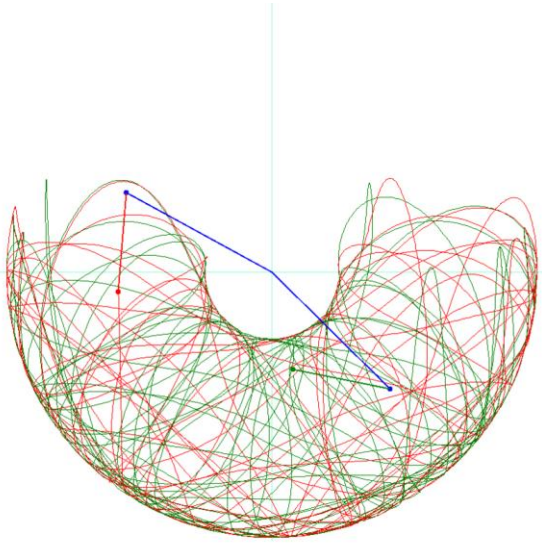
11. Shadow pendulum

To analyse the sensitivity, it is possible to add a pendulum whose starting position deviates only minimally from the original pendulum: so little that it is not visible in the diagram. We call this pendulum a "shadow pendulum".

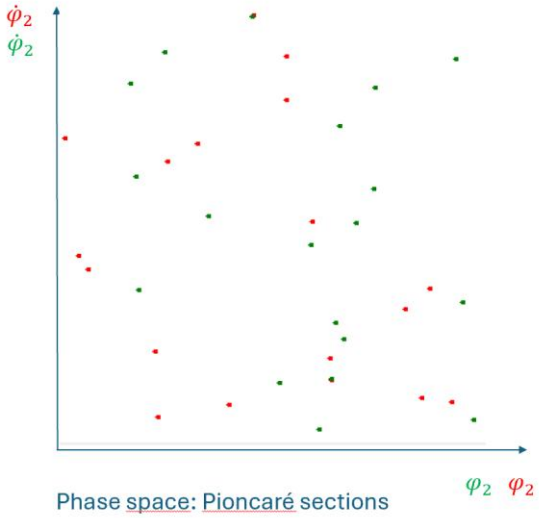
In certain cases, usually with small deflections and low energy, the movement of the shadow pendulum remains very close to the original pendulum. With larger deflections and greater energy, the shadow pendulum will behave completely differently to the original pendulum after a short time. Here are some examples.



Starting position of double pendulum and shadow pendulum (hidden, as practically identical)

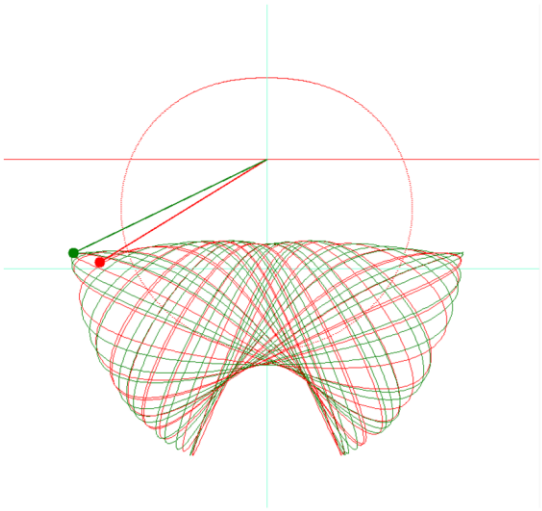


Double pendulum with shadow pendulum (green)

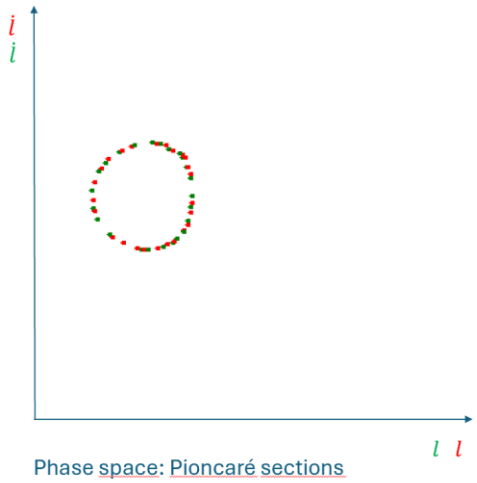


Phase space: Pioncaré sections

After a few oscillations, the pendulum (red) and shadow pendulum (green) diverge



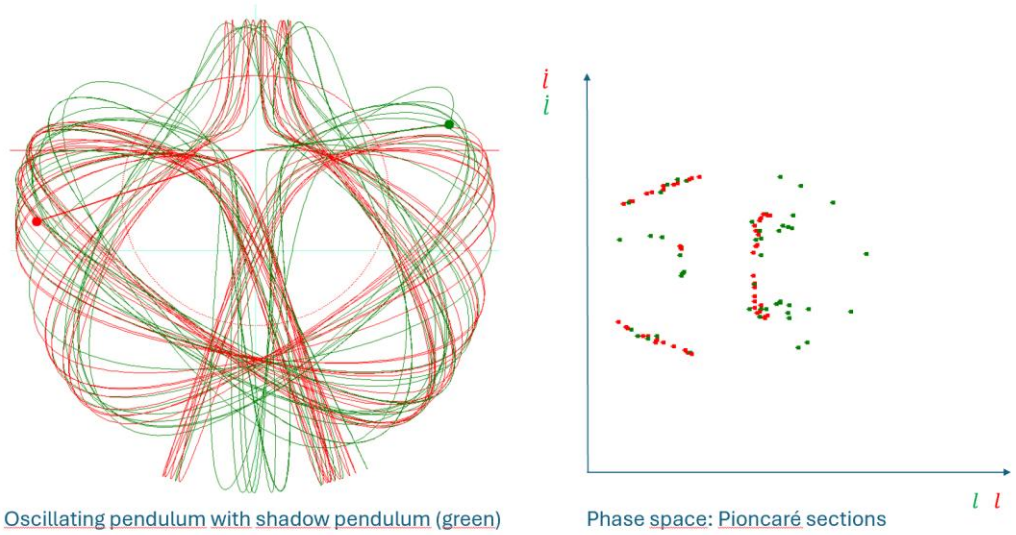
Oscillating pendulum with shadow pendulum (green)



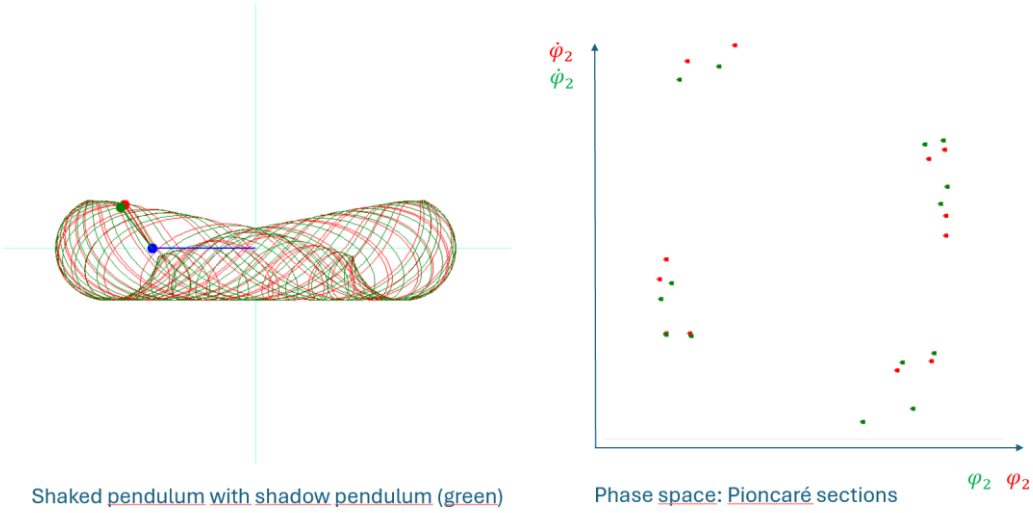
Phase space: Pioncaré sections

Shadow pendulum with oscillating spring pendulum

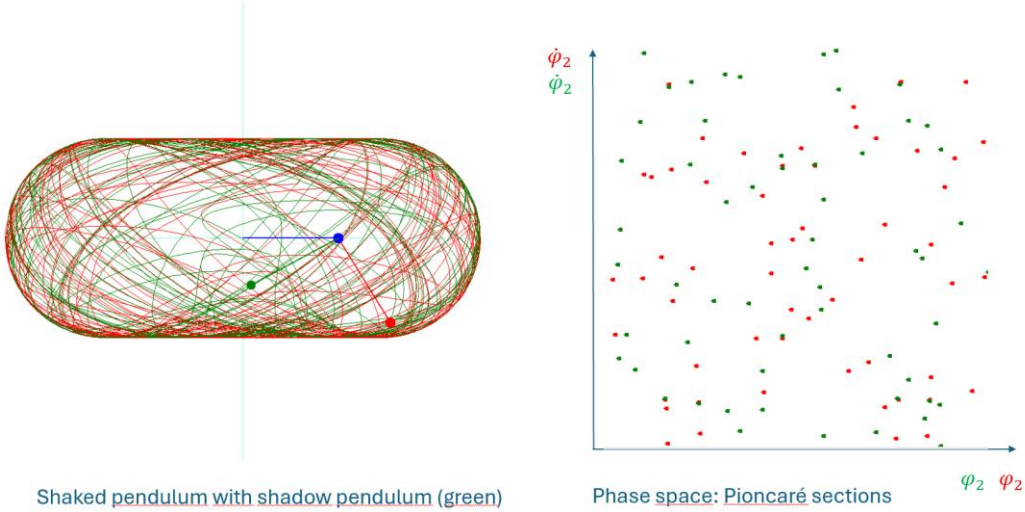
If the oscillations are small enough that no "reflections" occur at the suspension point of the spring, the pendulum and shadow pendulum remain close together. Their images also look similar in phase space. However, if reflections occur, this changes, as the following example shows, because tiny differences in position can play a decisive role in reflections.



Shadow pendulum with an oscillating spring pendulum



Shaked pendulum and shadow pendulum at low vibration frequency



At a high vibration frequency, the shadow pendulum soon goes its own way

12. Exercise examples

1. Consider the differential equation $y' = y$ with the initial condition $y(0) = 1$. Explicitly determine the recursion formulae for the explicit and implicit Euler method and for the centre point rule.
2. Create a table in Excel in which a list of the iterated values (t_n, y_n) is created for the spring pendulum with $\omega^2 = 1$ and the starting position $(t_1, y_1) = (0,1)$. The step size h can be freely selected. Use the various numerical methods for the calculation.
3. Is it possible for the double pendulum to oscillate like a simple thread pendulum at a suitable starting position? This means that the following applies with the initial condition $\varphi_2(0) = 0, \dot{\varphi}_2(0) = 0$: $\varphi_2(t) = 0, \forall t$. Tip: Investigate $\ddot{\varphi}_2$ with this initial condition.
4. For the oscillating spring pendulum, investigate for any angle φ for which values of $l(\varphi)$ the spring just cancels the gravitational force.
5. For the oscillating spring pendulum, there is a starting point $l(\varphi)$ of the pendulum for any angle φ at which the spring force just cancels out the gravitational force. Determine $l(\varphi)$.
6. Investigate whether the physically obvious (stable and unstable) equilibrium positions of the double pendulum and oscillating spring pendulum can also be calculated. To do this, consider the respective first-order differential equation system that describes the pendulum motion. In an equilibrium position, the following applies with the designations analogous to section 3: $\dot{u}_1 = 0, \dot{u}_2 = 0 \Rightarrow v_1 = v_2 = 0$. Furthermore $\dot{v}_1 = 0, \dot{v}_2 = 0$.
7. Examine a shaking pendulum in which the shaking pendulum swings up and down in a vertical direction. A i) thread pendulum ii) vertical spring pendulum is attached to this spring pendulum.
   a) Calculate the equation of motion for the pendulum in case i) and ii)
   b) Use suitable substitution to convert the respective differential equation into a system of first-order differential equations
   c) Use this to create the iteration equations for the Runge Kutta method

Further reading

[1] Bastian von Harrach: Numerics of Differential Equations, Lecture Notes, Goethe University Frankfurt am Main, Institute of Mathematics, 2022

[2] Robin Santra: Introduction to the Lagrange and Hamiltonian formalism, Springer Spektrum 2022

[3] Stefan Frei: Numerical Mathematics, Lecture Notes at the University of Konstanz, WS 2021,2022

[4] Urs Kirchgraber: Chaotisches Verhalten in einfachen Systemen, Berichte über Mathematik und Unterricht, ETHZ, 1992